Chapter 3

Examples

3.1 Orlicz sequence spaces

As it has been shown in Section 1.4 the spaces l_p $(1 \leq p < \infty,\ p \neq 2)$ and c_0 have the Lyapunov property. The next class of Banach spaces for which it is natural to consider this property is the class of Orlicz sequence spaces. The introduction of Orlicz functions has been inspired by the obvious role played by the functions t^p in the definition of the space l_p. It is quite natural to try to replace t^p by a more general function M and then to consider the set of scalars $\{a_n\}_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} M(|a_n|)$ converges. W. Orlicz [27] has checked the restrictions which have to be imposed on the function M in order to make this set of sequences into a suitable Banach space. His study led to the following definition.

Definition 3.1.1 *An Orlicz function M is a continuous, nondecreasing, and convex function defined for $t \geq 0$, with $M(0) = 0$.*

We shall consider vector - valued Orlicz sequence spaces. With any Orlicz function M and sequence $\{X_n\}$ of Banach spaces we associate the space $(X_1 \oplus X_2 \oplus ...)_M$ of all sequences $x = (x_1, x_2, ...)$, where $x_i \in X_i \quad (i = 1, 2, ...)$, such that $\sum\limits_{n=1}^{\infty} M\left(\frac{\|x_n\|}{\rho}\right) < \infty$ for some $\rho > 0$. The space $(X_1 \oplus X_2 \oplus ...)_M$ equipped with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{\|x_n\|}{\rho}\right) \leq 1\right\}$$

is a Banach space. If $X_n = \mathbf{R}$ $(n = 1, 2, ...)$ we get the Orlicz sequence space l_M.

We will show that all Orlicz sequence spaces containing no isomorphic copies of l_2 have the Lyapunov property. For this aim we recall some definitions and statements (that can be found in [24]) concerning them.

Proposition 3.1.2 ([24], p. 139) *Let M_1 and M_2 be Orlicz functions. Then the following assertions are equivalent:*

(i) $l_{M_1} = l_{M_2}$, i.e. both spaces consist of the same sequences and the identity mapping is an isomorphism between l_{M_1} and l_{M_2}.

(ii) M_1 and M_2 are equivalent at zero, i.e. there exist constants $k > 0$, $K > 0$, and $t_0 > 0$ such that for all $0 < t < t_0$, we have $K^{-1}M_2(k^{-1}t) \leq M_1(t) \leq KM_2(kt)$.

Anyway, without loss of generality we may assume that $M(1) = 1$.

We shall be interested in Orlicz spaces not containing isomorphic copies of l_2 space because l_2 is not a Lyapunov space.

Definition 3.1.3 *An Orlicz function M is said to satisfy the Δ_2-condition at zero if M is nondegenerate and $\sup\limits_{0<t\leq 1} \frac{M(2t)}{M(t)} < \infty$.*

Proposition 3.1.4 ([24], p. 138) *For an Orlicz function M the following conditions are equivalent:*

(i) M satisfies the Δ_2-condition at 0;

(ii) l_M contains no subspace isomorphic to l_∞.

In the sequel we suppose M to satisfy the Δ_2-condition.

Remark 3.1.5 *If M is an Orlicz function, $x \in (X_1 \oplus X_2 \oplus ...)_M$, and $\varepsilon > 0$, then evidently by the definition of the norm $\|x\| > \varepsilon$ if and only if $\sum\limits_{m=1}^{\infty} M\left(\frac{\|x_m\|}{\varepsilon}\right) > 1$.*

Proposition 3.1.6 ([24], p. 143) *The space l_p or c_0 if $p=\infty$ is isomorphic to a subspace of an Orlicz sequence space l_M if and only if $p\in[\alpha_M,\beta_M]$, where*

$$\begin{aligned}\alpha_M &= \sup\left\{q:\sup_{0<\lambda,\, t\leq 1}\tfrac{M(\lambda t)}{M(\lambda)t^q}<\infty\right\},\\ \beta_M &= \inf\left\{p:\inf_{0<\lambda,\, t\leq 1}\tfrac{M(\lambda t)}{M(\lambda)t^p}>0\right\}\end{aligned}$$

are the Boyd indices.

A detailed exposition of the basic properties of Orlicz sequence spaces is given in [24]. Now we prove some lemmas. The following lemma is a strengthening of Lemma 1.5.2.

Lemma 3.1.7 *Under the conditions of Lemma 1.5.2 for every $A\in\Sigma$, $\lambda(A)\neq 0$ there exist $G_n^{'}\in\Sigma|_A$, $G_n^{''}=A\backslash G_n^{'}$ $(n=1,2,...)$ such that*

(i) $\lambda\left(G_n^{'}\right)=\lambda\left(G_n^{''}\right)=\frac{1}{2}\lambda(A)$;

(ii) r_n=$\chi_{G_n^{'}}$-$\chi_{G_n^{''}}$ *are independent random variables on the measure space* $\left(A,\Sigma|_A,\frac{1}{\lambda(A)}\lambda(\cdot)\right)$;

(iii) $\left\|\mu\left(G_n^{'}\right)-\frac{1}{2}\mu(A)\right\|\leq\frac{1}{2^n}$.

Proof. We shall construct the sets $G_n^{'},G_n^{''}$ $(n=1,2,...)$ by induction on n. Let $n=1$. By Lemma 1.5.2 there is a $G_1^{'}\in\Sigma|_A$ such that

$$\left\|\mu\left(G_1^{'}\right)-\frac{1}{2}\mu(A)\right\|\leq\frac{1}{2}\text{ and }\lambda\left(G_1^{'}\right)=\frac{1}{2}\lambda(A). \tag{3.1}$$

Let $n=k+1$. Suppose $G_j^{'},G_j^{''}$ $(j=1,...,k)$, satisfying the required conditions, have been constructed. By independence of $\{r_j\}_{j=1}^k$ there are mutually disjoint sets $\{D_i\}_{i=1}^{2^k}\subset\Sigma|_A$ such that

$$\begin{aligned}r_1 &= \sum_{i=1}^{2^{k-1}}\chi_{D_i}-\sum_{i=2^{k-1}+1}^{2^k}\chi_{Di};\\ r_2 &= \sum_{i=1}^{2^{k-2}}\chi_{D_i}-\sum_{i=2^{k-2}+1}^{2^{k-1}}\chi_{D_i}+\sum_{i=2^{k-1}+1}^{3\cdot 2^{k-2}}\chi_{D_i}-\sum_{i=3\cdot 2^{k-2}+1}^{2^k}\chi_{D_i};\\ \dots &\ \dots\ \dots\dots\dots\dots\dots\dots\dots\dots\\ r_k &= \sum_{i=1}^{2^k}(-1)^{i+1}\chi_{D_i}.\end{aligned}$$

By Lemma 1.5.2, there exists $D_i^{'} \in \Sigma|_{D_i}$ such that

$$\left\| \mu\left(D_i^{'}\right) - \frac{1}{2}\mu\left(D_i\right) \right\| \leq \frac{1}{4^{k+1}} \text{ and } \lambda\left(D_i^{'}\right) = \frac{1}{2}\lambda\left(D_i\right) \quad \left(k = 1, ..., 2^k\right).$$

Put $G_{k+1}^{'} = \bigcup_{i=1}^{2^k} D_i^{'}$. It is easy to verify that $G_{k+1}^{'}, G_{k+1}^{''}$ satisfy conditions (i)-(iii). ■

Lemma 3.1.8 *Let X be a Banach space, $r > 0$, $F : X \to \mathbf{R}$ be a convex function, which is bounded in the ball $B(0,2r) = \{x \in X : \|x\| \leq 2r\}$. Then F is uniformly continuous in the ball $B(0,r)$ of the space X.*

Proof. Let $|F(x)| \leq C$ for any $x \in X$ with $\|x\| \leq 2r$. Take arbitrary $a, b \in B(0,r)$ with $F(b) \geq F(a)$. Consider the straight line $A = \{\theta a + (1-\theta) b : \theta \in \mathbf{R}\}$ passing through a and b. Choose $e \in A, \|e\| \leq 2r$, so that $\|a - e\| \geq r$ and b is between a and e. Since F is convex, we have

$$\frac{F(b) - F(a)}{\|b - a\|} \leq \frac{F(e) - F(a)}{\|e - a\|} \leq \frac{2C}{r}$$

and F is uniformly continuous. ■

Remark 3.1.9 *In the sequel we shall use Lemma 3.1.8 for the function $F(x) = \sum_{n=1}^{\infty} M(\|x_n\|)$, where $x = (x_1, x_2, ...) \in (X_1 \oplus X_2 \oplus ...)_M$, which is bounded in the ball $B(0,2)$ by the Δ_2-condition and the following lemma.*

Lemma 3.1.10 *Let M be an Orlicz function satisfying the Δ_2-condition, $0 < \delta \leq 1$, $x = (a_1, a_2, ...) \in (X_1 \oplus X_2 \oplus ...)_M$. Then we have the following properties:*

(i) if $\|x\| \leq \delta$, then $\sum_{m=1}^{\infty} M(\|a_m\|) \leq \delta$,

(ii) if $\sum_{m=1}^{\infty} M(\|a_m\|) \leq \delta$, then $\|x\| \leq 2\delta^{\frac{1}{\log_2 C}}$, where C is the constant from the Δ_2-condition.

Proof. (i) Using the definition of an Orlicz function we get

$$\sum_{m=1}^{\infty} M\left(\|a_m\|\right) = \sum_{m=1}^{\infty} M\left(\delta \frac{1}{\delta}\|a_m\|\right) \leq \delta \sum_{m=1}^{\infty} M\left(\frac{\|a_m\|}{\delta}\right) \leq \delta.$$

(ii) By the Δ_2-condition, it follows that there is a constant $C > 0$ such that $M(2t) \leq CM(t)$ for any $t \leq 1$. Define $\alpha = \|x\|_M$. Suppose $\frac{1}{2^n} \leq \alpha \leq \frac{1}{2^{n-1}}$. Let us remark that $1 = \sum\limits_{m=1}^{\infty} M\left(\frac{1}{\alpha}\|a_m\|\right)$, M is a nondecreasing function, and $M(1) = 1$. Hence,

$$1 = \sum_{m=1}^{\infty} M\left(\frac{\|a_m\|}{\alpha}\right) \leq C^n \sum_{m=1}^{\infty} M\left(\|a_m\|\right) \leq C^n \delta.$$

Therefore $1 \leq C^n \delta$, i.e. $n \geq \frac{\log_2 \frac{1}{\delta}}{\log_2 C}$. So we have obtained $\alpha \leq 2\delta^{\frac{1}{\log_2 C}}$. ■

Theorem 3.1.11 *Let M be an Orlicz function satisfying the Δ_2-condition, $2 \notin [\alpha_M, \beta_M]$, $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces such that X_n has the Lyapunov property for all n. Then $(X_1 \oplus X_2 \oplus ...)_M$ has the Lyapunov property.*

Proof. Let us consider two cases.

Case 1: $\beta_M < 2$. First we fix $N \in \mathbf{N}$ and $k = \left[\frac{N}{\ln N}\right]$. We shall prove ad absurdum. Assume that $(X_1 \oplus X_2 \oplus ...)_M$ fails the Lyapunov property. By Lemma 1.4.7 there are $(\Omega, \Sigma, \lambda)$, $\varepsilon > 0$ and $T : L_\infty \to X$ satisfying the conditions (a) and (b) of the condition (ii) of this lemma. Without loss of generality we may assume that $\|T\| \leq \frac{1}{2}$ and $\varepsilon \leq 1$. Denote $Tx = (T_1 x, T_2 x, ...)$. Let us show by induction on j that there exist functions $\{t_i\}_{i=1}^{\infty} \in L_\infty$ such that for every j the functions $\{t_i\}_{i=1}^{j}$ are jointly equidistributed with $\{s_i\}_{i=1}^{j}$ from Lemma 1.4.9 and

$$\begin{aligned} &\sum_{m=1}^{\infty} M\left(\left\|T_m\left(\sum_{i=1}^{j} t_i\right)\right\|\right) > \sum_{i=1}^{j}\sum_{m=1}^{\infty} M\left(\|T_m t_i\|\right) - 1, \\ &\left\|T_m\left(\sum_{i=1}^{j} t_i\right)\right\| < 1 \text{ for any } m = 1, 2, ... \end{aligned} \tag{3.2}$$

For $j = 1$ there is nothing to prove. Suppose that for $j = n$ $\{t_i\}_{i=1}^{n}$ satisfying (3.2) have been constructed. Now consider $A =$

$\left\{\omega \in \Omega : \left|\sum\limits_{i=1}^{n} t_i\right| < \sqrt{N}\right\}$ and put $a_m = T_m\left(\sum\limits_{i=1}^{n} t_i\right)$ $(m = 1, 2, ...)$. Take an arbitrary $\delta > 0$. In accordance with Lemmas 3.1.8, 3.1.10, and Remark 3.1.9 there is $\theta \in (0,1)$ such that

$$\left|\sum_{m=1}^{\infty} M(\|x_m\|) - \sum_{m=1}^{\infty} M(\|y_m\|)\right| \leq \delta \tag{3.3}$$

if $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \leq \theta$. Combining the Δ_2-condition and Lemma 3.1.10, we obtain

$$\sum_{m=N_1+1}^{\infty} M(\|a_m\|) \leq \delta,\ \|a_m\| \leq \delta \quad (m = N_1 + 1, ...), \tag{3.4}$$

$$\|(0, ..., 0, a_{N_1+1}, a_{N_1+2}, ...)\|_M \leq \theta, \tag{3.5}$$

for some $N_1 \in \mathbf{N}$. By Theorem 1.5.6, $(X_1 \oplus X_2 \oplus ... \oplus X_{N_1})_M$ has the Lyapunov property. It is clear that $\mu(B) = (T_1\chi_B, ..., T_{N_1}\chi_B)$ is a nonatomic measure. Hence, by Lemma 3.1.7 there exists a sequence $G_n' \in \Sigma|_A$, $G_n'' = A \backslash G_n'$, $\lambda\left(G_n'\right) = \lambda\left(G_n''\right) = \frac{1}{2}\lambda(A)$ such that the functions $z_n = \chi_{G'} - \chi_{G''}$ are independent on A and $\|(T_1 z_n, ..., T_{N_1} z_n)\|_M \underset{n\to\infty}{\longrightarrow} 0$. By continuity of M there is a $\theta_1 > 0$ such that if $|t|, |u| \leq 1$ and $|t - u| \leq \theta_1$ then $|M(t) - M(u)| \leq \frac{\delta}{N_1}$. Select $n_0 \in \mathbf{N}$ so that

$$\|(T_1 z_{n_0}, ..., T_{N_1} z_{n_0})\|_M \leq \delta, \tag{3.6}$$

$$\|T_i z_{n_0}\| \leq \theta_1,\ i = 1, 2, ..., N_1. \tag{3.7}$$

Combining (3.3)-(3.7) and Lemma 3.1.8, and Remark 3.1.9, we infer

$$\begin{aligned}
&\left|\sum_{m=1}^{\infty} M(\|a_m + T_m z_{n_0}\|) - \sum_{m=1}^{\infty} M(\|a_m\|) - \sum_{m=1}^{\infty} M(\|T_m z_{n_0}\|)\right| \\
&\leq \left|\sum_{m=1}^{N_1} [M(\|a_m + T_m z_{n_0}\|) - M(\|a_m\|)]\right| \\
&+ \sum_{m=1}^{N_1} M(\|T_m z_{n_0}\|) + \sum_{m=N_1+1}^{\infty} M(\|a_m\|) \\
&+ \left|\sum_{m=N_1+1}^{\infty} [M(\|a_m + T_m z_{n_0}\|) - M(\|T_m z_{n_0}\|)]\right| \leq 4\delta.
\end{aligned}$$

It is evident that

$$\|a_m + T_m z_{n_0}\| \leq \left[\begin{array}{lll} \|a_m\| + \theta_1 & \text{if} & m = 1, ..., N_1 \\ \delta + \frac{1}{2} & \text{if} & m = N_1 + 1, ... \end{array}\right.$$

Putting $t_{n+1} = z_{n_0}$ and fitting δ sufficiently small we arrive at (3.2) for $j = n+1$.

Notice that the condition $\beta_M < 2$ means the existence of $1 < p < 2$ and $C > 0$ such that

$$Cx^p M(t) \leq M(tx) \tag{3.8}$$

for any $t,\, x \in (0,1]$.

Now we employ the proved fact and this inequality. Let $\{t_i\}_{i=1}^k$ be jointly equidistributed with $\{s_i\}_{i=1}^k$ and meet requirement (3.2). Introduce the sets of indices

$$\begin{aligned} J_1^i &= \{m \in N : \|T_m t_i\| \leq \varepsilon\lambda(\operatorname{supp} t_k)\}, \\ J_2^i &= \{m \in N : \|T_m t_i\| > \varepsilon\lambda(\operatorname{supp} t_k)\}, \end{aligned}$$

where $i = 1, ..., k$. Then employing (3.2), we get

$$\begin{aligned} \sum_{m=1}^{\infty} M\left(\left\|T_m \sum_{i=1}^{k} t_i\right\|\right) &\geq \sum_{i=1}^{k}\sum_{m=1}^{\infty} M(\|T_m t_i\|) - 1 \\ &= \sum_{i=1}^{k}\left(\sum_{m\in J_1^i} M(\|T_m t_i\|) + \sum_{m\in J_2^i} M(\|T_m t_i\|)\right) - 1. \end{aligned}$$

If $J_2^i = \emptyset$, then by (3.8) we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} M(\|T_m t_i\|) &\geq \sum_{m=1}^{\infty} C\varepsilon^p \lambda(\operatorname{supp} t_k)^p\, M\left(\tfrac{\|T_m t_i\|}{\varepsilon\lambda(\operatorname{supp} t_k)}\right) \\ &\geq C\varepsilon^p\lambda(\operatorname{supp} t_k)^p. \end{aligned}$$

In the last inequality we used Remark 3.1.5 and property (b) of the map T. If $J_2^i \neq \emptyset$, then

$$\begin{aligned} \sum_{m=1}^{\infty} M(\|T_m t_i\|) &\geq \sum_{m\in J_2^i} M(\|T_m t_i\|) \\ &\geq \varepsilon\lambda(\operatorname{supp} t_k) \geq C\varepsilon^p\lambda(\operatorname{supp} t_k)^p. \end{aligned}$$

Thus, we get

$$\sum_{m=1}^{\infty} M\left(\left\|T_m\left(\sum_{i=1}^{k} t_i\right)\right\|\right) \geq C\left[\frac{N}{\ln N}\right]\varepsilon^p\left(\lambda\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right)\right)^p - 1. \tag{3.9}$$

Now we consider two cases. Let $1 < \left\| T\left(\sum_{i=1}^{k} t_i\right)\right\|$. Applying the $\|.\|_M$ -definition, (3.8), and (3.9), we get

$$\begin{aligned} 1 &= \sum_{m=1}^{\infty} M\left(\left\|T_m\left(\sum_{i=1}^{k} t_i\right)\right\| \cdot \frac{1}{\left\|T\left(\sum_{i=1}^{k} t_i\right)\right\|}\right) \geq \\ &\geq C\frac{1}{\left\|T\left(\sum_{i=1}^{k} t_i\right)\right\|^p}\left(\sum_{m=1}^{\infty} M\left(\left\|T_m\left(\sum_{i=1}^{k} t_i\right)\right\|\right)\right) \geq \\ &\geq C\frac{1}{\left\|T\left(\sum_{i=1}^{k} t_i\right)\right\|^p}\left(C\varepsilon^p\left[\frac{N}{\ln N}\right]\left(\lambda\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right)\right)^p - 1\right). \end{aligned}$$

Whence,

$$\begin{aligned} \|T\|^p\left(\sqrt{N}+1\right)^p + C &\geq \left\|T\left(\sum_{i=1}^{k} t_i\right)\right\|^p + C \\ &\geq \left(\frac{N}{\ln N} - 1\right)C^2\varepsilon^p\lambda\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right)^p \end{aligned}$$

for all $N \in \mathbf{N}$. By Lemma 1.4.9, the multiplier $\lambda\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right) \underset{N\to\infty}{\longrightarrow} 1$. It follows that this inequality is not true for large N. Let $\left\|T\left(\sum_{i=1}^{k} t_i\right)\right\| \leq 1$. Then $\sum_{m=1}^{\infty} M\left(\left\|T_m\left(\sum_{i=1}^{k} t_i\right)\right\|\right) \leq 1$. Employing (3.9), we arrive at a contradiction in the same way. Thus case 1 is finished.

Case 2: $\alpha_M > 2$. Equivalently, there exist $p > 2$ and $C > 0$ such that

$$M(tx) \leq Cx^p M(t) \tag{3.10}$$

for any $0 < t, x \leq 1$.

By analogy with case 1 we fix $N \in \mathbf{N}$ and $k = [N \ln N]$, choose the mapping T, construct functions $\{t_i\}_{i=1}^{k}$ jointly equidistributed with $\{s_i\}_{i=1}^{k}$ such that

$$\sum_{m=1}^{\infty} M\left(\left\|T_m\left(\sum_{i=1}^{k} t_i\right)\right\|\right) \leq \sum_{i=1}^{k}\sum_{m=1}^{\infty} M(\|T_m t_i\|) + 1 \leq k+1. \tag{3.11}$$

Let f_N be as in the proof of Theorem 1.4.11:

$$f_N = \begin{cases} 1 & \text{if } \sum_{i=1}^{k} t_i \geq \sqrt{N} \\ -1 & \text{if } \sum_{i=1}^{k} t_i \leq -\sqrt{N} \\ 0 & \text{for the rest} \end{cases}.$$

Let us stress that

$$\left\| T\left(\sum_{i=1}^{k} t_i\right)\right\| \leq \|T\| \left(\sqrt{N}+1\right) \leq \frac{\sqrt{N}+1}{2}. \tag{3.12}$$

Put $\alpha = \left\| T\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{k} t_i\right)\right\|$, $\alpha_m = \left\| T_m\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{k} t_i\right)\right\|$. Then $\alpha < 1$ and $\alpha_m < 1$ for large N. If we combine (1.8) ,(3.12) and property (b) of the operator T, we get $\alpha > \varepsilon\lambda\left(\operatorname{supp} f_N\right) \geq \frac{\varepsilon}{2}$ for large N. By the Δ_2-condition there is a constant $C > 0$ such that $M\left(\frac{1}{\alpha}\alpha_m\right) \leq C_1 M\left(\alpha_m\right)$. Consequently, by (3.10) and (3.11)

$$\begin{aligned} 1 &= \sum_{m=1}^{\infty} M\left(\frac{\alpha_m}{\alpha}\right) \leq C_1 \sum_{m=1}^{\infty} M\left(\alpha_m\right) = C_1 \sum_{m=1}^{\infty} M\left(\left\| T_m\left(\sum_{i=1}^{k} t_i\right)\frac{1}{\sqrt{N}}\right\|\right) \\ &\leq C_1 \sum_{m=1}^{\infty} M\left(\left\| T_m\left(\sum_{i=1}^{k} t_i\right)\right\|\right) C\left(\frac{1}{\sqrt{N}}\right)^p \\ &\leq C_1 C\left(\frac{1}{\sqrt{N}}\right)^p (k+1) = C_1 C\left(\frac{1}{\sqrt{N}}\right)^p \left([N\ln N]+1\right) \end{aligned}$$

for sufficiently large $N \in \mathbf{N}$. This contradiction concludes the proof. ■

Remark 3.1.12 *Let $\{X_n\}$ be a sequence of Banach spaces and $\{M_n\}_{n=1}^{\infty}$ be Orlicz functions satisfying the uniform Δ_2-condition (i.e., there exists a constant C such that $\sup_{0<t\leq 1} \frac{M_n(2t)}{M_n(t)} < C$ for all $n \in \mathbf{N}$). Then the previous theorem is valid for the modular space $\left(\sum X_n\right)_{M_n}$ of all sequences $x = (x_1, x_2, ...)$, where $x_n \in X_n$ $(n = 1, 2, ...)$, such that $\sum_{n=1}^{\infty} M_n\left(\frac{\|x_n\|}{\rho}\right) < \infty$ for some $\rho > 0$ and $\|x\| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M_n\left(\frac{\|x_n\|}{\rho}\right) \leq 1\right\}$.*

3.2 The A_p-property

In this section we shall show that Lorentz sequence spaces, Schreier's space, and Baernstein's spaces not containing isomorphic copies of l_2 have the Lyapunov property.

First let us summarize some material concerning these spaces.

The Lorentz sequence spaces were introduced in connection with some problems of analysis and interpolation theory. Let $1 \leq p < \infty$. For any $a = (a_1, a_2, ...) \in c_0 \backslash l_1$, $a_1 \geq a_2 \geq ... \geq 0$, let $d(a,p) = \{x = (x_1, x_2, ...) \in c_0 : \sup_{\sigma\in\pi} \sum_{n=1}^{\infty} \left|x_{\sigma(n)}\right|^p a_n < \infty\}$, where π is the set of all permutations of the natural numbers $\mathbf{N}$. Then $d(a,p)$ with the norm $\|x\| = \left(\sup_{\sigma\in\pi} \sum_{n=1}^{\infty} \left|x_{\sigma(n)}\right|^p a_n\right)^{\frac{1}{p}}$ for $x \in d(a,p)$ is a Banach space and the sequence of unit vectors $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of $d(a,p)$. For the basic properties of the Lorentz spaces $d(a,p)$ we refer to ([9], [10]). In particular, it is known that every infinite-dimensional subspace of $d(a,p)$ has a complemented subspace isomorphic to l_p [1].

In 1930, J. Schreier [30] introduced the notion of "admissibility" while producing a counterexample to a question of S. Banach and S. Saks. These two had just shown [3] that in the spaces $L_p[0,1]$ (where $p > 1$) each weakly convergent sequence contains a subsequence whose arithmetic means converges in norm. They went on to ask whether such a thing held in the space of continuous functions $C[0,1]$. J. Schreier showed that this was not the case. A slight variation in his original concept produces the space which is called Schreier's space in [4].

Definition 3.2.1 *A finite subset $E = \{n_1 < n_2 < ... < n_k\}$ of $\mathbf{N}$ is said to be admissible if $k \leq n_1$. We denote by L the class of all admissible subsets of* $\mathbf{N}$.

Let c_{00} denote the vector space of all real sequences which are eventually 0. Schreier's space S is the $\|.\|_S$-completion of c_{00}, where $\|x\|_S = \sup_{E\in L} \sum_{k\in E} |x(k)|$, where $x(k)$ is a k-th coordinate of x, and $x \in c_{00}$. It is known [4] that this space is not reflexive with a 1-unconditional basis.

In 1972, A. Baernstein [2] produced a reflexive variant of S which contains a weakly null sequence such that no subsequence is strongly Cesaro convergent.

If E and F are finite non-void subsets of $\mathbf{N}$, we write "$E < F$" for $\max E < \min F$. For $x \in \mathbf{c}_{00}$ we write Ex to indicate the vector defined by:

$$(Ex)(k) = \begin{cases} x(k), & \text{if } k \in E \\ 0, & \text{otherwise.} \end{cases}$$

Fix $1 < p < \infty$. For $x \in c_{00}$, we define

$$\|x\|_{B_p} = \sup\left\{ \left(\sum_{k=1}^{n} \|E_k x\|_{l_1}^p \right)^{\frac{1}{p}} : E_k \in L,\ E_1 < E_2 < ... < E_n,\ n \in \mathbf{N} \right\}.$$

Baernstein's space B_p is $\|.\|_{B_p}$-completion of c_{00}. It is known that every infinite-dimensional subspace of B_p has a complemented subspace isomorphic to l_p. See [4] for details.

The idea of the proofs here is very close to Theorem 1.4.11. But the new trick which we employ, namely considering an auxiliary c_0-valued operator, allows us to generalize the Lyapunov theorem for more spaces.

Definition 3.2.2 *A Banach space X with a basis is said to have the A_p-property ($X \in A_p$) if for some basis $\{e_n\}_{n=1}^{\infty}$ in X there exists a map $\widetilde{T} \in L(X, c_0)$ such that for every $x = \sum_{i=1}^{N} x(i)\, e_i \in X$, with $x(i) \neq 0$, $(1 \leq i \leq N)$, and every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y = \sum_{i=N+1}^{M} y(i)\, e_i \in X$ with $\left\|\widetilde{T}y\right\|_{\infty} \leq \delta$*

$$|\|x+y\|^p - \|x\|^p - \|y\|^p| < \varepsilon$$

if $1 \leq p < \infty$, or

$$|\|x+y\| - \max\{\|x\|, \|y\|\}| < \varepsilon$$

if $p = \infty$.

Lemma 3.2.3 *If $X \in A_p$ and $\widetilde{T}$ is from Definition 3.2.2 then for every $x \in X$, $x \neq 0$, for every sequence $x_n \xrightarrow[n\to\infty]{w} 0$ in X with $\left\|\widetilde{T}x_n\right\|_{\infty} \underset{n\to\infty}{\longrightarrow} 0$, and for every $\varepsilon > 0$ there exists n such that*

$$|\|x+x_n\|^p - \|x\|^p - \|x_n\|^p| < \varepsilon$$

if $1 \leq p < \infty$, *or*

$$\left| \|x + x_n\| - \max\left\{ \|x\|, \|x_n\| \right\} \right| < \varepsilon$$

if $p = \infty$.

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be a basis in X and let $\widetilde{T} \in L(X, c_0)$, for which the conditions of Definition 3.2.2 are satisfied. Fix $x = \sum\limits_{i=1}^{\infty} x(i) e_i \in X$, $x \neq 0$, $x_n = \sum\limits_{i=1}^{\infty} x_n(i) e_i \in X$, $(n = 1, 2, ...)$, $x_n \underset{n\to\infty}{\overset{w}{\to}} 0$, $\left\| \widetilde{T} x_n \right\|_{\infty} \underset{n\to\infty}{\to} 0$, $\varepsilon > 0$. Put $C = \sup\limits_n \|x_n\|$. Choose $\theta \in (0, 1)$ such that for any $a, b \in [0, \|x\| + 1 + C]$ with $|a - b| \leq \theta$ the inequality $|a^p - b^p| < \frac{\varepsilon}{4}$ holds. Select $N \in \mathbf{N}$ such that $\left\| \sum\limits_{i=N+1}^{\infty} x(i) e_i \right\| \leq \frac{\theta}{4}$. Denote $x^{\varepsilon} = \sum\limits_{i=1}^{N} x^{\varepsilon}(i) e_i$, where

$$x^{\varepsilon}(i) = \begin{cases} x(i) & \text{if } x(i) \neq 0 \\ \frac{\theta}{4N} & \text{if } x(i) = 0 \end{cases}.$$

It is obvious that

$$\left| \|x\|^p - \|x^{\varepsilon}\|^p \right| < \frac{\varepsilon}{4}. \tag{3.13}$$

Find $\delta > 0$ for x^{ε} and $\frac{\varepsilon}{4}$ according to Definition 3.2.2. There is $n \in \mathbf{N}$ such that $\left\| \widetilde{T} x_n \right\|_{\infty} \leq \delta$ and $\left\| \sum\limits_{i=1}^{N} x_n(i) e_i \right\| \leq \frac{\theta}{4}$. Select $M \in \mathbf{N}$ such that $\left\| \sum\limits_{i=M+1}^{\infty} x_n(i) e_i \right\| \leq \frac{\theta}{4}$. Put $x_n^{\varepsilon} = \sum\limits_{i=N+1}^{M} x_n(i) e_i$. It is easy to see that $\|x_n - x_n^{\varepsilon}\| \leq \frac{\theta}{2}$ and $\left\| \widetilde{T} x_n^{\varepsilon} \right\|_{\infty} \leq \delta$ and consequently

$$\left| \|x_n\|^p - \|x_n^{\varepsilon}\|^p \right| < \frac{\varepsilon}{4}, \tag{3.14}$$

$$\left| \|x^{\varepsilon} + x_n^{\varepsilon}\|^p - \|x^{\varepsilon}\|^p - \|x_n^{\varepsilon}\|^p \right| < \frac{\varepsilon}{4}. \tag{3.15}$$

Note that $\left| \|x + x_n\| - \|x^{\varepsilon} + x_n^{\varepsilon}\| \right| \leq \theta$. Therefore

$$\left| \|x + x_n\|^p - \|x^{\varepsilon} + x_n^{\varepsilon}\|^p \right| \leq \frac{\varepsilon}{4}. \tag{3.16}$$

Combining (3.13)-(3.16), we get

$$\begin{aligned} |\|x+x_n\|^p - \|x\|^p - \|x_n\|^p| &\le |\|x^\varepsilon + x_n^\varepsilon\|^p - \|x^\varepsilon\|^p - \|x_n^\varepsilon\|^p| \\ &+ |\|x+x_n\|^p - \|x^\varepsilon + x_n^\varepsilon\|^p| + |\|x\|^p - \|x^\varepsilon\|^p| \\ &+ |\|x_n\|^p - \|x_n^\varepsilon\|^p| < \varepsilon. \end{aligned}$$

The case $p = \infty$ can be shown in the same way. ■

Theorem 3.2.4 *If $X \in A_p$ $(p \neq 2)$, then there exist a Lyapunov topology τ on X, $n' \in \mathbf{N}$, and if $1 \le p < 2$ there is $p' \in [p, 2)$ such that*

$$C^\tau\left(n', X\right) > (n')^{1/p'},$$

if $2 < p < \infty$ there is $p' \in (2, p]$ such that

$$b^\tau\left(n', X\right) < (n')^{1/p'}.$$

Proof. Case $1 \le p < 2$. Let τ be the $\widetilde{T} - c_0$ weak topology on X, where $\widetilde{T}$ is from Definition 3.2.2, and $\{f_i\}_{i=1}^\infty$ be the coordinate functionals. Take an arbitrary $n \in \mathbf{N}$, a Lyapunov tree X^n with $\|x_{m_1,\dots,m_i}\| > 1$. We will show that there exist $\{m_i^0\}_{i=1}^n$ such that

$$\left\|\sum_{i=1}^n x_{m_1^0,\dots,m_i^0}\right\|^p \ge \sum_{i=1}^n \left\|x_{m_1^0,\dots,m_i^0}\right\|^p - 1 \qquad (3.17)$$

Let $m_1^0 = m_1^1$. Consider the sequence $\left\{x_{m_1^0,m_2}\right\}_{m_2=1}^\infty$. By the definition of a Lyapunov tree $x_{m_1^0,m_2} \underset{m_2\to\infty}{\overset{w}{\longrightarrow}} 0$ and $\left\|\widetilde{T}x_{m_1^0,m_2}\right\|_\infty \underset{m_2\to\infty}{\longrightarrow} 0$. Then in accordance with Lemma 3.2.3 we can choose m_2^0 such that

$$\left\|\sum_{i=1}^2 x_{m_1^0,m_i^0}\right\|^p > \sum_{i=1}^2 \left\|x_{m_1^0,m_i^0}\right\|^p - \frac{1}{2}.$$

Now consider the sequence $\left\{x_{m_1^0,m_2^0,m_3}\right\}_{m_3=1}^\infty$. Analogously choose m_3^0 such that

$$\left\|\sum_{i=1}^3 x_{m_1^0,m_2^0,m_i^0}\right\|^p > \sum_{i=1}^2 \left\|x_{m_1^0,m_i^0}\right\|^p + \left\|x_{m_3^0}\right\|^p - \frac{1}{4}.$$

It follows that

$$\left\| \sum_{i=1}^{3} x_{m_1^0, m_2^0, m_i^0} \right\|^p > \sum_{i=1}^{3} \left\| x_{m_1^0, m_2^0, m_i^0} \right\|^p - \left(\frac{1}{2} + \frac{1}{4} \right).$$

Continuing this process we obtain $\{m_i^0\}_{i=1}^n$ satisfying (3.17).

As $\|x_{m_1,\dots,m_i}\| > 1$ $(i = 1, \dots, n)$ we get

$$\left\| \sum_{i=1}^{n} x_{m_1^0,\dots,m_i^0} \right\|^p \geq n - 1,$$

and consequently

$$\left\| \sum_{i=1}^{n} x_{m_1^0,\dots,m_i^0} \right\|^p \geq (n-1)^{1/p}.$$

It follows that $C^\tau(n, X) \geq (n-1)^{1/p}$ for all n. It is not difficult to see that there exist $p' \in [p, 2)$ and n' such that $\left(n' - 1\right)^{1/p} > (n')^{1/p'}$ and consequently $C^\tau\left(n', X\right) > n'^{\frac{1}{p'}}$.

Case $2 < p < \infty$ is proved by analogy with the previous one. ■

Corollary 3.2.5 *If $X \in A_p$ $(p \neq 2)$, then X has the Lyapunov property.*

Proof. The proof follows immediately from the last theorem and Theorems 2.1.8, 2.1.9, 2.2.5, and 2.2.6. ■

In Lemmas 3.2.6, 3.2.7, 3.2.8 below the natural coordinate embedding of the corresponding sequence space into c_0 plays the role of $\widetilde{T}$ from Definition 3.2.2.

Lemma 3.2.6 $d(a, p) \in A_p$.

Proof. Note that if $x = (x_1, x_2, \dots) \in d(a, p)$, $\|x\| = \left(\sum\limits_{n=1}^{\infty} \hat{x}_n^p a_n \right)^{\frac{1}{p}}$, where $(\hat{x}_1, \hat{x}_2, \dots)$ is an enumeration of $\{|x_n|\}_{n=1}^{\infty}$ such that $\hat{x}_1 \geq \hat{x}_2 \geq \dots$. Fix $x = \sum\limits_{i=1}^{N} x_i e_i \in d(a, p)$, where $x_i \neq 0$ $(i = 1, 2, \dots, N)$, $\varepsilon \in (0, 1)$. Without loss of generality we may assume that $x_1 \geq x_2 \geq \dots > 0$. Since

$a = (a_1, a_2, ...) \in c_0$, there exists $k > N$ such that $\sum\limits_{i=k}^{k+N} a_i < \frac{\varepsilon}{2}$. Take $\delta = \min\left\{\frac{\varepsilon}{2k}, \min\limits_{1\leq i\leq N} x_i\right\}$. Let $y = \sum\limits_{i=N+1}^{M} y_i e_i \in d(a,p)$ with $\|y\|_\infty \leq \delta$. We may assume that $y_{N+1} \geq y_{N+2} \geq ... \geq 0$. Further we estimate the number $|\|x+y\|^p - \|x\|^p - \|y\|^p|$:

$$\begin{aligned} |\|x+y\|^p - \|x\|^p - \|y\|^p| &= \sum_{i=N+1}^{M} y_i^p (a_{i-N} - a_i) \\ &= \sum_{i=N+1}^{N+k} y_i^p (a_{i-N} - a_i) + \sum_{i=N+k+1}^{M} y_i^p (a_{i-N} - a_i). \end{aligned}$$

Since $1 \geq a_1 \geq a_2 \geq ... \geq 0$, we obtain

$$\sum_{i=N+1}^{N+k} y_i^p (a_{i-N} - a_i) \leq \sum_{i=N+1}^{N+k} y_i^p \leq \frac{\varepsilon}{2}.$$

An application of $\|y\|_\infty \leq 1$ yields

$$\begin{aligned} \sum_{i=N+k+1}^{M} y_i^p (a_{i-N} - a_i) &\leq \sum_{i=N+k+1}^{M} (a_{i-N} - a_i) \\ &= \sum_{i=k+1}^{N+k} a_i - \sum_{i=M-N+1}^{M} a_i < \tfrac{\varepsilon}{2}. \end{aligned}$$

So, $|\|x+y\|^p - \|x\|^p - \|y\|^p| < \varepsilon$. ■

Lemma 3.2.7 $S \in A_\infty$.

Proof. Let $x = \sum\limits_{i=1}^{N} x_i e_i \in S$, where $x_i \neq 0$ $(i = 1, 2, ..., N)$, $\varepsilon > 0$. Denote $\delta = \frac{1}{N} \min\limits_{1\leq i\leq N} |x_i|$. Choose $y = \sum\limits_{i=N+1}^{M} y_i e_i \in S$ with $\|y\|_\infty \leq \delta$. It is evident that $\|x\|, \|y\| \leq \|x+y\|$, consequently $\max\{\|x\|, \|y\|\} \leq \|x+y\|$.

We shall prove that $\|x+y\| \leq \max\{\|x\|, \|y\|\}$. Let $k \leq N$, $E = \{n_1 < ... < n_k\}$ be an admissible set, i.e. $k \leq n_1$. If $n_k \leq N$, then

$$\sum_{i\in E} |x_i| + \sum_{i\in E} |y_i| = \sum_{i\in E} |x_i| \leq \|x\|.$$

If $n_k > N$, then

$$\sum_{i\in E}|x_i| + \sum_{i\in E}|y_i| \le \sum_{i\in E}|x_i| + \min_{1\le i\le N}|x_i| \le \sum_{i\in E'}|x_i| \le \|x\|,$$

where $E' = \{k_0\}\cup\{i\in E : i\le N\}$, $k_0 \in \overline{k,N}\backslash E$. Let $k > N$, then

$$\sum_{i\in E}|x_i| + \sum_{i\in E}|y_i| = \sum_{i\in E}|y_i| \le \|y\|.$$

Thus $\|x+y\| \le \max\{\|x\|, \|y\|\}$. ■

Lemma 3.2.8 $B_p \in A_p$.

Proof. Let $x = \sum_{i=1}^{N} x_i e_i \in B_p$, $x \ne 0$, $\varepsilon > 0$. Select $\theta \in (0,1)$ such that for any $a, b \in \left[0, \|x\|_{l_1} + 1\right]$ with $|a-b| \le \theta$ the inequality $|a^p - b^p| < \varepsilon$ holds. Take $\delta = \frac{\theta}{N}$. Choose $y = \sum_{i=N+1}^{M} y_i e_i \in S$ with $\|y\|_\infty \le \delta$. Evidently $\|x\|^p + \|y\|^p \le \|x+y\|^p$. We shall prove that $\|x+y\|^p \le \|x\|^p + \|y\|^p + \varepsilon$. Let $E_1 < E_2 < ... < E_n$, $E_i \in L$, and

$$\|x+y\|^p = \sum_{i=1}^{n} \|E_i(x+y)\|_{l_1}^p.$$

We introduce the sets of indices

$$\begin{aligned} I_1 &= \left\{ i \in \overline{1,n} : E_i \cap \overline{N+1,M} = \emptyset \right\}, \\ I_2 &= \left\{ i \in \overline{1,n} : E_i \cap \overline{1,N} = \emptyset \right\}, \\ i_0 &= \overline{1,n}\backslash (I_1 \cup I_2). \end{aligned}$$

Then

$$\|x+y\|^p = \sum_{i\in I_1}\|E_i x\|_{l_1}^p + \sum_{i\in I_2}\|E_i y\|_{l_1}^p + \|E_{i_0}(x+y)\|_{l_1}^p.$$

Let us estimate the last item

$$\begin{aligned} \|E_{i_0}(x+y)\|_{l_1}^p &= \left(\left\| \left(E_{i_0} \cap \overline{1,N}\right) x \right\|_{l_1} + \left\| \left(E_{i_0} \cap \overline{N+1,M}\right) y \right\|_{l_1} \right)^p \\ &< \|E_{i_0} x\|_{l_1}^p + \varepsilon. \end{aligned}$$

This implies that

$$\|x+y\|^p = \sum_{i\in I_1\cup i_0} \|E_i x\|_{l_1}^p + \sum_{i\in I_2} \|E_i y\|_{l_1}^p + \varepsilon \leq \|x\|^p + \|y\|^p + \varepsilon.$$

The lemma is proved. ■

Corollary 3.2.9 *The Lorentz sequence spaces $d(a,p)$ $(p \neq 2)$, the Baernstein spaces B_p $(p \neq 2)$ and the Schreier space S have the Lyapunov property.*

3.3 Tsirelson-type spaces

The results of this section are contained in [34].

One of the historical concerns of the structure theory of Banach spaces has been whether there are any "fundamental" spaces which embed isomorphically in every infinite dimensional Banach space. The spaces c_0 or l_p $(1 \leq p < \infty)$ were hoped to have this property, because all classical Banach spaces do indeed contain a copy of c_0 or l_p $(1 \leq p < \infty)$. Also Orlicz spaces have this property despite the fact that the definition of an Orlicz space is not a priori connected to any l_p or c_0. This hope was destroyed by B. S. Tsirelson's construction of a reflexive Banach space with a monotone unconditional Schauder basis and no embedded copies of c_0 or any l_p [33]. T. Figiel and W. B. Johnson continued the research and obtained a reflexive Banach space with a symmetric basis that has the same property. The following construction of Tsirelson's space is due to Figiel and Johnson [4].

We inductively define a sequence of norms $\{\|\cdot\|_N\}_{N=1}^{\infty}$ as follows: for $x \in c_{00}$, let

$$\begin{cases} \|x\|_0 = \|x\|_\infty, \\ \|x\|_{N+1} = \|x\|_N \vee \frac{1}{2}\max\left\{\sum_{j=1}^n \|E_j x\|_N : n \in \mathbf{N},\ n \leq E_1 < ... < E_n\right\} \end{cases}$$

for $N \geq 0$. It is easily seen that the $\|\cdot\|_N$ are norms on c_{00}, that they increase with N, and that

$$\|x\|_N \leq \|x\|_{l_1}$$

for all $x \in c_{00}$ and for all N. Thus, for each $x \in c_{00}$, $\lim_N \|x\|_N$ exists and is majorized by $\|x\|_{l_1}$. We denote $\lim_N \|x\|_N$ by $\|x\|$ and easily confirm that it norms c_{00}. Tsirelson's space T is the $\|\cdot\|$-completion of c_{00}.

Note that from the definition of the norm it follows that

$$\|x\| = \max\left\{\|x\|_\infty, \frac{1}{2}\sup\left\{\sum_{j=1}^{n}\|E_j x\| : n \in \mathbf{N},\ n \leq E_1 < ... < E_n\right\}\right\} \tag{3.18}$$

for each $x \in T$.

This construction has been developed further and some Banach spaces of Tsirelson-type were obtained in order to solve some long-standing problems of Banach space theory. Below we present the results of Schlumprecht, Maurey, and Gowers.

Definition 3.3.1 *A space $(Y, \|\cdot\|)$ is said to be $\lambda -$ distortable if there exists an equivalent norm $|\|\cdot\||$ such that for every infinite dimensional subspace $Z \subset Y$ the quantity*

$$\sup\{|\|y\|| \,/\, |\|z\|| : \|y\| = \|z\| = 1\}$$

is at least λ.

In 1991 T. Schlumprecht [30] constructed an example of a space that is λ-distortable for every λ. Before giving his definition let us fix some notation.

T. Schlumprecht defines a class of functions $f : [1, \infty) \to [1, \infty)$, which we call $\mathcal{F}$, as follows. The function f is a member of $\mathcal{F}$ if it satisfies the following five conditions:

(i) $f(1) = 1$ and $f(x) < x$ for every $x > 1$.

(ii) f is strictly increasing and tends to infinity.

(iii) $\lim_{x\to\infty} x^{-q} f(x) = 0$ for every $q > 0$.

(iv) The function $x/f(x)$ is concave and not decreasing.

(v) $f(xy) \leq f(x) f(y)$for every $x, y \geq 1$.

It is easily verified that $f(x) = \log_2(x+1)$ satisfies these conditions, as does the function $\sqrt{\log_2(x+1)}$.

The *support* of a vector $x = \sum_{n=1}^{\infty} a_n e_n \in c_{00}$ is the set of $n \in \mathbf{N}$ for which $a_n \neq 0$. An *interval* of integers is a subset of $\mathbf{N}$ of the form

$\{a, a+1, ..., b\}$ for some $a, b \in \mathbf{N}$. We shall also define the *range* of a vector, written by the ran (x), to be a smallest interval containing its support. We shall write $x < y$ to mean ran $(x) <$ ran (y). If $x_1 < ... < x_n$, we shall say that $x_1, ..., x_n$ are successive.

Now let f be the function $x \mapsto \log_2(x+1)$ as above. Define the sequence of norms $\{\|\cdot\|_N\}_{N=0}^{\infty}$ on c_{00} as follows. Fix $x \in c_{00}$ and put

$$\begin{cases} \|x\|_0 = \|x\|_\infty, \\ \|x\|_{N+1} = \|x\|_N \vee \sup \left\{ f(n)^{-1} \sum_{j=1}^n \|E_j(x)\|_N : n \geq 2,\ E_1 < ... < E_n \right\}. \end{cases}$$

Schlumprecht's space S is the completion of the space c_{00} equipped with the norm $\|x\| = \lim\limits_N \|x\|_N$ for every $x \in c_{00}$. It is not difficult to show that the constructed norm satisfies the following condition

$$\|x\| = \max \left\{ \|x\|_\infty, \sup f(n)^{-1} \sum_{j=1}^{n} \|E_j(x)\| : n \geq 2,\ E_1 < ... < E_n \right\}. \tag{3.19}$$

In 1993 W. T. Gowers and B. Maurey [12] constructed a Banach space that does not contain any unconditional basic sequence. The definition of the space resembles that of Schlumprecht's space. First we shall need a certain amount of preliminary notation.

Let $\mathbf{Q}$ be the set of scalar sequences with finite support, rational coordinates, and maximum at most one in modulus. Let $J \subset \mathbf{N}$ be an infinite set such that, if $m < n$ and $m, n \in J$, then $\log \log \log n \geq 4m^2$. Let us write J in increasing order as $\{j_1, j_2, ...\}$. We shall also assume that $f(j_1) > 256$. (Recall that $f(x) = \log_2(x+1)$.) Now let $K \subset J$ be the set $\{j_1, j_3, ...\}$, and let $L \subset \mathbf{N}$ be the set of integers $j_2, j_4, ...$

Let σ be an injection from the collection of finite sequences of successive elements of $\mathbf{Q}$ to L such that, if $z_1, ..., z_s$ is such a sequence, $S = \sigma(z_1, ..., z_s)$, and $z = \sum\limits_{i=1}^{s} z_i$, then $(1/20) f\left(S^{1/40}\right)^{1/2} \geq |\text{supp}(z)|$.

We shall use the injection σ to define special functionals in an arbitrary normed space of the form $(c_{00}, \|\cdot\|)$.

If $(c_{00}, \|\cdot\|)$ is a normed space on the finitely supported sequences and $m \in \mathbf{N}$, let $A_m^*(X)$ be the set of functionals of the form $f(m)^{-1} \sum\limits_{i=1}^{m} f_i$ such that $f_1 < ... < f_m$ and $\|f_i\| \leq 1$ for each i. If

$k \in \mathbf{N}$, let Γ_k^X be the set of sequences $g_1, ..., g_k$ such that $g_i \in \mathbf{Q}$ for each i, $g_1 \in A_{j_{2k}}^* (X)$ and $g_{i+1} \in A_{\sigma(g_1,...,g_i)}^* (X)$ for each $1 \leq i \leq k-1$. We call these *special sequences*. Let $B_k^* (X)$ be the set of functionals of the form $f(k)^{-1/2} \sum_{j=1}^{k} g_j$ such that $(g_1, ..., g_k) \in \Gamma_k^X$. These are *specials functionals*.

The definition of the norm is as the limit of sequences of norms. Define $X_0 = (c_{00}, \|\cdot\|_0)$ by $\|x\|_0 = \|x\|_\infty$, and for $N \geq 0$ let

$$\begin{aligned} \|x\|_{X_{N+1}} = \; & \sup \left\{ f(n)^{-1} \textstyle\sum_{i=1}^n \|E_i x\|_{X_N} : 2 \leq n \in \mathbf{N},\ E_1 < ... < E_n \right\} \\ & \vee \sup \left\{ |g(Ex)| : k \in K,\ g \in B_k^* (X_N),\ E \subset \mathbf{N} \right\} \end{aligned}$$

Note that this is an increasing sequence of norms, because the sets $B_k^* (X_N)$ increase as N increases (and more generally, if $\|x\|_Y \leq \|x\|_Z$ for every $x \in c_{00}$, then $B_k^* (Y) \subset B_k^* (Z)$). They are also all bounded above by the l_1-norm. Define $\|x\| = \lim_{N \to \infty} \|x\|_{X_N}$. The $\|\cdot\|$-completion of the c_{00} is called Gowers-Maurey's space GM.

The constructed norm satisfies the following condition

$$\begin{aligned} \|x\| = \; & \|x\|_\infty \vee \sup \left\{ f(n)^{-1} \textstyle\sum_{i=1}^n \|E_i x\| : 2 \leq n \in \mathbf{N},\ E_1 < ... < E_n \right\} \\ & \vee \sup \left\{ |g(Ex)| : k \in K,\ g \in B_k^* (X),\ E \subset \mathbf{N} \right\}. \end{aligned} \tag{3.20}$$

W. T. Gowers modified the construction of GM to give an example of an infinite-dimensional Banach space that does not contain c_0, l_1 or an infinite-dimensional reflexive Banach space [11].

Let $\mathbf{Q}$ be as above. Let $J \subset \mathbf{N}$ be a set such that if $m < n$ and $m, n \subset J$, then $\log\log\log\log\log n \geq 1000m$. Let us also suppose that $f(j) > 10^{103}$ for every $j \in J$, where $f(x) = \sqrt{\log_2 (x+1)}$. Let σ be an injection from the set of finite sequences of successive elements of $\mathbf{Q}$ to J.

Let $X = (c_{00}, \|\cdot\|)$ be a normed space such that the standard basis is bimonotone. For every $m \in \mathbf{N}$ define $A_m^* (X)$ to be the set of functionals of the form $f(m)^{-1} \sum_{i=1}^{m} x_i^*$, where $x_1^*, ..., x_m^*$ are successive members of c_{00} and $\|x_i^*\| \leq 1$ for each i. A *special sequence* of functionals on X is defined to be a sequence of the form $z_1^*, ..., z_M^*$, where $M \in \mathbf{N}$, the z_i^* are successive, $z_i^* \in A_m^* \cap \mathbf{Q}$ for some $m \in J$ and for $2 \leq i \leq M$, we have $z_i^* \in A^*_{\sigma\left(z_1^*,...,z_{i-1}^*\right)} \cap \mathbf{Q}$. A *special functional* on X is defined to be a functional of the form $E\left(\sum\limits_{i=1}^{M} z_i^*\right)$ such that $z_1^*, ..., z_M^*$ is a special sequence. To any special sequence we associate a sequence of integers $n_1, ..., n_M \in J$ such that $z_1^* \in A^*_{n_1}$ and $n_i = \sigma\left(z_1^*, ..., z_{i-1}^*\right)$ for $2 \leq i \leq M$. The first number n_1 is not necessary uniquely determined, but $n_2, ..., n_M$ certainly are. Given a special functional $z^* = E\left(\sum\limits_{i=1}^{M} z_i^*\right)$, we say that $Z \subset J$ is associated to z^* if we can pick such a sequence $n_1, ..., n_M$ associated to the sequence $z_1^*, ..., z_M^*$ and Z consists of those n_i for which $E \cap \operatorname{ran}(z_i^*) \neq \emptyset$. A collection of special functionals $w_1^*, ..., w_N^*$ is called *disjoint* if we can choose for them disjoint associated sets $Z_1, ..., Z_N$.

We are now ready to define our norm. We shall define it as a limit of a sequence of norms on c_{00}. First let X_0 be defined by $\|x\|_0 = \|x\|_\infty$. For $N \geq 0$, define X_N by

$$\begin{aligned} \|x\|_{X_{N+1}} = \ & \sup\left\{f(n)^{-1} \textstyle\sum_{j=1}^{n} \|E_j x\|_{X_N} : n \geq 2,\ E_1 < ... < E_n\right\} \\ & \vee \sup\left(\textstyle\sum_{j=1}^{M} \left|x_j^*(x)\right|^2\right)^{1/2}, \end{aligned}$$

where the second supremum ranges over all sequences $x_1^*, ..., x_M^*$ of disjoint special functionals on X_N.

It is easy to check that every $x \in G$ satisfies the equation

$$\begin{aligned} \|x\| = \ & \|x\|_\infty \vee \sup\left\{f(n)^{-1} \textstyle\sum_{j=1}^{n} \|E_j x\| : n \geq 2,\ E_1 < ... < E_n\right\} \\ & \vee \sup\left(\textstyle\sum_{j=1}^{M} \left|x_j^*(x)\right|^2\right)^{1/2}. \end{aligned} \tag{3.21}$$

We will show that the above mentioned spaces have the Lyapunov property. To this end we will give some preliminary information.

Definition 3.3.2 *If X is a $\|\cdot\|$-completion of c_{00} such that (e_i) is a normalized monotone basis of X, $f \in \mathcal{F}$ or $f = const$, and every $x \in X$ satisfies the inequality*

$$\|x\| \geq \sup\left\{f(n)^{-1} \sum_{j=1}^{n} \|E_j x\| : n \in \mathbf{N},\ n \leq E_1 < ... < E_n\right\}$$

then we shall say that X satisfies an admissible lower f-estimate.

Note that by (3.18)-(3.21) Tsirelson's space has an admissible lower $\frac{1}{2}$-estimate, Schlumprecht's and Gowers-Maurey's spaces have a $\log_2(x+1)$-estimate, Gowers's space has a $\sqrt{\log_2(x+1)}$-estimate.

Theorem 3.3.3 *If X is a Banach space that satisfies an admissible lower f-estimate then there exist a Lyapunov topology τ on X and n such that $C^{\tau}(n, X) > n^{1/p}$ for some $p \in [1, 2)$.*

Proof. Let τ be the weak topology on X. Define the coordinate functionals by f_j. Take an arbitrary $n \in \mathbf{N}$ and consider a Lyapunov tree X^n with $\|x_{m_1,\ldots,m_i}\| \geq 1$ and take an arbitrary $\varepsilon > 0$. Then by the definition of a Lyapunov tree we can find numbers $m_1^0, \ldots, m_n^0$ such that

$$\left\| x_{m_1^0,\ldots,m_k^0} - \sum_{i=p_k+1}^{p_{k+1}} f_i\left(x_{m_1^0,\ldots,m_k^0}\right) e_i \right\| < \frac{\varepsilon}{n},$$

where $n - 1 \leq p_1 < p_2 < \ldots < p_{n+1}$, for $k = 1, \ldots, n$. Then

$$\begin{aligned} \left\| \sum_{k=1}^n x_{m_1^0,\ldots,m_k^0} \right\| &> \left\| \sum_{k=1}^n \sum_{i=p_k+1}^{p_{k+1}} f_i\left(x_{m_1^0,\ldots,m_k^0}\right) e_i \right\| - \varepsilon \\ &\geq f(n)^{-1} \sum_{k=1}^n \left\| \sum_{i=p_k+1}^{p_{k+1}} f_i\left(x_{m_1^0,\ldots,m_k^0}\right) e_i \right\| - \varepsilon \\ &\geq f(n)^{-1} \sum_{k=1}^n \left(\left\| x_{m_1^0,\ldots,m_k^0} \right\| - \frac{\varepsilon}{n} \right) - \varepsilon \\ &\geq f(n)^{-1} (n - \varepsilon) - \varepsilon. \end{aligned}$$

As ε is arbitrary, we get

$$\sup_{m_1,\ldots,m_i} \left\| \sum_{k=1}^n x_{m_1,\ldots,m_k} \right\| > n \cdot f(n)^{-1}.$$

For $f = const$ everything is obvious. If $f \in \mathcal{F}$, then from the condition (iii), it follows that there is n such that

$$C^{\tau}(n, X) > n^{1/p},$$

where $p < 2$. This completes the proof. ■

From this theorem and Theorems 2.1.9 and 2.1.8 we have the main result of this section.

Corollary 3.3.4 *Tsirelson's space, Schlumprecht's space, Gowers-Maurey's space, Gowers's space have the Lyapunov property.*

3.4 Asymptotic l_p spaces

In 1993 Vitali D. Milman and Nicole Tomczak-Jaegermann introduced the class of asymptotic l_p-spaces and showed that every Banach space with bounded distortions contains a subspace from this class [26]. We will prove that the asymptotic l_p spaces (where $1 \leq p < \infty,\ p \neq 2$) have the Lyapunov property.

Definition 3.4.1 *A Banach space X with a normalized basis $\{x_i\}$ is said to be an asymptotic l_p space, for some $1 \leq p < \infty$ (resp., an asymptotic c_0 space) if there exists a constant C such that for any $n \in \mathbf{N}$ there is $N \in \mathbf{N}$ such that normalized successive blocks $N < z_1 < z_2 < ... < z_n$ of $\{x_i\}$ are C-equivalent to the unit vector basis in l_p^n (resp., in c_0).*

Let us mention that we will say that two basis sequences $\{x_i\}$ and $\{e_i\}$ are C-equivalent, for some constant C, if for any finite sequence of scalars $\{a_i\}$ we have

$$C^{-1}\left\|\sum_i a_i x_i\right\| \leq \left\|\sum_i a_i e_i\right\| \leq C\left\|\sum_i a_i x_i\right\|.$$

Theorem 3.4.2 *Let X be an asymptotic l_p space. Then if $1 \leq p < 2$, then for every $p < p^{'} < 2$ there exists $n \in \mathbf{N}$ such that*

$$C^w(n, X) > n^{\frac{1}{p^{'}}};$$

and if $2 < p < \infty$, then for every $2 < p^{'} < p$ there exists $n \in \mathbf{N}$ such that

$$b^w(n, X) < n^{\frac{1}{p^{'}}}.$$

Proof. Case 1: $1 \leq p < 2$. Let w be the weak topology of X, $\{x_i\}$ be a normalized basis of X, $\{e_i\}$ be the unit vector basis in l_p. Take $p^{'} \in (p, 2)$ and fix for the present $n \in \mathbf{N}$ and a Lyapunov tree Y^n with $\|y_{m_1,\dots,m_i}\| \geq 1$ and an arbitrary $\varepsilon > 0$. For n choose $N = N(n) \in \mathbf{N}$ from Definition 3.4.1. By the definition of a Lyapunov tree we can find numbers $m_1^0, ..., m_n^0$ and blocks $N < z_1 < z_2 < ... < z_n$ such that $\{x_i\}$

$$\left\|y_{m_1^0,\dots,m_k^0} - z_k\right\| < \frac{\varepsilon}{n}$$

for $k = 1, ..., n$. Then we have

$$\left\| \sum_{k=1}^{n} y_{m_1^0,\dots,m_k^0} \right\| > \left\| \sum_{k=1}^{n} z_k \right\| - \varepsilon \geq C^{-1} \left\| \sum_{k=1}^{n} e_k \right\| - \varepsilon = C^{-1} n^{\frac{1}{p}} - \varepsilon.$$

As ε is arbitrary, we obtain

$$C^w(n, X) \geq C^{-1} n^{\frac{1}{p}}$$

and, consequently,

$$C^{-1} n^{\frac{1}{p}} > n^{\frac{1}{p'}}$$

for large enough n.

The case $2 < p < \infty$ is proved in analogy with the previous one. ■

Corollary 3.4.3 *If $1 \leq p < \infty$, $p \neq 2$ and X is an asymptotic l_p space, then X has the Lyapunov property.*

Proof. The proof follows immediately from Theorems 2.1.8, 2.1.9, 2.2.5, 2.2.6. ■

3.5 Tokarev's space

We have seen that most of the constructed examples obey the scheme: if a Banach space in some class contains no isomorphic copies of l_2 then it has the Lyapunov property. However, in this section we present an example of a Banach space that does not contain isomorphic copies of l_2 and does not have the Lyapunov property.

In 1984 by E. V. Tokarev obtained an example of a symmetric function space that contains no isomorphic copies of l_p $(1 \leq p < \infty)$ or c_0 [32]. We will recall this construction.

Definition 3.5.1 *A symmetric function space E is a Banach space of measurable functions such that:*

1. *if $x \in E$, $|y| \leq |x|$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$,*
2. *if the functions $|x|$ and $|y|$ are equimeasurable, then $\|x\|_E = \|y\|_E$.*

Let us consider the Marcinkiewicz space M_p defined by the function $\psi_p(t) = t(-\ln t)^{1/p}$, i.e., the space of functions $x(t)$ which are summable on $[0,1]$ with finite norm

$$\|x\|_{M_p} = \sup_{0<h\leq 1} A_h(x); \quad A_h(x) = \frac{1}{\psi_p(h)} \int_0^h x^*(t)\,dt,$$

where x^* is the nondecreasing rearrangement of x and such that $\lim_{h\to 0} A_h(x) = 0$ for all $x \in M_p$.

Further we will suppose that $p > 2$. Denote $(-\ln t)^{-1/p}$ by $\varphi_p(t)$ and note that $\|\chi_A\|_{M_p} = \varphi_p(\operatorname{mes} A)$.

Choose a sequence of natural numbers n_k; $n_1 < n_2 < ...$ so that for every k the lacunarity inequality $n_k^{-1}\sum_{i=1}^{k-1} n_i + n_k \sum_{1=k+1}^{\infty} n_i^{-1} < 2^{-k-1}$ holds and consider the space $Y_k = n_k M_p + n_k^{-1} L_\infty$ that consists of all summable functions x such that

$$\|x\|_k = \inf\left\{n_k \|f\|_{M_p} + n_k^{-1}\|g\|_{L_\infty} : f + g = x\right\} < \infty.$$

It is known (see [13] and [22]) that $\|\chi_A\|_k = \min\left\{n_k\varphi_p(\operatorname{mes} A);\, n_k^{-1}\right\}$.

Denote by E_W the set of all summable functions f with finite norm $|||f||| = \|\sum \|f\|_k e_k\|_W$, where W is an arbitrary reflexive Banach space with an unconditional basis (e_k) not containing isomorphic copies of l_p $(1 < p < \infty)$, for example Tsirelson's space.

For us it is important that E_W contains no isomorphic copies of l_2. From the proof of the result of Tokarev it follows that if we take $W = l_p$ $(p \neq 2)$, then E_W contains no isomorphic copies of l_2.

Theorem 3.5.2 *E_W does not have the Lyapunov property.*

Proof. Consider the measure $\mu : \Sigma \to E_W$ such that $\mu(A) = \chi_A$ and check that it is countably additive. Indeed, let $A_1 \supset A_2 \supset ...$ and $\operatorname{mes}(A_n) \underset{n\to\infty}{\longrightarrow} 0$. Take an arbitrary $\varepsilon > 0$. The series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ converges, so we can find $k_0 \in \mathbf{N}$ such that $\sum_{k=k_0+1}^{\infty} \frac{1}{n_k} \leq \frac{\varepsilon}{2}$. Select $N_0 \in \mathbf{N}$ so that $\sum_{k=1}^{k_0} n_k \varphi_p(\operatorname{mes}(A_n)) \leq \frac{\varepsilon}{2}$ for all $N \geq N_0$. Then it is plain that

$$|||\chi_{A_N}||| \leq \varepsilon$$

for all $N \geq N_0$.

Now we will show that the closure of $\mu(\Sigma)$ is not convex. Actually, it is clear that $\mu([0,1]) \in \overline{\mu(\Sigma)}$. On the other hand, $\frac{1}{2}\mu([0,1]) = \frac{1}{2}\chi_{[0,1]} \notin \overline{\mu(\Sigma)}$, as $\mu(\Sigma)$ involves only functions with values ± 1. This completes the proof. ■

Theorem 3.5.3 *There exists a Banach space X such that $l_2 \not\subset X$ and X fails the Lyapunov property.*

Proof. By Tokarev's results and Theorem 3.5.2, E_W is such a Banach space. ■