On the late-time asymptotics of the non-minimally coupled Einstein-scalar field system

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Summary

The aim of this thesis is to study the late-time asymptotics of cosmological models with accelerated expansion in which the acceleration is caused by a non-minimally coupled scalar field. While the dynamics of such models is well-understood in the case of spatially homogeneous and isotropic solutions, only a few mathematical results exist in cases with less or no symmetry at all. The method used here, originally developed by Rendall, is based on formal power series solutions and requires a positive lower bound on the potential of the scalar field.

In a first step, after making precise the notion of generalized formal power series solutions and the sense in which they are supposed to solve the Einstein-scalar field system, their existence and uniqueness is proven inductively and some of their basic properties are established. In a second step it is shown that, given a solution of the Einstein-scalar field system which exists globally towards the future with respect to a Gaussian time coordinate and satisfies certain decay conditions, there exists precisely one formal power series solution of the type considered that is asymptotic to the given solution. The fact that there actually exists a large class of solutions of the Einstein-scalar field system fulfilling the above hypotheses is proven in a final step. For this, the system is reduced to first order and, using a formal series solution, put into Fuchsian form. This system can then be solved in the analytic setting to yield a solution that exists globally towards the future with respect to a Gaussian time coordinate and with its late-time asymptotics given by the formal power series.

In addition, spatially homogeneous scalar field models are considered, where the extra symmetry allows more general potentials without a positive lower bound and the presence of matter to be treated. By establishing conditions for accelerated expansion and isotropization for models with a direct coupling to the matter, statements about the late-time behaviour of curvature-coupled scalar field models with exponential potentials can be deduced using a conformal transformation. It turns out that any arbitrarily small positive coupling of the field to the scalar curvature of space-time results in a late-time asymptotics which, in the minimally coupled case, can only be expected in the presence of a positive cosmological constant.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Langzeitasymptotik von Lösungen kosmologischer Modelle mit beschleunigter Expansion, im Besonderen solcher, bei denen ein nichtminimal gekoppeltes Skalarfeld für die Beschleunigung ursächlich ist. Während die in derartigen Modellen auftretende Dynamik in Fällen hoher Symmetrie, das heißt im Speziellen von räumlich homogenen und isotropen Lösungen, ausreichend gut verstanden ist, gibt es in allgemeineren Situationen nur wenige mathematische Resultate. Mit Hilfe einer von Rendall zur Behandlung der Asymptotik von Lösungen mit positiver kosmologischer Konstante entwickelten Methode soll dazu ein Beitrag geleistet werden.

Ausgangspunkt dafür sind verallgemeinerte formale Potenzreihenlösungen der Einstein-Skalarfeldgleichungen unter Voraussetzung einer positiven unteren Schranke an das Potential des Skalarfeldes. Nach einer geeigneten Präzisierung des Begriffs dieser formalen Reihen und des Sinns, in welchem solche Reihen die Einstein-Skalarfeldgleichungen lösen sollen, werden Existenz und Eindeutigkeit formaler Lösungen induktiv bewiesen und einige ihrer grundlegenden Eigenschaften festgestellt.

In einem nächsten Schritt wird gezeigt, dass es zu einer gegebenen tatsächlichen Lösung dieser Gleichungen unter geeigneten Voraussetzungen eine formale Lösung gibt, welche dazu asymptotisch ist. Genauer wird, unter Annahme der Existenz einer Gauß'schen Zeitkoordinate global in die Zukunft sowie minimaler Abfallbedingungen entlang dieser, nachgewiesen, dass es genau eine asymptotische formale Reihe von der betrachteten Form gibt und diese die Einstein-Skalarfeldgleichungen formal löst. Aufgrund der Tatsache, dass damit eine vollständige und darüber hinaus beliebig oft differenzierbare asymptotische Entwicklung der gegebenen Lösung für späte Zeiten vorliegt, kann ihr Langzeitverhalten ausschließlich anhand der verallgemeinerten Potenzreihe diskutiert werden.

Letztlich wird festgestellt, dass in der Tat eine große Klasse von Lösungen mit der betrachteten Asymptotik existiert. Dazu wird das Einstein-Skalarfeldsystem auf erste Ordnung reduziert und vermöge der formalen Lösungen in Fuchs'sche Form gebracht. Unter Anwendung eines Satzes über die Existenz und Eindeutigkeit von Lösungen Fuchs'scher Systeme werden Lösungen der Einstein-Skalarfeldgleichungen erhalten, welche eine Gauß'sche Zeitkoordinate zulassen, bezüglich dieser global in die Zukunft existieren und eine vorgeschriebene Langzeitasymptotik besitzen. Das vorgeschriebene Langzeitverhalten spielt dabei die Rolle von "Anfangsdaten" auf dem konformen Rand der zu erhaltenden Raumzeit und ist, ebenso wie das Potential, als analytisch vorauszusetzten.

Darüber hinausgehend werden auch räumlich homogene Modelle betrachtet, in denen es die verlangte Symmetrie gestattet, allgemeinere Potentiale, namentlich solche ohne positive untere Schranke, im Beisein zusätzlicher Materiefelder zu behandeln. Nachdem für Modelle, die eine direkte Kopplung des Skalarfeldes zur restlichen Materie erlauben, Kriterien für asymptotische beschleunigte Expansion und Isotropisierung festgestellt wurden, werden diese Resultate benutzt, um mit Hilfe einer konformen Transformation Aussagen über die Langzeitasymptotik eines krümmungsgekoppelten Skalarfeldes in einem exponentiellen Potential abzuleiten. Es zeigt sich, dass eine beliebig kleine, positive direkte Kopplung zur skalaren Krümmung der Raumzeit in einem Langzeitverhalten resultiert, das bei minimaler Kopplung nur in Gegenwart einer positiven kosmologischen Konstanten zu erwarten wäre.

Contents

Introduction and main results

1. General Relativity and Cosmology

Cosmology is the study of the universe as a whole and its description at the largest observable scales. On these scales the dynamics is predominantly governed by gravity and so cosmology is based on a theory of gravity which is often taken to be general relativity. There, the basic unknown is the gravitational field described by a Lorentzian metric $\tilde{g}$ on an $n+1$-dimensional space-time manifold $\tilde{M}$. The gravitational field is related to the matter content through Einstein's equations

$$\mathrm{Ric}_{\tilde{g}} - \Big(\frac{1}{2}R_{\tilde{g}} - \Lambda\Big)\tilde{g} = T$$

where $\mathrm{Ric}_{\tilde{g}}$ and $R_{\tilde{g}}$ are the Ricci tensor and scalar curvature of the metric $\tilde{g}$, Λ is the cosmological constant and T is the energy-momentum tensor of the matter fields ψ. The Einstein equations have to be supplemented by equations of motion for the individual matter fields. Some matter fields can be described in terms of a Lagrangian L_{m} that determines the energy-momentum tensor

$$T^{\mu\nu} = 2\frac{\partial L_{\mathrm{m}}}{\partial \tilde{g}_{\mu\nu}} + L_{\mathrm{m}}\tilde{g}^{\mu\nu}$$

as well as the equations of motion

$$\tilde{\nabla}_{\mu}\frac{\partial L_{\mathrm{m}}}{\partial(\tilde{\nabla}_{\mu}\psi)} - \frac{\partial L_{\mathrm{m}}}{\partial \psi} = 0$$

as the corresponding Euler-Lagrange equation. Here, the function L_{m} is supposed to depend on the matter fields ψ and their first derivatives $\tilde{\nabla}_{\mu}\psi$ where $\tilde{\nabla}$ denotes the covariant derivative with respect to the metric $\tilde{g}$.

Taking $\tilde{M} = M \times I$, where (M,h) is a simply connected space form and I an interval in the reals, together with the warped product metric

$$\tilde{g} = a^2(t)\pi^* h - dt \otimes dt$$

and a perfect fluid matter source $T_{\mathrm{m}} = p\,a^2(t)\pi^* h + \rho\, dt \otimes dt$ yields the spatially homogeneous and isotropic Friedmann-Lemaître models where $\pi : \tilde{M} \to M$ and $t : \tilde{M} \to I$ are the canonical projections and $\rho \geq 0$ and p are the density and pressure of the fluid connected by an equation of state. The Einstein equations then reduce to ordinary differential equations for the scale factor a, namely to the Friedmann equation

$$\frac{1}{2}n(n-1)\frac{\dot{a}^2 + K}{a^2} - \Lambda = \rho$$

from the normal component and

$$(n-1)\left[\frac{\ddot{a}}{a^2} + \frac{n-2}{2}\frac{\dot{a}^2 + K}{a^2}\right] - \Lambda = -p$$

from the components tangential to the t-hypersurfaces. The constant K is the sectional curvature of (M,h). If t is interpreted as cosmological time and a as a length scale within the space-like surfaces of homogeneity it is intuitive to say that the universe is expanding if $\dot{a} > 0$, static if $\dot{a} = 0$ and contracting if $\dot{a} < 0$. Moreover, in an expanding universe, the expansion is said to be accelerated if $\ddot{a} > 0$ and decelerated if $\ddot{a} < 0$. Observe that

$$n(n-1)\frac{\ddot{a}}{a^2} = 2\Lambda - \big[np + (n-2)\rho\big]$$

and so, within the class of Friedmann-Lemaître models and for $n \geq 3$, accelerated expansion is possible only if the cosmological constant is positive or the perfect fluid violates the strong energy condition $\rho + np \geq 0$.

To get an overview of the qualitatively different types of solutions of the Friedmann-Lemaître models one can for simplicity restrict to the case $n = 3$ and assume the perfect fluid to be pressure-less dust, $p = 0$. Furthermore $\rho > 0$ is supposed to hold and initial data shall be such that the solution is expanding initially. Then for $\Lambda < 0$ all solutions develop a singularity towards the past (big bang) as well as towards the future (recollapse) where the scale factor a vanishes. During the whole evolution $\ddot{a} < 0$. If $\Lambda = 0$ the models with $K > 0$ behave in the same way and are called closed models. For $K \leq 0$ the solutions again have a finite-time singularity towards the past but exist globally towards the future where the scale factor a grows without bound. They are called open models and are, while expanding forever, always decelerated too. Finally let $\Lambda > 0$. In this case the qualitative dynamics depend on the exact measures of Λ, K and a third constant, the conserved quantity $E := \rho a^3/3$. The solutions with $K^3 < E^2\Lambda$ have a big bang singularity in the past and expand forever towards the future without bound. For small values of the scale factor, $a^3 < E/\Lambda$, the expansion is decelerated while for large values, $a^3 > E/\Lambda$, it is accelerated. If $K^3 > E^2\Lambda$ solutions with $a^3 < E/\Lambda$ exhibit finite-time singularities both in the past and in the future where the scale factor actually vanishes. Solutions with $a^3 > E/\Lambda$ are singularity free. Their scale factor goes to infinity both for $t \to -\infty$ as well as for $t \to \infty$ while attaining a minimum (bounce) in between. During all of the evolution $\ddot{a} > 0$. What remains is the peculiar value $K^3 = E^2\Lambda$ where, depending on the initial data, there is a solution with a big bang singularity in the past and which exists globally towards the future but now with the scale factor approaching the value M/K for late times, one static solution with $a = M/K$ and a global accelerating solution which tends to M/K in the past and grows without bound towards the future.

2. Scalar fields in cosmology

In contemporary models for the evolution of the universe there are commonly two phases of accelerated expansion, one in the very early stages known as inflation and one in more recent epochs called late-time acceleration. For the purpose of this work accelerated expansion shall be understood as the existence of a foliation of the space-time manifold by level hypersurfaces of a Gaussian time coordinate t such that the mean curvature $\operatorname{tr} k/n$ of the leaves with respect to the normal ∂_t is negative (expansion) and its Lie derivative along the normal satisfies the condition $\partial_t \operatorname{tr} k < (\operatorname{tr} k)^2/n$ (acceleration). Note that in the case of Friedmann-Lemaître models one has $\operatorname{tr} k = -n\dot{a}/a$ and so these definitions comply with those stated in the previous section. The motivations for considering these two accelerated epochs are quite different. Inflation is introduced on theoretical grounds in an attempt to "explain" the initial data for the cosmological models. For it turns out that in order to obtain a solution with characteristics comparable to what is observed now the initial data at the beginning of, say, radiation domination had to be extremely homogeneous, isotropic and almost exactly flat. This need of "highly specialized" data may be considered unsatisfactory and inflation is a mechanism which can produce such out of "less special" data dynamically. It further yields a characteristic spectrum of primordial inhomogeneities which are to evolve to the present day large scale structure and it is this spectrum that allows inflationary models to be constrained observationally. Evidence for late-time acceleration primarily comes from recent measurements of the apparent magnitude–redshift relation of distant supernovae. If these observations are interpreted within spatially homogeneous and isotropic cosmological models accelerated expansion can be considered an experimental fact. The situation is far less clear in inhomogeneous cosmologies. The models used to produce inflation on the one hand and late-time acceleration on the other hand look superficially rather similar. One of the most popular on grounds of simplicity is a non-linear scalar field. It is a smooth real-valued function $\tilde{\phi}$ on the space-time manifold $(\tilde{M}, \tilde{g})$ satisfying the equation of motion

$$\Box_{\tilde{g}}\tilde{\phi} - U'(\tilde{\phi}) = 0$$

for a non-negative potential U and couples to the Einstein equations through its energy-momentum tensor

$$T_0 = d\tilde{\phi} \otimes d\tilde{\phi} - \left[\frac{1}{2}|d\tilde{\phi}|^2_{\tilde{g}} + U(\tilde{\phi})\right]\tilde{g}$$

where d is the exterior derivative on $\tilde{M}$ and $\Box_{\tilde{g}}$ the covariant wave operator associated with the metric $\tilde{g}$. If U is chosen to have a critical point at zero, say, then specifying trivial data for the field reduces the model to Einstein's equations with a cosmological constant $\Lambda = U(0)$. The dynamics of such scalar field models is well-understood in the spatially homogeneous and isotropic setting where a lot of work has been done mostly aiming at the extraction of observational signatures. Rigorous results in more general situations are less numerous. For solutions which expand initially Wald [**32**] obtained late-time asymptotics for a general class of spatially homogeneous models, namely those of Bianchi type I–VIII, in the presence of a positive cosmological constant and found in particular that accelerated expansion happens at an exponential rate eventually. These findings were extended by Rendall [**25**] to a scalar field evolving in a potential with a positive lower bound. If such a lower bound is missing the dynamics is much more involved. For the same class of solutions Kitada and Maeda [**17, 18**] as well as Lee [**19**] proved accelerated expansion for exponential potentials falling off not too steeply and Rendall [**26**] generalized this to a large class of potentials merely assumed positive. More precisely, supposing the fall-off condition $\varkappa = \limsup_{x\to\infty}(|U'|/U)(x) < \sqrt{2}$ on the potential to hold they concluded that acceleration occurs eventually but generally no longer at an exponential rate. One the other hand for exponential potentials with $\varkappa > \sqrt{2}$ late-time acceleration was shown to be absent. If no symmetry assumptions are made at all Rendall [**24**] proved full late-time asymptotic expansions for suitable solutions of the Einstein equations with a positive cosmological constant. Using Kaluza-Klein reductions Heinzle and Rendall [**12**] were able to obtain future stability and asymptotics for scalar fields in a countable set of exponential potentials having initial data close to that of spatially homogeneous and isotropic solutions. Recently, Ringström proved future stability of spatially homogeneous solutions as well as asymptotics first for scalar fields in potentials with a positive non-degenerate minimum [**27**] and afterwards for slowly decaying, i.e. $\varkappa < \sqrt{2}$, exponential potentials also.

3. Non-minimally coupled scalar fields

The models considered here are somewhat more general than those described in the preceding section in that the field itself is complex valued and allowed to couple directly to the space-time scalar curvature. This non-minimal coupling manifests itself both in the equation of motion for the scalar field

$$\Box_{\tilde{g}}\tilde{\phi} - \xi R_{\tilde{g}}\tilde{\phi} - 2\tilde{\phi}V'(\tilde{\phi}^*\tilde{\phi}) = 0$$

through the presence of the term $-\xi R_{\tilde{g}}\phi$ involving the space-time scalar curvature $R_{\tilde{g}}$ and as additional terms in the energy-momentum tensor. The real number ξ is called the coupling constant and the potential is written as a function V of the squared modulus of $\tilde{\phi}$. Setting ξ equal to zero recovers the usual minimally coupled model. The additional terms in the energy-momentum tensor contain second order derivatives of the metric and the field so that the question whether the non-minimally coupled system is still well-posed immediately arises. Noakes [**21**] proved the Cauchy problem well-posed for a specific value of the coupling constant known as conformal coupling and later Salgado [**28**] extended this result including in particular the case at hand.

The method to obtain full asymptotic expansions followed in this work has been developed by Rendall [**24**] for Einstein's equations with a positive cosmological constant based on a paper by Starobinskiĭ [**30**]. Assuming the potential $V(\tilde{\phi}^*\tilde{\phi})$ to have a positive critical point at $\tilde{\phi} = 0$ it is first shown that there are formal series solutions of the coupled Einstein-scalar field system of the form

$$\tilde{g}(x,t) = \sum_{m,s,l} \pi^* g_{m,s,l}(x)\, t^l e^{-(m+is)Ht} - dt \otimes dt$$

on $\tilde{M} = M \times I$ and similarly for the scalar field in the sense that if these series are plugged into the equations and acted upon term-wise, the equations are satisfied coefficient by coefficient. Here, I is an interval in the real numbers and π and t the projections from $\tilde{M}$ onto its factors respectively. In the series m and s take real values, l natural ones and $-H$ is a constant representing the asymptotic value of the mean curvature of the foliation generated by t. Making the notion of such formal series precise and providing existence and uniqueness of formal solutions is done in chapter 2 and by theorem 2.11. It turns out that the data which can be specified comprise of a Riemannian metric, a symmetric 2-tensor and two complex functions on the future conformal boundary of the resulting space-time. The number of free functions is thus the same as it was for an ordinary Cauchy problem where the same data, although subject to different constraints, can be specified. From this crude perspective the solution is therefore general. In a second step it is shown that to a given genuine solution of the Einstein-scalar field system a formal solution can be found which is actually asymptotic. For this to be true the genuine solution is supposed to allow for a Gaussian time coordinate t with respect to which it exists globally towards the future and satisfies some basic decay conditions as t goes to infinity. Then in chapter 3 and there in particular with the theorems 3.15 and 3.18 it is shown that one can find a formal solution which, truncated at any order, approximates the genuine solution up to an error of higher order asymptotically and that the approximation remains valid even when differentiated as often as desired. This permits the late-time asymptotics to be studied solely at the level of the formal series. The assertion that there in fact is a large class of solutions with the asymptotics just described is validated in chapter 4. Note that the de Sitter solution

$$\tilde{g} = \frac{1}{H^2}\cosh^2(Ht)\,\pi^* h - dt \otimes dt$$

with $\tilde{\phi} = 0$ and $\pi^* h$ the pullback of the standard metric h on the three-sphere S^3 by the canonical projection $\pi : S^3 \times \mathbb{R} \to S^3$ is a particular member of this class provided $2V(0) = n(n-1)H^2$. It is demonstrated that the approximate solutions indeed satisfy the Einstein-scalar field equations approximately and that the difference to an actual solution satisfies a Fuchsian equation. In the analytic setting it is therefore possible to prove existence and uniqueness of a solutions to this equation using a result by Kichenassamy and Rendall [**16**] and hence of solutions of the Einstein-scalar field system existing globally towards the future and with prescribed late-time asymptotics with respect to a Gaussian time coordinate. This is summarized in theorem 4.7. In view of chapter 2 it thus suffices to give analytic data on an analytic manifold in order to obtain a unique analytic solution with the corresponding late-time behaviour. The analyticity assumption is certainly undesirable but to go beyond that the Fuchsian system should necessarily be hyperbolic in some sense for solvability to be expected. The system considered in chapter 4 is not symmetric hyperbolic and so it is not obvious how to go to less regular data. It may be pointed out that series of the kind studied here have also been used to analyze the asymptotics of negative Einstein manifolds in Riemannian geometry by Fefferman and Graham [**9**] for instance since there is a correspondence between Riemannian metrics with negative Einstein constant and Lorentzian metrics with a positive cosmological constant.

With the assumption of spatial homogeneity the problem of determining the late-time behaviour of solutions becomes simplified such that a larger class of potentials for the scalar field can be considered and other forms of matter can be present. The presence of matter opens up the possibility for the scalar field to couple non-minimally to the matter. This direct coupling shall, for $\tilde{\phi}$ real, be described by two coupling functions C and c that modify the equation of motion for the scalar field according to

$$\Box_{\tilde{g}}\tilde{\phi} - U'(\tilde{\phi}) = -c(\tilde{\phi})\,\mathrm{tr}_{\tilde{g}}\,T_{\mathrm{m}}$$

while the energy-momentum tensor T_{m} for the matter goes as $C(\tilde{\phi})T_{\mathrm{m}}$ into the Einstein equations. The second Bianchi identity then requires the matter to satisfy

$$\mathrm{div}_{\tilde{g}}\big[C(\tilde{\phi})T_{\mathrm{m}}\big] = c(\tilde{\phi})\big(\mathrm{tr}_{\tilde{g}}\,T_{\mathrm{m}}\big)d\tilde{\phi}.$$

In chapter 5 it is shown that in particular for initially expanding solutions of all Bianchi types other than IX with matter satisfying the dominant and strong energy condition late-time acceleration and isotropization occur both for potentials with a positive lower bound (corollary 5.4) as well as for positive potentials which are flat at infinity meaning $\varkappa = 0$ (corollary 5.9). Isotropization refers to the fact that the trace-free part of the second fundamental form of the hypersurfaces of homogeneity vanishes faster than its trace. The result means that the basic asymptotics is not substantially altered by the direct coupling compared to the minimally coupled case $C = 1$ and $c = 0$ but the flatness condition imposed on the potential is much more restrictive. Due to possible recollapse Bianchi type IX is excluded for simplicity but it is expected that it can be handled by an argument similar to that in [**32**]. Finally, corollary 5.13 carries over these results to the case with a direct coupling to the curvature through a conformal transformation. It shows that for any exponential potential and any positive coupling constant $\xi > 0$ late-time acceleration and isotropization happens at an exponential rate in stark contrast to the minimally coupled case $\xi = 0$. This thereby gives an example of what has been proposed by Tsujikawa [**31**] under the term curvature-assisted inflation. Another fact provided by corollary 5.13 worth mentioning is the assertion that if the scalar field fulfills the inequality $1 - \xi\tilde{\phi}^*\tilde{\phi} > 0$, which is usually assumed and guarantees for instance smoothness of the employed conformal transformation, initially, then the quantity $1 - \xi\tilde{\phi}^*\tilde{\phi}$ stays bounded away from zero for all future times. It implies boundedness of the effective gravitational constant $(1-\xi\tilde{\phi}^*\tilde{\phi})^{-1}$, an issue addressed by Starobinskiĭ in [**29**].

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CHAPTER 1

The Einstein-scalar field-matter system

1. Direct coupling to the scalar curvature

This section introduces the system of equations arising when the Einstein equations are sourced by a complex scalar field in addition to some other, not further specified matter. A peculiarity of the model under consideration is the presence of an explicit direct coupling of the scalar field to the scalar curvature of space-time. If there is a Lagrangian $L_{\rm m}$ for the matter content, a corresponding Lagrangian for the Einstein-scalar field-matter system is

$$L := \frac{1}{2}R_{\tilde g} - \frac{1}{2}|d\tilde\phi|^2_{\tilde g} - V(\tilde\phi^*\tilde\phi) - \frac{1}{2}\xi R_{\tilde g}\tilde\phi^*\tilde\phi + L_{\rm m}.$$

Here, $R_{\tilde g}$ denotes the scalar curvature of an $n+1$-dimensional smooth Lorentzian space-time manifold $(\tilde M, \tilde g)$, $\tilde\phi \in C^\infty(\tilde M, \mathbb{C})$ a smooth complex scalar field on $\tilde M$ with potential $V(\tilde\phi^*\tilde\phi)$ for a $V \in C^\infty(\mathbb{R},\mathbb{R})$ and $|\cdot|^2_{\tilde g}$ the pseudo-norm given by the fibre metrics $\langle\cdot,\cdot\rangle_{\tilde g}$ induced by $\tilde g$ on any tensor bundle $T^r_s\tilde M$ or their extension to the complexification $T^r_s\tilde M \otimes \mathbb{C}$ where appropriate. (For the case at hand, $|d\tilde\phi|^2_{\tilde g} = \tilde g^{\mu\nu}\partial_\mu\tilde\phi^*\partial_\nu\tilde\phi$ in local coordinates.) The direct coupling of the scalar field to the space-time curvature is provided by the additional term $-\frac{1}{2}\xi R_{\tilde g}\tilde\phi^*\tilde\phi$ when the coupling constant $\xi \neq 0$. The case where $\xi = 0$ is referred to as the minimally coupled case. Another case which is sometimes of special interest is $\xi = (n-1)/4n$ and is called the conformally coupled case because for real fields in certain potentials, in particular those for which $\tilde\phi V'(\tilde\phi^*\tilde\phi)$ is positively homogeneous of degree $(n+3)/(n-1)$ in $\tilde\phi$, the equation of motion for the scalar field becomes conformally invariant. Examples of such potentials include $V(\tilde\phi^*\tilde\phi) = \Lambda + \lambda|\tilde\phi|^{2(n+1)/(n-1)}$ with $\lambda, \Lambda \in \mathbb{R}$ which is $V(\tilde\phi^*\tilde\phi) = \Lambda + \lambda|\tilde\phi|^4$ in four space-time dimensions.

The relevance of the Lagrangian L stated above is that if the metric $\tilde g$, the scalar field $\tilde\phi$ and the other matter described effectively by its energy-momentum tensor $T_{\rm m}$ satisfy the equations of motion

$$(1.1)\qquad (1-\xi\tilde\phi^*\tilde\phi)\Big(\mathrm{Ric}_{\tilde g} - \frac{1}{2}R_{\tilde g}\tilde g\Big) = \frac{1}{2}\bigg[(1-2\xi)d\tilde\phi^*\otimes d\tilde\phi + \Big(2\xi - \frac{1}{2}\Big)|d\tilde\phi|^2_{\tilde g}\tilde g - 2\xi\tilde\phi^*\nabla^2_{\tilde g}\tilde\phi + 2\xi\big(\tilde\phi^*\Box_{\tilde g}\tilde\phi\big)\tilde g - V(\tilde\phi^*\tilde\phi)\tilde g + \text{c.c.}\bigg] + T_{\rm m}$$

$$(1.2)\qquad \Box_{\tilde g}\tilde\phi - \xi R_{\tilde g}\tilde\phi - 2\tilde\phi V'(\tilde\phi^*\tilde\phi) = 0$$

$$(1.3)\qquad \mathrm{div}_{\tilde g}\, T_{\rm m} = 0$$

then they satisfy the Euler-Lagrange equations corresponding to L provided, as assumed in the sequel, that $\mathrm{div}_{\tilde g}\, T_{\rm m} = 0$ is equivalent to the equations of motion for the matter. Additional notation used at this point include the Hessian $\nabla^2_{\tilde g}$ and its metric trace, the wave operator $\Box_{\tilde g}$, the covariant divergence $\mathrm{div}_{\tilde g}$ as well as the abbreviation c.c. for the complex conjugate of all the preceding terms in the bracket. As can be seen immediately from equation (1.1) a non-minimal coupling $\xi \neq 0$ has interesting consequences in that it produces new second order terms and thus affects the principal part of the differential equation as well as introducing a potentially singular factor $1-\xi\tilde{\phi}^*\tilde{\phi}$ in front of the Einstein tensor. Note that in contrast to real scalar fields where the potential energy density is commonly written as $V(\tilde{\phi})$ it is defined here as $V(\tilde{\phi}^*\tilde{\phi})$ for still having V a function of a real variable. This is the origin of the probably strangely looking factor $2\tilde{\phi}$ in front of the potential term in the wave equation (1.2).

2. Einstein-scalar field system

The system which most of this work is concerned with is the Einstein-scalar field system that is obtained from (1.1)–(1.3) by setting $T_{\mathrm{m}}=0$ and thus supposing that no other matter apart from the scalar field is present. A possible cosmological constant shall be considered part of the potential. If it is further assumed that $1-\xi\tilde{\phi}^*\tilde{\phi}>0$ it is possible to divide equation (1.1) by that term and obtain the Einstein equation in its usual form

$$\mathrm{Ric}_{\tilde{g}}-\frac{1}{2}R_{\tilde{g}}\tilde{g}=T \tag{1.4}$$

with an effective energy-momentum tensor for the scalar field

$$\begin{aligned} T:=\frac{1}{2}\big(1-\xi\tilde{\phi}^*\tilde{\phi}\big)^{-1}\Big[&(1-2\xi)d\tilde{\phi}^*\otimes d\tilde{\phi}+\Big(2\xi-\frac{1}{2}\Big)|d\tilde{\phi}|^2_{\tilde{g}}\tilde{g} \\ &-2\xi\tilde{\phi}^*\nabla^2_{\tilde{g}}\tilde{\phi}+2\xi\big(\tilde{\phi}^*\Box_{\tilde{g}}\tilde{\phi}\big)\tilde{g}-V(\tilde{\phi}^*\tilde{\phi})\tilde{g}+\text{c.c.}\Big]. \end{aligned} \tag{1.5}$$

The validity of the Einstein- (1.4) *and* the scalar field equation (1.2) implies that this tensor is covariantly conserved since

$$\begin{aligned} \mathrm{div}_{\tilde{g}}\,T=\big(1-\xi\tilde{\phi}^*\tilde{\phi}\big)^{-1}\Big\{&\frac{1}{2}\Big[\Box_{\tilde{g}}\tilde{\phi}-\xi R_{\tilde{g}}\tilde{\phi}-2\tilde{\phi}V(\tilde{\phi}^*\tilde{\phi})\Big]\tilde{g} \\ &-\xi\tilde{\phi}\Big(\mathrm{Ric}_{\tilde{g}}-\frac{1}{2}R_{\tilde{g}}\tilde{g}-T\Big)\Big\}\cdot\nabla_{\tilde{g}}\tilde{\phi}^*+\text{c.c.} \end{aligned} \tag{1.6}$$

For $\xi=0$ and real fields it reduces to the familiar energy-momentum tensor for minimally coupled scalar fields

$$T_0=d\tilde{\phi}\otimes d\tilde{\phi}-\frac{1}{2}|d\tilde{\phi}|^2_{\tilde{g}}\tilde{g}-U(\tilde{\phi})\tilde{g} \tag{1.7}$$

with $U(x)=V(x^2)$ for $x\in\mathbb{R}$.

Suppose that there exists a Gaussian time coordinate $t:\tilde{M}\to I$ such that the space-time is decomposed into a product of a smooth n-dimensional manifold M and an open real interval I, $\tilde{M}=M\times I$. The metric hence decomposes into a smooth family of Riemannian metrics $g(t)\in\mathcal{T}^0_2(M)$ according to

$$\tilde{g}=\pi^*g-dt\otimes dt \tag{1.8}$$

with unit lapse and vanishing shift whilst the scalar field decomposes into a smooth family $\phi(t)\in C^\infty(M,\mathbb{C})$ of complex valued functions

$$\tilde{\phi}=\pi^*\phi=\phi(t)\circ\pi. \tag{1.9}$$

The pullback of a family of metrics by the canonical projection $\pi:\tilde{M}\to M$ is thereby defined as $(\pi^*g)(x,t):=(\pi^*g(t))(x,t)$ for all $x\in M$ and $t\in I$. The second fundamental form $k(t)\in\mathcal{T}^0_2(M)$ of a t-hypersurface in $\tilde{M}$ with respect to the normal ∂_t is then given by

$$k=-\frac{1}{2}\partial_t g. \tag{1.10}$$

The effective energy-momentum tensor (1.5) can be decomposed similarly into a smooth family of energy densities $\rho(t) \in C^\infty(M, \mathbb{R})$, flow forms $j(t) \in \mathcal{T}^0_1(M)$ and symmetric pressure tensors $S(t) \in \mathcal{T}^0_2(M)$ as

$$T = \pi^* S - \pi^* j \otimes dt - dt \otimes \pi^* j + \rho\, dt \otimes dt \tag{1.11}$$

with

$$\begin{aligned}\rho = \frac{1}{2}\big(1 - \xi\phi^*\phi\big)^{-1}\Big[&\frac{1}{2}\partial_t\phi^*\partial_t\phi - \Big(2\xi - \frac{1}{2}\Big)|\nabla\phi|^2 \\ &- 2\xi\phi^*\big[\Delta\phi + (\operatorname{tr} k)\partial_t\phi\big] + V(\phi^*\phi) + \text{c.c.}\Big]\end{aligned} \tag{1.12}$$

$$j = \frac{1}{2}\big(1 - \xi\phi^*\phi\big)^{-1}\Big[-(1-2\xi)\nabla\phi^*\partial_t\phi + 2\xi\phi^*\big[\partial_t\nabla\phi + k\cdot\nabla\phi\big] + \text{c.c.}\Big] \tag{1.13}$$

$$\begin{aligned}S = \frac{1}{2}\big(1 - \xi\phi^*\phi\big)^{-1}\Big[&(1-2\xi)\nabla\phi^*\otimes\nabla\phi - 2\xi\phi^*\big[\nabla^2\phi + k\partial_t\phi\big] \\ &+ \Big\{\Big(2\xi - \frac{1}{2}\Big)\big[|\nabla\phi|^2 - \partial_t\phi^*\partial_t\phi\big] + 2\xi\phi^*\big[\Delta\phi + (\operatorname{tr} k)\partial_t\phi - \partial_t^2\phi\big] \\ &- V(\phi^*\phi)\Big\}g + \text{c.c.}\Big]\end{aligned} \tag{1.14}$$

Here ∇ denotes the Levi-Civita connection induced by $g(t)$ on any t-hypersurface in $\tilde{M}$ where, like in all other operators, the metric g is understood implicitly. Likewise, Δ is the covariant Laplacian of $g(t)$ and $k \cdot l$ is the metric contraction between the last index of k and the first index of l for suitable tensors k, l on M.

Imposing the Einstein-scalar field equations on $\tilde{M}$ is equivalent to requiring the $n+1$-decomposed tensor families to satisfy the evolution equation

$$\partial_t k = \operatorname{Ric} - \frac{1}{2}k\cdot k + (\operatorname{tr} k)k - S + \frac{\operatorname{tr} S - \rho}{n-1}g \tag{1.15}$$

together with the Hamiltonian and momentum constraints

$$R - |k|^2 + (\operatorname{tr} k)^2 = 2\rho \tag{1.16}$$

$$\operatorname{div} k - \nabla \operatorname{tr} k = j \tag{1.17}$$

respectively as well as the scalar field equation

$$\Delta\phi + (\operatorname{tr} k)\partial_t\phi - \partial_t^2\phi - \xi R_{\tilde{g}}\phi - 2\phi V'(\phi^*\phi). \tag{1.18}$$

In the last equation the ambient scalar curvature is given by

$$R_{\tilde{g}} = R - 2\partial_t \operatorname{tr} k + |k|^2 + (\operatorname{tr} k)^2$$

and is used in the following as an abbreviation for this expression. A common way to deal with this system is to solve the evolution equation (1.15) first and then show that the constraints (1.16) and (1.17), if satisfied initially, remain satisfied during the evolution. For doing this it turns out to be useful to define the following evolution and constraint quantities

$$\mathsf{e} := \partial_t \operatorname{tr} k - \Big[R + (\operatorname{tr} k)^2 + \frac{\operatorname{tr} S - n\rho}{n-1}\Big] \tag{1.19}$$

$$\mathsf{E} := \partial_t\sigma - \Big[\widehat{\operatorname{Ric}} + (\operatorname{tr} k)\sigma - \hat{S}\Big] \tag{1.20}$$

$$\mathsf{c} := R - |k|^2 + (\operatorname{tr} k)^2 - 2\rho \tag{1.21}$$

$$\mathsf{C} := \operatorname{div} k - \nabla \operatorname{tr} k - j \tag{1.22}$$

where a hat stands for the trace-free part of the tensor with the first index raised, in local coordinates e.g. $\hat{S}^a{}_b = S^a{}_b - (\operatorname{tr} S/n)\delta^a{}_b$, and $\sigma := \hat{k}$ is the shear tensor. It follows that the evolution equation (1.15) is satisfied if and only if E and e

both vanish while the Hamiltonian (1.16) and momentum constraint (1.17) hold exactly when c and C vanish respectively. The above-mentioned propagation of the constraints is seen explicitly by calculating the time derivatives of the quantities c and C as

$$\begin{aligned}\partial_t \mathsf{c} &= 2(\operatorname{tr} k)\mathsf{c} - 2\operatorname{div}\mathsf{C} - 2\langle\sigma,\mathsf{E}\rangle + 2\Big(1-\frac{1}{n}\Big)(\operatorname{tr} k)\mathsf{e} \\ &\quad - 2\big[\partial_t\rho + \operatorname{div} j - \langle k, S\rangle - (\operatorname{tr} k)\rho\big]\end{aligned} \tag{1.23}$$

$$\begin{aligned}\partial_t \mathsf{C} &= (\operatorname{tr} k)\mathsf{C} - \frac{1}{2}\nabla\mathsf{c} + \operatorname{div}\mathsf{E} - \Big(1-\frac{1}{n}\Big)\nabla\mathsf{e} \\ &\quad - \big[\partial_t j + \operatorname{div} S - (\operatorname{tr} k) j\big].\end{aligned} \tag{1.24}$$

Note that the expressions in the square brackets above are the temporal and spatial projections of the covariant divergence of an energy-momentum tensor (1.11) respectively. Analogously the vanishing of the quantity

$$\mathsf{S} := \partial_t^2\phi - (\operatorname{tr} k)\partial_t\phi - \Delta\phi + \xi R_{\bar g}\phi + 2\phi V'(\phi^*\phi) \tag{1.25}$$

is synonymous for the scalar field equation to hold.

3. Direct coupling to matter

Another form of non-minimal coupling is realized if the scalar field couples directly to the matter content described by T_{m}. If the scalar field is real and furthermore the inequality $1-\xi\tilde\phi^2>0$ holds then the two forms of non-minimal coupling are related by a conformal transformation. This will now be explained in some detail. Suppose two smooth coupling functions $C, c \in C^\infty(\mathbb{R},\mathbb{R})$ and a real scalar field $\bar\phi \in C^\infty(\tilde M,\mathbb{R})$ with potential $U \in C^\infty(\mathbb{R},\mathbb{R})$ are given on a smooth Lorentzian space-time manifold $(\tilde M, \bar g)$. Then the matter coupled Einstein-scalar field-matter system is defined as

$$\operatorname{Ric}_{\bar g} - \frac{1}{2}R_{\bar g}\bar g = d\bar\phi\otimes d\bar\phi - \frac{1}{2}|d\bar\phi|^2_{\bar g}\bar g - U(\bar\phi)\bar g + C(\bar\phi)T_{\mathrm{m}} \tag{1.26}$$

$$\Box_{\bar g}\bar\phi - U'(\bar\phi) = -c(\bar\phi)\operatorname{tr}_{\bar g} T_{\mathrm{m}} \tag{1.27}$$

$$\operatorname{div}_{\bar g}[C(\bar\phi)T_{\mathrm{m}}] = c(\bar\phi)\big(\operatorname{tr}_{\bar g} T_{\mathrm{m}}\big)d\bar\phi. \tag{1.28}$$

Minimal coupling corresponds to the case where $C=1$ and $c=0$. The coupling between equations (1.27) and (1.28) involving the metric trace of the matter energy-momentum tensor only is of course not the most general form conceivable but general enough to cover many cases studied in the literature, e.g. in [**2**] or [**11**], and for demonstrating the relation to the curvature-coupled case aimed at here. Since equations (1.27) and (1.28) imply that $\operatorname{div}_{\bar g} T_0 = -\operatorname{div}_{\bar g}[C(\bar\phi)T_{\mathrm{m}}]$ it follows that despite the fact that the energy-momentum tensors T_0 for the scalar field and $C(\bar\phi)T_{\mathrm{m}}$ for the matter content are not conserved separately they are nonetheless conserved jointly as is required by the second Bianchi identity.

Let a real solution $(\tilde g,\tilde\phi)$ of the curvature coupled Einstein-scalar field-matter system (1.1)–(1.3) on $\tilde M$ be given such that $1-\xi\tilde\phi^2>0$. Then the function

$$\Omega := \sqrt[n-1]{1-\xi\tilde\phi^2}$$

is positive too and with $\eta := 1-4n\xi/(n-1)$ the map

$$\Phi := y \mapsto \int_0^y \frac{\sqrt{1-\eta\xi x^2}}{1-\xi x^2}dx$$

is a smooth diffeomorphism between the open interval

$$J := \begin{cases}]-1/\sqrt{\xi}, 1/\sqrt{\xi}[& \text{for } \xi>0 \\ \mathbb{R} & \text{for } \xi<0 \end{cases}$$

and $\mathbb{R}$ with inverse Ψ. Moreover the conformal metric

$$\bar{g} := \Omega^2 \tilde{g}$$

and a rescaled scalar field

$$\bar{\phi} := \Phi \circ \tilde{\phi}$$

solve the matter coupled Einstein-scalar field-matter equations (1.26)–(1.28) with potential

$$U = \frac{V(\Psi^2)}{\left(1 - \xi\Psi^2\right)^{\frac{n+1}{n-1}}} \tag{1.29}$$

and coupling functions

$$C = \frac{1}{1 - \xi\Psi^2}, \qquad c = \frac{1}{n-1} C' = \frac{1}{n-1} \frac{2\xi\Psi}{\sqrt{1 - \eta\xi\Psi^2}} C.$$

Observe that a conformal coupling is characterized by $\eta = 0$ and that in this case the diffeomorphism Φ is simply

$$\Phi(t) = \frac{1}{\sqrt{\xi}} \tanh^{-1} \sqrt{\xi} t.$$

The solution $(\tilde{g}, \tilde{\phi})$ where the coupling to the matter T_{m} is minimal is said to be in the Jordan frame. As the transformation above shows it is possible, under the assumptions stated, to get rid of a non-minimal curvature coupling but, in the presence of matter, at the expense of introducing a direct coupling to that matter. The transformed solution $(\bar{g}, \bar{\phi})$ with a minimal curvature coupling is said to be in the Einstein frame. It is emphasized that the transformation above is stated for real scalar fields only.

4. Spatially homogeneous solutions

One way to simplify the Einstein-scalar field-matter system is to consider spatially homogeneous solutions which are, in the most elementary case, modelled by an n-dimensional Lie group G acting simply transitively on the t-hypersurfaces of

$$\tilde{M} = G \times I$$

by left translation. Here $I \subset \mathbb{R}$ is an interval with non-empty interior and $t : \tilde{M} \to I$ the canonical projection as before. A spatially homogeneous solution of the Einstein-scalar field-matter system in the simplest case is then one invariant under this action. Note that in three dimensions this includes in particular all Bianchi models but not the Kantowski-Sachs space-times [**14**] where the four-dimensional symmetry group contains no three-dimensional subgroup acting transitively on the hypersurfaces of homogeneity. Requiring t a Gaussian time coordinate, a spatially homogeneous solution of the matter coupled system (1.26)–(1.28) is then given by a family of Riemannian metrics $g \in C^2(I, T_2^0\mathfrak{g})$ on the Lie algebra $\mathfrak{g}$ of G, a scalar field $\phi \in C^2(I)$ and projections $\rho_{\mathrm{m}} \in C^1(I)$, $j_{\mathrm{m}} \in C^1(I, T_1^0\mathfrak{g})$, $S_{\mathrm{m}} \in C^1(I, T_2^0\mathfrak{g})$ of the energy momentum tensor of the matter by (1.8), (1.9) and (1.11) after identifying the elements of $T_s^r\mathfrak{g}$ with their $C^\infty(G)$-multilinear left-invariant extension to $\mathcal{T}_s^r G$. It follows that $k \in C^1(I, T_2^0\mathfrak{g})$ and hence $H := -\operatorname{tr} k/n \in C^1(I)$ for the negative of the mean curvature of the t-hypersurfaces in $\tilde{M}$. The Hamiltonian and momentum constraints read

$$n(n-1)H^2 = \dot{\phi}^2 + 2U(\phi) + |\sigma|^2 - R + 2C(\phi)\rho_{\mathrm{m}} \tag{1.30}$$

$$\operatorname{div} \sigma = C(\phi) j_{\mathrm{m}} \tag{1.31}$$

and the evolution equations

$$n(n-1)\dot{H} = -n\dot{\phi}^2 - n|\sigma|^2 + R - C(\phi)(n\rho_{\mathrm{m}} + \operatorname{tr} S_{\mathrm{m}}) \tag{1.32}$$

$$\dot{\sigma} = -nH\sigma + \widehat{\mathrm{Ric}} - C(\phi)\hat{S}_{\mathrm{m}}. \tag{1.33}$$

For the scalar field equation one obtains

$$\ddot{\phi} + nH\dot{\phi} + U'(\phi) = c(\phi)\operatorname{tr}_{\tilde{g}} T_{\mathrm{m}} \tag{1.34}$$

and the equations of motion for the ordinary matter finally

$$\begin{aligned}\big[C(\phi)\rho_{\mathrm{m}}\big]^{\cdot} + \operatorname{div}[C(\phi)j_{\mathrm{m}}] - C(\phi)\langle\sigma, \hat{S}\rangle \\ + HC(\phi)(n\rho_{\mathrm{m}} + \operatorname{tr} S_{\mathrm{m}}) = -c(\phi)(\operatorname{tr}_{\tilde{g}} T_{\mathrm{m}})\dot{\phi}\end{aligned}$$

as well as

$$\big[C(\phi)j_{\mathrm{m}}\big]^{\cdot} + \operatorname{div}[C(\phi)\hat{S}_{\mathrm{m}}] + nHC(\phi)j_{\mathrm{m}} = 0.$$

Here, a dot stands for ∂_t. This yields in particular the Hamiltonian constraint (1.30), the evolution equation (1.32) and the scalar field equation (1.34) as ordinary differential equations which will turn out to be suitable for an analysis of the asymptotics.

CHAPTER 2

Formal solutions

In this chapter, a notion of formal, algebraic solutions of the Einstein-scalar field system is developed. It is an extended version of what has been published in [**5**].

1. Generalized power series

The aim of this section is to construct an algebra of generalized power series suitable for a formal solution of the Einstein-scalar field system. Generalized power series are usually considered with exponents from an arbitrary strictly ordered commutative monoid. Here attention will be focussed on the the case where this monoid is the group of real numbers and the term "generalized" merely refers to the fact that non-integer powers in the formal series are allowed.

The formal series constructed will be supported on sets belonging to the system

$$\mathfrak{A} := \{A \subset \mathbb{R} \mid A \text{ is bounded below and has no limit points in } \mathbb{R}\}.$$

It is immediate that every element of $\mathfrak{A}$ is well-ordered and at most countable. The system itself is closed under taking subsets, finite unions and pointwise sums, so for $A, B \in \mathfrak{A}$ and $C \subset A$ it follows that $A \cup B, A + B, C \in \mathfrak{A}$. The same is true for the subsystem $\mathfrak{A}_0 := \{A \in \mathfrak{A} \mid A \subset \mathbb{R}_+\}$ consisting of all elements of $\mathfrak{A}$ that are subsets of the non-negative reals $\mathbb{R}_+$. A property of fundamental importance for the inductive construction of solutions will be the following

Proposition 2.1. *For every $A \in \mathfrak{A}_0$ there is a $B \in \mathfrak{A}_0$ such that $B \supset A$ and B is $+$-stable, i.e. $B + B = B$.*

Proof. One can assume $0 \in A$. If $A = \{0\}$ then $B := A$ is sufficient, otherwise there exists $a := \min A \setminus \{0\} > 0$. Now define an increasing sequence $B_0 := \{0\}$, $B_{n+1} := A + B_n$ in $\mathfrak{A}_0$. It follows that for any non-negative integers $m, n \in \mathbb{N}$ $B_n \cap [0, an] = B_{n+1} \cap [0, an]$ and $B_m + B_n = B_{m+n}$. But this implies that $B := \bigcup_{n \in \mathbb{N}} B_n$ is in $\mathfrak{A}_0$ and has the desired properties. □

For a ring R the set of generalized finite Laurent series $R\langle\langle X\rangle\rangle := \{f : \mathbb{R} \to R \mid \operatorname{supp} f \in \mathfrak{A}\}$ in the unknown X is defined as those functions from the real numbers into R with support in $\mathfrak{A}$. On this set the pointwise sum and the Cauchy product (convolution)

$$(fg)(t) = \sum_{t=r+s} f(r)g(s)$$

are well-defined owing to the properties of $\mathfrak{A}$ stated in the preceding paragraph and turn $R\langle\langle X\rangle\rangle$ into a ring itself. In particular, for $f, g \in R\langle\langle X\rangle\rangle$, $\operatorname{supp}(f+g) \subset \operatorname{supp} f \cup \operatorname{supp} g$ and $\operatorname{supp} fg \subset \operatorname{supp} f + \operatorname{supp} g$. If R contains a multiplicative identity 1, then the map

$$1(t) := \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (t \in \mathbb{R})$$

is a multiplicative identity in $R\langle\langle X\rangle\rangle$. Likewise, $R\langle\langle X\rangle\rangle$ is commutative if R is. As usual, the ring R is embedded in $R\langle\langle X\rangle\rangle$ by $r \mapsto r1$ and is identified with its image. Maps into R that are defined on an subset of $\mathbb{R}$ only will be identified with their trivial extension to all of $\mathbb{R}$. The unknown X or, more general, arbitrary real powers X^α thereof, can be identified with the elements

$$X^\alpha(t) := 1(t-\alpha) \qquad (t \in \mathbb{R})$$

in $R\langle\langle X\rangle\rangle$. This allows any generalized Laurent series $f \in R\langle\langle X\rangle\rangle$ to be written as

$$f = \sum_{\alpha\in\mathbb{R}} f(\alpha)X^\alpha.$$

To end with suggestive notation for the moment a series $f \in R\langle\langle X\rangle\rangle$ will be called of at least the order $\alpha \in \mathbb{R}$, $f = O(X^\alpha)$, if $\alpha \leq \operatorname{supp} f$, it will be called of order higher than α, $f = o(X^\alpha)$, if $\alpha < \operatorname{supp} f$.

2. Substitution homomorphisms

Because of the nonlinearities in the field equation it is necessary, in an attempt to solve these equations formally, to make sense of what a "smooth function of a formal power series" means. This can be done by substitution homomorphisms. Let $R(\!(X)\!) := \{f \in R\langle\langle X\rangle\rangle \mid f = O(1)\}$ denote the subring of generalized power series and $R[\![X]\!] := \{f \in R\langle\langle X\rangle\rangle \mid \operatorname{supp} f \subset \mathbb{N}\}$ the subring of formal power series on R.

Proposition 2.2. *If R is commutative then for any $g \in R(\!(X)\!)$ with $g = o(1)$ the map*

$$\varphi_g : R[\![X]\!] \to R(\!(X)\!), \quad f \mapsto \sum_{\nu\in\mathbb{N}} f(\nu)g^\nu$$

is a ring homomorphism.

Proof. Since $g = o(1)$ there is a $0 < c \leq \operatorname{supp} g$ and hence $\nu c \leq \operatorname{supp} g^\nu$ for any $\nu \in \mathbb{N}$. This implies that the sequence $\nu \mapsto f(\nu)g^\nu$ is summable and therefore $\varphi_g f$ is well-defined for every $f \in R[\![X]\!]$. The fact that φ_g respects the ring structure follows by direct calculation on a $+$-stable superset of $\operatorname{supp} g$ according to proposition 2.1. □

The map φ_g is called the substitution homomorphism induced by g. If R has a multiplicative identity 1 then $\varphi_g 1 = 1$ and $\varphi_g X = g$. For $g \in R[\![X]\!]$, $h \in R(\!(X)\!)$ with $g, h = o(1)$ the relation

$$\varphi_{\varphi_h g} = \varphi_h \circ \varphi_g$$

holds.

Remark 2.3. The assumption of commutativity in the proposition above is essential!

Proof. Let $R = \operatorname{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ be the endomorphism ring of $\mathbb{R}^2$, (b_1, b_2) a basis of $\mathbb{R}^2$ with dual basis (β_1, β_2) and define $A := b_1\beta_1$, $B := b_2\beta_2$ elements of R, then $(A - X)(A + X) = A^2 - X^2$ but $\varphi_{BX}(A - X)\varphi_{BX}(A + X) = (A - BX)(A + BX) = A^2 + [A, B]X - B^2X^2 \neq A^2 - B^2X^2 = \varphi_{BX}(A^2 - X^2)$ since $[A, B] := AB - BA \neq 0$. □

Two other subrings of $R\langle\langle X\rangle\rangle$ will be used later on and shall be defined at this point. These are the ring of generalized polynomials $R\langle X\rangle := \{f \in R\langle\langle X\rangle\rangle \mid \operatorname{supp} f$ is finite$\}$ as well as the ring of polynomials $R[X] := \{f \in R\langle\langle X\rangle\rangle \mid \operatorname{supp} f \subset \mathbb{N}$ and finite$\}$ over R.

3. Invertibility

In order to be able to raise tensor indices with respect to a metric given as a generalized power series the existence and uniqueness of an "inverse" of such a metric will be established in this section. For this, let R be a ring with multiplicative identity 1. Recall that an element r of R is called a unit if there is an $s \in R$ with $rs = sr = 1$. In this case, s is unique and usually written as $s = r^{-1}$.

Proposition 2.4. *If for $f \in R(\!(X)\!)$ $f(0)$ is a unit in R then f is a unit in $R(\!(X)\!)$.*

Proof. Let $A \in \mathfrak{A}_0$ be a $+$-stable superset of $\operatorname{supp} f$ containing zero according to proposition 2.1. The inverse will be constructed by induction over the set A. Assume that for an $a \in A$ coefficients $g(r) \in R$ are given for all $r < a$ such that

$$\sum_{s \leq t} f(t-s)g(s) = 1(t)$$

holds for any $t < a$. If $a = 0$ the assumption allows for setting $g(0) := f(0)^{-1}$, if, on the other hand, $a > 0$ let

$$g(a) := g(0) \sum_{s \leq a} f(a-s)g(s) \in R.$$

In both cases it follows that

$$\sum_{s \leq a} f(a-s)g(s) = 1(a),$$

so transfinite recursion yields the existence of a map $g : A \to R$ with $(fg)(a) = 1(a)$ for all $a \in A$. Since $\operatorname{supp} fg \subset A$ it is even true that $fg = 1$. Finally, by construction, $g(0)$ is a unit and applying the same argument again proves $gf = 1$. □

In view of the applications to follow a series being a unit is too weak a notion of invertibility if coefficients with negative powers in X are allowed as in the ring $R\langle\!\langle X\rangle\!\rangle$ and the following stronger (see remark 2.6) version of it can be useful.

Definition 2.5. An $f \in R\langle\!\langle X\rangle\!\rangle$ is called invertible if there is an $m \in \mathbb{R}$ such that $f(m) \in R$ is a unit and $m = \min \operatorname{supp} f$.

Remark 2.6. Not every unit is invertible!

Proof. Let again $R = \operatorname{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ with identity I, (b_1, b_2) be a basis of $\mathbb{R}^2$ with dual basis (β_1, β_2) and $N := b_1\beta_2 \in R$. Then $NX^{-1} + I \in R\langle\!\langle X\rangle\!\rangle$ and $(NX^{-1} + I)(-NX^{-1} + I) = I = (-NX^{-1} + I)(NX^{-1} + I)$, since $N^2 = 0$ and so $NX^{-1} + I$ is a unit in $R\langle\!\langle X\rangle\!\rangle$ but is not invertible. □

Proposition 2.7. *If $f \in R\langle\!\langle X\rangle\!\rangle$ is invertible then f is a unit and f^{-1} is invertible.*

Proof. Let $f \in R\langle\!\langle X\rangle\!\rangle$ be invertible, then there is $m \in \mathbb{R}$ with $f(m)$ a unit in R and $f = O(X^m)$. Consider $F := X^{-m} f \in R(\!(X)\!)$. By definition $F(0)$ is a unit in R and proposition 2.4 yields the existence of $F^{-1} \in R(\!(X)\!)$. But then $g := X^{-m}F^{-1}$ is in $R\langle\!\langle X\rangle\!\rangle$ with $fg = gf = 1$, so f is a unit. Furthermore, $g(-m) = f(m)^{-1}$ is a unit in R and $g = O(X^{-m})$ and so g is invertible. □

4. Formal derivatives

A crucial ingredient for an algebraic formulation of the Einstein-scalar field equations is to give a meaning to partial and covariant derivatives acting on generalized power series. The general ring structures employed so far are no longer sufficient to achieve this and will be specialized to complex algebras. The rings and homomorphisms introduced so far carry over this additional structure. Moreover, instead of dealing with an arbitrary number of unknowns, attention is concentrated on three of them in a form suitable for the task aimed at.

This said, let $\mathcal{A}$ be a $\mathbb{C}$-algebra and D a derivation on $\mathcal{A}$. Then $\mathcal{A}[Z]\langle\!\langle Y,X\rangle\!\rangle = \big((\mathcal{A}[Z])\langle\!\langle Y\rangle\!\rangle\big)\langle\!\langle X\rangle\!\rangle$ is a complex algebra of generalized power series in the three unknowns X,Y and Z. Furthermore, fix $H\in\mathbb{R}$.

Proposition 2.8. *The maps $\partial_t : f\mapsto \partial_t f$ and $\partial_D : f\mapsto \partial_D f$ defined by*

$$\begin{aligned}\partial_t f(x,y)(z) &:= -(x+iy)Hf(x,y)(z)+(z+1)f(x,y)(z+1)\\ \partial_D f(x,y)(z) &:= Df(x,y)(z)\end{aligned}$$

for $f\in\mathcal{A}[Z]\langle\!\langle Y,X\rangle\!\rangle$, $x,y\in\mathbb{R}$ and $z\in\mathbb{N}$ are derivations on $\mathcal{A}[Z]\langle\!\langle Y,X\rangle\!\rangle$.

Proof. Clearly, the right hand sides are elements of $\mathcal{A}$ and the supports of the corresponding series are not enlarged, so that the maps are well-defined. Their $\mathbb{C}$-linearity is immediate. Let $f,g\in\mathcal{A}[Z]\langle\!\langle Y,X\rangle\!\rangle$ and $x,y\in\mathbb{R}$, $z\in\mathbb{N}$, then

$$\begin{aligned}\partial_t(fg)(x,y)(z) &= \sum_{(x,y,z)} -[(r+u)+i(s+v)]Hf(r,s)(m)g(u,v)(n)\\ &\quad+\sum_{(x,y,z)}(m+1)f(r,s)(m+1)g(u,v)(n)\\ &\quad+\sum_{(x,y,z)}(n+1)f(r,s)(m)g(u,v)(n+1)\\ &= \left[(\partial_t f)g+f\partial_t g\right](x,y)(z)\end{aligned}$$

as well as

$$\begin{aligned}\partial_D(fg)(x,y)(z) &= D\sum_{(x,y,z)} f(r,s)(m)g(u,v)(n)\\ &= \left[(\partial_D f)g+f\partial_D g\right](x,y)(z),\end{aligned}$$

where $\sum_{(x,y,z)}$ is an abbreviation for $\sum_{r+u=x}\sum_{s+v=y}\sum_{m+n=z}$ ☐

For derivations C and D on $\mathcal{A}$ their commutator $[C,D]:=C\circ D-D\circ C$ is a derivation on $\mathcal{A}$ as well. For the derivations defined above

$$\begin{aligned}[\partial_t,\partial_D]&=0,\\ [\partial_C,\partial_D]&=\partial_{[C,D]}\end{aligned}$$

hold true. Since there is little risk of confusion the derivation ∂_D on $\mathcal{A}[Z]\langle\!\langle Y,X\rangle\!\rangle$ induced by D will be denoted simply by D in the sequel.

5. Algebraic Einstein-scalar field equations

The prerequisites introduced so far will now be used to postulate algebraic Einstein-scalar field equations and to find formal, generalized power series solutions for them. Fix $n\in\mathbb{N}$, $n\geq 2$ and let (M,γ) be a smooth, n-dimensional Riemannian manifold. The covariant derivative associated to the background metric γ shall be D. Denote by $\mathcal{T}^r_s M$ and $\mathbb{C}\mathcal{T}^r_s M$ the smooth sections of the tensor bundles $T^r_s M$

and their complexifications $T^r_s M \otimes \mathbb{C}$ over M respectively. These sections form, together with the tensor product, a graded algebra

$$\mathbb{C}\mathcal{T}(M) = \bigoplus_{r,s\in\mathbb{N}} \mathbb{C}\mathcal{T}^r_s M$$

over $\mathbb{C}$. Therefore, it is possible to define the complex algebra of formal tensors

$$\mathfrak{T}(M) := \mathbb{C}\mathcal{T}(M)[Z]\langle Y\rangle\langle\!\langle X\rangle\!\rangle.$$

The homogeneous elements $(\mathbb{C}\mathcal{T}^r_s M)[Z]\langle Y\rangle\langle\!\langle X\rangle\!\rangle$ form the subspaces of formal (r,s)-tensors that are denoted by $\mathfrak{T}^r_s(M)$. The contraction of one co- and one contravariant index in the tensor algebra $T(M) = \bigoplus_{r,s\in\mathbb{N}} T^r_s M$ canonically extends in $\mathfrak{T}(M)$ and commutes with the product there. To have a notion of reality for formal tensors in $\mathfrak{T}(M)$ define complex conjugation as a ring homomorphism $f \mapsto f^*$ by $(f^*)_{m,s,l} := (f)^*_{m,-s,l}$ for any $f \in \mathfrak{T}(M)$, $m, s \in \mathbb{R}$, $l \in \mathbb{N}$ with $(f)_{m,s,l}$ an abbreviation for the coefficient $f(m)(s)(l) \in \mathbb{C}\mathcal{T}(M)$, and call f real if $f^* = f$.

The elements of $\mathfrak{T}^r_s(M)$ can be thought of as representing an asymptotic expansion

$$a(x,t) = \sum_m \sum_s \sum_l a_{m,s,l}(x)\, t^l e^{-(m+is)Ht}$$

of a family of tensors $t \mapsto a(t) \in \mathbb{C}\mathcal{T}^r_s M$ on M for large values of the parameter $t \in \mathbb{R}$. The unknown X thereby takes the role of the factor e^{-Ht}, Y that of e^{-iHt} and Z that of the pre-factor t. With this in mind the peculiar definitions given so far, particularly those for ∂_t and the conjugation $f \mapsto f^*$, become plausible since they do exactly what one would expect the operators to do when acting coefficient-wise on such sums. In this expansion the asymptotics is expected to be dominated by the factors e^{-mHt}. The contribution of the t^l-factor is only logarithmic with respect to this whereas the e^{-isHt}-factors give rise to oscillations. Non-vanishing coefficients with $l \neq 0$ are hence suggestively called *logarithmic terms*, such with $s \neq 0$ *oscillatory terms*.

Let $g \in \mathfrak{T}^0_2(M)$ be given such that g is real, symmetric, $g = O(X^m)$ for an $m \in \mathbb{R}$ and $g(m) \in \mathcal{T}^0_2 M$ is a Riemannian metric on M. (Note that $g(m) \in \mathcal{T}^0_2 M$ means that $(g)_{m,s,l} = 0$ for all $s, l \neq 0$.) By virtue of proposition 2.7 it is possible to invert the metric g in $(\mathbb{C}\mathcal{T}^1_1 M)[Z]\langle Y\rangle\langle\!\langle X\rangle\!\rangle$ and thus construct induced fibre metrics on any of the homogeneous subspaces of the formal tensor space $\mathfrak{T}(M)$ in the usual way. Furthermore, this together with the formal derivatives introduced in proposition 2.8 allows one to define a formal connection difference tensor

$$A^a_{bc} := \frac{1}{2} g^{ic}(D_a g_{ib} + D_b g_{ai} - D_i g_{ab}),$$

from this a formal Levi-Civita connection ∇ on all $\mathfrak{T}^r_s(M)$ and thus formal Riemann, Ricci and scalar curvature tensors $\mathrm{Riem}, \mathrm{Ric}$ and $R \in \mathfrak{T}(M)$ respectively associated with g where in particular

$$(\mathrm{Ric} - \mathrm{Ric}_\gamma)_{ab} = D_i A^i_{ab} - D_a A^i_{ib} + A^j_{ab} A^i_{ij} - A^i_{aj} A^j_{ib}.$$

Fix a coupling constant $\xi \in \mathbb{R}$, a cosmological constant $\Lambda > 0$, a squared field mass $\mu^2 \in \mathbb{R}$ as well as a potential $V \in C^\infty(\mathbb{R},\mathbb{R})$ with $V(0) = \Lambda$ and $V'(0) = \mu^2/2$. Let further $\phi \in \mathfrak{T}^0_0(M)$ be a formal scalar field with $\phi = o(1)$ and assume $H > 0$. Then $\phi^*\phi = o(1)$ and the substitution homomorphisms (c.f. section 2) induced by the Maclaurin expansions of V and V' can be applied to define formal equivalents of the non-linear terms involving the potential

$$V(\phi^*\phi) := \varphi_{(\partial^n V(0)/n!)}\phi^*\phi = \sum_{n=0}^{\infty} \frac{1}{n!}\partial^n V(0)(\phi^*\phi)^n$$

$$V'(\phi^*\phi) := \varphi_{(\partial^n V'(0)/n!)}\phi^*\phi = \sum_{n=0}^{\infty} \frac{1}{n!}\partial^{n+1} V(0)(\phi^*\phi)^n$$

respectively. For the same reason the element $1 - \xi\phi^*\phi$ has a multiplicative inverse in $\mathfrak{T}^0_0(M)$ which is exactly of order zero. With these constituents available equations (1.10), (1.12)–(1.14), (1.19)–(1.22) and (1.25) define algebraic analogues of the second fundamental form k, the projections of the energy-momentum tensor of the scalar field ρ, j and S, of the evolution and constraint quantities $\mathsf{e}, \mathsf{E}, \mathsf{c}$ and C as well as of the scalar field quantity S.

It is now obvious how a definition of a formal solution to the Einstein-scalar field system can be given. Denote by $\mathfrak{I}(M) := \{f \in \mathfrak{T}(M) \mid f = O(1) \text{ and } f(0) \in \mathbb{C}\mathcal{T}(M)\}$ those formal tensors which are at least of order zero and have neither logarithmic nor oscillatory terms at order zero. The homogeneous subspaces of such (r,s)-tensors shall again be distinguished by $\mathfrak{I}^r_s(M)$. Finally, for later reference, fix constants $\mu_c^2 := \left[n^2/4 - \xi n(n+1)\right] H^2$,

$$k_1 := \begin{cases} n/2 - \sqrt{\mu_c^2 - \mu^2}/H & \text{for } \mu^2 \leq \mu_c^2 \\ n/2 & \text{for } \mu^2 > \mu_c^2 \end{cases}$$

$$k_2 := \begin{cases} n/2 + \sqrt{\mu_c^2 - \mu^2}/H & \text{for } \mu^2 \leq \mu_c^2 \\ n/2 & \text{for } \mu^2 > \mu_c^2 \end{cases}$$

and call $g(m-2)$, $\phi(m+k_1)$ the coefficients of g and ϕ at *relative* order m respectively.

DEFINITION 2.9. Let $g \in \mathfrak{I}^0_2(M)X^{-2}$ be real symmetric with $g(-2) = g_0 \in \mathcal{T}^0_2 M$ a Riemannian metric on M and $\phi \in \mathfrak{T}^0_0(M)$ of order greater than zero. Then the pair (g, ϕ) is said to be a formal solution of the Einstein-scalar field system at relative order $m \in \mathbb{R}$ if the quantities $\mathsf{e}, \mathsf{E}, \mathsf{c}, \mathsf{C}$ vanish at order m and S vanishes at order $m+k_1$. It is said to be a formal solution if those quantities vanish everywhere, i.e.

$$(\mathsf{e}, \mathsf{E}, \mathsf{c}, \mathsf{C}, \mathsf{S}) = 0.$$

It might be appropriate at this point to remark on some restrictions made by the definition above. Firstly, the formal metric g is required to have no logarithmic or oscillatory terms in leading order. This is sufficient for an inverse series to have finite support with respect to the Y and Z unknowns and thus ensures invertibility. Secondly, g is supposed to be exactly of order -2 because this corresponds to de Sitter-like asymptotics and such is expected in the models considered here. Lastly, $\phi = o(1)$ guarantees the solvability of the equations in leading order with a constant parameter H, as is explained in the following section.

6. Properties of solutions

Given a solution (g, ϕ) of the algebraic Einstein-scalar field system several interesting necessary properties are obtained easily. They are presented in the following. To start with observe that the conditions on g imply that the $(0,1)$-fibre metric lies in $\mathfrak{I}^2_0(M)X^2$ which implies further $\mathrm{Ric} \in \mathfrak{I}^0_2(M)$ and $\widehat{\mathrm{Ric}} \in \mathfrak{I}^1_1(M), R \in \mathfrak{I}^0_0(M)$, all real. On the other hand, equation (1.10) shows $k \in \mathfrak{I}^0_2(M)X^{-2}$ and so $\sigma \in$

$\mathfrak{I}^1_1(M), \operatorname{tr} k \in \mathfrak{I}^0_0(M)$, all real as well. The same equation (1.10) already determines the lowest order term of the second fundamental form to $k(-2) = -Hg_0$ and therefore

$$\begin{aligned} \sigma(0) &= 0 \\ \operatorname{tr} k(0) &= -nH \end{aligned}$$

The scalar field equation (1.18) considerably constrains the lowest order term of the formal scalar field ϕ which was left quite unrestricted by the definition 2.9. To see this assume $\phi \neq 0$ so that $m := \min \operatorname{supp} \phi$ exists. The scalar field equation (1.18) at order m then reads $\mathsf{S}(m) = 0$ which is

$$\begin{aligned} &(l+1)(l+2)(\phi)_{m,s,l+2} + (l+1)(n-2m-2is)H(\phi)_{m,s,l+1} + \\ &\Big\{\mu^2 - s^2H^2 + \big[m(m-n) + \xi n(n+1)\big]H^2 + is(2m-n)H^2\Big\}(\phi)_{m,s,l} = 0 \end{aligned}$$

for all $s \in \mathbb{R}$ and $l \in \mathbb{N}$. By choosing l large enough initially and proceeding towards zero while keeping s fixed shows that the coefficient in curly braces has to vanish, $\phi = O(X^{k_1})$ and there holds the complete alternative

$$\begin{aligned} &\phi(m) \in C^\infty(M,\mathbb{C}) && \text{for } \mu^2 < \mu_{\mathrm{c}}^2 \\ &\phi(m) \in C^\infty(M,\mathbb{C}) + C^\infty(M,\mathbb{C})Z && \text{for } \mu^2 = \mu_{\mathrm{c}}^2 \\ &\phi(m) \in C^\infty(M,\mathbb{C})Y^{-\omega} + C^\infty(M,\mathbb{C})Y^{\omega} && \text{for } \mu^2 > \mu_{\mathrm{c}}^2. \end{aligned}$$

It says that for subcritical field masses ($\mu^2 < \mu_{\mathrm{c}}^2$) there are neither logarithmic nor oscillatory terms present at lowest order, for critical field mass ($\mu^2 = \mu_{\mathrm{c}}^2$) at most one logarithmic term and for supercritical field masses ($\mu^2 > \mu_{\mathrm{c}}^2$) no logarithmic but in general two oscillatory terms at frequency $\pm\omega H$ with

$$\omega := \begin{cases} \sqrt{\mu^2 - \mu_{\mathrm{c}}^2}/H & \text{for } \mu^2 > \mu_{\mathrm{c}}^2 \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $\phi = O(X^{k_1})$ one infers that the same is true of all covariant and time derivatives, that $\phi^*\phi$, $V(\phi^*\phi)$ and $V'(\phi^*\phi)$ are in $\mathfrak{I}(M)$ and with them the projections of the effective energy-momentum tensor $\hat{S}$, $\operatorname{tr} S$, j, ρ together with the evolution and constraint quantities e, E, c, C. For the scalar field quantity S it follows that $\mathsf{S} = O(X^{k_1})$. More precisely the equations (1.12)–(1.14) yield $S = -\Lambda g_0 X^{-2} + O(X^{2k_1-2})$ and so

$$\begin{aligned} \hat{S} &= O(X^{2k_1}) \\ \operatorname{tr} S &= -n\Lambda + O(X^{2k_1}) \\ j &= O(X^{2k_1}) \\ \rho &= \Lambda + O(X^{2k_1}). \end{aligned}$$

The zero order coefficient of e reads $(\mathsf{e})_{0,0,0} = 2n\Lambda/(n-1) - n^2H^2$ so that its vanishing implies

$$H = \sqrt{\frac{2\Lambda}{n(n-1)}}.$$

Likewise, the validity of the algebraic Einstein equations on the open interval $]0, k_0[$ where $0 < k_0 \leq \min\{2, 2k_1\}$ implies the vanishing of g and thus k on $]-2, 2k_0-2[$, of the induced $(0,1)$-fibre metric on $]2, 2k_0+2[$ and so of σ, $\operatorname{tr} k$, Ric on $]0, k_0[$ and $\widehat{\operatorname{Ric}}, R$ on $]2, k_0+2[$.

Taking the results of the preceding paragraph into account the coefficients of the evolution quantities fulfill the relations

$$\begin{aligned}(\mathsf{e})_{m,s,l} &= (2n-m-is)H(\operatorname{tr}k)_{m,s,l} + (l+1)(\operatorname{tr}k)_{m,s,l+1} - (R)_{m,s,l} \\ &- \sum_{p+q=m}\sum_{u+v=s}\sum_{\kappa+\lambda=l}(\operatorname{tr}k)_{p,u,\kappa}(\operatorname{tr}k)_{q,v,\lambda} \\ &- \frac{1}{n-1}\Big[(\operatorname{tr}S)_{m,s,l} + n(\rho)_{m,s,l}\Big]\end{aligned} \tag{2.1}$$

$$\begin{aligned}(\mathsf{E})_{m,s,l} &= (n-m-is)H(\sigma)_{m,s,l} + (l+1)(\sigma)_{m,s,l+1} - (\widehat{\mathrm{Ric}})_{m,s,l} \\ &- \sum_{p+q=m}\sum_{u+v=s}\sum_{\kappa+\lambda=l}(\sigma)_{p,u,\kappa}(\operatorname{tr}k)_{q,v,\lambda} + (\hat{S})_{m,s,l}\end{aligned} \tag{2.2}$$

and those of the Hamiltonian constraint quantity

$$\begin{aligned}(\mathsf{c})_{m,s,l} &= -(2n-2)H(\operatorname{tr}k)_{m,s,l} + (R)_{m,s,l} - 2(\rho)_{m,s,l} \\ &+ \sum_{p+q=m}\sum_{u+v=s}\sum_{\kappa+\lambda=l}\Big[-\langle(\sigma)_{p,u,\kappa},(\sigma)_{q,v,\lambda}\rangle \\ &+ \Big(1-\frac{1}{n}\Big)(\operatorname{tr}k)_{p,u,\kappa}(\operatorname{tr}k)_{q,v,\lambda}\Big],\end{aligned} \tag{2.3}$$

for $m \geq k_0$ and all $s \in \mathbb{R}$, $l \in \mathbb{N}$ whereas the indices p and q shall take values never less than k_0. The evolution equation for the metric (1.10) implies the validity of

$$\begin{aligned}(m+is)H(g)_{m-2,s,l} - (l+1)(g)_{m-2,s,l+1} = 2(g)_{-2,0,0}\Big[(\sigma)_{m,s,l} \\ +\frac{\delta}{n}(\operatorname{tr}k)_{m,s,l}\Big] + 2\sum_{p+q=m}\sum_{u+v=s}\sum_{\kappa+\lambda=l}(g)_{p-2,u,\kappa}\Big[(\sigma)_{q,v,\lambda} + \frac{\delta}{n}(\operatorname{tr}k)_{q,v,\lambda}\Big]\end{aligned} \tag{2.4}$$

for the corresponding coefficients, with δ denoting the identity element in $\mathcal{T}^1_1M$. The scalar field quantity at relative order $m > 0$ satisfies

$$\begin{aligned}(\mathsf{S})_{m+k_1,s,l} &= (l+1)(l+2)(\phi)_{m+k_1,s,l+2} \\ &- (l+1)(2m+k_1-k_2+2is)H(\phi)_{m+k_1,s,l+1} \\ &+ \big[m+i(s-\omega)\big]\big[m+k_1-k_2+i(s+\omega)\big]H^2(\phi)_{m+k_1,s,l} - \mathrm{LOT}_{m+k_1,s,l}\end{aligned} \tag{2.5}$$

with $\mathrm{LOT}_{m+k_1,s,l}$ containing lower order scalar field coefficients only, i.e.

$$\begin{aligned}\mathrm{LOT}_{m+k_1,s,l} &= (\Delta\phi)_{m+k_1,s,l} \\ &- \sum_{r<m}\sum_{u+v=s}\sum_{\kappa+\lambda=l}\Big[(r+k_1+iv)H(\operatorname{tr}k)_{m-r,u,\kappa} \\ &+ \xi(R_{\tilde{g}})_{m-r,u,\kappa} + (V'(\phi^*\phi))_{m-r,u,\kappa}\Big](\phi)_{r+k_1,v,\lambda} \\ &+ \sum_{r<m}\sum_{u+v=s}\sum_{\kappa+\lambda=l}(\lambda+1)(\operatorname{tr}k)_{m-r,u,\kappa}(\phi)_{r+k_1,v,\lambda+1}.\end{aligned} \tag{2.6}$$

Lastly, if the validity of the Einstein-scalar field equations up to but not including relative order $m \geq k_0$ is assumed the propagation equations (1.23), (1.24) for the constraint quantities yield for the coefficients at relative order m

$$(2n-m-is)H(\mathsf{c})_{m,s,l} + (l+1)(\mathsf{c})_{m,s,l+1} = -2(n-1)H(\mathsf{e})_{m,s,l} \tag{2.7}$$

and

$$\begin{aligned}(n-m-is)H(\mathsf{C})_{m,s,l} + (l+1)(\mathsf{C})_{m,s,l+1} = \\ -\frac{1}{2}d(\mathsf{c})_{m,s,l} + \operatorname{div}_{g_0}(\mathsf{E})_{m,s,l} - \Big(1-\frac{1}{n}\Big)d(\mathsf{e})_{m,s,l}\end{aligned} \tag{2.8}$$

by virtue of equation (1.6).

In the previous section the question whether the assumption of $\phi = o(1)$ could be relaxed was left open. It is now argued that already $\phi = O(1)$ can, in general, not be treated within the class of series under consideration. Assume $\phi \in \mathfrak{I}^0_0(M)$ and $\phi(0) = \phi_0 \in C^\infty(M, \mathbb{C})$ not identically zero with $1 - \xi\phi_0^*\phi_0 > 0$. Then $1 - \xi\phi^*\phi$ is invertible and the analysis outlined above goes through. For the Einstein equations to be satisfied at leading order it is necessary that $\mathsf{e}(0) = 0$ which now is true if and only if

$$H^2 = \frac{2}{n(n-1)} \frac{V(\phi_0^*\phi_0)}{1 - \xi\phi_0^*\phi_0},$$

a condition which is not met in general by a constant H. Requiring $\phi_0^*\phi_0$ constant on the other hand is a seemingly artificial restriction on the asymptotic data that can be prescribed and so ensuring $\phi = o(1)$ by hypothesis was chosen instead.

7. Existence of initial solutions

After noting some necessary properties of formal solutions of the Einstein-scalar field system such solutions shall now be constructed. Regarding the asymptotic initial data it turns out that the situation is somewhat analogous to a Cauchy problem at conformal infinity [**10**] in that the metric g_0 (as well as data for the scalar field ϕ) can be prescribed arbitrarily while the remaining piece, a symmetric 2-tensor h, has to satisfy constraints on its trace and divergence. In the case of a pure cosmological constant, $T = -\Lambda g$, this conditions demand h transverse-traceless [**24**]. In the presence of a scalar field the right hand sides of these conditions become non-trivial much as in the case of a perfect fluid [**24**]. Since the tensor h and hence the asymptotic constraints come into play only at relative order n of g initial solutions will first be constructed which allow then a convenient formulation of the asymptotic constraints and an extension to full solutions in the next section.

In the following the inequality

$$\mu^2 > -\xi n(n+1)H^2 \tag{2.9}$$

shall be assumed. This has the consequence of k_1 being positive and hence the resulting ϕ being of order greater than zero. The condition can be understood when considering a real scalar field. As was shown in chapter 1 the transformation of the system into the Einstein frame changes the potential according to equation (1.29) with the effect that the above condition is satisfied if and only if the squared mass $U''(0)$ of the minimally coupled field is positive. In other words the inequality ensures that the minimum at the origin of the transformed potential is non-degenerate.

The construction of solutions relies again heavily on proposition 2.1. Let A be a $+$-stable superset of $\{0, 1, k_1, k_2 - k_1\}$ in $\mathfrak{A}_0$. Suppose that the support of g is contained in $A - 2$ and that of ϕ in $A + k_1$. Then the support of the $(0,1)$-fibre metric is in $A + 2$ and since $\operatorname{supp} k \subset A - 2$ it follows that the supports of $\hat{S}, \operatorname{tr} S, j$ and ρ are subsets of A. It follows that $\operatorname{supp} \mathsf{e}, \operatorname{supp} \mathsf{E}, \operatorname{supp} \mathsf{c}, \operatorname{supp} \mathsf{C} \subset A$ as well as $\operatorname{supp} \mathsf{S} \subset A$. Therefore it is sufficient for such a solution to satisfy the algebraic Einstein-scalar field equations for all relative orders in the set A. Conversely, this observation can be used to construct a solution inductively for relative orders in the set A.

PROPOSITION 2.10. *Let $g_0 \in \mathcal{T}^0_2 M$ be a smooth Riemannian metric on M and $\phi_0, \phi_1 \in C^\infty(M, \mathbb{C})$ smooth complex-valued functions on M. Then there exists a formal solution (g, ϕ) of the algebraic Einstein equations up to but not including* *relative order n such that*

(G1) $\qquad g(-2) = g_0$

(P1) $$\phi(k_1) = \begin{cases} \phi_0 & \text{for } \mu^2 < \mu_\mathrm{c}^2 \\ \phi_0 + \phi_1 Z & \text{for } \mu^2 = \mu_\mathrm{c}^2 \\ \phi_0 Y^{-\omega} + \phi_1 Y^{\omega} & \text{for } \mu^2 > \mu_\mathrm{c}^2 \end{cases}$$

(P2) $\qquad \phi(k_2) \in \phi_1 + C^\infty(M,\mathbb{C})Z \quad \text{if } \mu^2 < \mu_\mathrm{c}^2.$

The solution is unique up to relative order n exclusive. There are neither logarithmic nor oscillatory terms in g and ϕ before relative order n and order k_2 respectively.

Proof. The proof goes by induction over relative orders m in the set $A \cap [0, n[$. The argument is given as how to extend solutions of relative orders $[0, m[$ to solutions of relative orders $[0, m]$. More precisely, for $0 \leq m < n$ suppose a real symmetric $g : [-2, m-2[\to (\mathbb{C}\mathcal{T}_2^0 M)[Z]\langle Y\rangle$ as well as a $\phi : [k_1, m+k_1[\to C^\infty(M,\mathbb{C})[Z]\langle Y\rangle$ both with finite support have been constructed such that the algebraic Einstein-scalar field equations hold for relative orders up to m exclusive. Assume further that g vanishes on $]-2, k_0-2[$, that for m positive (G1), (P1) and, if $m + k_1 > k_2$, additionally (P2) are satisfied. Finally, there shall be neither logarithmic nor oscillatory terms present in g before relative order m and in ϕ prior to both order $m+k_1$ and k_2. Then the coefficients $g(m-2)$ and $\phi(m+k_1)$ amount to:

$m = 0$: Define $g(-2)$ and $\phi(k_1)$ according to (G1) and (P1), then $\mathsf{e}(0) = 0$, $\mathsf{E}(0) = 0$ and $\mathsf{c}(0) = 0$ due to the choice of H. By (2.8) it then follows that $\mathsf{C}(0) = 0$. Further, k_1 and ω where defined such that they guarantee $\mathsf{S}(k_1) = 0$. Hence the Einstein-scalar field equations are satisfied at relative order zero.

$0 < m < k_0$: Set $g(m-2) := 0$. It follows that σ and $\operatorname{tr} k$ vanish at order m. The same is true for the projections $\hat{S}, \operatorname{tr} S, j$ and ρ of the energy-momentum tensor because $k_0 \leq 2k_1$. Then $k_0 \leq 2$ implies the Einstein equations to be satisfied at relative order m. If $\mu^2 > \mu_\mathrm{c}^2$ or $\mu^2 \leq \mu_\mathrm{c}^2$ and $m+k_1 \neq k_2$ relation (2.5) provides the existence of a coefficient $\phi(m+k_1) \in C^\infty(M,\mathbb{C})[Z]\langle Y\rangle$ which yields $\mathsf{S}(m+k_1) = 0$. Note that for $m+k_1 < k_2$ it is even true that $\phi(m+k_1) \in C^\infty(M,\mathbb{C})$. In the case $\mu^2 \leq \mu_\mathrm{c}^2$ and $m+k_1 = k_2$ the induction hypothesis together with (2.5) ensure the existence of a $\chi \in C^\infty(M,\mathbb{C})$ such that, if $\phi(m+k_1) := \phi_1 + \chi Z$, the scalar field quantity S vanishes at $m+k_1$. So in any case the scalar field equation is satisfied at relative order m as well. In particular, (P2) is true for $m+k_1 = k_2$.

$k_0 \leq m < n$: The evolution equations (2.1), (2.2) and (2.4) together with the induction hypothesis show the existence of a symmetric $g(m-2) \in \mathcal{T}_2^0 M$ which causes $\mathsf{e}(m)$ and $\mathsf{E}(m)$ to vanish. By (2.7) it follows then that $\mathsf{c}(m) = 0$ and (2.8) finally yields $\mathsf{C}(m) = 0$ and so the Einstein equations hold at relative order m. The field coefficient $\phi(m+k_1)$ which cancels $\mathsf{S}(m+k_1)$ is obtained exactly as in the case above. In particular, $\phi(m+k_1) \in C^\infty(M,\mathbb{C})$ for $m+k_1 < k_2$ and (P2) holds for $m+k_1 = k_2$.

The existence part of the statement now follows by transfinite recursion over $A \cap [0, n[$. Uniqueness holds as a consequence of the necessary conditions a solution has to fulfill established in section 6 and the construction. ☐

8. Asymptotic constraints and existence of solutions

To formulate asymptotic constraints it is useful to define two functionals z and Z on the space of freely choosable asymptotic initial data. This data consists of a Riemannian metric $g_0 \in \mathcal{T}_2^0 M$ and two smooth complex-valued functions $\phi_0, \phi_1 \in C^\infty(M,\mathbb{C})$. For any such datum there is, by virtue of proposition 2.10, a solution

(g, ϕ) of the algebraic Einstein-scalar field system up to but not including relative order n satisfying (G1), (P1) and (P2) and it is thus possible to set

$$\mathsf{z}(g_0, \phi_0, \phi_1) := \operatorname{tr}_{g_0}(g)_{n-2,0,0} - \frac{2}{n^2 H^2}(\mathsf{e})_{n,0,0}$$
$$\mathsf{Z}(g_0, \phi_0, \phi_1) := \operatorname{div}_{g_0}(g)_{n-2,0,0} - \frac{2}{n^2 H^2}\left[d(\mathsf{e})_{n,0,0} + nH(\mathsf{C})_{n,0,0}\right]$$

That this indeed defines functionals can be seen by noting that the right-hand sides are independent of the actual choice of the initial solution (g, ϕ) and are therefore uniquely determined by the corresponding asymptotic initial datum (g_0, ϕ_0, ϕ_1) due to proposition 2.10.

A necessary condition on the coefficient $(g)_{n-2,0,0}$ of any formal solution is now obvious: suppose that (g, ϕ) is a solution to the algebraic Einstein-scalar field system which fulfills (G1), (P1) and (P2), then it is true that $\operatorname{tr}_{g_0}(g)_{n-2,0,0} = \mathsf{z}(g_0, \phi_0, \phi_1)$ and $\operatorname{div}_{g_0}(g)_{n-2,0,0} = \mathsf{Z}(g_0, \phi_0, \phi_1)$. That this condition is also sufficient for the existence of formal solutions is proven in the next theorem.

THEOREM 2.11. *Let $g_0 \in \mathcal{T}^0_2 M$ be a smooth Riemannian metric, $\phi_0, \phi_1 \in C^\infty(M, \mathbb{C})$ smooth complex-valued functions and $h \in \mathcal{T}^0_2 M$ a smooth symmetric 2-tensor on M satisfying*

(AC1) $\quad \operatorname{tr}_{g_0} h = \mathsf{z}(g_0, \phi_0, \phi_1)$

(AC2) $\quad \operatorname{div}_{g_0} h = \mathsf{Z}(g_0, \phi_0, \phi_1)$.

Then there exists a unique formal solution (g, ϕ) of the algebraic Einstein equations such that

(G1) $\quad g(-2) = g_0$

(G2) $\quad (g)_{n-2,0,0} = h$

(P1) $$\phi(k_1) = \begin{cases} \phi_0 & \text{for } \mu^2 < \mu_c^2 \\ \phi_0 + \phi_1 Z & \text{for } \mu^2 = \mu_c^2 \\ \phi_0 Y^{-\omega} + \phi_1 Y^{\omega} & \text{for } \mu^2 > \mu_c^2 \end{cases}$$

(P2) $\quad \phi(k_2) \in \phi_1 + C^\infty(M, \mathbb{C})Z \quad \text{if } \mu^2 < \mu_c^2$

hold true. There are no logarithmic or oscillatory terms present in g and ϕ before relative order n and $k_2 - k_1$ respectively.

PROOF. The proof goes again by induction over relative orders m in A, a $+$-stable superset of $\{0, 1, k_1, k_2 - k_1\}$ in $\mathfrak{A}_0$. Since the construction has already been carried out up to relative order n, exclusive, in proposition 2.10 $m \geq n$ will be assumed. Suppose $g : [-2, m-2[\to (\mathbb{C}\mathcal{T}^0_2 M)[Z]\langle Y\rangle$ real symmetric and $\phi : [k_1, m+k_1[\to C^\infty(M, \mathbb{C})[Z]\langle Y\rangle$ being constructed such that their support is finite, they solve the algebraic Einstein-scalar field equations up to but not including relative order m and satisfy conditions (G1), (P1) and (P2). Assume further that g vanishes on $]-2, k_0 - 2[$ and that for $m > n$ (G2) holds. Finally, there shall be neither logarithmic nor oscillatory terms present in g before relative order n and in ϕ prior to order k_2. Then the metric is extended as follows:

$m = n$: The evolution equations (2.1), (2.2) and (2.4) yield the existence of a real symmetric $g(n-2) \in (\mathbb{C}\mathcal{T}^0_2 M)[Z]\langle Y\rangle$ with $(g)_{n-2,0,0} = h$ such that $\mathsf{E}(n) = 0$ and $\mathsf{e}(n) \in C^\infty(M, \mathbb{C})$. Calculating $(\mathsf{e})_{n,0,0}$ and using the trace condition (AC1) shows that this coefficient actually vanishes and so $\mathsf{e}(n) = 0$ too. But this then implies $\mathsf{c}(n) = 0$ by (2.7) and consequentially $\mathsf{C}(n) \in \mathbb{C}\mathcal{T}^1_0 M$ by (2.8). The divergence condition (AC2) then ensures $(\mathsf{C})_{n,0,0} = 0$ so that the Einstein equations are satisfied at relative order n.

$m = 2n$: In this case the relations (2.2), (2.3) and (2.4) can be used to obtain a real symmetric $g(m-2) \in (\mathbb{C}\mathcal{T}_2^0 M)[Z]\langle Y\rangle$ which cancels the quantities E and c at order m. It follows then from equation (2.7) that $\mathsf{e}(m) = 0$ and hence further from equation (2.8) that $\mathsf{C}(m) = 0$. This solves the algebraic Einstein equations at relative order m.

$m \notin \{n, 2n\}$: The equations (2.1), (2.2) and (2.4) guarantee the existence of a real symmetric $g(m-2) \in (\mathbb{C}\mathcal{T}_2^0 M)[Z]\langle Y\rangle$ such that the evolution quantities e and E vanish at order m. Using once again the propagation equations (2.7) and (2.8) subsequently shows that $\mathsf{c}(m) = 0$ and $\mathsf{C}(m) = 0$ too so that the Einstein equations are fulfilled at relative order m.

The scalar field is then extended in any case by means of equation (2.5) which provides a coefficient $\phi(m + k_1) \in C^\infty(M, \mathbb{C})[Z]\langle Y\rangle$ such that $S(m + k_1) = 0$, i.e. the scalar field equation as well is satisfied at relative order m.

The existence of a solution is now obtained by transfinite recursion over A. Uniqueness thereof follows from the properties stated in section 2.10, the hypotheses of the theorem and the construction. □

Theorem 2.11 shows that the conditions (AC1) and (AC2) are not only necessary but also sufficient for a solution with the properties (G1), (G2), (P1) and (P2) to exist. Due to their analogy with the constraint equations for the Cauchy problem, in particular with the momentum constraint (1.17), they are called the asymptotic constraints.

Note that for non-subcritical field masses the exponents in the formal solution become especially simple, more precisely, because of $k_1 = k_2 = n/2$ it is $A = \mathbb{N}$ for even n and $A = (1/2)\mathbb{N}$ for odd n, where $\mathbb{N}$ denotes the set of natural numbers. Oscillatory terms always occur at integer multiples of ω with no such terms at all if the field mass is not supercritical.

CHAPTER 3

Asymptotics of solutions

1. Exponentially bounded functions

The formal series solutions obtained in chapter 2 were motivated as representing a certain asymptotic expansion of a smooth tensor family. This notion is of course to be made precise and it will be shown that it applies to solutions of the Einstein-scalar field system thus providing full information about their asymptotics.

Let M be a smooth manifold and $I \subset \mathbb{R}$ an open interval not bounded from above.

Definition 3.1. Let E be a vector bundle over M together with a positive definite bundle metric $\langle\cdot,\cdot\rangle$. A family $f : I \to \Gamma E$ of sections of E is called σ-exponentially bounded, $\sigma > 0$, if

$$|f|(t) = \langle f, f\rangle^{1/2}(t) = O(\sigma^t) \qquad (t \to \infty)$$

is true locally uniformly, i.e. for any $p \in M$ there exists a neighbourhood U of p, a time $t_0 \in I$ and a constant $C > 0$ such that $|f|(t) \leq C\sigma^t$ on U for all $t > t_0$. Denote the space of all σ-exponentially bounded smooth families by $\mathfrak{B}_\sigma(\Gamma E)$.

Fix a Riemannian metric γ on M and a finite-dimensional complex vector space V with a positive definite inner product. The metric γ and the inner product on V induce fibre metrics $\langle\cdot,\cdot\rangle_\gamma$ on any bundle $T^r_s M \otimes V^{\otimes n}$ of $V^{\otimes n}$-valued (r, s)-tensors. Consider the spaces of exponentially bounded smooth (r, s)-tensors

$$\mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n}) := \bigcup_{\sigma>0} \mathfrak{B}_\sigma\big(\Gamma(T^r_s M \otimes V^{\otimes n})\big)$$

together with the map defined by

$$\mathfrak{v}(f) := \inf\big\{\sigma > 0 \mid f \in \mathfrak{B}_\sigma\big(\Gamma(T^r_s M \otimes V^{\otimes n})\big)\big\}$$

for all $f \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ and $r, s, n \in \mathbb{N}$. This map satisfies

(i) $\quad \mathfrak{v}(f) \geq 0$

(ii) $\quad \mathfrak{v}(f + g) \leq \max\{\mathfrak{v}(f), \mathfrak{v}(g)\}$

(iii) $\quad \mathfrak{v}(f \otimes f') \leq \mathfrak{v}(f)\mathfrak{v}(f')$

for any $f, g \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$, $f' \in \mathfrak{B}(\mathcal{T}^{r'}_{s'} M, V^{\otimes n'})$ where $r, s, n, r', s', n' \in \mathbb{N}$. Furthermore $\mathfrak{v}(0) = 0$ and $\mathfrak{v}(a) = 1$ for any unit a in $C^\infty(M, \mathbb{C})$ are valid. From this the inverse triangle inequality

(iv) $\quad |\mathfrak{v}(f) - \mathfrak{v}(g)| \leq \mathfrak{v}(f + g)$

is inferred. Note that for any $f \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$, $l \in \mathbb{N}$ and $c = \Re c + i\Im c \in \mathbb{C}$ the equalities $\mathfrak{v}(t^l e^{ct}) = e^{\Re c}$ and $\mathfrak{v}(fe^{ct}) = \mathfrak{v}(f)e^{\Re c}$ hold.

To make contact with the formal tensors it is useful to define an evaluation homomorphism $\mathcal{F} : (\mathcal{V}^n \mathcal{T}^r_s M)[Z]\langle Y, X\rangle \to \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ by

$$\mathcal{F}\,\boldsymbol{f} := \sum_{m,s\in\mathbb{R}} \sum_{l\in\mathbb{N}} (\boldsymbol{f})_{m,s,l}\, t^l e^{-(m+is)Ht}$$

where $\mathcal{V}^n\mathcal{T}^r_s M$ abbreviates the space $\Gamma(T^r_s M \otimes V^{\otimes n})$. It satisfies $\mathfrak{v}(\mathcal{F}\, X^\alpha) = e^{-\alpha H}$ for $\alpha \in \mathbb{R}$ and so $\mathfrak{v}(\mathcal{F}\,\boldsymbol{f}) \leq e^{-\alpha H}$ if $\boldsymbol{f} = O(X^\alpha)$ as well as $\mathfrak{v}(\mathcal{F}\,\boldsymbol{f}) < e^{-\alpha H}$ if $\boldsymbol{f} = o(X^\alpha)$. The image of $\mathcal{F}$ distinguishes the linear subspaces $\mathfrak{E}^r_s(M, V^{\otimes n}) := \mathcal{F}\big((\mathcal{V}^n\mathcal{T}^r_s M)[Z]\langle Y, X\rangle\big)$ of $\mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ in which the following important identity theorem is valid.

Proposition 3.2. *Let* $\boldsymbol{f} \in (\mathcal{V}^n\mathcal{T}^r_s M)[Z]\langle Y, X\rangle$ *be given and take* $\alpha \geq \operatorname{supp} \boldsymbol{f}$*. If* $\mathfrak{v}\,(\mathcal{F}\,\boldsymbol{f}) < e^{-\alpha H}$ *then* $\boldsymbol{f} = 0$*.*

Proof. Assume $\boldsymbol{f} \neq 0$ so that $m := \min \operatorname{supp} \boldsymbol{f}$ exists in $\mathbb{R}$ and $0 \neq \boldsymbol{f}(m) \in (\mathcal{V}^n\mathcal{T}^r_s M)[Z]\langle Y\rangle$. Now $\boldsymbol{f}(m) = X^{-m}\boldsymbol{f} - \big[X^{-m}\boldsymbol{f} - \boldsymbol{f}(m)\big]$ with $X^{-m}\boldsymbol{f} - \boldsymbol{f}(m) = o(1)$ so that $\mathfrak{v}\big(\mathcal{F}[X^{-m}\boldsymbol{f} - \boldsymbol{f}(m)]\big) < 1$ as well as $\mathfrak{v}(\mathcal{F}\,X^{-m}\boldsymbol{f}) < 1$ by hypothesis. But this implies $\mathfrak{v}\big(\mathcal{F}\,\boldsymbol{f}(m)\big) < 1$. For a $p \in M$ set

$$f(t) := (\mathcal{F}\,\boldsymbol{f}(m))(t)(p) = \sum_{s\in\mathbb{R}}\sum_{l\in\mathbb{N}} (\boldsymbol{f})_{m,s,l}(p) t^l e^{-isHt}$$

then it follows from what has been stated that there are $0 < \epsilon < 1$, $C > 0$ and $1 < t_0 \in I$ with $|f(t)|_\gamma \leq Ce^{-\epsilon t}$ for all $t > t_0$. Since $\boldsymbol{f}(m) \neq 0$ there exists $s_0 \in \operatorname{supp} \boldsymbol{f}(m)$ and $l_0 \in \operatorname{supp} \boldsymbol{f}(m, s_0)$ such that $\max_{s\in\mathbb{R}} \operatorname{supp} \boldsymbol{f}(m, s) \leq l_0$. Consider now the integral

$$\frac{1}{t - t_0}\int_{t_0}^{t} \tau^{-n_0} e^{-is_0 H\tau} f(\tau) d\tau$$

for $t > t_0$. From the estimate above it can be concluded that

$$\Big|\frac{1}{t - t_0}\int_{t_0}^{t} \tau^{-n_0} e^{-is_0 H\tau} f(\tau) d\tau\Big|_\gamma \leq \frac{C}{t - t_0}\int_{t_0}^{t} e^{-\epsilon\tau} d\tau \to 0 \quad (t \to \infty),$$

on the other hand

$$\frac{1}{t - t_0}\int_{t_0}^{t} \tau^{-n_0} e^{-is_0 H\tau} f(\tau) d\tau = \frac{1}{t - t_0}\int_{t_0}^{t} (\boldsymbol{f})_{m,s_0,l_0}(p)$$
$$+ \sum_{(s,l)\neq(s_0,l_0)} \frac{1}{t - t_0}\int_{t_0}^{t} (\boldsymbol{f})_{m,s,l}(p)\tau^{l-l_0} e^{-i(s-s_0)H\tau} d\tau \to (\boldsymbol{f})_{m,s_0,l_0}(p)$$

as $t \to \infty$ so that $(\boldsymbol{f})_{m,s_0,l_0}(p) = 0$. Since $p \in M$ was arbitrary $(\boldsymbol{f})_{m,s_0,l_0}$ vanishes identically which is a contradiction to the choice of l_0 in the support of $\boldsymbol{f}(m, s_0)$. □

Corollary 3.3. *For any* $\boldsymbol{f} \in (\mathcal{V}^n\mathcal{T}^r_s M)[Z]\langle Y, X\rangle$ *and* $\alpha \in \mathbb{R}$ *one has*

$$\begin{matrix} \mathfrak{v}(\mathcal{F}\,\boldsymbol{f}) < e^{-\alpha H} & & \boldsymbol{f} = o(X^\alpha) \\ & \textit{if and only if} & \\ \mathfrak{v}(\mathcal{F}\,\boldsymbol{f}) \leq e^{-\alpha H} & & \boldsymbol{f} = O(X^\alpha). \end{matrix}$$

Corollary 3.4. *The evaluation homomorphism* $\mathcal{F}$ *is injective and hence the vector spaces* $\mathfrak{E}^r_s(M, V^{\otimes n})$ *and* $(\mathcal{V}^n\mathcal{T}^r_s M)[Z]\langle Y, X\rangle$ *isomorphic.*

By virtue of this last corollary it is possible and will be useful to define a function on $\mathfrak{E}^r_s(M, V^{\otimes n})$ measuring the polynomial degree. This is done by setting

$$\deg f := -H \max \operatorname{supp} \mathcal{F}^{-1} f$$

for non-zero $f \in \mathfrak{E}^r_s(M, V^{\otimes n})$ and $\deg 0 := \infty$ where $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$ on $\mathfrak{E}^r_s(M, V^{\otimes n})$. The assertion of proposition 3.2 is then simply stated as $\mathfrak{v}(f) < e^{\deg f}$ implies $f = 0$ for $f \in \mathfrak{E}^r_s(M, V^{\otimes n})$. From the behaviour of the supports of formal series under addition and multiplication it follows easily that

$$\deg(f + g) \geq \min\{\deg f, \deg g\}$$
$$\deg(f \otimes f') \geq \deg f + \deg f'$$

hold for arbitrary $f, g \in \mathfrak{E}^r_s(M, V^{\otimes n})$, $f' \in \mathfrak{E}^{r'}_{s'}(M, V^{\otimes n'})$.

2. Asymptotic approximation

In order to show that the formal series indeed approximate solutions to the Einstein-scalar field system asymptotically it is helpful to have estimates of how the errors behave under non-linear substitution, differentiation and solution of ordinary differential equations. A couple of such tools shall be provided now.

Let again (M,γ) be a Riemannian manifold, V a finite-dimensional complex scalar product space and I an open real interval not bounded from above. To emphasize the role of γ as a fixed background metric only, the covariant derivative induced by it will again be denoted by D. On smooth tensor families $I \to \mathcal{V}^n\mathcal{T}^r_s M$ this covariant derivative commutes with ∂_t, the derivative along I, i.e. $[\partial_t, D] = 0$. It is thus appropriate to define

$$\mathfrak{B}^p(\mathcal{T}^r_s M, V^{\otimes n}) := \{ f \mid \partial_t^i D^j f \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n}) \text{ for all } i \le p, j \in \mathbb{N} \}$$

together with

$$\mathfrak{v}_p(f) := \sup_{i\le p}\sup_{j\in\mathbb{N}} \mathfrak{v}(\partial_t^i D^j f)$$

for $f \in \mathfrak{B}^p(\mathcal{T}^r_s M, V^{\otimes n})$ for any $p \in \mathbb{N}\cup\{\infty\}$. It is immediate that the properties (i)–(iv) of $\mathfrak{v}$ stated in section 1 remain valid for $\mathfrak{v}_p$ as well. Moreover, due to the fact that for any $\boldsymbol{f} \in (\mathcal{V}^n\mathcal{T}^r_s M)[Z]\langle Y, X\rangle$

$$\partial_t\,\mathcal{F}\,\boldsymbol{f} = \mathcal{F}\,\partial_t \boldsymbol{f}$$
$$D\,\mathcal{F}\,\boldsymbol{f} = \mathcal{F}\,\partial_D \boldsymbol{f}$$

are true it is clear that $\mathfrak{E}^r_s(M, V^{\otimes n}) \subset \mathfrak{B}^p(\mathcal{T}^r_s M, V^{\otimes n})$ and $\mathfrak{v}_p \le \mathfrak{v}$ on $\mathfrak{E}^r_s(M, V^{\otimes n})$.

Suppose for the following that f and g are functions in $\mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ that are approximated by elements $\mathcal{F}\,\boldsymbol{f}$ and $\mathcal{F}\,\boldsymbol{g} \in \mathfrak{E}^r_s(M, V^{\otimes n})$ of degree greater than $-\alpha H \in \mathbb{R}$ in the sense that $\mathfrak{v}_p(f - \mathcal{F}\,\boldsymbol{f}), \mathfrak{v}_p(g - \mathcal{F}\,\boldsymbol{g}) \le e^{-\alpha H}$. It is then obvious that the sum $f + g$ is approximated equally well by the element $\mathcal{F}(\boldsymbol{f} + \boldsymbol{g})$, which is of degree greater than $-\alpha H$ too, namely $\mathfrak{v}_p[(f+g) - \mathcal{F}(\boldsymbol{f}+\boldsymbol{g}) \le e^{-\alpha H}]$. Furthermore, such an element is unique due to proposition 3.2. Similar results can be obtained for non-linear operations such as multiplication and substitution.

PROPOSITION 3.5. *Assume* $\mathcal{F}\,\boldsymbol{f} \in \mathfrak{E}^r_s(M, V^{\otimes n})$, $\mathcal{F}\,\boldsymbol{g} \in \mathfrak{E}^{r'}_{s'}(M, V^{\otimes n'})$ *such that* $\mathfrak{v}(f), \mathfrak{v}(g) \le e^{-\beta H}$ *for a* $\beta \le \alpha$ *then there is a unique* $\mathcal{F}\,\boldsymbol{h} \in \mathfrak{E}^{r+r'}_{s+s'}(M, V^{\otimes(n+n')})$ *of degree greater than* $-(\alpha+\beta)H$ *with* $\mathfrak{v}_p(f\otimes g - \mathcal{F}\,\boldsymbol{h}) \le e^{-(\alpha+\beta)H}$.

PROOF. If $\boldsymbol{h} := \sum_{m<\alpha+\beta} (\boldsymbol{f}\otimes\boldsymbol{g})(m) X^m$ then clearly $\deg \mathcal{F}\,\boldsymbol{h} > -(\alpha+\beta)H$ and $\mathfrak{v}_p[\mathcal{F}(\boldsymbol{f}\otimes\boldsymbol{g} - \boldsymbol{h})] \le e^{-(\alpha+\beta)H}$. Further, $\mathfrak{v}_p(\mathcal{F}\,\boldsymbol{f}) \le \max\{\mathfrak{v}_p(f - \mathcal{F}\,\boldsymbol{f}), \mathfrak{v}(f)\} \le e^{-\beta H}$ as well as $\mathfrak{v}_p(\mathcal{F}\,\boldsymbol{g}) \le e^{-\beta H}$ analogously. But then $f\otimes g - \mathcal{F}\,\boldsymbol{h} = (f - \mathcal{F}\,\boldsymbol{f}) \otimes (g - \mathcal{F}\,\boldsymbol{g}) + (f - \mathcal{F}\,\boldsymbol{f}) \otimes (\mathcal{F}\,\boldsymbol{g}) + (\mathcal{F}\,\boldsymbol{f}) \otimes (g - \mathcal{F}\,\boldsymbol{g}) + \mathcal{F}(\boldsymbol{f}\otimes\boldsymbol{g} - \boldsymbol{h})$ shows that $\mathfrak{v}_p(f\otimes g - \mathcal{F}\,\boldsymbol{h}) \le e^{-(\alpha+\beta)H}$. □

Let $\Psi \in C^\infty(G, \mathbb{C})$ be a smooth function defined on an open neighbourhood of zero in the real numbers. Assume $f \in \mathfrak{B}^p(C^\infty(M, G))$ such that $\mathfrak{v}(f) < 1$ and $\mathfrak{v}_p(f - \mathcal{F}\,\boldsymbol{f}) \le e^{-\alpha H}$ for an $\alpha > 0$. Then there is a positive $\epsilon < 1$ with $\mathfrak{v}_p(f), \mathfrak{v}_p(f - \mathcal{F}\,\boldsymbol{f}) < \epsilon$ and the following lemma can be proven.

LEMMA 3.6. *Fix* $n \le m \in \mathbb{N}$ *then* $\mathfrak{v}(\Psi^{(\nu)} \circ f) \le \epsilon^{m-\nu}$ *for all* $\nu \le n$ *implies* $\max_{i\le p, i+j\le n} \mathfrak{v}[\partial_t^i D^j(\Psi\circ f)] \le \epsilon^m$.

PROOF. The assertion is trivial for $n = 0$ and shall be supposed to hold for an $n \in \mathbb{N}$ and any $m \ge n$. Assume $m \ge n+1$ and $\mathfrak{v}(\Psi^{(\nu)}\circ f) \le \epsilon^{m-\nu}$ for all $\nu \le n+1$. It follows then from the induction hypothesis that $\max_{i\le p, i+j\le n} \mathfrak{v}[\partial_t^i D^j(\Psi\circ f)] \le \epsilon^m$.

Take ∂ either ∂_t or D then

$$\begin{aligned}\mathfrak{v}[\partial\partial_t^k D^l(\Psi\circ f)] &= \mathfrak{v}\big[\partial_t^k D^l\big((\Psi'\circ f)\partial f\big)\big]\\ &\le \max_{i\le p, i+j\le n} \mathfrak{v}[\partial_t^i D^j(\Psi'\circ f)]\mathfrak{v}_p(f)\end{aligned}$$

for $k+l\le n$ and $k<p$ if $\partial=\partial_t$ or $k\le p$ otherwise. Since $\mathfrak{v}\big[(\Psi')^{(\nu)}\circ f\big] = \mathfrak{v}(\Psi^{(\nu+1)}\circ f)\le \epsilon^{(m-1)-\nu}$ the induction hypothesis ensures that the first factor above is not larger than ϵ^{m-1} and so the claim holds for $n+1$ and hence on $\mathbb{N}$ by induction. □

This allows for the construction of an asymptotic approximation of the composition $\Psi\circ f$ in $\mathfrak{E}_0^0(M,\mathbb{C})$.

Proposition 3.7. *There exists a unique* $\mathcal{F}\boldsymbol{g}\in\mathfrak{E}_0^0(M,\mathbb{C})$ *of degree greater than* $-\alpha H$ *with* $\mathfrak{v}_p(\Psi\circ f-\mathcal{F}\boldsymbol{g})\le e^{-\alpha H}$.

Proof. Let $n\in\mathbb{N}$ be arbitrary and choose $n<m\in\mathbb{N}$ such that $\epsilon^m\le e^{-\alpha H}$. For a compact interval neighbourhood $K\subset G$ of zero there are coefficients $c\in\mathbb{C}^m$ and constants $C\in\mathbb{R}^m$ such that

$$\Big|\Psi^{(\nu)}(k)-\Big(\sum_{i\in m} c_i x^i\Big)^{(\nu)}(k)\Big|\le C_\nu |k|^{m-\nu}$$

for all $\nu\in m$ and $k\in K$. According to proposition 3.5 there exists an $\mathcal{F}\boldsymbol{g}\in\mathfrak{E}_0^0(M,\mathbb{C})$ of degree greater than $-\alpha H$ such that

$$\mathfrak{v}_p\Big(\sum_{i\in m} c_i f^i-\mathcal{F}\boldsymbol{g}\Big)\le e^{-\alpha H}.$$

By concentrating on a suitable neighbourhood in M and sufficiently large times it can be assumed that $|f|_\gamma\le\epsilon^t$ and that the image of f lies in K. Defining $\Omega:=\Psi-\sum_{i\in m}c_i x^i\in C^\infty(G,\mathbb{C})$ one finds $|\Omega^{(\nu)}\circ f|\le C_\nu|f|_\gamma^{m-\nu}\le\epsilon^{(m-\nu)t}$ and so $\mathfrak{v}(\Omega^{(\nu)}\circ f)\le\epsilon^{m-\nu}$ for all $\nu\le n$. By lemma 3.6 one infers that

$$\max_{i\le p, i+j\le n}\mathfrak{v}[\partial_t^i D^j(\Omega\circ f)]\le\epsilon^m\le e^{-\alpha H}$$

so that indeed $\max_{i\le p,i+j\le n}\mathfrak{v}(\Psi\circ f-\mathcal{F}\boldsymbol{g})\le e^{-\alpha H}$. Finally, because $\deg\mathcal{F}\boldsymbol{g}>-\alpha H$ independent of n and n was arbitrary, it follows from proposition 3.2 that actually $\mathfrak{v}_p(\Psi\circ f-\mathcal{F}\boldsymbol{g})\le e^{-\alpha H}$. □

The construction in the proof further shows

Remark 3.8. If $f=\mathcal{F}\boldsymbol{f}$ then $\boldsymbol{g}=\sum_{m<\alpha}\Psi(\boldsymbol{f})(m)$.

where $\Psi(\boldsymbol{f})$ is defined using the substitution homomorphism induced by Ψ as described in section 2.

Corollary 3.9. *If* $\mathfrak{v}_p(f)<1$ *and* $\Psi(0)=0$ *then* $\mathfrak{v}_p(\Psi\circ f)\le\mathfrak{v}_p(f)$.

Proof. One can choose $\alpha>0$ such that $\mathfrak{v}_p(f)\le e^{-\alpha H}$, i.e. $\mathcal{F}\boldsymbol{f}=0$ approximates f suitably well. As in the proof of proposition 3.7 the constant c_0 vanishes one can actually choose $\mathcal{F}\boldsymbol{g}=0$ and get $\mathfrak{v}_p(\Psi\circ f)\le e^{-\alpha H}$. □

In order to construct fibre metrics out of a given family of Riemannian metrics on M the following propositions will be useful. Like in the statements on non-linear substitution above attention is focussed on settings which will actually be of use later on rather than aimed at the greatest possible generality for the sake of not overloading the notation. Let $f\in\mathfrak{B}(\mathcal{T}_r^r M,\mathbb{C})$ and $g\in\mathfrak{B}(\mathcal{T}_r^r M,\mathbb{C})$ be given such that $\mathfrak{v}(f-\mathcal{F}\boldsymbol{f})\le e^{-\alpha H}\mathfrak{v}(f)$ for an $\mathcal{F}\boldsymbol{f}\in\mathfrak{E}_r^r(M,\mathbb{C})$ and $\alpha>0$ and consider the contraction of r co- and r contravariant indices $f\cdot g\in\mathfrak{B}_r^r(M,\mathbb{C})$. Denote by δ the identity element in $\mathbb{C}\mathcal{T}_r^r M$.

Proposition 3.10. *If $f \cdot g = \delta$ and $\boldsymbol{f}$ is invertible then there is a unique $\mathcal{F}\boldsymbol{g} \in \mathfrak{E}^r_r(M,\mathbb{C})$ of maximal degree with $\mathfrak{v}(g - \mathcal{F}\boldsymbol{g}) \leq e^{-\alpha H}\mathfrak{v}(g)$. In particular, $\mathfrak{v}(f)\mathfrak{v}(g) = 1$ and $\boldsymbol{g} = \sum_{m<\alpha+\beta} \boldsymbol{f}^{-1}(m)$ with $\mathfrak{v}(f) = e^{\beta H}$.*

Proof. Since $\boldsymbol{f}$ is invertible there is $\beta \in \mathbb{R}$ with $\mathfrak{v}(\mathcal{F}\boldsymbol{f}) \leq e^{\beta H}$ and an $\mathcal{F}\boldsymbol{g} \in \mathfrak{E}^r_r(M,\mathbb{C})$ with $\mathfrak{v}(\mathcal{F}\boldsymbol{g}) \leq e^{-\beta H}$ such that $\mathfrak{v}(\mathcal{F}\boldsymbol{f} \cdot \mathcal{F}\boldsymbol{g} - \delta) \leq e^{-\alpha H}$. It follows that $\mathfrak{v}(f) \leq e^{\beta H}$ because $\alpha > 0$. From there one calculates

$$\begin{aligned}\mathfrak{v}(g - \mathcal{F}\boldsymbol{g}) &= \mathfrak{v}\big[g \cdot (f - \mathcal{F}\boldsymbol{f} + \boldsymbol{f}) \cdot (g - \mathcal{F}\boldsymbol{g})\big] \\ &= \mathfrak{v}\Big(g \cdot \big[\delta - (f - \mathcal{F}\boldsymbol{f}) \cdot \mathcal{F}\boldsymbol{g} - \mathcal{F}\boldsymbol{f} \cdot \mathcal{F}\boldsymbol{g}\big]\Big) \\ &\leq \mathfrak{v}\big[g \cdot (\delta - \mathcal{F}\boldsymbol{f} \cdot \mathcal{F}\boldsymbol{g})\big] + \mathfrak{v}\big[g \cdot (f - \mathcal{F}\boldsymbol{f}) \cdot \mathcal{F}\boldsymbol{g}\big] \leq e^{-\alpha H}\mathfrak{v}(g)\end{aligned}$$

as asserted. Uniqueness is obtained as usual by proposition 3.2. For the auxiliary statement observe that $\mathfrak{v}(g) \leq e^{-\beta H}$ and therefore $1 = \mathfrak{v}(\delta) \leq \mathfrak{v}(f)\mathfrak{v}(g) \leq e^{\beta H}e^{-\beta H} = 1$. □

Corollary 3.11. *If $\mathfrak{v}_p(f - \mathcal{F}\boldsymbol{f}) \leq e^{-\alpha H}\mathfrak{v}(f)$ then $\mathfrak{v}_p(g - \mathcal{F}\boldsymbol{g}) \leq e^{-\alpha H}\mathfrak{v}(g)$ is true as well.*

Proof. The estimate is true for zero derivatives according to proposition 3.10 and it shall hold up to $n \in \mathbb{N}$ derivatives. Take some $i, j \in \mathbb{N}$ with $i \leq p$ and $i + j = n + 1$. Then

$$\begin{aligned}0 &= g \cdot \partial_t^i D^j \delta = g \cdot \partial_t^i D^j (f \cdot g) \\ &= \partial_t^i D^j g - \partial_t^i D^j \mathcal{F}\boldsymbol{g} + g \cdot \partial_t^i D^j (\mathcal{F}\boldsymbol{f} \cdot \mathcal{F}\boldsymbol{g}) + g \cdot (f - \mathcal{F}\boldsymbol{f}) \cdot \partial_t^i D^j \mathcal{F}\boldsymbol{g} \\ &\quad + g \cdot \sum_{(k,l)\neq 0} \binom{i}{k}\binom{j}{l} \Big[\partial_t^k D^l (f - \mathcal{F}\boldsymbol{f}) \cdot \partial_t^{i-k} D^{j-l}(g - \mathcal{F}\boldsymbol{g}) \\ &\quad + \partial_t^k D^l \mathcal{F}\boldsymbol{f} \cdot \partial_t^{i-k} D^{j-l}(g - \mathcal{F}\boldsymbol{g}) + \partial_t^k D^l (f - \mathcal{F}\boldsymbol{f}) \cdot \partial_t^{i-k} D^{j-l} \mathcal{F}\boldsymbol{g}\Big]\end{aligned}$$

and hence the induction hypothesis implies $\mathfrak{v}(\partial_t^i D^j g - \partial_t^i D^j \mathcal{F}\boldsymbol{g}) \leq e^{-\alpha H}\mathfrak{v}(g)$. □

As stated at the beginning of this section it will be necessary to control the error terms of the asymptotic approximation not only through multiplications and substitutions but also under solving systems of linear ordinary differential equations. Before detailing this, let $t_0 \in I$ be fixed and consider an $f \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$. If $\mathfrak{v}(f) \geq 1$ then the family

$$g(t) := m \mapsto \int_{t_0}^{t} f(\tau)(m)d\tau, \qquad t \in I$$

is well-defined, lies in $\mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ as well and satisfies $\mathfrak{v}(g) \leq \mathfrak{v}(f)$. On the other hand, if $\mathfrak{v}(f) < 1$, the family

$$h(t) := m \mapsto \int_{t}^{\infty} f(\tau)(m)d\tau, \qquad t \in I$$

is also well-defined, lies in $\mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ and satisfies $\mathfrak{v}(h) \leq \mathfrak{v}(f)$ too.

An endomorphism $A \in \mathrm{End}(V)$ of V naturally induces smooth bundle endomorphisms

$$T^r_s M \otimes V^{\otimes n} \to T^r_s M \otimes V^{\otimes n}, \quad f \mapsto A^{\otimes n} \circ f$$

by composition with the $V^{\otimes n}$-valued tensors in $T^r_s M \otimes V^{\otimes n}$ which will be identified with A. Denote by Ξ the maximal distance between the real parts of the eigenvalues of the endomorphism $A^{\otimes n}$ of $V^{\otimes n}$.

Proposition 3.12. *For an endomorphism $A \in \operatorname{End}(V)$ and an inhomogeneity $f \in \mathfrak{E}^r_s(M, V^{\otimes n})$ assume there is a function $u \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ fulfilling*

$$\mathfrak{v}(\partial_t u + Au - f) \leq e^{-\alpha H}$$

for some $\alpha \in \mathbb{R}$. Then there exists a unique $\mathcal{F}\,\boldsymbol{u} \in \mathfrak{E}^r_s(M, V^{\otimes n})$ of degree greater than $-\alpha H + \Xi$ such that $\mathfrak{v}(u - \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}$.

Proof. The one-parameter group $t \to e^{At}$ in $\operatorname{End}(V^{\otimes n})$ generates a smooth family of bundle endomorphisms in $\operatorname{End}(T^r_s M \otimes V^{\otimes n})$ satisfying $\partial_t e^{At} = Ae^{At}$. With this, define

$$u_0 := \int_{t_0}^{\infty} e^{A\tau}(\partial_\tau u + Au - f)(\tau) d\tau \in \mathcal{V}^n \mathcal{T}^r_s M$$

if $\mathfrak{v}\big[e^{At}(\partial_t u + Au - f)\big] < 1$ and $u_0 := 0$ otherwise. Then the function v given by

$$v(t) := u_0 + e^{At_0} u(t_0) + \int_{t_0}^{t} e^{A\tau} f(\tau) d\tau$$

is in $\mathfrak{E}^r_s(M, V^{\otimes n})$. In the case $\mathfrak{v}\big[e^{At}(\partial_t u + Au - f)\big] < 1$ it follows that

$$\begin{aligned}
\mathfrak{v}(e^{At}u - v) &= \mathfrak{v}\bigg(\int_{t_0}^{t} \Big[\partial_\tau\big(e^{A\tau}u(\tau)\big) - e^{A\tau} f(\tau)\Big] d\tau - u_0\bigg) \\
&= \mathfrak{v}\bigg(-\int_{t}^{\infty} \Big[\partial_\tau\big(e^{A\tau}u(\tau)\big) - e^{A\tau} f(\tau)\Big] d\tau\bigg) \\
&\leq \mathfrak{v}\Big[\partial_t(e^{At}u) - e^{At} f\Big] = \mathfrak{v}\Big[e^{At}(\partial_t u + Au - f)\Big].
\end{aligned}$$

On the other hand, if $\mathfrak{v}\big[e^{At}(\partial_t u + Au - f)\big] \geq 1$ one calculates

$$\begin{aligned}
\mathfrak{v}(e^{At}u - v) &= \mathfrak{v}\bigg(\int_{t_0}^{t} \Big[\partial_\tau\big(e^{A\tau}u(\tau)\big) - e^{A\tau} f(\tau)\Big] d\tau\bigg) \\
&\leq \mathfrak{v}\Big[\partial_t(e^{At}u) - e^{At} f\Big] = \mathfrak{v}\Big[e^{At}(\partial_t u + Au - f)\Big]
\end{aligned}$$

again by what was stated in the previous paragraph. So in any case if $\mathcal{F}\,\boldsymbol{u} := e^{-At}v \in \mathfrak{E}^r_s(M, V^{\otimes n})$ then

$$\mathfrak{v}(u - \mathcal{F}\,\boldsymbol{u}) \leq \mathfrak{v}\big[e^{-At}(e^{At}u - v)\big] \leq e^{-\alpha H + \Xi}$$

as claimed. The condition on the degree of $\mathcal{F}\,\boldsymbol{u}$ can be accounted for by truncating $\boldsymbol{u}$ at and beyond order $\alpha - \Xi/H$ and uniqueness then follows from proposition 3.2. □

It is now shown that these approximations remain valid when being differentiated with respect to space and time. For shortness only the case $\alpha H > \Xi$ that is used later on is considered.

Lemma 3.13. *If u satisfies $\mathfrak{v}(\partial_t^{i+1} u + A\partial_t^i u - \partial_t^i f) \leq e^{-\alpha H}$ for all $i \leq m$ for some $m \in \mathbb{N}$ and $\alpha H > \Xi$ then in fact $\mathfrak{v}(\partial_t^i u - \partial_t^i \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}$ for $i \leq m$.*

Proof. The assertion holds for $m = 0$ by virtue of proposition 3.12 and shall be true for an $m \in \mathbb{N}$. Assume that $\mathfrak{v}(\partial_t^{i+1} u + A\partial_t^i u - \partial_t^i f) \leq e^{-\alpha H}$ for all $i \leq m+1$ then by the induction hypothesis one concludes $\mathfrak{v}(\partial_t^i u - \partial_t^i \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}$ for $i \leq m$. Moreover, $\partial_t^{m+1} f \in \mathfrak{E}^r_s(M, V^{\otimes n})$ and $\mathfrak{v}\big[\partial_t(\partial_t^{m+1} u) + A\partial_t^{m+1} u - \partial_t^{m+1} f\big] \leq e^{-\alpha H}$, so that there is $\mathcal{F}\,\boldsymbol{h} \in \mathfrak{E}^r_s(M, V^{\otimes n})$ of degree greater than $-\alpha H + \Xi$ with $\mathfrak{v}(\partial_t^{m+1} u - \mathcal{F}\,\boldsymbol{h}) \leq e^{-\alpha H + \Xi} < 1$ by proposition 3.12. Take

$$h(t) := \int_{t}^{\infty} \Big(\partial_\tau^{m+1} u - \mathcal{F}\,\boldsymbol{h}\Big)$$

and

$$v(t) := \int_{t_0}^{t} \mathcal{F}\,\boldsymbol{h} - \partial_t^m \mathcal{F}\,\boldsymbol{u} + \partial_t^m u(t_0) + h(t_0) \in \mathfrak{E}_s^r(M, V^{\otimes n})$$

then $\mathfrak{v}(v) = \mathfrak{v}(\partial_t^m u - \partial_t^m \mathcal{F}\,\boldsymbol{u} + h) \leq e^{-\alpha H + \Xi}$ while $\deg v > -\alpha H + \Xi$ which shows that $v = 0$ and therefore $\partial_t^{m+1}\boldsymbol{u} = \boldsymbol{h}$ and $\mathfrak{v}(\partial_t^{m+1} u - \partial_t^{m+1} \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}$. □

Corollary 3.14. *If u satisfies $\mathfrak{v}_p(\partial_t u + Au - f) \leq e^{-\alpha H}$ for some $\alpha H > \Xi$ then $\mathfrak{v}_p(u - \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}$.*

Proof. First note that in particular it is true that $\mathfrak{v}(\partial_t u + Au - f) \leq e^{-\alpha H}$ and so by proposition 3.12 there is an $\mathcal{F}\,\boldsymbol{u} \in \mathfrak{E}_s^r(M, V^{\otimes n})$ of degree greater than $-\alpha H + \Xi$ such that

$$\mathfrak{v}(u - \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}.$$

Let $i, j \in \mathbb{N}$ with $i \leq p$ be arbitrary. According to the assumptions one even has $\mathfrak{v}(\partial_t D^j u + AD^j u - D^j f) \leq e^{-\alpha H}$ and proposition 3.12 assures the existence of an $\mathcal{F}\,\boldsymbol{v} \in \mathfrak{E}_{s+j}^r(M, V^{\otimes n})$ of degree greater than $-\alpha H + \Xi$ with

$$\mathfrak{v}(D^j u - \mathcal{F}\,\boldsymbol{v}) \leq e^{-\alpha H + \Xi}.$$

The construction in the proof there shows that one can choose $\boldsymbol{v} = D^j \boldsymbol{u}$. Because $\mathfrak{v}\big[\partial_t^i(\partial_t D^j u + AD^j u - D^j f)\big] \leq e^{-\alpha H}$ holds as well, by lemma 3.13 there is an $\mathcal{F}\,\boldsymbol{w} \in \mathfrak{E}_{s+j}^r(M, V^{\otimes n})$ of degree greater than $-\alpha H + \Xi$ such that

$$\mathfrak{v}(\partial_t^k D^j u - \partial_t^k \mathcal{F}\,\boldsymbol{w}) \leq e^{-\alpha H + \Xi}$$

for all $k \leq i$. But then

$$\mathfrak{v}(\mathcal{F}\,\boldsymbol{v} - \mathcal{F}\,\boldsymbol{w}) = \mathfrak{v}\big[D^j u - \mathcal{F}\,\boldsymbol{w} - (D^j u - \mathcal{F}\,\boldsymbol{v})\big] \leq e^{-\alpha H + \Xi}$$

while $\deg(\mathcal{F}\,\boldsymbol{v} - \mathcal{F}\,\boldsymbol{w}) > -\alpha H + \Xi$ which proves that $\boldsymbol{w} = \boldsymbol{v} = D^j \boldsymbol{u}$ and so

$$\mathfrak{v}(\partial_t^i D^j u - \partial_t^i D^j \mathcal{F}\,\boldsymbol{u}) \leq e^{-\alpha H + \Xi}$$

follows. Since $i \leq p$ and j were arbitrary this proves the claim. □

Observe that if the vector space V is one-dimensional any endomorphism A of V has only a single eigenvalue and therefore Ξ vanishes which means that there is no loss of asymptotic decay in integrating the ordinary differential equations considered above.

3. Asymptotic expansion of a solution

After having provided some useful tools in the previous sections full asymptotic expansions will be constructed for solutions meeting some mild boundedness or decay assumptions in the direction of a Gaussian time coordinate. Restating from the previous chapters let $V \in C^\infty(J, \mathbb{R})$ be a smooth function defined on an open neighbourhood J of zero in the real numbers satisfying $\Lambda = V(0) > 0$, take $\mu^2/2 = V'(0)$ and fix a coupling constant $\xi \in \mathbb{R}$. Assume the non-degeneracy condition 2.9, $\mu^2 > -\xi n(n+1)H^2$, to hold and define the constants $H = \sqrt{2\Lambda/n(n-1)}$ and $\mu_c^2, k_0, k_1, k_2, \omega$ as in chapter 2. Let a smooth solution of the Einstein-scalar field system on $\tilde{M} = M \times I$ be given as a family of Riemannian metrics $g : I \to \mathcal{T}_2^0 M$ and scalar fields $\phi : I \to C^\infty(M, \mathbb{C})$ in the sense of chapter 1 with $1 - \xi\phi^*\phi > 0$.

Denote by $g^\sharp : I \to \mathcal{T}^2_0 M$ the family of metrics induced by g on the cotangent bundle and suppose the following decay conditions

(M) $\quad \mathfrak{v}_0(g) \le e^{2H},\ \mathfrak{v}_0(g^\sharp) \le e^{-2H}$

(K) $\quad \mathfrak{v}_0(\sigma) \le e^{-k_0 H}$

(F) $\quad \mathfrak{v}_0(\phi) \le e^{-k_1 H},\ \mathfrak{v}_0(\partial_t\phi) \le e^{-k_1 H}$

so that the quantities and all their spatial derivatives have the same corresponding decay for large times. It is first shown that these conditions determine the late-time asymptotics to first order. For this consider the family of difference tensors $A := \nabla - D \in \mathcal{T}^1_2 M$ between the connections ∇ of $g(t)$ and the background connection D of γ that is given by

$$2g(A(X,Y),Z) = D_X g(Z,Y) + D_Y g(X,Z) - D_Z g(X,Y)$$

for vector fields $X, Y, Z \in \mathcal{T}^1_0 M$. From condition (M) it is inferred that $\mathfrak{v}_0(A) \le 1$. Since the curvature endomorphism $R(\cdot,\cdot) \in \mathcal{T}^1_3 M$ of ∇ relates to that of D by

$$\begin{aligned} R(X,Y)Z = R_\gamma(X,Y)Z + D_X A(Y,Z) - D_Y A(X,Z) \\ + A\big(X, A(Y,Z)\big) - A\big(Y, A(X,Z)\big) \end{aligned}$$

it follows that $\mathfrak{v}_0(\mathrm{Ric}) \le 1$ and therefore $\mathfrak{v}_0(R) = \mathfrak{v}_0[\mathrm{tr}(g^\sharp \cdot \mathrm{Ric})] \le e^{-2H}$ as well as $\mathfrak{v}_0(\widehat{\mathrm{Ric}}) \le e^{-2H}$. Recall that $\widehat{\mathrm{Ric}}$ was defined as the trace-free part of the Ricci tensor with the first index raised. To get control on the components of the energy-momentum tensor it will prove useful to consider the quantities

$$\varpi := \frac{\xi\phi^*\partial_t\phi}{1 - \xi\phi^*\phi} + \text{c.c.}$$

and

$$\check\rho := \rho - \Lambda + \varpi\,\mathrm{tr}\,k.$$

From equation (1.12) it can be seen that

$$\begin{aligned} \check\rho = \frac{1}{2}\big(1 - \xi\phi^*\phi\big)^{-1}\Big[\frac{1}{2}\partial_t\phi^*\partial_t\phi - \Big(2\xi - \frac{1}{2}\Big)|\nabla\phi|^2 \\ - 2\xi\phi^*\Delta\phi + (V - \Lambda)(\phi^*\phi) + \Lambda\xi\phi^*\phi + \text{c.c.}\Big]. \end{aligned}$$

By hypothesis $\mathfrak{v}_0(\partial_t\phi^*\partial_t\phi) \le e^{-2k_1 H}$, $\mathfrak{v}_0(|\nabla\phi|^2) \le e^{(2-2k_1)H}$, $\mathfrak{v}_0(\nabla^2\phi) = \mathfrak{v}_0(D^2\phi - D\phi\cdot A) \le e^{-k_1 H}$ and so $\mathfrak{v}_0(\phi^*\Delta\phi) \le e^{(2-2k_1)H}$. Applying corollary 3.9 with $\mathfrak{v}_0(\phi^*\phi) \le e^{-2k_1 H}$ gives $\mathfrak{v}_0[(V-\Lambda)(\phi^*\phi)] \le e^{-2k_1 H}$ as well as $\mathfrak{v}_0[(1-\xi\phi^*\phi)^{-1} - 1] \le e^{-2k_1 H}$. This then implies $\mathfrak{v}_0(\varpi) \le e^{-2k_1 H}$ and $\mathfrak{v}_0(\check\rho) \le e^{-2k_1 H}$. By what has been stated above it then follows that the quantity

$$\vartheta := 2\check\rho - R + |\sigma|^2$$

satisfies $\mathfrak{v}_0(\vartheta) \le e^{-k_0 H}$. The Hamiltonian constraint $\mathsf{c} = 0$ reads

$$(\mathrm{tr}\,k)^2 + \frac{2n\varpi}{n-1}\,\mathrm{tr}\,k - \frac{n\vartheta}{n-1} - n^2H^2 = 0$$

which yields the relation

$$\mathrm{tr}\,k \pm nH = -\frac{n\varpi}{n-1} \mp nH\left(\sqrt{1 + \frac{1}{H^2}\Big[\Big(\frac{\varpi}{n-1}\Big)^2 + \frac{\vartheta}{n(n-1)}\Big]} - 1\right). \tag{3.1}$$

At first the ambiguous sign might change from point to point in $M \times I$. It is due to the decay of ϖ and ϑ that it must be locally constant at large times for $\mathrm{tr}\,k$ to remain smooth. Concentrate on such a neighbourhood and sufficiently large times and assume for the moment that the "lower" signs in (3.1) were true. It follows that $\mathfrak{v}_0(\operatorname{tr} k - nH) \leq e^{-k_0 H}$ and because of $\partial_t g + 2Hg = -2g\cdot\sigma - (2/n)(\operatorname{tr} k - nH)g$ that $\mathfrak{v}(\partial_t g + 2Hg) \leq e^{(2-k_0)H}$. Proposition 3.12 then gives $\mathfrak{v}(g) \leq e^{(2-k_0)H}$ which leads with (M) to the contradiction $1 = \mathfrak{v}(g\cdot g^\sharp) \leq \mathfrak{v}(g)\mathfrak{v}(g^\sharp) \leq e^{-k_0 H} < 1$. Thus the “upper” signs must hold and hence (3.1) implies $\mathfrak{v}_0(\operatorname{tr} k + nH) \leq e^{-k_0 H}$.

For $\bar g := e^{-2Ht} g$ equation (1.10) implies

$$\partial_t \bar g = -2\bar g\cdot\sigma - \frac{2}{n}(\operatorname{tr} k + nH)\bar g \tag{3.2}$$

and thus $\mathfrak{v}(\partial_t D^j \bar g) = \mathfrak{v}(D^j\partial_t \bar g) \leq \mathfrak{v}_0(\bar g)e^{-k_0 H} \leq e^{-k_0 H}$ for any $j \in \mathbb{N}$. It follows that the functions

$$g_0^{(j)} := D^j \bar g(t_0) + \int_{t_0}^\infty \partial_\tau D^j \bar g(\tau) d\tau \in \mathcal{T}_2^0 M$$

are well-defined with the property

$$\mathfrak{v}(D^j \bar g - g_0^{(j)}) = \mathfrak{v}\left(\int_t^\infty \partial_\tau D^j \bar g(\tau) d\tau\right) \leq e^{-k_0 H}.$$

This means in particular that $D^j \bar g(t) \to g_0^{(j)}$ locally uniformly as $t \to \infty$ which in turn implies $g_0^{(j)} = D^j g_0$ with $g_0 := g_0^{(0)}$ and hence $\mathfrak{v}_0(\bar g - g_0) \leq e^{-k_0 H}$. The tensor g_0 is symmetric, as a consequence of (M) positive definite and so a Riemannian metric on M.

As previously it turns out to be helpful to examine some modified matter quantity which is now taken to be

$$\begin{aligned}\check S := \frac{1}{2}\big(1-\xi\phi^*\phi\big)^{-1}\Big[&(1-2\xi)\nabla\phi^*\otimes\nabla\phi - 2\xi\phi^*\big[\nabla^2\phi + k\partial_t\phi\big]\\ &+ \Big\{\Big(2\xi - \frac{1}{2}\Big)\big(|\nabla\phi|^2 - \partial_t\phi^*\partial_t\phi\big) - (V-\Lambda)(\phi^*\phi) - \Lambda\xi\phi^*\phi\\ &+ 2\xi\phi^*\phi\big[\xi\big(R + |k|^2 + (\operatorname{tr} k)^2\big) + 2V'(\phi^*\phi)\big]\Big\}g + \text{c.c.}\Big].\end{aligned}$$

With equation (1.14) and (1.18) one gets

$$\check S = S + \Lambda g + 4(1-\xi\phi^*\phi)^{-1}\xi^2\phi^*\phi(\partial_t \operatorname{tr} k)g$$

while one has $\mathfrak{v}_0(\nabla\phi^*\otimes\nabla\phi) \leq e^{-2k_1 H}$, $\mathfrak{v}_0(\phi^*\nabla^2\phi) \leq e^{-2k_1 H}$ and by corollary 3.9 also $\mathfrak{v}_0[4(1-\xi\phi^*\phi)^{-1}\xi^2\phi^*\phi] \leq e^{-2k_1 H}$, $\mathfrak{v}_0[2V'(\phi^*\phi)] = \mathfrak{v}_0[(2V'-\mu^2)(\phi^*\phi)+\mu^2] \leq 1$. Therefore $\mathfrak{v}_0(\check S) \leq e^{(2-2k_1)H}$ and $\mathfrak{v}_0(\operatorname{tr}\check S) \leq e^{-2k_1 H}$. The evolution equation $\mathsf{e} = 0$ can be written in terms of this modified quantity as

$$\begin{aligned}\partial_t(\operatorname{tr} k + nH) = \frac{1-\xi\phi^*\phi}{1-\eta\xi\phi^*\phi}\Big[&R + (\operatorname{tr} k + nH)(\operatorname{tr} k - nH)\\ &+ \frac{1}{n-1}\operatorname{tr}\check S - \frac{n}{n-1}(\rho - \Lambda)\Big].\end{aligned} \tag{3.3}$$

Recall that η was defined as $\eta = 1 - 4n\xi/(n-1)$ and that $1 - \xi\phi^*\phi > 0$ implies $1 - \eta\xi\phi^*\phi > 0$ thence $\mathfrak{v}_0[(1-\eta\xi\phi^*\phi)^{-1}] \leq 1$ also by corollary 3.9. It follows from this equation that $\mathfrak{v}_0(\partial_t \operatorname{tr} k) \leq e^{-k_0 H}$ and with it $\mathfrak{v}_0[R_{\bar g} - n(n+1)H^2] \leq e^{-k_0 H}$.

Finally, to get the first order field asymptotics, consider $\bar\phi := e^{k_1 Ht}\phi$. From (F) it is clear that $\mathfrak{v}_0(\bar\phi)$ and $\mathfrak{v}_0(\partial_t\bar\phi)$ are both less than or equal to one and the scalar field equation $\mathsf{S} = 0$ is equivalent to

$$\partial_t^2\bar\phi + (n-2k_1)H\partial_t\bar\phi + \big[k_1(k_1-n) + \xi n(n+1) + \omega^2\big]H^2\bar\phi = E \tag{3.4}$$

where

$$E = \Delta\bar\phi + (\operatorname{tr} k + nH)(\partial_t\bar\phi - k_1 H\bar\phi) - \xi[R_{\bar g} - n(n+1)H^2]\bar\phi - (2V'-\mu^2)(\phi^*\phi)\bar\phi.$$

The three cases of supercritical, critical and subcritical field masses are treated separately. So firstly, assume $\mu^2 > \mu_c^2$. Equation (3.4) reduces to $\partial_t^2\bar\phi + \omega^2 H^2\bar\phi = E$

with $\mathfrak{v}_0(E) \le e^{-k_0 H}$. Because of $\omega > 0$ corollary 3.14 ensures the existence of two functions $\phi_0, \phi_1 \in C^\infty(M, \mathbb{C})$ such that $\mathfrak{v}_0(\bar\phi - \phi_0 e^{-i\omega H t} - \phi_1 e^{i\omega H t}) \le e^{-k_0 H}$. Secondly, assuming $\mu^2 = \mu_{\mathrm{c}}^2$, the equation (3.4) simplifies even further to $\partial_t^2 \bar\phi = E$ and again by corollary 3.14 there are functions $\phi_0, \phi_1 \in C^\infty(M, \mathbb{C})$ such that $\mathfrak{v}_0(\bar\phi - \phi_0 - \phi_1 t) \le e^{-k_0 H}$. Thirdly, for $\mu^2 < \mu_{\mathrm{c}}^2$, the equation (3.4) reads $\partial_t^2 \bar\phi + (k_2 - k_1) H \partial_t \bar\phi = E$ and corollary 3.14 applied for $\partial_t \bar\phi$ yields $\phi_0, \phi_1 \in C^\infty(M, \mathbb{C})$ with $\mathfrak{v}_0(\bar\phi - \phi_0 - \phi_1 e^{-(k_2-k_1)Ht}) \le e^{-k_0 H}$ since $k_2 - k_1 > 0$. In any case, if one defines

$$\Phi := \begin{cases} \phi_0 e^{-i\omega H t} + \phi_1 e^{i\omega H t} & \text{for } \mu^2 > \mu_{\mathrm{c}}^2 \\ \phi_0 + \phi_1 t & \text{for } \mu^2 = \mu_{\mathrm{c}}^2 \\ \phi_0 & \text{for } \mu^2 < \mu_{\mathrm{c}}^2 \text{ and } k_0 \le k_2 - k_1 \\ \phi_0 + \phi_1 e^{-(k_2-k_1)Ht} & \text{for } \mu^2 < \mu_{\mathrm{c}}^2 \text{ and } k_0 > k_2 - k_1 \end{cases}$$

then $\mathfrak{v}_0(\bar\phi - \Phi) \le e^{-k_0 H}$ as well as $\mathfrak{v}_0[\partial_t(\bar\phi - \Phi)] \le e^{-k_0 H}$.

The first-order expansions obtained so far are now shown to hold too when differentiated with respect to time as often as desired. The argument goes by inductively improving the estimates from the preceding paragraph. It was observed that

$$\mathfrak{v}_0(\bar g - g_0),\ \mathfrak{v}_0(\sigma),\ \mathfrak{v}_0(\operatorname{tr} k + nH),\ \mathfrak{v}_1(\bar\phi - \Phi) \le e^{-k_0 H}.$$

It shall then be assumed that for a $p \in \mathbb{N}$ it is true that

$$\mathfrak{v}_p(\bar g - g_0),\ \mathfrak{v}_p(\sigma),\ \mathfrak{v}_p(\operatorname{tr} k + nH),\ \mathfrak{v}_{p+1}(\bar\phi - \Phi) \le e^{-k_0 H}.$$

From there one infers $\mathfrak{v}_p(g) \le e^{2H}$ and hence $\mathfrak{v}_p(g^\sharp) \le e^{-2H}$ from corollary 3.11. It follows that $\mathfrak{v}_p(A) \le 1$, $\mathfrak{v}_p(\mathrm{Ric}) \le 1$ and accordingly $\mathfrak{v}_p(\widehat{\mathrm{Ric}}) \le e^{-2H}$, $\mathfrak{v}_p(R) \le e^{-2H}$. Furthermore the induction hypothesis gives $\mathfrak{v}_{p+1}(\phi) \le e^{-k_1 H}$ and so $\mathfrak{v}_{p+1}[(1-\xi\phi^*\phi)^{-1}] \le 1$ by corollary 3.9 which entails $\mathfrak{v}_{p+1}[4(1-\xi\phi^*\phi)^{-1}\xi^2\phi^*\phi] \le e^{-2k_1 H}$, $\mathfrak{v}_{p+1}[(V-\Lambda)(\phi^*\phi)] \le e^{-2k_1 H}$ and $\mathfrak{v}_{p+1}[2V'(\phi^*\phi)] \le 1$. Since both $\mathfrak{v}_p(\nabla\phi^* \otimes \nabla\phi)$ and $\mathfrak{v}_p(\phi^* \nabla^2 \phi)$ are not larger than $e^{-2k_1 H}$ one finds that $\mathfrak{v}_p(\rho - \Lambda) \le e^{-2k_1 H}$ and $\mathfrak{v}_p(\check S) \le e^{(2-2k_1)H}$. The trace-free parts of S and $\check S$ are equal thus $\mathfrak{v}_p(\hat S) \le e^{-2k_1)H}$. But then the evolution equations (1.20), i.e. $\mathsf{E} = 0$, and (3.3) immediately yield $\mathfrak{v}_{p+1}(\sigma) \le e^{-k_0 H}$ as well as $\mathfrak{v}_{p+1}(\operatorname{tr} k + nH) \le e^{-k_0 H}$ respectively. From relation (3.2) it is clear that $\mathfrak{v}_{p+1}(\bar g - g_0) \le e^{-k_0 H}$ and one certainly has $\mathfrak{v}_p[R_{\bar g} - n(n+1)H^2] \le e^{-k_0 H}$. As a consequence $\mathfrak{v}_p(E) \le e^{-k_0 H}$ and because of

$$\partial_t^2(\bar\phi - \Phi) + (k_2 - k_1) H \partial_t(\bar\phi - \Phi) + \omega^2 H^2 (\bar\phi - \Phi) = E$$

one finally gets $\mathfrak{v}_{p+2}(\bar\phi - \Phi) \le e^{-k_0 H}$. With this, the induction hypothesis has been improved from p to $p+1$ which then shows that in fact

$$\mathfrak{v}_\infty(\bar g - g_0),\ \mathfrak{v}_\infty(\sigma),\ \mathfrak{v}_\infty(\operatorname{tr} k + nH),\ \mathfrak{v}_\infty(\bar\phi - \Phi) \le e^{-k_0 H} \tag{3.5}$$

is true.

It is now proven that there exist such expansion not only to first order but to any arbitrary high order. The coupling constant and the potential that specify the model shall fulfill the conditions given at the beginning of this section.

THEOREM 3.15. *Suppose that (g, ϕ) is a smooth solution to the Einstein-scalar field system with respect to a Gaussian time coordinate $M \times I \to I$, existing globally towards the future and satisfying the decay conditions* (M), (K) *and* (F). *Then for any $\alpha > 0$ there are unique asymptotic approximants $\mathcal{F}\bar{\boldsymbol g} \in \mathfrak{E}_2^0(M, \mathbb{C})$ and $\mathcal{F}\bar{\boldsymbol\phi} \in \mathfrak{E}_0^0(M, \mathbb{C})$ of degree greater than $-\alpha H$ such that*

$$\mathfrak{v}_\infty(\bar g - \mathcal{F}\bar{\boldsymbol g}),\ \mathfrak{v}_\infty(\bar\phi - \mathcal{F}\bar{\boldsymbol\phi}) \le e^{-\alpha H}$$

are valid for the rescaled quantities $\bar g = e^{-2Ht} g$ and $\bar\phi = e^{k_1 H t}\phi$.

PROOF. It follows directly from the identity theorem 3.2 that such approximants, if they exist, are unique due to the assumptions concerning their degree. To show existence it is sufficient and will be convenient to consider orders α that are integer multiples of $k_0 H$ only. The assertion was shown to hold for $\alpha = 0$ and $\alpha = k_0 H$ in (3.5) and shall now be assumed to hold for an $0 < \alpha \in k_0 H\mathbb{N}$. It follows from the details given in this section that $\bar{\boldsymbol{g}} = g_0 + O(X^{k_0})$ with a Riemannian metric g_0 and is as such invertible. By corollary 3.11 there is an $\mathcal{F}\,\bar{\boldsymbol{g}}^\sharp \in \mathfrak{E}^2_0(M,\mathbb{C})$ with $\mathfrak{v}_\infty(\bar{g}^\sharp - \mathcal{F}\,\bar{\boldsymbol{g}}^\sharp) \le e^{-\alpha H}$ where $\bar{g}^\sharp = e^{2Ht}g^\sharp$. From proposition 3.5 and $\mathfrak{v}_\infty(A) \le 1$ it is then inferred that there is $\mathcal{F}\,\mathbf{Ric} \in \mathfrak{E}^0_2(M,\mathbb{C})$ with $\mathfrak{v}_\infty(\mathrm{Ric} - \mathcal{F}\,\mathbf{Ric}) \le e^{-\alpha H}$ and therefore $\mathcal{F}\,\widehat{\mathbf{Ric}} \in \mathfrak{E}^1_1(M,\mathbb{C})$, $\mathcal{F}\,\boldsymbol{R} \in \mathfrak{E}^0_0(M,\mathbb{C})$ such that $\mathfrak{v}_\infty(\widehat{\mathrm{Ric}} - \mathcal{F}\,\widehat{\mathbf{Ric}})$, $\mathfrak{v}_\infty(R - \mathcal{F}\,\boldsymbol{R}) \le e^{-(\alpha+2)H}$.

Clearly, from equation (1.10), there are $\mathcal{F}\,\boldsymbol{\sigma} \in \mathfrak{E}^1_1(M,\mathbb{C})$ and $\mathcal{F}\,\mathbf{tr}\,\boldsymbol{k} \in \mathfrak{E}^0_0(M,\mathbb{C})$ approximating σ and $\mathrm{tr}\,k$ as $\mathfrak{v}_\infty(\sigma - \mathcal{F}\,\boldsymbol{\sigma})$, $\mathfrak{v}_\infty(\mathrm{tr}\,k - \mathcal{F}\,\mathbf{tr}\,\boldsymbol{k}) \le e^{-\alpha H}$. Further, with $\boldsymbol{\phi} := \bar{\boldsymbol{\phi}}X^{k_1}$, one has $\mathfrak{v}_\infty(\phi - \mathcal{F}\,\boldsymbol{\phi}) \le e^{-(\alpha+k_1)H}$ whereas $\mathfrak{v}_\infty(\phi)$, $\mathfrak{v}_\infty(\mathcal{F}\,\boldsymbol{\phi}) \le e^{-k_1 H}$. From the equations (1.12) and (1.14) the existence of $\mathcal{F}\,\boldsymbol{\rho} \in \mathfrak{E}^0_0(M,\mathbb{C})$ and $\mathcal{F}\,\hat{\boldsymbol{S}} \in \mathfrak{E}^1_1(M,\mathbb{C})$ with $\mathfrak{v}_\infty(\rho - \mathcal{F}\,\boldsymbol{\rho}) \le e^{-(\alpha+k_0)H}$ and $\mathfrak{v}_\infty(\hat{S} - \mathcal{F}\,\hat{\boldsymbol{S}}) \le e^{-(\alpha+k_0)H}$ is obtained. Because both $\mathfrak{v}(\mathrm{tr}\,k + nH)$ and $\mathfrak{v}(\sigma)$ are not larger than $e^{-k_0 H}$ according to (3.5) there are $\mathcal{F}\,\boldsymbol{F} \in \mathfrak{E}^1_1(M,\mathbb{C})$ and $\mathcal{F}\,\boldsymbol{f} \in \mathfrak{E}^0_0(M,\mathbb{C})$ such that

$$\mathfrak{v}_\infty\Big[\widehat{\mathrm{Ric}} + (\mathrm{tr}\,k + nH)\sigma - \hat{S} - \mathcal{F}\,\boldsymbol{F}\Big] \le e^{-(\alpha+k_0)H}$$

as well as

$$\mathfrak{v}_\infty\left[R + (\mathrm{tr}\,k + nH)^2 + \frac{\mathrm{tr}\,S + n\Lambda}{n-1} - \frac{n(\rho - \Lambda)}{n-1} - \mathcal{F}\,\boldsymbol{f}\right] \le e^{-(\alpha+k_0)H}.$$

The evolution equations (1.20) and (1.19) imply

$$\partial_t\sigma + nH\sigma = \widehat{\mathrm{Ric}} + (\mathrm{tr}\,k + nH)\sigma - \hat{S}$$

and

$$\partial_t(\mathrm{tr}\,k + nH) + 2nH(\mathrm{tr}\,k + nH) = R + (\mathrm{tr}\,k + nH)^2 + \frac{\mathrm{tr}\,S + n\Lambda}{n-1} - \frac{n(\rho - \Lambda)}{n-1}$$

so that corollary 3.14 yields improved approximants denoted again by $\mathcal{F}\,\boldsymbol{\sigma}$ and $\mathcal{F}\,\mathbf{tr}\,\boldsymbol{k}$ that actually fulfill $\mathfrak{v}_\infty(\sigma - \mathcal{F}\,\boldsymbol{\sigma})$, $\mathfrak{v}_\infty(\mathrm{tr}\,k - \mathcal{F}\,\mathbf{tr}\,\boldsymbol{k}) \le e^{-(\alpha+k_0)H}$. According to equation (1.10) the rescaled metric $\bar{g}$ obeys the relation

$$\partial_t\bar{g} = -2\bar{g}\cdot\sigma - \frac{2}{n}(\mathrm{tr}\,k + nH)\bar{g}.$$

The right hand side of this equation can now be approximated by an $\mathcal{F}\,\boldsymbol{G} \in \mathfrak{E}^0_2(M,\mathbb{C})$ as

$$\mathfrak{v}_\infty\big[-2\bar{g}\cdot\sigma - (2/n)(\mathrm{tr}\,k + nH)\bar{g} - \mathcal{F}\,\boldsymbol{G}\big] \le e^{-(\alpha+k_0)H}$$

too and corollary 3.14 once more allows then for an improved estimate $\mathfrak{v}_\infty(\bar{g} - \mathcal{F}\,\bar{\boldsymbol{g}}) \le e^{-(\alpha+k_0)H}$. Moreover there is $\mathcal{F}\,\boldsymbol{R}_{\bar{\boldsymbol{g}}} \in \mathfrak{E}^0_0(M,\mathbb{C})$ with $\mathfrak{v}_\infty(R_{\bar{g}} - \mathcal{F}\,\boldsymbol{R}_{\bar{\boldsymbol{g}}}) \le e^{-(\alpha+k_0)H}$ so that the right hand side of the rescaled scalar field equation (3.4) is approximated by an $\mathcal{F}\,\boldsymbol{E} \in \mathfrak{E}^0_0(M,\mathbb{C})$ as $\mathfrak{v}_\infty(E - \mathcal{F}\,\boldsymbol{E}) \le e^{-(\alpha+k_0)H}$. As before by corollary 3.14 the estimate on $\bar{\phi} - \mathcal{F}\,\bar{\boldsymbol{\phi}}$ can then be improved to $\mathfrak{v}_\infty(\bar{\phi} - \mathcal{F}\,\bar{\boldsymbol{\phi}}) \le e^{-(\alpha+k_0)H}$. In summary, given approximations of $\bar{g}$ and $\bar{\phi}$ to order $e^{-\alpha H}$ it was possible to construct improved approximations $\mathcal{F}\,\bar{\boldsymbol{g}}$ and $\mathcal{F}\,\bar{\boldsymbol{\phi}}$ such that

$$\mathfrak{v}_\infty(\bar{g} - \mathcal{F}\,\bar{\boldsymbol{g}}),\ \mathfrak{v}_\infty(\bar{\phi} - \mathcal{F}\,\bar{\boldsymbol{\phi}}) \le e^{-(\alpha+k_0)H}.$$

The existence statement aimed at follows hence by recursion. □

4. Asymptotic character of formal solutions

After having obtained full asymptotic expansions of solutions of the Einstein-scalar field system out of weak decay assumptions in the previous section the question arises whether these expansions correspond to the formal, algebraic solutions studied in chapter 2. This would enable one to study the asymptotics of solutions solely at the level of formal series where proofs are usually much simpler and more intuitive. It is shown in the following that this question can be answered in the affirmative.

In contrast to the propositions 3.5 and 3.7 that were concerned with the approximation of products and compositions of functions by elements in $\mathfrak{E}^r_s(M, V^{\otimes n})$ what is needed here additionally are crude error estimates under addition, multiplication and substitution only. So let $f, g, a, b \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$ be given with $\mathfrak{v}(f - a) \leq e^{-\alpha H}\mathfrak{v}(f)$ and $\mathfrak{v}(g - b) \leq e^{-\alpha H}\mathfrak{v}(g)$ for an $\alpha \in \mathbb{R}$. It is then immediate that

$$\mathfrak{v}[(f + g) - (a + b)] \leq e^{-\alpha H} \max\{\mathfrak{v}(f), \mathfrak{v}(g)\}.$$

For the product a similar result is true.

PROPOSITION 3.16. *Assume* $f, a \in \mathfrak{B}(\mathcal{T}^r_s M, V^{\otimes n})$, $f', a' \in \mathfrak{B}(\mathcal{T}^{r'}_{s'} M, V^{\otimes n'})$ *with* $\mathfrak{v}(f - a) \leq e^{-\alpha H}\mathfrak{v}(f)$ *and* $\mathfrak{v}(f' - a') \leq e^{-\alpha H}\mathfrak{v}(f')$ *for an* $\alpha > 0$. *Then* $\mathfrak{v}(f \otimes f' - a \otimes a') \leq e^{-\alpha H}\mathfrak{v}(f)\mathfrak{v}(f')$.

PROOF. First note that $\mathfrak{v}(a) \leq \mathfrak{v}[f - (f - a)] \leq \mathfrak{v}(f)$ since $\alpha > 0$ and analogously $\mathfrak{v}(a') \leq \mathfrak{v}(f')$. But this implies immediately $\mathfrak{v}(f \otimes f' - a \otimes a') = \mathfrak{v}[(f - a) \otimes (f' - a') + a \otimes (f' - a') + (f - a) \otimes a'] \leq e^{-\alpha H}\mathfrak{v}(f)\mathfrak{v}(f')$ again as α is non-negative. □

For the estimate under substitution take again $\Phi \in C^\infty(G, \mathbb{C})$ a smooth function defined on an open neighbourhood G of zero in the real numbers. Again only the case $f, a \in \mathfrak{B}(C^\infty(M, G))$ is considered.

PROPOSITION 3.17. *Suppose* $\mathfrak{v}(f - a) \leq e^{-\alpha H}\mathfrak{v}(f)$ *for an* $\alpha > 0$ *and* $\mathfrak{v}(f) < 1$. *Then the estimate* $\mathfrak{v}[\Psi(f) - \Psi(a)] \leq e^{-\alpha H}\mathfrak{v}(f)$ *holds.*

PROOF. The proof goes much along the lines of that of proposition 3.7. Fix $\mathfrak{v}(f) < \epsilon < 1$ and $n \in \mathbb{N}$ with $\epsilon^n \leq e^{-\alpha H}\epsilon$. For a compact interval neighbourhood $K \subset G$ of zero there are constants $c \in \mathbb{C}^n$ and $C > 0$ such that

$$\left|\Psi(k) - \sum_{i \in n} c_i k^i\right| \leq C|k|^n$$

for all $k \in K$. By restricting to a suitable neighbourhood in M and sufficiently large times it can be assumed that the images of f and a lie within K and $|f|_\gamma$, $|a|_\gamma \leq \epsilon^t$. For $\Omega := \Psi - \sum_{i \in n} c_i x^i$ it hence follows that $|\Omega \circ f| \leq C|f|^n_\gamma \leq C\epsilon^{nt}$ which proves that $\mathfrak{v}(\Omega \circ f) \leq \epsilon^n \leq e^{-\alpha H}\epsilon$. In the same way $\mathfrak{v}(\Omega \circ a) \leq e^{-\alpha H}\epsilon$ is obtained. On the other hand, from proposition 3.16 one gets $\mathfrak{v}(f^i - a^i) \leq e^{-\alpha H}\epsilon^i$ for all $i \in \mathbb{N}$ so that

$$\mathfrak{v}\left[\sum_{i \in n} c_i f^i - \sum_{i \in n} c_i a^i\right] = \mathfrak{v}\left[\sum_{0 < i \in n} c_i(f^i - a^i)\right] \leq e^{-\alpha H}\epsilon.$$

But then

$$\mathfrak{v}\big[\Psi(f) - \Psi(a)\big] = \mathfrak{v}\left[\Omega \circ f + \sum_{i \in n} c_i f^i - \Omega \circ a - \sum_{i \in n} c_i a^i\right] \leq e^{-\alpha H}\epsilon.$$

Since $\epsilon > \mathfrak{v}(f)$ was arbitrary this proves the claim. □

Now let (g, ϕ) be a solution to the Einstein-scalar field system given as as tensor families with respect to a Gaussian time coordinate. The solution is assumed to exist globally towards the future and the conditions stated at the beginning of section 3, in particular (M), (K) and (F), shall be satisfied.

THEOREM 3.18. *For an $\alpha > 0$ let $\mathcal{F}\boldsymbol{g} \in \mathfrak{E}^0_2(M, \mathbb{C})$ and $\mathcal{F}\boldsymbol{\phi} \in \mathfrak{E}^0_0(M, \mathbb{C})$ be of degree greater than $-(\alpha-2)H$ and $-(\alpha+k_1)H$ respectively such that*

$$\mathfrak{v}_\infty(g - \mathcal{F}\boldsymbol{g}) \le e^{-(\alpha-2)H} \quad \text{and} \quad \mathfrak{v}_\infty(\phi - \mathcal{F}\boldsymbol{\phi}) \le e^{-(\alpha+k_1)H}$$

hold. Then $(\boldsymbol{g}, \boldsymbol{\phi})$ is a solution to the algebraic Einstein-scalar field system up to but not including relative order α.

PROOF. The idea of the proof is to compare quantities calculated "analytically" from g and ϕ to those calculated "algebraically" from $\boldsymbol{g}$ and $\boldsymbol{\phi}$. All the latter shall be marked as bold symbols. Starting from $\mathfrak{v}_\infty(g - \mathcal{F}\boldsymbol{g}) \le e^{-(\alpha-2)H}$ one obtains due to (1.10) $\mathfrak{v}_\infty(k - \mathcal{F}\boldsymbol{k}) \le e^{-(\alpha-2)H}$ and from that $\mathfrak{v}_\infty(\sigma - \mathcal{F}\boldsymbol{\sigma})$, $\mathfrak{v}_\infty(\operatorname{tr} k - \mathcal{F}\operatorname{tr}\boldsymbol{k}) \le e^{-\alpha H}$ due to proposition 3.10, corollary 3.11 and proposition 3.16. For the same reason $\mathfrak{v}_\infty(A - \mathcal{F}\boldsymbol{A}) \le e^{-\alpha H}$, $\mathfrak{v}_\infty(\mathrm{Ric} - \mathcal{F}\,\mathbf{Ric}) \le e^{-\alpha H}$ and hence $\mathfrak{v}_\infty(\widehat{\mathrm{Ric}} - \mathcal{F}\,\widehat{\mathbf{Ric}})$, $\mathfrak{v}_\infty(R - \mathcal{F}\boldsymbol{R}) \le e^{-(\alpha+2)H}$. Furthermore $\mathfrak{v}_\infty(\nabla\phi - \mathcal{F}\,\nabla\boldsymbol{\phi})$, $\mathfrak{v}_\infty(\nabla^2\phi - \mathcal{F}\,\nabla^2\boldsymbol{\phi}) \le e^{-(\alpha+k_1)H}$ and so $\mathfrak{v}_\infty(\Delta\phi - \mathcal{F}\,\Delta\boldsymbol{\phi}) \le e^{-(\alpha+k_1+2)H}$. Finally, combining the propositions 3.7 and 3.17 yields

$$\mathfrak{v}\big(V(\phi^*\phi) - \mathcal{F}[V(\boldsymbol{\phi}^*\boldsymbol{\phi})|(\alpha+2k_1)]\big) \le e^{-(\alpha+2k_1)H}$$
$$\mathfrak{v}\big(V'(\phi^*\phi) - \mathcal{F}[V'(\boldsymbol{\phi}^*\boldsymbol{\phi})|(\alpha+2k_1)]\big) \le e^{-(\alpha+2k_1)H}$$

together with

$$\mathfrak{v}\big[(1-\xi\phi^*\phi)^{-1} - \mathcal{F}[(1-\xi\boldsymbol{\phi}^*\boldsymbol{\phi})^{-1}|(\alpha+2k_1)]\big] \le e^{-(\alpha+2k_1)H}$$

where $\boldsymbol{f}|\alpha$ is an abbreviation for the restriction of $\boldsymbol{f} \in \mathfrak{T}(M)$ to the interval $]-\infty, \alpha[$. Now this implies that $\mathfrak{v}[\rho - \mathcal{F}(\boldsymbol{\rho}|\alpha)]$ and $\mathfrak{v}[j - \mathcal{F}(\boldsymbol{j}|\alpha)]$ are less than or equal to $e^{-\alpha H}$ whereas $\mathfrak{v}[S - \mathcal{F}(\boldsymbol{S}|(\alpha-2))] \le e^{-(\alpha-2)H}$. It follows that the same is true for the evolution and constraint quantities $\mathfrak{v}[\mathsf{e} - \mathcal{F}(\mathbf{e}|\alpha)]$, $\mathfrak{v}[\mathsf{E} - \mathcal{F}(\mathbf{E}|\alpha)]$, $\mathfrak{v}[\mathsf{c} - \mathcal{F}(\mathbf{c}|\alpha)]$ and $\mathfrak{v}[\mathsf{C} - \mathcal{F}(\mathbf{C}|\alpha)]$. Since the "analytic" quantities e, E, c and C vanish it follows from the above inequalities and the identity theorem 3.2 that their "algebraic" counterparts have to vanish up to order α exclusively. Analogously one infers that $\mathfrak{v}[\mathsf{S} - \mathcal{F}(\mathbf{S}|(\alpha+k_1))] \le e^{-(\alpha+k_1)H}$ and thence $\mathbf{S}|(\alpha+k_1) = 0$. But this says exactly that the algebraic Einstein-scalar field equations are satisfied up to but not including relative order α. □

The tools and the methods introduced in this chapter for obtaining full asymptotic expansions were presented in a quite general setting with the hope of being directly applicable to models other than the scalar fields considered here. In [**24**] Rendall proved existence and uniqueness of formal solutions of Einstein's equations coupled to a perfect fluid with linear equation of state. The formal series for the metric and the fluid variables given there lie well within the algebra $\mathfrak{T}(M)$ here as they lack oscillatory terms entirely. It could thus be presumed that for this case the formal solutions could be turned into asymptotic expansions rather analogously.

CHAPTER 4

Solutions with prescribed asymptotics

1. An existence and uniqueness theorem for Fuchsian systems

In this chapter existence and uniqueness of solutions of the Einstein-scalar field system with prescribed asymptotics of the form studied in the previous chapter will be proven for *analytic* asymptotic data at the future conformal boundary. This will be achieved by reducing the field equations to first order Fuchsian form thereby making use of the formal series solutions obtained in chapter 2. Attention is restricted to analytic data as the most powerful results on Fuchsian systems are available in this case. To begin with, this section introduces a certain class of analytic functions and states the basic existence and uniqueness theorem for Fuchsian equations in this class.

Let T be a topological space, Y a Banach space and X an open subset of $\mathbb{R}^n$.

Definition 4.1. A function $f : X \times T \to Y$ is called uniformly analytic (on X) if it is continuous on $X \times T$ and for any x_0 in X there is a neighbourhood U of x_0 and continuous functions $a_i \in C(T, Y)$ for every multi-index $i \in \mathbb{N}^n$, such that

$$f(x, \tau) = \sum_{|i|=0}^{\infty} a_i(\tau)(x - x_0)^i \qquad \tau \in T, x \in U$$

converges uniformly on $U \times T$.

A careful introduction and a detailed discussion of this class of functions can be found in [**22**]. An obvious generalization for families of maps between analytic manifolds is the following.

Definition 4.2. Let M and N be analytic manifolds. A function $f : M \times T \to N$ is called uniformly analytic (on M) if for the projections τ onto T, π onto M and any charts x and y of M and N respectively the function

$$y \circ f \circ [(x \circ \pi) \times \tau]^{-1}$$

is uniformly analytic on the image of x.

Suppose now that E is an analytic vector bundle over an analytic Riemannian manifold (M, γ_0) with a bundle metric $\langle \cdot, \cdot \rangle$ and a linear connection D and that T is an interval of the form $T =]0, \tau_0[$ for some positive τ_0. Let U be an open subset of E containing the zero-section of E. Henceforth, under a Fuchsian system it is understood a first order partial differential equation for a family of analytic sections $u(\tau)$ of E of the form

$$\tau \partial_\tau u + Lu = \tau^\epsilon f(u, Du, \tau) \tag{4.1}$$

for an $\epsilon > 0$ where $L \in \mathrm{End}(E)$ is an analytic vector bundle endomorphism and $f : U \times_M E \otimes T^0_1 M \times T \to E$ is a family of bundle maps from $U \times_M E \otimes T^0_1 M$ to E which is uniformly analytic on $U \times_M E \otimes T^0_1 M$. The linear part L and the non-linearity f are further subject to the condition that the bundle endomorphisms τ^L as well as the maps $f(\cdot, \tau)$ are locally uniformly bounded independently of $\tau \in T$ and that $f(\cdot,\tau)$ is locally Lipschitz uniformly in $\tau \in T$. Then a result by Kichenassamy and Rendall [**16**] states that the following theorem holds.

THEOREM A. *Given that the regularity and boundedness assumptions on* L *and* f *are satisfied the Fuchsian system* (4.1) *has a unique solution* u *which is defined on an open neighbourhood of* $M \times \{0+\}$ *in* $M \times T$*, is uniformly analytic on* M *and tends to zero locally uniformly as* τ *approaches zero.*

The proof of theorem A provided in [**16**] goes by a fixed point argument along the lines of [**3**]. The analyticity assumptions are needed to estimate the gradient of the solution by the Cauchy integral theorem in terms of the function itself as well as to control the derivatives when taking the limit. There are also results in the non-analytic setting [**8, 15**] where control on the gradient is gained by the assumption of hyperbolicity of the system. In [**23**] results obtained previously in the analytic setting [**16**] on the initial singularity in Gowdy spacetimes were generalized to the smooth case. It is interesting that this generalization could be obtained without resorting to the quite technical machinery of [**8**] and [**15**].

2. The first order Einstein-scalar field system

As a initial step towards bringing the Einstein-scalar field system to Fuchsian form and thereby making theorem A applicable it is now written as a first order equation similar to equation (4.1). For this it is recalled that the system is determined by the choice of a coupling constant $\xi \in \mathbb{R}$ and a potential V which shall now be assumed to be a holomorphic function on some open neighbourhood J of zero in $\mathbb{C}$ that takes real values if the argument is real. Define a vector bundle

$$E := \mathbb{C}S^0_2M \oplus \mathbb{C}T^1_1M \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}T^0_3M \oplus \mathbb{C}T^0_1M \oplus \mathbb{C}T^0_1M$$

and denote the projections onto the ten factors by γ, ς, λ, ψ, $\psi^\dagger$, ψ_τ, $\psi^\dagger_\tau$, $\partial\gamma$, $\partial\psi$ and $\partial\psi^\dagger$ respectively. The background metric γ_0 on M induces a bundle metric $\langle\cdot,\cdot\rangle_{\gamma_0}$ and a linear connection D on E. Let further U_1 be the open subset of non-degenerate elements of S^0_2M and U_2 an open ball around zero in $\mathbb{C}^2$ such that firstly $|(1+|\eta|)\xi xy| < 1$ and secondly $xy \in J$ are valid for all $(x,y) \in \tau_0^{-k_1}U_2$. With this the preimages $\gamma^{-1}(U_1)$ and $(\psi\times\psi^\dagger)^{-1}(U_2)$ are open in E and thus their intersection

$$U := \gamma^{-1}(U_1) \cap (\psi\times\psi^\dagger)^{-1}(U_2)$$

as well.

To write down the expression for the non-linearity f eventually it is useful to consider the following analytic maps defined on $U \times_M E \otimes T^0_1M \times T$. With $\mathrm{inv} : \mathrm{Aut}(E) \to \mathrm{Aut}(E)$ assigning to any bundle automorphism its inverse and using the background metric γ_0 to identify metrically equivalent tensors as usual one may define

$$\gamma^\sharp := \mathrm{inv}\circ\gamma$$
$$(\partial\gamma^\sharp)^{ab}{}_c := -(\gamma^\sharp)^{ai}\partial\gamma_{ijc}(\gamma^\sharp)^{jb}$$

and further

$$A^c_{ab} := \frac{1}{2}(\gamma^\sharp)^{ci}(\partial\gamma_{iba} + \partial\gamma_{aib} - \partial\gamma_{abi})$$
$$\partial A^c_{abd} := \frac{1}{2}(\partial\gamma^\sharp)^{ic}{}_d(\partial\gamma_{iba} + \partial\gamma_{aib} - \partial\gamma_{abi}) + \frac{1}{2}(\gamma^\sharp)^{ci}(\partial\gamma_{iba,d} + \partial\gamma_{aib,d} - \partial\gamma_{abi,d})$$

where the projections of $E \otimes T^0_1 M$ are distinguished from those of E by a comma. This leads to

$$R^a{}_b := (\gamma^\sharp)^{ia}\big[(\mathrm{Ric}_{\gamma_0})_{ib} + \partial A^i_{abi} - \partial A^i_{iba} + A^j_{ab}A^i_{ij} - A^i_{aj}A^j_{ib}\big]$$

and consequently $R := R^i{}_i$ and $\hat{R}^a{}_b := R^a{}_b - (R/n)\delta^a{}_b$. The possible presence of logarithmic or oscillatory terms in leading order of the scalar field calls for a special treatment of the corresponding variables. Let $\phi_1 \in C^\omega(M, \mathbb{C})$ be an analytic function, arbitrary for the moment, with its complex conjugate ϕ_1^*. Later, this function will be taken from the asymptotic initial data of the scalar field and will so help to improve the asymptotics of the field variable ϕ. As an abbreviation and with the notation already introduced

$$\varUpsilon(x) := \begin{cases} 0 & \text{if } \mu^2 < \mu_\mathrm{c}^2 \\ -x/H & \text{if } \mu^2 = \mu_\mathrm{c}^2 \\ 1 & \text{if } \mu^2 > \mu_\mathrm{c}^2 \end{cases} \qquad \text{for } x \in \mathbb{R}$$

is analytic with constant derivative $\varUpsilon'$. Using this, one may proceed by defining

$$\begin{aligned} \bar\psi &:= \tau^{-i\omega}\psi + \phi_1\varUpsilon(\log\tau)\tau^{i\omega} \\ \bar\psi^\dagger &:= \tau^{i\omega}\psi^\dagger + \phi_1^*\varUpsilon(\log\tau)\tau^{-i\omega} \\ \bar\psi_{;a} &:= \tau^{-i\omega}\partial\psi_a + D_a\phi_1\varUpsilon(\log\tau)\tau^{i\omega} \\ \bar\psi^\dagger_{;a} &:= \tau^{i\omega}\partial\psi^\dagger_a + D_a\phi_1^*\varUpsilon(\log\tau)\tau^{-i\omega} \end{aligned}$$

and

$$\begin{aligned} \bar\psi_{a,b} &:= \tau^{-i\omega}\partial\psi_{a,b} + D_aD_b\phi_1\varUpsilon(\log\tau)\tau^{i\omega} \\ \bar\psi^\dagger_{a,b} &:= \tau^{i\omega}\partial\psi^\dagger_{a,b} + D_aD_b\phi_1^*\varUpsilon(\log\tau)\tau^{-i\omega}. \end{aligned}$$

It will further be convenient to have

$$\begin{aligned} \bar\psi^{;a} &:= (\gamma^\sharp)^{ia}\bar\psi_{;i} & \bar\psi^{\dagger;a} &:= (\gamma^\sharp)^{ia}\bar\psi^\dagger_{;i} \\ \bar\psi_{;ab} &:= \bar\psi_{a,b} - A^i_{ab}\bar\psi_{;i} & \bar\psi^\dagger_{;ab} &:= \bar\psi^\dagger_{a,b} - A^i_{ab}\bar\psi^\dagger_{;i} \\ \bar\psi^{;a}{}_b &:= (\gamma^\sharp)^{ia}\bar\psi_{;ib} & \bar\psi^{\dagger;a}{}_b &:= (\gamma^\sharp)^{ia}\bar\psi_{;ib}. \end{aligned}$$

It might be pointed out here explicitly that the commas and semicolons in the indices are just intended as mnemonics and do not indicate partial and covariant derivatives of the corresponding tensors. The variables representing time derivatives of the scalar field are rescaled analogously according to

$$\begin{aligned} \bar\psi_\tau &:= -H\tau^{k_0-i\omega}\psi_\tau - (k_1 - i\omega)H\bar\psi - \phi_1(\varUpsilon' + 2i\omega)H\tau^{i\omega} \\ \bar\psi^\dagger_\tau &:= -H\tau^{k_0+i\omega}\psi^\dagger_\tau - (k_1 + i\omega)H\bar\psi^\dagger - \phi_1^*(\varUpsilon' - 2i\omega)H\tau^{-i\omega} \\ \bar\psi_{\tau;a} &:= -H\tau^{k_0-i\omega}\psi_{\tau,a} - (k_1 - i\omega)H\bar\psi_{;a} - D_a\phi_1(\varUpsilon' + 2i\omega)H\tau^{i\omega} \\ \bar\psi^\dagger_{\tau;a} &:= -H\tau^{k_0+i\omega}\psi^\dagger_{\tau,a} - (k_1 + i\omega)H\bar\psi^\dagger_{;a} - \phi_1^*(\varUpsilon' - 2i\omega)H\tau^{-i\omega} \end{aligned}$$

and

$$\bar\psi^{;a}_\tau := (\gamma^\sharp)^{ia}\bar\psi_{\tau;i} \qquad\qquad \bar\psi^{\dagger;a}_\tau := (\gamma^\sharp)^{ia}\bar\psi^\dagger_{\tau;i}.$$

Since the potential V is holomorphic with $V(0) = \Lambda$ and $V'(0) = \mu^2/2$ the functions specified by $[V(x) - \Lambda]/x$ and $[V'(x) - \mu^2/2]/x$ for $x \in J \setminus \{0\}$ have analytic extensions denoted by

$$\left[\frac{V - \Lambda}{\cdot}\right] \quad \text{and} \quad \left[\frac{V' - \mu^2/2}{\cdot}\right]$$

to J respectively. With their help one may set

$$W := \bar\psi^\dagger\bar\psi\left[\frac{V-\Lambda}{\cdot}\right](\tau^{2k_1}\bar\psi^\dagger\bar\psi) \qquad W' := \bar\psi^\dagger\bar\psi\left[\frac{V'-\mu^2/2}{\cdot}\right](\tau^{2k_1}\bar\psi^\dagger\bar\psi)$$

as well as

$$\kappa := \tau^{2k_0}\Big(\varsigma + \frac{\lambda}{n}\delta\Big) - H\delta$$

for brevity. Another peculiarity, besides possible occurrence of logarithmic or oscillatory terms in leading order, is that for non-minimal coupling $\xi \neq 0$ the energy-momentum tensor (1.14) contains second order time derivatives of the scalar field while the scalar field equation (1.18) contains second order time derivatives of the metric through $R_{\tilde g}$. This fact has played a role already in chapter 3 for the determination of the asymptotics and requires also here the introduction of some additional quantities, namely $\check\Sigma$, Q and $\dot\lambda$ as follows.

$$\begin{aligned}
\varepsilon := \frac{1}{2}\big(1-\tau^{2k_1}\xi\bar\psi^\dagger\bar\psi\big)^{-1}\Big[&\bar\psi^\dagger_\tau\bar\psi_\tau - \Big(2\xi-\frac12\Big)\tau^2\big(\bar\psi^\dagger_{;i}\bar\psi^{;i}+\bar\psi^{\dagger;i}\bar\psi_{;i}\big)\\
&-2\xi\bar\psi^\dagger\big[\tau^2\bar\psi^{;i}{}_i + (\tau^{2k_0}\lambda - nH)\bar\psi_\tau\big]\\
&-2\xi\bar\psi\big[\tau^2\bar\psi^{\dagger;i}{}_i + (\tau^{2k_0}\lambda - nH)\bar\psi^\dagger_\tau\big]\\
&+2W + 2\Lambda\xi\bar\psi^\dagger\bar\psi\Big]\\
\iota_a := \frac{1}{2}\big(1-\tau^{2k_1}\xi\bar\psi^\dagger\bar\psi\big)^{-1}\Big[&-(1-2\xi)\big(\bar\psi^\dagger_{;a}\bar\psi_\tau + \bar\psi_{;a}\bar\psi^\dagger_\tau\big)\\
&+2\xi\bar\psi^\dagger\big(\bar\psi_{\tau;a} + \kappa^i{}_a\bar\psi_{;i}\big) + 2\xi\bar\psi\big(\bar\psi^\dagger_{\tau;a} + \kappa^i{}_a\bar\psi^\dagger_{;i}\big)\Big]\\
\check\Sigma^a{}_b := \frac{1}{2}\big(1-\tau^{2k_1}\xi\bar\psi^\dagger\bar\psi\big)^{-1}\Big[&(1-2\xi)\tau^2\big(\bar\psi^{\dagger;a}\bar\psi_{;b} + \bar\psi^{;a}\bar\psi^\dagger_{;b}\big)\\
&-2\xi\bar\psi^\dagger\big(\tau^2\bar\psi^{;a}{}_b + \kappa^a{}_b\bar\psi_\tau\big) - 2\xi\bar\psi\big(\tau^2\bar\psi^{\dagger;a}{}_b + \kappa^a{}_b\bar\psi^\dagger_\tau\big)\\
&+\Big\{\Big(2\xi-\frac12\Big)\big(\tau^2\bar\psi^\dagger_{;i}\bar\psi^{;i} + \tau^2\bar\psi^{\dagger;i}\bar\psi_{;i}\big)\\
&+4\xi\bar\psi^\dagger\bar\psi\Big(\xi\big[\tau^2R + \kappa^i{}_j\kappa^j{}_i + (\tau^{2k_0}\lambda - nH)^2\big] + 2\tau^{2k_1}W' + \mu^2\Big)\\
&-2W - 2\Lambda\xi\bar\psi^\dagger\bar\psi\Big\}\delta^a{}_b\Big].
\end{aligned}$$

Consider the function given by $[(1-\xi x)(1-\eta\xi x)^{-1}-1]/x$ for all non-zero x satisfying $(1+|\eta|)\xi|x| < 1$. It is analytic and has an analytic extension

$$\left[\frac{(1-\xi\cdot)(1-\eta\xi\cdot)^{-1}-1}{\cdot}\right]$$

to zero. Therefore it is possible to set

$$Q := \bar\psi^\dagger\bar\psi\left[\frac{(1-\xi\cdot)(1-\eta\xi\cdot)^{-1}-1}{\cdot}\right](\tau^{2k_1}\xi\bar\psi^\dagger\bar\psi)$$

and using this

$$\begin{aligned}
\dot\lambda := (1+\tau^{2k_1}Q)\Big[&\tau^{2(1-k_0)}R + \tau^{2k_0}\lambda^2 + \frac{1}{n-1}\tau^{2(k_1-k_0)}\Big(\check\Sigma^i{}_i - n\varepsilon\Big)\Big]\\
&-2nH\tau^{2k_1}Q\lambda
\end{aligned}$$

as well as

$$\tilde{R} := \tau^{2(1-k_0)} R - 2\dot{\lambda} + \tau^{2k_0} \varsigma^i{}_j \varsigma^j{}_i + \Big(1 + \frac{1}{n}\Big) \tau^{2k_0} \lambda^2 + 2(n-1) H \lambda.$$

In order to compare the quantities defined here to the projections of the energy-momentum tensor of the solution to be constructed it is convenient to have

$$\Sigma^a{}_b := \check{\Sigma}^a{}_b - \frac{4\xi^2 \tau^{2k_0} \bar{\psi}^\dagger \bar{\psi}}{1 - \xi \tau^{2k_1} \bar{\psi}^\dagger \bar{\psi}} (\dot{\lambda} - 2nH\lambda) \delta^a{}_b$$

at one's disposal. Lastly, as a shorthand, one may write

$$\begin{aligned} \ddot{\psi} &:= \tau^{2(1-k_0)} \bar{\psi}^{;i}{}_i + \lambda \bar{\psi}_\tau - \xi \tilde{R} \bar{\psi} - 2\tau^{2(k_1-k_0)} W' \bar{\psi} \\ \ddot{\psi}^\dagger &:= \tau^{2(1-k_0)} \bar{\psi}^{\dagger;i}{}_i + \lambda \bar{\psi}^\dagger_\tau - \xi \tilde{R} \bar{\psi}^\dagger - 2\tau^{2(k_1-k_0)} W' \bar{\psi}^\dagger. \end{aligned}$$

Then the non-linearity f which maps $U \times_M E \otimes T^0_1 M \times T$ to E analytically shall be given as components by

$$\begin{aligned} \gamma_{ab} \circ f &:= \frac{\tau^{2k_0}}{H} \Big[\gamma_{ai} \varsigma^i{}_b + \gamma_{bi} \varsigma^i{}_a + \frac{1}{n}(\gamma_{ab} + \gamma_{ba}) \Big] \\ \varsigma^a{}_b \circ f &:= -\frac{1}{2H} \Big[\tau^{2(1-k_0)} \big(\hat{R}^a{}_b + \hat{R}_a{}^b \big) + \tau^{2k_0} \lambda (\varsigma^a{}_b + \varsigma_a{}^b) \\ &\qquad - \tau^{2(k_1-k_0)} \big(\hat{\Sigma}^a{}_b + \hat{\Sigma}_a{}^b \big) - \frac{2}{n} \tau^{2k_0} \lambda (\varsigma^a{}_b - \varsigma_a{}^b) \\ &\qquad + \tau^{2k_0} (\varsigma^a{}_i \varsigma^i{}_b - \varsigma_b{}^i \varsigma_i{}^a) \Big] \\ \lambda \circ f &:= -\frac{1}{H} \dot{\lambda} \\ \psi \circ f &:= \tau^{k_0} \psi_\tau \\ \psi^\dagger \circ f &:= \tau^{k_0} \psi^\dagger_\tau \\ \psi_\tau \circ f &:= \frac{\tau^{k_0 + i\omega}}{H^2} \ddot{\psi} \\ \psi^\dagger_\tau \circ f &:= \frac{\tau^{k_0 - i\omega}}{H^2} \ddot{\psi}^\dagger \\ \partial\gamma_{abc} \circ f &:= \frac{2\tau^{2k_0}}{H} \Big[\partial\gamma_{aic} \varsigma^i{}_b + \gamma_{ai} \varsigma^i{}_{b,c} + \frac{1}{n} \gamma_{ab} \lambda_{,c} + \frac{1}{n} \lambda \partial\gamma_{abc} \Big] \\ \partial\psi_a \circ f &:= \tau^{k_0} \psi_{\tau,a} \\ \partial\psi^\dagger_a \circ f &:= \tau^{k_0} \psi^\dagger_{\tau,a}. \end{aligned}$$

As mentioned before, a hat is used to denote the trace-free part of the corresponding tensor and the notation $X_a{}^b$ is used as a shorthand for $\gamma_{ai} (\gamma^\sharp)^{jb} X^i{}_j$ for any $X \in \mathbb{C}T^1_1 M$.

What remains to be defined is the main part of equation (4.1), the bundle endomorphism $L \in \mathrm{End}(E)$. It is taken to be the following direct sum on the factors

$$\begin{aligned} L := {} & 0 \oplus (2k_0 - n) \oplus 2(k_0 - n) \oplus 0 \oplus 0 \\ & \oplus (k_0 - 2i\omega - k_2 + k_1) \oplus (k_0 + 2i\omega - k_2 + k_1) \oplus 0 \oplus 0 \oplus 0 \end{aligned}$$

from which the eigenvalues can directly be read off. Note that the endomorphism is constant and the real part of its spectrum is bounded below by $-2(n - k_0)$. The first order equation (4.1) obtained with these definitions of L and f will be referred to as the *reduced* system.

3. Approximate solutions of the reduced system

The system obtained so far is of the form of equation (4.1) but it is not Fuchsian since there are eigenvalues of the endomorphism L with a negative real part. The properties of the non-linearity already fit the assumptions of theorem A though. The plan now is to shift the eigenvalues up by using the formal series solutions obtained in chapter 2. As a step in that direction it is shown in this section that truncations of the formal solutions "approximately" solve the reduced system.

The notion of asymptotic character developed in chapter 3 can be carried over to the situation at hand by means of the bianalytic diffeomorphism

$$t :]0, \tau_0[\to]t_0, \infty[\,, \ \tau \mapsto -\frac{1}{H} \log \tau \tag{4.2}$$

where $-Ht_0 = \log \tau_0$ so that a smooth family $f : T \to \mathbb{C}\mathcal{T}^r_s M$ of (r, s)-tensors is exponentially bounded if and only if $f \circ t :]t_0, \infty[\to \mathbb{C}\mathcal{T}^r_s M$ is. As $0 < \tau_0 < \infty$ there is no risk of confusion and the families f and $f \circ t$ will simply be identified, i.e. $f \in \mathfrak{B}(\mathcal{T}^r_s M, \mathbb{C})$ will be written instead of $f \circ t \in \mathfrak{B}(\mathcal{T}^r_s M, \mathbb{C})$ or just $\mathfrak{v}(f)$ instead of $\mathfrak{v}(f \circ t)$.

Definition 4.3. Fix a $\theta > 0$. Two smooth tensor families $f, g : T \to \mathbb{C}\mathcal{T}^r_s M$ are said to be θ-asymptotically equivalent, $f \simeq g$, if $f - g \in \mathfrak{B}(\mathcal{T}^r_s M, \mathbb{C})$ and satisfies $\mathfrak{v}_\infty(f - g) \le e^{-\theta H}$.

For a fixed $\theta > 0$ the truncated evaluation map

$$\mathcal{F}_\theta\, \boldsymbol{f} := \mathcal{F}(\boldsymbol{f}|\theta) = \mathcal{F}\left(\sum_{m<\theta} \boldsymbol{f}(m) X^m\right), \qquad \boldsymbol{f} \in \mathfrak{T}(M)$$

has the following properties which immediately follow from the results of chapter 3. Let $\kappa \in \mathbb{C}$, $f, g : T \to \mathbb{C}\mathcal{T}^r_s M$, $f' : T \to \mathbb{C}\mathcal{T}^{r'}_{s'} M$ smooth tensor families and $\boldsymbol{f}, \boldsymbol{g} \in (\mathbb{C}\mathcal{T}^r_s M)[Z]\langle Y \rangle(\!(X)\!)$, $\boldsymbol{f}' \in (\mathbb{C}\mathcal{T}^{r'}_{s'} M)[Z]\langle Y \rangle(\!(X)\!)$, then

$$\begin{aligned} &\mathcal{F}_\theta(\kappa \boldsymbol{f}) = \kappa\, \mathcal{F}_\theta\, \boldsymbol{f} \\ &\mathcal{F}_\theta(\boldsymbol{f} + \boldsymbol{g}) = \mathcal{F}_\theta\, \boldsymbol{f} + \mathcal{F}_\theta\, \boldsymbol{g} \\ &\mathcal{F}_\theta(\boldsymbol{f} \otimes \boldsymbol{f}') \simeq (\mathcal{F}_\theta\, \boldsymbol{f}) \otimes (\mathcal{F}_\theta\, \boldsymbol{f}'). \end{aligned}$$

It means that $\mathcal{F}_\theta$, while not quite an algebra homomorphism, has compatible properties with respect to the equivalence relation $\simeq$. Furthermore, one has the approximation relations

$$\begin{aligned} &f' \simeq \mathcal{F}_\theta\, \boldsymbol{f}' \implies f + f' \simeq f + \mathcal{F}_\theta\, \boldsymbol{f}' \\ &g \simeq \mathcal{F}_\theta\, \boldsymbol{g} \wedge \mathfrak{v}_\infty(f) \le 1 \implies f \otimes g \simeq f \otimes \mathcal{F}_\theta\, \boldsymbol{g} \\ &f \simeq \mathcal{F}_\theta\, \boldsymbol{f} \wedge g \simeq \mathcal{F}_\theta\, \boldsymbol{g} \implies f + g \simeq \mathcal{F}_\theta(\boldsymbol{f} + \boldsymbol{g}) \\ &f \simeq \mathcal{F}_\theta\, \boldsymbol{f} \wedge f' \simeq \mathcal{F}_\theta\, \boldsymbol{f}' \implies f \otimes f' \simeq \mathcal{F}_\theta(\boldsymbol{f} \otimes \boldsymbol{f}') \end{aligned}$$

as well as, for a holomorphic function Ψ defined on an open neighbourhood of zero and provided the image of $f \in \mathfrak{B}(C^\infty(M, \mathbb{C}))$ lies within the domain of Ψ,

$$f \simeq \mathcal{F}_\theta\, \boldsymbol{f} \wedge \boldsymbol{f} = o(1) \implies \Psi \circ f \simeq \mathcal{F}_\theta\, \Psi(\boldsymbol{f}).$$

Finally it is noted that in the case where $f \simeq \mathcal{F}_\theta\, \boldsymbol{f}$ it is true that

$$\begin{aligned} &Df \simeq \mathcal{F}_\theta\, \partial_D \boldsymbol{f} \\ &\tau \partial_\tau f \simeq -\frac{1}{H}\, \mathcal{F}_\theta\, \partial_t \boldsymbol{f} \end{aligned}$$

and for any $\alpha \ge 0$ non-negative, $\beta \in \mathbb{R}$, $l \in \mathbb{N}$

$$(-H)^{-l} (\log \tau)^l \tau^{\alpha + i\beta} f \simeq \mathcal{F}_\theta(X^\alpha Y^\beta Z^l \boldsymbol{f}).$$

Let any formal solution $(\boldsymbol{g}, \boldsymbol{\phi})$ of the algebraic Einstein-scalar field system be given and fix ϕ_1 according to the conditions (P1) and (P2) of theorem 2.11. Assume the solution to be analytic, that is, all coefficients of $\boldsymbol{g}$ and $\boldsymbol{\phi}$ are analytic. With the rescaled metric $\bar{\boldsymbol{g}} := X^2 \boldsymbol{g}$ and scalar field $\check{\boldsymbol{\phi}} := X^{-k_1} Y^{\omega} \boldsymbol{\phi} - \phi_1 Y^{2\omega} \Upsilon(-HZ)$ define a family v of analytic sections of E as

$$
\begin{aligned}
(4.3) \qquad v := {} & \mathcal{F}_\theta \, \bar{\boldsymbol{g}} \oplus \mathcal{F}_\theta \, X^{-2k_0} \boldsymbol{\sigma} \oplus \mathcal{F}_\theta \, X^{-2k_0} (\mathbf{tr}\, \boldsymbol{k} + nH) \oplus \mathcal{F}_\theta \, \check{\boldsymbol{\phi}} \oplus \mathcal{F}_\theta \, \check{\boldsymbol{\phi}}^* \\
& \oplus \left(-\frac{1}{H} \, \mathcal{F}_\theta \, X^{-k_0} \partial_t \check{\boldsymbol{\phi}} \right) \oplus \left(-\frac{1}{H} \, \mathcal{F}_\theta \, X^{-k_0} \partial_t \check{\boldsymbol{\phi}}^* \right) \\
& \oplus \mathcal{F}_\theta \, D\bar{\boldsymbol{g}} \oplus \mathcal{F}_\theta \, D\check{\boldsymbol{\phi}} \oplus \mathcal{F}_\theta \, D\check{\boldsymbol{\phi}}^*
\end{aligned}
$$

for some $\theta > n$. One may assume, without loss of generality, that the images of $v(\tau)$ all lie within U. Then, from corollary 3.11 it follows that

$$
\begin{aligned}
\gamma^\sharp(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, \bar{\boldsymbol{g}}^\sharp \\
\partial\gamma^\sharp(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, D\bar{\boldsymbol{g}}^\sharp \\
A(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, \boldsymbol{A} \\
\partial A(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, D\boldsymbol{A}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\hat{R}^a{}_b(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2} \widehat{\mathbf{Ric}}^a{}_b \\
R(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2} \boldsymbol{R}
\end{aligned}
$$

are inferred. For the scalar field variables one has

$$
\begin{aligned}
\bar{\psi}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \boldsymbol{\phi} & \bar{\psi}^\dagger(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \boldsymbol{\phi}^* \\
\bar{\psi}_{;a}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \nabla_a \boldsymbol{\phi} & \bar{\psi}^\dagger_{;a}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \nabla_a \boldsymbol{\phi}^*
\end{aligned}
$$

because of $D\boldsymbol{\phi} = \nabla\boldsymbol{\phi}$, as well as

$$
\begin{aligned}
\bar{\psi}_{a,b}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} D_a D_b \boldsymbol{\phi} & \bar{\psi}^\dagger_{a,b}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} D_a D_b \boldsymbol{\phi}^* \\
\bar{\psi}_{;ab}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \nabla_a \nabla_b \boldsymbol{\phi} & \bar{\psi}^\dagger_{;ab}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \nabla_a \nabla_b \boldsymbol{\phi}^*.
\end{aligned}
$$

The quantities representing time derivatives of the scalar field satisfy

$$
\begin{aligned}
\bar{\psi}_\tau(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \partial_t \boldsymbol{\phi} & \bar{\psi}^\dagger_\tau(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \partial_t \boldsymbol{\phi}^* \\
\bar{\psi}_{\tau;a}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \nabla_a \partial_t \boldsymbol{\phi} & \bar{\psi}^\dagger_{\tau;a}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-k_1} \nabla_a \partial_t \boldsymbol{\phi}^*
\end{aligned}
$$

giving rise to

$$
\begin{aligned}
\bar{\psi}^{;a}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2-k_1} \nabla^a \boldsymbol{\phi} & \bar{\psi}^{\dagger;a}(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2-k_1} \nabla^a \boldsymbol{\phi}^* \\
\bar{\psi}^{;a}_\tau(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2-k_1} \nabla^a \partial_t \boldsymbol{\phi} & \bar{\psi}^{\dagger;a}_\tau(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2-k_1} \nabla^a \partial_t \boldsymbol{\phi}^*
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\psi}^{;a}{}_b(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2-k_1} \nabla^a \nabla_b \boldsymbol{\phi} \\
\bar{\psi}^{\dagger;a}{}_b(v, Dv, \tau) &\simeq \mathcal{F}_\theta \, X^{-2-k_1} \nabla^a \nabla_b \boldsymbol{\phi}^*.
\end{aligned}
$$

By noting that $(\bar\psi^\dagger\bar\psi)(v,Dv,\tau) \simeq \mathcal{F}_\theta\, X^{-2k_1}\boldsymbol{\phi}^*\boldsymbol{\phi}$ and so $(\tau^{2k_1}\bar\psi^\dagger\bar\psi)(v,Dv,\tau) \simeq \mathcal{F}_\theta\, \boldsymbol{\phi}^*\boldsymbol{\phi}$ and due to the choice of U_2 one concludes that

$$\begin{aligned} W(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}\Big[V(\boldsymbol{\phi}^*\boldsymbol{\phi}) - \Lambda\Big] \\ W'(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}\Big[V'(\boldsymbol{\phi}^*\boldsymbol{\phi}) - \mu^2/2\Big] \\ \big(1-\xi\tau^{2k_1}\bar\psi^\dagger\bar\psi\big)^{-1} &\simeq \mathcal{F}_\theta\big(1-\xi\boldsymbol{\phi}^*\boldsymbol{\phi}\big)^{-1} \\ \big(1-\eta\xi\tau^{2k_1}\bar\psi^\dagger\bar\psi\big)^{-1} &\simeq \mathcal{F}_\theta\big(1-\eta\xi\boldsymbol{\phi}^*\boldsymbol{\phi}\big)^{-1} \end{aligned}$$

and hence

$$Q(v,Dv,\tau) \simeq \mathcal{F}_\theta\, X^{-2k_1}\left[\frac{1-\xi\boldsymbol{\phi}^*\boldsymbol{\phi}}{1-\eta\xi\boldsymbol{\phi}^*\boldsymbol{\phi}} - 1\right].$$

Obviously,

$$\kappa(v,Dv,\tau) \simeq \mathcal{F}_\theta\Big(\boldsymbol{\sigma} + \frac{\operatorname{tr}\boldsymbol{k}}{n}\delta\Big)$$

and this implies

$$\begin{aligned} \varepsilon(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}(\boldsymbol{\rho} - \Lambda) \\ \iota_a(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}\boldsymbol{j}_a \\ \check\Sigma^a{}_b(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}\check{\boldsymbol{S}}^a{}_b. \end{aligned}$$

With this one calculates that

$$\begin{aligned} \dot\lambda(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-k_0}\dot{\boldsymbol{\lambda}} \\ \check S^a{}_b(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}\left[\check{\boldsymbol{S}}^a{}_b - \frac{4\xi^2\boldsymbol{\phi}^*\boldsymbol{\phi}}{1-\xi\boldsymbol{\phi}^*\boldsymbol{\phi}}\big(\dot{\boldsymbol{\lambda}} - 2nH(\operatorname{tr}\boldsymbol{k} + nH)\big)\right] \end{aligned}$$

and

$$\begin{aligned} \tilde R(v,Dv,\tau) \simeq \mathcal{F}_\theta\, X^{-2k_0}\Big[&\boldsymbol{R} - 2\dot{\boldsymbol{\lambda}} + \boldsymbol{\sigma}^i{}_j\boldsymbol{\sigma}^j{}_i + \Big(1+\frac{1}{n}\Big)(\operatorname{tr}\boldsymbol{k} + nH)^2 \\ &+ 2(n-1)H(\operatorname{tr}\boldsymbol{k}+nH)\Big] \end{aligned}$$

where the abbreviation

$$\begin{aligned} \dot{\boldsymbol{\lambda}} := \frac{1-\xi\boldsymbol{\phi}^*\boldsymbol{\phi}}{1-\eta\xi\boldsymbol{\phi}^*\boldsymbol{\phi}}\Big[&\boldsymbol{R} + (\operatorname{tr}\boldsymbol{k}+nH)(\operatorname{tr}\boldsymbol{k}-nH) + \frac{1}{n-1}\big(\operatorname{tr}\check{\boldsymbol{S}} - n\boldsymbol{\rho} + n\Lambda\big)\Big] \\ &+ 2nH(\operatorname{tr}\boldsymbol{k}+nH) \end{aligned}$$

was used. It will now be exploited that the algebraic quantities for the metric and the scalar field, $\boldsymbol{g}$ and $\boldsymbol{\phi}$ respectively, satisfy the Einstein-scalar field equations. The fact that the algebraic scalar field equation holds implies

$$\operatorname{tr}\boldsymbol{S} + n\Lambda - \operatorname{tr}\check{\boldsymbol{S}} = -\frac{4n\xi^2\boldsymbol{\phi}^*\boldsymbol{\phi}}{1-\xi\boldsymbol{\phi}^*\boldsymbol{\phi}}\partial_t\operatorname{tr}\boldsymbol{k}$$

and from there the algebraic Einstein equations yield

$$\dot{\boldsymbol{\lambda}} - 2nH(\operatorname{tr}\boldsymbol{k}+nH) = \partial_t\operatorname{tr}\boldsymbol{k}.$$

But then it is actually true that

$$\begin{aligned} \Sigma^a{}_b(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_1}(\boldsymbol{S}^a{}_b + \Lambda\delta^a{}_b) \\ (\dot\lambda - 2nH\lambda)(v,Dv,\tau) &\simeq \mathcal{F}_\theta\, X^{-2k_0}\partial_t\operatorname{tr}\boldsymbol{k} \end{aligned}$$

as well as

$$\tilde{R}(v, Dv, \tau) \simeq \mathcal{F}_\theta\, X^{-2k_0}\big[\boldsymbol{R}_{\bar{\boldsymbol{g}}} - n(n+1)H^2\big]$$

and this in turn shows again by the algebraic scalar field equation that

$$\begin{aligned}\ddot{\psi}(v, Dv, \tau) &\simeq \mathcal{F}_\theta\, X^{-2k_0} Y^{-\omega}\big[\partial_t^2\check{\boldsymbol{\phi}} + (k_2 - k_1 + 2 + 2i\omega)H\partial_t\check{\boldsymbol{\phi}}\big]\\ \ddot{\psi}^\dagger(v, Dv, \tau) &\simeq \mathcal{F}_\theta\, X^{-2k_0} Y^{\omega}\big[\partial_t^2\check{\boldsymbol{\phi}}^* + (k_2 - k_1 + 2 - 2i\omega)H\partial_t\check{\boldsymbol{\phi}}^*\big].\end{aligned}$$

Putting all this together finally yields

$$\begin{aligned}\gamma \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t\bar{\boldsymbol{g}}\\ \varsigma \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t(X^{-2k_0}\boldsymbol{\sigma}) - (n-2k_0)\,\mathcal{F}_\theta\, X^{-2k_0}\boldsymbol{\sigma}\\ \lambda \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t\big[X^{-2k_0}(\operatorname{tr}\boldsymbol{k} + nH)\big]\\ &\quad - 2(n-k_0)\,\mathcal{F}_\theta\, X^{-2k_0}(\operatorname{tr}\boldsymbol{k} + nH)\\ \psi \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t\check{\boldsymbol{\phi}}\\ \psi^\dagger \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t\check{\boldsymbol{\phi}}^*\\ \psi_\tau \circ f(v, Dv, \tau) &\simeq \frac{1}{H^2}\,\mathcal{F}_\theta\,\partial_t\big(X^{-k_0}\partial_t\check{\boldsymbol{\phi}}\big)\\ &\quad + \frac{1}{H}(k_2 - k_1 - k_0 + 2i\omega)\,\mathcal{F}_\theta\, X^{-k_0}\partial_t\check{\boldsymbol{\phi}}\\ \psi_\tau^\dagger \circ f(v, Dv, \tau) &\simeq \frac{1}{H^2}\,\mathcal{F}_\theta\,\partial_t\big(X^{-k_0}\partial_t\check{\boldsymbol{\phi}}^*\big)\\ &\quad + \frac{1}{H}(k_2 - k_1 - k_0 - 2i\omega)\,\mathcal{F}_\theta\, X^{-k_0}\partial_t\check{\boldsymbol{\phi}}^*\\ \partial\gamma \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t D\bar{\boldsymbol{g}}\\ \partial\psi \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t D\check{\boldsymbol{\phi}}\\ \partial\psi^\dagger \circ f(v, Dv, \tau) &\simeq -\frac{1}{H}\,\mathcal{F}_\theta\,\partial_t D\check{\boldsymbol{\phi}}^*\end{aligned}$$

meaning that indeed

$$\tau\partial_\tau v + Lv \simeq f(v, Dv, \tau) \tag{4.4}$$

holds.

Instead of putting the result of this section with the quite technical definitions for the rescaled quantities $\bar{\boldsymbol{g}}$ and $\check{\boldsymbol{\phi}}$ and the approximate solution (4.3) into a proposition they will be summarized in a rather informal and imprecise manner. So assume that a solution of the algebraic Einstein-scalar field system is given. For any arbitrary high order, the functions defined by some finite truncation of the formal series solutions solve the reduced system asymptotically to at least that order. This is expressed by the asymptotic equivalence (4.4).

4. Solutions of the reduced system

The formal solutions or more precisely the approximate solutions (4.3) obtained in the previous section can now be used to solve the reduced system exactly. This is done by showing that the "difference" u between an actual solution $v + \tau^\zeta u$ and the approximate solution v satisfies a Fuchsian system with linear part $L + \zeta$, thus yielding positive eigenvalues thereof by only choosing ζ large enough. The non-trivial task thereby is to verify that the non-linearity remains well-behaved for an application of theorem A. The corresponding estimates will therefore be provided in the following.

Let E, F and G be vector bundles over a Riemannian manifold (M, γ_0) together with (positive definite) bundle metrics and linear connections D^E, D^F and D^G on E, F and G respectively. Consider a smooth family of bundle maps $u : E \times T \to F$ over M where $T =]0, \tau_0[$ for some positive τ_0. By identifying the maps $u(\cdot, \tau)$ with sections of the pullback bundle $\pi_E^* F$ by the projection $\pi_E : E \to M$ and using the splitting of $TE = E \times_M E \oplus TM$ provided by the connection D^E to introduce the fibre metric $\langle \cdot, \cdot \rangle_E \oplus \gamma_0$ on $T_pE = E_m \oplus T_mM$ for any $p \in E$, $m = \pi_E(p)$, the definitions of chapter 3, section 1 become applicable to u and analogously to any smooth family of bundle maps $f : F \times T \to G$ over M.

PROPOSITION 4.4. *Suppose* $\mathring{u} : E \to F$ *and* $u_\tau, v_\tau : E \to F$ *are smooth (families of) bundle maps over* M *with* $\mathfrak{v}(u - \mathring{u}) < 1$ *and* $\mathfrak{v}(u - v) < 1$. *If* $\mathfrak{v}(D^G f) < 1$ *then the families* $f(u) : \tau \mapsto f_\tau(u_\tau)$ *and* $f(v) : \tau \mapsto f_\tau(v_\tau)$ *satisfy*

$$\mathfrak{v}[f(u) - f(v)] \leq \mathfrak{v}(D^G f)\mathfrak{v}(u - v).$$

PROOF. Choose $\alpha, \beta > 0$ such that $\mathfrak{v}(D^G f) < e^{-\alpha H} < 1$ and $\mathfrak{v}(u - v) < e^{-\beta H} < 1$. For a fixed $x_0 \in E$ there is a fibre-wise convex neighbourhood P of $p_0 = \mathring{u}(x_0)$ such that for τ small $|D^G f_\tau(p)|_{\mathrm{Hom}(F_m \oplus T_mM, G_m)} < \tau^\alpha$ for all $p \in P$. Due to the fact that $\mathfrak{v}(u - \mathring{u}) < 1$ as well as $\mathfrak{v}(v - \mathring{u}) \leq \max\{\mathfrak{v}(v - u), \mathfrak{v}(v - \mathring{u})\} < 1$ there is a neighbourhood U of x_0 in E with $\mathring{u}_\tau(U)$, $u_\tau(U)$ and $v_\tau(U)$ are all contained in P for small τ. It can further be assumed that $|u_\tau - v_\tau|_F \leq \tau^\beta$ on U as well. But then for τ sufficiently small one has on U

$$\frac{d}{ds} f\big[(1 - s)u_\tau + sv_\tau\big] = D^G f\big((1 - s)u_\tau + sv_\tau\big)[v_\tau - u_\tau, 0]$$

for $s \in [0, 1]$. Since $(1 - s)u_\tau(x) + sv_\tau(x) \in P$ for all $s \in [0, 1]$, $x \in U$ and τ sufficiently small it implies

$$|f(v) - f(u)|_G \leq C|D^G f|_{\mathrm{Hom}(F \oplus TM, G)}|v - u|_F \tag{4.5}$$

on U and for small τ where C is a constant. This shows $\mathfrak{v}[f(u) - f(v)] \leq e^{-(\alpha+\beta)H}$ which proves the claim. □

COROLLARY 4.5. *If* $\mathfrak{v}_p(u - \mathring{u}) < 1$, $\mathfrak{v}_p(u - v) < 1$ *and* $\mathfrak{v}_p(f) < 1$ *then* $\mathfrak{v}_p[f(u) - f(v)] \leq \mathfrak{v}_p(f)\mathfrak{v}_p(u - v)$.

PROOF. Consider the statement that if for α, β positive $\mathfrak{v}\big[(\tau\partial_\tau)^i (D^F)^j (u - \mathring{u})\big] < 1$, $\mathfrak{v}\big[(\tau\partial_\tau)^i (D^F)^j (u - v)\big] \leq e^{-\beta H}$ and $\mathfrak{v}\big[(\tau\partial_\tau)^i (D^G)^{j+1} f\big] \leq e^{-\alpha H}$ were valid for $i + j \leq n$ then so is $\mathfrak{v}\big((\tau\partial_\tau)^i (D^G)^j [f(u) - f(v)]\big) \leq e^{-(\alpha+\beta)H}$. Due to proposition 4.4 this holds for $n = 0$. Assume that it holds for an $n \in \mathbb{N}$ and that the hypothesis is fulfilled for $i + j \leq n + 1$. Then, for any vector field $X \in \mathcal{T}_0^1 E$,

$$\begin{aligned}
D^G_X[f(u) - f(v)] &= D^G f(u)[D^F_X u, T\pi_E X] - D^G f(v)[D^F_X v, T\pi_E X] \\
&= \Big(D^G f(u) - D^G f(v)\Big)[D^F_X u, T\pi_E X] \\
&\quad + D^G f(v)[D^F_X u - D^F_X v, 0]
\end{aligned}$$

as well as

$$\begin{aligned}
\tau\partial_\tau[f(u) - f(v)] &= D^G f(u)[\tau\partial_\tau u, 0] - D^G f(v)[\tau\partial_\tau v, 0] \\
&\quad + (\tau\partial_\tau f)(u) - (\tau\partial_\tau f)(v) \\
&= \Big(D^G f(u) - D^G f(v)\Big)[\tau\partial_\tau u, 0] \\
&\quad + D^G f(v)[\tau\partial_\tau u - \tau\partial_\tau v, 0] \\
&\quad + (\tau\partial_\tau f)(u) - (\tau\partial_\tau f)(v).
\end{aligned}$$

Using the induction hypothesis together with $\mathfrak{v}\big[(\tau\partial_\tau)^i(D^G)^{j+1}D^Gf\big] \le e^{-\alpha H}$ and $\mathfrak{v}\big[(\tau\partial_\tau)^i(D^G)^{j+1}\tau\partial_\tau f\big] \le e^{-\alpha H}$ for all $i+j \le n$ it can be concluded that

$$\mathfrak{v}\Big((\tau\partial_\tau)^i(D^G)^j[D^Gf(u) - D^Gf(v)]\Big) \le e^{-(\alpha+\beta)H},$$

$$\mathfrak{v}\Big((\tau\partial_\tau)^i(D^G)^j[(\tau\partial_\tau f)(u) - (\tau\partial_\tau f)(v)]\Big) \le e^{-(\alpha+\beta)H}$$

and

$$\mathfrak{v}\big[(\tau\partial_\tau)^i(D^G)^jD^Gf(u)\big] \le \max\Big\{\mathfrak{v}\big((\tau\partial_\tau)^i(D^G)^j[D^Gf(u) - D^Gf(\mathring{u})]\big), \mathfrak{v}\big[(\tau\partial_\tau)^i(D^G)^jD^Gf(\mathring{u})\big]\Big\} \le e^{-\alpha H}.$$

Now certainly $\mathfrak{v}\big[(\tau\partial_\tau)^i(D^F)^j(D^Fu - D^Fv)\big] \le e^{-\beta H}$ and $\mathfrak{v}\big[(\tau\partial_\tau)^i(D^F)^j(\tau\partial_\tau u - \tau\partial_\tau v)\big] \le e^{-\beta H}$ so that the statement holds for $i+j \le n+1$ and this completes the proof. □

Remark 4.6. Due to the local nature of the proofs proposition 4.4 and corollary 4.5 remain valid if E and F are merely open subsets of vector bundles over M containing a smooth section of the respective bundle. Note also that equation (4.5) establishes local Lipschitz continuity of f_τ uniformly in τ for τ sufficiently small provided $\mathfrak{v}(D^Gf) < 1$.

After procuring these important estimates the notation introduced in section 2 shall be adopted again and so in particular the spaces E, U and the maps L, f and v shall be defined as detailed there. It is now straightforward to show that f has the regularity required by theorem A. In fact, much more is ascertained. Fix a positive k_0 such that $2k_0$ is not greater than both one and k_1. To start with note that all the projections from $U \times_M E \otimes T^0_1 M$ are independent of τ and so have an $\mathfrak{v}_\infty$-value of at most one. Since the map inv is analytic the same is true of $\gamma^\sharp$. Then it follows immediately that $\mathfrak{v}_\infty(\partial\gamma^\sharp)$, $\mathfrak{v}_\infty(A)$, $\mathfrak{v}_\infty(\partial A)$, $\mathfrak{v}_\infty(\hat{R})$ and $\mathfrak{v}_\infty(R)$ are all less than or equal to one. Because $\mathfrak{v}_\infty[\Upsilon(\log\tau)\tau^{\pm i\omega}] \le 1$ it is clear that $\mathfrak{v}_\infty(\bar\psi)$, $\mathfrak{v}_\infty(\bar\psi^\dagger)$, $\mathfrak{v}_\infty(\bar\psi_{;a})$, $\mathfrak{v}_\infty(\bar\psi^\dagger_{;a})$, $\mathfrak{v}_\infty(\bar\psi_{a,b})$, $\mathfrak{v}_\infty(\bar\psi^\dagger_{a,b})$, $\mathfrak{v}_\infty(\bar\psi^{;a})$, $\mathfrak{v}_\infty(\bar\psi^{\dagger;a})$, $\mathfrak{v}_\infty(\bar\psi_{;ab})$, $\mathfrak{v}_\infty(\bar\psi^\dagger_{;a})$, $\mathfrak{v}_\infty(\bar\psi^{;a}{}_b)$, $\mathfrak{v}_\infty(\bar\psi^{\dagger;a}{}_b)$, $\mathfrak{v}_\infty(\bar\psi_\tau)$, $\mathfrak{v}_\infty(\bar\psi^\dagger_\tau)$, $\mathfrak{v}_\infty(\bar\psi_{\tau;a})$, $\mathfrak{v}_\infty(\bar\psi^\dagger_{\tau;a})$ and $\mathfrak{v}_\infty(\bar\psi^{;a}_\tau)$, $\mathfrak{v}_\infty(\bar\psi^{\dagger;a}_\tau)$ are not greater than one either. Now $\mathfrak{v}_\infty(\tau^{2k_1}\bar\psi^\dagger\bar\psi) \le e^{-2k_1H} < 1$ and so by corollary 3.9 one has $\mathfrak{v}_\infty(W)$, $\mathfrak{v}_\infty(W')$, $\mathfrak{v}_\infty(Q) \le 1$. Obviously, $\mathfrak{v}_\infty(\kappa) \le 1$ and therefore $\mathfrak{v}_\infty(\check\Sigma)$, $\mathfrak{v}_\infty(\iota)$, $\mathfrak{v}_\infty(\varepsilon) \le 1$. This then implies, by the choice of k_0, that $\mathfrak{v}_\infty(\dot\lambda) \le e^{-2k_0H}$ as well as $\mathfrak{v}_\infty(\tilde{R})$, $\mathfrak{v}_\infty(\Sigma) \le 1$ and hence $\mathfrak{v}_\infty(\ddot\psi)$, $\mathfrak{v}_\infty(\ddot\psi^\dagger) \le 1$. Putting all together yields

$$\mathfrak{v}_\infty(f) \le e^{-k_0H}. \tag{4.6}$$

Let A be the smallest $+$-stable set in $\mathfrak{A}_0$ containing the supports of $X^2\boldsymbol{g}$ and $X^{-k_1}\boldsymbol{\phi}$ and fix $\theta > 2n$ with $\theta \notin A$. It can be assumed that $2k_0 \le [(A-\theta)\cup A]\cap]0,\infty[$ and there is a positive ϵ with $2\epsilon < 2k_0 \le \max\{1, k_1\}$. It will further be assumed that the image of the approximate solution v is compactly contained in U. With this define $\zeta := \theta - \epsilon$ and an analytic map $\mathring{f} : \mathring{U} \times_M E \otimes T^0_1 M \times T \to E$ by

$$\mathring{f}(z, z', \tau) := \tau^{-\zeta}\Big[f(v + \tau^\zeta z, Dv + \tau^\zeta z', \tau) - (\tau\partial_\tau + L)v\Big]$$

where $\mathring{U}$ is a suitable open subset of E containing the zero-section of E. For simplicity compositions with the bundle projection from $\mathring{U} \times_M E \otimes T^0_1 M$ to M are not written out explicitly. To see that the regularity of f carries over to $\mathring{f}$ one observes that there is $\mathring{v} \in C^\omega(M, U)$ with $\mathfrak{v}_\infty(v - \mathring{v}) < 1$ which is given by the zero-order coefficients of the formal series occurring in equation (4.3). Then the maps $\mathring{v} \times D\mathring{v}$, $v \times Dv$ and $h(z, z', \tau) := v \times Dv + \tau^\zeta(z, z')$ are analytic and fulfill $\mathfrak{v}_\infty(v \times Dv - \mathring{v} \times D\mathring{v}) < 1$ and $\mathfrak{v}_\infty(v \times Dv - h) \le e^{-\zeta H}$. With the estimates (4.6) on the function f and (4.4) on the approximate solution v it follows by corollary 4.5 that

$$\mathfrak{v}_\infty(\tau^\zeta \mathring{f}) \leq \max\{\mathfrak{v}_\infty[f(h) - f(v)], \mathfrak{v}_\infty[(\tau\partial_\tau + L)v - f(v)]\} \leq \max\{e^{-(\zeta+k_0)H}, e^{-\theta H}\} \leq e^{-(\zeta+\epsilon)H}$$

and therefore

$$\mathfrak{v}_\infty(\mathring{f}) \leq e^{-\epsilon H}. \tag{4.7}$$

This together with the fact that all eigenvalues of $L + \zeta$ have strictly positive real part implies that the system

$$\tau\partial_\tau u + (L + \zeta)u = \mathring{f}(u, Du, \tau) \tag{4.8}$$

is indeed of Fuchsian form and theorem A yields the existence of a unique solution $u : \mathfrak{M} \to \mathring{U}$ defined on some open neighbourhood $\mathfrak{M}$ of $M \times \{0+\}$ in $M \times T$ which tends to zero locally uniformly as $\tau \to 0$ and for which $\pi_E \circ u$ is the canonical projection $\mathfrak{M} \to M$. As $\mathring{f}$ is actually analytic so is u. It is then clear that

$$w = v + \tau^\zeta u$$

is an analytic solution of the reduced system with $w - v = o(\tau^\zeta)$ locally uniformly as $\tau \to 0$. Conversely, if $\tilde{w} : \mathfrak{M} \to U$ is any solution of the reduced system uniformly analytic on M with $\tilde{w} - v = o(\tau^\zeta)$ locally uniformly then $\tilde{u} = \tau^{-\zeta}(\tilde{w} - v)$ solves the Fuchsian system above and so $\tilde{u} = u$ and therefore $\tilde{w} = w$. In this sense the Fuchsian system and the reduced system are equivalent.

In the rest of this section some basic properties of the solution w of the reduced system are collected that will allow for an interpretation of certain components of w as a solution to the Einstein-scalar field system. According to the definition of f one has $(\gamma_{ab} - \gamma_{ba}) \circ f(w, Dw, \tau) = 0$ as well as $\gamma_{ab} \circ L = 0$ so $\tau\partial_\tau[\gamma_{ab}(w) - \gamma_{ba}(w)] = (\gamma_{ab} - \gamma_{ba}) \circ f(w, Dw, \tau) = 0$ and hence readily

$$\gamma_{ab}(w) = \gamma_{ba}(w). \tag{R1}$$

This already ensures $X_i{}^i = X^i{}_i$ for $X \in \mathbb{C}T^1_1 M$. Using the fact that $\boldsymbol{\sigma}$ is trace-less and therefore $\varsigma^i{}_i(v) = 0$ it follows that the quantity $\varsigma^i{}_i(u)$ satisfies the Fuchsian equation

$$\tau\partial_\tau[\varsigma^i{}_i(u)] + [\zeta - (n - 2k_0)]\varsigma^i{}_i(u) = -\frac{1}{H}\tau^{2k_0}\lambda(w)\varsigma^i{}_i(u).$$

The uniqueness statement of theorem A thus implies that $\varsigma^i{}_i(u)$ is identically zero and so

$$\varsigma^i{}_i(w) = 0. \tag{R2}$$

Also with (R1) it is inferred that $[\tau\partial_\tau - (n-2k_0)][\gamma_{ai}(w)\varsigma^i{}_b(w) - \gamma_{bi}(w)\varsigma^i{}_a(w)] = 0$. From the algebraic solution it is known that $\gamma_{ai}(v)\varsigma^i{}_b(v) \simeq \gamma_{bi}(v)\varsigma^i{}_a(v)$ which shows that $y := \tau^{-\zeta}[\gamma_{ai}(w)\varsigma^i{}_b(w) - \gamma_{bi}(w)\varsigma^i{}_a(w)] = o(1)$ locally uniformly as $\tau \to 0$ while satisfying $\tau\partial_\tau y + [\zeta - (n - 2k_0)]y = 0$. Again by uniqueness it is concluded that

$$\gamma_{ai}(w)\varsigma^i{}_b(w) = \gamma_{bi}(w)\varsigma^i{}_a(w). \tag{R3}$$

Apart from these symmetries it is crucial to show that the "daggered" quantities used in the formulation of the reduced system as representing the corresponding complex conjugate variables do in fact have this meaning while all the other quantities remain real. For this the bundle automorphism

$$J := (\gamma, \varsigma, \lambda, \psi^\dagger, \psi, \psi^\dagger_\tau, \psi_\tau, \partial\gamma, \partial\psi^\dagger, \partial\psi) \in \mathrm{Aut}(E)$$

interchanging respectively the daggered and un-daggered variables is introduced. It maps the set U^* into U and by direct calculation it is seen that it satisfies

$$f\big(Jz^*, (J\otimes\delta)z'^*, \tau\big) = Jf(z, z', \tau)^*$$

for all $(z, z', \tau) \in U \times_M E \otimes T^0_1 M \times T$. Additionally,

$$LJ = JL^*$$

and for the approximate solution the relation $v = Jv^*$ is true. It follows then that Ju^* is analytic on $\mathfrak{M}$, tends to zero locally uniformly as τ goes to zero and satisfies the same equation u does,

$$\begin{aligned}\tau\partial_\tau(Ju^*) + (L+\zeta)Ju^* &= J\big[\tau\partial_\tau u + (L+\zeta)u\big]^* \\ &= \tau^{-\zeta}\big[f(Jw^*, DJw^*, \tau) - (\tau\partial_\tau + L)Jv^*\big] \\ &= \mathring{f}(Ju^*, DJu^*, \tau),\end{aligned}$$

where $(J\otimes\delta)Dw^* = DJw^*$ was used. Once again uniqueness implies

(R4) $\qquad w = Jw^*.$

As a result of the reduction of the Einstein-scalar field system to first order the auxiliary variables $\partial\gamma$, $\partial\psi$ and $\partial\psi^\dagger$ representing spatial derivatives of the corresponding quantities had to be introduced. That they really serve this purpose will now be established. Certainly, $\varsigma^a{}_{b,c}(w, Dw) = D_c\varsigma^a{}_b(w)$ and $\lambda_a(w, Dw) = D_a\lambda(w)$ so that with (R1)–(R3) one gets

$$\tau\partial_\tau\big[D_c\gamma_{ab}(w) - \partial\gamma_{abc}(w)\big] = \\ \frac{2}{H}\tau^{2k_0}\Big(\varsigma^i{}_b(w) + \frac{1}{n}\lambda(w)\delta^i{}_b\Big)\big[D_c\gamma_{ai}(w) - \partial\gamma_{aic}(w)\big].$$

Thus $y_{ab} := \tau^{-\zeta}[D_c\gamma_{ab}(w) - \partial\gamma_{abc}(w)] = D_c\gamma_{ab}(u) - \partial\gamma_{abc}(u)$ fulfills the equation

$$\tau\partial_\tau y_{ab} + \zeta y_{ab} = \frac{2}{H}\tau^{2k_0}\Big(\varsigma^i{}_b(w) + \frac{1}{n}\lambda(w)\delta^i{}_b\Big)y_{ai}$$

while vanishing locally uniformly in the limit $\tau \to 0$ and so $y_{ab} = 0$ is inferred which proves that

(R5) $\qquad D_c\gamma_{ab}(w) = \partial\gamma_{abc}(w).$

Along the same lines it is observed that $\tau\partial_\tau[D\psi(w) - \partial\psi(w)] = 0$ and $D\psi(w) - \partial\psi(w)$ vanishes for $\tau \to 0$ hence implying with (R4)

(R6) $\qquad D\psi(w) = \partial\psi(w) \qquad\qquad D\psi^\dagger(w) = \partial\psi^\dagger(w).$

The last property which shall be remarked here is that, given asymptotic data corresponding to a real scalar field, the solution will in fact be real. For this it is first noted that $\psi(u) = o(\tau^{k_0})$ locally uniformly as $\tau \to 0$ since it satisfies

$$\tau\partial_\tau\psi(u) + \zeta\psi(u) = \tau^{k_0}\psi_\tau(u)$$

and the assertion follows by integration. Then an analytic map $u^{\mathfrak{c}}$ on $\mathfrak{M}$ shall be given by requiring that

$$\begin{aligned}\psi(u^{\mathfrak{c}}) &:= \tau^{2i\omega}\psi^\dagger(u)\\ \psi^\dagger(u^{\mathfrak{c}}) &:= \tau^{-2i\omega}\psi(u)\\ \psi_\tau(u^{\mathfrak{c}}) &:= \tau^{2i\omega}\big[\psi^\dagger_\tau(u) + 2i\omega\tau^{-k_0}\psi^\dagger(u)\big]\\ \psi^\dagger_\tau(u^{\mathfrak{c}}) &:= \tau^{-2i\omega}\big[\psi_\tau(u) - 2i\omega\tau^{-k_0}\psi(u)\big]\\ \partial\psi(u^{\mathfrak{c}}) &:= \tau^{2i\omega}\partial\psi^\dagger(u)\\ \partial\psi^\dagger(u^{\mathfrak{c}}) &:= \tau^{-2i\omega}\partial\psi(u)\end{aligned}$$

and all other projections of $u^{\mathfrak{c}}$ agree with those of u. Because of the decay estimate for $\psi(u)$ just established it is obvious that $u^{\mathfrak{c}} = o(1)$ locally uniformly as τ approaches zero. Now suppose that the data for the scalar field is real, $\boldsymbol{\phi}^* = \boldsymbol{\phi}$, then $v^* = v$ and consider the function $w^{\mathfrak{c}} := v + \tau^{\zeta} u^{\mathfrak{c}}$. By a direct calculation it can be seen that $\bar{\psi}(w^{\mathfrak{c}}, Dw^{\mathfrak{c}}, \tau) = \bar{\psi}^{\dagger}(w, Dw, \tau)$ and $\bar{\psi}^{\dagger}(w^{\mathfrak{c}}, Dw^{\mathfrak{c}}, \tau) = \bar{\psi}(w, Dw, \tau)$ and that the analogues are true for all the other "barred" field variables. Furthermore $\ddot{\psi}(w^{\mathfrak{c}}, Dw^{\mathfrak{c}}, \tau) = \ddot{\psi}^{\dagger}(w, Dw, \tau)$ and $\ddot{\psi}^{\dagger}(w^{\mathfrak{c}}, Dw^{\mathfrak{c}}, \tau) = \ddot{\psi}(w, Dw, \tau)$. This finally implies the validity of

$$\tau\partial_\tau u^{\mathfrak{c}} + (L + \zeta)u^{\mathfrak{c}} = \mathring{f}(u^{\mathfrak{c}}, Du^{\mathfrak{c}}, \tau)$$

from which $u^{\mathfrak{c}} = u$ is deduced due to uniqueness of solutions of Fuchsian systems. In particular $\psi^{\dagger}(u) = \tau^{-2i\omega}\psi(u)$ and because of (R4) $\psi(w^*) = \psi^{\dagger}(w)$ so that one obtains

$$\text{(R7)} \qquad \bar{\psi}(w, Dw, \tau)^* = \bar{\psi}(w, Dw, \tau).$$

5. Solutions of the Einstein-scalar field system

An analytic formal solution $(\boldsymbol{g}, \boldsymbol{\phi})$ of the algebraic Einstein-scalar field system on an analytic Riemannian manifold (M, γ_0) determines an approximate solution v up to some order θ of the reduced system and it has been shown so far that this approximate solution can be "corrected" to yield a unique exact solution w of the reduced system which is $(\theta - \epsilon)$-asymptotically equivalent to v. It is demonstrated in this section that such a solution of the reduced system gives rise to a unique solution of the Einstein-scalar field equations. In particular it is shown that the constraints, which did not go into the reduced system, are satisfied.

The domain of the solution w is an open neighbourhood $\mathfrak{M}$ of $M \times \{0+\}$ in $M \times T$. Using the diffeomorphism (4.2) $t : T \to I$ this can be mapped bianalytically onto an open neighbourhood $\mathcal{M}$ of $M \times \{+\infty\}$ in $M \times I$,

$$\varphi : \mathfrak{M} \to \mathcal{M}, \quad (x, \tau) \mapsto (x, t(\tau)).$$

As done hitherto, compositions with φ and its inverse will be understood implicitly so for instance w and $w \circ \varphi^{-1}$ are identified. It is further convenient to denote by π and t also the canonical projections $\mathcal{M} \to M$ and $\mathcal{M} \to I$ respectively. On the manifold $\mathcal{M}$ define a Lorentzian metric

$$\tilde{g} := e^{2Ht}\pi^*\gamma(w, Dw, t) - dt \otimes dt$$

and a complex scalar field

$$\tilde{\phi} := e^{-k_1 Ht}\pi^*\bar{\psi}(w, Dw, t).$$

This metric indeed is real by (R4) and symmetric by (R1). Using the notation from chapter 1 and the fact that w is a solution of the reduced system it follows with (1.10) that

$$\sigma + \frac{\operatorname{tr} k + nH}{n}\delta = e^{-2k_0 Ht}\Big[\varsigma(w, Dw, t) + \frac{1}{n}\lambda(w, Dw, t)\delta\Big]$$

and therefore by (R2)

$$\operatorname{tr} k + nH = e^{-2k_0 Ht}\lambda(w, Dw, t)$$
$$\sigma = e^{-2k_0 Ht}\varsigma(w, Dw, t).$$

If $g^\sharp$ is the metric induced by g on the cotangent bundle it follows that

$$g^\sharp = e^{-2Ht}\gamma^\sharp(w, Dw, t),$$

from (R5) that

$$\begin{aligned} Dg &= e^{2Ht}\partial\gamma(w, Dw, t) \\ Dg^\sharp &= e^{-2Ht}\partial\gamma^\sharp(w, Dw, t) \end{aligned}$$

and thus for the scalar curvature and the trace-free Ricci tensor

$$\begin{aligned} R_g &= e^{-2Ht}R(w, Dw, t) \\ \widehat{\mathrm{Ric}} &= e^{-2Ht}\hat{R}^a{}_b(w, Dw, t)\,\partial_a \otimes dx^b. \end{aligned}$$

For the scalar field and its covariant derivatives one gets using (R4) and (R6)

$$\begin{aligned} \phi &= e^{-k_1 Ht}\bar{\psi}(w, Dw, t) \\ \phi^* &= e^{-k_1 Ht}\bar{\psi}^\dagger(w, Dw, t) \\ \nabla\phi &= e^{-k_1 Ht}\bar{\psi}_{;a}(w, Dw, t)\,dx^a \\ \nabla\phi^* &= e^{-k_1 Ht}\bar{\psi}^\dagger_{;a}(w, Dw, t)\,dx^a \\ \nabla^2\phi &= e^{-k_1 Ht}\bar{\psi}_{;ab}(w, Dw, t)\,dx^a \otimes dx^b \\ \nabla^2\phi^* &= e^{-k_1 Ht}\bar{\psi}^\dagger_{;ab}(w, Dw, t)\,dx^a \otimes dx^b, \end{aligned}$$

as well as for its time derivative

$$\begin{aligned} \partial_t\phi &= e^{-k_1 Ht}\bar{\psi}_\tau(w, Dw, t) \\ \partial_t\phi^* &= e^{-k_1 Ht}\bar{\psi}^\dagger_\tau(w, Dw, t) \\ \nabla\partial_t\phi &= e^{-k_1 Ht}\bar{\psi}_{\tau;a}(w, Dw, t)\,dx^a \\ \nabla\partial_t\phi^* &= e^{-k_1 Ht}\bar{\psi}^\dagger_{\tau;a}(w, Dw, t)\,dx^a. \end{aligned}$$

It is clear that

$$\begin{aligned} V(\phi^*\phi) &= \Lambda + e^{-2k_1 Ht}W(w, Dw, t) \\ V'(\phi^*\phi) &= \frac{\mu^2}{2} + e^{-2k_1 Ht}W'(w, Dw, t) \\ \frac{1-\xi\phi^*\phi}{1-\eta\xi\phi^*\phi} &= 1 + e^{-2k_1 Ht}Q(w, Dw, t) \end{aligned}$$

and hence

$$\begin{aligned} \partial_t \operatorname{tr} k &= e^{-2k_0 Ht}\big[\dot{\lambda}(w, Dw, t) - 2nH\lambda(w, Dw, t)\big] \\ R_{\tilde{g}} &= n(n+1)H^2 + e^{-2k_0 Ht}\tilde{R}(w, Dw, t). \end{aligned}$$

From here it can already be concluded that the scalar field equation (1.2) holds true for S vanishes identically. As the projections of the energy-momentum tensor satisfy

$$\begin{aligned} \rho &= \Lambda + e^{-2k_1 Ht}\varepsilon(w, Dw, t) \\ j &= e^{-2k_1 Ht}\iota_a(w, Dw, t)\,dx^a \\ \operatorname{tr} S &= -n\Lambda + e^{-2k_1 Ht}\Sigma^i{}_i(w, Dw, t) \\ \hat{S} &= e^{-2k_1 Ht}\hat{\Sigma}^a{}_b(w, Dw, t)\,\partial_a \otimes dx^b \end{aligned}$$

it is seen that the evolution quantities e and E also vanish. To establish the same statement for the constraint quantities consider the analytic functions

$$\chi := \tau^{2-k_0}R - \tau^{3k_0}\varsigma^i{}_j\varsigma^j{}_i + \tau^{k_0}\Big(1 - \frac{1}{n}\Big)\lambda(\tau^{2k_0}\lambda - 2nH) - 2\tau^{2k_1-k_0}\varepsilon$$

and

$$\chi_a := \tau^{2k_0}\big(\varsigma^i{}_{a,i} + A^i_{ij}\varsigma^j{}_a - A^j_{ia}\varsigma^i{}_j\big) - \Big(1 - \frac{1}{n}\Big)\tau^{2k_0}\lambda_{,a} - \tau^{2k_1}\iota_a$$

on $U \times_M E \otimes T^0_1 M \times T$. They satisfy $\mathfrak{v}_\infty(\chi) < 1$ and $\mathfrak{v}_\infty(\chi_a) < 1$. From section 3 one obtains additionally

$$\chi(v, Dv, \tau) \simeq \mathcal{F}_\theta\, X^{-k_0} \mathsf{c} = 0$$

as well as

$$\chi_a(v, Dv, \tau) \simeq \mathcal{F}_\theta\, \mathsf{C} = 0.$$

Using this together with corollary 4.5 shows that $\mathfrak{v}[\chi(w, Dw, \tau)] < e^{-\zeta H}$ and $\mathfrak{v}[\chi_a(w, Dw, \tau)] < e^{-\zeta H}$. On the other hand

$$\begin{aligned} \mathsf{c} &= e^{-k_0 H t}\chi(w, Dw, t) \\ \mathsf{C} &= \chi_a(w, Dw, t)\, dx^a \end{aligned}$$

so that according to the equations (1.23) and (1.24) and since $\mathsf{e} = 0$, $\mathsf{E} = 0$ and $\mathsf{S} = 0$ the uniformly analytic functions $C := \chi(w, Dw, \tau)$, $C_a := \chi_a(w, Dw, \tau)$ satisfy the system

$$\begin{aligned} \tau\partial_\tau C - \quad (2n - k_0)C &= -\frac{2}{H}\Big[\tau^{2k_0 H}\lambda(w)C - \tau^{2-k_0}(\gamma^\sharp)^{ij}(w)\nabla_i C_j\Big] \\ \tau\partial_\tau C_a - \quad nC_a &= -\frac{1}{H}\Big[\tau^{2k_0 H}\lambda(w)C_a - \frac{1}{2}\tau^{k_0}\nabla_a C\Big]. \end{aligned}$$

Because of $\mathfrak{v}(C), \mathfrak{v}(C_a) < e^{-\zeta H}$ and $\mathfrak{v}[\lambda(w, Dw, \tau)] \leq 1$ while $2n - k_0$ and n are both less than ζ it follows from the uniqueness theorem for Fuchsian systems that C and C_a actually have to vanish but this implies the constraints to be satisfied, $\mathsf{c} = 0$ and $\mathsf{C} = 0$. Thus, the space-time $\mathcal{M}$ equipped with the metric $\tilde{g}$ and the scalar field $\tilde{\phi}$ is a solution of the Einstein-scalar field system with t a Gaussian time coordinate and such that $\bar{g} = e^{-2Ht} g$ and $\mathcal{F}_\zeta\, \bar{\boldsymbol{g}}$ as well as $\bar{\phi} = e^{k_1 H t}\phi$ and $\mathcal{F}_\zeta\, \bar{\boldsymbol{\phi}}$ are ζ-asymptotically equivalent.

In order to establish uniqueness assume that a uniformly analytic solution $(\tilde{g}, \tilde{\phi})$ of the Einstein-scalar field system on $\mathcal{M}$ is given such that the projection t is a Gaussian time coordinate, $1 - \xi\tilde{\phi}^*\tilde{\phi} > 0$ and $\mathfrak{v}_\infty[e^{k_1 H t}\phi - \mathcal{F}\,\boldsymbol{\phi}(k_1)] < 1$. Define then a uniformly analytic function $\tilde{w}$ on $\mathfrak{M}$ by requiring

$$\begin{aligned} \gamma(\tilde{w}) &:= \tau^2 g \\ \varsigma(\tilde{w}) &:= \tau^{-2k_0}\sigma \\ \lambda(\tilde{w}) &:= \tau^{-2k_0}(\operatorname{tr} k + nH) \\ \psi(\tilde{w}) &:= \tau^{-k_1 + i\omega}\phi - \phi_1 \Upsilon(\log\tau)\tau^{2i\omega} \\ \psi^\dagger(\tilde{w}) &:= \tau^{-k_1 - i\omega}\phi^* - \phi_1^* \Upsilon(\log\tau)\tau^{-2i\omega} \end{aligned}$$

as well as

$$\begin{aligned} \psi_\tau(\tilde{w}) &:= \tau^{-k_0}\tau\partial_\tau\psi(\tilde{w}) \\ \psi^\dagger_\tau(\tilde{w}) &:= \tau^{-k_0}\tau\partial_\tau\psi^\dagger(\tilde{w}) \\ \partial\gamma(\tilde{w}) &:= D\gamma(\tilde{w}) \\ \partial\psi(\tilde{w}) &:= D\psi(\tilde{w}) \\ \partial\psi^\dagger(\tilde{w}) &:= D\psi^\dagger(\tilde{w}). \end{aligned}$$

Due to the assumptions this function $\tilde{w}$ takes values in the set U and it follows from the definition that

$$\begin{aligned} \gamma^\sharp(\tilde{w}) &= \tau^{-2} g^\sharp \\ R(\tilde{w}, D\tilde{w}, \tau) &= \tau^{-2} R_g \\ \hat{R}^a{}_b(\tilde{w}, D\tilde{w}, \tau) &= \tau^{-2}(\widehat{\mathrm{Ric}})^a{}_b. \end{aligned}$$

Moreover the field quantities satisfy

$$\begin{aligned} \bar\psi(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}\phi & \bar\psi^\dagger(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}\phi^* \\ \bar\psi_{;a}(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}D_a\phi & \bar\psi^\dagger_{;a}(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}D_a\phi^* \\ \bar\psi_{;ab}(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}D_aD_b\phi & \bar\psi^\dagger_{;ab}(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}D_aD_b\phi^* \end{aligned}$$

and their time derivatives

$$\begin{aligned} \bar\psi_\tau(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}\partial_t\phi & \bar\psi^\dagger_\tau(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}\partial_t\phi^* \\ \bar\psi_{\tau;a}(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}D_a\partial_t\phi & \bar\psi^\dagger_{\tau;a}(\tilde w, D\tilde w, \tau) &= \tau^{-k_1}D_a\partial_t\phi^*. \end{aligned}$$

From there one clearly gets

$$\begin{aligned} W(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}\big[V(\phi^*\phi) - \Lambda\big] \\ W'(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}\Big[V'(\phi^*\phi) - \frac{\mu^2}{2}\Big] \\ Q(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}\Big[\frac{1-\xi\phi^*\phi}{1-\eta\xi\phi^*\phi} - 1\Big] \end{aligned}$$

and hence for the projections of the energy-momentum tensor

$$\begin{aligned} \varepsilon(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}(\rho - \Lambda) \\ \iota_a(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}j_a \\ \check\Sigma^a{}_b(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}\check S^a{}_b. \end{aligned}$$

The validity of the Einstein-scalar field equations then imply

$$(\dot\lambda - 2nH\lambda)(\tilde w, D\tilde w, \tau) = \tau^{-2k_0}\partial_t \operatorname{tr} k$$

and therefore

$$\begin{aligned} \tilde R(\tilde w, D\tilde w, \tau) &= \tau^{-2k_0}\big[R_{\tilde g} - n(n+1)H^2\big] \\ \Sigma^a{}_b(\tilde w, D\tilde w, \tau) &= \tau^{-2k_1}(S^a{}_b + \Lambda\delta^a{}_b). \end{aligned}$$

Putting all this together and using the validity of the Einstein-scalar field equations shows that $\tilde w$ is a solution of the reduced system

$$\tau\partial_\tau\tilde w + L\tilde w = f(\tilde w, D\tilde w, \tau).$$

By imposing suitable conditions on the late-time asymptotics of $\tilde g$ and $\tilde\phi$ that guarantee $\tilde w - v = o(\tau^\zeta)$ and $D\tilde w - Dv = o(\tau^\zeta)$ it can then be concluded that in fact $\tilde w = w$ and therefore the solution $(\tilde g, \tilde\phi)$ coincides with the one obtained from w.

The findings of this chapter can be summarized by the following theorem. Recall that the potential shall be holomorphic on a neighbourhood of zero assuming real values when the argument is real and the usual conditions are supposed to hold, namely that $\Lambda = V(0) > 0$ and $\mu^2 = 2V'(0)$ fulfill the inequality (2.9).

THEOREM 4.7. *Let M be an analytic manifold and $(\boldsymbol{g}, \boldsymbol{\phi})$ a formal analytic solution of the algebraic Einstein-scalar field equations. Given any $\zeta > 2n$ there exists a unique analytic solution $(\tilde g, \tilde\phi)$ of the Einstein-scalar field equations on an open neighbourhood $\mathcal{M}$ of $M \times \{+\infty\}$ in $M \times \mathbb{R}$ such that the projection $t : \mathcal{M} \to \mathbb{R}$ is a Gaussian time coordinate and the asymptotics of the solution with respect to t is given by $(\boldsymbol{g}, \boldsymbol{\phi})$ up to relative order ζ in the sense that*

$$\mathfrak{v}_\infty(\bar g - \mathcal{F}_\theta\,\bar{\boldsymbol{g}}),\ \mathfrak{v}_\infty(\bar\phi - \mathcal{F}_\theta\,\bar{\boldsymbol{\phi}}) < e^{-\zeta H}$$

is true for the rescaled metric $\bar g = e^{-2Ht}g$, the rescaled field $\bar\phi = e^{k_1 Ht}\phi$ and a $\theta > \zeta$. If the asymptotic data is real, $\boldsymbol{\phi}^* = \boldsymbol{\phi}$, so is the solution.

PROOF. For any $\zeta > 2n$ there is a $\theta > \zeta$ such that $3(\theta-\zeta) \leq [(A-\zeta)\cup A]\cap]0,\infty[$, where A is the smallest $+$-stable set in $\mathfrak{A}_0$ containing the supports of $\bar{\boldsymbol{g}} = X^2\boldsymbol{g}$ and $\bar{\boldsymbol{\phi}} = X^{-k_1}\boldsymbol{\phi}$. Then for $\zeta < \zeta' < \theta$ it has been shown that there is a solution $(\tilde{g}, \tilde{\bar{\phi}})$ such that $\bar{g} - \mathcal{F}_\theta\,\bar{\boldsymbol{g}} = o(e^{-\zeta' H t})$ and $\bar{\phi} - \mathcal{F}_\theta\,\bar{\boldsymbol{\phi}} = o(e^{-\zeta' H t})$ locally uniformly as $t \to \infty$ which shows that $\mathfrak{v}(\bar{g} - \mathcal{F}_\theta\,\bar{\boldsymbol{g}}) < e^{-\zeta H}$ and $\mathfrak{v}(\bar{\phi} - \mathcal{F}_\theta\,\bar{\boldsymbol{\phi}}) < e^{-\zeta H}$. As the non-linearity f of the reduced system is in fact analytic so is the solution $(\tilde{g}, \tilde{\bar{\phi}})$. Moreover, the estimate (4.7) shows that the Fuchsian system (4.8) stays Fuchsian even when differentiated with respect to space and time as often as desired, which then yields in particular the asserted decay estimates. Uniqueness on the other hand, as it has been pointed out above, stems from the fact that the decay estimates imply $\bar{g} - \mathcal{F}_\theta\,\bar{\boldsymbol{g}} = o(e^{-\zeta' H t})$, $\partial_t \bar{g} - \mathcal{F}_\theta\,\partial_t \bar{\boldsymbol{g}} = o(e^{-(\zeta' + 2k_0) H t})$, $\bar{\phi} - \mathcal{F}_\theta\,\bar{\boldsymbol{\phi}} = o(e^{-(\zeta' + k_0) H t})$ and $\partial_t \bar{\phi} - \mathcal{F}_\theta\,\partial_t \bar{\boldsymbol{\phi}} = o(e^{-(\zeta' + k_0) H t})$ locally uniformly as $t \to \infty$ and that the same is true of all spatial derivatives for a ζ' with $\zeta < \zeta' < \theta$ and k_0 sufficiently small. The remainder follows from (R7).

□

The same comment as at the end of chapter 3 might be appropriate here in that approximate formal solutions together with proposition 4.4 and corollary 4.5 could possibly be useful in other situations too to prove existence and uniqueness of solutions with prescribed asymptotics.

CHAPTER 5

Spatially homogeneous models

The aim of this chapter is to prove late-time asymptotics for a class of solutions of the Einstein-scalar field-matter system which is, compared to the previous results, far more restricted in that a strong symmetry assumption is made, but also more general as it allows for a much wider class of potentials and, notably, the presence of other forms of matter. No assumptions on that matter are made except that it has to satisfy certain energy conditions and so the result will not be a global existence statement with detailed asymptotics like before but rather crude estimates on the late-time acceleration and isotropization of solutions that exist globally towards the future. To have the conformal transformation between matter coupled and curvature coupled models available only the case of a real scalar field is considered. The contents of this chapter were published in [**6**].

1. Potentials with a positive lower bound

To analyse the asymptotics of global solutions of the matter coupled Einstein-scalar field system some assumptions on the Lie group G, the potential U, the coupling functions c and C as well as on the energy-momentum tensor T_{m} describing the ordinary matter content will now be made. Two cases are then considered separately, namely one where the potential has a positive lower bound that leads asymptotically to exponential acceleration, and a second in the next section where the potential energy density of the scalar field approaches zero, which allows, in general, for intermediate late-time acceleration only. The argument closely follows that of [**25**].

The Lie group G is assumed such that every left invariant Riemannian metric has non-positive scalar curvature. In the three-dimensional simply connected case, this corresponds to groups of Bianchi type other than IX [**32**]. In general Bergery [**4**] proved that a connected Lie group admits only left invariant Riemannian metrics of zero or strictly negative scalar curvature if and only if its universal cover is diffeomorphic to $\mathbb{R}^n$. On the other hand Milnor [**20**] has shown that the latter is not the case if the Lie group contains a compact non-commutative subgroup. Anyway, the assumption guarantees $R \leq 0$. The energy-momentum tensor T_{m} of the additional matter is supposed to satisfy the dominant and strong energy condition, which imply

(DEC) $\quad |j_{\mathrm{m}}| \leq \rho_{\mathrm{m}}, \; |\operatorname{tr} S_{\mathrm{m}}| \leq n\rho_{\mathrm{m}}$

(SEC) $\quad \rho_{\mathrm{m}} + \operatorname{tr} S_{\mathrm{m}} \geq 0.$

Furthermore, it is assumed that the coupling function C is non-negative, so $C(\phi)T_{\mathrm{m}}$ fulfills every energy condition T_{m} does, and that there is a constant C_0 such that

(C) $\quad |c| \leq C_0 C$

holds. The potential U shall be positive and satisfy the following, more technical conditions:

(U1) There exists a positive lower semi-continuous minorant $\bar{U}$ on the closure $\bar{J}$ of J in $\mathbb{R}$, i.e. $\bar{U} : \bar{J} \to \mathbb{R}$ is lower semi-continuous, $\bar{U} > 0$ and $U(x) \geq \bar{U}(x)$ for all $x \in J$.

(U2) The derivative U' is bounded on any subset of J on which U is bounded.

(U3) U' extends to a continuous function on the closure of J in $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with values in $\bar{\mathbb{R}}$.

Condition (U1) ensures that U is bounded away from zero on bounded subsets of J whereas (U3) yields the existence of limits $\lim_{x \to J_\pm} U'(x) = U'_\pm \in \bar{\mathbb{R}}$ when x approaches the endpoints $J_\pm$ of J in $\bar{\mathbb{R}}$. Finally, suppose that the model is initially expanding, so $t_0 = \min I \in \mathbb{R}$ and $H(t_0) > 0$, where $H = -\operatorname{tr} k/n$ shall now be used as an abbreviation for the negative of the mean curvature. Lemma 5.1 below collects some immediate consequences of these assumptions.

LEMMA 5.1. *The following properties hold:*

(1) *H is positive, bounded and monotonically decreasing*
(2) *$\dot\phi$ is bounded and square-integrable with $\|\dot\phi\|_{L^\infty} \leq \sqrt{n(n-1)}\|H\|_{L^\infty}$ and $\|\dot\phi\|^2_{L^2} \leq (n-1)\|H\|_{L^\infty}$*
(3) *$\ddot\phi$, R, $|\sigma|^2$, $U(\phi)$, $U'(\phi)$, $C(\phi)\rho_{\mathrm{m}}$ and $c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}}$ are bounded with*

$$\|R\|_{L^\infty}, \||\sigma|^2\|_{L^\infty}, 2\|U(\phi)\|_{L^\infty}, 2\|C(\phi)\rho_{\mathrm{m}}\|_{L^\infty} \leq n(n-1)\|H\|^2_{L^\infty}$$
$$\|c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}}\|_{L^\infty} \leq \frac{1}{2}n(n-1)(n+1)C_0\|H\|_{L^\infty}$$

PROOF. Since U is positive so is H^2 due the Hamiltonian constraint (1.30) and so is H itself because it is positive initially. The evolution equation (1.32) and (DEC) render $\dot H$ non-positive in the same way, so that H is monotonically decreasing and bounded by $0 < H \leq H(t_0)$. In fact, the inequality $\dot\phi^2 \leq -(n-1)\dot H$ holds, so integration over I yields $\|\dot\phi\|^2_{L^2} \leq (n-1)\|H\|_{L^\infty}$. The L^∞ bounds on $\dot\phi$, R, $|\sigma|^2$, $U(\phi)$ and $C(\phi)\rho_{\mathrm{m}}$ follow from (1.30) directly, $U'(\phi) \in L^\infty(I)$ is then a consequence of (U2) and the estimate for $c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}}$ is obtained from the inequality $|c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}}| = |c(\phi)||\operatorname{tr} S_{\mathrm{m}} - \rho_{\mathrm{m}}| \leq C_0 C(\phi)(n+1)\rho_{\mathrm{m}}$ using (C). The scalar field equation (1.34) finally gives a L^∞ bound for $\ddot\phi$. □

Since U is a positive function, the infimum of the potential energy density of the field $U_0 := \inf (U \circ \phi)(I)$ is either positive or zero. The rest of this section is concerned with the late-time asymptotics in the first case. It is shown that expansion, isotropization and decay of matter terms take place exponentially in time. Note that the solution is assumed to exist globally towards the future and so I is not bounded from above.

PROPOSITION 5.2. *For any $\delta > 0$ the spatial curvature R, the shear $|\sigma|^2$, the matter terms $C(\phi)\rho_{\mathrm{m}}$, $C(\phi)|j_{\mathrm{m}}|$ and $C(\phi)\operatorname{tr} S_{\mathrm{m}}$ as well as the coupling term $c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}}$ decay at least as $e^{-(2-\delta)H_0 t}$ for $t \to \infty$,*

$$R,\ |\sigma|^2,\ C(\phi)\rho_{\mathrm{m}},\ C(\phi)|j_{\mathrm{m}}|,\ C(\phi)\operatorname{tr} S_{\mathrm{m}},\ c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}} = O\big(e^{-(2-\delta)H_0 t}\big),$$

where $H_0 := \sqrt{2U_0/n(n-1)}$.

PROOF. (i) From the square-integrability of $\dot\phi$ and the boundedness of $\ddot\phi$ established in lemma 5.1 it follows that $\dot\phi(t) \to 0$ as $t \to \infty$. (ii) With $E := 1/2\,\dot\phi^2 + U(\phi) \in C^1(I)$ the field energy density, define a quantity [**25**]

$$Z := n(n-1)H^2 - 2E \in C^1(I).$$

Because of (1.34) $\dot{E} = c(\phi)(\operatorname{tr}_{\tilde{g}} T_{\mathrm{m}})\dot{\phi} - nH\dot{\phi}^2$, which implies with (SEC)

$$\begin{aligned}\dot{Z} &= -2H\big(Z + (n-1)|\sigma|^2 + C(\phi)\big[(n-2)\rho_{\mathrm{m}} + \operatorname{tr} S_{\mathrm{m}}\big]\big) - 2c(\phi)(\operatorname{tr}_{\tilde{g}} T_{\mathrm{m}})\dot{\phi} \\ &\le -2HZ - 2c(\phi)(\operatorname{tr}_{\tilde{g}} T_{\mathrm{m}})\dot{\phi}.\end{aligned}$$

(iii) Fix a $0 < \delta < 1$, then

$$\delta HC(\phi)\rho_{\mathrm{m}} + 2c(\phi)(\operatorname{tr}_{\tilde{g}} T_{\mathrm{m}})\dot{\phi} \ge \big[\delta H - 2(n+1)C_0|\dot{\phi}|\big]C(\phi)\rho_{\mathrm{m}}$$

holds and with $-2HZ \le -(2-\delta)HZ - \delta HC(\phi)\rho_{\mathrm{m}}$ and $H \ge H_0$ by (1.30) one obtains

$$Z \le -(2-\delta)H_0 Z$$

eventually from (i). Integrating this yields

$$Z(t) = O\big(e^{-(2-\delta)H_0 t}\big) \qquad (t \to \infty).$$

Noting again that $Z = |\sigma|^2 - R + C(\phi)\rho_{\mathrm{m}}$ and using (DEC) and (C) now yields the claims. □

With this information at hand, it is possible to obtain late-time limits for the mean curvature $-H$, the potential energy density of the field $U(\phi)$, its derivative, and for the derivatives of the field ϕ itself.

PROPOSITION 5.3. *As t goes to infinity, the following limits are attained:*

(1) $H(t) \to H_\infty$, *with* $H_\infty \ge H_0 > 0$
(2) $(U \circ \phi)(t) \to U_\infty := \frac{1}{2}n(n-1)H_\infty^2 > 0$
(3) $(U' \circ \phi)(t) \to 0$
(4) $\dot{\phi}(t)$, $\ddot{\phi}(t) \to 0$

PROOF. (i) By lemma 5.1 H is monotonically decreasing and bounded from below by $H_0 > 0$, so convergence follows immediately. (ii) Using the decay of the matter terms and of $\dot{\phi}$ from proposition 5.2 as well as the convergence of H just obtained, the Hamiltonian constraint (1.30) yields $2(U \circ \phi)(t) \to n(n-1)H_\infty^2$ $(t \to \infty)$. (iii) Let $\phi_- := \liminf_{t\to\infty} \phi(t)$ and $\phi_+ := \limsup_{t\to\infty} \phi(t)$ be the limes inferior and limes superior of ϕ in $\bar{\mathbb{R}}$ respectively. If $\phi_- = \phi_+$, then $\phi(t)$ converges for $t \to \infty$ and the limit lies in the closure of J in $\bar{\mathbb{R}}$, so $(U' \circ \phi)(t)$ converges in $\bar{\mathbb{R}}$ by assumption (U3). Thus, suppose $\phi_- < \phi_+$ and choose ϕ_0 in the open interval $]\phi_-, \phi_+[$. Then, there exists a sequence (s_n) in I with $s_n \to \infty$ and $\phi(s_n) \to \phi_0$ for $n \to \infty$ because ϕ is continuous. But then (ii) implies

$$U_\infty = \lim_{t\to\infty} (U \circ \phi)(t) = \lim_{n\to\infty} U(\phi(s_n)) = U(\phi_0),$$

so U is constant on $]\phi_-, \phi_+[$ and thus U' vanishes on $]\phi_-, \phi_+[$. This means that for every neighborhood N of zero, $G := (U')^{-1}(N)$ is a neighborhood of the closure of $]\phi_-, \phi_+[$ in $\mathbb{R}$. By construction, it follows that $\phi(t) \in G$ for t sufficiently large and therefore $(U' \circ \phi)(t) \in N$ eventually. This shows the convergence of $(U' \circ \phi)(t)$ in $\bar{\mathbb{R}}$ for $t \to \infty$ in any case. Employing the scalar field equation (1.34) and using proposition 5.2 it follows that $-\ddot{\phi}$ converges to the same limit, so boundedness of $\dot{\phi}$ requires that limit to vanish. □

COROLLARY 5.4. *Consider a solution of Bianchi type I–VIII of the Einstein equations together with a non-linear scalar field evolving in a positive potential and coupled directly to ordinary matter satisfying the dominant and strong energy condition. Suppose the potential to possess a positive lower bound and the assumptions (U1)–(U3) to be fulfilled. If the solution is expanding initially and exists globally in the future then exponential acceleration and isotropization occur asymptotically.*

PROOF. This follows directly from proposition 5.2 and 5.3. □

The results of this section show in particular that if the potential is bounded below away from zero, the deceleration parameter $q := -1 - \dot{H}/H^2$ approaches -1 at late times and so the expansion of the solution is accelerated exponentially. Moreover, the density of ordinary matter $C(\phi)\rho_\mathrm{m}$ as well as any anisotropy $|\sigma|/H$ vanish exponentially fast.

2. Intermediate acceleration

In the case in which the potential energy density of the scalar field $U(\phi)$ can become arbitrarily small, $\inf\,(U \circ \phi)(I) = U_0 = 0$, the dynamics is more subtle. Nevertheless, for potentials falling off not too steeply, it is possible to prove late-time acceleration as well as decay estimates for the curvature and matter terms. In contrast to the findings in the case of positive $U_0 > 0$, the acceleration is in general no longer exponential but may be asymptotically power-law. This will be referred to as *intermediate acceleration*. Of course, the solution again is supposed to exist globally in forward time direction.

In general without having a positive lower bound U_0 on the field's potential energy density, proposition 5.2 is weakened to the following proposition 5.5.

PROPOSITION 5.5. *The spatial curvature R, the shear $|\sigma|^2$, the matter terms $C(\phi)\rho_\mathrm{m}$, $C(\phi)|j_\mathrm{m}|$ and $C(\phi)\operatorname{tr} S_\mathrm{m}$ as well as the coupling term $c(\phi)\operatorname{tr}_{\tilde{g}} T_\mathrm{m}$ decay at least as t^{-2} so*

$$R,\ |\sigma|^2,\ C(\phi)\rho_\mathrm{m},\ C(\phi)|j_\mathrm{m}|,\ C(\phi)\operatorname{tr} S_\mathrm{m},\ c(\phi)\operatorname{tr}_{\tilde{g}} T_\mathrm{m} = O\big(t^{-2}\big) \quad (t \to \infty).$$

PROOF. Using the same quantity $Z = n(n-1)H^2 - \dot{\phi}^2 - 2U(\phi)$ as in the proof of proposition 5.2 it follows by lemma 5.1 that $\dot{\phi}(t) \to 0$ as $t \to \infty$ and so both $Z \geq 0$ and $\dot{Z} \leq -HZ$ eventually are valid. The Hamiltonian constraint (1.30) in turn gives $Z \leq n(n-1)H^2$, so

$$\dot{Z} \leq -\frac{1}{\sqrt{n(n-1)}} Z^{\frac{3}{2}}$$

holds eventually. Integrating this yields

$$Z(t) = O\big(t^{-2}\big) \qquad (t \to \infty),$$

which establishes the claimed decay by (DEC) and (C). □

Since $U(\phi)$ is positive and continuous on I, the condition $U_0 = 0$ is equivalent to $\liminf_{t\to\infty}(U \circ \phi)(t) = 0$. Monotonicity of H and the constraint equation (1.30) then give $\lim_{t\to\infty} H(t) = 0$ and, together with the proposition 5.5 just proven, that actually $\lim_{t\to\infty}(U \circ \phi)(t) = 0$. By assumption (U1) it follows that J is unbounded and either $\lim_{t\to\infty} \phi(t) = -\infty$ or $\lim_{t\to\infty} \phi(t) = \infty$. Without loss of generality the latter will be assumed for the rest of this section.

Restricting the asymptotic steepness of the potential

$$\varkappa := \limsup_{x\to\infty} \frac{|U'|}{U}(x)$$

will allow more detailed asymptotics to be obtained and late-time accelerated expansion to be shown. For this, consider the quantity

$$Y := \frac{n(n-1)H^2}{2U(\phi)}.$$

According to the equations (1.30) and (1.32) it fulfills $Y \geq 1$,

$$\frac{\dot{\phi}^2}{H^2} = n(n-1)\Big(1 - \frac{1}{Y}\Big) - \frac{|\sigma|^2}{H^2} + \frac{R}{H^2} - \frac{2C(\phi)\rho_\mathrm{m}}{H^2} \tag{5.1}$$

and its time derivative is given by

$$
\begin{aligned}
\dot{Y} = -\frac{2}{n-1}\Bigg[&\frac{\dot{\phi}^2}{H} + \frac{1}{2}(n-1)\frac{U'}{U}(\phi)\dot{\phi} \\
&+ \frac{|\sigma|^2}{H} - \frac{1}{n}\frac{R}{H} + \frac{1}{n}\frac{C(\phi)}{H}(n\rho_{\mathrm{m}} + \operatorname{tr} S_{\mathrm{m}})\Bigg] Y
\end{aligned} \tag{5.2}
$$

respectively. Also note that with the elementary inequality $\sqrt{a-b} \geq \sqrt{a} - b/\sqrt{a}$ for all $a > b \geq 0$ the relation

$$
\begin{aligned}
-\frac{\dot{\phi}^2}{H} - \frac{|\sigma|^2}{H} + \frac{1}{n}\frac{R}{H} - \frac{1}{n}\frac{C(\phi)}{H}(n\rho_{\mathrm{m}} + \operatorname{tr} S_{\mathrm{m}}) &\leq -\sqrt{n(n-1)}\Big(1 - \frac{1}{Y}\Big)|\dot{\phi}| \\
- \Bigg(1 - \frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H}\Bigg)\frac{|\sigma|^2}{H} &+ \Bigg(\frac{1}{n} - \frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H}\Bigg)\frac{R}{H} \\
&- \Bigg(\frac{1}{2}\frac{n-1}{n} - \frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H}\Bigg)\frac{2C(\phi)}{H}
\end{aligned} \tag{5.3}
$$

follows from (5.1).

Proposition 5.6. *If $\varkappa < 2/\sqrt{n(n-1)}$ then*

$$
\limsup_{t\to\infty} Y(t) \leq \Bigg(1 - \frac{1}{2}\sqrt{\frac{n-1}{n}}\varkappa\Bigg)^{-1}.
$$

Proof. Fix $\epsilon > 0$ with $\varkappa + 2\epsilon < 2/\sqrt{n(n-1)}$, then there is a $T \in I$ with $(|U'|/U)(\phi(t)) \leq \varkappa + \epsilon$ for all $t \geq T$. Define

$$
1 < \delta := \Bigg(1 - \sqrt{\frac{n-1}{n}}\frac{\varkappa + 2\epsilon}{2}\Bigg)^{-1} < \frac{n}{n-1}
$$

and let $T \leq L \subset I$ be any subinterval of I bounded below by T on which Y is not less than δ, $Y|L \geq \delta$. For an arbitrary $t \in L$ distinguish two cases. First, assume that $(|\dot{\phi}|/H)(t) \geq \sqrt{(n-1)/n}$, then (5.2) directly gives

$$
\dot{Y}(t) \leq \Bigg(-\frac{2}{\sqrt{n(n-1)}} + \varkappa + \epsilon\Bigg)|\dot{\phi}(t)|Y(t) \leq -\epsilon|\dot{\phi}(t)|Y(t).
$$

Second, assume $(|\dot{\phi}|/H)(t) \leq \sqrt{(n-1)/n}$ instead, so

$$
\frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H} \leq \frac{1}{n} \leq \frac{1}{2}\frac{n-1}{n} \leq 1,
$$

then (5.3) yields

$$
\dot{Y}(t) \leq \Bigg[-2\sqrt{\frac{n}{n-1}}\Big(1 - \frac{1}{\delta}\Big) + \varkappa + \epsilon\Bigg]|\dot{\phi}(t)|Y(t) \leq -\epsilon|\dot{\phi}(t)|Y(t).
$$

This, together, shows that on L the inequality $\dot{Y} \leq -\epsilon Y|\dot{\phi}|$ holds, so L is bounded because $\dot{\phi}$ is non-integrable. Since Y is in particular monotonically decreasing on any such L the set where Y is bigger than δ is itself bounded, but this means that $\limsup_{t\to\infty} Y(t) \leq \delta$. □

Combining the evolution and constraint equations (1.30), (1.32) with the relation (5.1) results in an upper bound on the deceleration parameter q, namely

$$
q \leq n\Big(1 - \frac{1}{Y}\Big) - 1, \tag{5.4}
$$

leading to the following sufficient condition for accelerated expansion.

PROPOSITION 5.7. *If $\varkappa < 2/\sqrt{n(n-1)}$ then accelerated expansion occurs eventually.*

PROOF. From proposition 5.6 it is known that

$$\limsup_{t\to\infty} Y(t) \le \Big(1 - \frac{1}{2}\sqrt{\frac{n-1}{n}}\varkappa\Big)^{-1} < \frac{n}{n-1},$$

so $Y < n/(n-1)$ eventually. By (5.4) this means that $q < 0$ eventually. □

Isotropization can be proven by requiring the potential to be "flat" at infinity, which says that $\varkappa = 0$.

PROPOSITION 5.8. *If the potential is flat at infinity, $\lim_{x\to\infty}(U'/U)(x) = 0$, then the curvature and matter terms vanish faster than H^2, more precisely, as $t \to \infty$,*

$$\frac{\dot\phi}{H}(t), \frac{|\sigma|^2}{H^2}(t), \frac{R}{H^2}(t) \to 0$$

as well as

$$\frac{C(\phi)\rho_{\mathrm{m}}}{H^2}(t), \frac{C(\phi)|j_{\mathrm{m}}|}{H^2}(t), \frac{C(\phi)\operatorname{tr} S_{\mathrm{m}}}{H^2}(t), \frac{c(\phi)\operatorname{tr}_{\tilde g} T_{\mathrm{m}}}{H^2}(t) \to 0.$$

PROOF. When $\varkappa = 0$, proposition 5.6 implies $Y(t) \to 1$ as $t \to \infty$ and equation (5.1) together with (DEC) and (C) makes the claims evident immediately. □

COROLLARY 5.9. *Consider the same situation as in corollary 5.4 above but suppose instead of a positive lower bound that the the asymptotic steepness $\varkappa$ of the potential is less than $\sqrt{2/3}$. If the solution is expanding initially and exists globally in the future then accelerated expansion takes place eventually. Moreover if the potential is actually flat at infinity, $\varkappa = 0$, the model isotropizes as well.*

PROOF. This follows from propositions 5.7 and 5.8 with $n = 3$. □

A question that arises in the treatment outlined so far is whether it is possible to decide *a priori*, without referring to the actual solution, to which of the two cases ($U_0 > 0$ or $U_0 = 0$) the evolution belongs. As an answer to this question in general not only depends on the form of the potential U but also on the initial data, no simple necessary and sufficient criterion can be expected. Instead, two rather rough conditions leading to each of the cases will be presented, that might be useful in some situations though. They can be employed to make contact with the findings of [**26**].

Take a threshold value $B > 0$ and consider a component G of the set $\{x \in J \mid U(x) \le B\}$. By monotonicity of H it is clear that if $H^2(t_0) \le 2B/n(n-1)$ and ϕ belongs to G initially, $\phi(t_0) \in G$, then ϕ stays in G, $\phi(I) \subset G$. Now the conditions can be stated as

LEMMA 5.10. *If G is bounded then $U_0 > 0$.*

PROOF. With G its closure $\bar G$ in $\mathbb{R}$ is bounded too and thus compact. Because of $U(\phi) \ge \bar U(\phi)$ and $(\bar U \circ \phi)(I) \subset \bar U(\bar G)$ it follows that $U_0 = \inf(U \circ \phi)(I) \ge \inf(\bar U \circ \phi)(I) \ge \inf \bar U(\bar G) > 0$ from (U1). □

LEMMA 5.11. *If $U' < 0$ on G and G is bounded below within J, then*

$$\lim_{t\to\infty} \phi(t) = J_+.$$

In particular, if the potential has negative derivative everywhere and $U(x)$ decays as $x \to \infty$ then $U_0 = 0$.

PROOF. Let $G_\pm$ denote the endpoints of G in $\bar{\mathbb{R}}$. By assumption, G_- is an element of J and thus of G itself. It will now be shown that actually $G_+ = J_+$. Assume that this is not the case, so $G_+ < J_+$, then G_+ lies in the interior of J and $U' < 0$ on G implies that G can be extended in J beyond G_+ while remaining a connected subset of $\{x \in J \mid U(x) \leq B\}$. But this is a contradiction to G being a component of that set. So indeed $G_+ = J_+$. If $U_0 > 0$ proposition 5.3 gives $(U' \circ \phi)(t) \to 0$ for $t \to \infty$ and therefore $\phi(t) \to J_+$ $(t \to \infty)$. If $U_0 = 0$ nothing is left to be shown. The additional statement is immediate. □

3. Curvature-assisted acceleration

The results of the previous sections shall now be applied to the example of a scalar field with an explicit coupling to the scalar curvature of space-time to demonstrate the mechanism of curvature-assisted acceleration. For simplicity an exponential potential is assumed although the method is not restricted to that case by any means. By a conformal transformation the direct coupling of the field to the scalar curvature is first moved to the energy-momentum tensor where results are obtained by virtue of the propositions proved so far. Transforming back to the Jordan frame will then yield the desired statements.

On an interval $\acute{I} := [\acute{t}_0, \infty[$ suppose functions $\acute{\phi} \in C^2(\acute{I})$, $\acute{\rho}_{\mathrm{m}} \in C^1(\acute{I})$, $\acute{\jmath}_{\mathrm{m}} \in C^1(I, T^0_1\mathfrak{g})$ and families $\acute{g} \in C^2(\acute{I}, T^0_2\mathfrak{g})$ and $\acute{S}_{\mathrm{m}} \in C^1(\acute{I}, T^0_2\mathfrak{g})$ of Riemannian metrics and symmetric tensors on the Lie algebra $\mathfrak{g}$ respectively are given such that

$$\acute{T}_{\mathrm{m}} := \acute{\pi}^* \acute{S}_{\mathrm{m}} - \acute{\pi}^* \acute{\jmath}_{\mathrm{m}} \otimes d\acute{t} - d\acute{t} \otimes \acute{\pi}^* \acute{\jmath}_{\mathrm{m}} + \acute{\rho}_{\mathrm{m}}\, d\acute{t} \otimes d\acute{t}$$

satisfies the dominant and strong energy conditions (DEC) and (SEC) while

$$\tilde{g} := \acute{\pi}^* \acute{g} - d\acute{t} \otimes d\acute{t}$$

and

$$\tilde{\phi} := \acute{\pi}^* \acute{\phi}$$

are a solution to the curvature-coupled Einstein-scalar field-matter equations on $\acute{M} := G \times \acute{I}$ with potential $\acute{U}$ of class C^1. Like before, $\acute{\pi} : \acute{M} \to G$ and $\acute{t} : \acute{M} \to \acute{I}$ are the canonical projections, compositions with which are understood implicitly when appropriate. The coupling constant ξ is taken to be any non-zero real number.

Assume $1 - \xi \acute{\phi}^2 > 0$ and define according to chapter 1 a conformal factor

$$\Omega := \sqrt[n-1]{1 - \xi \acute{\phi}^2} \in C^2(\acute{I}).$$

The transformation functions

$$p : \acute{I} \to I, \quad \acute{t} \mapsto \int_{\acute{t}_0}^{\acute{t}} \Omega$$

$$\Phi : \acute{J} \to \mathbb{R}, \quad \acute{x} \mapsto \int_0^{\acute{x}} \frac{\sqrt{1 - \eta \xi x^2}}{1 - \xi x^2} dx$$

are smooth diffeomorphisms from $\acute{I}$ onto $I := p(\acute{I})$ and from

$$\acute{J} := \begin{cases}]-1/\sqrt{\xi}, 1/\sqrt{\xi}[& \xi > 0 \\ \mathbb{R} & \xi < 0 \end{cases}$$

onto $\mathbb{R}$ whose inverses will be denoted by p^{-1} and Ψ respectively. Then the quantities

$$g := (\Omega^2 \acute{g}) \circ p^{-1}, \qquad \phi := \Psi(\acute{\phi}) \circ p^{-1}$$

$$\rho_{\mathrm{m}} := (\Omega^{-2} \acute{\rho}_{\mathrm{m}}) \circ p^{-1}, \qquad j_{\mathrm{m}} := (\Omega^{-1} \acute{\jmath}_{\mathrm{m}}) \circ p^{-1}, \qquad S_{\mathrm{m}} := \acute{S}_{\mathrm{m}} \circ p^{-1}$$

define a solution of the matter-coupled Einstein-scalar field-matter equations (1.26)–(1.28) on $\tilde{M} = G \times I$ with the smooth coupling functions

$$C = \frac{1}{1 - \xi\Psi^2}, \qquad c = \frac{1}{n-1}C' = \frac{1}{n-1}\frac{2\xi\Psi}{\sqrt{1-\eta\xi\Psi^2}}C$$

and the transformed potential $U = \acute{U}(\Psi)C^{\frac{n+1}{n-1}}$. Apart from a conformal change of the metric and a redefinition of the scalar field a change of the time coordinate is employed to keep it Gaussian. It is easily seen that (SEC) and (DEC) hold for T_{m} as well and the condition (C) is fulfilled due to the boundedness of $2\xi\Psi/\sqrt{1-\eta\xi\Psi^2}$ on $\mathbb{R}$.

Attention shall now be restricted to positive coupling constants $\xi > 0$ and fields evolving in exponential potentials $\acute{U}(x) = \lambda e^{-\kappa x}$ for positive $\kappa, \lambda > 0$ and all $x \in \mathbb{R}$. Suppose $\acute{H}(\acute{t}_0) > -(\log\Omega)\dot{}(\acute{t}_0)$ and define

$$\acute{\phi}_\infty := \sqrt{\frac{1}{\xi} + \frac{1}{\kappa^2}\Big(\frac{n+1}{n-1}\Big)^2} - \frac{1}{\kappa}\frac{n+1}{n-1}$$

then $\phi_\infty := \Phi(\acute{\phi}_\infty)$ is the critical point of the transformed potential $U = \acute{U}(\Psi)C^{\frac{n+1}{n-1}}$. The following proposition 5.12 holds true.

Proposition 5.12. *In the limit $\acute{t} \to \infty$, the field $\acute{\phi}$ and its derivatives converge,*

$$\acute{\phi}(\acute{t}) \to \acute{\phi}_\infty, \quad \dot{\acute{\phi}}(\acute{t}) \to 0, \quad \ddot{\acute{\phi}}(\acute{t}) \to 0,$$

so does the mean curvature $-\acute{H}$,

$$\acute{H}(\acute{t}) \to \acute{H}_\infty := \sqrt{\frac{2}{n(n-1)}\frac{\acute{U}(\acute{\phi}_\infty)}{1-\xi\acute{\phi}_\infty^2}}$$

and the curvature and matter terms vanish exponentially,

$$R_{\acute{g}},\ |\acute{\sigma}|^2_{\acute{g}},\ \acute{\rho}_{\mathrm{m}},\ |\acute{j}_{\mathrm{m}}|_{\acute{g}},\ \mathrm{tr}_{\acute{g}}\,\acute{S}_{\mathrm{m}} = O\big(e^{-(2-\delta)\acute{H}_\infty\acute{t}}\big)$$

for any $\delta > 0$. Moreover, asymptotically, the expansion is accelerated exponentially

$$\acute{q} = -1 - \frac{\dot{\acute{H}}}{\acute{H}^2} \to -1.$$

Proof. (i) The assumption $\acute{H}(\acute{t}_0) > -(\log\Omega)\dot{}(\acute{t}_0)$ is equivalent to $H(0) > 0$ while $0 = \min I$. Furthermore, the transformed potential U clearly fulfills the requirements (U1), (U2) and (U3) and $U(x)$ goes to infinity when $|x|$ does. Hence, it possesses a positive lower bound and lemma 5.10 ensures the image $\phi(I)$ being relatively compact which implies that Ω is bounded away from zero. This renders $\Omega \notin L^1(\acute{I})$ non-integrable and so the transformed evolution exists globally too, $I = [0, \infty[$. The results of section 1 thus apply. (ii) From proposition 5.3 convergence of $\phi(t)$ to ϕ_∞ and therefore of $\acute{\phi}(\acute{t})$ to $\acute{\phi}_\infty$ follows. The decay of $\dot{\phi}$ and $\ddot{\phi}$ give those for the derivatives of $\acute{\phi}$ as well as convergence of $\acute{H}$ and $\acute{q}$. The exponential decay of the curvature and matter terms

$$|\acute{\sigma}|^2_{\acute{g}} = \Omega^6(|\sigma|^2 \circ p) \qquad R_{\acute{g}} = \Omega^2(R \circ p)$$
$$\acute{\rho}_{\mathrm{m}} = \Omega^{n+1}[C(\phi)\rho_{\mathrm{m}} \circ p] \qquad |\acute{j}_{\mathrm{m}}|_{\acute{g}} = \Omega^{n+1}[C(\phi)|j_{\mathrm{m}}| \circ p]$$
$$\mathrm{tr}_{\acute{g}}\,\acute{S}_{\mathrm{m}} = \Omega^{n+1}[C(\phi)(\mathrm{tr}\,S_{\mathrm{m}}) \circ p]$$

is then directly obtained from proposition 5.2. □

Corollary 5.13. *Consider a solution of Bianchi type I–VIII of the Einstein equations together with a non-linear scalar field coupled directly to the scalar curvature of space-time and evolving in an arbitrary exponential potential in the presence of ordinary matter satisfying the dominant and strong energy condition. Suppose that for a positive coupling constant* ξ *the conditions* $1 - \xi\phi^2 > 0$ *and* $H > -[\log(1-\xi\phi^2)]^{\cdot}/(n-1)$ *on the field* ϕ *and the expansion factor* H *are fulfilled initially. If the solution exists globally in the future then* $1 - \xi\phi^2 > 0$ *holds at any time and exponential acceleration and isotropization take place asymptotically.*

This result shows that an arbitrarily small positive coupling constant ξ establishes a dynamics similar to the presence of a cosmological constant with value

$$\Lambda_{\mathrm{dyn}} = \frac{\acute{U}(\acute{\phi}_\infty)}{1 - \xi\acute{\phi}_\infty^2}$$

although the potential lacks a positive lower bound. It causes exponential acceleration to occur asymptotically as well as exponentially fast isotropization and decay of matter independent of the steepness κ of the potential.

In the proof given above the particular form of the potential $\acute{U}$ is used mainly to ensure the existence of exactly one critical point of U and thus to obtain convergence of the field $\acute{\phi}$ and the conformal factor Ω at late times. For assumptions (U1), (U2) and (U3) to hold it is sufficient for instance to require the potential only to allow for a C^1-extension to the closure of $\acute{J}$ in $\mathbb{R}$. Invoking lemma 5.10 it can still be concluded that the field $\acute{\phi}$ cannot approach the boundary of $\acute{J}$ and hence the presumption $1 - \xi\acute{\phi}^2 > 0$ does not restrict the dynamics even in this more general situation once it is fulfilled initially.

Concluding remarks

In this section some possible extensions of the presented work are sketched. For the construction of formal solutions in chapter 2 asymptotic initial data could be prescribed that consists of a Riemannian metric g_0, two scalar fields ϕ_0 and ϕ_1 and a symmetric 2-tensor h subject to the asymptotic constraints (AC1) and (AC2) stated in theorem 2.11. While the asymptotic constraints look similar to the momentum constraint of the ordinary Cauchy problem there is no analogue of the much harder Hamiltonian constraint. One can thus try to solve (AC1) and (AC2) using the conformal method for the constraints developed by Choquet-Bruhat and York [**7**]. Following Isenberg [**13**] consider the conformal Killing operator

$$\mathcal{D}X := \mathcal{L}_X\, g_0 - \frac{2}{n}\operatorname{div}_{g_0} X$$

on vector fields X on (M, g_0). It has the formal adjoint $\mathcal{D}^* = -\operatorname{div}_{g_0}$. In the absence of conformal Killing vectors, that is if the kernel of $\mathcal{D}$ is trivial, the linear elliptic system

$$\mathcal{D}^*\mathcal{D}X = -\mathsf{Z} + \frac{1}{n}d\mathsf{z}$$

is expected to be solvable for any inhomogeneity on the right hand side. But then evidently

$$h := \mathcal{D}X + \frac{\mathsf{z}}{n}g_0$$

is a particular solution of the asymptotic constraints (AC1) and (AC2) and the general solution is obtained by adding arbitrary symmetric transverse-traceless 2-tensors.

When a space-time metric $\tilde{g}$ as considered in chapter 3 is expressed as its asymptotic series and in terms of the time coordinate $\tau = e^{-Ht}$ it fulfills

$$\tau^2\tilde{g} = g_0 + \sum_{m\geq k_0}\sum_{s,l}\pi^*(g)_{m-2,s,l}(-H)^{-l}(\log\tau)^l\tau^{m+is} - \frac{1}{H^2}d\tau\otimes d\tau$$

and is thus conformal to a Lorentzian metric that extends continuously to $\tau = 0$ and is non-degenerate in the limit. Due to the presence of logarithmic and oscillatory terms as well as the non-integer powers of τ even in the case $n = 3$ the extension will, however, in general not be smooth. This shows that the existence of a smooth conformal boundary necessary for the applicability of Friedrichs conformal method [**10**] is rather special already within the class of scalar field models taken into account here.

For the conformal transformation between the curvature-coupled and matter-coupled Einstein-scalar field system to work it was assumed that the field fulfills the inequality $1-\xi\tilde{\phi}^2 > 0$ which is non-trivial for $\xi > 0$. One can ask what happens if $1-\xi\tilde{\phi}^2 < 0$. Evidently, it is possible to choose

$$\Omega = \sqrt[n-1]{\xi\tilde{\phi}^2 - 1}$$

as a conformal factor. But unlike before $1 - \eta\xi\tilde{\phi}^2 \neq 0$ is no longer guaranteed automatically and a sign has to be assumed in order for the field rescaling

$$d\bar{\phi} = \frac{\sqrt{|1 - \eta\xi\tilde{\phi}^2|}}{\xi\tilde{\phi}^2 - 1} d\tilde{\phi}$$

to remain a smooth diffeomorphism $\Phi = \Psi^{-1}$. Hence, suppose $1 - \eta\xi\tilde{\phi}^2 > 0$, then the transformed solution $(\bar{g}, \bar{\phi})$ satisfies

$$\begin{aligned} &\mathrm{Ric}_{\bar{g}} - \frac{1}{2} R_{\bar{g}} \bar{g} = d\bar{\phi} \otimes d\bar{\phi} - \frac{1}{2} |d\bar{\phi}|^2_{\bar{g}} \bar{g} + U(\bar{\phi}) \bar{g} - C(\bar{\phi}) T_{\mathrm{m}} \\ &\Box_{\bar{g}} \bar{\phi} + U'(\bar{\phi}) = c(\bar{\phi}) \operatorname{tr}_{\bar{g}} T_{\mathrm{m}} \end{aligned}$$

with $U = V(\Psi^2)/(\xi\Psi^2 - 1)^{\frac{n+1}{n-1}}$, $C = (\xi\Psi^2 - 1)^{-1}$ and $c = C'/(n-1)$. It means that the potential U and the energy-momentum tensor for the matter have switched their signs. On the other hand, if $1 - \eta\xi\tilde{\phi}^2 < 0$, the transformed equations read

$$\begin{aligned} &\mathrm{Ric}_{\bar{g}} - \frac{1}{2} R_{\bar{g}} \bar{g} = -\Big[d\bar{\phi} \otimes d\bar{\phi} - \frac{1}{2} |d\bar{\phi}|^2_{\bar{g}} \bar{g} - U(\bar{\phi}) \bar{g} \Big] - C(\bar{\phi}) T_{\mathrm{m}} \\ &\Box_{\bar{g}} \bar{\phi} - U'(\bar{\phi}) = -c(\bar{\phi}) \operatorname{tr}_{\bar{g}} T_{\mathrm{m}}. \end{aligned}$$

This can happen only if the coupling constant ξ lies properly between the values for minimal and conformal coupling respectively. In that case the right hand side of Einstein's equations acquires an overall negative sign. In both cases it is doubtful whether such equations constitute a reasonable theory. A more elaborate discussion of this issue can be found in Abramo, Brenig and Gunzig [**1**].

As mentioned in chapter 4 it would be desirable to relax the analyticity assumption on the prescribed asymptotic data. But even if this could be done the method presented in this work is not directly suited to obtain a stability statement similar to that of Ringström [**27**]. On the other hand the existence of asymptotic expansions only is already quite useful in situations where the perturbation series do not actually converge.

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