Chapter 2

Lyapunov Trees

In this chapter we introduce a generalization of the well known notions of type and cotype, b- and C-convexity for a special type of trees. Applying the methods known from the theory of type and cotype (see [15]) we will establish the connection between generalizations of type and b-convexity, cotype and C-convexity. From these properties we will see that if in a Banach space there exists any norm restriction from above (below) on weakly null sequences then there exists a "p-type estimation" from above (below), which is necessary for the proving of the Lyapunov property. The results obtained in this chapter will be used in Chapter 3.

First we introduce the notion of Lyapunov convergence. It will be convenient for proving a generalization of the Lyapunov theorem to some Banach spaces that will be considered in the next chapter.

Definition 2.0.1 *Let X be a Banach space, $\widetilde{T} \in L(X, c_0)$. By the c_0-$\widetilde{T}$-weak topology on X we will mean the topology with the following determining system of neighborhoods of zero:*

$$\left\{x \in X : \left\|\widetilde{T}x\right\|_\infty < \varepsilon,\ |f_i(x)| < \varepsilon,\ i = 1, ..., n\right\},$$

where $\varepsilon > 0$, $n \in \mathbf{N}$, and $f_i \in X^*$ $(i = 1, ..., n)$.

It is not difficult to see that this topology determines the convergence: x_n converges to zero if and only if $x_n \xrightarrow[n\to\infty]{w} 0$ and $\left\|\widetilde{T}x_n\right\|_\infty \xrightarrow[n\to\infty]{} 0$. Denote this convergence by c_0-$\widetilde{T}$-weak convergence.

Definition 2.0.2 *Let X be a Banach space. By a Lyapunov topology on X we will mean the weak topology or a c_0–$\widetilde{T}$-weak topology on X.*

In the sequel we fix once and for all a Lyapunov topology τ.

Definition 2.0.3 *A Lyapunov tree X^n of length n in a Banach space X is a family of sequences $\{x_{m_1,\ldots,m_i}\}_{i=1\ m_i\in\mathbf{N}}^{n} \subset X$ such that for given $0 \leq i_i < i_2 \leq n$ for every fixed $m_1, \ldots, m_{i_1}$*

$$x_{m_1,\ldots,m_{i_1},m_{i_1+1},\ldots,m_{i_2}} \xrightarrow[m_{i_1+1},\ldots,m_{i_2}\to\infty]{\tau} 0.$$

2.1 Lyapunov tree cotype

Definition 2.1.1 *We will say that a Banach space X has Lyapunov tree cotype p with constant C if for every $n \in \mathbf{N}$ and for every Lyapunov tree X^n there exist numbers $m_1^0, \ldots, m_n^0$ such that*

$$\left\| \sum_{i=1}^{n} x_{m_1^0,\ldots,m_i^0} \right\| \geq C \left(\sum_{i=1}^{n} \inf_{m_1,\ldots,m_i} \|x_{m_1,\ldots,m_i}\|^p \right)^{1/p}.$$

Example 2.1.2 *By Lemma 1.4.10 for every $1 \leq p < \infty$ the space l_p has Lyapunov tree cotype p with constant 1 for the weak topology.*

Definition 2.1.3 *A Banach space X is called Lyapunov tree C-convex if there exists $n \in \mathbf{N}$ such that*

$$C^{\tau}(n,\, X) = \inf \left\{ \sup_{m_1,\ldots,m_n} \left\| \sum_{i=1}^{n} x_{m_1,\ldots,m_i} \right\| \right\} > 1,$$

where the infimum is taken over all Lyapunov trees of length n with $\|x_{m_1,\ldots,m_i}\| \geq 1$. (Recall that τ denotes a Lyapunov topology.)

Remark 2.1.4 *If $\|x_{m_1,\ldots,m_i}\| \geq \alpha$ for all elements of a Lyapunov tree X^n, then we have*

$$\sup_{m_1,\ldots,m_n} \left\| \sum_{i=1}^{n} x_{m_1,\ldots,m_i} \right\| \geq \alpha \cdot C^{\tau}(n,\, X).$$

Lemma 2.1.5 *The numbers $C^{\tau}(n, X)$ generate a semimultiplicative sequence, i.e.*

$$C^{\tau}(n \cdot l, X) \geq C^{\tau}(n, X) \cdot C^{\tau}(l, X).$$

Proof. Let us consider a Lyapunov tree $\{x_{m_1,\ldots,m_i}\}_{i=1\ m_i \in \mathbf{N}}^{n \cdot l}$ with $\|x_{m_1,\ldots,m_i}\| \geq 1$ and take an arbitrary $\delta > 0$. Introduce the Lyapunov tree of length n

$$\{x_{m_1,\ldots,m_i}\}_{i=1\ m_i \in \mathbf{N}}^{n}.$$

In accordance with the definition of $C^{\tau}(n, X)$ we can choose numbers $m_1^1, \ldots, m_n^1$ such that $\left\| \sum\limits_{i=1}^{n} x_{m_1^1,\ldots,m_i^1} \right\| \geq C^{\tau}(n, X) - \delta$. Denote $y_1 = \sum\limits_{i=1}^{n} x_{m_1^1,\ldots,m_i^1}$. Further, consider the Lyapunov tree of length n

$$\{x_{m_1,\ldots,m_i}\}_{i=1\ m_i = m_i^1 + 1}^{n\ \ \infty}$$

and select $m_1^2, \ldots, m_n^2$ such that $\left\| \sum\limits_{i=1}^{n} x_{m_1^2,\ldots,m_i^2} \right\| \geq C^{\tau}(n, X) - \delta$. Put $y_2 = \sum\limits_{i=1}^{n} x_{m_1^2,\ldots,m_i^2}$. Continuing this process, we obtain the sequence $y_{s_1} = \sum\limits_{i=1}^{n} x_{m_1^{s_1},\ldots,m_i^{s_1}}$ such that

$$\|y_{s_1}\| \geq C^{\tau}(n, X) - \delta \text{ for all } s_1 \text{ and } y_{s_1} \xrightarrow[s_1 \to \infty]{\tau} 0.$$

For every s_1 we consider the Lyapunov tree

$$\left\{x_{m_1^{s_1},\ldots,m_n^{s_1},\, m_{n+1},\ldots,m_i}\right\}_{i=n+1\ m_i \in \mathbf{N}}^{2n}.$$

By analogy with above we construct a sequence

$$y_{s_1,s_2} = \sum_{i=n+1}^{2n} x_{m_1^{s_1},\ldots,m_n^{s_1},\, m_{n+1}^{s_2},\ldots,m_i^{s_2}} \quad (s_2 \in \mathbf{N})$$

such that

$$\|y_{s_1,s_2}\| \geq C^{\tau}(n, X) - \delta \text{ for all } s_2 \text{ and } y_{s_1,s_2} \xrightarrow[s_2 \to \infty]{\tau} 0.$$

It is not difficult to see that $y_{s_1,s_2} \xrightarrow[s_1,s_2\to\infty]{\tau} 0$. Continuing this process, we get the Lyapunov tree

$$\{y_{s_1,\ldots,s_i}\}_{i=1\ s_i\in\mathbf{N}}^{l}$$

satisfying the condition

$$\|y_{s_1,\ldots,s_i}\| \geq C^{\tau}(n,\,X)-\delta.$$

Then, in accordance with the Remark 2.1.4, we infer

$$\sup_{s_1,\ldots,s_l}\left\|\sum_{i=1}^{l} y_{s_1,\ldots,s_i}\right\| \geq \left(C^{\tau}(n,\,X)-\delta\right)\cdot C^{\tau}(l,\,X)\,.$$

As δ is arbitrary, the required inequality is proved. ■

Lemma 2.1.6 *Let X^n be a Lyapunov tree of a Banach space X. Then for every $k=1,\ldots,n$ the following inequality holds*

$$\sup_{m_1,\ldots,m_n}\left\|\sum_{i=1}^{n} x_{m_1,\ldots,m_i}\right\| \geq \sup_{m_1,\ldots,m_k}\left\|\sum_{i=1}^{k} x_{m_1,\ldots,m_i}\right\|.$$

Proof. Take $k\in\{1,\ldots,n\}$ and an arbitrary set of indices $m_1^0,\ldots,m_k^0$. Applying Mazur's theorem (which states that if $a_i, a\in X$ and $a_i \xrightarrow[i\to\infty]{\tau} a$ then $\underline{\lim}\|a_i\|\geq\|a\|$) we obtain

$$\begin{aligned}
&\sup_{m_1,\ldots,m_n}\left\|\sum_{i=1}^{n} x_{m_1,\ldots,m_i}\right\| \\
&\quad\geq \sup_{m_{k+1},\ldots,m_n}\left\|\sum_{i=1}^{k} x_{m_1^0,\ldots,m_i^0}+\sum_{i=k+1}^{n} x_{m_1^0,\ldots,m_k^0,m_{k+1},\ldots,m_i}\right\| \\
&\quad\geq \underline{\lim}_{m_{k+1},\ldots,m_n\to\infty}\left\|\sum_{i=1}^{k} x_{m_1^0,\ldots,m_i^0}+\sum_{i=k+1}^{n} x_{m_1^0,\ldots,m_k^0,m_{k+1},\ldots,m_i}\right\| \\
&\quad\geq \left\|\sum_{i=1}^{k} x_{m_1^0,\ldots,m_i^0}\right\|.
\end{aligned}$$

This completes the proof. ■

Lemma 2.1.7 *Let X be a Banach space, $\{x_n\}_{n=1}^{\infty}$, $x \in X$, and $x_n \xrightarrow[n\to\infty]{\tau} 0$. Then for every $\varepsilon > 0$ there exists an $n \in \mathbf{N}$ such that*

$$\|x + x_n\| \geq \frac{1}{2}\|x_n\| - \varepsilon.$$

Proof. Let $\varepsilon > 0$. Take a functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Choose $n \in \mathbf{N}$ such that $f(x_n) \leq 2\varepsilon$, then

$$\|x + x_n\| \geq |f(x + x_n)| \geq \|x\| - 2\varepsilon.$$

It follows that

$$\|x_n\| \leq \|x + x_n\| + \|x\| \leq 2\|x + x_n\| + 2\varepsilon.$$

The lemma is proved. ■

Theorem 2.1.8 *Let X be a Banach space. If there exists some n_1 such that $C^{\tau}(n_1, X) > n_1^{1/p}$, where $1 \leq p < \infty$, then X has Lyapunov tree cotype p.*

Proof. Let us choose $\varepsilon > 0$ such that $C^{\tau}(n_1, X) \geq n_1^{1/(p-\varepsilon)}$. Put $n_k = n_1^k$ for $k = 0, 1, 2....$ By Lemma 2.1.5, $C^{\tau}(n_k, X) \geq n_1^{k/(p-\varepsilon)}$. Denote $D = \sum\limits_{k=0}^{\infty} n_1^{-\varepsilon k}$. We will show that X has Lyapunov tree cotype p with the constant $\frac{1}{2}D^{-1/(p-\varepsilon)}$.

Take an arbitrary Lyapunov tree X^n. Define

$$\alpha_i = \inf_{m_1,\ldots,m_i} \|x_{m_1,\ldots,m_i}\| \quad (i{=}1, ..., n).$$

Decompose the set of indices $\{1, ..., n\}$ into the union of mutually disjoint sets

$$A_k = \left\{ j : \frac{\left(\sum\limits_{i=1}^{n} \alpha_i^p\right)^{1/p}}{n_k} \geq \alpha_j > \frac{\left(\sum\limits_{i=1}^{n} \alpha_i^p\right)^{1/p}}{n_{k+1}} \right\},$$

where $k = 0, 1, 2, ...$ Define a_k as the number of elements of A_k. Then

$$\sum_{j=1}^{n} \alpha_j^p = \sum_{k=0}^{\infty} \sum_{j \in A_k} \alpha_j^p \leq \sum_{k=0}^{\infty} a_k \cdot \frac{\sum\limits_{i=1}^{n} \alpha_i^p}{n_k^p}.$$

Consequently

$$\sum_{k=0}^{\infty} \frac{a_k}{n_k^p} \geq 1.$$

It follows that there exists a number $k = k_0$ such that $a_{k_0} \geq \left[\frac{n_{k_0}^{p-\varepsilon}}{D}\right]$. Indeed, if $a_k < \left[\frac{n_k^{p-\varepsilon}}{D}\right]$ for all k, then

$$\frac{a_k}{n_k^p} < \frac{n_k^{p-\varepsilon}}{D \cdot n_k^p} = \frac{1}{D \cdot n_1^{\varepsilon k}} \text{ and } \sum_{k=0}^{\infty} \frac{a_k}{n_k^p} < 1.$$

Fix $\theta > 0$ and construct a tree of length a_{k_0} as follows. Reindex the elements of the set $A_{k_0} = \left\{j_1 < ... < j_{a_{k_0}}\right\}$. Fix arbitrary $j_1 - 1$ indices: $m_1^1, ..., m_{j_1-1}^1$. By Lemma 2.1.7 we can choose a number $m_{j_1}^1$ such that

$$\left\| \sum_{i=1}^{j_1} x_{m_1^1,...,m_i^1} \right\| \geq \frac{1}{2} \left\| x_{m_1^1,...,m_{j_1}^1} \right\| - \theta.$$

Put $y_1 = \sum_{i=1}^{j_1} x_{m_1^1,...,m_i^1}$. Take $m_1^2, ..., m_{j_1-1}^2$ satisfying the condition $m_1^2 > m_1^2$. Using Lemma 2.1.7, we find a number $m_{j_1}^2$ such that

$$\left\| \sum_{i=1}^{j_1} x_{m_1^2,...,m_i^2} \right\| \geq \frac{1}{2} \left\| x_{m_1^2,...,m_{j_1}^2} \right\| - \theta.$$

Denote $y_2 = \sum_{i=1}^{j_1} x_{m_1^2,...,m_i^2}$. Continuing this process we obtain the sequence $\{y_{s_1}\}_{s_1=1}^{\infty}$ such that $y_{s_1} \xrightarrow[s_1\to\infty]{\tau} 0$. For each s_1 we fix arbitrary $m_{j_1+1}^1, ..., m_{j_2-1}^1$ and select $m_{j_2}^1$ so that

$$\left\| \sum_{i=j_1+1}^{j_2} x_{m_1^{s_1},...,m_{j_1}^{s_1},m_{j_1+1}^1,...,m_i^1} \right\| \geq \frac{1}{2} \left\| x_{m_1^{s_1},...,m_{j_1}^{s_1},m_{j_1+1}^1,...,m_{j_2}^1} \right\| - \theta.$$

Define $y_{s_1,1} = \sum_{i=j_1+1}^{j_2} x_{m_1^{s_1},...,m_{j_1}^{s_1},m_{j_1+1}^1,...,m_i^1}$. By the same method as above we get the sequence $y_{s_1,s_2} \xrightarrow[s_2\to\infty]{\tau} 0$. It is clear that

$y_{s_1,s_2} \underset{s_1,s_2\to\infty}{\overset{\tau}{\longrightarrow}} 0$. Continuing this process we construct the Lyapunov tree $\{y_{s_1,\dots,s_i}\}_{i=1}^{a_{k_0}}{}_{s_i\in\mathbf{N}}$. Applying Lemma 2.1.6, we have

$$\begin{aligned}
\sup_{m_1,\dots,m_n}\left\|\sum_{i=1}^{n} x_{m_1,\dots,m_i}\right\| &\geq \sup_{m_1,\dots,m_{i_{a_{k_0}}}}\left\|\sum_{i=1}^{i_{a_{k_0}}} x_{m_1,\dots,m_i}\right\| \\
&\geq \sup_{s_1,\dots,s_{a_{k_0}}}\left\|\sum_{i=1}^{a_{k_0}} y_{s_1,\dots,s_i}\right\| \\
&\geq \inf_{s_1,\dots,s_{a_{k_0}}}\|y_{s_1,\dots,s_i}\|\cdot C^{\tau}\left(a_{k_0},X\right) \\
&\geq \left(\tfrac{1}{2}\min_{j\in A_{k_0}}\alpha_j-\theta\right)\cdot C^{\tau}\left(\left[\frac{n_{k_0}^{p-\varepsilon}}{D}\right],X\right).
\end{aligned}$$

Note that $D = n_1^{\ln D/\ln n_1}$ and put $\gamma = \ln D/\ln n_1$, then

$$\begin{aligned}
C^{\tau}\left(\left[\frac{(n_{k_0})^{p-\varepsilon}}{D}\right],X\right) &= C^{\tau}\left(\left[n_1^{k_0(p-\varepsilon)-\gamma}\right],X\right)\geq C^{\tau}\left(n_1^{[k_0(p-\varepsilon)-\gamma]},X\right) \\
&\geq n_1^{\frac{[k_0(p-\varepsilon)-\gamma]}{p-\varepsilon}}\geq n_1^{\frac{[-\gamma]}{p-\varepsilon}}\geq D^{-1/(p-\varepsilon)}.
\end{aligned}$$

As θ is arbitrary and by the definition of A_k, we obtain the required inequality

$$\sup_{m_1,\dots,m_n}\left\|\sum_{i=1}^{n} x_{m_1,\dots,m_i}\right\| \geq \frac{1}{2}D^{-1/(p-\varepsilon)}\left(\sum_{i=1}^{n}\alpha_i^p\right)^{1/p}.$$

The theorem is proved. ■

Theorem 2.1.9 *If a Banach space X has Lyapunov tree cotype $p\in[1,2)$, then X has the Lyapunov property.*

Proof. Fix $N\in\mathbf{N}$ for the present, and let $k=[N/\ln N]$. We assume that X fails the Lyapunov property and use Lemma 1.4.7. Let $\Omega,\tau,\lambda,\varepsilon$, and $T: L_\infty\to X$ be as in that lemma, with $\lambda(\Omega)=1$ and $\|T\|\leq 1$. Note that we can select a σ-algebra Σ (see the proof of Lemma 1.5.3) such that $L_1(\Omega,\Sigma,\lambda)$ is separable. We shall prove by induction that there exist sequences $\{r_{n_1,\dots,n_i}\}_{n_i\in\mathbf{N}}\subset L_\infty(\Omega,\Sigma,\lambda)$ $(i=1,2,\dots)$ such that for every j $\{Tr_{n_1,\dots,n_i}\}_{i=1}^{j}{}_{n_i\in\mathbf{N}}$ generates a Lyapunov tree and for any $\{n_i,\dots,n_j\}$ the functions $\{r_{n_1,\dots,n_i}\}_{i=1}^{j}$ are jointly equidistributed with the $\{s_i\}_{i=1}^{j}$ in Lemma 1.4.9.

Indeed, everything is obvious for $j = 1$. Suppose that the assertion is already proved for $j = m$, and that $\{r_{n_1,...,n_i}\}_{i=1\ n_i\in\mathbf{N}}^{j}$ have been constructed that satisfy the required condition with $j = m$. For every $\{n_1, ..., n_m\}$ we consider the set $A_{n_1,...,n_m} = \left\{\omega \in \Omega : \left|\sum\limits_{i=1}^{m} r_{n_1,...,n_i}(\omega)\right| < \sqrt{N}\right\}$ and a sequence $\left\{z_{n_1,...,n_m,n_{m+1}}\right\}_{n_{m+1}=1}^{\infty}$ of functions independent on $A_{n_1,...,n_m}$, equal to zero off $A_{n_1,...,n_m}$, and taking the values ± 1 with probability $\lambda(A_{n_1,...,n_m})/2$ such that $Tz_{n_1,...,n_m,n_{m+1}} \underset{n_{m+1}\to\infty}{\xrightarrow{\tau}} 0$. For any $\{n_1, ..., n_{m+1}\}$ the system $\left\{\{r_{n_1,...,n_i}\}_{i=1}^{m}, z_{n_1,...,n_m,n_{m+1}}\right\}$ is equidistributed with $\{s_i\}_{i=1}^{m+1}$. The sequence $z_{n_1,...,n_m,n_{m+1}}$ tends to zero in the weak* topology as $n_{m+1}\to\infty$; hence $Tz_{n_1,...,n_m,n_{m+1}} \underset{n_{m+1}\to\infty}{\xrightarrow{w}} 0$. As $L_1(\Omega, \Sigma, \lambda)$ is separable, the weak* topology on $L_\infty(\Omega, \Sigma, \lambda)$ is metrizable on bounded sets. Let ρ be a metric defining the weak* convergence on the unit ball of $L_\infty(\Omega, \Sigma, \lambda)$. Then for every set $\{n_1, ..., n_m\}$ we can choose a subsequence $\left\{r_{n_1,...,n_m,n_{m+1}}\right\}_{n_{m+1}=1}^{\infty}$ of $\left\{z_{n_1,...,n_m,n_{m+1}}\right\}_{n_{m+1}=1}^{\infty}$ such that

$$\sup_{n_{m+1}} \rho\left(r_{n_1,...,n_m,n_{m+1}}, 0\right) \leq \frac{1}{\sum_{i=1}^{m} n_i}$$

(and $\left\|\widetilde{T}Tr_{n_1,...,n_m,n_{m+1}}\right\|_\infty \leq 1/\sum_{i=1}^{m} n_i$ in the case τ is a c_0-$\widetilde{T}$-weak topology). Hence, for given $0 \leq i_1 < i_2 \leq m+1$ and for every fixed $n_1, ..., n_{i_1}$, we obtain

$$\rho\left(r_{n_1,...,n_{i_1},n_{i_1+1},...,n_{i_2}}, 0\right) \underset{n_{i_1+1},...,n_{i_2}\to\infty}{\longrightarrow} 0$$

(and $\left\|\widetilde{T}Tr_{n_1,...,n_m,n_{m+1}}\right\|_\infty \underset{n_{i_1+1},...,n_{i_2}\to\infty}{\longrightarrow} 0$ in the case τ is a c_0-$\widetilde{T}$-weak topology). It follows that $\{Tr_{n_1,...,n_i}\}_{i=1\ n_i\in\mathbf{N}}^{m+1}$ is a Lyapunov tree.

We now use the fact just proved. Suppose that $j = k$. Since the Banach space X has Lyapunov tree cotype p with constant C, there exist numbers $n_1^0, ..., n_k^0$ such that

$$\left\|\sum_{i=1}^{k} Tr_{n_1^0,...,n_i^0}\right\|^p \geq C^p \left(\sum_{i=1}^{k} \inf_{n_1,...,n_i} \|Tr_{n_1,...,n_i}\|^p\right).$$

In view of the condition (ii) from the Lemma1.4.7, which T satisfies by construction, it follows that $\|Tr_{n_1,...,n_i}\| \geq \varepsilon \cdot \lambda(\operatorname{supp} r_{n_1,...,n_i}) \geq$

$\varepsilon \cdot \lambda\,(\operatorname{supp} r_{n_1,\ldots,n_k})$ for $i \le k$. Since $\{r_{n_1,\ldots,n_i}\}_{i=1}^k$ and $\{s_i\}_{i=1}^k$ are equidistributed, this gives us by the choice of k that

$$\left(\sqrt{N}+1\right)^p \|T\|^p + 1 \ge \left\|\sum_{i=1}^k T r_{n_1^0,\ldots,n_i^0}\right\|^p \ge k\varepsilon^p C^p \left(P\left(s_k \ne 0\right)\right)^p \ge$$
$$\ge \left(N/\ln N - 1\right)\varepsilon^p C^p \left(P\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right)\right)^p.$$

By Lemma 1.4.9, the last factor tends to 1; hence this inequality cannot hold for large N. This contradiction completes the proof. ■

2.2 Lyapunov tree type

Definition 2.2.1 *We will say that a Banach space X has Lyapunov tree type $p > 1$ with constant C if for every $n \in \mathbf{N}$ and for every Lyapunov tree X^n there exist numbers $m_1^0, \ldots, m_n^0$ such that*

$$\left\|\sum_{i=1}^n x_{m_1^0,\ldots,m_i^0}\right\| \le C\left(\sum_{i=1}^n \sup_{m_1,\ldots,m_i} \|x_{m_1,\ldots,m_i}\|^p\right)^{1/p}.$$

Definition 2.2.2 *A Banach space X is called Lyapunov tree b-convex if there exists $n \in \mathbf{N}$ such that*

$$b^\tau(n,\,X) = \sup\left\{\inf_{m_1,\ldots,m_n}\left\|\sum_{i=1}^n x_{m_1,\ldots,m_i}\right\|\right\} < n,$$

where the supremum is taken over all Lyapunov trees X^n of length n with $\|x_{m_1,\ldots,m_i}\| \le 1$.

Remark 2.2.3 *If $\|x_{m_1,\ldots,m_i}\| \le \alpha$ for all elements of a Lyapunov tree X^n, then we have*

$$\inf_{m_1,\ldots,m_n}\left\|\sum_{i=1}^n x_{m_1,\ldots,m_i}\right\| \le \alpha \cdot b^\tau(n,\,X).$$

Lemma 2.2.4 *The numbers $b^\tau(n,\,X)$ generate a semimultiplicative sequence, i.e.*

$$b^\tau(n \cdot l,\,X) \le b^\tau(n,\,X) \cdot b^\tau(l,\,X).$$

Proof. The lemma is proved by analogy with Lemma 2.1.5. ■

Theorem 2.2.5 *Let X be a Lyapunov tree b-convex Banach space. If there exists n_1 such that $b^\tau(n_1, X) < n_1^{1/p}$, where $1 < p < \infty$, then X has the Lyapunov tree type p.*

Proof. Let us choose $n_1 > 1$ and $\varepsilon > 0$ such that $b^\tau(n_1, X) \leq n_1^{1/(p+\varepsilon)}$. Put $n_k = (n_1)^k$ $(k = 0, 1, 2, ...)$. By Lemma 2.2.4, $b^\tau(n_k, X) \leq n_1^{k/(p+\varepsilon)}$.

Take an arbitrary Lyapunov tree X^n. Define

$$\alpha_i = \sup_{m_1,...,m_i} \|x_{m_1,...,m_i}\| \quad (i = 1, ..., n).$$

Decompose the set of indices $\{1, ..., n\}$ into the union of mutually disjoint sets

$$A_k = \left\{ j : \frac{\left(\sum\limits_{i=1}^{n} \alpha_i^p\right)^{1/p}}{n_k} > \alpha_j \geq \frac{\left(\sum\limits_{i=1}^{n} \alpha_i^p\right)^{1/p}}{n_{k+1}} \right\},$$

where $k = 0, 1, 2, ...$ Define a_k as the number of elements of A_k. Then

$$\sum_{j=1}^{n} \alpha_j^p = \sum_{k=0}^{\infty} \sum_{j \in A_k} \alpha_j^p \geq \sum_{k=0}^{\infty} a_k \cdot \frac{\sum\limits_{i=1}^{n} \alpha_i^p}{n_{k+1}^p}.$$

Consequently,

$$\sum_{k=0}^{\infty} \frac{a_k}{n_{k+1}^p} \leq 1.$$

It follows that for every k the inequality $a_k \leq n_1^{[p(k+1)]+1}$ holds. Indeed, if there exists a number k_0 such that $a_{k_0} > n_{k_0}^{[p(k_0+1)]+1}$, then

$$\frac{a_{k_0}}{n_{k_0+1}^p} > 1 \text{ and } \sum_{k=0}^{\infty} \frac{a_k}{n_{k+1}^p} > 1.$$

It is not difficult to see that for every k and for every fixed numbers $\{m_i\}_{i \notin A_k}$ the sequences $\{x_{m_1,...,m_i}\}_{i \in A_k \; m_i \in \mathbf{N}}$ generate a Lyapunov tree of length a_k. It follows that for every fixed set $\{m_i\}_{i \notin A_k}$ and for every $\delta > 0$ we can select $\{m_i\}_{i \in A_k}$ such that

$$\left\| \sum_{i \in A_k} x_{m_1,...,m_i} \right\| \leq \sup_{m_i : i \in A_k} \|x_{m_1,...,m_i}\| \cdot (b^\tau(a_k, X) + \delta) \qquad (2.1)$$

Consequently we can find numbers $m_1^0, ..., m_n^0$ such that condition (2.1) holds for all k. Then we have

$$\begin{aligned}\inf_{m_1,\ldots,m_n}\left\|\sum_{i=1}^{n} x_{m_1,\ldots,m_i}\right\| &\le \left\|\sum_{i=1}^{n} x_{m_1^0,\ldots,m_n^0}\right\| \\ &\le \sum_{k=0}^{\infty}\frac{\left(\sum_{i=1}^{n}\alpha_i^p\right)^{1/p}}{n_k}\cdot\left(b^{\tau}\left(a_k,X\right)+\delta\right).\end{aligned}$$

Estimate

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{b^{\tau}(a_k,X)}{n_k} &\le \sum_{k=0}^{\infty}\frac{b^{\tau}\left(n_1^{[p(k+1)]+1},X\right)}{n_k} \le \sum_{k=0}^{\infty} n_1^{\frac{[p(k+1)]+1}{p+\varepsilon}-k} \\ &\le n_1^{1+1/(p+\varepsilon)}\sum_{k=0}^{\infty} n_1^{-\varepsilon/(p+\varepsilon)(k+1)}.\end{aligned}$$

It is clear that the series converges. Denote its sum by C. As δ is arbitrary, we obtain the required inequality

$$\inf_{m_1,\ldots,m_n}\left\|\sum_{i=1}^{n} x_{m_1,\ldots,m_i}\right\| \le C\left(\sum_{i=1}^{n}\alpha_i^p\right)^{1/p}.$$

The theorem is proved. ■

Theorem 2.2.6 *If a Banach space X has Lyapunov tree type $p > 2$ with constant C, then X has the Lyapunov property.*

Proof. Fix $N \in \mathbf{N}$ for the present, and let $k = [N \ln N]$. We assume that X has not the Lyapunov property and use Lemma 1.4.7. Let $\Omega, \tau, \lambda, \varepsilon$, and $T : L_\infty \to X$ be as in that lemma, with $\lambda(\Omega) = 1$ and $\|T\| \le 1$. By analogy with the proof of Theorem 2.1.9 we construct a Lyapunov tree of length k and choose $n_1^0, ..., n_k^0$ such that

$$\left\|\sum_{i=1}^{k} Tr_{n_1^0,\ldots,n_i^0}\right\|^p \le C^p\left(\sum_{i=1}^{k}\inf_{n_1,\ldots,n_i}\left\|Tr_{n_1,\ldots,n_i}\right\|^p\right) \le C^p\left\|T\right\|^p\cdot k.$$

The rest of the proof is similar to the proof of Theorem 1.4.11. ■