Appendix D

Integration of the pendulum equation

The first integral of the Eq. [2.19] reads as:

$$\mathcal{L}_1 - \dot{\theta}\frac{\partial \mathcal{L}_1}{\partial \dot{\theta}} = -\lambda \dot{\theta}^2(t)/\mu^2 + \sin^2(\theta(t))/T = C, \tag{D.1}$$

where $C = -\lambda V^2(0)$ is a constant of integration and $V(0)$ corresponds to the initial amplitude of the control field. After the second integration we obtain:

$$\sqrt{\frac{-\lambda}{C\mu^2}} \int_0^{\theta} \frac{d\theta'}{\sqrt{1 - \frac{1}{CT}\sin^2(\theta')}} = t, \tag{D.2}$$

and, finally

$$\theta = am(\mu V(0)t, \frac{1}{CT}). \tag{D.3}$$

Here am is the amplitude of the Jacobian elliptic function. The shape of the optimal control field $V(t)$ is obtained through differentiation:

$$V(t) \equiv \dot{\theta}(t)/\mu = V(0)\, dn(\mu V(0)t, -\frac{1}{\lambda V^2(0)T}), \tag{D.4}$$

where dn is the Jacobian elliptic function. With a help of the condition on the pulse energy E_0 (see Eq. [2.6]) we can determine the Lagrange multiplier λ.