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Some Basic Boundary Value Problems for Complex Partial Differential Equations in Quarter Ring and Half Hexagon

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Abstract

This dissertation is devoted to the investigation of some boundary value problems for complex partial differential equations in a quarter ring and a half hexagon. The method of reflection is used among the main tools to obtain the Schwarz-Poisson representation formula and the harmonic Green function for both domains. For the quarter ring the related Schwarz, Dirichlet and Neumann problems for the Cauchy-Riemann equation are solved explicitly. From the harmonic Green function for this domain the Neumann function is derived satisfying certain prescribed properties. By use of the Green and Neumann functions the corresponding Dirichlet and Neumann problems for the Poisson equation are solved.

Similarly, using the reflection points the Schwarz-Poisson representation formula is found for the half hexagon and the solution of the Schwarz problem for the Cauchy-Riemann equation is provided. The harmonic Green function obtained for this domain allows to solve the related harmonic Dirichlet problem.

Due to the fact that both domains are non-regular, special attention is paid to the boundary behavior in the corner points.

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Introduction

A method of solution is perfect if we can foresee
from the start, and even prove, that following
that method we shall attain our aim
Gotfried Wilhelm von Leibniz

The theory of complex boundary value problems originating from the work of B.Riemann [35] and D.Hilbert [31] and developed by F.D.Gakhov [28], I.N.Muskhelishvili [33], I.N.Vekua [42], W.Haack and W.Wendland [30] and others is still investigated. On one hand a theory for complex model equations of arbitrary order is explored after the basic fundamental solutions were approached [15]. On the other hand explicit solutions are found in many different particular domains. The explicit solutions are important not only for applications in engineering, mathematical physics, fluid dynamics etc., but also influence the general theory for arbitrary domains. Complex model equations are simple inhomogeneous equations with a differential operator being the product of powers of the Cauchy-Riemann $\partial_{\overline{z}}$ and anti-Cauchy-Riemann operator ∂_z, i.e. $\partial_z^k \partial_{\overline{z}}^l w = f,\ k, l \in \mathbb{N}$. There are three different basic model operators of this kind, the Cauchy-Riemann operator $\partial_{\overline{z}}$, the Laplace operator $\partial_z \partial_{\overline{z}}$ and the Bitsadze operator $\partial_{\overline{z}}^2$.

The basic boundary value problems for complex partial differential equations have been considered for different particular domains, see [2, 3, 4, 5, 6, 7, 8, 12, 13, 14, 17, 18, 19, 20, 21, 22, 25, 26, 27, 29, 32, 34, 41, 43, 44].

They are explicitly solved as well for the inhomogeneous Cauchy-Riemann equation as for the Poisson equation. Among these particular domains the case of the unit disc is mostly considered and explained in classical textbooks on complex analysis and partial differential equations. In particular, the Schwarz and Poisson kernels for the unit disc are used to generalize the concept of the kernel functions and integral representations for solutions. The Green and Neumann functions for the unit disc, see e.g [5, 6], are given explicitly and their general concepts have been developed. The conformal invariance of these functions allows to calculate the harmonic Green and Neumann functions for other domains, which are conformal equivalent to the unit disc. The method of conformal invariance is mainly of theoretical value. In order to find the Green and Neumann functions in explicit form the conformal mapping has to be expressed. Its existence is assured by the Riemann mapping theorem, but a general method for its computation is not available. For certain polygonal domains the Schwarz-Christoffel formula provides a method to compute the conformal mapping to the unit disc. However, this formula is not proper for practical problems since it involves elliptic functions.

Polygonal domains which include corner points are irregular. Boundary value problems in such domains are always delicate and the studying of the behavior of solutions in the neighborhood of the corner points is in general difficult. Among the irregular domains studied recently are those the boundaries of which consist of pieces of circles and the straight lines. They are related to polygonal domains, where some of the boundary sides are replaced by circular arcs e.g. half discs, quarter discs, half rings, disc sectors, lenses and lunes, see [18, 22, 26, 43]. Some polygonal domains are studied in [1, 2, 14, 19, 29, 44] etc. Multiply connected domains are also considered, e.g. ring domains, in particular a concentric ring domain [20, 41].

For certain of the domains mentioned above a method for constructing the Schwarz kernel and the Green and Neumann functions exists. It is well explained e.g. in [20, 21, 22]. The method uses reflection of the domain at all parts of the boundary. It can be applied if the whole complex plane can be covered by continuously repeated reflections, see e.g. [1, 18, 19, 20, 22, 41, 43, 44]. This method is applicable for a quarter ring and a half hexagon and they are subjects of investigation of the present thesis.

The main tools for treating the boundary value problems for the Cauchy-Riemann equation is the Cauchy-Pompeiu representation formula. This formula is to be adjusted to the boundary value conditions. For the boundary value problems for the Poisson equation the Green and Neumann representation formulas are exploited. The method is based on the explicit form of the Green and Neumann functions. The solutions of boundary value problems are considered in the distributional sense. The reason for this fact is that a particular solution of the inhomogeneous Cauchy-Riemann equation is given by the Pompeiu operator which has weak derivatives with respect to z and $\overline{z}$ in proper functions spaces, see [42] for the extensive explanation of the operator. This Pompeiu operator is a kind of potential operator of the Cauchy-Riemann operator. The particular solutions of the Poisson equation are given by area integral operators with the Green and Neumann functions as kernels. These are also potential operators. Other potential operators of this kind for higher order model operators are developed for certain simple domains and the boundary value problems are solved, see [10, 11, 23, 24].

In the present thesis the Cauchy-Riemann equation $\partial_{\overline{z}}w = f$ and the Poisson equation $\partial_z\partial_{\overline{z}}w = f$ in two particular domains are studied, namely a quarter ring and a half hexagon. The boundary of the quarter ring consists of two straight segments and two circular arcs. Reflections at the segments gives the covering of a ring domain and continued reflections at the boundary circles of the ring produces a covering of the punctured complex plane. Similarly, the half hexagon is reflected to the whole hexagon which provides a parqueting of the whole plane as well. In this way for both domains the required kernel functions are obtained and thus the Schwarz, Dirichlet and Neumann problems are explicitly solved. Particular attention is paid to the behavior of the solutions in the corner points. While the area integral, the Pompeiu operator, under suitable assumptions is continuous in the entire plane, the boundary behavior of the boundary integral has to be investigated, especially in the corner points. Thus the Poisson kernel is needed. It turns out to have different forms on the different parts of the boundary of these domains. It is shown that continuity of the boundary function produces solutions which behave continuously in these points.

In the first part of the thesis the boundary value problems for the quarter ring are considered. At first the Schwarz-Pompeiu representation formula and the Schwarz kernel are obtained. For the inhomogeneous Cauchy-Riemann equation the Schwarz and Dirichlet problems and also the Neumann problem for the homogeneous Cauchy-Riemann equation are solved. Further, the harmonic Green function is found and the solution of the related Dirichlet problem for the Poisson equation is presented. The harmonic Neumann function is constructed by multiplying all the terms from the Green function adding some factor in order to get convergence. By the Neumann representation formula the related Neumann problem for the quarter ring is explicitly solved and the solubility condition is explicitly given.

The second part of the thesis is devoted to boundary value problems for the half hexagon. By use of the reflection points the Schwarz-Pompeiu representation formula is reached and due to the different behavior on the boundary parts, three equivalent forms of the representation are presented and used to solve the Schwarz problem for the inhomogeneous Cauchy-Riemann equation. Similarly, the Green function is constructed with respect to the different boundary parts and the equality of the three forms obtained is shown. The related Dirichlet problem for the Poisson equation is solved and the solution is given in explicit form.

Chapter 1

Preliminaries

Let $\mathbb{C}$ be the complex plane of the variable $z = x + iy$. The real number x is called the *real part* of z and is written $x = \operatorname{Re} z$. The real number y is called the *imaginary part* of z and is written $y = \operatorname{Im} z$. The complex number $\overline{z} = x - iy$ is by definition the complex *conjugate* of z.

A *function* of the complex variable z is a rule that assigns a complex number to each z within some specified set D, D is called the *domain of definition* of the function. The collection of all possible values of the function is called the *range* of the function. A curve γ is a continuous complex-valued function $\gamma(t)$ defined for t in some interval $[a, b]$ in the real axis. The curve γ is *simple* if $\gamma(t_1) \neq \gamma(t_2)$, $a \leq t_1 < t_2 \leq b$ and it is *closed* if $\gamma(a) = \gamma(b)$. A curve is *smooth* if for $\gamma(t)$ exists $\gamma'(t)$ and it is continuous on $[a, b]$. A curve is *piecewise smooth* if it consists of a finite number of smooth curves and the end of one coincides with the beginning of the next. A domain D in $\mathbb{C}$ is called *regular* if it is bounded and its boundary ∂D is being a smooth curve.

Let the complex-valued function w be defined in $D \in \mathbb{C}$ and let u and v denote its real and imaginary parts: $w = u + iv$, where $u(x, y)$ and $v(x, y)$ are real-valued functions. The two partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.1}$$

are called the *Cauchy-Riemann* equations for the pair of functions u, v.

The function w is called *differentiable* at each point where the partial derivatives of u, v are continuous and satisfy the Cauchy-Riemann equations. A complex-valued function defined in $D \subset \mathbb{C}$ differentiable at every point of D is said to be *analytic (or holomorphic)* in D.
The partial differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are applied to the function w as

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \; \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

Defining the complex partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \; \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{1.2}$$

then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \; \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right).$$

The Cauchy-Riemann equation can be written as $\frac{\partial w}{\partial \overline{z}} = 0$. As this is the condition for w to be analytic, then w is independent of $\overline{z}$ in $D \in \mathbb{C}$.
The complex-valued function w, defined in $D \in \mathbb{C}$, is said to be *harmonic* if it is of the class C^2 and satisfies the Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

An analytic (holomorphic) function is harmonic. As it is from C^2 this follows from

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) = -\frac{\partial^2 u}{\partial y^2},$$

which proves that u is harmonic. Similarly can be proved for v to be harmonic and thus w is harmonic. As

$$\frac{\partial^2}{\partial \bar{z} \partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.3}$$

the Laplace equation can be written as $\frac{\partial^2 w}{\partial \bar{z} \partial z} = 0$.

The following formulas are used to solve the boundary value problems.

Definition 1.0.1. *[28]*
Let γ be a smooth closed contour in $\mathbb{C}$ and denote the domain within the contour as D^+, the interior domain; the exterior domain denoted as D^-. If $w(z)$ is an analytic function in D^+ and continuous in $D^+ \cup \gamma$, then

$$\frac{1}{2\pi i}\int\limits_{\gamma} w(\zeta)\frac{d\zeta}{\zeta - z} = \begin{cases} w(z), & z \in D^+, \\ 0, & z \in D^-. \end{cases} \tag{1.4}$$

And if $w(z)$ is analytic in D^- and continuous in $D^- \cup \gamma$, then

$$\frac{1}{2\pi i}\int\limits_{\gamma} w(\zeta)\frac{d\zeta}{\zeta - z} = \begin{cases} w(\infty), & z \in D^+, \\ -w(z) + w(\infty), & z \in D^-. \end{cases} \tag{1.5}$$

The integral presented in the left-hand side of (1.4) and (1.5) is the Cauchy integral.

Theorem 1.0.1. *(Cauchy's Theorem) [36]*
Let D be an open subset of $\mathbb{C}$, and let Γ be a contour contained with its interior in D. Then

$$\int\limits_{\Gamma} w(z)dz = 0$$

for every function w that is holomorphic in D.

This equality is valid as well for simply connected domains.

The definitions of some classes of functions are needed [42].

Let a function $f(z)$ and its partial derivatives up to the mth order be continuous in a domain D. The set of these functions is denoted by $C^m(D)$ and $C^m(\overline{D})$, where $\overline{D}$ is the closure of the domain D. By $C_\alpha(\overline{D})$ denote the set of all bounded functions $f(z)$ satisfying the inequality

$$|f(z_1) - f(z_2)| \leq H(f)|z_1 - z_2|^\alpha, \ 0 < \alpha \leq 1, \tag{1.6}$$

$$H(f) = sup_{z_1, z_2 \in \overline{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}, \tag{1.7}$$

where α is called the *Hölder* index of the function f.
Let a function $f(z)$ given in the domain D satisfy the inequality

$$\int\limits_{D'} |f(z)|^p dxdy < M_{D'}, \ p \geq 1,$$

where D' is an arbitrary closed (bounded) subset of the domain D and $M_{D'}$ is a constant depending on D'. The set of such functions is denoted by $L_p(D)$. The set of functions satisfying

$$L_p(f) \equiv L_p(f, \overline{D}) = \left(\int\limits_{D} |f(z)|^p dxdy\right)^{1/p} < \infty$$

is denoted by $L_p(\overline{D})$ and is being the Banach type space.
Let $f \in C^m(G)$, and let there exist a closed subset G_f of the set G, such that $f = 0$ outside G_f. The set of such functions is denoted by $D^0_m(G)$.

Definition 1.0.2. *[42] Let $f, g \in L_1(G)$. If f and g satisfy the relation*

$$\int\limits_G g\frac{\partial\varphi}{\partial\overline{z}}dxdy + \int\limits_G f\varphi dxdy = 0$$

$$\left(\int\limits_G g\frac{\partial\varphi}{\partial z}dxdy + \int\limits_G f\varphi dxdy = 0\right), \tag{1.8}$$

where φ is an arbitrary function of the class $D_1^0(G)$, f is said to be the generalized (weak) derivative of g with respect to $\overline{z}$ (to z).

This definition is a particular case of the general one, which can be found in e.g. [38]. We take the definition of the Sobolev space as

Definition 1.0.3. *[38] The linear manifold of all summable functions $\varphi(x_1, x_2, ..., x_n)$ having on a finite domain D all generalized derivatives of order l summable to power $p > 1$, are called $W^{l,p}$:*

$$\frac{\partial^l\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}} \in L_p; \sum \alpha_i = l.$$

Theorem 1.0.2. *(Gauss Theorem (real form)) [5]*
Let $(f, g) \in C^1(D; \mathbb{R}^2) \cap C(\overline{D}; \mathbb{R}^2)$ be a differentiable real vector field in a regular domain $D \subset \mathbb{R}^2$ then

$$\int\limits_D (f_x(x, y) + g_y(x, y))dxdy = -\int\limits_{\partial D}(f(x, y)dy - g(x, y)dx).$$

Theorem 1.0.3. *(Gauss theorem (complex form)) [5], [6]*
Let $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$ in a regular domain D of the complex plane $\mathbb{C}$ then

$$\int\limits_D w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int\limits_{\partial D} w(z)dz, \tag{1.9}$$

$$\int\limits_D w_z(z)dxdy = -\frac{1}{2i}\int\limits_{\partial D} w(z)d\overline{z}. \tag{1.10}$$

Theorem 1.0.4. *(Cauchy-Pompeiu representation formula)*
Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$. Then using $\zeta = \xi + i\eta$ for $z \in D$

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{1.11}$$

$$w(z) = -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{\overline{d\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \tag{1.12}$$

hold.

The Cauchy-Pompeiu representation formula for any function $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$ in a bounded domain D of the entire complex plane $\mathbb{C}$ with piecewise smooth boundary is complemented with the relation

$$0 = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \mathbb{C}\backslash\overline{D}. \tag{1.13}$$

Let $\mathbb{D}$ be the unit disc defined as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The kernel [5]

$$\frac{\zeta + z}{\zeta - z},\ z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}$$

is called the *Schwarz kernel* and its real part

$$\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1$$

is the *Poisson kernel* for the unit disc with a property proved by H.A.Schwarz [37]

$$\lim_{|z|\to 1, |z|<1} \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right) \frac{d\zeta}{\zeta} = \gamma(\zeta), \ \zeta \in \partial\mathbb{D} \tag{1.14}$$

for $\gamma \in C(\partial\mathbb{D}; \mathbb{R})$.

The Poisson kernel for the upper half plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ studied in [29] and it is

$$\frac{1}{t - z} - \frac{1}{t - \overline{z}}, \ z \in \mathbb{H}, \ \zeta \in \partial\mathbb{H} \tag{1.15}$$

with the property

$$\lim_{z\to t, z\in\mathbb{H}} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \gamma(\zeta) \left(\frac{1}{t - z} - \frac{1}{t - \overline{z}} \right) d\zeta = \gamma(t), \ \ z \in \mathbb{H}, \ \ t \in \partial\mathbb{H} \tag{1.16}$$

for proper $\gamma \in C(\partial\mathbb{H}; \mathbb{R})$.

For the non-homogeneous Cauchy-Riemann equation $\frac{\partial w}{\partial \overline{z}} = f, \ w = u + iv$ the solution is presented in [42]. If $f \in C^1(D)$ then $w(z) = \Phi(z) + Tf$, where

$$\begin{aligned} \Phi(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{w(\zeta) d\zeta}{\zeta - z}, \\ Tf &= -\frac{1}{\pi} \int_{D} \frac{f(\zeta) d\xi d\eta}{\zeta - z}. \end{aligned} \tag{1.17}$$

The integral Tf, which is called the *Pompeiu operator,* exists for all points z in $\mathbb{C}$ and is holomorphic outside $\overline{G}$ with respect to z and vanishes at infinity.

Theorem 1.0.5. *[42] Let D be a bounded domain. If $f \in L_1(\overline{D})$, then Tf, regarded as a function of a point z of the domain D, exists almost everywhere and belongs to an arbitrary class $L_p(\overline{D^*})$ where p is an arbitrary number satisfying the condition $1 \le p < 2$ and D^* is an arbitrary bounded domain of the complex plane.*

Theorem 1.0.6. *[42] If $f \in L_1(\overline{D})$, then*

$$\int_D Tf \frac{\partial \varphi}{\partial \overline{z}} dx dy + \int_D f \varphi dx dy = 0, \tag{1.18}$$

where φ is an arbitrary function of the class $D_1^0(G)$.

By Theorem 1.0.6 the operator $Tf \in D_{\overline{z}}(G)$ if $f \in L_1(\overline{G})$ and

$$\frac{\partial Tf}{\partial \overline{z}} = f \tag{1.19}$$

in the weak sense.

The method of reflection used in the present thesis allows to attain the Schwarz-Pompeiu representation formula and construct the harmonic Green function for a given domain D. As an example, let us consider the upper half plane H^+, $\operatorname{Im} z > 0$ with a fixed point ζ, $\operatorname{Im}\zeta > 0$. Reflection of ζ at the boundary, i.e. real axis, gives a point $\overline{\zeta}$. The function $\overline{\zeta} - z$ is analytic in $z \in \mathbb{C}$ and has a zero at $z = \overline{\zeta}$. The difference $\log |\overline{\zeta} - z| - \log |\zeta - z|$ is a harmonic function in $z \in H^+$ for any ζ, except for the case $z = \zeta$, and it vanishes on the boundary. Then, this difference $\log |\overline{\zeta} - z| - \log |\zeta - z|$, satisfying the properties of the Green function, is the harmonic Green function for the upper half plane. The method is applicable to certain regular and non regular domains if the reflection provides a parqueting of the complex plane.

Part I

Boundary Value Problems in a Quarter Ring Domain

Chapter 2

Boundary Value Problems for the Inhomogeneous Cauchy-Riemann Equation

In this Chapter, the Schwarz-Poisson representation formula is obtained in a quarter ring domain and the Schwarz and Dirichlet problems as well for the inhomogeneous as the Neumann problem for the homogeneous Cauchy-Riemann equation are solved explicitly.

2.1 Schwarz problem

Let R^* be the upper right quarter ring domain (see Fig.1) in the complex plane $\mathbb{C}$ defined by

$$R^* = \{z \in \mathbb{C} : r < z < 1, \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}.$$

The boundary ∂R^* is piecewise smooth and oriented counter-clockwise. It is non-regular and contains four corner points r, 1, i, ir. Thus the Cauchy-Pompeiu formula holds for proper functions. For attaining the Poisson kernel the quarter ring is reflected across its boundary parts on the real and the imaginary axes to the entire ring. The ring itself is repeatedly reflected across its two boundary circles such that the entire punctured plane is reached. Applying the Cauchy-Pompeiu representation formula in the points where z from the quarter ring is mapped to, leads to the adjusted Schwarz-Poisson formula obtained by some modifications. The latter enables to solve the related Schwarz problem.

2.1.1 Schwarz-Poisson representation formula

To get the Cauchy-Pompeiu representation due to the boundary conditions a meromorphic kernel function is constructed as follows. A point $z \in R^*$ chosen to be a simple pole is reflected across the parts of the boundary:

$$\{|z| = 1, \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}; \qquad \{|z| = r, \operatorname{Re} z > 0, \operatorname{Im} z > 0\};$$
$$\{0 < \operatorname{Re} z < 1,\ \operatorname{Im} z = 0\}; \qquad \{0 < \operatorname{Im} z < 1,\ \operatorname{Re} z = 0\}.$$

The direct reflection of the pole gives zeros. The reflected points are

$$\frac{1}{\overline{z}}, \overline{z}, -\overline{z}, \frac{r^2}{\overline{z}}.$$

These zeros in turn are also reflected at the parts of the boundaries $|z| = 1, |z| = r, |z| = \frac{1}{r}, |z| = r^2$ and these reflection points

$$\frac{1}{z}, \frac{r^2}{z}, -z, -\frac{1}{z}, zr^2, \frac{z}{r^2}$$

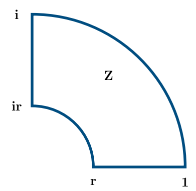

Fig.1: Quarter ring

as the direct reflection of the zeros become poles. Continuing this process, the new points can be expressed in general form as

$$\pm zr^{2n}, \pm\overline{z}r^{2n}, \pm\frac{z}{r^{2n}}, \pm\frac{\overline{z}}{r^{2n}}, \pm\frac{r^{2n}}{z}, \pm\frac{r^{2n}}{\overline{z}}, \pm\frac{1}{\overline{z}r^{2n}}, \pm\frac{1}{zr^{2n}}, \tag{2.1}$$

where $\pm zr^{2n}, \pm\frac{z}{r^{2n}}, \pm\frac{r^{2n}}{z}, \pm\frac{1}{zr^{2n}}$ are poles and $\pm\overline{z}r^{2n}, \pm\frac{\overline{z}}{r^{2n}}, \pm\frac{r^{2n}}{\overline{z}}, \pm\frac{1}{\overline{z}r^{2n}}$ are zeros. As it was done in [18] the Cauchy-Pompeiu formula is to be modified.

Theorem 2.1.1. *Any $w \in C^1(R^*;\mathbb{C}) \cap C(\overline{R^*};\mathbb{C})$ for the domain $R^* \subset \mathbb{C}$ can be represented as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial R^*} w(\zeta)\left\{\frac{\zeta}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n}z}\right]\right\}\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{\pi}\int_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{1}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\zeta - z} + \frac{z}{\zeta(\zeta - r^{2n}z)}\right]\right\}d\xi d\eta \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} w(z) &= \frac{1}{\pi i}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Re} w(\zeta)\left\{\frac{\zeta^2 + z^2}{\zeta^2 - z^2} - \frac{\overline{\zeta}^2 + z^2}{\overline{\zeta}^2 - z^2}\right. \\ &\quad \left. +2\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2 - z^2} - \frac{z^2}{r^{4n}z^2 - \zeta^2} + \frac{z^2}{r^{4n}z^2 - \overline{\zeta}^2} - \frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2 - z^2}\right]\right\}\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{\pi i}\int_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Re} w(\zeta)\left\{\frac{\zeta^2 + z^2}{\zeta^2 - z^2} - \frac{\overline{\zeta}^2 + z^2}{\overline{\zeta}^2 - z^2}\right. \\ &\quad \left. +2\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2 - z^2} - \frac{z^2}{r^{4n}z^2 - \zeta^2} + \frac{z^2}{r^{4n}z^2 - \overline{\zeta}^2} - \frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2 - z^2}\right]\right\}\frac{d\zeta}{\zeta} \\ &\quad +\frac{2}{\pi i}\int_r^1 \operatorname{Re} w(t)\left\{\frac{t}{t^2 - z^2} - \frac{tz^2}{1 - t^2z^2}\right. \\ &\quad \left. +\sum_{n=1}^{\infty} r^{4n}\left[\frac{t}{r^{4n}t^2 - z^2} - \frac{z^2}{t(r^{4n}z^2 - t^2)} + \frac{tz^2}{r^{4n}z^2t^2 - 1} - \frac{1}{t(r^{4n} - z^2t^2)}\right]\right\}dt \\ &\quad +\frac{2}{\pi i}\int_1^r \operatorname{Re} w(it)\left\{\frac{t}{t^2 + z^2} + \frac{tz^2}{1 + t^2z^2}\right. \\ &\quad \left. +\sum_{n=1}^{\infty} r^{4n}\left[\frac{t}{r^{4n}t^2 + z^2} - \frac{z^2}{t(r^{4n}z^2 + t^2)} + \frac{tz^2}{r^{4n}z^2t^2 + 1} - \frac{1}{t(r^{4n} + z^2t^2)}\right]\right\}dt \\ &\quad +\frac{2}{\pi}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} - \frac{2}{\pi}\int_{R^*}\left\{w_{\overline{\zeta}}(\zeta)\left[\frac{z^2\zeta}{z^2\zeta^2 - 1} + \frac{\zeta}{\zeta^2 - z^2}\right.\right. \\ &\quad \left. +\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta}{r^{4n}\zeta^2 - z^2} + \frac{z^2}{\zeta(\zeta^2 - r^{4n}z^2)} + \frac{\zeta z^2}{r^{4n}\zeta^2z^2 - 1} + \frac{1}{\zeta(\zeta^2z^2 - r^{4n})}\right)\right] \\ &\quad -\overline{w_{\overline{\zeta}}(\zeta)}\left[\frac{z^2\overline{\zeta}}{z^2\overline{\zeta}^2 - 1} + \frac{\overline{\zeta}}{\overline{\zeta}^2 - z^2}\right. \\ &\quad \left.\left. +\sum_{n=1}^{\infty} r^{4n}\left(\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2 - z^2} + \frac{z^2}{\overline{\zeta}(\overline{\zeta}^2 - r^{4n}z^2)} + \frac{\overline{\zeta}z^2}{r^{4n}\overline{\zeta}^2z^2 - 1} + \frac{1}{\overline{\zeta}(\overline{\zeta}^2z^2 - r^{4n})}\right)\right]\right\}d\xi d\eta \end{aligned} \tag{2.3}$$

Proof. From the Cauchy-Pompeiu representation formulas (1.11), (1.13) and the relation

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n}z}\right]\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\zeta - z} + \frac{z}{\zeta(\zeta - r^{2n}z)}\right]d\xi d\eta,\ z\in R^* \end{aligned} \tag{2.4}$$

the equality

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\zeta}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n}z}\right]\right\}\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{1}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\zeta - z} + \frac{z}{\zeta(\zeta - r^{2n}z)}\right]\right\}d\xi d\eta,\ z\in R^* \end{aligned} \tag{2.5}$$

follows. Substituting the reflection points

$$-z, \frac{1}{z}, -\frac{1}{z}, \overline{z}, -\overline{z}, \frac{1}{\overline{z}}, -\frac{1}{\overline{z}}$$

in (1.13) and (2.4) gives the following equalities

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\zeta}{\zeta + z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta + z} - \frac{z}{\zeta + r^{2n}z}\right]\right\}\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{1}{\zeta + z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\zeta + z} - \frac{z}{\zeta(\zeta + r^{2n}z)}\right]\right\}d\xi d\eta,\ z\in R^*. \end{aligned} \tag{2.6}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{z\zeta}{z\zeta - 1} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{z\zeta}{r^{2n}\zeta z - 1} + \frac{1}{z\zeta - r^{2n}}\right]\right\}\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{z}{z\zeta - 1} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{z}{r^{2n}z\zeta - 1} + \frac{1}{\zeta(z\zeta - r^{2n})}\right]\right\}d\xi d\eta,\ z\in R^*. \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{z\zeta}{z\zeta + 1} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{z\zeta}{r^{2n}\zeta z + 1} - \frac{1}{z\zeta + r^{2n}}\right]\right\}\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{z}{z\zeta + 1} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{z}{r^{2n}z\zeta + 1} - \frac{1}{\zeta(z\zeta + r^{2n})}\right]\right\}d\xi d\eta,\ z\in R^*. \end{aligned} \tag{2.8}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\zeta}{\zeta - \overline{z}} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - \overline{z}} + \frac{z}{\zeta - r^{2n}\overline{z}}\right]\right\}\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{1}{\zeta - \overline{z}} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\zeta - \overline{z}} + \frac{z}{\zeta(\zeta - r^{2n}\overline{z})}\right]\right\}d\xi d\eta,\ z\in R^*. \end{aligned} \tag{2.9}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\zeta}{\zeta + \overline{z}} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta + \overline{z}} - \frac{z}{\zeta + r^{2n}\overline{z}}\right]\right\}\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{1}{\zeta + \overline{z}} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{1}{r^{2n}\zeta + \overline{z}} - \frac{z}{\zeta(\zeta + r^{2n}\overline{z})}\right]\right\}d\xi d\eta,\ z\in R^*. \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\overline{z}\zeta}{\overline{z}\zeta-1}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\overline{z}\zeta}{r^{2n}\zeta\overline{z}-1}+\frac{1}{\overline{z}\zeta-r^{2n}}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{\overline{z}}{\overline{z}\zeta-1}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{z}{r^{2n}\overline{z}\zeta-1}+\frac{1}{\zeta(\overline{z}\zeta-r^{2n})}\right]\right\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
0 &= \frac{1}{2\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\overline{z}\zeta}{\overline{z}\zeta+1}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{\overline{z}\zeta}{r^{2n}\zeta\overline{z}+1}-\frac{1}{\overline{z}\zeta+r^{2n}}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{1}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{\overline{z}}{\overline{z}\zeta+1}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{z}{r^{2n}\overline{z}\zeta+1}-\frac{1}{\zeta(\overline{z}\zeta+r^{2n})}\right]\right\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.12}
$$

To simplify the following calculations the evaluations above are composed, namely (2.5) and (2.6), (2.7) and (2.8), (2.9) and (2.10), (2.11) and (2.12). Then

$$
\begin{aligned}
w(z) &= \frac{1}{\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\zeta^2}{\zeta^2-z^2}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{2}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{\zeta}{\zeta^2-z^2}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{\zeta}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta(\zeta^2-r^{4n}z^2)}\right]\right\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
0 &= \frac{1}{\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{z^2\zeta^2}{z^2\zeta^2-1}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{z^2\zeta^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{z^2\zeta^2-r^{4n}}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{2}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{z^2\zeta}{z^2\zeta^2-1}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{z^2\zeta}{r^{4n}z^2\zeta^2-1}+\frac{1}{\zeta(z^2\zeta^2-r^{4n})}\right]\right\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
0 &= \frac{1}{\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{2}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{\zeta}{\zeta^2-\overline{z}^2}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{\zeta}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta(\zeta^2-r^{4n}\overline{z}^2)}\right]\right\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
0 &= \frac{1}{\pi i}\int\limits_{\partial R^*} w(\zeta)\left\{\frac{\overline{z}^2\zeta^2}{\overline{z}^2\zeta^2-1}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{\overline{z}^2\zeta^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\overline{z}^2\zeta^2-r^{4n}}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{2}{\pi}\int\limits_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{\overline{z}^2\zeta}{\overline{z}^2\zeta^2-1}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{\overline{z}^2\zeta}{r^{4n}\overline{z}^2\zeta^2-1}+\frac{1}{\zeta(\overline{z}^2\zeta^2-r^{4n})}\right]\right\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.16}
$$

Taking the complex conjugation of (2.15) and (2.16) and considering the integrals over the different boundary parts give their new forms

$$
\begin{aligned}
0 &= -\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \overline{w(\zeta)}\left\{\frac{1}{\zeta^2z^2-1}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{1}{\zeta^2z^2-r^{4n}}+\frac{z^2\zeta^2}{\zeta^2r^{4n}z^2-1}\right]\right\}\frac{d\zeta}{\zeta} \\
&-\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \overline{w(\zeta)}\left\{\frac{r^4}{\zeta^2z^2-r^4}+\sum_{n=1}^{\infty}r^{4n}\left[\frac{r^4}{\zeta^2z^2-r^{4(n+1)}}+\frac{z^2\zeta^2}{\zeta^2r^{4n}z^2-r^4}\right]\right\}\frac{d\zeta}{\zeta}
\end{aligned}
$$

$$-\frac{1}{\pi i}\int_r^1 \overline{w(t)}\left\{\frac{t^2}{t^2-z^2}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t^2}{t^2r^{4n}-z^2}+\frac{z^2}{t^2-r^{4n}z^2}\right]\right\}\frac{dt}{t} \tag{2.17}$$
$$-\frac{1}{\pi i}\int_1^r \overline{w(it)}\left\{\frac{t^2}{t^2+z^2}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t^2}{t^2r^{4n}+z^2}-\frac{z^2}{t^2+r^{4n}z^2}\right]\right\}\frac{dt}{t}$$
$$-\frac{2}{\pi}\int_{R^*} \overline{w_{\overline{\zeta}}(\zeta)}\left\{\frac{\overline{\zeta}}{\overline{\zeta}^2-z^2}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2-z^2}+\frac{z^2}{\overline{\zeta}(\overline{\zeta}^2-r^{4n}z^2)}\right]\right\}d\xi d\eta.$$

$$0 = -\frac{1}{\pi i}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \overline{w(\zeta)}\left\{\frac{z^2}{\zeta^2-z^2}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{z^2}{\zeta^2-z^2r^{4n}}+\frac{\zeta^2}{\zeta^2r^{4n}-z^2}\right]\right\}\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi i}\int_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \overline{w(\zeta)}\left\{\frac{r^4z^2}{\zeta^2-z^2r^4}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{r^4z^2}{\zeta^2-z^2r^{4(n+1)}}+\frac{\zeta^2}{\zeta^2r^{4n}-z^2r^4}\right]\right\}\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi i}\int_r^1 \overline{w(t)}\left\{\frac{t^2z^2}{t^2z^2-1}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t^2z^2}{t^2r^{4n}z^2-1}+\frac{1}{t^2z^2-r^{4n}}\right]\right\}\frac{dt}{t} \tag{2.18}$$
$$-\frac{1}{\pi i}\int_1^r \overline{w(it)}\left\{\frac{t^2z^2}{t^2z^2+1}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t^2z^2}{t^2r^{4n}z^2+1}-\frac{1}{t^2z^2+r^{4n}}\right]\right\}\frac{dt}{t}$$
$$-\frac{2}{\pi}\int_{R^*} \overline{w_{\overline{\zeta}}(\zeta)}\left\{\frac{\overline{\zeta}z^2}{\overline{\zeta}^2z^2-1}+\sum_{n=1}^{\infty} r^{4n}\left[\frac{\overline{\zeta}z^2}{r^{4n}\overline{\zeta}^2z^2-1}+\frac{1}{\overline{\zeta}(\overline{\zeta}^2z^2-r^{4n})}\right]\right\}d\xi d\eta.$$

Then, subtracting (2.18) from (2.13), (2.17) from(2.14) and adding (2.18) and (2.13), (2.17) and (2.14) lead to

$$w(z) = b_1 + b_2 + b_3 + b_4 + b_5 + b_6,$$

where

$$b_1 = \frac{1}{\pi i}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Re} w(\zeta)\left\{\frac{\zeta^2+z^2}{\zeta^2-z^2}-\frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2}\right.$$
$$\left.+2\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-z^2}-\frac{z^2}{r^{4n}z^2-\zeta^2}+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}-\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}\right]\right\}\frac{d\zeta}{\zeta};$$

$$b_2 = -\frac{1}{\pi i}\int_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Re} w(\zeta)\left\{\frac{\zeta^2+z^2}{\zeta^2-z^2}-\frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2}\right.$$
$$\left.+2\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-z^2}-\frac{z^2}{r^{4n}z^2-\zeta^2}+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}-\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}\right]\right\}\frac{d\zeta}{\zeta};$$

$$b_3 = \frac{2}{\pi i}\int_r^1 \operatorname{Re} w(t)\left\{\frac{t}{t^2-z^2}+\frac{tz^2}{t^2z^2-1}\right.$$
$$\left.+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t}{r^{4n}t^2-z^2}-\frac{z^2}{t(r^{4n}z^2-t^2)}+\frac{tz^2}{r^{4n}z^2t^2-1}-\frac{1}{t(r^{4n}-t^2z^2)}\right]\right\}dt;$$

$$\begin{aligned}
b_4 &= -\frac{2}{\pi i}\int_r^1 \operatorname{Re} w(it)\Bigg\{\frac{t}{t^2+z^2}+\frac{tz^2}{t^2z^2+1} \\
&+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t}{r^{4n}t^2+z^2}-\frac{z^2}{t(r^{4n}z^2+t^2)}+\frac{tz^2}{r^{4n}z^2t^2+1}-\frac{1}{t(r^{4n}+t^2z^2)}\right]\Bigg\}dt;
\end{aligned}$$

$$\begin{aligned}
b_5 &= \frac{1}{\pi i}\Bigg\{\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} w(\zeta)\Bigg[\frac{\zeta^2}{\zeta^2-z^2}+\frac{z^2\zeta^2}{z^2\zeta^2-1}+\sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2} \\
&+\frac{\zeta^2z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta^2z^2-r^{4n}}\Bigg)\Bigg]\frac{d\zeta}{\zeta}-\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \overline{w(\zeta)}\Bigg[\frac{z^2}{\zeta^2-z^2}+\frac{1}{z^2\zeta^2-1} \\
&+\sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta^2z^2-r^{4n}}\Bigg)\Bigg]\frac{d\zeta}{\zeta}\Bigg\} = \\
&\frac{1}{\pi}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Im} w(\zeta)\left[\frac{\zeta^2}{\zeta^2-z^2}-\frac{z^2}{\zeta^2-z^2}+\frac{z^2\zeta^2}{z^2\zeta^2-1}-\frac{1}{z^2\zeta^2-1}\right]\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta};
\end{aligned}$$

$$\begin{aligned}
b_6 &= -\frac{2}{\pi}\Bigg\{\int_{R^*} w_{\overline{\zeta}}(\zeta)\Bigg[\frac{\zeta}{\zeta^2-z^2}+\frac{z^2\zeta}{z^2\zeta^2-1} \\
&+\sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{\zeta}{\zeta^2r^{4n}-z^2}+\frac{z^2}{\zeta(\zeta^2-r^{4n}z^2)}+\frac{\zeta z^2}{r^{4n}z^2\zeta^2-1}+\frac{1}{\zeta(\zeta^2z^2-r^{4n})}\Bigg)\Bigg] \\
&-\overline{w_{\overline{\zeta}}(\zeta)}\Bigg[\frac{\overline{\zeta}}{\overline{\zeta}^2-z^2}+\frac{z^2\overline{\zeta}}{z^2\overline{\zeta}^2-1} \\
&+\sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{z^2\overline{\zeta}}{r^{4n}z^2\overline{\zeta}^2-1}+\frac{1}{\overline{\zeta}(\overline{\zeta}^2z^2-r^{4n})}+\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2-z^2}+\frac{z^2}{\overline{\zeta}(\overline{\zeta}^2-r^{4n}z^2)}\Bigg)\Bigg]\Bigg\}d\xi d\eta.
\end{aligned}$$

Hence, adding all of these formulas for $b_1,...,b_6$ the representation formula (2.3) is obtained. □

Theorem 2.1.1 enables to solve the related Schwarz problem.

2.1.2 Schwarz problem

In order to solve the Schwarz problem on the basis of Theorem 2.1.1, the boundary behavior of the boundary integral is studied. Let the parts of the boundary be denoted by

$$\partial_1R^*=\{|\zeta|=1,\,\operatorname{Re}\zeta\ge 0,\,\operatorname{Im}\zeta\ge 0\};\qquad \partial_2R^*=\{|\zeta|=r,\,\operatorname{Re}\zeta\ge 0,\,\operatorname{Im}\zeta\ge 0\};$$
$$\partial_3R^*=\{\zeta=t:r\le t\le 1\};\qquad \partial_4R^*=\{\zeta=it:r\le t\le 1\}$$

and new functions be expressed by

$$\begin{aligned}
K_1(z,\zeta) &= \Bigg\{\left(\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta}^2-\overline{z}^2}-1\right)-\left(\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}+\frac{\zeta^2}{\zeta^2-\overline{z}^2}-1\right) \\
&+\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-\overline{z}^2}-\frac{z^2}{r^{4n}z^2-\zeta^2}-\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}\right. \qquad (2.19) \\
&\left.+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}+\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}-\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}\right]\Bigg\}\frac{1}{\zeta},
\end{aligned}$$

$$
\begin{aligned}
K_2(z,\zeta) \;=\; & \left\{ \frac{\zeta^2}{\zeta^2 - z^2} - \frac{\overline{\zeta}^2}{\overline{\zeta}^2 - \overline{z}^2} - \frac{\zeta^2 z^2}{1-\zeta^2 z^2} + \frac{\overline{\zeta}^2\overline{z}^2}{1-\overline{\zeta}^2\overline{z}^2} \right. \\
& + \sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2 - z^2} - \frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2 - \overline{z}^2} - \frac{z^2}{r^{4n}z^2-\zeta^2} + \frac{\overline{z}^2}{r^{4n}\overline{z}^2 - \overline{\zeta}^2} \right. \\
& \left.\left. + \frac{\zeta^2 z^2}{r^{4n}z^2\zeta^2 - 1} - \frac{\overline{\zeta}^2\overline{z}^2}{r^{4n}\overline{z}^2\overline{\zeta}^2 - 1} - \frac{1}{r^{4n} - \zeta^2 z^2} + \frac{1}{r^{4n} - \overline{\zeta}^2\overline{z}^2} \right]\right\}\frac{1}{\zeta}.
\end{aligned}
\tag{2.20}
$$

Theorem 2.1.2. *If $\gamma \in C(\partial R^*;\mathbb{C})$, then for $\zeta_0 \neq r, 1, i, ir$*

$$
\lim_{z\in R^*, z\to\zeta_0} \frac{1}{\pi i} \int\limits_{\partial_1 R^* \cup \partial_2 R^*} \gamma(\zeta) K_1(z,\zeta) d\zeta = \gamma(\zeta_0),\ \zeta_0 \in \partial_1 R^* \cup \partial_2 R^*, \tag{2.21}
$$

$$
\lim_{z\in R^*, z\to\zeta_0} \frac{1}{\pi i} \int\limits_{\partial_3 R^* \cup \partial_4 R^*} \gamma(\zeta) K_2(z,\zeta) d\zeta = \gamma(\zeta_0),\ \zeta_0 \in \partial_3 R^* \cup \partial_4 R^*, \tag{2.22}
$$

where $K_1(z,\zeta)$ and $K_2(\zeta,z)$ are given in (2.19), (2.20) respectively.

Proof. Studying the boundary behavior of the boundary integral implies computations on the different parts of the boundary ∂R^*. Taking the boundary relations into account, the following sum is obtainable easily

$$
\begin{aligned}
& \frac{1}{\pi i} \int\limits_{\partial_1 R^* \cup \partial_2 R^*} \gamma(\zeta) K_1(z,\zeta) d\zeta + \frac{1}{\pi i} \int\limits_{\partial_3 R^* \cup \partial_4 R^*} \gamma(\zeta) K_2(z,\zeta) d\zeta = \\
& \frac{1}{\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}} \gamma(\zeta)\left\{ \left(\frac{\zeta^2}{\zeta^2 - z^2} + \frac{\overline{\zeta}^2}{\overline{\zeta^2 - z^2}} - 1 \right) - \left(\frac{\overline{\zeta}^2}{\overline{\zeta}^2 - z^2} + \frac{\zeta^2}{\zeta^2 - \overline{z}^2} - 1 \right) \right. \\
& + \sum_{n=1}^{\infty} r^{4n}\left(\frac{1}{r^{4n} - z^2\overline{\zeta}^2} + \frac{1}{r^{4n} - \overline{z}^2\zeta^2} - \frac{|z|^4}{r^{4n}|z|^4 - \zeta^2\overline{z}^2} - \frac{|z|^4}{r^{4n}|z|^4 - \overline{\zeta}^2 z^2} \right. \\
& \left.\left. + \frac{|z|^4}{r^{4n}|z|^4 - \overline{\zeta^2 z^2}} + \frac{|z|^4}{r^{4n}|z|^4 - \zeta^2 z^2} - \frac{1}{r^{4n} - z^2\zeta^2} - \frac{1}{r^{4n} - \overline{z\zeta}^2} \right) \right\} \frac{d\zeta}{\zeta} \\
& - \frac{1}{\pi i} \int\limits_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}} \gamma(\zeta)\left\{ \left(\frac{\zeta^2}{\zeta^2 - z^2} + \frac{\overline{\zeta}^2}{\overline{\zeta^2 - z^2}} - 1 \right) - \left(\frac{\overline{\zeta}^2}{\overline{\zeta}^2 - z^2} + \frac{\zeta^2}{\zeta^2 - \overline{z}^2} - 1 \right) \right. \\
& + \sum_{n=1}^{\infty} r^{4n}\left(\frac{r^4}{r^{4(n+1)} - z^2\overline{\zeta}^2} + \frac{r^4}{r^{4(n+1)} - \overline{z}^2\zeta^2} - \frac{|z|^4}{r^{4n}|z|^4 - \zeta^2\overline{z}^2} - \frac{|z|^4}{r^{4n}|z|^4 - \overline{\zeta}^2 z^2} \right. \\
& \left.\left. + \frac{|z|^4}{r^{4n}|z|^4 - \overline{\zeta^2 z^2}} + \frac{|z|^4}{r^{4n}|z|^4 - \zeta^2 z^2} - \frac{r^4}{r^{4(n+1)} - z^2\zeta^2} - \frac{r^4}{r^{4(n+1)} - \overline{z^2\zeta}^2} \right) \right\} \frac{d\zeta}{\zeta} \\
& + \frac{(z^2 - \overline{z}^2)}{\pi i} \int_r^1 \gamma(t) \left\{ \frac{t}{|t^2 - z^2|^2} - \frac{t}{|1 - t^2 z^2|^2} \right. \\
& \left. + \sum_{n=1}^{\infty} r^{4n}\left(\frac{t}{|r^{4n}t^2 - z^2|^2} + \frac{t}{|r^{4n}z^2 - t^2|^2} - \frac{t}{|r^{4n}z^2t^2 - 1|^2} - \frac{t}{|r^{4n} - t^2z^2|^2} \right) \right\} dt \\
& + \frac{(z^2 - \overline{z}^2)}{\pi i} \int_r^1 \gamma(it) \left\{ \frac{t}{|t^2 + z^2|^2} - \frac{t}{|1 + t^2 z^2|^2} \right. \\
& \left. + \sum_{n=1}^{\infty} r^{4n}\left(\frac{t}{|r^{4n}z^2 + t^2|^2} - \frac{t}{|r^{4n}z^2t^2 + 1|^2} + \frac{t}{|r^{4n}t^2 + z^2|^2} - \frac{t}{|r^{4n} + t^2z^2|^2} \right) \right\} dt.
\end{aligned}
\tag{2.23}
$$

Let $|\zeta_0| = 1$, $\operatorname{Re}\zeta_0 > 0$, $\operatorname{Im}\zeta_0 > 0$, $z \in R^*$, then

$$\lim_{z\to\zeta_0}\left\{\frac{1}{\pi i}\int\limits_{\partial_1 R^*\cup\partial_2 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta + \frac{1}{\pi i}\int\limits_{\partial_3 R^*\cup\partial_4 R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta\right\} =$$

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}}\gamma(\zeta)\left\{\left(\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta^2-z^2}}-1\right)-\left(\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}+\frac{\zeta^2}{\zeta^2-\overline{z}^2}-1\right)\right\}\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}}\gamma(\zeta)\sum_{n=1}^{\infty}r^{4n}\left\{\frac{1}{r^{4n}-\overline{\zeta}^2{\zeta_0}^2}+\frac{1}{r^{4n}-\overline{\zeta_0}^2\zeta^2}-\frac{|\zeta_0|^4}{r^{4n}|\zeta_0|^4-\overline{\zeta_0}^2\zeta^2}-\frac{|\zeta_0|^4}{r^{4n}-\overline{\zeta}^2\zeta_0^2}\right.$$

$$\left.+\frac{|\zeta_0|^4}{r^{4n}|\zeta_0|^4-\overline{{\zeta_0}^2\zeta^2}}+\frac{|\zeta_0|^4}{r^{4n}|\zeta_0|^4-{\zeta_0}^2\zeta^2}-\frac{1}{r^{4n}-\zeta^2{\zeta_0}^2}-\frac{1}{r^{4n}-\overline{\zeta^2{\zeta_0}^2}}\right\}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}}\gamma(\zeta)\left\{\frac{\zeta^2}{\zeta^2-{\zeta_0}^2}+\frac{\overline{\zeta}^2}{\overline{\zeta^2-{\zeta_0}^2}}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-{\zeta_0}^2}-\frac{\zeta^2}{\zeta^2-\overline{\zeta_0}^2}\right\}\frac{d\zeta}{\zeta}$$

$$-\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}}\gamma(\zeta)\sum_{n=1}^{\infty}\left\{\frac{r^{4(n+1)}}{r^{4(n+1)}-z^2\overline{\zeta}^2}+\frac{r^{4(n+1)}}{r^{4(n+1)}-\overline{z}^2\zeta^2}-\frac{r^{4(n+1)}}{r^{4(n+1)}-z^2\zeta^2}\right.$$

$$\left.-\frac{r^{4(n+1)}}{r^{4(n+1)}-\overline{z^2\zeta^2}}-\frac{|z|^4r^{4n}}{|z|^4r^{4n}-\zeta^2\overline{z}^2}-\frac{|z|^4r^{4n}}{|z|^4r^{4n}-\overline{\zeta}^2z^2}+\frac{|z|^4r^{4n}}{|z|^4r^{4n}-\overline{\zeta^2z^2}}+\frac{|z|^4r^{4n}}{|z|^4r^{4n}-\zeta^2z^2}\right\}\frac{d\zeta}{\zeta}$$

$$+\lim_{z\to\zeta_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int\limits_r^1\gamma(t)\left\{\frac{t}{|t^2-z^2|^2}-\frac{t}{|1-t^2z^2|^2}\right.$$

$$\left.+\frac{1}{2}\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}t^2-z^2|^2}+\frac{t}{|r^{4n}z^2-t^2|^2}-\frac{t}{|r^{4n}z^2t^2-1|^2}-\frac{t}{|r^{4n}-t^2z^2|^2}\right)\right\}dt$$

$$+\lim_{z\to\zeta_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int\limits_r^1\gamma(it)\left\{\frac{t}{|t^2+z^2|^2}-\frac{t}{|1+t^2z^2|^2}\right.$$

$$\left.+\frac{1}{2}\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}z^2+t^2|^2}-\frac{t}{|r^{4n}z^2t^2+1|^2}+\frac{t}{|r^{4n}t^2+z^2|^2}-\frac{t}{|r^{4n}+t^2z^2|^2}\right)\right\}dt.$$

Since for $|\zeta_0| = 1$, $\operatorname{Re}\zeta_0 > 0$, $\operatorname{Im}\zeta_0 > 0$ the relations

$$\begin{aligned}
&|t^2-{\zeta_0}^2|^2 = |1-t^2{\zeta_0}^2|^2 \neq 0,\ |r^{4n}t^2-{\zeta_0}^2|^2 = |r^{4n}t^2{\zeta_0}^2-1|^2 \neq 0,\\
&|r^{4n}{\zeta_0}^2-t^2|^2 = |r^{4n}-t^2{\zeta_0}^2|^2 \neq 0,\ |1+t^2{\zeta_0}^2|^2 = |t^2+{\zeta_0}^2|^2 \neq 0,\\
&|r^{4n}t^2{\zeta_0}^2+1|^2 = |r^{4n}t^2+{\zeta_0}^2|^2,\ |r^{4n}{\zeta_0}^2+t^2|^2 = |r^{4n}+t^2{\zeta_0}^2|^2
\end{aligned} \tag{2.24}$$

are valid, then one has to calculate

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_1 R^*\cup\partial_2 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta =$$

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}}\gamma(\zeta)\left\{\left(\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta^2-z^2}}-1\right)-\left(\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}+\frac{\zeta^2}{\zeta^2-\overline{z}^2}-1\right)\right\}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}}\gamma(\zeta)\left\{\frac{\zeta^2}{\zeta^2-{\zeta_0}^2}+\frac{\overline{\zeta^2}}{\overline{\zeta^2-{\zeta_0}^2}}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-{\zeta_0}^2}-\frac{\zeta^2}{\zeta^2-\overline{\zeta_0}^2}+\sum_{n=1}^{\infty}\left[\frac{r^{4n}}{r^{4n}-{\zeta_0}^2\overline{\zeta}^2}-\frac{r^4}{r^4-{\zeta_0}^2\overline{\zeta}^2}\right.\right.$$

$$
+\frac{r^{4n}}{r^{4n}-\overline{\zeta_0}^2\zeta^2}-\frac{r^4}{r^4-\overline{\zeta_0}^2\zeta^2}-\frac{r^{4n}}{r^{4n}-\zeta_0{}^2\zeta^2}+\frac{r^4}{r^4-\zeta_0{}^2\zeta^2}-\frac{r^{4n}}{r^{4n}-\overline{\zeta_0{}^2\zeta^2}}+\frac{r^4}{r^4-\overline{\zeta_0{}^2\zeta^2}}
$$

$$
-\frac{r^{4n}}{r^{4n}-\zeta^2\overline{\zeta_0}^2}-\frac{r^{4n}}{r^{4n}-\zeta^2\overline{\zeta_0}^2}-\frac{r^{4n}}{r^{4n}-\overline{\zeta}^2\zeta_0{}^2}+\frac{r^{4n}}{r^{4n}-\overline{\zeta_0{}^2\zeta^2}}+\frac{r^{4n}}{r^{4n}-\zeta^2\zeta_0{}^2}\Bigg]\Bigg\}\frac{d\zeta}{\zeta}=
$$

$$
\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta^2-z^2}}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}\Bigg\}\frac{d\zeta}{\zeta}.
$$

Finally, decomposing the integral so that the entire unit disc $\mathbb{D}$ is reached,

$$
\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_1 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta=\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\Bigg\{\bigg(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\bigg)
$$

$$
+\bigg(\frac{\zeta}{\zeta+z}+\frac{\overline{\zeta}}{\overline{\zeta}+\overline{z}}-1\bigg)-\bigg(\frac{\overline{\zeta}}{\overline{\zeta}-z}+\frac{\zeta}{\zeta-\overline{z}}-1\bigg)-\bigg(\frac{\overline{\zeta}}{\overline{\zeta}+z}+\frac{\zeta}{\zeta+\overline{z}}-1\bigg)\Bigg\}\frac{d\zeta}{\zeta}=
$$

$$
\lim_{z\to\zeta_0}\frac{1}{2\pi i}\Bigg\{\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\bigg(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\bigg)\frac{d\zeta}{\zeta}+\int\limits_{\substack{|\zeta|=1,\\ 0>\mathrm{Im}\zeta,\\ 0>\mathrm{Re}\,\zeta}}\gamma(-\zeta)\bigg(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\bigg)\frac{d\zeta}{\zeta}
$$

$$
-\int\limits_{\substack{|\zeta|=1,\\ 0>\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}}\gamma(\overline{\zeta})\bigg(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\bigg)\frac{d\overline{\zeta}}{\overline{\zeta}}-\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0>\mathrm{Re}\,\zeta}}\gamma(-\overline{\zeta})\bigg(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\bigg)\frac{d\overline{\zeta}}{\overline{\zeta}},
$$

the property (1.14) of the Poisson kernel for $\mathbb{D}$ is used. Thus

$$
\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_1 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta=\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Gamma_1(\zeta)\bigg(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\bigg)\frac{d\zeta}{\zeta}=\gamma(\zeta_0),\tag{2.25}
$$

where

$$
\Gamma_1(\zeta)=\begin{cases}\gamma(\zeta),\ \mathrm{Re}\,\zeta\ge 0,\mathrm{Im}\,\zeta\ge 0\\ \gamma(-\zeta),\ \mathrm{Re}\,\zeta\le 0,\mathrm{Im}\,\zeta\le 0\\ \gamma(\overline{\zeta}),\ \mathrm{Re}\,\zeta\ge 0,\mathrm{Im}\,\zeta\le 0\\ \gamma(-\overline{\zeta}),\mathrm{Re}\,\zeta\le 0,\mathrm{Im}\,\zeta\ge 0\end{cases}\tag{2.26}
$$

is a continuous function on $|\zeta|=1$.

Similarly, for $|\zeta_0|=r$, $\mathrm{Re}\,\zeta_0>0,\mathrm{Im}\,\zeta_0>0$, $z\in R^*$

$$
\lim_{z\to\zeta_0}\Bigg\{\frac{1}{\pi i}\int\limits_{\partial_1 R^*\cup\partial_2 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta+\frac{1}{\pi i}\int\limits_{\partial_3 R^*\cup\partial_4 R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta\Bigg\}=
$$

$$
\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta^2-z^2}}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}
$$

$$
+\lim_{z\to\zeta_0}\sum_{n=1}^{\infty}r^{4n}\bigg(\frac{\overline{z}^2}{r^{4n}\overline{z}^2-|z|^4\overline{\zeta}^2}+\frac{z^2}{r^{4n}z^2-|z|^4\zeta^2}-\frac{\overline{z}^2}{r^{4n}\overline{z}^2-|z|^4\zeta^2}-\frac{z^2}{r^{4n}z^2-|z|^4\overline{\zeta}^2}
$$

$$
+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}+\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}-\frac{z^2}{r^{4n}z^2-\zeta^2}-\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}\bigg)\Bigg\}\frac{d\zeta}{\zeta}
$$

$$-\lim_{z\to\zeta_0}\int\limits_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\Bigg\{\left(\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta^2-z^2}}-1\right)-\left(\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}+\frac{\zeta^2}{\zeta^2-\overline{z}^2}-1\right)\Bigg\}\frac{d\zeta}{\zeta}$$

$$+\sum_{n=1}^{\infty}r^{4n}\Bigg(\frac{r^4}{r^{4(n+1)}-{\zeta_0}^2\overline{\zeta}^2}+\frac{r^4}{r^{4(n+1)}-\overline{\zeta_0}^2\zeta^2}-\frac{r^4}{r^{4(n+1)}-\overline{\zeta_0}^2\zeta^2}-\frac{r^4}{r^{4(n+1)}-{\zeta_0}^2\overline{\zeta}^2}$$

$$+\frac{r^4}{r^{4(n+1)}-\overline{{\zeta_0}^2\zeta^2}}+\frac{r^4}{r^{4(n+1)}-{\zeta_0}^2\zeta^2}-\frac{r^4}{r^{4(n+1)}-{\zeta_0}^2\zeta^2}-\frac{r^4}{r^{4(n+1)}-\overline{{\zeta_0}^2\zeta^2}}\Bigg)\frac{d\zeta}{\zeta}$$

$$+\lim_{z\to\zeta_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int_r^1\gamma(t)\Bigg\{\frac{t}{|t^2-{\zeta_0}^2|^2}-\frac{t}{|1-t^2{\zeta_0}^2|^2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}t^2-{\zeta_0}^2|^2}+\frac{t}{|r^{4n}{\zeta_0}^2-t^2|^2}-\frac{t}{|r^{4n}{\zeta_0}^2t^2-1|^2}-\frac{t}{|r^{4n}-t^2{\zeta_0}^2|^2}\right)\Bigg\}dt$$

$$+\lim_{z\to\zeta_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int_r^1\gamma(it)\Bigg\{\frac{t}{|t^2+\zeta^2|^2}-\frac{t}{|1+t^2\zeta^2|^2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}\zeta^2+t^2|^2}-\frac{t}{|r^{4n}\zeta^2t^2+1|^2}+\frac{t}{|r^{4n}t^2+\zeta^2|^2}-\frac{t}{|r^{4n}+t^2\zeta^2|^2}\right)\Bigg\}dt,$$

then

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_1R^*\cup\partial_2R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta=-\lim_{\substack{z\to\zeta_0,\\ z\in R^*}}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Gamma_1(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta}=\gamma(\zeta_0),\qquad(2.27)$$

where $\Gamma_1(\zeta)$ is defined as in (2.26) for $|\zeta|=r$.

For $\mathrm{Re}\,\zeta_0>0$, $\mathrm{Im}\,\zeta_0=0$, $\zeta_0=t_0$, $r<t_0<1$ after similar computations one gets

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_3R^*\cup\partial_4R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta=\lim_{z\to t_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int_r^1\gamma(t)\Bigg\{\frac{t}{|t^2-z^2|^2}-\frac{t}{|1-t^2z^2|^2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}t^2-z^2|^2}+\frac{t}{|r^{4n}z^2-t^2|^2}-\frac{t}{|r^{4n}z^2t^2-1|^2}-\frac{t}{|r^{4n}-t^2z^2|^2}\right)\Bigg\}dt.$$

Here

$$|r^{4n}t^2-t_0^2|^2\neq0,\quad |r^{4n}t_0^2-t^2|^2\neq0,\ |r^{4n}t_0^2t^2-1|^2\neq0,\quad |r^{4n}-t^2t_0^2|^2\neq0,$$

then

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_3R^*\cup\partial_4R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta=\lim_{z\to t_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int_r^1\gamma(t)\left\{\frac{t}{|t^2-z^2|^2}-\frac{t}{|1-t^2z^2|^2}\right\}dt=$$

$$\lim_{z\to\zeta_0}\left\{\frac{z-\overline{z}}{2\pi i}\int_r^1\gamma(t)\left[\frac{1}{|t-z|^2}-\frac{1}{|t+z|^2}\right]dt-\frac{z-\overline{z}}{2\pi i}\int_r^1\gamma(t)\left[\frac{1}{|1-tz|^2}-\frac{1}{|1+tz|^2}\right]dt\right\}=$$

$$\lim_{z\to\zeta_0}\left\{\frac{z-\overline{z}}{2\pi i}\left[\int_r^1\gamma(t)\frac{dt}{|t-z|^2}+\int_{-r}^{-1}\gamma(-t)\frac{dt}{|t-z|^2}\right]-\frac{z-\overline{z}}{2\pi i}\left[-\int_{\frac{1}{r}}^{1}\gamma(\tfrac{1}{t})\frac{dt}{|t-z|^2}-\int_{-\frac{1}{r}}^{-1}\gamma(-\tfrac{1}{t})\frac{dt}{|t-z|^2}\right]\right\}=$$

$$\lim_{z\to\zeta_0}\frac{z-\overline{z}}{2\pi i}\left\{\int_r^1\gamma(t)\frac{dt}{|t-z|^2}-\int_{-1}^{-r}\gamma(-t)\frac{dt}{|t-z|^2}-\int_1^{\frac{1}{r}}\gamma(\tfrac{1}{t})\frac{dt}{|t-z|^2}+\int_{-\frac{1}{r}}^{-1}\gamma(-\tfrac{1}{t})\frac{dt}{|t-z|^2}\right\},$$

then

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_3R^*\cup\partial_4R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta=\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\Gamma_2(t)\frac{z-\overline{z}}{|t-z|^2}dt=\gamma(\zeta_0),\qquad(2.28)$$

where

$$\Gamma_2(t) = \begin{cases} \gamma(t), & r \le t \le 1 \\ -\gamma(\frac{1}{t}), & 1 \le t \le \frac{1}{r} \\ 0, & -r < t < r,\ |t| > \frac{1}{r} \\ -\gamma(-t), & -1 \le t \le -r \\ \gamma(-\frac{1}{t}), & -\frac{1}{r} \le t \le -1 \end{cases} \tag{2.29}$$

is piecewise continuous on $\mathbb{R}$.

Finally, for $\operatorname{Re}\zeta_0 = 0$, $\operatorname{Im}\zeta_0 > 0$, $\zeta_0 = it_0$, $r < t_0 < 1$

$$\lim_{z\to\zeta_0}\left\{\frac{1}{\pi i}\int\limits_{\partial_1 R^*\cup\partial_2 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta + \frac{1}{\pi i}\int\limits_{\partial_3 R^*\cup\partial_4 R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta\right\} =$$

$$\lim_{z\to\zeta_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int\limits_r^1\gamma(it)\Bigg\{\frac{t}{|t^2+z^2|^2} - \frac{t}{|1+t^2z^2|^2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}z^2+t^2|^2} - \frac{t}{|r^{4n}z^2t^2+1|^2} + \frac{t}{|r^{4n}t^2+z^2|^2} - \frac{t}{|r^{4n}+t^2z^2|^2}\right)\Bigg\}dt.$$

Since

$$|t^2+{t_0}^2|^2 \ne 0,\quad |1+t^2{t_0}^2|^2 \ne 0,\quad |r^{4n}t^2+{t_0}^2|^2 \ne 0,$$
$$|r^{4n}{t_0}^2+t^2|^2 \ne 0,\quad |r^{4n}{t_0}^2t^2+1|^2 \ne 0,\quad |r^{4n}+t^2{t_0}^2|^2 \ne 0,$$

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial R^*\cap i\mathbb{R}}\gamma(\zeta)K_2(z,\zeta)d\zeta = \lim_{z\to it_0}\frac{(z^2-\overline{z}^2)}{\pi i}\int\limits_r^1\gamma(it)\left\{\frac{t}{|t^2+z^2|^2} - \frac{t}{|1+t^2z^2|^2}\right\}dt =$$

$$\lim_{z\to\zeta_0}\left\{\frac{z+\overline{z}}{2\pi}\int\limits_r^1\gamma(it)\left[\frac{1}{|it-z|^2} - \frac{1}{|it+z|^2}\right]dt + \frac{z+\overline{z}}{2\pi}\int\limits_r^1\gamma(it)\left[\frac{1}{|1+itz|^2} - \frac{1}{|1-itz|^2}\right]dt\right\} =$$

$$\lim_{z\to\zeta_0}\left\{\frac{z+\overline{z}}{2\pi}\left[\int\limits_r^1\gamma(it)\frac{dt}{|it-z|^2} + \int\limits_{-r}^{-1}\gamma(-it)\frac{dt}{|it-z|^2}\right] + \frac{z+\overline{z}}{2\pi}\left[-\int\limits_{\frac{1}{r}}^1\gamma(\frac{i}{t})\frac{dt}{|t+iz|^2} - \int\limits_{-\frac{1}{r}}^{-1}\gamma(-\frac{i}{t})\frac{dt}{|t+iz|^2}\right]\right\}=$$

$$\lim_{z\to\zeta_0}\frac{z+\overline{z}}{2\pi}\left\{\int\limits_r^1\gamma(it)\frac{dt}{|it-z|^2} - \int\limits_{-1}^{-r}\gamma(-it)\frac{dt}{|it-z|^2} + \int\limits_1^{\frac{1}{r}}\gamma(-\frac{1}{it})\frac{dt}{|it-z|^2} - \int\limits_{-\frac{1}{r}}^{-1}\gamma(\frac{1}{it})\frac{dt}{|it-z|^2}\right\},$$

then

$$\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial R^*\cap i\mathbb{R}}\gamma(\zeta)K_2(z,\zeta)d\zeta = \lim_{z\to\zeta_0}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Gamma_2(it)\frac{z+\overline{z}}{|it-z|^2}dt = \gamma(\zeta_0) \tag{2.30}$$

where

$$\Gamma_2(it) = \begin{cases} \gamma(it), & r \le t \le 1 \\ \gamma(-\frac{1}{it}), & 1 \le t \le \frac{1}{r} \\ 0, & -r < t > r,\ |t| > \frac{1}{r} \\ -\gamma(-it), & -1 \le t \le -r \\ -\gamma(\frac{1}{it}), & -\frac{1}{r} \le t \le -1 \end{cases} \tag{2.31}$$

is piecewise continuous on $i\mathbb{R}$. Hence, the equalities (2.21), (2.22) are valid for $\gamma \in C(\partial R^*;\mathbb{C})$. $\square$

The following lemma enables to see the boundary behavior at the corner points of ∂R^*.

According to the representation formula (2.3) the function $w(z) \equiv 1$ can be presented by

$$
\begin{aligned}
1 &= \frac{1}{\pi i} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \left\{ \frac{\zeta^2+z^2}{\zeta^2-z^2} - \frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2} + 2\sum_{n=1}^{\infty}\left[\frac{r^{4n}\zeta^2}{r^{4n}\zeta^2-z^2} - \frac{r^{4n}z^2}{r^{4n}z^2-\zeta^2} + \frac{r^{4n}z^2}{r^{4n}z^2-\overline{\zeta}^2} - \frac{r^{4n}\zeta^2}{r^{4n}\overline{\zeta}^2-z^2} \right]\right\} \frac{d\zeta}{\zeta} \\
&- \frac{1}{\pi i} \int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \left\{ \frac{\zeta^2+z^2}{\zeta^2-z^2} - \frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2} + 2\sum_{n=1}^{\infty}\left[\frac{r^{4n}\zeta^2}{r^{4n}\zeta^2-z^2} - \frac{r^{4n}z^2}{r^{4n}\zeta^2-\zeta^2} + \frac{r^{4n}z^2}{r^{4n}\zeta^2-\overline{\zeta}^2} - \frac{r^{4n}\zeta^2}{r^{4n}\overline{\zeta}^2-z^2} \right]\right\} \frac{d\zeta}{\zeta} \\
&+ \frac{2}{\pi i} \int_r^1 \left\{ \frac{t}{t^2-z^2} - \frac{tz^2}{1-t^2z^2} + \sum_{n=1}^{\infty}\left[\frac{r^{4n}t}{r^{4n}t^2-z^2} - \frac{r^{4n}z^2}{t(r^{4n}z^2-t^2)} + \frac{r^{4n}tz^2}{r^{4n}z^2t^2-1} - \frac{r^{4n}}{t(r^{4n}-z^2t^2)} \right]\right\} dt \\
&- \frac{2}{\pi i} \int_r^1 \left\{ \frac{t}{t^2+z^2} + \frac{tz^2}{1+t^2z^2} + \sum_{n=1}^{\infty}\left[\frac{r^{4n}t}{r^{4n}t^2+z^2} - \frac{r^{4n}z^2}{t(r^{4n}z^2+t^2)} + \frac{r^{4n}tz^2}{r^{4n}z^2t^2+1} - \frac{r^{4n}}{t(r^{4n}+z^2t^2)} \right]\right\} dt.
\end{aligned}
\tag{2.32}
$$

Lemma 2.1.1. *If* $\gamma \in C([r,1];\mathbb{C})$, *then*

$$\lim_{z\in R^*,\, z\to t_0} \frac{1}{\pi i}\int_r^1 [\gamma(\zeta)-\gamma(r)]K_2(z,\zeta)d\zeta = \gamma(t_0)-\gamma(r),\ t_0\in\partial_3R^*\backslash\{1\}, \tag{2.33}$$

$$\lim_{z\in R^*,\, z\to t_0} \frac{1}{\pi i}\int_r^1 [\gamma(\zeta)-\gamma(r)]K_2(z,\zeta)d\zeta = 0,\ t_0\in\partial R^*\backslash[r,1], \tag{2.34}$$

where $K_2(z,\zeta)$ *is given in (2.20).*

Proof. Taking the real part of (2.32) and multiplying the both sides of the equality by $\gamma(r)$ give

$$\gamma(r) = \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(r)K_1(z,\zeta)d\zeta - \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(r)K_1(z,\zeta)d\zeta + \frac{1}{\pi i}\int_r^1 \gamma(r)K_2(z,t)dt - \frac{1}{\pi i}\int_r^1 \gamma(r)K_2(z,it)dt.$$

Let $t_0 \neq 1$ be from the boundary part ∂_3R^*. We consider the difference

$$
\begin{aligned}
\lim_{z\to t_0}\Bigg\{ &\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} [\gamma(\zeta)-\gamma(r)]K_1(z,\zeta)d\zeta - \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} [\gamma(\zeta)-\gamma(r)]K_1(z,\zeta)d\zeta \\
&+ \frac{1}{\pi i}\int_r^1 [\gamma(\zeta)-\gamma(r)]K_2(z,t)dt - \frac{1}{\pi i}\int_r^1 [\gamma(\zeta)-\gamma(r)]K_2(z,it)dt.
\end{aligned}
$$

It is seen that for $t_0 \in \partial_3R^*$ we have to calculate only the following boundary integral

$$
\begin{aligned}
&\lim_{z\to t_0}\frac{1}{\pi i}\int_r^1 [\gamma(\zeta)-\gamma(r)]K_2(z,\zeta)d\zeta = \lim_{z\to t_0}\frac{1}{\pi i}\int_r^1 [\gamma(t)-\gamma(r)]\Bigg\{ \frac{t}{t^2-z^2} - \frac{t}{t^2-\overline{z}^2} \\
&- \frac{tz^2}{1-t^2z^2} + \frac{t\overline{z}^2}{1-t^2z^2} + \sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{t}{r^{4n}t^2-z^2} - \frac{t}{r^{4n}t^2-\overline{z}^2} - \frac{z^2}{t(r^{4n}z^2-t^2)} \\
&+ \frac{\overline{z}^2}{t(r^{4n}\overline{z}^2-t^2)} + \frac{tz^2}{r^{4n}z^2t^2-1} - \frac{t\overline{z}^2}{r^{4n}\overline{z}^2t^2-1} - \frac{1}{t(r^{4n}-t^2z^2)} + \frac{1}{t(r^{4n}-t^2\overline{z}^2)} \Bigg)\Bigg\} dt = \\
&\lim_{z\to t_0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \Lambda_2(t)\frac{z-\overline{z}}{|t-z|^2}dt,\ t_0\in[r,1],\ z\in R^*,
\end{aligned}
\tag{2.35}
$$

where $\Lambda_2(t)$ is defined on the basis of calculations (2.28) and $\Gamma_2(t)$ given in (2.29) and it is

$$\Lambda_2(t) = \begin{cases} \gamma(t) - \gamma(r), & r \le t \le 1 \\ -\gamma(\frac{1}{t}) + \gamma(r), & 1 \le t \le \frac{1}{r} \\ 0, & -r \le t \le r,\ |t| \ge \frac{1}{r} \\ -\gamma(-t) + \gamma(r), & -1 \le t \le -r \\ \gamma(-\frac{1}{t}) - \gamma(r), & -\frac{1}{r} \le t \le -1 \end{cases} \tag{2.36}$$

Because $\Lambda_2(t)$ defined on $\mathbb{R}$ is continuous in (-1;1), the relation (2.33) holds. Thus, in particular

$$\lim_{z \to r} \frac{1}{\pi i} \int_r^1 \gamma(\zeta) K_2(z,\zeta) d\zeta = \gamma(r) \tag{2.37}$$

follows. If $t_0 \in \partial R^* \backslash [r,1]$, then the boundary integral tends to 0 for $z \to t_0$ and then the equality (2.34) holds. □

Lemma 2.1.2. *If* $\gamma \in C(i[r,1]; \mathbb{C})$, *then*

$$-\lim_{z \in R^*,\, z \to it_0} \frac{1}{\pi i} \int_r^1 [\gamma(\zeta) - \gamma(ir)] K_2(z,\zeta) d\zeta = \gamma(it_0) - \gamma(ir),\ t_0 \in \partial_4 R^* \backslash \{1\}, \tag{2.38}$$

$$-\lim_{z \in R^*,\, z \to it_0} \frac{1}{\pi i} \int_r^1 [\gamma(\zeta) - \gamma(ir)] K_2(z,\zeta) d\zeta = 0,\ t_0 \in \partial R^* \backslash \partial_4 R^*. \tag{2.39}$$

Proof. Similarly as before, the boundary integral over the interval [r,1] of the imaginary axis remains

$$\begin{aligned} & -\lim_{z \to it_0} \frac{1}{\pi i} \int_r^1 [\gamma(\zeta) - \gamma(ir)] K_2(z,\zeta) d\zeta = \lim_{z \to it_0} \frac{1}{\pi} \int_r^1 [\gamma(it) - \gamma(ir)] \Bigg\{ \frac{it}{t^2 + z^2} - \frac{it}{t^2 + \overline{z}^2} \\ & + \frac{itz^2}{1 + t^2 z^2} - \frac{it\overline{z}^2}{1 + t^2 \overline{z}^2} + \sum_{n=1}^{\infty} r^{4n} \Bigg(\frac{it}{r^{4n} t^2 + z^2} - \frac{it}{r^{4n} t^2 + \overline{z}^2} - \frac{z^2}{it(r^{4n} z^2 + t^2)} \\ & + \frac{\overline{z}^2}{it(r^{4n} \overline{z}^2 + t^2)} + \frac{itz^2}{r^{4n} z^2 t^2 + 1} - \frac{it\overline{z}^2}{r^{4n} \overline{z}^2 t^2 + 1} - \frac{1}{it(r^{4n} + t^2 z^2)} + \frac{1}{it(r^{4n} + t^2 \overline{z}^2)} \Bigg) \Bigg\} dt = \\ & \lim_{z \to it_0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Lambda_2(it) \frac{z + \overline{z}}{|it - z|^2} dt,\ t_0 \in [r,1],\ z \in R^*, \end{aligned} \tag{2.40}$$

where $\Lambda_2(it)$ is obtained on the basis of (2.30) and $\Gamma_2(it)$ given in (2.31) and it is

$$\Lambda_2(it) = \begin{cases} \gamma(it) - \gamma(ir), & r \le t \le 1 \\ \gamma(-\frac{1}{it}) - \gamma(ir), & 1 \le t \le \frac{1}{r} \\ 0, & -r < t < r,\ |t| > \frac{1}{r} \\ -\gamma(-it) + \gamma(ir), & -1 \le t \le -r \\ -\gamma(\frac{1}{it}) + \gamma(ir), & -\frac{1}{r} \le t \le -1. \end{cases} \tag{2.41}$$

Because $\Lambda_2(it)$ defined for $t \in \mathbb{R}$ is continuous in (-1;1), the relation (2.38) holds. Thus, in particular

$$-\lim_{z \to ir} \frac{1}{\pi i} \int_r^1 \gamma(\zeta) K_2(z,\zeta) d\zeta = \gamma(ir) \tag{2.42}$$

follows. Finally, if $it_0 \in \partial R^* \backslash \partial_4 R^*$, then $K_2(\zeta, z)$ tends to 0 for $z \to it_0$. □

Theorem 2.1.3. *The Schwarz problem*

$$w_{\overline{z}} = f \text{ in } R^*, f \in L_p(R^*;\mathbb{C}), p > 2, \operatorname{Re} w = \gamma \text{ on } \partial R^*,$$
$$\gamma \in C(\partial R^*;\mathbb{C}), \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \operatorname{Im} w(e^{i\varphi})d\varphi = c,\ c \in \mathbb{R} \tag{2.43}$$

is uniquely solvable in the space of functions with generalized derivatives with respect to $\overline{z}$ *by*

$$\begin{aligned}
w(z) &= \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)\Bigg\{\frac{\zeta^2+z^2}{\zeta^2-z^2} - \frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2}\\
&+2\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-z^2} - \frac{z^2}{r^{4n}z^2-\zeta^2} + \frac{z^2}{r^{4n}z^2-\overline{\zeta}^2} - \frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}\right]\Bigg\}\frac{d\zeta}{\zeta}\\
&-\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)\Bigg\{\frac{\zeta^2+z^2}{\zeta^2-z^2} - \frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2}\\
&+2\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2-z^2} - \frac{z^2}{r^{4n}z^2-\zeta^2} + \frac{z^2}{r^{4n}z^2-\overline{\zeta}^2} - \frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}\right]\Bigg\}\frac{d\zeta}{\zeta}\\
&+\frac{2}{\pi i}\int_r^1 \gamma(t)\Bigg\{\frac{t}{t^2-z^2} - \frac{tz^2}{1-t^2z^2}\\
&+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t}{r^{4n}t^2-z^2} - \frac{z^2}{t(r^{4n}z^2-t^2)} + \frac{tz^2}{r^{4n}z^2t^2-1} - \frac{1}{t(r^{4n}-z^2t^2)}\right]\Bigg\}dt\\
&-\frac{2}{\pi i}\int_r^1 \gamma(it)\Bigg\{\frac{t}{t^2+z^2} + \frac{tz^2}{1+t^2z^2}\\
&+\sum_{n=1}^{\infty} r^{4n}\left[\frac{t}{r^{4n}t^2+z^2} - \frac{z^2}{t(r^{4n}z^2+t^2)} + \frac{tz^2}{r^{4n}z^2t^2+1} - \frac{1}{t(r^{4n}+z^2t^2)}\right]\Bigg\}dt + ic\\
&-\frac{2}{\pi}\int\limits_{R^*}\Bigg\{f(\zeta)\Bigg[\frac{\zeta}{\zeta^2-z^2} + \frac{z^2\zeta}{z^2\zeta^2-1}\\
&+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta}{r^{4n}\zeta^2-z^2} - \frac{z^2}{\zeta(r^{4n}z^2-\zeta^2)} + \frac{\zeta z^2}{r^{4n}\zeta^2z^2-1} - \frac{1}{\zeta(r^{4n}-\zeta^2z^2)}\right)\Bigg]\\
&-\overline{f(\zeta)}\Bigg[\frac{\overline{\zeta}}{\overline{\zeta}^2-z^2} + \frac{z^2\overline{\zeta}}{z^2\overline{\zeta}^2-1}\\
&+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2-z^2} - \frac{z^2}{\overline{\zeta}(r^{4n}z^2-\overline{\zeta}^2)} + \frac{\overline{\zeta}z^2}{r^{4n}\overline{\zeta}^2z^2-1} - \frac{1}{\overline{\zeta}(r^{4n}-\overline{\zeta}^2z^2)}\right)\Bigg]\Bigg\}d\xi d\eta
\end{aligned} \tag{2.44}$$

Proof. Theorem 2.1.1 provides representation forms for any smooth enough function. Therefore if the Schwarz problem has a solution, it has to be of the form (2.44) with an analytic function on the right-hand side up to the Pompeiu-type operator [42], [5]. Let us denote by

$$\begin{aligned}
T_{R^*}f(z) &= -\frac{2}{\pi}\int\limits_{R^*}\Bigg\{f(\zeta)\Bigg[\frac{\zeta}{\zeta^2-z^2} + \frac{z^2\zeta}{z^2\zeta^2-1}\\
&+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta}{r^{4n}\zeta^2-z^2} - \frac{z^2}{\zeta(r^{4n}z^2-\zeta^2)} + \frac{\zeta z^2}{r^{4n}\zeta^2z^2-1} - \frac{1}{\zeta(r^{4n}-\zeta^2z^2)}\right)\Bigg]
\end{aligned} \tag{2.45}$$

$$-\overline{f(\zeta)}\left[\frac{\overline{\zeta}}{\overline{\zeta}^2 - z^2} + \frac{z^2\overline{\zeta}}{z^2\overline{\zeta}^2 - 1}\right.$$

$$\left.\left.+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2 - z^2} - \frac{z^2}{\overline{\zeta}(r^{4n}z^2 - \overline{\zeta}^2)} + \frac{\overline{\zeta}z^2}{r^{4n}\overline{\zeta}^2 z^2 - 1} - \frac{1}{\overline{\zeta}(r^{4n} - \overline{\zeta}^2 z^2)}\right)\right]\right\}d\xi d\eta,$$

which can be easily represented by

$$\begin{aligned} T_{R^*}f(z) \;\; = \;\; & -\frac{1}{\pi}\int_{R^*} f(\zeta)\frac{d\xi d\eta}{\zeta - z} - \frac{2}{\pi}\int_{R^*}\left\{f(\zeta)\left[\frac{1}{2}\frac{1}{\zeta + z} + \frac{z^2\zeta}{z^2\zeta^2 - 1}\right.\right. \\ & \left.+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta}{r^{4n}\zeta^2 - z^2} - \frac{z^2}{\zeta(r^{4n}z^2 - \zeta^2)} + \frac{\zeta z^2}{r^{4n}\zeta^2 z^2 - 1} - \frac{1}{\zeta(r^{4n} - \zeta^2 z^2)}\right)\right] \\ & -\overline{f(\zeta)}\left[\frac{\overline{\zeta}}{\overline{\zeta}^2 - z^2} + \frac{z^2\overline{\zeta}}{z^2\overline{\zeta}^2 - 1}\right. \\ & \left.\left.+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2 - z^2} - \frac{z^2}{\overline{\zeta}(r^{4n}z^2 - \overline{\zeta}^2)} + \frac{\overline{\zeta}z^2}{r^{4n}\overline{\zeta}^2 z^2 - 1} - \frac{1}{\overline{\zeta}(r^{4n} - \overline{\zeta}^2 z^2)}\right)\right]\right\}d\xi d\eta. \end{aligned} \tag{2.46}$$

Thus, by the property of the Pompeiu operator $\partial_{\overline{z}}Tf = f$, the operator $T_{R^*}f(z)$ provides a solution of the Cauchy-Riemann equation $w_{\overline{z}} = f$ in a weak sense.
Then the side condition and boundary behavior are to be checked. Consider the proof piece by piece.

Part1. Side condition. At first the boundary integral has to be studied.
For $|\zeta| = 1$, $\operatorname{Re}\zeta > 0$, $\operatorname{Im}\zeta > 0$

$$\frac{1}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{\zeta^2 + z^2}{\zeta^2 - z^2} - \frac{\overline{\zeta}^2 + z^2}{\overline{\zeta}^2 - z^2}\right]\frac{dz}{z} = \frac{1}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{\zeta^2 + z^2}{\zeta^2 - z^2} - \frac{\overline{z}^2 + \zeta^2}{\overline{z}^2 - \zeta^2}\right]\frac{dz}{z} =$$

$$\frac{2}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{z^2}{\zeta^2 - z^2} + \frac{\overline{z}^2}{\zeta^2 - \overline{z}^2} + 1\right]\frac{dz}{z} =$$

$$\frac{1}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{1}{\zeta - z} - \frac{1}{\zeta + z}\right]dz - \frac{1}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{1}{\zeta - \overline{z}} - \frac{1}{\zeta + \overline{z}}\right]d\overline{z} + \frac{2}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\frac{dz}{z} =$$

$$\frac{1}{\pi i}\left\{\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\frac{dz}{\zeta - z} + \int_{\substack{|z|=1,\\0>\operatorname{Im} z,\\0>\operatorname{Re} z}}\frac{dz}{\zeta - z} + \int_{\substack{|z|=1,\\0>\operatorname{Im} z,\\0<\operatorname{Re} z}}\frac{dz}{\zeta - z} + \int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0>\operatorname{Re} z}}\frac{dz}{\zeta - z}\right\} + 1 = 1 - \frac{1}{\pi i}\int_{|z|=1}\frac{dz}{z - \zeta} = 0.$$

The sum

$$\frac{2}{\pi i}\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta^2}{r^{4n}\zeta^2 - z^2} - \frac{z^2}{r^{4n}z^2 - \zeta^2} + \frac{z^2}{r^{4n}z^2 - \overline{\zeta}^2} - \frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2 - z^2}\right]\frac{dz}{z}$$

is divided and two integrals are calculated:

$$\sum_{n=1}^{\infty}\frac{r^{2n}}{\pi i}\left\{\int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{\zeta}{r^{2n}\zeta + z}\right]\frac{dz}{z} - \int_{\substack{|z|=1,\\0<\operatorname{Im} z,\\0<\operatorname{Re} z}}\left[\frac{\zeta}{r^{2n}\zeta - \overline{z}} + \frac{\zeta}{r^{2n}\zeta + \overline{z}}\right]\frac{d\overline{z}}{\overline{z}}\right\} =$$

$$\sum_{n=1}^{\infty}\frac{r^{2n}}{\pi i}\int_{|z|=1}\frac{\zeta}{r^{2n}\zeta - z}\frac{dz}{z} = -\sum_{n=1}^{\infty}\frac{1}{\pi i}\int_{|z|=1}\left(\frac{1}{z - r^{2n}\zeta} - \frac{1}{z}\right)dz = -\sum_{n=1}^{\infty}\frac{1}{\pi i}(2\pi i - 2\pi i) = 0$$

and

$$-\sum_{n=1}^{\infty}\frac{r^{2n}}{\pi i}\left\{\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left[\frac{z}{r^{2n}z-\zeta}+\frac{z}{r^{2n}z+\zeta}\right]\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left[\frac{\overline{z}}{r^{2n}\overline{z}-\zeta}+\frac{\overline{z}}{r^{2n}\overline{z}+\zeta}\right]\frac{d\overline{z}}{\overline{z}}\right\}=$$

$$-\sum_{n=1}^{\infty}\frac{r^{2n}}{\pi i}\int\limits_{|z|=1}\frac{z}{r^{2n}z-\zeta}\frac{dz}{z}=0.$$

Calculating the boundary integral on $|\zeta|=r$, $\operatorname{Re}\zeta>0$, $\operatorname{Im}\zeta>0$ gives

$$\frac{2}{\pi i}\int\limits_{\substack{|z|=r,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left\{\frac{\zeta^2}{\zeta^2-z^2}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}-\frac{r^4\overline{z}^2}{r^4\overline{z}^2-\overline{\zeta}^2}+\frac{r^4\overline{z}^2}{r^4\overline{z}^2-\zeta^2}\right.$$

$$\left.+\sum_{n=1}^{\infty}\left[\frac{r^{4n}\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}-\frac{r^{4n}z^2}{r^{4n}z^2-\zeta^2}+\frac{r^{4n}z^2}{r^{4n}z^2-\overline{\zeta}^2}-\frac{r^{4n}\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}\right]\right\}\frac{dz}{z}=$$

$$\frac{2}{\pi i}\left\{\int\limits_{\substack{|z|=r,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left[\frac{\zeta^2}{\zeta^2-z^2}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}-\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}\right]\right.$$

$$\left.+\sum_{n=1}^{\infty}\left[\frac{r^{4n}\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}-\frac{r^{4n}z^2}{r^{4n}z^2-\zeta^2}+\frac{r^{4n}z^2}{r^{4n}z^2-\overline{\zeta}^2}-\frac{r^{4n}\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}\right]\right\}\frac{dz}{z}=$$

$$\frac{4}{\pi}\int\limits_{\substack{|z|=r,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\sum_{n=1}^{\infty}\operatorname{Im}\left[-\frac{r^{4n}z^2}{r^{4n}z^2-\zeta^2}-\frac{r^{4n}\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}\right]\frac{dz}{z}=\frac{2}{\pi}\left\{\int\limits_{\substack{|z|=r,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\sum_{n=1}^{\infty}\operatorname{Im}\left[\frac{z}{\zeta-r^{2n}z}-\frac{z}{\zeta+r^{2n}z}\right]\frac{dz}{z}\right.$$

$$\left.-\int\limits_{\substack{|z|=r,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\sum_{n=1}^{\infty}\operatorname{Im}\left[\frac{\overline{z}}{\zeta-r^{2n}\overline{z}}-\frac{\overline{z}}{\zeta+r^{2n}\overline{z}}\right]\frac{d\overline{z}}{\overline{z}}\right\}=-\sum_{n=1}^{\infty}\frac{2}{\pi r^{2n}}\operatorname{Im}\int\limits_{|z|=1}\frac{dz}{z-\frac{\zeta}{r^{2n}}}=0.$$

For $r<t<1,\ \zeta=t$

$$\frac{2}{\pi i}\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left(\frac{t}{t^2-z^2}-\frac{tz^2}{1-t^2z^2}\right)\frac{dz}{z}=\frac{1}{\pi i}\left\{\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{t}{t^2-z^2}\frac{dz}{z}+\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{t}{\overline{z}^2-t^2}\frac{d\overline{z}}{\overline{z}}\right\}=$$

$$\frac{1}{\pi i}\left\{\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{t-z}\frac{dz}{z}+\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}\zeta,\\0>\operatorname{Re}\zeta}}\frac{1}{t-z}\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{z-t}\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0>\operatorname{Re}z}}\frac{1}{z-t}\frac{dz}{z}\right\}=-\frac{1}{\pi i}\int\limits_{|z|=1}\frac{1}{z-t}\frac{dz}{z}=0.$$

The terms in the sum are rewritten as

$$\frac{2}{\pi i}\sum_{n=1}^{\infty}r^{4n}\left\{\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left[\frac{t}{r^{4n}t^2-z^2}-\frac{1}{t(r^{4n}-z^2t^2)}\right]\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\left[\frac{t}{r^{4n}t^2-\overline{z}^2}-\frac{1}{t(r^{4n}-\overline{z}^2t^2)}\right]\frac{d\overline{z}}{\overline{z}}\right\}=$$

$$\sum_{n=1}^{\infty}\frac{r^{2n}}{\pi i}\left\{\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{r^{2n}t-z}\frac{dz}{z}+\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0>\operatorname{Re}z}}\frac{1}{r^{2n}t-z}\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0<\operatorname{Im}\zeta,\\0<\operatorname{Re}\zeta}}\frac{1}{t(r^{2n}-tz)}\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0>\operatorname{Re}z}}\frac{1}{t(r^{2n}-tz)}\frac{dz}{z}\right.$$

$$\left.+\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{r^{2n}t-z}\frac{dz}{z}+\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{r^{2n}t+z}\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{t(r^{2n}-tz)}\frac{dz}{z}-\int\limits_{\substack{|z|=1,\\0>\operatorname{Im}z,\\0<\operatorname{Re}z}}\frac{1}{t(r^{2n}+tz)}\frac{dz}{z}\right\}=$$

$$\sum_{n=1}^{\infty} r^{2n} \frac{1}{\pi i} \int\limits_{|z|=1} \left[\frac{1}{r^{2n}t - z} - \frac{1}{t(r^{2n} - tz)} \right] \frac{dz}{z} = 0.$$

For $r < t < 1,\ \zeta = it$ similarly

$$\frac{2}{\pi i} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{t}{t^2 + z^2} + \frac{tz^2}{1 + t^2 z^2} \right] \frac{dz}{z} = \frac{2}{\pi i} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{t}{t^2 - (iz)^2} - \frac{t(iz)^2}{1 - t^2 (iz)^2} \right] \frac{dz}{z} =$$

$$\frac{2}{\pi i} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \frac{t}{t^2 - (iz)^2} \frac{dz}{z} + \frac{2}{\pi i} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \frac{t}{(i\overline{z})^2 - t^2} \frac{d\overline{z}}{\overline{z}} = \frac{1}{\pi} \int\limits_{|z|=1} \frac{1}{z + it} \frac{dz}{z} = 0$$

and the sum is

$$\frac{2}{\pi i} \sum_{n=1}^{\infty} r^{4n} \Bigg\{ \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{t}{r^{4n} t^2 - (iz)^2} - \frac{1}{t(r^{4n} - (iz)^2 t^2)} \right] \frac{dz}{z} + \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{1}{t(r^{4n} - (i\overline{z})^2 t^2)} \right.$$

$$\left. - \frac{t}{r^{4n} t^2 - (i\overline{z})^2} \right] \frac{d\overline{z}}{\overline{z}} \Bigg\} = \frac{1}{\pi i} \sum_{n=1}^{\infty} r^{2n} \int\limits_{|z|=1} \left[\frac{1}{iz - r^{2n} t} - \frac{1}{t(r^{2n} - itz)} \right] \frac{dz}{z} = 0.$$

Treating the side condition for the area integral, we observe

$$\frac{2}{\pi} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{z^2 \zeta}{z^2 \zeta^2 - 1} + \frac{\zeta}{\zeta^2 - z^2} \right] \frac{dz}{z} = \frac{2}{\pi} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{\zeta}{\zeta^2 - \overline{z}^2} + \frac{\zeta}{\zeta^2 - z^2} \right] \frac{dz}{z} =$$

$$\frac{1}{\pi} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{1}{\zeta - z} + \frac{1}{\zeta + z} \right] \frac{dz}{z} - \frac{1}{\pi} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{1}{\zeta - \overline{z}} + \frac{1}{\zeta + \overline{z}} \right] \frac{d\overline{z}}{\overline{z}} =$$

$$\frac{1}{\pi} \Bigg\{ \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \frac{1}{\zeta - z} \frac{dz}{z} + \int\limits_{\substack{|z|=1, \\ 0>\operatorname{Im} z, \\ 0>\operatorname{Re} z}} \frac{1}{\zeta - z} \frac{dz}{z} + \int\limits_{\substack{|z|=1, \\ 0>\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \frac{1}{\zeta - z} \frac{dz}{z} + \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0>\operatorname{Re} z}} \frac{1}{\zeta - z} \frac{dz}{z} \Bigg\} =$$

$$-\frac{1}{\pi} \int\limits_{|z|=1} \frac{1}{z - \zeta} \frac{dz}{z} = 0 \quad \text{for } r < |\zeta| < 1.$$

Integrating the terms of the first sum under the area integral, it is similarly reduced to the integral on the whole unit circle

$$\frac{2}{\pi i} \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \sum_{n=1}^{\infty} r^{4n} \left[\frac{\zeta}{r^{4n}\zeta^2 - z^2} - \frac{z^2}{\zeta(r^{4n} z^2 - \zeta^2)} + \frac{\zeta z^2}{r^{4n} z^2 \zeta^2 - 1} - \frac{1}{\zeta(r^{4n} - z^2 \zeta^2)} \right] \frac{dz}{z} =$$

$$\frac{2}{\pi i} \sum_{n=1}^{\infty} r^{4n} \Bigg\{ \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{\zeta}{r^{4n}\zeta^2 - z^2} - \frac{1}{\zeta(r^{4n} - z^2\zeta^2)} \right] \frac{dz}{z} - \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \left[\frac{\zeta}{r^{4n}\zeta^2 - \overline{z}^2} - \frac{1}{\zeta(r^{4n} - \overline{z}^2\zeta^2)} \right] \frac{d\overline{z}}{\overline{z}} \Bigg\} =$$

$$\sum_{n=1}^{\infty} \frac{r^{2n}}{\pi i} \Bigg\{ \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} z, \\ 0<\operatorname{Re} z}} \frac{1}{r^{2n}\zeta - z} \frac{dz}{z} + \int\limits_{\substack{|z|=1, \\ 0>\operatorname{Im} z, \\ 0>\operatorname{Re} z}} \frac{1}{r^{2n}\zeta - z} \frac{dz}{z} - \int\limits_{\substack{|z|=1, \\ 0<\operatorname{Im} \zeta, \\ 0<\operatorname{Re} \zeta}} \frac{1}{\zeta(r^{2n} - \zeta z)} \frac{dz}{z} - \int\limits_{\substack{|z|=1, \\ 0>\operatorname{Im} z, \\ 0>\operatorname{Re} z}} \frac{1}{\zeta(r^{2n} - \zeta z)} \frac{dz}{z}$$

$$+\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \frac{1}{r^{2n}\zeta - z}\frac{dz}{z} + \int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \frac{1}{r^{2n}\zeta + z}\frac{dz}{z} - \int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \frac{1}{\zeta(r^{2n} - \zeta z)}\frac{dz}{z} - \int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \frac{1}{\zeta(r^{2n} + \zeta z)}\frac{dz}{z}\Bigg\} =$$

$$\frac{1}{\pi i}\sum_{n=1}^{\infty} r^{2n} \int\limits_{|z|=1} \left[\frac{1}{r^{2n}\zeta - z} - \frac{1}{\zeta(r^{2n} - \zeta z)}\right]\frac{dz}{z} = 0.$$

Similarly, for $r < |\zeta| < 1$

$$-\frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \left[\frac{\overline{\zeta}}{\overline{\zeta}^2 - z^2} + \frac{z^2\overline{\zeta}^2}{z^2\overline{\zeta}^2 - 1}\right]\frac{dz}{z} = \frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \left[\frac{\zeta}{\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2\zeta}{\overline{z}^2\zeta^2 - 1}\right]\frac{dz}{z} =$$

$$-\frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \left[\frac{\zeta}{\zeta^2 - \overline{z}^2} + \frac{\zeta}{\zeta^2 - z^2}\right]\frac{dz}{z} = -\frac{1}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \left[\frac{1}{\zeta - z} + \frac{1}{\zeta + z}\right]\frac{dz}{z}$$

$$+\frac{1}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \left[\frac{1}{\zeta - \overline{z}} + \frac{1}{\zeta + \overline{z}}\right]\frac{d\overline{z}}{\overline{z}} = \frac{1}{\pi}\int\limits_{|z|=1} \frac{1}{z - \zeta}\frac{dz}{z} = 0.$$

And the complex conjugation at sum under the area integral is

$$-\frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \sum_{n=1}^{\infty} r^{4n}\left[\frac{\overline{\zeta}}{r^{4n}\overline{\zeta}^2 - z^2} - \frac{z^2}{\overline{\zeta}(r^{4n}z^2 - \overline{\zeta}^2)} + \frac{\overline{\zeta}z^2}{r^{4n}\overline{\zeta}^2 z^2 - 1} - \frac{1}{\overline{\zeta}(r^{4n} - \overline{\zeta}^2 z^2)}\right]\frac{dz}{z} =$$

$$-\frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta}{r^{4n}\zeta^2 - \overline{z}^2} - \frac{\overline{z}^2}{\zeta(r^{4n}\overline{z}^2 - \zeta^2)} + \frac{\zeta\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2 - 1} - \frac{1}{\zeta(r^{4n} - \zeta^2\overline{z}^2)}\right]\frac{dz}{z} =$$

$$-\frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \sum_{n=1}^{\infty} r^{4n}\left[\frac{\zeta}{r^{4n}\zeta^2 - z^2} - \frac{1}{\zeta(r^{4n} - \zeta^2 z^2)}\right]\frac{dz}{z} - \frac{2}{\pi}\int\limits_{\substack{|z|=1,\\ 0>\operatorname{Im} z,\\ 0>\operatorname{Re} z}} \sum_{n=1}^{\infty} r^{4n}\Bigg[\frac{\zeta}{r^{4n}\zeta^2 - \overline{z}^2}$$

$$-\frac{1}{\zeta(r^{4n} - \zeta^2\overline{z}^2)}\Bigg]\frac{d\overline{z}}{\overline{z}} = -\frac{1}{\pi}\sum_{n=1}^{\infty} r^{2n}\int\limits_{|z|=1} \left[\frac{1}{r^{2n}\zeta - z} - \frac{1}{\zeta(r^{2n} - \zeta z)}\right]\frac{dz}{z} = 0.$$

Hence, from Part 1 the following computations

$$\frac{1}{\pi i}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z,\\ 0<\operatorname{Re} z}} w(z)\frac{dz}{z} = \frac{c}{\pi}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \frac{dz}{z},$$

$$\frac{1}{\pi i}\int\limits_{\substack{|z|=1,\\ 0<\operatorname{Im} z,\\ 0<\operatorname{Re} z}} \operatorname{Im} w(z)\frac{dz}{z} = \frac{c}{2},\ \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \operatorname{Im} w(e^{i\varphi})d\varphi = c.$$

follow.

Part 2. Boundary behavior. Let s_1 denote the area integral, then

$$\operatorname{Re} s_1(z) \quad = \quad \frac{1}{\pi}\int\limits_{R^*}\left\{f(\zeta)K_2(z,\zeta) - \overline{f(\zeta)}K_2(\overline{z},\zeta)\right\}d\xi d\eta,$$

where $K_2(z,\zeta)$ is given in (2.20). The real part of the area integral turns on to be zero on the boundary ∂R^* as can be verified by simple calculations on the boundary parts:
$\{|z|=1, \operatorname{Re} z>0, \operatorname{Im} z>0\}$ and $\{|z|=r, \operatorname{Re} z>0, \operatorname{Im} z>0\}$,
$\{z=\overline{z}, \operatorname{Re} z>0, \operatorname{Im} z=0\}$ and $\{z=-\overline{z}, \operatorname{Re} z=0, \operatorname{Im} z>0\}$.

Let s_2 denote the sum of the boundary integrals. Then

$$\begin{aligned}\operatorname{Re} s_2(z) \;=\; & \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)K_1(\zeta,z)\frac{d\zeta}{\zeta} - \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)K_1(\zeta,z)\frac{d\zeta}{\zeta}\\ & +\frac{1}{\pi i}\int\limits_r^1 \gamma(t)K_2(t,z)dt - \frac{1}{\pi i}\int\limits_r^1 \gamma(it)K_2(it,z)dt.\end{aligned}$$

To complete the proof one has to study the relations for the Schwarz operator for the unit disc $\mathbb{D}$ and the half plane [5]

$$\operatorname{Re} S\varphi = \varphi \,\text{on}\, \partial D, \text{ i.e. } \lim_{z\to\zeta} S\varphi(z) = \varphi(\zeta),\ \zeta\in\partial D.$$

By the Theorem 2.1.2, this boundary condition holds for the domain R^*. The boundary behavior at the corner points is also studied in Lemmas (2.1.1) - (2.1.2). $\square$

2.2 Dirichlet problem

Using the Cauchy formula for the analytic functions in D and the results of the previous section, the Dirichlet problem is now solved as well for the homogeneous as for the inhomogeneous Cauchy-Riemann equations.

2.2.1 Dirichlet problem for analytic functions

Theorem 2.2.1. *The Dirichlet problem*

$$\begin{gathered} w_{\overline{z}} = 0 \text{ in } R^*,\ w=\gamma \text{ on } \partial R^*,\\ \gamma\in C(\partial R^*;\mathbb{C}) \end{gathered} \tag{2.47}$$

is uniquely solvable if and only if

$$\begin{aligned}&\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1}\\ &+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta}=0,\end{aligned} \tag{2.48}$$

and the solution can be presented as

$$\begin{aligned}w(z) \;=\; &\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\zeta^2 z^2}{\zeta^2 z^2-1}\\ &+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2 z^2}{r^{4n}\zeta^2 z^2-1}+\frac{1}{\zeta^2 z^2-r^{4n}}\right)\Bigg\}\frac{d\zeta}{\zeta},\ z\in R^*\end{aligned} \tag{2.49}$$

Proof. At first the condition in (2.48) is shown to be necessary.
Let w defined by (2.49) be a solution to the Dirichlet problem. This formula can be decomposed into the sum of Cauchy type integrals. Then the equality

$$\lim_{z\to\zeta, z\in R^*} w(z) = \gamma(\zeta) \,\text{for any}\, \zeta\in\partial R^*, \tag{2.50}$$

holds. Consider for $|z|<1$ the function (2.49) at the reflected point $\frac{1}{\overline{z}}$

$$\begin{aligned} w(\tfrac{1}{\overline{z}}) &= \frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1} \\ &\quad+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta} \end{aligned} \tag{2.51}$$

and take the difference

$$\begin{aligned} w(z)-w(\tfrac{1}{\overline{z}}) &= \frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\zeta^2z^2}{\zeta^2z^2-1}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}-\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1} \\ &\quad+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta^2z^2-r^{4n}}\right. \\ &\quad\left.-\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}-\frac{1}{\zeta^2\overline{z}^2-r^{4n}}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}-\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta}. \end{aligned} \tag{2.52}$$

Let $L(\zeta,z)$ denote the sum of integrands, then

$$\begin{aligned} w(z)-w(\tfrac{1}{\overline{z}}) &= \frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)L(\zeta,z)\frac{d\zeta}{\zeta}=\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{z^2}{z^2-\overline{\zeta}^2}-\frac{\overline{z}^2}{\overline{z}^2-\overline{\zeta}^2} \\ &\quad+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}+\frac{\overline{\zeta}^2}{z^2-r^{4n}\overline{\zeta}^2}\right. \\ &\quad\left.-\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}-\frac{\overline{\zeta}^2}{\overline{z}^2-r^{4n}\overline{\zeta}^2}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}-\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta} \\ &\quad-\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2|z|^4}{\zeta^2|z|^4-\overline{z}^2}-\frac{\zeta^2|z|^4}{\zeta^2|z|^4-z^2} \\ &\quad+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2|z|^4}{r^{4n}\zeta^2|z|^4-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2|z|^4-r^{4n}\overline{z}^2}\right. \\ &\quad\left.-\frac{\zeta^2|z|^4}{r^{4n}\zeta^2|z|^4-z^2}-\frac{z^2}{\zeta^2|z|^4-r^{4n}z^2}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}-\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta} \\ &\quad+\frac{1}{\pi i}\int_r^1\gamma(t)\Bigg\{\frac{t^2}{t^2-z^2}-\frac{t^2}{t^2-\overline{z}^2}+\frac{t^2|z|^4}{t^2|z|^4-\overline{z}^2}-\frac{t^2|z|^4}{t^2|z|^4-z^2} \\ &\quad+\sum_{n=1}^{\infty}r^{4n}\left(\frac{t^2}{r^{4n}t^2-z^2}+\frac{z^2}{t^2-r^{4n}z^2}+\frac{t^2|z|^4}{r^{4n}t^2|z|^4-\overline{z}^2}+\frac{\overline{z}^2}{t^2|z|^4-r^{4n}\overline{z}^2}\right. \\ &\quad\left.-\frac{t^2|z|^4}{r^{4n}t^2|z|^4-z^2}-\frac{z^2}{t^2|z|^4-r^{4n}z^2}-\frac{t^2}{r^{4n}t^2-\overline{z}^2}-\frac{\overline{z}^2}{t^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{dt}{t} \\ &\quad-\frac{1}{\pi i}\int_r^1\gamma(it)\Bigg\{\frac{t^2}{t^2+z^2}-\frac{t^2}{t^2+\overline{z}^2}+\frac{t^2|z|^4}{t^2|z|^4+\overline{z}^2}-\frac{t^2|z|^4}{t^2|z|^4+z^2} \\ &\quad+\sum_{n=1}^{\infty}r^{4n}\left(\frac{t^2}{r^{4n}t^2+z^2}-\frac{z^2}{t^2+r^{4n}z^2}+\frac{t^2|z|^4}{r^{4n}t^2|z|^4+\overline{z}^2}-\frac{\overline{z}^2}{t^2|z|^4+r^{4n}\overline{z}^2}\right. \\ &\quad\left.-\frac{t^2|z|^4}{r^{4n}t^2|z|^4+z^2}+\frac{z^2}{t^2|z|^4+r^{4n}z^2}-\frac{t^2}{r^{4n}t^2+\overline{z}^2}+\frac{\overline{z}^2}{t^2+r^{4n}\overline{z}^2}\right)\Bigg\}\frac{dt}{t}. \end{aligned}$$

Calculating the boundary integrals separately, letting $z \to \zeta$, $z \in R^*$, $\zeta \in \partial_1 R^*$, it is seen that solely the integral on the outer circle is to be studied.

$$\lim_{z\to\zeta,|z|<1}[w(z)-w(\frac{1}{\overline{z}})] = \lim_{z\to\zeta,|z|<1}\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{z^2}{z^2-\overline{\zeta}^2}-\frac{\overline{z}^2}{\overline{z}^2-\overline{\zeta}^2}$$
$$+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}-\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}+\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-\overline{z}^2}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}\right)$$
$$+\left(\frac{z^2r^{4n}}{\zeta^2-r^{4n}z^2}+1-\frac{z^2r^{4n}}{\overline{\zeta}^2-r^{4n}z^2}-1+\frac{\overline{z}^2r^{4n}}{\overline{\zeta}^2-r^{4n}\overline{z}^2}+1-\frac{\overline{z}^2r^{4n}}{\zeta^2-r^{4n}\overline{z}^2}-1\right)\Bigg\}\frac{d\zeta}{\zeta}$$

and

$$\lim_{z\to\zeta,|z|<1}[w(z)-w(\frac{1}{\overline{z}})] = \frac{1}{2\pi i}\Bigg\{\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}}\gamma(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta}$$
$$+\int\limits_{\substack{|\zeta|=1,\\ 0>\mathrm{Im}\zeta,\\ 0>\mathrm{Re}\zeta}}\gamma(-\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta}+\int\limits_{\substack{|\zeta|=1,\\ 0>\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}}\gamma(\overline{\zeta})\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta} \tag{2.53}$$
$$+\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0>\mathrm{Re}\zeta}}\gamma(-\overline{\zeta})\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta}=\lim_{z\to\zeta}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Gamma_1(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta}=\gamma(\zeta),$$

where $\Gamma_1(\zeta)$ is given in (2.26). Then

$$\lim_{z\to\zeta}w(\frac{1}{\overline{z}})=0.$$

Consider for $|z|>r$ the function

$$w(\tfrac{r^2}{\overline{z}}) = \frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-r^4}+\frac{\zeta^2r^4}{\zeta^2r^4-\overline{z}^2}$$
$$+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-r^4}+\frac{r^4}{\zeta^2\overline{z}^2-r^{4(n+1)}}+\frac{\zeta^2r^4}{r^{4(n+1)}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2r^4-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta}, \tag{2.54}$$

which can be easily rewritten as

$$\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-r^4}+\frac{\zeta^2r^4}{\zeta^2r^4-\overline{z}^2}$$
$$+\sum_{n=0}^{\infty}r^{4n}\left[\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right]+\sum_{n=2}^{\infty}r^{4n}\left[\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}\right]\Bigg\}\frac{d\zeta}{\zeta}=$$
$$\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-r^4}+\frac{\zeta^2r^4}{\zeta^2r^4-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1}+\frac{\overline{z}^2}{\zeta^2-\overline{z}^2}-\frac{r^4}{\zeta^2\overline{z}^2-r^4}-\frac{\zeta^2r^4}{r^4\zeta^2-\overline{z}^2}$$
$$+\sum_{n=1}^{\infty}\left[\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}}{\zeta^2-r^{4n}\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}\right]\frac{dz}{z}=$$
$$\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1}+\frac{\overline{z}^2}{\zeta^2-\overline{z}^2}+1$$
$$+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}}{\zeta^2-r^{4n}\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}\right)\Bigg\}\frac{d\zeta}{\zeta}.$$

Thus $w(\frac{r^2}{\overline{z}}) = w(\frac{1}{\overline{z}})$. Then the difference

$$\begin{aligned} w(z) - w(\tfrac{r^2}{\overline{z}}) &= \frac{1}{\pi i} \int\limits_{\partial R^*} \gamma(\zeta) \Bigg\{ \frac{\zeta^2}{\zeta^2 - z^2} + \frac{\zeta^2 z^2}{\zeta^2 z^2 - 1} - \frac{\zeta^2 \overline{z}^2}{\zeta^2 \overline{z}^2 - r^4} - \frac{\zeta^2 r^4}{\zeta^2 r^4 - \overline{z}^2} \\ &+ \sum_{n=1}^{\infty} r^{4n} \Bigg(\frac{\zeta^2}{r^{4n}\zeta^2 - z^2} + \frac{z^2}{\zeta^2 - r^{4n} z^2} + \frac{\zeta^2 z^2}{r^{4n}\zeta^2 z^2 - 1} + \frac{1}{\zeta^2 z^2 - r^{4n}} \\ &- \frac{\zeta^2 \overline{z}^2}{r^{4n}\zeta^2 \overline{z}^2 - r^4} - \frac{r^4}{\zeta^2 \overline{z}^2 - r^{4(n+1)}} - \frac{\zeta^2 r^4}{r^{4(n+1)}\zeta^2 - \overline{z}^2} - \frac{\overline{z}^2}{\zeta^2 r^4 - r^{4n}\overline{z}^2} \Bigg) \Bigg\} \frac{d\zeta}{\zeta} \end{aligned}$$

is reduced to the difference $w(z) - w(\frac{1}{\overline{z}})$ in (2.52).
If $z \to \zeta$, $|\zeta| = r$, $\operatorname{Im}\zeta > 0$, $\operatorname{Re}\zeta > 0$, then, similarly to (2.53)

$$\lim_{z \to \zeta, |z| > r} [w(z) - w(\frac{r^2}{\overline{z}})] = - \lim_{z \to \zeta, |z| > r} \int\limits_{|\zeta| = r} \Gamma_1(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1 \right) \frac{d\zeta}{\zeta} = \gamma(\zeta), \tag{2.55}$$

where $\Gamma_1(\zeta)$ is defined as in (2.26) for $|\zeta| = r$. Then again

$$\lim_{z \to \zeta} w(\frac{r^2}{\overline{z}}) = 0.$$

Consider next for $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, $|z| < 1$ the function

$$\begin{aligned} w(\overline{z}) &= \frac{1}{\pi i} \int\limits_{\partial R^*} \gamma(\zeta) \Bigg\{ \frac{\zeta^2}{\zeta^2 - \overline{z}^2} + \frac{\zeta^2 \overline{z}^2}{\zeta^2 \overline{z}^2 - 1} \\ &+ \sum_{n=1}^{\infty} r^{4n} \Bigg(\frac{\zeta^2}{r^{4n}\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\zeta^2 - r^{4n}\overline{z}^2} + \frac{\zeta^2 \overline{z}^2}{r^{4n}\zeta^2 \overline{z}^2 - 1} + \frac{1}{\zeta^2 \overline{z}^2 - r^{4n}} \Bigg) \Bigg\} \frac{d\zeta}{\zeta}, \end{aligned} \tag{2.56}$$

which is equal to $w(\frac{1}{\overline{z}})$. Then again

$$w(z) - w(\overline{z}) = w(z) - w(\frac{1}{\overline{z}}) = \frac{1}{\pi i} \int\limits_{\partial R^*} \gamma(\zeta) L(\zeta, z) \frac{d\zeta}{\zeta}$$

and taking $z \to \zeta$, $r < t < 1$, $\zeta = t$ the equality

$$\begin{aligned} &\lim_{z \to t} [w(z) - w(\overline{z})] = \lim_{z \to t} \frac{(z^2 - \overline{z}^2)}{\pi i} \int\limits_r^1 \gamma(t) \Bigg\{ \frac{t}{|t^2 - z^2|^2} - \frac{t}{|t^2 z^2 - 1|^2} \\ &+ \sum_{n=1}^{\infty} r^{4n} \Bigg(\frac{t}{|r^{4n} t^2 - z^2|^2} + \frac{t}{|t^2 - r^{4n} z^2|^2} - \frac{t}{|r^{4n} t^2 z^2 - 1|^2} - \frac{t}{|t^2 z^2 - r^{4n}|^2} \Bigg) \Bigg\} dt \\ &= \lim_{z \to t} \frac{1}{\pi i} \left[\int\limits_r^1 \gamma(t) + \int\limits_1^{\frac{1}{r}} \gamma(\frac{1}{t}) \right] \frac{t(z - \overline{z})(z + \overline{z})}{|t - z|^2 |t + z|^2} dt = \lim_{z \to t} \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \Gamma_2(t) \frac{z - \overline{z}}{|t - z|^2} dt = \gamma(t) \end{aligned} \tag{2.57}$$

follows, where $\Gamma_2(t)$ is determined in (2.29). Thus

$$\lim_{z \to t} w(\overline{z}) = \gamma(t), \ \ r < t < 1.$$

Consider now for $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, $|z| < 1$ the function

$$\begin{aligned} w(-\overline{z}) &= \frac{1}{\pi i} \int\limits_{\partial R^*} \gamma(\zeta) \Bigg\{ \frac{\zeta^2}{\zeta^2 - \overline{z}^2} + \frac{\zeta^2 \overline{z}^2}{\zeta^2 \overline{z}^2 - 1} \\ &+ \sum_{n=1}^{\infty} r^{4n} \Bigg(\frac{\zeta^2}{r^{4n}\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\zeta^2 - r^{4n}\overline{z}^2} + \frac{\zeta^2 \overline{z}^2}{r^{4n}\zeta^2 \overline{z}^2 - 1} + \frac{1}{\zeta^2 \overline{z}^2 - r^{4n}} \Bigg) \Bigg\} \frac{d\zeta}{\zeta} = w(\overline{z}) \end{aligned} \tag{2.58}$$

then similarly

$$w(z)-w(-\overline{z})=w(z)-w(\frac{1}{\overline{z}})=\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)L(\zeta,z)\frac{d\zeta}{\zeta}. \tag{2.59}$$

Letting $z\to\zeta$, $\zeta=it$, $r<t<1$, one gets

$$\lim_{z\to it}[w(z)-w(-\overline{z})]=\lim_{z\to it}\frac{(z^2-\overline{z}^2)}{\pi i}\int\limits_r^1\gamma(it)\Bigg\{\frac{t}{|t^2+z^2|^2}-\frac{t}{|t^2z^2+1|^2}$$
$$+\sum_{n=1}^{\infty}r^{4n}\left(\frac{t}{|r^{4n}t^2+z^2|^2}-\frac{t}{|r^{4n}t^2z^2+1|^2}+\frac{t}{|t^2+r^{4n}z^2|^2}-\frac{t}{|t^2z^2+r^{4n}|^2}\right)\Bigg\}dt= \tag{2.60}$$
$$\lim_{z\to it}\frac{1}{\pi i}\left[\int\limits_r^1\gamma(it)-\int\limits_{\frac{1}{r}}^1\gamma(\frac{i}{t})\right]\frac{t(z+\overline{z})(z-\overline{z})}{|it-z|^2|it+z|^2}dt=\lim_{z\to it}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Gamma_2(it)\frac{z+\overline{z}}{|it-z|^2}dt=\gamma(it),$$

where $\Gamma_2(it)$ is given in (2.31). Hence,

$$\lim_{z\to it}w(-\overline{z})=\gamma(it),\ \ r<t<1.$$

Denote for $z\in R^*$ the function $\phi(z)=w(\frac{1}{\overline{z}})=w(\overline{z})=w(-\overline{z})=w(\frac{r^2}{\overline{z}})$. From the formulas (2.53), (2.55), (2.57), (2.60) the equality

$$\lim_{z\to\zeta}[w(z)-\phi(z)]=\gamma(\zeta),\ \zeta\in\partial R^* \tag{2.61}$$

follows and thus, because of (2.50), the relation $\lim\phi(z)=0,\zeta\in\partial R^*$ is valid.

Because $\overline{\phi(z)}$ is an analytic function in R^* then from the maximum principle for analytic functions it follows that $w(\overline{z})=0$ in R^*, which is given as condition (2.48). On the other hand, if this solubility condition is valid, then, subtracting (2.48) from (2.49), the relation $w=\gamma$ on ∂R^* follows from (2.61). $\square$

2.2.2 Dirichlet problem for the inhomogeneous Cauchy-Riemann equation

Theorem 2.2.2. *The Dirichlet problem*

$$\begin{aligned}&w_{\overline{z}}=f,\ z\in R^*,\ f\in L_p(R^*,\mathbb{C}),2<p,\\ &w=\gamma,\ \gamma\in C(\partial R^*,\mathbb{C})\end{aligned} \tag{2.62}$$

is solvable if and only if for $z\in R^*$

$$\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1}$$
$$+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta}=$$
$$\frac{1}{\pi}\int\limits_{R^*}f(\zeta)\Bigg\{\frac{\zeta}{\zeta^2-\overline{z}^2}+\frac{\zeta\overline{z}^2}{\zeta^2\overline{z}^2-1} \tag{2.63}$$
$$+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta(\zeta^2\overline{z}^2-r^{4n})}+\frac{\zeta}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta(\zeta^2-r^{4n}\overline{z}^2)}\right)\Bigg\}d\xi d\eta.$$

The solution is unique and can be represented as

$$
\begin{aligned}
w(z) \quad = \quad & \frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\zeta^2z^2}{\zeta^2z^2-1} \\
& +\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta^2z^2-r^{4n}}\right)\Bigg\}\frac{d\zeta}{\zeta} \\
& -\frac{1}{\pi}\int\limits_{R^*}f(\zeta)\Bigg\{\frac{\zeta}{\zeta^2-z^2}+\frac{\zeta z^2}{\zeta^2z^2-1} \\
& +\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta(\zeta^2-r^{4n}z^2)}+\frac{\zeta z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta(\zeta^2z^2-r^{4n})}\right)\Bigg\}d\xi d\eta,\ z\in R^*.
\end{aligned}
\tag{2.64}
$$

Proof. By the Cauchy-Pompeiu representation formula, if a solution exists, it must have the form of (2.64). Let

$$
\begin{aligned}
&\varphi = w - Tf \text{ in } R^*,\ \varphi=\gamma-Tf \text{ on } \partial R^*,\ \text{ then} \\
&w_{\overline{z}} = f \text{ is } (\varphi+Tf)_{\overline{z}} = \varphi_{\overline{z}}+f \text{ and thus } \varphi_{\overline{z}}=0.
\end{aligned}
$$

Then consider the homogeneous Dirichlet problem

$$
\varphi_{\overline{z}} = 0 \,\text{in}\, R^*,\ \varphi=\gamma-Tf\,\text{on}\,\partial R^*. \tag{2.65}
$$

By the solvability condition (2.48) the new condition for (2.65) is

$$
\begin{aligned}
&\frac{1}{\pi i}\int\limits_{\partial R^*}[\gamma(\zeta)-Tf(\zeta)]\Bigg\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1} \\
&+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta}=0.
\end{aligned}
\tag{2.66}
$$

Calculation of the successive integral

$$
\begin{aligned}
&\frac{1}{\pi i}\int\limits_{\partial R^*}Tf(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1} \\
&+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta}= \\
&\frac{1}{\pi}\int\limits_{R^*}f(\tilde{\zeta})\int\limits_{\partial R^*}\frac{1}{\pi i}\Bigg\{\frac{\zeta^2}{\zeta^2-\overline{z}^2}+\frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2-1} \\
&+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta^2\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta^2\overline{z}^2-r^{4n}}+\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2}\right)\Bigg\}\frac{d\zeta}{\zeta(\zeta-\tilde{\zeta})}d\tilde{\xi}d\tilde{\eta}= \\
&\frac{2}{\pi}\int\limits_{R^*}f(\zeta)\Bigg\{\frac{\zeta}{\zeta^2-\overline{z}^2}+\frac{\zeta\overline{z}^2}{\zeta^2\overline{z}^2-1} \\
&+\sum_{n=1}^{\infty}r^{4n}\left(\frac{\zeta\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta(\zeta^2\overline{z}^2-r^{4n})}+\frac{\zeta}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta(\zeta^2-r^{4n}\overline{z}^2)}\right)\Bigg\}d\xi d\eta
\end{aligned}
$$

reduces (2.66) to the corresponding formula (2.63). That (2.64) under the condition (2.63) provides a solution to (2.62) follows because (2.64) can be written as

$$
w(z)=\frac{1}{\pi i}\int\limits_{\partial R^*}\gamma(\zeta)L(\zeta,z)\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{R^*}f(\zeta)L(\zeta,z)\frac{1}{\zeta}d\xi d\eta,
$$

where $L(\zeta, z)$ is defined as the term in (2.52). Calculations of the integrals on the different parts of the boundary in the manner of the problem (2.47) show that

$$-\frac{1}{\pi}\int\limits_{R^*} f(\zeta)L(\zeta,z)\frac{1}{\zeta}d\xi d\eta$$

tends to 0 when $z \to \zeta$, $\zeta \in \partial R^*$ and $w(z) \to \gamma(z)$ if $z \to \zeta_0 \in \partial R^*$, $z \in R^*$. The property of the Pompeiu operator gives the weak solution of the differential equation $w_{\overline{z}} = f$, $z \in R^*$. □

2.3 Neumann problem for the homogeneous Cauchy-Riemann equation

The Neumann problem for an analytic function in R^* means to find a function with prescribed outward normal derivatives. In this section this Neumann condition is modified in a proper way in order to adjust it to the concept of analyticity.

Definition 2.3.1. *Let on $\partial R^* \backslash \{1, i, ir, r\}$ denote*

$$\partial_\nu = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & |z| = 1,\ Re\, z > 0,\ Im\, z > 0, \\ -\frac{1}{r}(z\partial_z + \overline{z}\partial_{\overline{z}}), & |z| = r,\ Re\, z > 0,\ Im\, z > 0, \\ -i(\partial_z - \partial_{\overline{z}}), & Re\, z > 0,\ Im\, z = 0, \\ -(\partial_z + \partial_{\overline{z}}), & Re\, z = 0,\ Im\, z > 0. \end{cases} \tag{2.67}$$

Then the Neumann problem is to find an analytic function w in R^*, such that

$$\partial_\nu w(z) = \gamma(z) \text{ on } \partial R^* \backslash \{1, i, ir, r\}$$

for given $\gamma \in C(\partial R^*; \mathbb{C})$, $w(1) = 0$.

Theorem 2.3.1. *The Neumann problem*

$$\begin{aligned} & w_{\overline{z}} = 0,\ z \in R^*, \\ & \partial_{\nu_z} w = \gamma \text{ on} \partial R^*,\ \gamma \in C(\partial R^*; \mathbb{C}),\ w(1) = 0 \end{aligned} \tag{2.68}$$

where

$$\partial_\nu w = \begin{cases} zw', & |z| = 1,\ \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z > 0, \\ -\frac{1}{r}zw', & |z| = r,\ \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z > 0, \\ -iw', & \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z = 0, \\ -w', & \mathrm{Re}\, z = 0,\ \mathrm{Im}\, z > 0, \end{cases} \tag{2.69}$$

is solvable if and only if for $z \in R^*$

$$\begin{aligned} & \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\Bigg\{\frac{\zeta}{\zeta^2 - \overline{z}^2} + \frac{\zeta\overline{z}^2}{\zeta^2\overline{z}^2 - 1} \\ & + \sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2 - 1} + \frac{1}{\zeta(\zeta^2\overline{z}^2 - r^{4n})} + \frac{\zeta}{r^{4n}\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\zeta(\zeta^2 - r^{4n}\overline{z}^2)}\right)\Bigg\}\frac{d\zeta}{\zeta} \\ & + \frac{r}{\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\Bigg\{\frac{\zeta}{\zeta^2 - \overline{z}^2} + \frac{\zeta\overline{z}^2}{\zeta^2\overline{z}^2 - 1} \\ & + \sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2 - 1} + \frac{1}{\zeta(\zeta^2\overline{z}^2 - r^{4n})} + \frac{\zeta}{r^{4n}\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\zeta(\zeta^2 - r^{4n}\overline{z}^2)}\right)\Bigg\}\frac{d\zeta}{\zeta} \end{aligned}$$

$$+\frac{1}{\pi}\int_r^1 \gamma(t)\Bigg\{\frac{t}{t^2-\overline{z}^2}+\frac{t\overline{z}^2}{t^2\overline{z}^2-1} \tag{2.70}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{t\overline{z}^2}{r^{4n}t^2\overline{z}^2-1}+\frac{1}{t(t^2\overline{z}^2-r^{4n})}+\frac{t}{r^{4n}t^2-\overline{z}^2}+\frac{\overline{z}^2}{t(t^2-r^{4n}\overline{z}^2)}\right)\Bigg\}dt$$
$$-\frac{1}{\pi}\int_r^1 \gamma(it)\Bigg\{\frac{it}{t^2+\overline{z}^2}+\frac{it\overline{z}^2}{t^2\overline{z}^2+1}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{it\overline{z}^2}{r^{4n}t^2\overline{z}^2+1}+\frac{1}{it(t^2\overline{z}^2+r^{4n})}+\frac{it}{r^{4n}t^2+\overline{z}^2}+\frac{\overline{z}^2}{it(t^2+r^{4n}\overline{z}^2)}\right)\Bigg\}dt =$$
$$\frac{1}{\pi i}\int_{\partial R^*} \delta(\zeta)\Bigg\{\frac{\zeta}{\zeta^2-\overline{z}^2}+\frac{\zeta\overline{z}^2}{\zeta^2\overline{z}^2-1}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta\overline{z}^2}{r^{4n}\zeta^2\overline{z}^2-1}+\frac{1}{\zeta(\zeta^2\overline{z}^2-r^{4n})}+\frac{\zeta}{r^{4n}\zeta^2-\overline{z}^2}+\frac{\overline{z}^2}{\zeta(\zeta^2-r^{4n}\overline{z}^2)}\right)\Bigg\}ds_\zeta = 0,$$

where

$$\delta = \begin{cases} \gamma, & |z|=1,\ \mathrm{Re}\,z>0,\ \mathrm{Im}\,z>0,\\ -\gamma, & |z|=r,\ \mathrm{Re}\,z>0,\ \mathrm{Im}\,z>0,\\ \gamma, & \mathrm{Re}\,z>0,\ \mathrm{Im}\,z=0,\\ -\gamma, & \mathrm{Re}\,z=0,\ \mathrm{Im}\,z>0. \end{cases} \tag{2.71}$$

Then the solution is

$$w(z) = \frac{z-1}{\pi}\int_{\partial R^*}\frac{\gamma(\zeta)}{\zeta}ds_\zeta + \frac{1}{2\pi}\int_{\partial R^*}\gamma(\zeta)\sum_{-\infty}^{+\infty} r^{2n}\left[\log\frac{\zeta r^{2n}+z}{\zeta r^{2n}-z}\frac{r^{2n}\zeta-1}{r^{2n}\zeta+1}-\frac{1}{\zeta^2}\log\frac{z\zeta+r^{2n}}{z\zeta-r^{2n}}\frac{\zeta-r^{2n}}{\zeta+r^{2n}}\right]ds_\zeta$$

Proof. If w is a solution to the Neumann problem, then $\varphi = w'$ is a solution to the Dirichlet problem

$$\varphi_{\overline{z}} = 0 \text{ in } R^*,\ \varphi = w' \text{ on } \partial R^*, \tag{2.72}$$

where from (2.71) w' is presented by

$$w'(z) = \begin{cases} \overline{z}\gamma(z), & |z|=1,\ \mathrm{Re}\,z>0,\ \mathrm{Im}\,z>0,\\ -\frac{\overline{z}}{r}\gamma(z), & |z|=r,\ \mathrm{Re}\,z>0,\ \mathrm{Im}\,z>0,\\ i\gamma(z), & \mathrm{Re}\,z>0,\ \mathrm{Im}\,z=0,\\ -\gamma(z), & \mathrm{Re}\,z=0,\ \mathrm{Im}\,z>0. \end{cases} \tag{2.73}$$

Then the solution of the Dirichlet problem is

$$w'(z) = \frac{1}{\pi i}\int_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\overline{\zeta}\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\zeta^2z^2}{\zeta^2z^2-1}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta^2z^2-r^{4n}}\right)\Bigg\}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{\pi i r}\int_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\overline{\zeta}\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\zeta^2z^2}{\zeta^2z^2-1}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{z^2}{\zeta^2-r^{4n}z^2}+\frac{\zeta^2z^2}{r^{4n}\zeta^2z^2-1}+\frac{1}{\zeta^2z^2-r^{4n}}\right)\Bigg\}\frac{d\zeta}{\zeta} \tag{2.74}$$

$$+\frac{1}{\pi}\int_r^1 \gamma(t)\Bigg\{\frac{t^2}{t^2-z^2}+\frac{t^2z^2}{t^2z^2-1}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{t^2}{r^{4n}t^2-z^2}+\frac{z^2}{t^2-r^{4n}z^2}+\frac{t^2z^2}{r^{4n}t^2z^2-1}+\frac{1}{t^2z^2-r^{4n}}\right)\Bigg\}\frac{dt}{t}$$
$$-\frac{1}{\pi}\int_r^1 \gamma(it)\Bigg\{\frac{t^2}{t^2+z^2}+\frac{t^2z^2}{t^2z^2+1}$$
$$+\sum_{n=1}^{\infty} r^{4n}\left(\frac{t^2}{r^{4n}t^2+z^2}-\frac{z^2}{t^2+r^{4n}z^2}+\frac{t^2z^2}{r^{4n}t^2z^2+1}-\frac{1}{t^2z^2+r^{4n}}\right)\Bigg\}\frac{dt}{t}.$$

Let $w(z)=w_1(z)+w_2(z)+w_3(z)+w_4(z)+w(1)$, where $w(1)=w_1(1)+w_2(1)+w_3(1)+w_4(1)=0$, then

$$\begin{aligned}
w_1(z)-w_1(1) &= \int_1^z \frac{1}{2\pi i}\int_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\Bigg\{\frac{1}{\zeta-z}+\frac{1}{\zeta+z}+\frac{z}{\zeta z-1}+\frac{z}{\zeta z+1} \\
&+\sum_{n=1}^{\infty} r^{4n}\Bigg[\left(\frac{1}{r^{2n}\zeta-z}+\frac{1}{r^{2n}\zeta+z}\right)\frac{1}{r^{2n}}+\left(\frac{\overline{\zeta}z}{\zeta-r^{2n}z}-\frac{\overline{\zeta}z}{\zeta+r^{2n}z}\right)\frac{1}{r^{2n}} \\
&+\left(\frac{z}{r^{2n}\zeta z-1}+\frac{z}{r^{2n}\zeta z+1}\right)\frac{1}{r^{2n}}+\left(\frac{\overline{\zeta}}{\zeta z-r^{2n}}-\frac{\overline{\zeta}}{\zeta z+r^{2n}}\right)\frac{1}{r^{2n}}\Bigg]\Bigg\}\frac{d\zeta}{\zeta}dz = \\
&\int_1^z \frac{1}{2\pi i}\int_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\Bigg\{\frac{1}{\zeta-z}+\frac{1}{\zeta+z}-\frac{1}{\zeta}\frac{1}{(1-\zeta z)}-\frac{1}{\zeta}\frac{1}{(1+\zeta z)}+\frac{2}{\zeta} \\
&+\sum_{n=1}^{\infty} r^{2n}\Bigg[\frac{\overline{\zeta}}{r^{2n}-z\overline{\zeta}}+\frac{\overline{\zeta}}{r^{2n}+z\overline{\zeta}}-\frac{\overline{\zeta}}{r^{2n}-\zeta z}-\frac{\overline{\zeta}}{r^{2n}+\zeta z}-\frac{2\overline{\zeta}}{r^{2n}}+\frac{2}{r^{2n}\zeta} \\
&+\frac{\overline{\zeta}}{r^{2n}}\frac{1}{(1-r^{2n}z\overline{\zeta})}+\frac{\overline{\zeta}}{r^{2n}}\frac{1}{(1+r^{2n}z\overline{\zeta})}-\frac{1}{r^{2n}\zeta}\frac{1}{(1-r^{2n}\zeta z)}-\frac{1}{r^{2n}\zeta}\frac{1}{(1+r^{2n}\zeta z)}\Bigg]\Bigg\}\frac{d\zeta}{\zeta}dz
\end{aligned}$$

and

$$\begin{aligned}
w_1(z) &= \frac{1}{2\pi i}\int_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\Bigg\{\log\frac{\zeta+z}{\zeta+1}\frac{\zeta-1}{\zeta-z}-\frac{1}{\zeta^2}\log\frac{1+z\zeta}{1+\zeta}\frac{1-\zeta}{1-\zeta z}+\frac{2z}{\zeta}-\frac{2}{\zeta} \\
&+\sum_{n=1}^{\infty}\Bigg[r^{2n}\left(\log\frac{r^{2n}\zeta+z}{r^{2n}\zeta+1}\frac{r^{2n}\zeta-1}{r^{2n}\zeta-z}-\frac{1}{\zeta^2}\log\frac{r^{2n}+z\zeta}{r^{2n}+\zeta}\frac{r^{2n}-\zeta}{r^{2n}-z\zeta}\right) \\
&+\frac{1}{r^{2n}}\left(\log\frac{\zeta+r^{2n}z}{\zeta+r^{2n}}\frac{\zeta-r^{2n}}{\zeta-r^{2n}z}-\frac{1}{\zeta^2}\log\frac{1+r^{2n}z\zeta}{1+r^{2n}\zeta}\frac{1-r^{2n}\zeta}{1-r^{2n}z\zeta}\right)\Bigg]\Bigg\}\frac{d\zeta}{\zeta}+w_1(1).
\end{aligned} \tag{2.75}$$

Similarly

$$\begin{aligned}
w_2(z)-w_2(1) &= \int_1^z \frac{1}{2\pi i r}\int_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\overline{\zeta}\Bigg\{\frac{\zeta}{\zeta-z}+\frac{\zeta}{\zeta+z}+\frac{\zeta z}{\zeta z-1}+\frac{\zeta z}{\zeta z+1} \\
&+\sum_{n=1}^{\infty}\Bigg[\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}\zeta}{r^{2n}\zeta+z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{\zeta+r^{2n}z} \\
&+\frac{\zeta z r^{2n}}{\zeta r^{2n}z-1}+\frac{\zeta z r^{2n}}{\zeta r^{2n}z+1}+\frac{r^{2n}}{\zeta z-r^{2n}}-\frac{r^{2n}}{\zeta z+r^{2n}}\Bigg]\Bigg\}\frac{d\zeta}{\zeta}dz
\end{aligned}$$

and

$$
\begin{aligned}
w_2(z) &= \frac{r}{2\pi i} \int\limits_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\zeta}} \gamma(\zeta)\Bigg\{\log\frac{\zeta+z}{\zeta+1}\frac{\zeta-1}{\zeta-z} - \frac{1}{\zeta^2}\log\frac{1+z\zeta}{1+\zeta}\frac{1-\zeta}{1-\zeta z} + \frac{2z}{\zeta} - \frac{2}{\zeta} \\
&+ \sum_{n=1}^{\infty}\Bigg[r^{2n}\left(\log\frac{r^{2n}\zeta+z}{r^{2n}\zeta+1}\frac{r^{2n}\zeta-1}{r^{2n}\zeta-z} - \frac{1}{\zeta^2}\log\frac{r^{2n}+z\zeta}{r^{2n}+\zeta}\frac{r^{2n}-\zeta}{r^{2n}-z\zeta}\right) \\
&+ \frac{1}{r^{2n}}\left(\log\frac{\zeta+r^{2n}z}{\zeta+r^{2n}}\frac{\zeta-r^{2n}}{\zeta-r^{2n}z} - \frac{1}{\zeta^2}\log\frac{1+r^{2n}z\zeta}{1+r^{2n}\zeta}\frac{1-r^{2n}\zeta}{1-r^{2n}z\zeta}\right)\Bigg]\Bigg\}\frac{d\zeta}{\zeta} + w_2(1).
\end{aligned}
\tag{2.76}
$$

Next

$$
\begin{aligned}
w_3(z) - w_3(1) &= \int_1^z \frac{1}{2\pi}\int_r^1 \gamma(t)\Bigg\{\frac{t}{t-z} + \frac{t}{t+z} + \frac{tz}{tz-1} + \frac{tz}{tz+1} \\
&+ \sum_{n=1}^{\infty}\Bigg[\frac{r^{2n}t}{r^{2n}t-z} + \frac{r^{2n}t}{r^{2n}t+z} + \frac{r^{2n}z}{t-r^{2n}z} - \frac{r^{2n}z}{t+r^{2n}z} \\
&+ \frac{tzr^{2n}}{tr^{2n}z-1} + \frac{tzr^{2n}}{tr^{2n}z+1} + \frac{r^{2n}}{tz-r^{2n}} - \frac{r^{2n}}{tz+r^{2n}}\Bigg]\Bigg\}\frac{dt}{t}dz,
\end{aligned}
$$

$$
\begin{aligned}
w_3(z) &= \frac{1}{2\pi}\int_r^1 \gamma(t)\Bigg\{t\log\frac{t+z}{t+1}\frac{t-1}{t-z} - \frac{1}{t}\log\frac{1+zt}{1+t}\frac{1-t}{1-zt} + 2z - 2 \\
&+ \sum_{n=1}^{\infty}\Bigg(r^{2n}t\log\frac{r^{2n}t+z}{r^{2n}t+1}\frac{r^{2n}t-1}{r^{2n}t-z} - \frac{r^{2n}}{t}\log\frac{tz+r^{2n}}{t+r^{2n}}\frac{t-r^{2n}}{tz-r^{2n}} \\
&+ \frac{t}{r^{2n}}\log\frac{t+r^{2n}z}{t+r^{2n}}\frac{t-r^{2n}}{t-r^{2n}z} - \frac{1}{tr^{2n}}\log\frac{1+r^{2n}zt}{1+r^{2n}t}\frac{1-r^{2n}t}{1-r^{2n}zt}\Bigg)\Bigg\}\frac{dt}{t} + w_3(1).
\end{aligned}
\tag{2.77}
$$

Similarly for the last integral

$$
\begin{aligned}
w_4(z) - w_4(1) &= \int_1^z \frac{1}{2\pi i}\int_r^1 \gamma(it)\Bigg\{\frac{it}{it-z} + \frac{it}{it+z} + \frac{itz}{itz-1} - \frac{itz}{itz+1} \\
&+ \sum_{n=1}^{\infty}\Bigg[\frac{r^{2n}it}{r^{2n}it-z} + \frac{r^{2n}it}{r^{2n}it+z} + \frac{r^{2n}z}{it-r^{2n}z} - \frac{r^{2n}z}{it+r^{2n}z} \\
&+ \frac{itzr^{2n}}{itr^{2n}z-1} + \frac{itzr^{2n}}{itr^{2n}z+1} + \frac{r^{2n}}{itz-r^{2n}} - \frac{r^{2n}}{itz+r^{2n}}\Bigg]\Bigg\}\frac{dt}{t}dz
\end{aligned}
$$

and finally

$$
\begin{aligned}
w_4(z) &= \frac{1}{2\pi i}\int_r^1 \gamma(it)\Bigg\{it\log\frac{it+z}{it+1}\frac{it-1}{it-z} - \frac{1}{it}\log\frac{1+zit}{1+it}\frac{1-it}{1-zit} + 2z - 2 \\
&+ \sum_{n=1}^{\infty}\Bigg(r^{2n}it\log\frac{r^{2n}it+z}{r^{2n}it+1}\frac{r^{2n}it-1}{r^{2n}it-z} - \frac{r^{2n}}{it}\log\frac{itz+r^{2n}}{it+r^{2n}}\frac{it-r^{2n}}{itz-r^{2n}} \\
&+ \frac{it}{r^{2n}}\log\frac{it+r^{2n}z}{it+r^{2n}}\frac{it-r^{2n}}{it-r^{2n}z} - \frac{1}{itr^{2n}}\log\frac{1+r^{2n}zit}{1+r^{2n}it}\frac{1-r^{2n}it}{1-r^{2n}zit}\Bigg)\Bigg\}\frac{dt}{t} + w_4(1).
\end{aligned}
\tag{2.78}
$$

Thus, composing (2.75)-(2.78), the equality

$$
w(z) = \frac{z-1}{\pi}\int\limits_{\partial R^*}\frac{\gamma(\zeta)}{\zeta}ds_\zeta + \frac{1}{2\pi}\int\limits_{\partial R^*}\gamma(\zeta)\sum_{-\infty}^{+\infty} r^{2n}\left[\log\frac{\zeta r^{2n}+z}{\zeta r^{2n}-z}\frac{r^{2n}\zeta-1}{r^{2n}\zeta+1} - \frac{1}{\zeta^2}\log\frac{z\zeta+r^{2n}}{z\zeta-r^{2n}}\frac{\zeta-r^{2n}}{\zeta+r^{2n}}\right]ds_\zeta
$$

follows. Obviously, $w(z)$ is an analytic function in R^*. The boundary condition in (2.68) is satisfied due to the fact that the composition of (2.70) and (2.74) gives

$$w'(z) = \frac{1}{\pi} \int\limits_{\partial R^*} \gamma(\zeta) L(\zeta, z) ds_\zeta,$$

where $L(\zeta, z)$ is the kernel in the integral (2.52), and, consequently, $\gamma(\zeta)$ can be attained as its boundary value, i.e. as the boundary value of $\varphi = w'$. $\square$

Chapter 3

Boundary Value Problems for the Poisson Equation

3.1 Harmonic Dirichlet Problem

In this Chapter, the Green and Neumann functions for the quarter ring are constructed and the solutions to the Dirichlet and the Neumann problems for the Poisson equation are presented.

3.1.1 Harmonic Green function and the Green Representation formula

Definition 3.1.1. *[6] A real-valued function $G(z, z_0)$ in a domain D of $\mathbb{C}$ having the properties:*
1. $G(z, z_0)$ is harmonic in $z \in D \backslash \{z_0\}$,
2. $\log|z - z_0| + G(z, z_0)$ is harmonic in the neighborhood of z_0,
3. $\lim_{z \to \partial D} G(z, z_0) = 0$
is called the Green function of D, more exactly the Green function of D for the Laplace operator.

Theorem 3.1.1. *[6] The Green function of D has the following additional properties:*
1. $0 < G(z, z_0),\ z \in D \backslash \{z_0\}$,
2. $G(z, z_0) = G(z_0, z),\ z \neq z_0$,
3. it is uniquely given by the properties of Definition 3.1.1.

The harmonic Green function for the quarter ring domain can be obtained on the basis of the Green function $G(z, \zeta) = \frac{1}{2} G_1(z, \zeta)$ for the ring R [41]

$$G_{1R}(z, \zeta) = \frac{\log|z|^2 \log|\zeta|^2}{\log r^2} - \log\left| \frac{\zeta - z}{1 - z\overline{\zeta}} \prod_{k=1}^{\infty} \frac{(z - r^{2k}\zeta)(\zeta - r^{2k}z)}{(z\overline{\zeta} - r^{2k})(1 - r^{2k}\overline{z}\zeta)} \right|^2$$

and the upper half ring R^+ [18]

$$G_{1R^+}(z, \zeta) = \log\left| \frac{(1 - z\overline{\zeta})(\overline{\zeta} - z)}{(\zeta - z)(1 - z\zeta)} \prod_{n=1}^{\infty} \frac{(z - r^{2n}\overline{\zeta})(z\overline{\zeta} - r^{2n})(\overline{\zeta} - r^{2n}z)(1 - r^{2n}z\overline{\zeta})}{(z - r^{2n}\zeta)(z\zeta - r^{2n})(\zeta - r^{2n}z)(1 - r^{2n}z\zeta)} \right|^2$$

by observing the additional reflection points and using the method of reflection. Thus, for the quarter ring domain R^*

$$\begin{aligned} G_1(z, \zeta) &= \log\left| \frac{(\zeta - \overline{z})(\zeta + \overline{z})(\zeta - \frac{1}{\overline{z}})(\zeta + \frac{1}{\overline{z}})}{(\zeta - z)(\zeta + z)(\zeta - \frac{1}{z})(\zeta + \frac{1}{z})} \prod_{n=1}^{\infty} \frac{(\zeta - r^{2n}\overline{z})(\zeta + r^{2n}\overline{z})}{(\zeta - r^{2n}z)(\zeta + r^{2n}z)} \times \right. \\ &\left. \frac{(\zeta - \frac{\overline{z}}{r^{2n}})(\zeta + \frac{\overline{z}}{r^{2n}})(\zeta - \frac{r^{2n}}{\overline{z}})(\zeta + \frac{r^{2n}}{\overline{z}})(\zeta - \frac{1}{r^{2n}\overline{z}})(\zeta + \frac{1}{r^{2n}\overline{z}})}{(\zeta - \frac{z}{r^{2n}})(\zeta + \frac{z}{r^{2n}})(\zeta - \frac{r^{2n}}{z})(\zeta + \frac{r^{2n}}{z})(\zeta - \frac{1}{r^{2n}z})(\zeta + \frac{1}{r^{2n}z})} \right|^2, \end{aligned}$$

i.e.

$$G_1(z, \zeta) = \log\left| \frac{(\zeta^2 - \overline{z}^2)(\zeta^2\overline{z}^2 - 1)}{(\zeta^2 - z^2)(\zeta^2 z^2 - 1)} \prod_{n=1}^{\infty} \frac{(\zeta^2 - r^{4n}\overline{z}^2)}{(\zeta^2 - r^{4n}z^2)} \frac{(\zeta^2 r^{4n} - \overline{z}^2)(\zeta^2\overline{z}^2 - r^{4n})(\zeta^2\overline{z}^2 r^{4n} - 1)}{(\zeta^2 r^{4n} - z^2)(\zeta^2 z^2 - r^{4n})(\zeta^2 z^2 r^{4n} - 1)} \right|^2. \tag{3.1}$$

$G_1(z, \zeta)$ in (3.1) can be rewritten as

$$G_1(z, \zeta) = \log\left| \frac{(\overline{\zeta}^2 - z^2)(\overline{\zeta}^2 z^2 - 1)}{(\zeta^2 - z^2)(\zeta^2 z^2 - 1)} \prod_{n=1}^{\infty} \frac{(\overline{\zeta}^2 - r^{4n}z^2)}{(\zeta^2 - r^{4n}z^2)} \frac{(\overline{\zeta}^2 r^{4n} - z^2)(\overline{\zeta}^2 z^2 - r^{4n})(\overline{\zeta}^2 z^2 r^{4n} - 1)}{(\zeta^2 r^{4n} - z^2)(\zeta^2 z^2 - r^{4n})(\zeta^2 z^2 r^{4n} - 1)} \right|^2. \tag{3.2}$$

The properties of the Green function are investigated in the following Lemmas.

Lemma 3.1.1. *The infinite product*

$$\prod_{n=1}^{\infty}\left|\frac{(\overline{\zeta}^2-r^{4n}z^2)(\overline{\zeta}^2r^{4n}-z^2)(\overline{\zeta}^2z^2-r^{4n})(\overline{\zeta}^2z^2r^{4n}-1)}{(\zeta^2-r^{4n}z^2)(\zeta^2r^{4n}-z^2)(\zeta^2z^2-r^{4n})(\zeta^2z^2r^{4n}-1)}\right|^2 \tag{3.3}$$

converges for $z\in R^*$, $\zeta\in\partial R^*$.

Proof. Let us consider the terms of this product separately.
By the definition of the series convergence, the sum

$$\sum_{n=1}^{\infty}\left[\left|\frac{\overline{\zeta}^2-r^{4n}z^2}{\zeta^2-r^{4n}z^2}\right|^2-1\right]$$

must be observed. Since

$$\left|\frac{\overline{\zeta}^2-r^{4n}z^2}{\zeta^2-r^{4n}z^2}\right|^2=\frac{|\zeta|^4-r^{4n}\overline{\zeta}^2\overline{z}^2-r^{4n}\zeta^2z^2+|z|^4r^{8n}}{|\zeta|^4-r^{4n}\zeta^2\overline{z}^2-r^{4n}\overline{\zeta}^2z^2+|z|^4r^{8n}},$$

then

$$\sum_{n=1}^{\infty}\left[\left|\frac{\overline{\zeta}^2-r^{4n}z^2}{\zeta^2-r^{4n}z^2}\right|^2-1\right]=\sum_{n=1}^{\infty}\frac{r^{4n}(\zeta^2-\overline{\zeta}^2)(\overline{z}^2-z^2)}{r^{8n}|z|^4+|\zeta|^4-r^{4n}(\zeta^2\overline{z}^2+\overline{\zeta}^2z^2)}.$$

Similarly for the other terms one gets

$$\sum_{n=1}^{\infty}\left[\left|\frac{\overline{\zeta}^2r^{4n}-z^2}{\zeta^2r^{4n}-z^2}\right|^2-1\right]=\sum_{n=1}^{\infty}\frac{r^{4n}(\zeta^2-\overline{\zeta}^2)(\overline{z}^2-z^2)}{r^{8n}|\zeta|^4+|z|^4-r^{4n}(\zeta^2\overline{z}^2+\overline{\zeta}^2z^2)};$$

$$\sum_{n=1}^{\infty}\left[\left|\frac{\overline{\zeta}^2z^2-r^{4n}}{\zeta^2z^2-r^{4n}}\right|^2-1\right]=\sum_{n=1}^{\infty}\frac{r^{4n}(\zeta^2-\overline{\zeta}^2)(z^2-\overline{z}^2)}{r^{8n}+|\zeta z|^4-r^{4n}(\zeta^2z^2+\overline{\zeta}^2\overline{z}^2)};$$

$$\sum_{n=1}^{\infty}\left[\left|\frac{\overline{\zeta}^2z^2r^{4n}-1}{\zeta^2z^2r^{4n}-1}\right|^2-1\right]=\sum_{n=1}^{\infty}\frac{r^{4n}(\zeta^2-\overline{\zeta}^2)(z^2-\overline{z}^2)}{r^{8n}|\zeta z|^4+1-r^{4n}(\zeta^2z^2+\overline{\zeta^2}\overline{z}^2)}.$$

Thus, the convergence of the particular sums leads to the convergence of the whole product (3.3). □

Lemma 3.1.2. *The function $G_1(z,\zeta)$ has vanishing boundary values on ∂R^*, i.e.*

$$\lim_{z\to z_0\in\partial R^*}G_1(z,\zeta)=0.$$

Proof. Consider the function in (3.1) on the different parts of the boundary.
For $z\in R^*$

$$G_1(z,\zeta)=\log\left|\frac{(\zeta^2z^2-|z|^4)(\zeta^2|z|^4-z^2)}{z^4(\zeta^2-z^2)(\zeta^2z^2-1)}\times\right.$$
$$\left.\prod_{n=1}^{\infty}\frac{z^8(\zeta^2z^2-r^{4n}|z|^4)(\zeta^2r^{4n}z^2-|z|^4)(\zeta^2|z|^4-r^{4n}z^2)(\zeta^2|z|^4r^{4n}-z^2)}{(\zeta^2-r^{4n}z^2)(\zeta^2r^{4n}-z^2)(\zeta^2z^2-r^{4n})(\zeta^2z^2r^{4n}-1)}\right|^2,$$

then, taking $|z|\to 1$, yields

$$\lim_{|z|\to 1}G_1(z,\zeta)=0.$$

For $|z|=r$, $\operatorname{Re}z>0$, $\operatorname{Im}z>0$ after multiplying some terms of the numerator by z^2 and by $\overline{z}^2$ in the denominator, the function becomes:

$$\lim_{|z|\to r}G_1(z,\zeta)=\log\left|\frac{(\zeta^2z^2-r^4)(\zeta^2r^4-z^2)}{(\zeta^2\overline{z}^2-r^4)(\zeta^2r^4-\overline{z}^2)}\times\right.$$
$$\left.\prod_{n=1}^{\infty}\frac{(\zeta^2z^2-r^{4(n+1)})(\zeta^2r^{4n}-\overline{z}^2)(\zeta^2\overline{z}^2-r^{4n})(\zeta^2r^{4(n+1)}-z^2)}{(\zeta^2\overline{z}^2-r^{4(n+1)})(\zeta^2r^{4n}-z^2)(\zeta^2z^2-r^{4n})(\zeta^2r^{4(n+1)}-\overline{z}^2)}\right|^2.$$

Here for any $M \in \mathbb{N}$

$$\prod_{n=1}^{M}(\zeta^2 z^2 - r^{4(n+1)}) = \prod_{n=2}^{M+1}(\zeta^2 z^2 - r^{4n}) = \frac{1}{\zeta^2 z^2 - r^4}\prod_{n=1}^{M+1}(\zeta^2 z^2 - r^{4n}),$$
$$\prod_{n=1}^{M}(\zeta^2 r^{4(n+1)} - z^2) = \prod_{n=2}^{M+1}(\zeta^2 r^{4n} - z^2) = \frac{1}{(\zeta^2 r^4 - z^2)}\prod_{n=1}^{M+1}(\zeta^2 r^{4n} - z^2),$$
$$\prod_{n=1}^{M}(\zeta^2 \overline{z}^2 - r^{4(n+1)}) = \prod_{n=2}^{M+1}(\zeta^2 \overline{z}^2 - r^{4n}) = \frac{1}{(\zeta^2 \overline{z}^2 - r^4)}\prod_{n=1}^{M+1}(\zeta^2 \overline{z}^2 - r^{4n}),$$
$$\prod_{n=1}^{M}(\zeta^2 r^{4(n+1)} - \overline{z}^2) = \prod_{n=2}^{M+1}(\zeta^2 r^{4n} - \overline{z}^2) = \frac{1}{(\zeta^2 r^4 - \overline{z}^2)}\prod_{n=1}^{M+1}(\zeta^2 r^{4n} - \overline{z}^2),$$

then

$$\lim_{|z|\to r} G_1(z,\zeta) = 0.$$

Obviously, for the real $z = \overline{z}$ and imaginary $z = -\overline{z}$ axes it follows that

$$G_1(z,\zeta) = 0$$

since $z^2 = (\overline{-z})^2 = \overline{z}^2$ there. □

In the Green representation formula the harmonic Green function and the Poisson kernel are being used. The latter can be obtained as the outward normal derivative of the Green function on the boundary.

$$\partial_{\nu_z} = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & |z| = 1,\ \mathrm{Re}\, z > 0, \mathrm{Im}\, z > 0 \\ -\frac{1}{r}(z\partial_z + \overline{z}\partial_{\overline{z}}), & |z| = r,\ \mathrm{Re}\, z > 0, \mathrm{Im}\, z > 0 \\ -i(\partial_z - \partial_{\overline{z}}) & \mathrm{Re}\, z > 0, \mathrm{Im}\, z = 0 \\ -(\partial_z + \partial_{\overline{z}}), & \mathrm{Re}\, z = 0, \mathrm{Im}\, z > 0 \end{cases} \tag{3.4}$$

So, differentiating $G_1(z,\zeta)$ in (3.2), one gets

$$\begin{aligned} \partial_z G_1(z,\zeta) \quad &= 2\Bigg\{\frac{z}{\zeta^2 - z^2} - \frac{z}{\overline{\zeta}^2 - z^2} + \frac{z\overline{\zeta}^2}{\overline{\zeta}^2 z^2 - 1} - \frac{z\zeta^2}{\zeta^2 z^2 - 1} \\ &+ \sum_{n=1}^{\infty}\Bigg(-\frac{zr^{4n}}{\overline{\zeta}^2 - r^{4n}z^2} - \frac{z}{r^{4n}\overline{\zeta}^2 - z^2} + \frac{z\overline{\zeta}^2}{\overline{\zeta}^2 z^2 - r^{4n}} + \frac{z\overline{\zeta}^2 r^{4n}}{\overline{\zeta}^2 z^2 r^{4n} - 1} \\ &+ \frac{zr^{4n}}{\zeta^2 - r^{4n}z^2} + \frac{z}{\zeta^2 r^{4n} - z^2} - \frac{z\zeta^2}{\zeta^2 z^2 - r^{4n}} - \frac{z\zeta^2 r^{4n}}{\zeta^2 z^2 r^{4n} - 1}\Bigg)\Bigg\} \end{aligned} \tag{3.5}$$

$$\begin{aligned} \partial_{\overline{z}} G_1(z,\zeta) \quad &= 2\Bigg\{\frac{\overline{z}}{\overline{\zeta}^2 - \overline{z}^2} - \frac{\overline{z}}{\zeta^2 - \overline{z}^2} + \frac{\overline{z}\zeta^2}{\zeta^2 \overline{z}^2 - 1} - \frac{\overline{z}\overline{\zeta}^2}{\overline{\zeta}^2 \overline{z}^2 - 1} \\ &+ \sum_{n=1}^{\infty}\Bigg(-\frac{\overline{z}r^{4n}}{\zeta^2 - r^{4n}\overline{z}^2} - \frac{\overline{z}}{r^{4n}\zeta^2 - \overline{z}^2} + \frac{\overline{z}\zeta^2}{\zeta^2 \overline{z}^2 - r^{4n}} + \frac{\overline{z}\zeta^2 r^{4n}}{\zeta^2 \overline{z}^2 r^{4n} - 1} \\ &+ \frac{\overline{z}r^{4n}}{\overline{\zeta}^2 - r^{4n}\overline{z}^2} + \frac{\overline{z}}{\overline{\zeta}^2 r^{4n} - \overline{z}^2} - \frac{\overline{z}\overline{\zeta}^2}{\overline{\zeta}^2 \overline{z}^2 - r^{4n}} - \frac{\overline{z}\overline{\zeta}^2 r^{4n}}{\overline{\zeta}^2 \overline{z}^2 r^{4n} - 1}\Bigg)\Bigg\} \end{aligned}$$

and considering the outward normal derivatives on the different boundary parts, one gets the following: for $|z| = 1, \mathrm{Re}\, z > 0, \mathrm{Im}\, z > 0$

$$\begin{aligned} \partial_{\nu_z} G_1(z,\zeta) \quad &= 8\mathrm{Re}\Bigg\{\frac{\zeta^2}{\zeta^2 - z^2} - \frac{\overline{\zeta}^2}{\overline{\zeta}^2 - z^2} \\ &+ \sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{\zeta^2}{\zeta^2 r^{4n} - z^2} - \frac{\overline{\zeta}^2}{\overline{\zeta}^2 r^{4n} - z^2} - \frac{1}{r^{4n} - \overline{\zeta}^2 z^2} + \frac{1}{r^{4n} - \zeta^2 z^2}\Bigg)\Bigg\}. \end{aligned}$$

For $|z| = r, \operatorname{Re} z > 0, \operatorname{Im} z > 0$

$$\begin{aligned}\partial_{\nu_z} G_1(z,\zeta) &= -\frac{8}{r}\operatorname{Re}\left\{\frac{\zeta^2}{\zeta^2 - z^2} - \frac{\overline{\zeta}^2}{\overline{\zeta}^2 - z^2}\right. \\ &\left. + \sum_{n=1}^{\infty} r^{4n}\left(\frac{\zeta^2}{\zeta^2 r^{4n} - z^2} - \frac{\overline{\zeta}^2}{\overline{\zeta}^2 r^{4n} - z^2} + \frac{z^2}{r^{4n}z^2 - \overline{\zeta}^2} - \frac{z^2}{r^{4n}z^2 - \zeta^2}\right)\right\}.\end{aligned}$$

For $\operatorname{Re} z > 0, \operatorname{Im} z = 0$

$$\begin{aligned}\partial_{\nu_z} G_1(z,\zeta) &= 8\operatorname{Im}\left\{\frac{z(\zeta^2 - \overline{\zeta}^2)}{|1 - \zeta^2 z^2|^2} - \frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 - z^2|^2}\right. \\ &\left. + \sum_{n=1}^{\infty}\left(\frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 - r^{4n}z^2|^2} + \frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 r^{4n} - z^2|^2} - \frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 z^2 - r^{4n}|^2} - \frac{z(\zeta^2 - \overline{\zeta}^2)}{|1 - \zeta^2 z^2 r^{4n}|^2}\right)\right\}.\end{aligned}$$

For $\operatorname{Re} z = 0, \operatorname{Im} z > 0$

$$\begin{aligned}\partial_{\nu_z} G_1(z,\zeta) &= -8\operatorname{Re}\left\{\frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 - z^2|^2} - \frac{z(\zeta^2 - \overline{\zeta}^2)}{|1 - \zeta^2 z^2|^2}\right. \\ &\left. + \sum_{n=1}^{\infty}\left(\frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 - r^{4n}z^2|^2} - \frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 z^2 - r^{4n}|^2} - \frac{z(\zeta^2 - \overline{\zeta}^2)}{|\zeta^2 r^{4n} - z^2|^2} + \frac{z(\zeta^2 - \overline{\zeta}^2)}{|1 - \zeta^2 z^2 r^{4n}|^2}\right)\right\}.\end{aligned}$$

Theorem 3.1.2. *[6] Any $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ can be represented as*

$$w(z) = -\frac{1}{4\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta - \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta,$$

where s_ζ is the arc length parameter on ∂D with respect to the variable ζ and $G(z,\zeta) = \frac{1}{2}G_1(z,\zeta)$ is the harmonic Green function for D.

3.1.2 Harmonic Dirichlet Problem

The Green representation formula provides a solution to the Dirichlet problem.

Theorem 3.1.3. *The harmonic Dirichlet problem*

$$\begin{aligned}&w_{z\overline{z}} = f \text{ in } R^*,\ w = \gamma \text{ on } \partial R^* \\ &\text{for } f \in L_2(R^*;\mathbb{C}) \cap C(R^*;\mathbb{C}),\ \gamma \in C(\partial R^*;\mathbb{C})\end{aligned} \tag{3.6}$$

is uniquely solvable by

$$\begin{aligned}w(z) &= \frac{1}{\pi i}\int_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)K_1(z,\zeta)d\zeta - \frac{1}{\pi i}\int_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)K_1(z,\zeta)d\zeta + \frac{2}{\pi i}\int_r^1 \gamma(t)K_2(z,t)dt \\ &-\frac{2}{\pi}\int_r^1 \gamma(it)K_2(z,it)dt - \frac{1}{\pi}\int_{R^*} f(\zeta)G_1(z,\zeta)d\xi d\eta,\end{aligned} \tag{3.7}$$

where $K_1(z,\zeta)$ and $K_2(z,\zeta)$ are given in (2.19) and (2.20) respectively.

Proof. We consider first the boundary condition. Taking the property of the Green function to vanish on the boundary [6] into account, for checking the boundary behavior only the boundary integrals need to be considered.

Let

$$w(z)=w_1-w_2+w_3+w_4-\frac{1}{\pi}\int\limits_{R^*}f(\zeta)G_1(z,\zeta)d\xi d\eta,$$

then

$$\begin{aligned}w_1(z) &= \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\0<\mathrm{Im}\zeta,\\0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta}^2-\overline{z}^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}\\ &+\sum_{n=1}^{\infty}r^{4n}\Bigg(\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-\overline{z}^2}+\frac{\zeta^2}{r^{4n}\zeta^2-z^2}-\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}-\frac{z^2}{r^{4n}z^2-\zeta^2}\\ &-\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}+\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}\Bigg)\Bigg\}\frac{d\zeta}{\zeta}=\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\0<\mathrm{Im}\zeta,\\0<\mathrm{Re}\,\zeta}}\gamma(\zeta)K_1(\zeta,z)d\zeta,\end{aligned}$$

where $K_1(\zeta,z)$ is given in (2.19).
Letting $z\to\zeta_0$, $\{|\zeta_0|=1,\,\mathrm{Re}\,\zeta_0\geq 0,\,\mathrm{Im}\,\zeta_0\geq 0\}$, from the calculations in Theorem 2.1.2 and formula (2.25) the equality

$$\lim_{z\to\zeta_0,z\in R^*}w_1(z)=\frac{1}{2\pi i}\lim_{z\to\zeta_0,z\in R^*}\int\limits_{|\zeta|=1}\Gamma_1(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right)\frac{d\zeta}{\zeta}=\gamma(\zeta_0)$$

follows, where $\Gamma_1(\zeta,z)$ is defined in (2.26).

Similarly,

$$\begin{aligned}w_2(z) &= \frac{1}{\pi i}\int\limits_{\substack{|\zeta|=r,\\0<\mathrm{Im}\zeta,\\0<\mathrm{Re}\,\zeta}}\gamma(\zeta)\Bigg\{\frac{\zeta^2}{\zeta^2-z^2}+\frac{\overline{\zeta}^2}{\overline{\zeta}^2-\overline{z}^2}-\frac{\zeta^2}{\zeta^2-\overline{z}^2}-\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2}\\ &+\sum_{n=1}^{\infty}r^{4n}\Bigg(\frac{\zeta^2}{r^{4n}\zeta^2-z^2}+\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-\overline{z}^2}-\frac{\zeta^2}{r^{4n}\zeta^2-\overline{z}^2}-\frac{\overline{\zeta}^2}{r^{4n}\overline{\zeta}^2-z^2}-\frac{z^2}{r^{4n}z^2-\zeta^2}\\ &-\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\overline{\zeta}^2}+\frac{\overline{z}^2}{r^{4n}\overline{z}^2-\zeta^2}+\frac{z^2}{r^{4n}z^2-\overline{\zeta}^2}\Bigg)\Bigg\}\frac{d\zeta}{\zeta}=\frac{1}{\pi i}\int\limits_{\substack{|\zeta|=1,\\0<\mathrm{Im}\zeta,\\0<\mathrm{Re}\,\zeta}}\gamma(\zeta)K_1(\zeta,z)d\zeta.\end{aligned}$$

Letting $z\to\zeta_0$, $\{|\zeta_0|=r,\,\mathrm{Re}\,\zeta_0\geq 0,\,\mathrm{Im}\,\zeta_0\geq 0\}$, and by Theorem 2.1.2 one gets

$$\lim_{z\to\zeta_0,z\in R^*}w_2(z)=\frac{1}{2\pi i}\lim_{z\to\zeta_0,z\in R^*}\int\limits_{|\zeta|=r}\Gamma_1(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\right)\frac{d\zeta}{\zeta}=-\gamma(\zeta_0),$$

where $\Gamma_1(\zeta)$ is defined as in (2.26) for $|\zeta|=r$.

For $r<\mathrm{Re}\,\zeta\leq 1$, $\mathrm{Im}\,\zeta=0$

$$\begin{aligned}w_3(z) &= \frac{2(z^2-\overline{z}^2)}{\pi i}\int_r^1\gamma(t)\Bigg\{\frac{t}{|t^2-z^2|^2}-\frac{t}{|1-t^2z^2|^2}+\sum_{n=1}^{\infty}r^{4n}\Bigg(\frac{t}{|r^{4n}t^2-z^2|^2}+\frac{t}{|r^{4n}z^2-t^2|^2}\\ &-\frac{t}{|r^{4n}z^2t^2-1|^2}-\frac{t}{|r^{4n}-t^2z^2|^2}\Bigg)\Bigg\}dt=\frac{2}{\pi i}\int_r^1\gamma(t)K_2(t,z)dt,\end{aligned}$$

where $K_2(\zeta,z)$ is defined in (2.20). Taking $z\to\zeta_0$, $r<\zeta_0<1$, $\zeta_0=t_0$ on the basis of Theorem 2.1.2

$$\lim_{z\to t_0,z\in R^*}w_3(z)=\lim_{z\to t_0,z\in R^*}\frac{1}{\pi i}\int_r^1\gamma(t)K_2(t,z)dt=\lim_{z\to t_0,z\in R^*}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\Gamma_2(t)\frac{z-\overline{z}}{|t-z|^2}dt=\gamma(t_0),$$

where $\Gamma_2(t)$ is given in (2.29).

For $r < \operatorname{Im}\zeta \le 1$, $\operatorname{Re}\zeta = 0$

$$\begin{aligned} w_4 &= \frac{2(z^2-\overline{z}^2)}{\pi i}\int\limits_r^1 \gamma(it)\Bigg\{\frac{t}{|t^2+z^2|^2} - \frac{t}{|1+t^2z^2|^2} + \sum_{n=1}^{\infty} r^{4n}\Bigg(\frac{t}{|r^{4n}z^2+t^2|^2} - \frac{t}{|r^{4n}z^2t^2+1|^2} \\ &\quad + \frac{t}{|r^{4n}t^2+z^2|^2} - \frac{t}{|r^{4n}+t^2z^2|^2}\Bigg)\Bigg\}dt = \frac{2}{\pi i}\int\limits_1^r \gamma(it)K_2(it,z)dt, \end{aligned}$$

taking $z \to \zeta_0$, $r < t_0 < 1$, $\zeta_0 = it_0$

$$\lim_{z\to it_o} w_4(z) = \lim_{z\to it_o}\int\limits_r^1 \gamma(it)K_2(it,z)dt = \lim_{z\to it_0}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Gamma_2(it)\frac{z+\overline{z}}{|it-z|^2}dt = \gamma(it_0),$$

where $\Gamma_2(it)$ is defined in (2.31).

It is seen that the derivative of the Green function $\partial_z G_1$ in (3.5) gives the Pompeiu-type operator

$$\begin{aligned} \tilde{T}_{R^*}f(z) &= -\frac{1}{\pi}\int\limits_{R^*} f(\zeta)\frac{d\xi d\eta}{\zeta - z} + \frac{2}{\pi}\int\limits_{R^*} f(\zeta)\Bigg\{\frac{1}{2}\frac{1}{\zeta+z} - \frac{z}{\overline{\zeta}^2 - z^2} + \frac{z\overline{\zeta}^2}{\overline{\zeta}^2z^2-1} - \frac{z\zeta^2}{\zeta^2z^2-1} \\ &\quad + \sum_{n=1}^{\infty}\Bigg(-\frac{zr^{4n}}{\overline{\zeta}^2 - r^{4n}z^2} - \frac{z}{r^{4n}\overline{\zeta}^2 - z^2} + \frac{z\overline{\zeta}^2}{\overline{\zeta}^2z^2 - r^{4n}} + \frac{z\overline{\zeta}^2r^{4n}}{\overline{\zeta}^2z^2r^{4n}-1} \\ &\quad + \frac{zr^{4n}}{\zeta^2 - r^{4n}z^2} + \frac{z}{\zeta^2r^{4n}-z^2} - \frac{z\zeta^2}{\zeta^2z^2-r^{4n}} - \frac{z\zeta^2r^{4n}}{\zeta^2z^2r^{4n}-1}\Bigg)\Bigg\}d\xi d\eta \end{aligned} \tag{3.8}$$

and $\partial_{\overline{z}}\tilde{T}_{R^*}f(z) = f(z)$ provides the weak solution of the differential equation in (3.1.3). □

Remark 3.1.1. *The boundary behavior at the corner points is studied in the Lemmas (2.1.1)-(2.1.2). The following relations for the boundary behavior are valid*

$$\begin{aligned}
&\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_1 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta = \gamma(\zeta_0) \text{ for } \zeta_0\in\partial_1R^*,\\
&\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_1 R^*}\gamma(\zeta)K_1(z,\zeta)\frac{d\zeta}{\zeta} = 0 \text{ for } \zeta_0\in\partial R^*\backslash\partial_1R^*,\\
&\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_2 R^*}\gamma(\zeta)K_1(z,\zeta)d\zeta = \gamma(\zeta_0) \text{ for } \zeta_0\in\partial_2R^*,\\
&\lim_{z\to\zeta_0}\frac{1}{\pi i}\int\limits_{\partial_2 R^*}\gamma(\zeta)K_1(z,\zeta)\frac{d\zeta}{\zeta} = 0 \text{ for } \zeta_0\in\partial R^*\backslash\partial_2R^*,\\
&\lim_{z\to t_0}\frac{1}{\pi i}\int\limits_{\partial_3 R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta = \gamma(\zeta_0) \text{ for } \zeta_0\in\partial_3R^*\backslash\{r\},\ \zeta_0 = t_0,\\
&\lim_{z\to t_0}\frac{1}{\pi i}\int\limits_{\partial_3 R*}[\gamma(\zeta)-\gamma(r)]K_2(z,\zeta)d\zeta = \gamma(\zeta_0)-\gamma(r) \text{ for } \zeta_0\in\partial_3R^*,\ \zeta_0 = t_0,\\
&\lim_{z\to t_0}\frac{1}{\pi i}\int\limits_{\partial_3 R^*}\gamma(\zeta)K_2(z,\zeta)\frac{d\zeta}{\zeta} = 0 \text{ for } \zeta_0\in\partial R^*\backslash\partial_3R^*,\\
&\lim_{z\to it_0}\frac{1}{\pi i}\int\limits_{\partial_4 R^*}\gamma(\zeta)K_2(z,\zeta)d\zeta = \gamma(\zeta_0) \text{ for } \zeta_0\in\partial_4R^*\backslash\{r\},\ \zeta_0 = it_0,\\
&\lim_{z\to it_0}\frac{1}{\pi i}\int\limits_{\partial_4 R^*}[\gamma(\zeta)-\gamma(ir)]K_2(z,\zeta)d\zeta = \gamma(\zeta_0)-\gamma(ir) \text{ for } \zeta_0\in\partial_4R^*,\ \zeta_0 = it_0,\\
&\lim_{z\to it_0}\frac{1}{\pi i}\int\limits_{\partial_4}\gamma(\zeta)K_2(z,\zeta)\frac{d\zeta}{\zeta} = 0 \text{ for } \zeta_0\in\partial R^*\backslash\partial_4R^*.
\end{aligned}$$

3.2 Harmonic Neumann Problem

3.2.1 Harmonic Neumann function and the Neumann representation formula

Definition 3.2.1. *[6] A real-valued function N in a domain D is called Neumann function (for the Laplace operator) if it satisfies:*
1. $N(z,z_0)$ is harmonic in $z \in D\backslash\{z_0\}$, $z_0 \in D$
2. $\log|z-z_0| + N(z,z_0)$ is harmonic in the neighborhood of z_0,
3. $\frac{\partial}{\partial n}N(z,z_0) = \sigma(z)$ on ∂D, where σ is a piecewise constant function on ∂D, $z_0 \in D$,
4. $\int\limits_{\partial D} \sigma(z)N(z,z_0)ds_z = 0$.

The examples of the Neumann function $N_1(z,\zeta) = 2N(z,\zeta)$ are found for the circular ring domain in [41]

$$N_1(z,\zeta) = -\log\left|(\zeta - z)(1 - z\overline{\zeta})\prod_{k=1}^{\infty}\frac{(z - r^{2k}\zeta)(z\overline{\zeta} - r^{2k})(\zeta - r^{2k}z)(1 - r^{2k}z\overline{\zeta})}{|z|^2|\zeta|^2}\right|^2$$

with the observed boundary behavior

$$\partial_{\nu_z}N_1(z,\zeta) = \begin{cases} zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta) = -2, & |z| = 1, \\ -\frac{z}{r}N_{1z}(z,\zeta) - \frac{\overline{z}}{r}N_{1\overline{z}}(z,\zeta) = 0, & |z| = r \end{cases}$$

as well as for the upper half ring domain in [18]

$$\begin{aligned} N_1(z,\zeta) &= 2\log\frac{|z\zeta|^2}{r^2} - \log|(\zeta - z)(\overline{\zeta} - z)(1 - z\overline{\zeta})(1 - z\zeta)|^2 + \sum_{n=1}^{\infty}\Bigg[4\log|z\zeta|^2 \\ &- \log\left|(z - r^{2n}\zeta)(z - r^{2n}\overline{\zeta})(z\zeta - r^{2n})(z\overline{\zeta} - r^{2n})(\zeta - r^{2n}z)(\overline{\zeta} - r^{2n}z)(1 - r^{2n}z\zeta)(1 - r^{2n}z\overline{\zeta})\right|^2\Bigg] \end{aligned}$$

with the boundary behavior

$$\partial_{\nu_z}N_1(z,\zeta) = \begin{cases} zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta) = 0, & |z| = 1, \\ -\frac{z}{r}N_{1z}(z,\zeta) - \frac{\overline{z}}{r}N_{1\overline{z}}(z,\zeta) = 4, & |z| = r, \\ -i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = 0, & \operatorname{Re} z > 0, \operatorname{Im} z = 0. \end{cases}$$

The Neumann function for the quarter ring domain R^* is obtained from the Green function for R^* by multiplying the factors so that the properties are met. Thus

$$\begin{aligned} N_1(z,\zeta) &= \log\frac{|z\zeta|^8}{r^8} - \log|(\overline{\zeta}^2 - z^2)(\zeta^2 - z^2)(\overline{\zeta}^2z^2 - 1)(\zeta^2z^2 - 1)|^2 \\ &+ \sum_{n=1}^{\infty}\Bigg(\log|z\zeta|^{16} - \log|(\overline{\zeta}^2 - r^{4n}z^2)(\zeta^2 - r^{4n}z^2)(\overline{\zeta}^2r^{4n} - z^2)\times \\ &(\zeta^2r^{4n} - z^2)(\overline{\zeta}^2z^2 - r^{4n})(\zeta^2z^2 - r^{4n})(\overline{\zeta}^2z^2r^{4n} - 1)(\zeta^2z^2r^{4n} - 1)|^2\Bigg) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} N_1(z,\zeta) &= \log\frac{|z|^8}{r^8} - \log\left|(1 - \frac{z^2}{\zeta^2})(1 - \frac{z^2}{\overline{\zeta}^2})(1 - \zeta^2z^2)(1 - \overline{\zeta}^2z^2)\right|^2 \\ &- \sum_{n=1}^{\infty}\Bigg[\log\Bigg|(1 - \frac{r^{4n}z^2}{\zeta^2})(1 - \frac{r^{4n}z^2}{\overline{\zeta}^2})(1 - \frac{r^{4n}\zeta^2}{z^2})(1 - \frac{r^{4n}\overline{\zeta}^2}{z^2})\times \\ &(1 - \frac{r^{4n}}{\zeta^2z^2})(1 - \frac{r^{4n}}{\overline{\zeta}^2z^2})(1 - r^{4n}\zeta^2z^2)(1 - r^{4n}\overline{\zeta}^2z^2)\Bigg|^2\Bigg]. \end{aligned} \tag{3.10}$$

The function presented in (3.10) is seen to be convergent. This can be shown, for example, for the term in the sum

$$-\sum_{n=1}^{\infty}\log(1-\frac{r^{4n}z^2}{\zeta^2})=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}(\frac{r^{4n}z^2}{\zeta^2})^k,$$

since

$$\begin{aligned}&\sum_{n=1}^{\infty}(r^{4k})^n=\frac{1}{1-r^{4k}}-1=\frac{r^{4k}}{1-r^{4k}},\ |r^{4k}|\le 1,\\ &1-r^{4k}\ge 1-r^4,\ \frac{1}{1-r^{4k}}\le\frac{1}{1-r^4},\ |\frac{z^2}{\zeta^2}|^k\le\frac{1}{r^{2k}},\\ &\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{r^{2k}}\frac{r^{4k}}{1-r^4}=\frac{1}{1-r^4}\sum_{k=1}^{\infty}\frac{r^{2k}}{k}=\frac{1}{1-r^4}\log(\frac{1}{1-r^2}).\end{aligned}$$

The Neumann function $N_1(z,\zeta)$ on the boundary parts has the following forms.
For $r<|z|<1, \operatorname{Re} z>0, \operatorname{Im} z>0$

$$\begin{aligned}N_1(z,\zeta)\ &=2\log\frac{|z|^4}{r^4}-2\log|(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)|^2\\ &-2\sum_{n=1}^{\infty}\left(\log\left|(1-r^{4n}z^2\zeta^2)(1-r^{4n}z^2\overline{\zeta}^2)(1-\frac{r^{4n}\overline{\zeta}^2}{z^2})(1-\frac{r^{4n}\zeta^2}{z^2})\right|^2\right)\end{aligned}\tag{3.11}$$

on $|\zeta|=1$, and

$$\begin{aligned}N_1(z,\zeta)\ &=2\log\frac{|z\zeta|^4}{r^4}-2\log|(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)|^2\\ &-2\sum_{n=1}^{\infty}\left(\log\left|(1-r^{4n}z^2\zeta^2)(1-r^{4n}z^2\overline{\zeta}^2)(1-\frac{r^{4n}}{z^2\zeta^2})(1-\frac{r^{4n}}{z^2\overline{\zeta}^2})\right|^2\right)\end{aligned}\tag{3.12}$$

on $|\zeta|=r$,

for $r<\operatorname{Re}\zeta<1, \operatorname{Im}\zeta=0$

$$\begin{aligned}N_1(z,\zeta)\ &=2\log\frac{|z|^4}{r^4}-2\log|(1-\frac{z^2}{t^2})(1-z^2t^2)|^2\\ &-2\sum_{n=1}^{\infty}\left(\log\left|(1-\frac{r^{4n}z^2}{t^2})(1-\frac{r^{4n}t^2}{z^2})(1-\frac{r^{4n}}{t^2z^2})(1-r^{4n}z^2t^2)\right|^2\right)\end{aligned}\tag{3.13}$$

and for $r<\operatorname{Im}\zeta<1$, $\operatorname{Re}\zeta=0$

$$\begin{aligned}N_1(z,\zeta)\ &=\ 2\log\frac{|z|^4}{r^4}-2\log|(1+\frac{z^2}{t^2})(1+z^2t^2)|^2\\ &-2\sum_{n=1}^{\infty}\left(\log\left|(1+\frac{r^{4n}z^2}{t^2})(1+\frac{r^{4n}t^2}{z^2})(1+\frac{r^{4n}}{t^2z^2})(1+r^{4n}z^2t^2)\right|^2\right).\end{aligned}\tag{3.14}$$

Taking derivatives of (3.9), we have

$$\begin{aligned}\partial_z N_1(z,\zeta)\ &=2\Bigg\{\frac{2}{z}+\frac{z}{\overline{\zeta}^2-z^2}+\frac{z}{\zeta^2-z^2}+\frac{z\zeta^2}{1-z^2\zeta^2}+\frac{z\overline{\zeta}^2}{1-z^2\overline{\zeta}^2}\\ &+\sum_{n=1}^{\infty}\Bigg(\frac{4}{z}+\frac{r^{4n}z}{\overline{\zeta}^2-r^{4n}z^2}+\frac{r^{4n}z}{\zeta^2-r^{4n}z^2}-\frac{z}{z^2-\overline{\zeta}^2r^{4n}}-\frac{z}{z^2-r^{4n}\zeta^2}\\ &-\frac{\overline{\zeta}^2z}{\overline{\zeta}^2z^2-r^{4n}}-\frac{\zeta^2z}{\zeta^2z^2-r^{4n}}+\frac{\overline{\zeta}^2zr^{4n}}{1-\overline{\zeta}^2z^2r^{4n}}+\frac{\zeta^2zr^{4n}}{1-\zeta^2z^2r^{4n}}\Bigg)\Bigg\}\end{aligned}\tag{3.15}$$

$$\begin{aligned}\partial_{\overline{z}} N_1(z,\zeta) \quad &= 2\Bigg\{\frac{2}{\overline{z}} + \frac{\overline{z}}{\zeta^2-\overline{z}^2} + \frac{\overline{z}}{\overline{\zeta}^2-\overline{z}^2} + \frac{\overline{z}\overline{\zeta}^2}{1-\overline{z}^2\overline{\zeta}^2} + \frac{\overline{z}\zeta^2}{1-\overline{z}^2\zeta^2} \\ &+ \sum_{n=1}^{\infty}\Bigg(\frac{4}{\overline{z}} + \frac{r^{4n}\overline{z}}{\zeta^2-r^{4n}\overline{z}^2} + \frac{r^{4n}\overline{z}}{\overline{\zeta}^2-r^{4n}\overline{z}^2} - \frac{\overline{z}}{\overline{z}^2-\zeta^2 r^{4n}} - \frac{\overline{z}}{\overline{z}^2-r^{4n}\overline{\zeta}^2} \\ &- \frac{\zeta^2\overline{z}}{\zeta^2\overline{z}^2-r^{4n}} - \frac{\overline{\zeta}^2\overline{z}}{\overline{\zeta}^2\overline{z}^2-r^{4n}} + \frac{\zeta^2\overline{z}r^{4n}}{1-\zeta^2\overline{z}^2r^{4n}} + \frac{\overline{\zeta}^2\overline{z}r^{4n}}{1-\overline{\zeta}^2\overline{z}^2r^{4n}}\Bigg)\Bigg\}\end{aligned}$$

and since

$$\partial_{\nu_z} = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & |z|=1,\ \operatorname{Re} z>0, \operatorname{Im} z>0 \\ -\frac{1}{r}(z\partial_z+\overline{z}\partial_{\overline{z}}), & |z|=r,\ \operatorname{Re} z>0, \operatorname{Im} z>0 \\ -i(\partial_z-\partial_{\overline{z}}) & \operatorname{Re} z>0, \operatorname{Im} z=0 \\ -(\partial_z+\partial_{\overline{z}}), & \operatorname{Re} z=0, \operatorname{Im} z>0 \end{cases} \tag{3.16}$$

it follows that for any $\zeta \in R^*$

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} 0, & |z|=1,\ \operatorname{Re} z>0, \operatorname{Im} z>0 \\ -\frac{8}{r}, & |z|=r,\ \operatorname{Re} z>0, \operatorname{Im} z>0 \\ 0, & \operatorname{Re} z>0, \operatorname{Im} z=0 \\ 0, & \operatorname{Re} z=0, \operatorname{Im} z>0. \end{cases} \tag{3.17}$$

Then the third property in Definition 3.2.2 is satisfied.

For $|\zeta|=1$, $\operatorname{Re}\zeta>0$, $\operatorname{Im}\zeta>0$

$$\begin{aligned}&(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 2\Bigg\{12 + \frac{z^2}{\overline{\zeta}^2-z^2} + \frac{z^2}{\zeta^2-z^2} + \frac{z^2}{\overline{\zeta}^2-z^2} + \frac{z^2}{\zeta^2-z^2} \\ &+ \frac{\overline{z}^2}{\zeta^2-\overline{z}^2} + \frac{\overline{z}^2}{\overline{\zeta}^2-\overline{z}^2} + \frac{\overline{z}^2}{\overline{\zeta}^2-\overline{z}^2} + \frac{\overline{z}^2}{\zeta^2-\overline{z}^2} \\ &+ \sum_{n=1}^{\infty}\Bigg(\frac{r^{4n}z^2}{\overline{\zeta}^2-r^{4n}z^2} + \frac{r^{4n}z^2}{\zeta^2-r^{4n}z^2} + \frac{r^{4n}\overline{z}^2}{\zeta^2-r^{4n}\overline{z}^2} + \frac{r^{4n}\overline{z}^2}{\overline{\zeta}^2-r^{4n}\overline{z}^2} \\ &- \frac{r^{4n}\overline{\zeta}^2}{z^2-r^{4n}\overline{\zeta}^2} - 1 - \frac{r^{4n}\zeta^2}{z^2-r^{4n}\zeta^2} - 1 - \frac{r^{4n}\zeta^2}{\overline{z}^2-r^{4n}\zeta^2} - 1 - \frac{r^{4n}\overline{\zeta}^2}{\overline{z}^2-r^{4n}\overline{\zeta}^2} - 1 \\ &- \frac{\overline{\zeta}^2}{\overline{\zeta}^2-r^{4n}\overline{z}^2} - \frac{\zeta^2}{\zeta^2-r^{4n}\overline{z}^2} - \frac{\zeta^2}{\zeta^2-r^{4n}z^2} - \frac{\overline{\zeta}^2}{\overline{\zeta}^2-r^{4n}z^2} \\ &+ \frac{r^{4n}\overline{\zeta}^2}{\overline{z}^2-r^{4n}\overline{\zeta}^2} + \frac{r^{4n}\zeta^2}{\overline{z}^2-r^{4n}\zeta^2} + \frac{r^{4n}\zeta^2}{z^2-r^{4n}\zeta^2} + \frac{r^{4n}\overline{\zeta}^2}{z^2-r^{4n}\overline{\zeta}^2}\Bigg)\Bigg\} = \\ &2\Bigg\{4 + \frac{2z^2}{\overline{\zeta}^2-z^2} + \frac{2z^2}{\zeta^2-z^2} + \frac{2\overline{z}^2}{\overline{\zeta}^2-\overline{z}^2} + \frac{2\overline{z}^2}{\zeta^2-\overline{z}^2}\Bigg\} = \\ &2\Bigg\{4 + 2\Bigg(\frac{\overline{\zeta}^2}{\overline{\zeta}^2-z^2} - 1 + \frac{\zeta^2}{\zeta^2-z^2} - 1 + \frac{\overline{\zeta}^2}{\overline{\zeta}^2-\overline{z}^2} - 1 + \frac{\zeta^2}{\zeta^2-\overline{z}^2} - 1\Bigg)\Bigg\} = \\ &2\Bigg\{\Bigg(\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta-z}} - 1\Bigg) + \Bigg(\frac{\zeta}{\zeta+z} + \frac{\overline{\zeta}}{\overline{\zeta+z}} - 1\Bigg) \\ &+ \Bigg(\frac{\overline{\zeta}}{\overline{\zeta}-z} + \frac{\zeta}{\zeta-\overline{z}} - 1\Bigg) + \Bigg(\frac{\overline{\zeta}}{\overline{\zeta}+z} + \frac{\zeta}{\zeta+\overline{z}} - 1\Bigg)\Bigg\}\end{aligned}$$

when $|z|=1$ and

$$\begin{aligned} -\frac{1}{r}(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta) &= -\frac{8}{r}, \quad |z|=r, \\ -i(\partial_z-\partial_{\overline{z}})N_1(z,\zeta) &= 0, \quad z=\overline{z}, \\ -(\partial_z+\partial_{\overline{z}})N_1(z,\zeta) &= 0, \quad z=-\overline{z}. \end{aligned}$$

For $|\zeta| = r$, $\mathrm{Re}\,\zeta > 0$, $\mathrm{Im}\zeta > 0$

$$-\frac{1}{r}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) = -\frac{2}{r}\Bigg\{4 + \frac{z^2}{\overline{\zeta}^2 - z^2} + \frac{z^2}{\zeta^2 - z^2} + \frac{z^2\overline{\zeta}^2}{1 - z^2\overline{\zeta}^2}$$
$$+\frac{z^2\zeta^2}{1 - z^2\zeta^2} + \frac{\overline{z}^2}{\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\overline{\zeta}^2 - \overline{z}^2} + \frac{\overline{z}^2\zeta^2}{1 - \overline{z}^2\zeta^2} + \frac{\overline{z^2\zeta^2}}{1 - \overline{z^2\zeta^2}}$$
$$+\sum_{n=1}^{\infty}\Bigg(8 + \frac{r^{4n}z^2}{\overline{\zeta}^2 - r^{4n}z^2} + \frac{r^{4n}z^2}{\zeta^2 - r^{4n}z^2} - \frac{\overline{\zeta}^2 r^{4n}}{z^2 - \overline{\zeta}^2 r^{4n}} - 1 - \frac{\zeta^2 r^{4n}}{z^2 - \zeta^2 r^{4n}} - 1$$
$$-\frac{\overline{\zeta}^2 z^2}{\overline{\zeta}^2 z^2 - r^{4n}} - \frac{\zeta^2 z^2}{\zeta^2 z^2 - r^{4n}} + \frac{\overline{\zeta}^2 z^2 r^{4n}}{1 - \overline{\zeta}^2 z^2 r^{4n}} + \frac{\zeta^2 z^2 r^{4n}}{1 - \zeta^2 z^2 r^{4n}} + \frac{r^{4n}\overline{z}^2}{\zeta^2 - r^{4n}\overline{z}^2}$$
$$+\frac{r^{4n}\overline{z}^2}{\overline{\zeta}^2 - r^{4n}\overline{z}^2} - \frac{\zeta^2 r^{4n}}{\overline{z}^2 - \zeta^2 r^{4n}} - 1 - \frac{\overline{\zeta}^2 r^{4n}}{\overline{z}^2 - \overline{\zeta}^2 r^{4n}} - 1 - \frac{\zeta^2\overline{z}^2}{\zeta^2\overline{z}^2 - r^{4n}}$$
$$-\frac{\overline{\zeta}^2\overline{z}^2}{\overline{\zeta}^2\overline{z}^2 - r^{4n}} + \frac{\zeta^2\overline{z}^2 r^{4n}}{1 - \zeta^2\overline{z}^2 r^{4n}} + \frac{\overline{\zeta}^2\overline{z}^2 r^{4n}}{1 - \overline{\zeta}^2\overline{z}^2 r^{4n}}\Bigg)\Bigg\} =$$
$$-\frac{2}{r}\Bigg\{4 + \frac{z^2}{\overline{\zeta}^2 - z^2} + \frac{z^2}{\zeta^2 - z^2} + \frac{z^2\overline{\zeta}^2}{1 - z^2\overline{\zeta}^2} + \frac{z^2\zeta^2}{1 - z^2\zeta^2} + \frac{\overline{z}^2}{\zeta^2 - \overline{z}^2} + \frac{\overline{z}^2}{\overline{\zeta}^2 - \overline{z}^2}$$
$$+\frac{\overline{z}^2\zeta^2}{1 - \overline{z}^2\zeta^2} + \frac{\overline{z^2\zeta^2}}{1 - \overline{z^2\zeta^2}} + \sum_{n=1}^{\infty}\Bigg(\frac{r^{4n}z^2}{\overline{\zeta}^2 - r^{4n}z^2} + \frac{r^{4n}z^2}{\zeta^2 - r^{4n}z^2} - \frac{\overline{\zeta}^2 r^{4n}}{z^2 - \overline{\zeta}^2 r^{4n}}$$
$$-\frac{\zeta^2 r^{4n}}{z^2 - \zeta^2 r^{4n}} - \frac{z^2 r^4}{z^2 r^4 - r^{4n}\zeta^2} - \frac{z^2 r^4}{z^2 r^4 - r^{4n}\overline{\zeta}^2} + \frac{z^2 r^{4(n+1)}}{\zeta^2 - z^2 r^{4(n+1)}}$$
$$+\frac{z^2 r^{4(n+1)}}{\overline{\zeta}^2 - z^2 r^{4(n+1)}} + \frac{r^{4n}\overline{z}^2}{\zeta^2 - r^{4n}\overline{z}^2} + \frac{r^{4n}\overline{z}^2}{\overline{\zeta}^2 - r^{4n}\overline{z}^2} - \frac{\zeta^2 r^{4n}}{\overline{z}^2 - \zeta^2 r^{4n}} - \frac{\overline{\zeta}^2 r^{4n}}{\overline{z}^2 - \overline{\zeta}^2 r^{4n}}$$
$$-\frac{\overline{z}^2 r^4}{\overline{z}^2 r^4 - r^{4n}\overline{\zeta}^2} - \frac{\overline{z}^2 r^4}{\overline{z}^2 r^4 - r^{4n}\zeta^2} + \frac{\overline{z}^2 r^{4(n+1)}}{\zeta^2 - \overline{z}^2 r^{4(n+1)}} + \frac{\overline{z}^2 r^{4(n+1)}}{\overline{\zeta}^2 - \overline{z}^2 r^{4(n+1)}}\Bigg)\Bigg\} =$$
$$-\frac{2}{r}\Bigg\{\left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right) + \left(\frac{\zeta}{\zeta + z} + \frac{\overline{\zeta}}{\overline{\zeta + z}} - 1\right)$$
$$+\left(\frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1\right) + \left(\frac{\overline{\zeta}}{\overline{\zeta} + z} + \frac{\zeta}{\zeta + \overline{z}} - 1\right) + 4\Bigg\},$$

when $z = |r|$ and

$$(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 0, \quad |z| = 1,$$
$$-i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = 0, \quad z = \overline{z},$$
$$-(\partial_z + \partial_{\overline{z}})N_1(z,\zeta) = 0, \quad z = -\overline{z}.$$

For $\mathrm{Re}\,\zeta > 0$, $\mathrm{Im}\zeta = 0$

$$-i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = -2i\Bigg\{\frac{2}{z} - \frac{2}{\overline{z}} + \frac{z}{\overline{\zeta}^2 - z^2} + \frac{z}{\zeta^2 - z^2} + \frac{z\overline{\zeta}^2}{1 - z^2\overline{\zeta}^2}$$
$$+\frac{z\zeta^2}{1 - z^2\zeta^2} - \frac{\overline{z}}{\zeta^2 - \overline{z}^2} - \frac{\overline{z}}{\overline{\zeta}^2 - \overline{z}^2} - \frac{\overline{z}\zeta^2}{1 - \overline{z}^2\zeta^2} - \frac{\overline{z\zeta^2}}{1 - \overline{z^2\zeta^2}}$$
$$+\sum_{n=1}^{\infty}\Bigg(\frac{4}{z} - \frac{4}{\overline{z}} + \frac{r^{4n}z}{\overline{\zeta}^2 - r^{4n}z^2} + \frac{r^{4n}z}{\zeta^2 - r^{4n}z^2} - \left(\frac{r^{4n}\overline{\zeta}^2}{z^2 - r^{4n}\overline{\zeta}^2} + 1\right)\frac{1}{z} - \left(\frac{r^{4n}\zeta^2}{z^2 - r^{4n}\zeta^2} + 1\right)\frac{1}{z}$$
$$-\left(\frac{r^{4n}}{\overline{\zeta}^2 z^2 - r^{4n}} + 1\right)\frac{1}{z} - \left(\frac{r^{4n}}{\zeta^2 z^2 - r^{4n}} + 1\right)\frac{1}{z} + \frac{r^{4n}z\overline{\zeta}^2}{1 - r^{4n}z^2\overline{\zeta}^2} + \frac{r^{4n}z\zeta^2}{1 - r^{4n}z^2\zeta^2}$$
$$-\frac{r^{4n}\overline{z}}{\zeta^2 - r^{4n}\overline{z}^2} - \frac{r^{4n}\overline{z}}{\overline{\zeta}^2 - r^{4n}\overline{z}^2} + \left(\frac{r^{4n}\zeta^2}{\overline{z}^2 - r^{4n}\zeta^2} + 1\right)\frac{1}{\overline{z}} + \left(\frac{r^{4n}\overline{\zeta}^2}{\overline{z}^2 - r^{4n}\overline{\zeta}^2} + 1\right)\frac{1}{\overline{z}}$$

$$+\left(\frac{r^{4n}}{\zeta^2\overline{z}^2-r^{4n}}+1\right)\frac{1}{\overline{z}}+\left(\frac{r^{4n}}{\overline{\zeta^2 z^2}-r^{4n}}+1\right)\frac{1}{\overline{z}}-\frac{r^{4n}\overline{z}\zeta^2}{1-r^{4n}\overline{z}^2\zeta^2}-\frac{r^{4n}\overline{z}\overline{\zeta}^2}{1-r^{4n}\overline{z}^2\overline{\zeta}^2}\right)\Bigg\}=$$

$$-2i\Bigg\{2\frac{\overline{z}-z}{|z|^2}+2\frac{(z-\overline{z})(\zeta^2+|z|^2)}{|\zeta^2-z^2|^2}+2\frac{(z-\overline{z})(\zeta^2+|z|^2|\zeta|^4)}{|1-z^2\zeta^2|^2}$$

$$+2\sum_{n=1}^{\infty}\left(\frac{r^{4n}[(z-\overline{z})(\zeta^2+r^{4n}|z|^2)]}{|\overline{\zeta}^2-r^{4n}z^2|^2}-\frac{(z-\overline{z})r^{4n}[|\zeta|^4r^{4n}-\zeta^2(2z^2+|z|^2)]}{|z|^2|z^2-r^{4n}\zeta^2|^2}\right.$$

$$\left.-\frac{(z-\overline{z})r^{4n}[r^{4n}-\zeta^2r^{4n}(2z^2+|z|^2)]}{|z|^2|\zeta^2z^2-r^{4n}|^2}+\frac{r^{4n}[(z-\overline{z})(\zeta^2+r^{4n}|\zeta|^4|z|^2)]}{|1-r^{4n}\zeta^2z^2|^2}\right)\Bigg\}=$$

$$4i\left(\frac{z-\overline{z}}{|z|^2}-\frac{(z-\overline{z})(\zeta^2+|z|^2)}{|\zeta^2-z^2|^2}-\frac{(z-\overline{z})(\zeta^2+|z|^2|\zeta|^4)}{|1-z^2\zeta^2|^2}\right),\ z=\overline{z},$$

$$\begin{aligned}(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta)&=0, & |z|=1,\\ -\frac{1}{r}(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta)&=-\frac{8}{r}, & |z|=r,\\ -(\partial_z+\partial_{\overline{z}})N_1(z,\zeta)&=0, & z=-\overline{z}.\end{aligned}$$

For $\mathrm{Re}\,\zeta=0$, $\mathrm{Im}\zeta>0$

$$\begin{aligned}(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta)&=0, & |z|=1,\\ -\frac{1}{r}(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta)&=-\frac{8}{r}, & |z|=r,\\ -i(\partial_z-\partial_{\overline{z}})N_1(z,\zeta)&=0, & z=\overline{z},\end{aligned}$$

calculating in the same way, if $z=-\overline{z}$, one gets

$$-(\partial_z+\partial_{\overline{z}})N_1(z,\zeta)=-4\left(\frac{z+\overline{z}}{|z|^2}+\frac{(z+\overline{z})(\zeta^2-|z|^2)}{|\zeta^2+z^2|^2}+\frac{(z+\overline{z})(\zeta^2-|z|^2|\zeta|^4)}{|1+z^2\zeta^2|^2}\right).$$

$N_1(z,\zeta)$ is obviously harmonic in the domain up to $z=\zeta$.

The Neumann function also satisfies the normalization conditions

$$\frac{1}{\pi i}\int_{\partial R^*}\sigma(z)N_1(z,\zeta)\frac{dz}{z}=\frac{8}{r\pi i}\int_{\substack{|z|=r,\\0<Im\,z,\\0<Re\,z}}\Bigg\{\log\left|(1-\frac{z^2}{\zeta^2})(1-\frac{z^2}{\overline{\zeta}^2})(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)\right|^2$$

$$+\sum_{n=1}^{\infty}\log\left|(1-\frac{z^2r^{4n}}{\overline{\zeta}^2})(1-\frac{z^2r^{4n}}{\zeta^2})(1-\frac{r^{4n}\zeta^2}{z^2})(1-\frac{r^{4n}\overline{\zeta}^2}{z^2})\times\right.$$

$$\left.(1-\frac{r^{4n}}{\zeta^2z^2})(1-\frac{r^{4n}}{\overline{\zeta}^2z^2})(1-z^2\zeta^2r^{4n})(1-z^2\overline{\zeta}^2r^{4n})\right|^2\Bigg\}\frac{dz}{z}=$$

$$\frac{8}{r\pi i}\int_{|z|=r}\log\left|(1-\frac{z}{\zeta})(1-z\zeta)\right|^2\frac{dz}{z}$$

$$+\frac{1}{\pi i}\int_{|z|=r}\sum_{n=1}^{\infty}\log\left|(1-\frac{zr^{4n}}{\zeta})(1-\frac{\zeta r^{4n}}{z})(1-\frac{r^{4n}}{z\zeta})(1-z\zeta r^{4n})\right|^2\frac{dz}{z}=0.$$

Theorem 3.2.1. *Any $w\in C^2(R^*,C)\cap C^1(\overline{R^*},C)$ can be represented by*

$$w(z)=-\frac{1}{4\pi}\int_{\partial R^*}[w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)-\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)]ds_\zeta-\frac{1}{\pi}\int_{R^*}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta, \tag{3.18}$$

with $N_1=2N$, where N is the harmonic Neumann function for R^*.

Proof. The proof can be done similarly as for Theorem 13 (Green) in [6].
Let z be fixed in the domain $R^*\backslash\overline{K_\epsilon}(z)$, where $K_\epsilon(z)$ is the open disc with small enough radius ϵ and the center z

$$K_\epsilon(z) = \{\zeta \in \mathbb{C} : |\zeta - z| < \epsilon\}.$$

Let this domain be denoted as $R^*_\epsilon = R^*\backslash\overline{K_\epsilon}(z)$ and consider the area integral

$$\frac{1}{\pi}\int\limits_{R^*_\epsilon} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta = \frac{1}{2\pi}\int\limits_{R^*_\epsilon}\{\partial_{\overline{\zeta}}[w_\zeta N_1(z,\zeta)] + \partial_\zeta[w_{\overline{\zeta}}N_1(z,\zeta)] - w_\zeta N_{1\overline{\zeta}}(z,\zeta) - w_{\overline{\zeta}}N_{1\zeta}(z,\zeta)\}d\xi d\eta \tag{3.19}$$

Applying the Gauss theorem, one gets

$$\begin{aligned}
&\frac{1}{\pi}\int\limits_{R^*_\epsilon} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta = \frac{1}{4\pi i}\int\limits_{\partial R^*_\epsilon}\{w_\zeta N_1(z,\zeta)d\zeta - w_{\overline{\zeta}}N_1(z,\zeta)d\overline{\zeta}\}\\
&-\frac{1}{2\pi}\int\limits_{R^*_\epsilon}\{w_\zeta N_{1\overline{\zeta}}(z,\zeta) + w_{\overline{\zeta}}N_{1\zeta}(z,\zeta)\}d\xi d\eta = \frac{1}{4\pi i}\int\limits_{\partial R^*_\epsilon} N_1(z,\zeta)[w_\zeta d\zeta - w_{\overline{\zeta}}d\overline{\zeta}]\\
&-\frac{1}{2\pi}\int\limits_{R^*_\epsilon}\{\partial_\zeta[w(\zeta)N_{1\overline{\zeta}}(z,\zeta)] - 2w(\zeta)N_{1\zeta\overline{\zeta}}(z,\zeta) + \partial_{\overline{\zeta}}[w(\zeta)N_{1\zeta}(z,\zeta)]\}d\xi d\eta =\\
&\frac{1}{4\pi i}\int\limits_{\partial R^*} N_1(z,\zeta)[w_\zeta d\zeta - w_{\overline{\zeta}}d\overline{\zeta}] - \frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} N_1(z,\zeta)[w_\zeta d\zeta - w_{\overline{\zeta}}d\overline{\zeta}]\\
&+\frac{1}{4\pi i}\int\limits_{\partial R^*_\epsilon} w(\zeta)N_{1\overline{\zeta}}(z,\zeta)d\overline{\zeta} - \frac{1}{4\pi i}\int\limits_{\partial R^*_\epsilon} w(\zeta)N_{1\zeta}(z,\zeta)d\zeta =\\
&\frac{1}{4\pi}\int\limits_{\partial R^*} N_1(z,\zeta)\partial_{\nu_\zeta}w(\zeta)ds_\zeta - \frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} N_1(z,\zeta)[w_\zeta d\zeta - w_{\overline{\zeta}}d\overline{\zeta}]\\
&-\frac{1}{4\pi i}\int\limits_{\partial R^*} w(\zeta)[N_{1\zeta}(z,\zeta)d\zeta - N_{1\overline{\zeta}}(z,\zeta)d\overline{\zeta}] + \frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} w(\zeta)[N_{1\zeta}(z,\zeta) - N_{1\overline{\zeta}}(z,\zeta)d\overline{\zeta}] =\\
&\frac{1}{4\pi}\int\limits_{\partial R^*}[N_1(z,\zeta)\partial_{\nu_\zeta}w(\zeta) - w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)]ds_\zeta\\
&-\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon}\left\{N_1(z,\zeta)[(\zeta-z)w_\zeta + (\overline{\zeta-z})w_{\overline{\zeta}}] - w(\zeta)[(\zeta-z)N_{1\zeta}(z,\zeta) + (\overline{\zeta-z})N_{1\overline{\zeta}}(z,\zeta)]\right\}\frac{d\zeta}{\zeta-z}
\end{aligned}$$

since for the circumference $|\zeta - z| = \rho = \epsilon$

$$\frac{\partial}{\partial n_\zeta} = \frac{\partial}{\partial \rho}.$$

Introducing the polar coordinates $\zeta = z + \rho e^{i\varphi}$ for the circle $|\zeta - z| < \epsilon$, then

$$\begin{aligned}
&\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} N_1(z,\zeta)\{(\zeta-z)w_\zeta(\zeta) + (\overline{\zeta-z})w_{\overline{\zeta}}(\zeta)\}\frac{d\zeta}{\zeta-z} =\\
&\frac{1}{4\pi i}\int\limits_0^{2\pi} N_1(z, z+\rho e^{i\varphi})\{\rho e^{i\varphi}w_\zeta(z+\rho e^{i\varphi}) + \rho e^{-i\varphi}w_{\overline{\zeta}}(z+\rho e^{i\varphi})\}\frac{d\rho e^{i\varphi}}{\rho e^{i\varphi}} =\\
&\frac{1}{4\pi}\int\limits_0^{2\pi} N_1(z, z+\rho e^{i\varphi})\rho\{e^{i\varphi}w_\zeta(z+\rho e^{i\varphi}) + e^{-i\varphi}w_{\overline{\zeta}}(z+\rho e^{i\varphi})\}d\varphi,
\end{aligned}$$

letting $\rho = \epsilon$ tend to 0, the integral becomes 0. Consider next

$$\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} w(\zeta)\{(\zeta-z)N_{1\zeta}(z,\zeta)+(\overline{\zeta-z})N_{1\overline{\zeta}}(z,\zeta)\}\frac{d\zeta}{\zeta-z}=$$
$$\frac{1}{4\pi i}\int\limits_{0}^{2\pi} w(z+\rho e^{i\varphi})\{\rho e^{i\varphi}N_{1\zeta}(z,z+\rho e^{i\varphi})+\rho e^{-i\varphi}N_{1\overline{\zeta}}(z,z+\rho e^{i\varphi})\}d\varphi=$$
$$\frac{1}{4\pi}\int\limits_{0}^{2\pi} w(z+\rho e^{i\varphi})\rho\{e^{i\varphi}\partial_\zeta+e^{-i\varphi}\partial_{\overline{\zeta}}\}N_1(z,z+\rho e^{i\varphi})d\varphi.$$

By the property of the Neumann function, the function $h_1(z,\zeta)=\log|\zeta-z|^2+N_1(z,\zeta)$ is harmonic in the neighborhood of ζ and can be substituted into the integral as

$$\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} w(\zeta)\{(\zeta-z)N_{1\zeta}(z,\zeta)+(\overline{\zeta-z})N_{1\overline{\zeta}}(z,\zeta)\}\frac{d\zeta}{\zeta-z}=$$
$$\frac{1}{4\pi}\int\limits_{0}^{2\pi} w(z+\rho e^{i\varphi})\rho\{e^{i\varphi}\partial_\nu+e^{-i\varphi}\partial_{\overline{\nu}}\}[-\log|\zeta-z|^2+h_1(z,\zeta)]d\varphi,$$

where $r=\rho e^{i\varphi}$ and since

$$\rho\{e^{i\varphi}\partial_\nu+e^{-i\varphi}\partial_{\overline{\nu}}\}\log|\rho e^{i\varphi}|^2=\rho\left(\frac{e^{i\varphi}}{\rho e^{i\varphi}}+\frac{e^{-i\varphi}}{\rho e^{-i\varphi}}\right)=2,$$

then finally

$$\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} w(\zeta)\{(\zeta-z)N_{1\zeta}(z,\zeta)+(\overline{\zeta-z})N_{1\overline{\zeta}}(z,\zeta)\}\frac{d\zeta}{\zeta-z}=$$
$$\frac{1}{4\pi}\int\limits_{0}^{2\pi} w(z+\rho e^{i\varphi})\rho\{e^{i\varphi}\partial_\zeta+e^{-i\varphi}\partial_{\overline{\zeta}}\}h_1(z,z+\rho e^{i\varphi})d\varphi-\frac{1}{2\pi}\int\limits_{0}^{2\pi} w(z+\rho e^{i\varphi})d\varphi.$$

Letting again $\rho=\epsilon$ tend to 0, one gets

$$\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\epsilon} w(\zeta)\{(\zeta-z)N_{1\zeta}(z,\zeta)+(\overline{\zeta-z})N_{1\overline{\zeta}}(z,\zeta)\}\frac{d\zeta}{\zeta-z}=-\frac{1}{2\pi}\int\limits_{0}^{2\pi} w(z)d\varphi=-w(z).$$

Thus, the following equality yields (3.18)

$$\frac{1}{\pi}\int\limits_{R^*_\epsilon} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta=\frac{1}{4\pi}\int\limits_{\partial R^*}[N_1(z,\zeta)\partial_{\nu_\zeta}w(\zeta)-w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)]ds_\zeta-w(z).$$

□

Thus, by formula (3.18), the function $w(z)$ can be represented as

$$\begin{aligned} w(z)=&\frac{2}{\pi}\int\limits_{0}^{\frac{\pi}{2}} w(re^{i\varphi})d\varphi+\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<Im\zeta,\\ 0<Re\,\zeta}}\partial_{\nu_\zeta}w(\zeta)\Bigg[\log\frac{|z\zeta|^4}{r^4}-\log|(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)|^2\\ &-\sum_{n=1}^{\infty}\log|(1-r^{4n}z^2\zeta^2)(1-r^{4n}z^2\overline{\zeta}^2)(1-\frac{r^{4n}\overline{\zeta}^2}{z^2})(1-\frac{r^{4n}\zeta^2}{z^2})|^2\Bigg]\frac{d\zeta}{\zeta}\\ &-\frac{r}{2\pi i}\int\limits_{\substack{|\zeta|=r,\\ 0<Im\zeta,\\ 0<Re\,\zeta}}\partial_{\nu_\zeta}w(\zeta)\Bigg[\log\frac{|z\zeta|^4}{r^4}-\log|(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)|^2\\ &-\sum_{n=1}^{\infty}|(1-r^{4n}z^2\zeta^2)(1-r^{4n}z^2\overline{\zeta}^2)(1-\frac{r^{4n}}{z^2\zeta^2})(1-\frac{r^{4n}}{z^2\overline{\zeta}^2})|^2\Bigg]\frac{d\zeta}{\zeta} \end{aligned}\tag{3.20}$$

$$+\frac{1}{2\pi}\int_r^1 \partial_{\nu_t} w(t)\Bigg[\log\frac{|z|^4}{r^4} - \log|(1-\frac{z^2}{t^2})(1-z^2t^2)|^2$$

$$-\sum_{n=1}^{\infty}\log|(1-\frac{r^{4n}z^2}{t^2})(1-\frac{r^{4n}t^2}{z^2})(1-\frac{r^{4n}}{t^2z^2})(1-r^{4n}z^2t^2)|^2\Bigg]dt$$

$$-\frac{i}{2\pi}\int_r^1 \partial_{\nu_{it}} w(it)\Bigg[\log\frac{|z|^4}{r^4} - \log|(1+\frac{z^2}{t^2})(1+z^2t^2)|^2$$

$$-\sum_{n=1}^{\infty}\log|(1+\frac{r^{4n}z^2}{t^2})(1+\frac{r^{4n}t^2}{z^2})(1+\frac{r^{4n}}{t^2z^2})(1+r^{4n}z^2t^2)|^2\Bigg]dt$$

$$-\frac{1}{\pi}\int_{R^*} w_{\zeta\bar{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta.$$

The Neumann representation formula provides the solution of the Neumann problem, but first the outward normal derivatives at the corner points are to be interpreted. Let them be introduced as

$$\begin{aligned}
\partial^+_{\nu_z} w(r) &= \lim_{\substack{t\to r,\\ t\in(r,1)}} \partial_{\nu_z} w(t), & \partial^-_{\nu_z} w(r) &= \lim_{\substack{t\to r,\\ t\in\{|z|=r, 0<\operatorname{Im} z, 0<\operatorname{Re} z\}}} \partial_{\nu_z} w(t),\\
\partial^+_{\nu_z} w(1) &= \lim_{\substack{t\to 1,\\ t\in\{|z|=1, 0<\operatorname{Im} z, 0<\operatorname{Re} z\}}} \partial_{\nu_z} w(t), & \partial^-_{\nu_z} w(1) &= \lim_{\substack{t\to 1,\\ t\in(r,1)}} \partial_{\nu_z} w(t),\\
\partial^+_{\nu_z} w(i) &= \lim_{\substack{t\to 1,\\ t\in(r,1)}} \partial_{\nu_z} w(it), & \partial^-_{\nu_z} w(i) &= \lim_{\substack{t\to 1,\\ t\in\{|z|=1, 0<\operatorname{Im} z, 0<\operatorname{Re} z\}}} \partial_{\nu_z} w(it),\\
\partial^+_{\nu_z} w(ir) &= \lim_{\substack{t\to r,\\ t\in\{|z|=r, 0<\operatorname{Im} z, 0<\operatorname{Re} z\}}} \partial_{\nu_z} w(it), & \partial^-_{\nu_z} w(ir) &= \lim_{\substack{t\to r,\\ t\in(r,1)}} \partial_{\nu_z} w(it).
\end{aligned} \tag{3.21}$$

Definition 3.2.2. *If the partial outward derivatives, defined in (3.21), exist, then the normal derivatives at the corner points of R^* can be presented as*

$$\partial_{\nu_z} w(t) = \lambda\partial^+_{\nu_z} w(t) + (1-\lambda)\partial^-_{\nu_z} w(t),$$

where $t\in\{r,1,i,ir\}$, $\lambda=\frac{\alpha}{2\pi}$, $0\le\alpha\le 2\pi$ and α is the angle between the respective two normal vectors.

3.2.2 Harmonic Neumann Problem

Theorem 3.2.2. *The Neumann problem*

$$\begin{gathered}
w_{z\bar{z}} = f \text{ in } R^*,\ \partial_\nu w = \gamma \text{ on } \partial R^*,\ \frac{2}{\pi}\int_0^{\frac{\pi}{2}} w(re^{i\varphi})d\varphi = c,\\
f\in L_2(R^*;\mathbb{C})\cap C(R^*;\mathbb{C}),\ \gamma\in C(\partial R^*;\mathbb{C}),\ c\in\mathbb{C}
\end{gathered} \tag{3.22}$$

is uniquely solvable if and only if

$$\frac{1}{4\pi}\int_{\partial R^*}\gamma(\zeta)ds_\zeta = \frac{1}{\pi}\int_{R^*} f(\zeta)d\xi d\eta. \tag{3.23}$$

The solution is presented by

$$
\begin{aligned}
w(z) &= \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}} \gamma(\zeta)\Bigg[\log\frac{|z\zeta|^4}{r^4} - \log|(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)|^2 \\
&- \sum_{n=1}^{\infty}\log|(1-r^{4n}z^2\zeta^2)(1-r^{4n}z^2\overline{\zeta}^2)(1-\frac{r^{4n}\overline{\zeta}^2}{z^2})(1-\frac{r^{4n}\zeta^2}{z^2})|^2\Bigg]\frac{d\zeta}{\zeta} \\
&- \frac{1}{2\pi r i}\int\limits_{\substack{|\zeta|=r,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}} \gamma(\zeta)\Bigg[\log\frac{|z\zeta|^4}{r^4} - \log|(1-z^2\zeta^2)(1-z^2\overline{\zeta}^2)|^2 \\
&- \sum_{n=1}^{\infty}|(1-r^{4n}z^2\zeta^2)(1-r^{4n}z^2\overline{\zeta}^2)(1-\frac{r^{4n}}{z^2\zeta^2})(1-\frac{r^{4n}}{z^2\overline{\zeta}^2})|^2\Bigg]\frac{d\zeta}{\zeta} \qquad (3.24)\\
&+ \frac{1}{2\pi}\int\limits_r^1 \gamma(t)\Bigg[\log\frac{|z|^4}{r^4} - \log|(1-\frac{z^2}{t^2})(1-z^2t^2)|^2 \\
&- \sum_{n=1}^{\infty}\log|(1-\frac{r^{4n}z^2}{t^2})(1-\frac{r^{4n}t^2}{z^2})(1-\frac{r^{4n}}{t^2z^2})(1-r^{4n}z^2t^2)|^2\Bigg]dt \\
&- \frac{i}{2\pi}\int\limits_r^1 \gamma(it)\Bigg[\log\frac{|z|^4}{r^4} - \log|(1+\frac{z^2}{t^2})(1+z^2t^2)|^2 \\
&- \sum_{n=1}^{\infty}\log|(1+\frac{r^{4n}z^2}{t^2})(1+\frac{r^{4n}t^2}{z^2})(1+\frac{r^{4n}}{t^2z^2})(1+r^{4n}z^2t^2)|^2\Bigg]dt + c \\
&- \frac{1}{\pi}\int\limits_{R^*} f(\zeta)N_1(z,\zeta)d\xi d\eta.
\end{aligned}
$$

Proof. At first the boundary behavior is to be considered.

For $|z_0| = 1$, $\mathrm{Re}\, z_0 > 0$, $\mathrm{Im}\, z_0 > 0$

$$
\begin{aligned}
\partial_\nu w(z_0) &= \lim_{z\to z_0}(z\partial_z + \overline{z}\partial_{\overline{z}})w(z) = \\
&\lim_{z\to z_0}\Bigg\{\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\zeta,\\ 0<\mathrm{Re}\,\zeta}} \gamma(\zeta)\Bigg[\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)+\left(\frac{\zeta}{\zeta+z}+\frac{\overline{\zeta}}{\overline{\zeta}+\overline{z}}-1\right) \\
&+\left(\frac{\overline{\zeta}}{\overline{\zeta}-z}+\frac{\zeta}{\zeta-\overline{z}}-1\right)+\left(\frac{\overline{\zeta}}{\overline{\zeta}+z}+\frac{\zeta}{\zeta+\overline{z}}-1\right)\Bigg]\frac{d\zeta}{\zeta}\Bigg\} = \\
&\lim_{z\to z_0}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Gamma_1(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\right)\frac{d\zeta}{\zeta} = \gamma(z_0),
\end{aligned}
$$

where $\Gamma_1(\zeta)$ is defined in (2.26) for $|\zeta| = 1$.

Also on this part of the boundary

$$
\begin{aligned}
\partial^+_{\nu_z}w(1) &= \lim_{\substack{\zeta_0\to 1,\\ \zeta_0\in\{|\zeta|=1,0<\mathrm{Im}\zeta,0<\mathrm{Re}\,\zeta\}}}\partial_{\nu_z}w(\zeta_0) = \gamma(1), \\
\partial^-_{\nu_z}w(i) &= \lim_{\substack{\zeta_0\to i,\\ \zeta_0\in\{|\zeta|=1,0<\mathrm{Im}\zeta,0<\mathrm{Re}\,\zeta\}}}\partial_{\nu_z}w(\zeta_0) = \gamma(i).
\end{aligned} \qquad (3.25)
$$

For $|z_0| = r$, $Re\, z_0 > 0$, $\operatorname{Im} z_0 > 0$

$$\begin{aligned}
\partial_\nu w(z_0) &= -\lim_{z\to z_0} \frac{1}{r}(z\partial_z + \overline{z}\partial_{\overline{z}})w(z) = -\lim_{z\to z_0}\Bigg\{ \frac{2}{\pi i r} \int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)\frac{d\zeta}{\zeta} \\
&+\frac{1}{2\pi i r} \int\limits_{\substack{|\zeta|=r,\\ 0<\operatorname{Im}\zeta,\\ 0<\operatorname{Re}\zeta}} \gamma(\zeta)\Bigg[\left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}} - 1\right) + \left(\frac{\zeta}{\zeta + z} + \frac{\overline{\zeta}}{\overline{\zeta}+\overline{z}} - 1\right) \\
&+\left(\frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta-\overline{z}} - 1\right) + \left(\frac{\overline{\zeta}}{\overline{\zeta} + z} + \frac{\zeta}{\zeta+\overline{z}} - 1\right) + 4\Bigg]\frac{d\zeta}{\zeta} \\
&+\frac{2}{\pi r}\int_r^1 \gamma(t)dt - \frac{2i}{\pi}\int_r^1 \gamma(it)dt - \frac{8}{\pi r}\int\limits_{R^*} f(\zeta)d\xi d\eta\Bigg\} = \\
&\lim_{z\to z_0}\Bigg\{\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Gamma_1(\zeta)\left(\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}} - 1\right)\frac{d\zeta}{\zeta} - \frac{2}{\pi r}\int\limits_{\partial R^*}\gamma(\zeta)ds_\zeta + \frac{8}{\pi r}\int\limits_{R^*} f(\zeta)d\xi d\eta\Bigg\} = \\
&\gamma(z_0) - \frac{2}{\pi r}\int\limits_{\partial R^*}\gamma(\zeta)ds_\zeta + \frac{8}{\pi r}\int\limits_{R^*} f(\zeta)d\xi d\eta,
\end{aligned}$$

where $\Gamma_1(\zeta)$ is defined as in (2.26) on $|\zeta| = r$ and

$$\begin{aligned}
\partial^-_{\nu_z} w(r) &= \lim_{\substack{\zeta_0\to r,\\ \zeta_0\in\{|\zeta|=r, 0<\operatorname{Im}\zeta, 0<\operatorname{Re}\zeta\}}} \partial_{\nu_z} w(\zeta_0) = \gamma(r), \\
\partial^+_{\nu_z} w(ir) &= \lim_{\substack{\zeta_0\to ir,\\ \zeta_0\in\{|\zeta|=r, 0<\operatorname{Im}\zeta, 0<\operatorname{Re}\zeta\}}} \partial_{\nu_z} w(\zeta_0) = \gamma(ir).
\end{aligned} \tag{3.26}$$

For $r < t_0 \le 1$, $z_0 = t_0$

$$\begin{aligned}
\partial_\nu w(z_0) &= -i\lim_{z\to t_0}(\partial_z - \partial_{\overline{z}})w(z) = \\
&\lim_{z\to t_0}\frac{i}{\pi}\int_r^1\gamma(t)\left[\frac{z-\overline{z}}{|z|^2} - \frac{(z-\overline{z})(t^2+|z|^2)}{|t^2-z^2|^2} - \frac{(z-\overline{z})t^2(1+t^2|z|^2)}{|1-t^2z^2|^2}\right]dt = \\
&\lim_{z\to t_0}\Bigg\{-\frac{i}{2\pi}\int_r^1\gamma(t)\left[\frac{z-\overline{z}}{|t-z|^2} + \frac{z-\overline{z}}{|t+z|^2}\right]dt - \frac{i}{2\pi}\int_r^1\gamma(t)\left[\frac{(z-\overline{z})t^2}{|1-tz|^2} + \frac{(z-\overline{z})t^2}{|1+tz|^2}\right]dt\Bigg\} = \\
&\lim_{z\to t_0}\Bigg\{\frac{1}{2\pi i}\int_r^1\gamma(t)\frac{z-\overline{z}}{|t-z|^2}dt + \frac{1}{2\pi i}\int_{-1}^{-r}\gamma(-t)\frac{z-\overline{z}}{|t-z|^2}dt \\
&+\frac{1}{2\pi i}\int_1^{\frac{1}{r}}\frac{1}{t^2}\gamma(\frac{1}{t})\frac{z-\overline{z}}{|t-z|^2}dt + \frac{1}{2\pi i}\int_{-\frac{1}{r}}^{-1}\frac{1}{t^2}\gamma(\frac{1}{t})\frac{z-\overline{z}}{|t-z|^2}dt\Bigg\} = \\
&\lim_{z\to t_0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\Gamma_2^*(t)\frac{z-\overline{z}}{|t-z|^2}dt = \gamma(t_0),
\end{aligned}$$

where $\Gamma_2^*(t)$ is defined as

$$\Gamma_2^*(t) = \begin{cases} \gamma(t), & r \le t \le 1 \\ \frac{1}{t^2}\gamma(\frac{1}{t}), & 1 \le t \le \frac{1}{r} \\ 0, & -r < t < r,\ |t| > \frac{1}{r} \\ \gamma(-t), & -1 \le t \le -r \\ \frac{1}{t^2}\gamma(-\frac{1}{t}), & -\frac{1}{r} \le t \le -1, \end{cases} \tag{3.27}$$

Similarly, on this part of the boundary

$$\begin{aligned} \partial^+_{\nu_z} w(r) &= \lim_{\substack{\zeta_0 \to r, \\ \zeta_0 \in (r,1)}} \partial_{\nu_z} w(\zeta_0) = \gamma(r), \\ \partial^-_{\nu_z} w(1) &= \lim_{\substack{\zeta_0 \to 1, \\ \zeta_0 \in (r,1)}} \partial_{\nu_z} w(\zeta_0) = \gamma(1). \end{aligned} \tag{3.28}$$

For $r < t_0 < 1$, $z_0 = it_0$

$$\begin{aligned} \partial_\nu w(z_0) &= -\lim_{z \to it_0} (\partial_z + \partial_{\overline{z}}) w(z) = \\ & -\lim_{z \to it_0} \left\{ \frac{i}{\pi} \int_r^1 \gamma(it) \left[\frac{z+\overline{z}}{|z|^2} - \frac{(z+\overline{z})(t^2+|z|^2)}{|t^2+z^2|^2} + \frac{(z+\overline{z})(t^2+t^4|z|^2)}{|1+z^2t^2|^2} \right] dt \right\} = \\ & \lim_{z \to it_0} \left\{ \frac{1}{2\pi} \int_r^1 \gamma(it) \left[\frac{z+\overline{z}}{|it-z|^2} + \frac{z+\overline{z}}{|it+z|^2} \right] dt + \frac{1}{2\pi} \int_r^1 \gamma(it) \left[\frac{t^2(z+\overline{z})}{|1+itz|^2} + \frac{t^2(z+\overline{z})}{|1-itz|^2} \right] dt \right\} = \\ & \lim_{z \to it_0} \left\{ \frac{1}{2\pi} \int_r^1 \gamma(it) \frac{z+\overline{z}}{|it-z|^2} dt + \frac{1}{2\pi} \int_{-1}^{-r} \gamma(-it) \frac{z+\overline{z}}{|it-z|^2} dt \right. \\ & \left. + \frac{1}{2\pi} \int_1^{\frac{1}{r}} \frac{1}{t^2} \gamma(-\frac{1}{it}) \frac{z+\overline{z}}{|it-z|^2} dt + \frac{1}{2\pi} \int_{-1}^{-\frac{1}{r}} \frac{1}{t^2} \gamma(\frac{1}{it}) \frac{z+\overline{z}}{|it-z|^2} dt \right\} = \\ & \lim_{z \to it_0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Gamma_2^*(it) \frac{z+\overline{z}}{|it-z|^2} dt = \gamma(it_0), \end{aligned}$$

where $\Gamma_2^*(it)$ is defined as

$$\Gamma_2^*(it) = \begin{cases} \gamma(it), & r \le t \le 1 \\ \frac{1}{t^2}\gamma(-\frac{1}{it}), & 1 \le t \le \frac{1}{r} \\ 0, & -r < t < r,\ |t| > \frac{1}{r} \\ \gamma(-it), & -1 \le t \le -r \\ \frac{1}{t^2}\gamma(\frac{1}{it}), & -\frac{1}{r} \le t \le -1, \end{cases} \tag{3.29}$$

and

$$\begin{aligned} \partial^+_{\nu_z} w(i) &= \lim_{\substack{\zeta_0 \to i, \\ \zeta_0 \in (i,ir)}} \partial_{\nu_z} w(\zeta_0) = \gamma(i), \\ \partial^-_{\nu_z} w(ir) &= \lim_{\substack{\zeta_0 \to ir, \\ \zeta_0 \in (i,ir)}} \partial_{\nu_z} w(\zeta_0) = \gamma(ir). \end{aligned} \tag{3.30}$$

In consideration of (3.25)-(3.30) and Definition 3.2.2 it follows that

$$\partial_{\nu_z} w(t) = \gamma(t),\ t \in \{r, 1, i, ir\}.$$

Similarly, as it was done in the proof of the Theorem 3.1.3, the Pompeiu-type operator can be derived from $\partial_z N_1(z,\zeta)$ given in (3.15) and it provides the solution to the differential equation $w_{z\overline{z}} = f$ in a weak sense. □

Part II

Boundary Value Problems for a Half Hexagon

Chapter 4

Schwarz Problem for the Inhomogeneous Cauchy-Riemann Equation

In this Chapter, in order to get the Schwarz-Pompeiu representation formula for a half hexagon the method of reflection is used similarly as for the quarter ring in the preceding Part I and the related Schwarz problem for the inhomogeneous Cauchy-Riemann equation is solved explicitly.

4.1 Description of the domain

The half hexagon P^+ (see Fig.2) consists of the 4 corner points: 2, $1+i\sqrt{3}$, $-1+i\sqrt{3}$, -2.

A point $z \in P^+$, chosen to be a simple pole of a meromorphic function to be constructed, is reflected through the real axis. The entire set P^+ is reflected also so that the hexagon P is reached. The points z, $\overline{z}$ from P are reflected through all the sides of the hexagon, starting with the right upper side and continuing in positive direction. The successively reflected points, which become zeros of the mentioned meromorphic function in the entire complex plane $\mathbb{C}$, are

$$\begin{aligned}
&-\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3},\\
&\overline{z}+2i\sqrt{3},\\
&-\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3},\\
&-\frac{1}{2}(1+i\sqrt{3})\overline{z}-3-i\sqrt{3},\\
&\overline{z}-2i\sqrt{3},\\
&-\frac{1}{2}(1-i\sqrt{3})\overline{z}+3-i\sqrt{3}
\end{aligned}$$

Reflecting, in turn, these points through the sides of the new hexagons $P_1, P_2, P_3, P_4, P_5, P_6$ (see Fig.3), except for reflecting to the original hexagon P with the single pole z and a simple zero $\overline{z}$, shows that every of those hexagons includes 6 points: 3 poles are being reflections of zeros and 3 zeros are being reflections of poles. Continuation of these operations reveals that all the points have the same coefficients of rotation:

$$1, -\frac{1}{2}(1+i\sqrt{3}), -\frac{1}{2}(1-i\sqrt{3})$$

and displacement $3m+i\sqrt{3}n$.

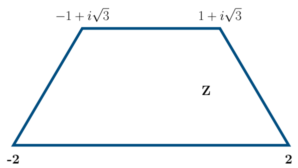

Fig.2: Half hexagon

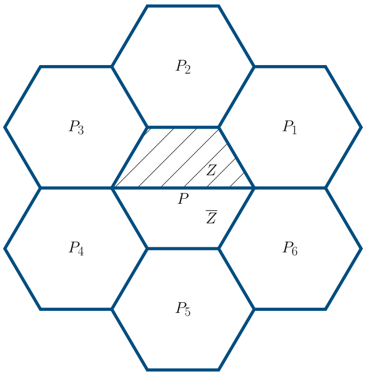

Fig.3: Hexagons

It should be noted that reflection includes rotation and shifting and the points from P_1 appear after two rotations on the left-hand side in P_3, the points from P_6- in P_4. The points in the hexagons P_2, P_5, which are translations of P in y-axis direction, are obtained by rotation. In each of the reflected hexagons 6 reflection points originating for z and $\overline{z}$ in P appear. They are:

$$\begin{aligned}
\text{in } P_1: \quad & -\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})\overline{z}+3+i\sqrt{3},\ \overline{z}+3+i\sqrt{3} \\
& -\frac{1}{2}(1+i\sqrt{3})z+3+i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})z+3+i\sqrt{3},\ z+3+i\sqrt{3}. \\
\text{in } P_3: \quad & -\frac{1}{2}(1+i\sqrt{3})\overline{z}-3+i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3},\ \overline{z}-3+i\sqrt{3} \\
& -\frac{1}{2}(1+i\sqrt{3})z-3+i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3},\ z-3+i\sqrt{3}.
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\text{in } P_6: \quad & -\frac{1}{2}(1+i\sqrt{3})\overline{z}+3-i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})\overline{z}+3-i\sqrt{3},\ \overline{z}+3-i\sqrt{3} \\
& -\frac{1}{2}(1+i\sqrt{3})z+3-i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})z+3-i\sqrt{3},\ z+3-i\sqrt{3}. \\
\text{in } P_4: \quad & -\frac{1}{2}(1+i\sqrt{3})\overline{z}-3-i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})\overline{z}-3-i\sqrt{3},\ \overline{z}-3-i\sqrt{3} \\
& -\frac{1}{2}(1+i\sqrt{3})z-3-i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})z-3-i\sqrt{3},\ z-3-i\sqrt{3}.
\end{aligned}$$

It can be seen that the points from these hexagons differ only by displacement $6m$. Consider next the points

$$\begin{aligned}
\text{in } P_2: \quad & \overline{z}+2i\sqrt{3},\ -\frac{1}{2}(1+i\sqrt{3})\overline{z}+2i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})\overline{z}+2i\sqrt{3}, \\
& -\frac{1}{2}(1+i\sqrt{3})z+2i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})z+2i\sqrt{3},\ z+2i\sqrt{3}. \\
\text{in } P_5: \quad & \overline{z}-2i\sqrt{3},\ -\frac{1}{2}(1+i\sqrt{3})\overline{z}-2i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})\overline{z}-2i\sqrt{3}, \\
& -\frac{1}{2}(1+i\sqrt{3})z-2i\sqrt{3},\ -\frac{1}{2}(1-i\sqrt{3})z-2i\sqrt{3},\ z-2i\sqrt{3},
\end{aligned}$$

which differ by displacement $2i\sqrt{3}n$ in the direction of the imaginary axis. Thus the main period is $\mu_{mn} = 6m + 2i\sqrt{3}n$.

Obviously, the repeated reflections of the point $z \in P^+$ are representable in different ways, using either the points

$$z_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3},\ \ z_2 = -\frac{1}{2}(1+i\sqrt{3})z+3+i\sqrt{3} \tag{4.2}$$

or

$$\check{z}_1 = -\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3},\ \ \check{z}_2 = -\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3}, \tag{4.3}$$

which are connected by the relations $\overline{\check{z}_2} = z_1 - 6 - 2i\sqrt{3}$, $\overline{\check{z}_1} = z_2 - 6 - 2i\sqrt{3}$.
In general, all reflection points are either given by

$$\begin{gathered} z + \omega_{mn},\ \overline{z}_1 + \omega_{mn},\ z_2 + \omega_{mn}, \\ \overline{z} + \omega_{mn},\ z_1 + \omega_{mn},\ \overline{z}_2 + \omega_{mn} \end{gathered} \tag{4.4}$$

or by

$$\begin{gathered} z + \omega_{mn},\ \overline{\check{z}_1} + \omega_{mn},\ \check{z}_2 + \omega_{mn}, \\ \overline{z} + \omega_{mn},\ \check{z}_1 + \omega_{mn},\ \overline{\check{z}_2} + \omega_{mn}, \end{gathered} \tag{4.5}$$

where $\omega_{mn} = 3m + i\sqrt{3}n$ with m and n both either even or both odd, i.e. $m + n \in 2\mathbb{Z}$.
With these reflections of z an entire meromorphic function is constructed later in Section 5.1, having a pole at the point z, zeros as direct reflection of poles and poles at direct reflection of zeros. By this principle the zeros are

$$\overline{z} + \omega_{mn},\ z_1 + \omega_{mn},\ \overline{z}_2 + \omega_{mn}, \tag{4.6}$$

or

$$\overline{z} + \omega_{mn},\ \check{z}_1 + \omega_{mn},\ \overline{\check{z}_2} + \omega_{mn}, \tag{4.7}$$

while the poles are

$$z + \omega_{mn},\ \overline{\check{z}_1} + \omega_{mn},\ z_2 + \omega_{mn} \tag{4.8}$$

or

$$z + \omega_{mn},\ \overline{\check{z}_1} + \omega_{mn},\ \check{z}_2 + \omega_{mn}. \tag{4.9}$$

These points will also be needed for constructing the Schwarz kernel for P^+.

The half hexagon is located in the complement of four half planes. Denote them:
H_1^- is the right-hand half plane with the boundary line passing through the points 2 and $1 + i\sqrt{3}$,
H_2^- is the upper half plane with the border line through the points $1 + i\sqrt{3}$ and $-1 + i\sqrt{3}$,
H_3^- is the left-hand half plane with the line passing through the corner points $-1 + i\sqrt{3}$ and -2
and H_4^- is the half plane which is below the real axis.
Let then H_1^+, H_2^+, H_3^+, H_4^+ be the complementary half planes of those listed above. We consider now the Green functions of these half planes in order to get their Poisson kernels which will be needed in the sequel.

For H_1^+ with the boundary where $\zeta = -\frac{1}{2}(1 + i\sqrt{3})\overline{\zeta} + 3 + i\sqrt{3}$ or $\zeta - 2 = -\frac{1}{2}(1 + i\sqrt{3})(\overline{\zeta} - 2)$

$$G_1(z, \zeta) = \log\left|\frac{\frac{1}{2}(1 + i\sqrt{3})(\overline{\zeta} - 2) + z - 2}{\zeta - z}\right|^2,\ z, \zeta \in H_1^+,$$

$$-\frac{1}{2}\partial_{\nu_\zeta} G_1(z, \zeta) = -\frac{(\sqrt{3} - i)}{4}\frac{z - z_1}{|\zeta - z|^2},\ z \in \partial H_1^+,\ \zeta \in H_1^+,$$

$z_1 = -\frac{1}{2}(1 + i\sqrt{3})\overline{z} + 3 + i\sqrt{3}$, $z \in H_1^+$.

For H_2^+ with the boundary described by $\zeta = \overline{\zeta} + 2i\sqrt{3}$

$$G_1(z, \zeta) = \log\left|\frac{\overline{\zeta} - z + 2i\sqrt{3}}{\zeta - z}\right|^2,\ \ z, \zeta \in H_2^+,$$

$$-\frac{1}{2}\partial_{\nu_\zeta} G_1(z, \zeta) = -\frac{1}{i}\frac{z - z_2}{|\zeta - z|^2},\ \ z \in \partial H_2^+,\ \zeta \in H_2^+,$$

$z_2 = \overline{z} + 2i\sqrt{3}$, $z \in H_2^+$.

For H_3^+ with the boundary given by $\zeta + 2 = -\frac{1}{2}(1 - i\sqrt{3})(\overline{\zeta} + 2)$

$$G_1(z,\zeta) = \log\left|\frac{\frac{1}{2}(1-i\sqrt{3})(\overline{\zeta}+2) + z + 2}{\zeta - z}\right|^2, \; z, \zeta \in H_3^+,$$
$$-\frac{1}{2}\partial_{\nu_\zeta} G_1(z,\zeta) = -\frac{\sqrt{3}+i}{4}\frac{z - \check{z}_1}{|\zeta - z|}, \; z \in \partial H_3^+, \; \zeta \in H_3^+,$$

where $\check{z}_1 = -\frac{1}{2}(1 - i\sqrt{3})\overline{z} - 3 + i\sqrt{3}$, $z \in H_3^+$.

Finally, for H_4^+ with the boundary described by $\zeta = \overline{\zeta}$

$$G_1(z,\zeta) = \log\left|\frac{\overline{\zeta} - z}{\zeta - z}\right|^2, \; z, \zeta \in H_4^+,$$
$$-\frac{1}{2}\partial_{\nu_\zeta} G_1(z,\zeta) = \frac{1}{i}\frac{z - \overline{z}}{|\zeta - z|^2}, \; z \in \partial H_4^+, \; \zeta \in H_4^+$$

for $z \in H_4^+$.

4.2 Schwarz-Poisson representation formula

The Schwarz-Poisson representation formula is derived from the Cauchy-Pompeiu representation formulas (1.11) and (1.13). By substituting the reflection points described above, different representation formulas are obtained. The Schwarz kernel is obtained in the same way as for the quarter ring R^*.

Theorem 4.2.1. *Any $w \in C^1(P^+;\mathbb{C}) \cap C(\overline{P^+};\mathbb{C})$ for the half hexagon $P^+ \subset \mathbb{C}$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \; z \in P^+, \tag{4.10}$$

and for $k = 1, 2, 3$

$$\begin{aligned} w(z) \;=\; & \frac{1}{2\pi i}\int_{\partial P^+} \operatorname{Re} w(\zeta) 2 \sum_{m+n\in 2\mathbb{Z}} [q^k_{mn}(\zeta,z) - q^k_{mn}(\zeta,0)]d\zeta \\ & -\frac{1}{2\pi i}\int_{\partial_1 P^+}\left[\operatorname{Re} w(\zeta)\frac{2(2\xi-3)}{(2\xi-3)^2+3} + \operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi-3)^2+3}\right]ds_\zeta \\ & +\frac{1}{2\pi i}\int_{\partial_2 P^+}\left[\operatorname{Re} w(\zeta)\frac{2\xi}{\xi^2+3} + \operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{\xi^2+3}\right]ds_\zeta \\ & -\frac{1}{2\pi i}\int_{\partial_3 P^+}\left[\operatorname{Re} w(\zeta)\frac{2(2\xi+3)}{(2\xi+3)^2+3} + \operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi+3)^2+3}\right]ds_\zeta \\ & +\frac{1}{2\pi i}\int_{\partial_4 P^+} \operatorname{Re} w(\zeta)\frac{2}{\xi}ds_\zeta - \frac{1}{\pi}\int_{P^+}\left\{w_{\overline{\zeta}}(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}}[q^k_{mn}(\zeta,z) - q^k_{mn}(\zeta,0)] + \frac{1}{\zeta}\right)\right. \\ & \left. -\overline{w_{\overline{\zeta}}(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}}[q^k_{mn}(\overline{\zeta},z) - q^k_{mn}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\right)\right\}d\xi d\eta, \end{aligned} \tag{4.11}$$

where $\partial_k P^+$, $k = 1, 2, 3, 4$, are the four boundary segments of P^+ and

$$
\begin{aligned}
q^1_{mn}(\zeta,z) &= \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3},\\
q^2_{mn}(\zeta,z) &= \frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3},\\
q^3_{mn}(\zeta,z) &= \frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}.
\end{aligned}
$$

Proof. Substitute first the reflection points into formula (1.13)

$$
0 = \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta-z-\omega_{mn}} - \frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta-z-\omega_{mn}}, \tag{4.12}
$$

$$
\begin{aligned}
0 = & \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1+i\sqrt{3})z-3-i\sqrt{3}-\omega_{mn}}\\
&-\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1+i\sqrt{3})z-3-i\sqrt{3}-\omega_{mn}},
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
0 = & \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3}-\omega_{mn}}\\
&-\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3}-\omega_{mn}}
\end{aligned} \tag{4.14}
$$

and

$$
0 = \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta-\overline{z}-\omega_{mn}} - \frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta-\overline{z}-\omega_{mn}}, \tag{4.15}
$$

$$
\begin{aligned}
0 = & \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1+i\sqrt{3})\overline{z}-3-i\sqrt{3}-\omega_{mn}}\\
&-\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta+\frac{1}{2}(1+i\sqrt{3})\overline{z}-3-i\sqrt{3}-\omega_{mn}},
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
0 = & \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3}-\omega_{mn}}\\
&-\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta+\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3}-\omega_{mn}}.
\end{aligned} \tag{4.17}
$$

Define

$$
\begin{aligned}
q^1_{mn}(\zeta,z) = & \frac{1}{\zeta-z-\omega_{mn}} + \frac{1}{\zeta+\frac{1}{2}(1+i\sqrt{3})z-3-i\sqrt{3}-\omega_{mn}}\\
&+\frac{1}{\zeta+\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3}-\omega_{mn}} = \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}
\end{aligned} \tag{4.18}
$$

and consider the double series

$$
\sum_{m+n\in 2\mathbb{Z}} [q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)] = \sum_{m+n\in 2\mathbb{Z}} \left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right],
$$

which is convergent since

$$\sum_{m+n\in 2\mathbb{Z}} 3\left[\frac{(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]=$$
$$\sum_{m+n\in 2\mathbb{Z}} 3\left\{\frac{8+(z-2)^3}{(\zeta-\omega_{mn}-2)^4\left[1+\frac{8-(z-2)^3}{(\zeta-\omega_{mn}-2)^3}-\frac{8(z-2)^3}{(\zeta-\omega_{mn}-2)^6}\right]}\right\}.$$

For m=n=0

$$\begin{aligned} q^1_{00}(\zeta,z)-q^1_{00}(\zeta,0) &= \frac{1}{\zeta-z}+\frac{1}{\zeta+\frac{1}{2}(1+i\sqrt{3})z-3-i\sqrt{3}}+\frac{1}{\zeta+\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3}}-\frac{1}{\zeta} \\ &\quad -\frac{1}{\zeta-3-i\sqrt{3}}-\frac{1}{\zeta-3+i\sqrt{3}}=\frac{3(\zeta-2)^2}{(\zeta-2)^3-(z-2)^3}-\frac{3(\zeta-2)^2}{(\zeta-2)^3+8}. \end{aligned}$$

Then by formula (1.11) the function $w(z)$ is also presented as

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)[q^1_{00}(\zeta,z)-q^1_{00}(\zeta,0)+\frac{1}{\zeta}]d\zeta-\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)[q^1_{00}(\zeta,z)-q^1_{00}(\zeta,0)+\frac{1}{\zeta}]d\xi d\eta \tag{4.19}$$

and

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]d\xi d\eta. \end{aligned}\tag{4.20}$$

Similarly, from the formulas (4.15)-(4.17)

$$0=\frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)[q^1_{00}(\zeta,\overline{z})-q^1_{00}(\zeta,0)+\frac{1}{\zeta}]d\zeta-\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)[q^1_{00}(\zeta,\overline{z})-q^1_{00}(\zeta,0)+\frac{1}{\zeta}]d\xi d\eta \tag{4.21}$$

and

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}[q^1_{mn}(\zeta,\overline{z})-q^1_{mn}(\zeta,0)]d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}[q^1_{mn}(\zeta,\overline{z})-q^1_{mn}(\zeta,0)]d\xi d\eta. \end{aligned}\tag{4.22}$$

Taking the complex conjugation of (4.21) and (4.22), the Cauchy formulas

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\right)d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\right)d\xi d\eta, \end{aligned}\tag{4.23}$$

$$\begin{aligned} 0 &= -\frac{1}{2\pi i}\int\limits_{\partial P^+} \overline{w(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\overline{\zeta},z)-q^1_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\right)d\overline{\zeta} \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} \overline{w_{\overline{\zeta}}(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\overline{\zeta},z)-q^1_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\right)d\xi d\eta, \end{aligned}\tag{4.24}$$

are obtained. It should be noted, that $\sum\limits_{m+n\in 2\mathbb{Z}} q_{m,-n}$ can be rewritten as $\sum\limits_{m+n\in 2\mathbb{Z}} q_{mn}$.

Subtracting next (4.24) from (4.23) gives

$$\begin{aligned} w(z) \;&=\; \frac{1}{2\pi i}\int\limits_{\partial P^+}\Bigg\{w(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\overline{w(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\overline{\zeta},z)-q^1_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\} \\ &-\frac{1}{\pi}\int\limits_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg) \\ &-\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\overline{\zeta},z)-q^1_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta. \end{aligned} \tag{4.25}$$

Let $w(z)$ be decomposed as

$$\begin{aligned} w(z) \;&=\; w_1(z)+w_2(z)+w_3(z)+w_4(z)-\frac{1}{\pi}\int\limits_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg) \\ &+\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\overline{\zeta},z)-q^1_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta, \end{aligned}$$

where $w_1(z),...,w_4(z)$ are the boundary integrals considered on the different parts of ∂P^+.

We compute first $w_1(z)$ on $\partial_1 P$, the side of P^+ between the points 2 and $1+i\sqrt{3}$, where $\zeta=-\frac{1}{2}(1+i\sqrt{3})\overline{\zeta}+3+i\sqrt{3}$ and the relations

$$\begin{gathered} \overline{\zeta}=-\frac{1}{2}(1-i\sqrt{3})\zeta+3-i\sqrt{3},\ d\overline{\zeta}=-\frac{1}{2}(1-i\sqrt{3})d\zeta, \\ (\zeta-2)^2=-\frac{1}{2}(1-i\sqrt{3})(\overline{\zeta}-2)^2,\ (\zeta-2)^3=(\overline{\zeta}-2)^3. \end{gathered}$$

are considered.

$$\begin{aligned} w_1(z) \;&=\; \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{w(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\Bigg]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\overline{w(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\overline{\zeta}-\overline{\omega_{mn}}-2)^2}{(\overline{\zeta}-\overline{\omega_{mn}}-2)^3-(z-2)3}-\frac{3(\overline{\zeta}-\overline{\omega_{mn}}-2)^2}{(\overline{\zeta}-\overline{\omega_{mn}}-2)^3+8}\Bigg]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\}, \end{aligned}$$

$$(\overline{\zeta}-\overline{w_{mn}}-2)^2=[-\frac{1}{2}(1-i\sqrt{3})\zeta+3-i\sqrt{3}-(3m-i\sqrt{3}n)-2]^2=-\frac{1}{2}(1+i\sqrt{3})(\zeta-\omega_{kl}-2)^2,$$

where if $k=-\frac{m+n}{2}$, $l=-\frac{3m-n}{2}$, $k+l\in 2\mathbb{Z}$, then $m=-\frac{k+l}{2}$, $n=-\frac{3k-l}{2}$, $m+n\in 2\mathbb{Z}$.
Then the terms with $\overline{\zeta}$ are representable by

$$\begin{aligned} &(\overline{\zeta}-\overline{\omega_{mn}}-2)^2 d\overline{\zeta}=\frac{1}{4}(1+i\sqrt{3})(\zeta-w_{kl}-2)^2(1-i\sqrt{3})d\zeta=(\zeta-\omega_{kl}-2)^2 d\zeta; \\ &\frac{d\overline{\zeta}}{\overline{\zeta}}=\frac{d\zeta}{\zeta-\frac{1}{2}(3-i\sqrt{3})(1+i\sqrt{3})}=\frac{d\zeta}{\zeta-3-i\sqrt{3}} \end{aligned}$$

and therefore

$$\begin{aligned} w_1(z) \;&=\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\mathrm{Re}\,w(\zeta)\Bigg[2\sum_{m+n\in 2\mathbb{Z}}\Bigg(\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\Bigg) \\ &+\frac{1}{\zeta}+\frac{1}{\zeta-3-i\sqrt{3}}\Bigg]+i\mathrm{Im}\,w(\zeta)\Bigg(\frac{1}{\zeta}-\frac{1}{\zeta-3-i\sqrt{3}}\Bigg)\Bigg\}d\zeta. \end{aligned}$$

Consider next the terms $(\frac{1}{\zeta} \pm \frac{1}{\zeta-3-i\sqrt{3}})d\zeta$. On $\partial_1 P^+$

$$\eta = -\sqrt{3}\xi + 2\sqrt{3},\ d\eta = -\sqrt{3}d\xi,\ d\zeta = (1-i\sqrt{3})d\xi = -\frac{1}{2}(1-i\sqrt{3})ds_\zeta,$$

then

$$\left(\frac{1}{\zeta} + \frac{1}{\zeta-3-i\sqrt{3}}\right)d\zeta = \frac{(1-i\sqrt{3})(2\xi-3)d\zeta}{[(1-i\sqrt{3})\xi+2i\sqrt{3}][(1-i\sqrt{3})\xi-3+i\sqrt{3}]} = \frac{-2(2\xi-3)ds_\zeta}{(2\xi-3)^2+3}; \tag{4.26}$$

$$(\frac{1}{\zeta} - \frac{1}{\zeta-3-i\sqrt{3}})d\zeta = \frac{\frac{1}{2}(1+i\sqrt{3})(3+i\sqrt{3})ds_\zeta}{(2\xi-3)^2+3} = \frac{2i\sqrt{3}ds_\zeta}{(2\xi-3)^2+3}. \tag{4.27}$$

Thus, the boundary integral on the first boundary part is

$$\begin{aligned} w_1(z) &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg[2\sum_{m+n\in 2\mathbb{Z}}\left(\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right)d\zeta \\ &\quad -\frac{2(2\xi-3)}{(2\xi-3)^2+3}ds_\zeta\Bigg] - \operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi-3)^2+3}ds_\zeta\Bigg\}. \end{aligned}$$

Consider next the second boundary part $\partial_2 P^+$ between the points $1+i\sqrt{3}$ and $-1+i\sqrt{3}$, where $\zeta = \overline{\zeta} + 2i\sqrt{3}$, $d\zeta = d\overline{\zeta}$.

$$\begin{aligned} w_2(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{w(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right] + \frac{1}{\zeta}\Bigg)d\zeta \\ &\quad +\overline{w(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta}-\overline{\omega_{mn}}-2)^2}{(\overline{\zeta}-\overline{\omega_{mn}}-2)^3-(z-2)^3} - \frac{3(\overline{\zeta}-\overline{\omega_{mn}}-2)^2}{(\overline{\zeta}-\overline{\omega_{mn}}-2)^3+8}\right] + \frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\}, \end{aligned}$$

or

$$\begin{aligned} w_2(z) = \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{&\operatorname{Re} w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z) - q^1_{mn}(\zeta,0)] + \frac{1}{\zeta}\Bigg)d\zeta \\ &+\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{m,n-2}(\overline{\zeta},z) - q^1_{m,n-2}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg] \\ &+i\operatorname{Im} w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z) - q^1_{mn}(\zeta,0)] + \frac{1}{\zeta}\Bigg)d\zeta \\ &-\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{m,n-2}(\overline{\zeta},z) - q^1_{m,n-2}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg]\Bigg\}, \end{aligned}$$

$$q^1_{m,n-2}(\overline{\zeta},z) = \frac{3(\overline{\zeta}-\omega_{m,n-2}-2)^2}{(\overline{\zeta}-\omega_{m,n-2}-2)^3-(z-2)^3} = \frac{3(\zeta-2i\sqrt{3}-\omega_{m,n-2}-2)^2}{(\zeta-2i\sqrt{3}-\omega_{m,n-2}-2)^3-(z-2)^3} = q^1_{mn}(\zeta,z).$$

Then

$$w_2(z) = \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z) - q^1_{mn}(\zeta,0)] + \frac{1}{\zeta} + \frac{1}{\overline{\zeta}}\Bigg)d\zeta + i\operatorname{Im} w(\zeta)\left(\frac{1}{\zeta} - \frac{1}{\overline{\zeta}}\right)d\zeta\Bigg\}.$$

Substituting the variables η, ξ, with $\eta = \sqrt{3}$ on $\partial_2 P^+$ gives $\zeta = \xi + i\sqrt{3}$, $d\zeta = d\xi = ds_\zeta$ and

$$\left(\frac{1}{\zeta} + \frac{1}{\zeta-2i\sqrt{3}}\right)d\zeta = \frac{2\xi}{\xi^2+3}ds_\zeta, \qquad \left(\frac{1}{\zeta} - \frac{1}{\zeta-2i\sqrt{3}}\right)d\zeta = -\frac{2i\sqrt{3}}{\xi^2+3}ds_\zeta. \tag{4.28}$$

Then the boundary integral gets the form

$$\begin{aligned} w_2(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg[2\sum_{m+n\in 2\mathbb{Z}}\left(\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right)d\zeta \\ &\quad +\frac{2\xi}{\xi^2+3}ds_\zeta\Bigg]+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{\xi^2+3}ds_\zeta\Bigg\}. \end{aligned}$$

On $\partial_3 P^+$, the side of the boundary between the points $-1+i\sqrt{3}$, -2, where

$$\zeta=-\frac{1}{2}(1-i\sqrt{3})\overline{\zeta}-3+i\sqrt{3},$$

$$d\overline{\zeta}=-\frac{1}{2}(1+i\sqrt{3})d\zeta$$

the boundary integral is

$$\begin{aligned} w_3(z)=\frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\Bigg\{&\operatorname{Re} w(\zeta)\Bigg[\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\right)d\zeta \\ &+\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{m,-n}(\overline{\zeta},z)-q^1_{m,-n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\right)d\overline{\zeta}\Bigg] \\ &+i\operatorname{Im} w(\zeta)\Bigg[\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\right)d\zeta \\ &-\left(\sum_{m+n\in 2\mathbb{Z}}[q^1_{m,-n}(\overline{\zeta},z)-q^1_{m,-n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\right)d\overline{\zeta}\Bigg]\Bigg\}. \end{aligned}$$

Rewriting the terms

$$(\overline{\zeta}-\omega_{mn}-2)=-\frac{1}{2}(1+i\sqrt{3})[\zeta-\omega_{kl}-2],$$

where if $k=-\frac{m+n+4}{2}$, $l=\frac{3m-n+4}{2}$, $k+l\in 2\mathbb{Z}$, then $m=-\frac{k-l+4}{2}$, $n=-\frac{3k+l+4}{2}$, $m+n\in 2\mathbb{Z}$, then the relations

$$(\overline{\zeta}-\omega_{mn}-2)^2 d\overline{\zeta}=(\zeta-\omega_{kl}-2)^2 d\zeta,\ \ q^1_{mn}(\overline{\zeta},z)d\overline{\zeta}=q^1_{kl}(\zeta,z)d\zeta$$

lead to

$$w_3(z)=\frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\Bigg\{\operatorname{Re} w(\zeta)\left(2\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]d\zeta+\left[\frac{d\zeta}{\zeta}+\frac{d\overline{\zeta}}{\overline{\zeta}}\right]\right)+i\operatorname{Im} w(\zeta)\left(\frac{d\zeta}{\zeta}-\frac{d\overline{\zeta}}{\overline{\zeta}}\right)\Bigg\}.$$

On this boundary part

$$\eta=\sqrt{3}\xi+2\sqrt{3},\ \ \zeta=(1+i\sqrt{3})\xi+2i\sqrt{3},\ \ d\zeta=(1+i\sqrt{3})d\xi=-\frac{1}{2}(1+i\sqrt{3})ds_\zeta.$$

$$\begin{aligned} \left(\frac{1}{\zeta}+\frac{1}{\zeta+3-i\sqrt{3}}\right)d\zeta &= \frac{(1+i\sqrt{3})(2\xi+3)d\zeta}{[(1+i\sqrt{3})\xi+2i\sqrt{3}][(1+i\sqrt{3})\xi+3+i\sqrt{3}]}=\frac{-2(2\xi+3)ds_\zeta}{(2\xi+3)^2+3}, \\ \left(\frac{1}{\zeta}-\frac{1}{\zeta+3-i\sqrt{3}}\right)d\zeta &= \frac{-\frac{1}{2}(1-i\sqrt{3})(3-i\sqrt{3})ds_\zeta}{(2\xi+3)^2+3}=\frac{2i\sqrt{3}ds_\zeta}{(2\xi+3)^2+3}. \end{aligned} \tag{4.29}$$

Then the third boundary integral becomes

$$\begin{aligned} w_3(z) &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg[2\sum_{m+n\in 2\mathbb{Z}}\left(\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right)d\zeta \\ &\quad -\frac{2(2\xi+3)^2}{(2\xi+3)^2+3}ds_\zeta\Bigg]-\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi+3)^2+3}ds_\zeta\Bigg\}. \end{aligned}$$

Finally, on the boundary part $\partial_4 P^+$, the segment between the points -2 and 2, with $\zeta = \overline{\zeta}$ the boundary integral is

$$w_4(z) = \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \operatorname{Re} w(\zeta) \left(2 \sum_{m+n \in 2\mathbb{Z}} [q^1_{mn}(\zeta, z) - q^1_{mn}(\zeta, 0)] d\zeta + \frac{2}{\zeta} \right) d\zeta,$$

because the equality $q^1_{m,n}(\overline{\zeta}, z) = q^1_{mn}(\zeta, z)$ holds. Since $\zeta = \xi$, $d\zeta = d\xi = ds_\zeta$ on $\partial_4 P^+$, then the final form of the fourth boundary integral is

$$w_4(z) = \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \operatorname{Re} w(\zeta) \left[2 \sum_{m+n \in 2\mathbb{Z}} \left(\frac{3(\zeta - \omega_{mn} - 2)^2}{(\zeta - \omega_{mn} - 2)^3 - (z-2)^3} - \frac{3(\zeta - \omega_{mn} - 2)^2}{(\zeta - \omega_{mn} - 2)^3 + 8} \right) + \frac{2}{\xi} \right] ds_\zeta.$$

Thus, composing all the boundary integrals, one gets

$$\begin{aligned} w(z) &= \frac{1}{2\pi i} \int\limits_{\partial P^+} \operatorname{Re} w(\zeta) 2 \sum_{m+n \in 2\mathbb{Z}} [q^1_{mn}(\zeta, z) - q^1_{mn}(\zeta, 0)] d\zeta \\ &- \frac{1}{2\pi i} \int\limits_{\partial_1 P^+} \left[\operatorname{Re} w(\zeta) \frac{2(2\xi - 3)}{(2\xi - 3)^2 + 3} + \operatorname{Im} w(\zeta) \frac{2\sqrt{3}}{(2\xi - 3)^2 + 3} \right] ds_\zeta \\ &+ \frac{1}{2\pi i} \int\limits_{\partial_2 P^+} \left[\operatorname{Re} w(\zeta) \frac{2\xi}{\xi^2 + 3} + \operatorname{Im} w(\zeta) \frac{2\sqrt{3}}{\xi^2 + 3} \right] ds_\zeta \\ &- \frac{1}{2\pi i} \int\limits_{\partial_3 P^+} \left[\operatorname{Re} w(\zeta) \frac{2(2\xi + 3)}{(2\xi + 3)^2 + 3} + \operatorname{Im} w(\zeta) \frac{2\sqrt{3}}{(2\xi + 3)^2 + 3} \right] ds_\zeta \\ &+ \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \operatorname{Re} w(\zeta) \frac{2}{\xi} ds_\zeta - \frac{1}{\pi} \int\limits_{P^+} \left\{ w_{\overline{\zeta}}(\zeta) \left(\sum_{m+n \in 2\mathbb{Z}} [q^1_{mn}(\zeta, z) - q^1_{mn}(\zeta, 0)] + \frac{1}{\zeta} \right) \right. \\ &\left. - \overline{w_{\overline{\zeta}}(\zeta)} \left(\sum_{m+n \in 2\mathbb{Z}} [q^1_{mn}(\overline{\zeta}, z) - q^1_{mn}(\overline{\zeta}, 0)] + \frac{1}{\overline{\zeta}} \right) \right\} d\xi d\eta. \end{aligned} \tag{4.30}$$

With respect to the boundary part $\partial_2 P^+$ another representation formula is appropriate. A variation of the choice of formulas from (4.12)-(4.14) and (4.15)-(4.17) allows to introduce

$$q^2_{mn}(\zeta, z) = \frac{1}{\zeta - z - \omega_{mn}} + \frac{1}{\zeta - \overline{z_1} - \omega_{m-1,n+3}} + \frac{1}{\zeta - z_2 - \omega_{m-2,n}}, \tag{4.31}$$

where

$$\overline{z_1} = -\frac{1}{2}(1 - i\sqrt{3})z + 3 - i\sqrt{3}, \; z_2 = -\frac{1}{2}(1 + i\sqrt{3})z + 3 + i\sqrt{3}.$$

Multiplying (4.31) by $-\frac{1}{2}(1 + i\sqrt{3})$ and denoting

$$\begin{aligned} &\acute{\zeta} = -\frac{1}{2}(1 + i\sqrt{3})\zeta, \; \acute{z} = -\frac{1}{2}(1 + i\sqrt{3})z, \\ &\overline{\acute{z}_1} = -\frac{1}{2}(1 - i\sqrt{3})\acute{z} + 3 - i\sqrt{3}, \; \acute{z}_2 = -\frac{1}{2}(1 + i\sqrt{3})\acute{z} + 3 + i\sqrt{3} \end{aligned}$$

lead to the following calculations

$$\begin{aligned} a. \quad & -\frac{1}{2}(1 + i\sqrt{3})(\zeta - z - \omega_{mn}) = \acute{\zeta} - \acute{z} + \frac{1}{2}(1 + i\sqrt{3})(3m + i\sqrt{3}n) = \acute{\zeta} - \acute{z} - \omega_{-\frac{m-n}{2}, -\frac{3m+n}{2}}; \\ b. \quad & -\frac{1}{2}(1 + i\sqrt{3})[\zeta + \frac{1}{2}(1 - i\sqrt{3})z - 3 + i\sqrt{3} - \omega_{m-1,n+3}] = \\ & \acute{\zeta} + \frac{1}{2}(1 - i\sqrt{3})\acute{z} + 3 + i\sqrt{3} + \frac{1}{2}(3(m - n - 4) + i\sqrt{3}(3m + n)) = \acute{\zeta} - \overline{\acute{z}_1} - \omega_{-\frac{m-n}{2}, -\frac{3m+n}{2}}; \\ c. \quad & -\frac{1}{2}(1 + i\sqrt{3})[\zeta + \frac{1}{2}(1 + i\sqrt{3})z - 3 - i\sqrt{3} - \omega_{m-2,n}] = \\ & \acute{\zeta} + \frac{1}{2}(1 + i\sqrt{3})\acute{z} + \frac{1}{2}(3(m - n - 2) + i\sqrt{3}(3m + n - 2)) = \acute{\zeta} - \acute{z}_2 - \omega_{-\frac{m-n}{2}, -\frac{3m+n}{2}}; \end{aligned}$$

If $-\frac{m-n}{2}=k$, $-\frac{3m+n}{2}=l$, $m+n\in 2\mathbb{Z}$ then $m=-\frac{k+l}{2}$, $n=\frac{3k-l}{2}$, $k+l\in 2\mathbb{Z}$, then

$$q^2_{mn}(\zeta,z)=-\frac{1}{2}(1+i\sqrt{3})\frac{3(\zeta'-\omega_{kl}-2)^2}{(\zeta'-\omega_{kl}-2)^3-(z'-2)^3}=\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}.$$

Therefore, formula (4.23) with q^2_{mn} rather than with q^1_{mn} holds.

Similar modifications for the reflection points with $\overline{z}$ are needed. Define

$$\tilde{q}^2_{mn}(\zeta,\overline{z}) \;=\; \frac{1}{\zeta-\overline{z}-\omega_{mn}}+\frac{1}{\zeta-z_1-\omega_{m-1,n-3}}+\frac{1}{\zeta-\overline{z_2}-\omega_{m-2,n}}. \tag{4.32}$$

Presenting new variables $\acute{z}=-\frac{1}{2}(1+i\sqrt{3})z$, $\zeta''=-\frac{1}{2}(1-i\sqrt{3})\zeta$ leads to

$$\begin{aligned}
a.\quad & -\frac{1}{2}(1-i\sqrt{3})\zeta+\frac{1}{2}(1-i\sqrt{3})\overline{z}+\frac{1}{2}(1-i\sqrt{3})\omega_{mn}=\zeta''-\overline{\acute{z}}-\omega_{-\frac{m+n}{2},\frac{3m-n}{2}};\\
b.\quad & -\frac{1}{2}(1-i\sqrt{3})[\zeta+\frac{1}{2}(1-i\sqrt{3})\overline{z}-(3+i\sqrt{3})-\omega_{m-1,n-3}]=\\
& \zeta''+\frac{1}{2}(1+i\sqrt{3})\overline{\acute{z}}-3-i\sqrt{3}+\frac{1}{2}(3(m+n)-i\sqrt{3}(3m-n))=\zeta''-\acute{z_1}-\omega_{-\frac{m+n}{2},\frac{3m-n}{2}};\\
c\quad & -\frac{1}{2}(1-i\sqrt{3})[\zeta+\frac{1}{2}(1+i\sqrt{3})\overline{z}-(3-i\sqrt{3})-\omega_{m-2,n}]=\\
& \zeta''+\frac{1}{2}(1-i\sqrt{3})\overline{\acute{z}}-3+i\sqrt{3}+\frac{1}{2}(3(m+n)-i\sqrt{3}(3m-n))=\zeta''-\overline{\acute{z_2}}-\omega_{-\frac{m+n}{2},\frac{3m-n}{2}}.
\end{aligned}$$

And if $-\frac{m+n}{2}=k$, $\frac{3m-n}{2}=l$, $m+n\in 2\mathbb{Z}$ then $m=-\frac{k-l}{2}$, $n=-\frac{3k+l}{2}$, $k+l\in 2\mathbb{Z}$. Thus

$$\tilde{q}^2_{mn}(\zeta,\overline{z})=-\frac{1}{2}(1-i\sqrt{3})\frac{3(\zeta''-\omega_{kl}-2)^2}{(\zeta''-\omega_{kl}-2)^3-(\overline{z'}-2)^3}=\frac{3(\zeta-\omega_{mn}+1+i\sqrt{3})^2}{(\zeta-\omega_{mn}+1+i\sqrt{3})^3-(\overline{z}+1+i\sqrt{3})^3}.$$

Obviously, by changing summation, one has

$$\sum_{m+n\in 2\mathbb{Z}}\overline{[\tilde{q}^2_{mn}(\zeta,\overline{z})-\tilde{q}^2_{mn}(\zeta,0)]}=\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{mn}(\overline{\zeta},z)-\tilde{q}^2_{mn}(\overline{\zeta},0)],$$

$$\overline{\tilde{q}^2_{mn}(\zeta,\overline{z})}=\frac{3(\overline{\zeta}-\omega_{m,-n}+1-i\sqrt{3})^2}{(\overline{\zeta}-\omega_{m,-n}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}.$$

Therefore, besides the formula (4.25), the function $w(z)$ can be represented also by

$$\begin{aligned}
w(z) \;=\; & \frac{1}{2\pi i}\int_{\partial P^+}\Bigg\{w(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta\\
& +\overline{w(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{mn}(\overline{\zeta},z)-\tilde{q}^2_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\}\\
& -\frac{1}{\pi}\int_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)\\
& -\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{mn}(\overline{\zeta},z)-\tilde{q}^2_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta.
\end{aligned} \tag{4.33}$$

Let $w(z)$ be again the sum of the four boundary integrals and the area integral

$$\begin{aligned}
w(z) \; & =\tilde{w_1}(z)+\tilde{w_2}(z)+\tilde{w_3}(z)+\tilde{w_4}(z)-\frac{1}{\pi}\int_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)\\
& +\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{mn}(\overline{\zeta},z)-\tilde{q}^2_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta.
\end{aligned}$$

Consider first the integral on $\partial_1 P^+$, where $\zeta = -\frac{1}{2}(1+i\sqrt{3})\overline{\zeta} + 3 + i\sqrt{3}$, $d\overline{\zeta} = -\frac{1}{2}(1-i\sqrt{3})d\zeta$

$$\begin{aligned}\tilde{w_1}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{m,n}(\overline{\zeta},z)-\tilde{q}^2_{m,n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg] \\ &+i\operatorname{Im} w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &-\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{m,n}(\overline{\zeta},z)-\tilde{q}^2_{m,n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg]\Bigg\}.\end{aligned}$$

Here

$$(\overline{\zeta}-\omega_{mn}+1-i\sqrt{3}) = -\frac{1}{2}(1-i\sqrt{3})[\zeta-\omega_{-\frac{m-n-4}{2},-\frac{3m+n}{2}}+1-i\sqrt{3}].$$

If $-\frac{m-n-4}{2}=k$, $-\frac{3m+n}{2}=l$, $m+n\in 2\mathbb{Z}$ then $m=-\frac{k+l-2}{2}$, $n=\frac{3k-l-2}{2}$, $k+l\in 2\mathbb{Z}$ and

$$(\overline{\zeta}-\omega_{mn}+1-i\sqrt{3})^2 d\overline{\zeta} = (\zeta-\omega_{kl}+1-i\sqrt{3})^2 d\zeta;\ \frac{d\overline{\zeta}}{\overline{\zeta}} = \frac{d\zeta}{\zeta-3-i\sqrt{3}},$$

then, using relations in (4.26), (4.27), one gets

$$\begin{aligned}\tilde{w_1}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}+\frac{1}{\zeta-3-i\sqrt{3}}\Bigg) \\ &+i\operatorname{Im} w(\zeta)\Bigg(\frac{1}{\zeta}-\frac{1}{\zeta-3-i\sqrt{3}}\Bigg)\Bigg\}d\zeta = \\ &\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\ &-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8}\Bigg]d\zeta-\frac{2(2\xi-3)}{(2\xi-3)^2+3}ds_\zeta\Bigg)+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi-3)^2+3}ds_\zeta\Bigg\}.\end{aligned}$$

On the boundary $\partial_2 P^+$, where $\zeta = \overline{\zeta}+2i\sqrt{3}$, $d\overline{\zeta} = d\zeta$

$$\begin{aligned}\tilde{w_2}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\overline{w(\zeta)}\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{m,n-2}(\overline{\zeta},z)-q^2_{m,n-2}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\}.\end{aligned}$$

Rewriting the term

$$\tilde{q}^2_{m,n-2}(\overline{\zeta},z) = \frac{3(\overline{\zeta}-\omega_{m,n-2}+1-i\sqrt{3})^2}{(\overline{\zeta}-\omega_{m,n-2}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} = \frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}$$

and, using the formulas in (4.28), one gets the following boundary integral

$$\begin{aligned}\tilde{w_2}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}+\frac{1}{\zeta-2i\sqrt{3}}\Bigg) \\ &+i\operatorname{Im} w(\zeta)\Bigg(\frac{1}{\zeta}-\frac{1}{\zeta-2i\sqrt{3}}\Bigg)\Bigg\}d\zeta = \\ &\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\ &-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8}\Bigg]d\zeta-\frac{2\xi}{\xi^2+3}ds_\zeta\Bigg)+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{\xi^2+3}ds_\zeta\Bigg\}.\end{aligned}$$

On the part $\partial_3 P^+$, where $\zeta = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} - 3 + i\sqrt{3}$ similarly as before

$$\begin{aligned}\tilde{w_3}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\left\{w(\zeta)\left(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\right)d\zeta\right.\\ &\left.+\overline{w(\zeta)}\left(2\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{m,n}(\overline{\zeta},z)-\tilde{q}^2_{m,n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\right)d\overline{\zeta}\right\}.\end{aligned}$$

Since the terms in the second sum are representable as

$$(\overline{\zeta}-\omega_{mn}+1-i\sqrt{3}) = -\frac{1}{2}(1+i\sqrt{3})[\zeta-\omega_{-\frac{m+n+2}{2},\frac{3m-n-2}{2}}+1-i\sqrt{3}],$$

where if $-\frac{m+n+2}{2}=k$, $\frac{3m-n-2}{2}=l$, $m+n\in 2\mathbb{Z}$ then $m=-\frac{k-l}{2}$, $n=-\frac{3k+l+4}{2}$, $k+l\in 2\mathbb{Z}$ and

$$(\overline{\zeta}-\omega_{mn}+1-i\sqrt{3})^2 d\overline{\zeta} = (\zeta-\omega_{kl}+1-i\sqrt{3})d\zeta,\ \frac{d\overline{\zeta}}{\overline{\zeta}} = \frac{d\zeta}{\zeta+3-i\sqrt{3}}$$

then the boundary integral $\tilde{w_3}$ is

$$\begin{aligned}\tilde{w_3}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\left\{\operatorname{Re} w(\zeta)\left(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}+\frac{1}{\zeta+3-i\sqrt{3}}\right)\right.\\ &\left.+i\operatorname{Im} w(\zeta)\left(\frac{1}{\zeta}-\frac{1}{\zeta+3-i\sqrt{3}}\right)\right\}d\zeta.\end{aligned}$$

Taking the formula (4.29) into account, one gets

$$\begin{aligned}\tilde{w_3}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\left\{\operatorname{Re} w(\zeta)\left(2\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}\right.\right.\right.\\ &\left.\left.\left.-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(1-i\sqrt{3})^3}\right]d\zeta-\frac{2(2\xi+3)}{(2\xi+3)^2+3}ds_\zeta\right)-\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi+3)^2+3}ds_\zeta\right\}.\end{aligned}$$

On the boundary part $\partial_4 P^+$, where $\zeta = \overline{\zeta}$, similar calculations give the boundary integral in the form

$$\begin{aligned}\tilde{w_4}(z) &= \frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\left\{w(\zeta)\left(2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\right)d\zeta\right.\\ &\left.+\overline{w(\zeta)}\left(2\sum_{m+n\in 2\mathbb{Z}}[\tilde{q}^2_{m,n}(\overline{\zeta},z)-\tilde{q}^2_{m,n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\right)d\overline{\zeta}\right\} =\\ &\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\left\{\operatorname{Re} w(\zeta)\left(2\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}\right.\right.\right.\\ &\left.\left.\left.-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8}\right]d\zeta-\frac{2}{\xi}\right)\right\}ds_\zeta.\end{aligned}$$

Thus, the computations for the second form of the representation formula yield

$$\begin{aligned}w(z) &= \frac{1}{2\pi i}\int\limits_{\partial P^+}\operatorname{Re} w(\zeta)2\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]d\zeta\\ &-\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\left[\operatorname{Re} w(\zeta)\frac{2(2\xi-3)}{(2\xi-3)^2+3}+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi-3)^2+3}\right]ds_\zeta\\ &+\frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\left[\operatorname{Re} w(\zeta)\frac{2\xi}{\xi^2+3}+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{\xi^2+3}\right]ds_\zeta\\ &-\frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\left[\operatorname{Re} w(\zeta)\frac{2(2\xi+3)}{(2\xi+3)^2+3}+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi+3)^2+3}\right]ds_\zeta\end{aligned} \tag{4.34}$$

$$+\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\operatorname{Re} w(\zeta)\frac{2}{\xi}ds_\zeta-\frac{1}{\pi}\int\limits_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)$$
$$-\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\overline{\zeta},z)-q^2_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta.$$

In the same manner the representation formula for the left-hand side of P^+ can be easily obtained by use of the other reflection points

$$\check{z}_1=-\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3},\qquad \overline{\check{z}_1}=-\frac{1}{2}(1+i\sqrt{3})z-3-i\sqrt{3},$$
$$\check{z}_2=-\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3},\qquad \overline{\check{z}_2}=-\frac{1}{2}(1+i\sqrt{3})\overline{z}-3-i\sqrt{3}$$

by substituting them into (1.13), so for $m^2+n^2>0$

$$0=\frac{1}{2\pi i}\int\limits_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta-z-\omega_{mn}}-\frac{1}{\pi}\int\limits_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta-z-\omega_{mn}},$$

$$\begin{aligned}0=&\frac{1}{2\pi i}\int\limits_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1+i\sqrt{3})z+3+i\sqrt{3}-\omega_{mn}}\\&-\frac{1}{\pi}\int\limits_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1+i\sqrt{3})z+3+i\sqrt{3}-\omega_{mn}},\end{aligned}$$

$$\begin{aligned}0=&\frac{1}{2\pi i}\int\limits_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1-i\sqrt{3})z+3-i\sqrt{3}-\omega_{mn}}\\&-\frac{1}{\pi}\int\limits_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1-i\sqrt{3})z+3-i\sqrt{3}-\omega_{mn}}\end{aligned}$$

and

$$\begin{aligned}0=&\frac{1}{2\pi i}\int\limits_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta-\overline{z}-\omega_{mn}}-\frac{1}{\pi}\int\limits_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta-\overline{z}-\omega_{mn}},\\
0=&\frac{1}{2\pi i}\int\limits_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3}-\omega_{mn}}\\&-\frac{1}{\pi}\int\limits_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta+\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3}-\omega_{mn}},\\
0=&\frac{1}{2\pi i}\int\limits_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta+\frac{1}{2}(1-i\sqrt{3})\overline{z}+3-i\sqrt{3}-\omega_{mn}}\\&-\frac{1}{\pi}\int\limits_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta+\frac{1}{2}(1-i\sqrt{3})\overline{z}+3-i\sqrt{3}-\omega_{mn}}.\end{aligned}$$

Define the new function

$$\begin{aligned}q^3_{mn}(\zeta,z)=&\frac{1}{\zeta-z-\omega_{mn}}+\frac{1}{\zeta+\frac{1}{2}(1+i\sqrt{3})z+3+i\sqrt{3}-\omega_{mn}}\\&+\frac{1}{\zeta+\frac{1}{2}(1-i\sqrt{3})z+3-i\sqrt{3}-\omega_{mn}}=\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}\end{aligned}\tag{4.35}$$

and the double series

$$\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]=\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\right].$$

Then

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)[q^3_{00}(\zeta,z) - q^3_{00}(\zeta,0) + \frac{1}{\zeta}]d\zeta - \frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)[q^3_{00}(\zeta,z) - q^3_{00}(\zeta,0) + \frac{1}{\zeta}]d\xi d\eta, \quad (4.36)$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta) \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}} [q^3_{mn}(\zeta,z) - q^3_{mn}(\zeta,0)]d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta) \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}} [q^3_{mn}(\zeta,z) - q^3_{mn}(\zeta,0)]d\xi d\eta. \end{aligned} \quad (4.37)$$

Also

$$0 = \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)[q^3_{00}(\zeta,\overline{z}) - q^3_{00}(\zeta,0) + \frac{1}{\zeta}]d\zeta - \frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)[q^3_{00}(\zeta,\overline{z}) - q^3_{00}(\zeta,0) + \frac{1}{\zeta}]d\xi d\eta, \quad (4.38)$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta) \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}} [q^3_{mn}(\zeta,\overline{z}) - q^3_{mn}(\zeta,0)]d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta) \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}} [q^3_{mn}(\zeta,\overline{z}) - q^3_{mn}(\zeta,0)]d\xi d\eta. \end{aligned} \quad (4.39)$$

The complex conjugation of (4.38) and (4.39) gives the following Cauchy formulas

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{\partial P^+} w(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\zeta,z) - q^3_{mn}(\zeta,0)] + \frac{1}{\zeta}\right)d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} w_{\overline{\zeta}}(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\zeta,z) - q^3_{mn}(\zeta,0)] + \frac{1}{\zeta}\right)d\xi d\eta, \end{aligned} \quad (4.40)$$

$$\begin{aligned} 0 &= -\frac{1}{2\pi i}\int\limits_{\partial P^+} \overline{w(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\overline{\zeta},z) - q^3_{mn}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\right)d\overline{\zeta} \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} \overline{w_{\overline{\zeta}}(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\overline{\zeta},z) - q^3_{mn}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\right)d\xi d\eta. \end{aligned} \quad (4.41)$$

Subtracting (4.41) from (4.40) leads to the representation formula

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{\partial P^+} \left\{ w(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\zeta,z) - q^3_{mn}(\zeta,0)] + \frac{1}{\zeta}\right)d\zeta \right. \\ &\quad \left. +\overline{w(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\overline{\zeta},z) - q^3_{mn}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\right)d\overline{\zeta}\right\} \\ &\quad -\frac{1}{\pi}\int\limits_{P^+} \left\{ w_{\overline{\zeta}}(\zeta)\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\zeta,z) - q^3_{mn}(\zeta,0)] + \frac{1}{\zeta}\right) \right. \\ &\quad \left. -\overline{w_{\overline{\zeta}}(\zeta)}\left(\sum_{m+n\in 2\mathbb{Z}} [q^3_{mn}(\overline{\zeta},z) - q^3_{mn}(\overline{\zeta},0)] + \frac{1}{\overline{\zeta}}\right)\right\}d\xi d\eta. \end{aligned} \quad (4.42)$$

To study the boundary behavior of the function $w(z)$ it is decomposed as

$$\begin{aligned} w(z) &= \breve{w}_1(z)+\breve{w}_2(z)+\breve{w}_3(z)+\breve{w}_4(z)-\frac{1}{\pi}\int\limits_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg) \\ &\quad +\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\overline{\zeta},z)-q^3_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta \end{aligned}$$

and the equality

$$\overline{q^3_{mn}(\zeta,\overline{z})}=\frac{3(\overline{\zeta-\omega_{mn}+2})^2}{(\overline{\zeta-\omega_{mn}+2})^3-(\overline{z}+2)^3}=q^3_{m,-n}(\overline{\zeta},z)$$

is used.

Consider first $\zeta\in\partial_1 P^+$, where $\zeta=-\frac{1}{2}(1+i\sqrt{3})\overline{\zeta}+3+i\sqrt{3}$

$$\begin{aligned} \breve{w}_1(z)=\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}&\Bigg\{w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\overline{w(\zeta)}\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{m,-n}(\overline{\zeta},z)-q^3_{m,-n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\}. \end{aligned}$$

The terms are rewritten as $(\overline{\zeta}-\omega_{mn}+2)=-\frac{1}{2}(1+i\sqrt{3})(\zeta-\omega_{kl}+2)$, where if $k=-\frac{m-n-4}{2}$, $l=-\frac{3m+n-4}{2}$, $m+n\in 2\mathbb{Z}$, then $m=-\frac{k+l-4}{2}$, $n=\frac{3k-l-4}{2}$, $k+l\in 2\mathbb{Z}$. Therefore

$$(\overline{\zeta}-\omega_{mn}+2)^2 d\overline{\zeta}=(\zeta-\omega_{kl}+2)^2 d\zeta;\ \frac{d\overline{\zeta}}{\overline{\zeta}}=\frac{d\zeta}{\zeta-3-i\sqrt{3}}.$$

Then

$$\begin{aligned} \breve{w}_1(z) &=\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &\quad +\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\overline{\zeta},z)-q^3_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg] \\ &\quad +i\operatorname{Im} w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &\quad -\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\overline{\zeta},z)-q^3_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg]\Bigg\}, \end{aligned}$$

$$\begin{aligned} \breve{w}_1(z) &=\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}+\frac{1}{\zeta-3-i\sqrt{3}}\Bigg) \\ &\quad +i\operatorname{Im} w(\zeta)\Bigg(\frac{1}{\zeta}-\frac{1}{\zeta-3-i\sqrt{3}}\Bigg)\Bigg\}d\zeta= \\ &\quad \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg\{\operatorname{Re} w(\zeta)\Bigg[2\sum_{m+n\in 2\mathbb{Z}}\Bigg(\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\Bigg)d\zeta \\ &\quad -\frac{2(2\xi-3)^2}{(2\xi-3)^2+3}ds_\zeta\Bigg]-\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi-3)^2+3}ds_\zeta\Bigg\} \end{aligned}$$

in view of the formulas (4.26), (4.27).

On the boundary part $\partial_2 P^+$ for $\zeta = \overline{\zeta} + 2i\sqrt{3}$

$$\begin{aligned}\breve{w}_2(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{\mathrm{Re}\, w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{m,n-2}(\overline{\zeta},z)-q^3_{m,n-2}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg] \\ &+i\mathrm{Im}\, w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &-\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{m,n-2}(\overline{\zeta},z)-q^3_{m,n-2}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg]\Bigg\}.\end{aligned}$$

Here the equality

$$q^3_{m,n-2}(\overline{\zeta},z)=\frac{3[\zeta-2i\sqrt{3}-(3m+i\sqrt{3}n-2i\sqrt{3})+2]^2}{[\zeta-2i\sqrt{3}-(3m+i\sqrt{3}n-2i\sqrt{3})+2]^3-(z+2)^3}=q^3_{mn}(\zeta,z)$$

holds, then

$$\begin{aligned}\breve{w}_2(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{\mathrm{Re}\, w(\zeta)\Bigg[\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}+\frac{1}{\zeta-2i\sqrt{3}}\Bigg) \\ &+i\mathrm{Im}\, w(\zeta)\Bigg(\frac{1}{\zeta}-\frac{1}{\zeta-2i\sqrt{3}}\Bigg)\Bigg\}d\zeta.\end{aligned}$$

Using again the relations (4.28), one has

$$\begin{aligned}\breve{w}_2(z) &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg\{\mathrm{Re}\, w(\zeta)(2\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\Bigg] \\ &+\frac{2\xi}{\xi^2+3}\Bigg)+\mathrm{Im}\, w(\zeta)\frac{2\sqrt{3}}{\xi^2+3}\Bigg\}ds_\zeta.\end{aligned}$$

For $\zeta\in\partial_3 P^+$, where $\zeta=-\frac{1}{2}(1-i\sqrt{3})\overline{\zeta}-3+i\sqrt{3}$ the boundary integral is

$$\begin{aligned}\breve{w}_3(z) &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\Bigg\{\mathrm{Re}\, w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &+\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\overline{q^3_{mn}(\zeta,z)}-\overline{q^3_{mn}(\zeta,0)}]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg] \\ &+i\mathrm{Im}\, w(\zeta)\Bigg[\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\ &-\Bigg(\sum_{m+n\in 2\mathbb{Z}}[\overline{q^3_{mn}(\zeta,z)}-\overline{q^3_{mn}(\zeta,0)}]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg]\Bigg\}.\end{aligned}$$

Consider again the term $(\overline{\zeta}-\omega_{m,-n}+2)=-\frac{1}{2}(1+i\sqrt{3})(\zeta-\omega_{-\frac{m-n}{2},\frac{3m+n}{2}}+2)$, where if $-\frac{m-n}{2}=k$, $\frac{3m+n}{2}=l$, $m+n\in 2\mathbb{Z}$ then $m=-\frac{k-l}{2}$, $n=\frac{3k+l}{2}$, $k+l\in 2\mathbb{Z}$ and

$$(\overline{\zeta}-\omega_{m,-n}+2)^2 d\overline{\zeta}=(\zeta-\omega_{kl}+2)^2 d\zeta;\ \frac{d\overline{\zeta}}{\overline{\zeta}}=\frac{d\zeta}{\zeta+3-i\sqrt{3}}.$$

Then, on the basis of the formulas in (4.29), it follows that

$$\begin{aligned}\breve{w}_3(z) &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\Bigg\{\mathrm{Re}\, w(\zeta)\Bigg[2\sum_{m+n\in 2\mathbb{Z}}\Bigg(\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\Bigg)d\zeta \\ &-\frac{2(2\xi+3)^2}{(2\xi+3)^2+3}ds_\zeta\Bigg]-\mathrm{Im}\, w(\zeta)\frac{2\sqrt{3}}{(2\xi+3)^2+3}ds_\zeta\Bigg\}.\end{aligned}$$

Similarly for $\zeta \in \partial_4 P^+$, where $\zeta = \overline{\zeta}$

$$\begin{aligned}
\check{w}_4(z) &= \frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\Bigg\{w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)d\zeta \\
&\quad +\overline{w(\zeta)}\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{m,-n}(\overline{\zeta},z)-q^3_{m,-n}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)d\overline{\zeta}\Bigg\} = \\
&\quad \frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{2}{\zeta}\Bigg)d\zeta
\end{aligned}$$

and the boundary integral on $\partial_4 P^+$ becomes

$$\check{w}_4(z) = \frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\operatorname{Re} w(\zeta)\Bigg(2\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\Bigg]+\frac{2}{\xi}\Bigg)ds_\zeta.$$

Thus, the use of $q^3_{mn}(\zeta,z)$ gives the representation formula for the function $w(z)$ in the form

$$\begin{aligned}
w(z) &= \frac{1}{2\pi i}\int\limits_{\partial P^+}\operatorname{Re} w(\zeta) 2\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]d\zeta \\
&\quad -\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\Bigg[\operatorname{Re} w(\zeta)\frac{2(2\xi-3)}{(2\xi-3)^2+3}+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi-3)^2+3}\Bigg]ds_\zeta \\
&\quad +\frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\Bigg[\operatorname{Re} w(\zeta)\frac{2\xi}{\xi^2+3}+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{\xi^2+3}\Bigg]ds_\zeta \\
&\quad -\frac{1}{2\pi i}\int\limits_{\partial_3 P^+}\Bigg[\operatorname{Re} w(\zeta)\frac{2(2\xi+3)}{(2\xi+3)^2+3}+\operatorname{Im} w(\zeta)\frac{2\sqrt{3}}{(2\xi+3)^2+3}\Bigg]ds_\zeta \\
&\quad +\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\operatorname{Re} w(\zeta)\frac{2}{\xi}ds_\zeta - \frac{1}{\pi}\int\limits_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\zeta,z)-q^3_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg) \\
&\quad -\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^3_{mn}(\overline{\zeta},z)-q^3_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta.
\end{aligned} \tag{4.43}$$

One can see that the form of the representation formula for the function $w(z)$ is taken subject to which boundary part the point z goes to. For the case $z \to \partial_4 P^+$, since $z = \overline{z}$ there, any of the representations can be used and treated in the same way. □

4.3 Schwarz problem for the inhomogeneous Cauchy-Riemann equation

These three representation formulas in the preceding section will be used to prove the boundary behavior for solution of the Schwarz problem on the respective parts of ∂P^+. These proofs are based on the Poisson kernel of half planes, see Section 4.1 and e.g. [19].

The essential part of the boundary integrals in the representation formulas (4.30), (4.34) and (4.43) are the Schwarz operators for P^+

$$S_k\gamma(z) = \frac{1}{2\pi i}\int\limits_{\partial P^+}\gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}2[q^k_{mn}(\zeta,z)-q^k_{mn}(\zeta,0)]d\zeta,\ k=1,2,3. \tag{4.44}$$

For any real-valued function on ∂P^+ it defines an analytic function in P^+, the real part of which coincides on ∂P^+ with γ. This fact is shown in the following Lemmas.

Lemma 4.3.1. *For $\zeta_0 \in \partial_1 P^+$, $\gamma \in C(\partial P^+;\mathbb{R})$*

$$\lim_{z\to\zeta_0} S_1\gamma(z) = \gamma(\zeta_0), \tag{4.45}$$

where the Schwarz operator for P^+ is given in (4.44) for $k=1$.

Proof. We consider the representation formula (4.30) for the right-hand boundary part of P^+ and take the real part of the boundary integral

$$\begin{aligned}\operatorname{Re} S_1\gamma(z) &= \frac{1}{2\pi i}\int\limits_{\partial P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]d\zeta \\ &-\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3-(\overline{z}-2)^3}-\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3+8}\right]d\overline{\zeta}\Bigg\}.\end{aligned}$$

We decompose it into the integrals with respect to the different parts of ∂P^+, so $\operatorname{Re} S_1\gamma(z) = b_1+b_2+b_3+b_4$.
Let ζ_0 be a fixed point on $\partial_1 P^+$, where for $\zeta_0 = -\frac{1}{2}(1+i\sqrt{3})\overline{\zeta_0}+3+i\sqrt{3}$ the relations

$$(\zeta_0-2)^2 = -\frac{1}{2}(1-i\sqrt{3})(\overline{\zeta_0}-2)^2;\ (\zeta_0-2)^3 = (\overline{\zeta_0}-2)^3$$

hold.

Consider now the boundary integral for ζ on the boundary part $\partial_1 P^+$.

$$\begin{aligned}b_1 &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]d\zeta \\ &-\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{m,-n}-2})^2}{(\overline{\zeta-\omega_{m,-n}-2})^3-(\overline{z}-2)^3}-\frac{3(\overline{\zeta-\omega_{m,-n}-2})^2}{(\overline{\zeta-\omega_{m,-n}-2})^3+8}\right]d\overline{\zeta}\Bigg\}.\end{aligned}$$

If $k=-\frac{m-n}{2}$, $l=-\frac{3m+n}{2}$ then $m=-\frac{k+l}{2}$, $n=\frac{3k-l}{2}$, $m+n\in 2\mathbb{Z}$, $k+l\in 2\mathbb{Z}$ and

$$(\overline{\zeta}-\omega_{mn}-2)^2 d\overline{\zeta} = [-\frac{1}{2}(1-i\sqrt{3})(\zeta-\omega_{kl}-2)]^2(-\frac{1}{2}(1-i\sqrt{3})d\zeta) = (\zeta-\omega_{kl}-2)^2 d\zeta.$$

As z tends to ζ_0, the term for $m=n=0$ becomes singular. Consider this case in detail

$$\begin{aligned}b_1^0 &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\Bigg\{\left(\frac{3(\zeta-2)^2}{(\zeta-2)^3-(z-2)^3}-\frac{3(\zeta-2)^2}{(\zeta-2)^3+8}\right)d\zeta \\ &-\left(\frac{3(\overline{\zeta}-2)^2}{(\overline{\zeta}-2)^3-(\overline{z}-2)^3}-\frac{3(\overline{\zeta}-2)^2}{(\overline{\zeta}-2)^3+8}\right)d\overline{\zeta}\Bigg\} = \\ &\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\left[\frac{3(\zeta-2)^2}{(\zeta-2)^3-(z-2)^3}-\frac{3(\zeta-2)^2}{(\zeta-2)^3-(\overline{z}-2)^3}\right]d\zeta = \\ &\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\left[\frac{3(\zeta-2)^2[(z-2)^3-(\overline{z}-2)^3]}{|(\zeta-2)^3-(z-2)^3|^2}\right]d\zeta.\end{aligned}$$

The term

$$(\zeta-2)^3-(z-2)^3 = (\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]$$

and

$$(\overline{z}-2)^3 = [-\frac{1}{2}(1+i\sqrt{3})(\overline{z}-2)]^3 = [-\frac{1}{2}(1+i\sqrt{3})\overline{z}+1+i\sqrt{3}]^3 = (z_1-2)^3,$$
$$(z-2)^3-(\overline{z}-2)^3 = (z-z_1)[(z-2)^2+(z-2)(z_1-2)+(z_1-2)^2].$$

Then

$$b_0^1 = \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\rho(\zeta,z)\frac{z-z_1}{|\zeta-z|^2}d\zeta,$$

where

$$\rho(\zeta,z) = \frac{3(\zeta-2)^2[(z-2)^2+(z-2)(z_1-2)+(z_1-2)^2]}{|(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2|^2}.$$

If $z\to\zeta_0$ then $z_1\to\zeta_0$

$$\lim_{z\to\zeta_0}\rho(\zeta,z) = \frac{(\zeta_0-2)^4}{|\zeta_0-2|^2} = -\frac{1}{2}(1-i\sqrt{3}).$$

Then, using the property of the Poisson kernel for the upper half plane with the boundary line passing through the boundary side $\partial_1 P^+$, one gets

$$\lim_{z\to\zeta_0}\left\{\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\rho(\zeta,z)\frac{z-z_1}{|\zeta-z|^2}(-\frac{1}{2}(1-i\sqrt{3}))ds_\zeta\right\} =$$
$$\lim_{z\to\zeta_0}\left\{\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\rho(\zeta,z)\frac{z-z_1}{|\zeta-z|^2}(-\frac{1}{2}(1+i\sqrt{3}))^2ds_\zeta\right\} =$$
$$\lim_{z\to\zeta_0}\left\{\frac{(-\sqrt{3}+i)}{4\pi}\int\limits_{\partial_1 P^+}\gamma(\zeta)(-\frac{1}{2}(1+i\sqrt{3}))\rho(\zeta,z)\frac{z-z_1}{|\zeta-z|^2}ds_\zeta\right\} =$$
$$-\frac{1}{2}(1+i\sqrt{3})\gamma(\zeta_0)\rho(\zeta_0,\zeta_0) = \gamma(\zeta_0)$$

This is also true for the corner points $2, 1+i\sqrt{3}$ if $\gamma(\zeta_0)$ vanishes at these points.
For the other terms of b_1

$$\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z_1-2)^3}\right]d\zeta =$$
$$\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}\frac{3(\zeta-\omega_{mn}-2)^2[(z-2)^3-(z_1-2)^3]}{[(\zeta-\omega_{mn}-2)^3-(z-2)^3][(\zeta-\omega_{mn}-2)^3-(z_1-2)^3]} = 0.$$

Thus, for $\zeta_0\in\partial_1 P^+$ the equality $\lim_{z\to\zeta_o} b_1 = \gamma(\zeta_0)$ holds.

On the boundary part $\partial_2 P^+$ with $\zeta = \overline{\zeta}+2i\sqrt{3}$

$$(\overline{\zeta-\omega_{m,-n}-2}) = [\zeta-2i\sqrt{3}-(3m+i\sqrt{3}n)-2] = [\zeta-\omega_{m,n+2}-2],$$
$$\text{if } n+2=l,\ n=l-2,\ m+n\in 2\mathbb{Z},\ m+l\in 2\mathbb{Z},$$

then

$$b_2 = \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\left\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]d\zeta\right.$$
$$\left.-\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{m,-n}-2})^2}{(\overline{\zeta-\omega_{m,-n}-2})^3-(\overline{z}-2)^3}-\frac{3(\overline{\zeta-\omega_{m,-n}-2})^2}{(\overline{\zeta-\omega_{m,-n}-2})^3+8}\right]d\overline{\zeta}\right\}.$$

Letting $z \to \zeta_0 \in \partial_1 P^+$

$$\lim_{z\to\zeta_0} b_2 = \lim_{z\to\zeta_0} \frac{1}{2\pi i}\int\limits_{\partial_2 P^+} \gamma(\zeta)\Big\{\sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z_1-2)^3}\Big]\Big\}d\zeta$$

the integral becomes 0 for z_1 tending to ζ_0.

Consider next the boundary part $\partial_3 P^+$, where

$$\overline{(\zeta-\omega_{m,-n}-2)}^2 d\overline{\zeta} = (\zeta-\omega_{kl}-2)^2 d\zeta,$$

if $k = -\frac{m-n+4}{2}$, $l = \frac{m+n+4}{2}$, then $m = -\frac{k-l+4}{2}$, $n = \frac{3k+l}{2}$, $m+n \in 2\mathbb{Z}$, $k+l \in 2\mathbb{Z}$. Then

$$\begin{aligned} b_3 &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\Big\{\sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\Big]d\zeta \\ &- \sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3\overline{(\zeta-\omega_{m,-n}-2)}^2}{\overline{(\zeta-\omega_{m,-n}-2)}^3-(\overline{z}-2)^3} - \frac{3\overline{(\zeta-\omega_{m,-n}-2)}^2}{\overline{(\zeta-\omega_{m,-n}-2)}^3+8}\Big]d\overline{\zeta}\Big\}. \end{aligned}$$

Letting $z \to \zeta_0$, $z_1 \to \zeta_0 \in \partial_1 P^+$, the sum under the integral tends to 0 for $\zeta \in \partial_3 P^+$.

For the boundary part $\partial_4 P^+$, where $\zeta = \overline{\zeta}$

$$\begin{aligned} b_4 &= \frac{1}{2\pi i}\int\limits_{\partial_4 P^+} \gamma(\zeta)\Big\{\sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\Big]d\zeta \\ &- \sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3\overline{(\zeta-\omega_{m,-n}-2)}^2}{\overline{(\zeta-\omega_{m,-n}-2)}^3-(z_1-2)^3} - \frac{3\overline{(\zeta-\omega_{m,-n}-2)}^2}{\overline{(\zeta-\omega_{m,-n}-2)}^3+8}\Big]d\overline{\zeta}\Big\} \end{aligned}$$

and it is seen that if $z \to \zeta_0$ then the integral becomes 0.
Thus, from the calculations of the boundary integral on all parts of ∂P^+ the equality (4.45) follows. $\square$

Lemma 4.3.2. *For $\zeta_0 \in \partial_2 P^+$, $\gamma \in C(\partial P^+;\mathbb{R})$*

$$\lim_{z\to\zeta_0} S_2\gamma(z) = \gamma(\zeta_0), \tag{4.46}$$

where the Schwarz operator for P^+ is given in (4.44) for $k = 2$.

Proof. Let $\zeta_0 \in \partial_2 P^+$, where $\zeta_0 = \overline{\zeta_0} + 2i\sqrt{3}$. Consider now the representation formula (4.34) and take as before its real part of the boundary integral

$$\operatorname{Re} S_2\gamma(z) = \frac{1}{2\pi i}\int\limits_{\partial P^+} \gamma(\zeta)\Big(\sum_{m+n\in 2\mathbb{Z}} [q^2_{mn}(\zeta,z) - q^2_{mn}(\zeta,0)]d\zeta - [\overline{q^2_{mn}(\zeta,z)} - \overline{q^2_{mn}(\zeta,0)}]d\overline{\zeta}\Big).$$

Similarly, we decompose the boundary integral into the sum $\operatorname{Re} S_2\gamma(z) = b_1' + b_2' + b_3' + b_4'$ and calculate it on the part $\partial_2 P^+$.

For $\zeta \in \partial_1 P^+$

$$\begin{aligned} b_1' &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+} \gamma(\zeta)\Big\{\sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\ &- \frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8}\Big]d\zeta - \sum_{m+n\in 2\mathbb{Z}}\Big[\frac{3\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2}{\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3-\overline{(z+1-i\sqrt{3})}^3} \\ &- \frac{3\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2}{\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3+8}\Big]d\overline{\zeta}\Big\}. \end{aligned}$$

Again

$$\begin{aligned} &(\overline{\zeta}-\omega_{mn}+1+i\sqrt{3})^2 d\overline{\zeta} = (\zeta-\omega_{kl}+1-i\sqrt{3})^2 d\zeta, \\ &(\overline{\zeta}-\omega_{mn}+1+i\sqrt{3})^3 = (\zeta-\omega_{kl}+1-i\sqrt{3})^3, \end{aligned}$$

where if $k=-\frac{m-n-2}{2}$, $l=-\frac{3m+n-2}{2}$, then $m=-\frac{k+l-2}{2}$, $n=\frac{3k-l-2}{2}$, $m+n\in 2\mathbb{Z}$, $k+l\in 2\mathbb{Z}$.
And since

$$(1\pm i\sqrt{3})^3=-8,\ \overline{(z+1-i\sqrt{3})}=\overline{z}+1+i\sqrt{3}=z_2+1-i\sqrt{3}, \tag{4.47}$$

where $z_2=\overline{z}+2i\sqrt{3}$

$$\begin{aligned} b_1' &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\ &-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3}\Bigg]d\zeta\Bigg\}= \\ &\frac{1}{2\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2[(z+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3]}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3][(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3]}\Bigg\}d\zeta \end{aligned}$$

Then, letting z and z_2 tend to $\zeta_0\in\partial_2 P^+$, the integral becomes 0.

For ζ from the boundary part $\partial_2 P^+$, where $\zeta=\overline{\zeta}+2i\sqrt{3}$

$$\begin{aligned} &\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2 d\overline{\zeta}=(\zeta-2i\sqrt{3}-(3m+i\sqrt{3}n)+1+i\sqrt{3})^2 d\zeta=(\zeta-\omega_{mn}+1-i\sqrt{3})^2 d\zeta, \\ &\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3=(\zeta-\omega_{mn}+1-i\sqrt{3})^3, \end{aligned}$$

then

$$\begin{aligned} b_2' &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\ &-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8}\Bigg]d\zeta-\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2}{\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3-(\overline{z+1-i\sqrt{3}})^3} \\ &-\frac{3\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2}{\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3+8}\Bigg]d\overline{\zeta}\Bigg\}= \\ &\frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\ &-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3}\Bigg\}d\zeta. \end{aligned}$$

As $z\to\zeta_0\in\partial_2 P^+$, where $\zeta=\zeta_0$, the term of the sum for case $m=n=0$ becomes singular and therefore we observe it carefully.

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\Bigg\{\frac{3(\zeta+1-i\sqrt{3})^2}{(\zeta+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}-\frac{3(\zeta+1-i\sqrt{3})^2}{(\zeta+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3}\Bigg\}d\zeta= \\ &\frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\Bigg\{\frac{3(\zeta+1-i\sqrt{3})^2[(z+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3]}{|(\zeta+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3|^2}\Bigg\}d\zeta, \end{aligned}$$

since

$$\begin{aligned} &(z+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3=(z+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3= \\ &(z-z_2)[(z+1-i\sqrt{3})^2+(z+1-i\sqrt{3})(z_2+1-i\sqrt{3})+(z_2+1-i\sqrt{3})^2] \end{aligned}$$

then

$$b_2'^{0}=\frac{1}{2\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\frac{(z-z_2)}{|\zeta-z|^2}\rho'(\zeta,z)d\zeta,$$

where

$$\rho'(\zeta,z)=\frac{3(\zeta+1-i\sqrt{3})^2[(z+1-i\sqrt{3})^2+(z+1-i\sqrt{3})(z_2+1-i\sqrt{3})+(z_2+1-i\sqrt{3})^2]}{|(\zeta+1-i\sqrt{3})^2+(\zeta+1-i\sqrt{3})(z+1-i\sqrt{3})+(z+1-i\sqrt{3})^2|^2}.$$

Letting $z\to\zeta_0$, also $z_2\to\zeta_0$ for $\zeta=\zeta_0$

$$\rho'(\zeta_0,\zeta_0)=\lim_{z\to\zeta_0}\rho'(\zeta,z)=\frac{(\zeta_0+1-i\sqrt{3})^4}{|\zeta_0+1-i\sqrt{3}}|^4=\frac{(\zeta_0+1-i\sqrt{3})^2}{(\zeta_0-2i\sqrt{3}+1+i\sqrt{3})^2}=1,$$

then for $\zeta_0\in\partial_2P^+$

$$\begin{aligned}\lim_{z\to\zeta_0}b_2' &= \lim_{z\to\zeta_0}\left[\frac{1}{2\pi i}\int\limits_{\partial_2P^+}\gamma(\zeta)\rho'(\zeta,z)\frac{z-z_2}{|\zeta-z|^2}\right]d\zeta=\lim_{z\to\zeta_0}\left[-\frac{1}{2\pi i}\int\limits_{\partial_2P^+}\gamma(\zeta)\rho'(\zeta,z)\frac{z-z_2}{|\zeta-z|^2}\right]ds_\zeta=\\ &\rho'(\zeta_0,\zeta_0)\gamma(\zeta_0)=\gamma(\zeta_0).\end{aligned}$$

This holds also at the corner points $\pm1+i\sqrt{3}$ if γ vanishes there. For the other terms of b_2'

$$\begin{aligned}&\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_2P^+}\gamma(\zeta)\sum_{\substack{m+n\in2\mathbb{Z},\\ m^2+n^2\neq0}}\left\{\frac{(\zeta+1-i\sqrt{3})^2}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3]}\times\right.\\ &\left.\frac{(z+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3]}d\zeta\right\}=0.\end{aligned}$$

On the boundary part ∂_3P^+

$$\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})^2d\zeta}=(\zeta-\omega_{kl}+1-i\sqrt{3})^2d\zeta,$$

where if $-\frac{m+n}{2}=k$, $\frac{3m-n}{2}=l$, then $m=-\frac{k-l}{2}$, $n=-\frac{3k+l}{2}$, $m+n\in2\mathbb{Z}$, $k+l\in2\mathbb{Z}$. Then, using (4.47)

$$\begin{aligned}b_3' &=\frac{1}{2\pi i}\int\limits_{\partial_3P^+}\gamma(\zeta)\left\{\sum_{m+n\in2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}\right.\right.\\ &-\left.\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(1-i\sqrt{3})^3}\right]d\zeta-\sum_{m+n\in2\mathbb{Z}}\left[\frac{3\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2}{\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3-\overline{(z+1-i\sqrt{3})}^3}\right.\\ &-\left.\left.\frac{3\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^2}{\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})}^3-(1+i\sqrt{3})^3}\right]d\overline{\zeta}\right\}=\\ &\frac{1}{2\pi i}\int\limits_{\partial_3P^+}\gamma(\zeta)\sum_{m+n\in2\mathbb{Z}}\left(\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}\right.\\ &-\left.\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-\overline{(z+1-i\sqrt{3})}^3}\right)d\zeta.\end{aligned}$$

Letting $z\to\zeta_0$

$$\begin{aligned}&\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_3P^+}\gamma(\zeta)\left\{\sum_{m+n\in2\mathbb{Z}}\frac{(\zeta+1-i\sqrt{3})^2}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3]}\times\right.\\ &\left.\frac{(z+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3]}\right\}d\zeta=0.\end{aligned}$$

Finally, for $\zeta\in\partial_4P^+$

$$\overline{(\zeta-\omega_{m,-n}+1-i\sqrt{3})^2d\zeta}=[\zeta-(3m+i\sqrt{3}n+2i\sqrt{3})+1+i\sqrt{3}]^2d\zeta=(\zeta-\omega_{m,n+2}+1-i\sqrt{3})d\zeta,$$

where $m+n \in 2\mathbb{Z}$.

Similarly as before

$$\begin{aligned}
b_4' &= \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \gamma(\zeta) \Bigg\{ \sum_{m+n\in 2\mathbb{Z}} \Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\
&- \frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8} \Bigg] d\zeta - \sum_{m+n\in 2\mathbb{Z}} \Bigg[\frac{3(\overline{\zeta-\omega_{m,-n-2}+1-i\sqrt{3}})^2}{(\overline{\zeta-\omega_{m,-n-2}+1-i\sqrt{3}})^3-(\overline{z+1-i\sqrt{3}})^3} \\
&- \frac{3(\overline{\zeta-\omega_{m,-n-2}+1-i\sqrt{3}})^2}{(\overline{\zeta-\omega_{m,-n-2}+1-i\sqrt{3}})^3+8} \Bigg] d\overline{\zeta} \Bigg\} = \\
&\frac{1}{2\pi i} \int\limits_{\partial_3 P^+} \gamma(\zeta) \sum_{m+n\in 2\mathbb{Z}} \Bigg(\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} \\
&- \frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3} \Bigg) d\zeta.
\end{aligned}$$

Therefore, for $z \to \zeta_0$

$$\begin{aligned}
&\lim_{z\to\zeta_0} \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \gamma(\zeta) \Bigg\{ \sum_{m+n\in 2\mathbb{Z}} \frac{(\zeta+1-i\sqrt{3})^2}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3]} \times \\
&\frac{(z+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3}{[(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z_2+1-i\sqrt{3})^3]} \Bigg\} d\zeta = 0.
\end{aligned}$$

Thus, the boundary condition (4.46) for $\zeta_0 \in \partial_2 P^+$ is valid. □

Lemma 4.3.3. *For $\zeta_0 \in \partial_3 P^+$, $\gamma \in C(\partial P^+;\mathbb{R})$*

$$\lim_{z\to\zeta_0} S_3\gamma(z) = \gamma(\zeta_0), \tag{4.48}$$

where the Schwarz operator for P^+ is given in (4.44) for $k=3$.

Proof. For $\zeta_0 = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta_0} - 3 + i\sqrt{3}$ the relations

$$(\zeta_0+2)^2 = -\frac{1}{2}(1+i\sqrt{3})(\overline{\zeta_0}+2)^2,\ (\zeta_0+2)^3 = (\overline{\zeta_0}+2)^3$$

are being used. We consider the third representation formula (4.43) and take its real part

$$\operatorname{Re} S_3\gamma(z) = \frac{1}{2\pi i} \int\limits_{\partial P^+} \gamma(\zeta) \Bigg(\sum_{m+n\in 2\mathbb{Z}} [q_{mn}^3(\zeta,z) - q_{mn}^3(\zeta,0)] d\zeta - [\overline{q_{mn}^3(\zeta,z)} - \overline{q_{mn}^3(\zeta,0)}] d\overline{\zeta} \Bigg).$$

Presenting again this boundary integral as $\operatorname{Re} S_3\gamma(z) = b_1'' + b_2'' + b_3'' + b_4''$, we compute it on the different parts of ∂P^+.

We consider at first the boundary part $\partial_1 P^+$, where

$$(\overline{\zeta-\omega_{mn}+2})^2 d\overline{\zeta} = (\zeta-\omega_{kl}+2)^2 d\zeta$$

if $k = -\frac{m+n-4}{2}$, $l = -\frac{3m-n-4}{2}$, then $m = -\frac{k+l-4}{2}$, $n = -\frac{3k-l-4}{2}$, $m+n \in 2\mathbb{Z}$, $k+l \in 2\mathbb{Z}$. Also

$$(\overline{z}+2)^3 = [-\frac{1}{2}(1-i\sqrt{3})(\overline{z}+2)]^3 = [-\frac{1}{2}(1-i\sqrt{3})\overline{z} - 1 + i\sqrt{3}]^3 = (\check{z}_1+2)^3. \tag{4.49}$$

Then

$$
\begin{aligned}
b_1'' &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+} \gamma(\zeta)\left[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\right]d\zeta \\
&- \sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{mn}+2})^2}{(\overline{\zeta-\omega_{mn}+2})^3-(\overline{z+2})^3}-\frac{3(\overline{\zeta-\omega_{mn}+2})^2}{(\overline{\zeta-\omega_{mn}+2})^3-8}\right]d\overline{\zeta}\Bigg\} = \\
&\frac{1}{2\pi i}\int\limits_{\partial_1 P^+} \gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(\overline{z}+2)^3}\right]d\zeta = \\
&\frac{1}{2\pi i}\int\limits_{\partial_1 P^+} \gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left(\frac{3(\zeta-\omega_{mn}+2)^2[(z+2)^3-(\check{z}_1+2)^3]}{[(\zeta-\omega_{mn}+2)^3-(z+2)^3][(\zeta-\omega_{mn}+2)^3-(\check{z}_1+2)^3]}\right)d\zeta
\end{aligned}
$$

and taking $z\to\zeta_0$ and $\check{z}_1\to\zeta_0$, $\zeta_0\in\partial_3 P^+$, the sum tends to 0.

Similarly on $\partial_2 P^+$

$$
(\overline{\zeta-\omega_{m,-n+2}+2})^2 d\overline{\zeta} = (\zeta-\omega_{mn}+2)^2 d\zeta,
$$

$$
\begin{aligned}
b_2'' &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+} \gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\right]d\zeta \\
&- \sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{m,-n+2}+2})^2}{(\overline{\zeta-\omega_{m,-n+2}+2})^3-(\overline{z+2})^3}-\frac{3(\overline{\zeta-\omega_{m,-n+2}+2})^2}{(\overline{\zeta-\omega_{m,-n+2}+2})^3-8}\right]d\overline{\zeta}\Bigg\}.
\end{aligned}
$$

Using (4.49) and letting $z\to\zeta_0\in\partial_3 P^+$

$$
\lim_{z\to\zeta_0} b_2'' = \lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_2 P^+} \gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left(\frac{3(\zeta-\omega_{mn}+2)^2[(z+2)^3-(\check{z}_1+2)^3]}{[(\zeta-\omega_{mn}+2)^3-(z+2)^3][(\zeta-\omega_{mn}+2)^3-(\check{z}_1+2)^3]}\right)d\zeta
$$

the integral tends to 0 for $\zeta\in\partial_2 P^+$.

Consider next the boundary part $\partial_3 P^+$, where $\zeta = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta}-3+i\sqrt{3}$. Here

$$
(\overline{\zeta-\omega_{mn}+2})^2 d\overline{\zeta} = [-\frac{1}{2}(1+i\sqrt{3})\zeta-3-i\sqrt{3}-(3m-i\sqrt{3}n)+2]^2 d\overline{\zeta} = (\zeta-\omega_{kl}+2)^2 d\zeta,
$$

where if $k=-\frac{m-n}{2}$, $l=\frac{3m+n}{2}$, then $m=-\frac{k-l}{2}$, $n=\frac{3k+l}{2}$, $m+n\in 2\mathbb{Z}$, $k+l\in 2\mathbb{Z}$. So

$$
\begin{aligned}
b_3'' &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}-\frac{3(\zeta-\omega_{mn}+2)^2}{(\zeta-\omega_{mn}+2)^3-8}\right]d\zeta \\
&- \sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{mn}+2})^2}{(\overline{\zeta-\omega_{mn}+2})^3-(\overline{z+2})^3}-\frac{3(\overline{\zeta-\omega_{mn}+2})^2}{(\overline{\zeta-\omega_{mn}+2})^3-8}\right]d\overline{\zeta}\Bigg\}.
\end{aligned}
$$

To take care of the potential singularity, the term of the integral for the case $m=n=0$ is to be studied separately.

For $m=n=0$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\left[\frac{3(\zeta+2)^2}{(\zeta+2)^3-(z+2)^3}-\frac{3(\zeta+2)^2}{(\zeta+2)^3-(\overline{z}+2)^3}\right]d\zeta = \\
&\frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\left(\frac{3(\zeta+2)^2[(z+2)^3-(\check{z}_1+2)^3]}{|(\zeta+2)^3-(z+2)^3|^2}\right)d\zeta = \frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\frac{z-\check{z}_1}{|\zeta-z|^2}\check{\rho}(\zeta,z)d\zeta,
\end{aligned}
$$

where

$$
\check{\rho}(\zeta,z) = 3(\zeta+2)^2\frac{[(z+2)^2+(z+2)(\check{z}_1+2)+(\check{z}_1+2)^2]}{|(\zeta+2)^2+(\zeta+2)(z+2)+(z+2)^2|^2}.
$$

When $z \to \zeta_0 \in \partial_3 P^+$ also $\check{z}_1 \to \zeta_0$ and then for $\zeta = \zeta_0$

$$\check{\rho}(\zeta_0, \zeta_0) = \lim_{z \to \zeta_0} \check{\rho}(\zeta, z) = \frac{(\zeta_0 + 2)^4}{|\zeta_0 + 2|^4} = \frac{-\frac{1}{2}(1 + i\sqrt{3})(\overline{\zeta_0} + 2)^2}{(\overline{\zeta_0} + 2)^2} = -\frac{1}{2}(1 + i\sqrt{3}).$$

Then

$$\begin{aligned}
&\lim_{z \to \zeta_0} \left[\frac{1}{2\pi i} \int\limits_{\partial_3 P^+} \gamma(\zeta) \check{\rho}(\zeta, z)(-\frac{1}{2}(1 + i\sqrt{3})) \frac{z - \check{z}_1}{|\zeta - z|^2} ds_\zeta \right] = \\
&\lim_{z \to \zeta_0} \left[\frac{1}{2\pi i} \int\limits_{\partial_3 P^+} \gamma(\zeta)(-\frac{1}{2}(1 - i\sqrt{3}))^2 \check{\rho}(\zeta, z) \frac{z - \check{z}_1}{|\zeta - z|^2} ds_\zeta \right] = \\
&\lim_{z \to \zeta_0} \left[\frac{\sqrt{3} + i}{4\pi} \int\limits_{\partial_3 P^+} \gamma(\zeta)(-\frac{1}{2}(1 - i\sqrt{3})) \check{\rho}(\zeta, z) \frac{z - \check{z}_1}{|\zeta - z|^2} ds_\zeta \right] = \\
&-\frac{1}{2}(1 - i\sqrt{3}) \check{\rho}(\zeta_0, \zeta_0) \gamma(\zeta_0) = \gamma(\zeta_0).
\end{aligned}$$

This is also valid at the corner points $-2, -1 + \sqrt{3}$ if $\gamma(-2) = \gamma(-1 + i\sqrt{3}) = 0$.
For the other terms of b_3''

$$\lim_{z \to \zeta_0} \left\{ \frac{1}{2\pi i} \int\limits_{\partial_3 P^+} \gamma(\zeta) \sum_{\substack{m+n \in 2\mathbb{Z}, \\ m^2+n^2 \neq 0}} \left(\frac{3(\zeta - \omega_{mn} + 2)^2 [(z+2)^3 - (\check{z}_1 + 2)^3]}{[(\zeta - \omega_{mn} + 2)^3 - (z+2)^3][(\zeta - \omega_{mn} + 2)^3 - (\check{z}_1 + 2)^3]} \right) d\zeta \right\} = 0.$$

Finally, for $\zeta = \overline{\zeta}$ on $\partial_4 P^+$

$$(\overline{\zeta - \omega_{m,-n} + 2})^2 d\overline{\zeta} = (\zeta - \omega_{mn} + 2)^2 d\zeta,$$

$$\begin{aligned}
b_4'' &= \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \gamma(\zeta) \Bigg\{ \sum_{m+n \in 2\mathbb{Z}} \left[\frac{3(\zeta - \omega_{mn} + 2)^2}{(\zeta - \omega_{mn} + 2)^3 - (z+2)^3} - \frac{3(\zeta - \omega_{mn} + 2)^2}{(\zeta - \omega_{mn} + 2)^3 - 8} \right] d\zeta \\
&- \sum_{m+n \in 2\mathbb{Z}} \left[\frac{3(\overline{\zeta - \omega_{m,-n} + 2})^2}{(\overline{\zeta - \omega_{m,-n} + 2})^3 - (\overline{z+2})^3} - \frac{3(\overline{\zeta - \omega_{m,-n} + 2})^2}{(\overline{\zeta - \omega_{m,-n} + 2})^3 - 8} \right] d\overline{\zeta} \Bigg\}.
\end{aligned}$$

Then, similarly as before

$$\lim_{z \to \zeta_0} \frac{1}{2\pi i} \int\limits_{\partial_4 P^+} \gamma(\zeta) \sum_{m+n \in 2\mathbb{Z}} \left(\frac{3(\zeta - \omega_{mn} + 2)^2 [(z+2)^3 - (\check{z}_1 + 2)^3]}{[(\zeta - \omega_{mn} + 2)^3 - (z+2)^3][(\zeta - \omega_{mn} + 2)^3 - (\check{z}_1 + 2)^3]} \right) d\zeta = 0.$$

Therefore, the equality (4.48) for $\zeta_0 \in \partial_3 P^+$ and representation formula (4.43) hold. □

Lemma 4.3.4. *For $\zeta_0 \in \partial_4 P^+$, $\gamma \in C(\partial P^+; \mathbb{R})$*

$$\lim_{z \to \zeta_0} S_k \gamma(z) = \gamma(\zeta_0), \tag{4.50}$$

where the Schwarz operator for P^+ is given in (4.44) for any $k = 1, 2, 3$.

Proof. For z tending to $\zeta_0 \in \partial_4 P^+$, $\zeta_0 = \overline{\zeta_0} + 2i\sqrt{3}$ any of the representation formulas for the function $w(z)$ can be considered.
Let us take, for example, the form of (4.30) and consider the real part of the boundary integral

$$\begin{aligned}
\operatorname{Re} S_1 \gamma(z) &= \frac{1}{2\pi i} \int\limits_{\partial P^+} \gamma(\zeta) \Bigg\{ \sum_{m+n \in 2\mathbb{Z}} \left[\frac{3(\zeta - \omega_{mn} - 2)^2}{(\zeta - \omega_{mn} - 2)^3 - (z-2)^3} - \frac{3(\zeta - \omega_{mn} - 2)^2}{(\zeta - \omega_{mn} - 2)^3 + 8} \right] d\zeta \\
&- \sum_{m+n \in 2\mathbb{Z}} \left[\frac{3(\overline{\zeta - \omega_{m,n} - 2})^2}{(\overline{\zeta - \omega_{m,n} - 2})^3 - (\overline{z} - 2)^3} - \frac{3(\overline{\zeta - \omega_{m,n} - 2})^2}{(\overline{\zeta - \omega_{m,n} - 2})^3 + 8} \right] d\overline{\zeta} \Bigg\}.
\end{aligned}$$

Decompose as before into the sum $\operatorname{Re} S_4\gamma(z) = b_1''' + b_2''' + b_3''' + b_4'''$.

On the first boundary part $\partial_1 P^+$ with $\zeta = -\frac{1}{2}(1+i\sqrt{3})\overline{\zeta} + 3 + i\sqrt{3}$ the following relations are valid.

$$(\overline{\zeta - \omega_{mn} - 2})^2 d\overline{\zeta} = (\zeta - \omega_{kl} - 2)^2 d\zeta,$$

where if $k = -\frac{m+n}{2}$, $l = -\frac{3m-n}{2}$, then $m = -\frac{k+l}{2}$, $n = -\frac{3k-l}{2}$, $m+n \in 2\mathbb{Z}$, $k+l \in 2\mathbb{Z}$.
Also here $(\overline{z-2})^3 = (\overline{z}-2)^3 = (z_4-2)^3$, $z_4 \in \partial_4 P^+$. Then $z_4 = \overline{z}$ and

$$\begin{aligned} b_1''' &= \frac{1}{2\pi i}\int\limits_{\partial_1 P^+} \gamma(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z_4-2)^3}\right]\Bigg)d\zeta = \\ & \frac{1}{2\pi i}\int\limits_{\partial_1 P^+} \gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2[(z-2)^3-(z_4-2)^3]}{[(\zeta-\omega_{mn}-2)^3-(z-2)^3][(\zeta-\omega_{mn}-2)^3-(z_4-2)^3]}\right]d\zeta, \end{aligned}$$

letting $z \to \zeta_0 \in \partial_4 P^+$, $z_4 \to \zeta_0$, then

$$\lim_{z\to\zeta_0} b_1''' = 0.$$

For $\zeta \in \partial_2 P^+$, where $\zeta = \overline{\zeta} + 2i\sqrt{3}$

$$(\overline{\zeta - \omega_{m,2-n} - 2})^2 d\overline{\zeta} = (\zeta - \omega_{mn} - 2)^2 d\zeta,$$

then

$$\begin{aligned} b_2''' &= \frac{1}{2\pi i}\int\limits_{\partial_2 P^+} \gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]d\zeta \\ & - \sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{m,2-n}}-2)^2}{(\overline{\zeta-\omega_{m,2-n}}-2)^3-(\overline{z-2})^3} - \frac{3(\overline{\zeta-\omega_{m,2-n}}-2)^2}{(\overline{\zeta-\omega_{m,2-n}}-2)^3+8}\right]d\overline{\zeta} = \\ & \frac{1}{2\pi i}\int\limits_{\partial_2 P^+} \gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2[(z-2)^3-(z_4-2)^3]}{[(\zeta-\omega_{mn}-2)^3-(z-2)^3][(\zeta-\omega_{mn}-2)^3-(z_4-2)^3]}\right]d\zeta \end{aligned}$$

and letting z and z_4 simultaneously tend to $\zeta_0 \in \partial_4 P^+$, $\lim_{z\to\zeta_0} b_2''' = 0$.

On the third boundary part $\partial_3 P^+$ for $\zeta = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} - 3 + i\sqrt{3}$

$$(\overline{\zeta - \omega_{mn} - 2}) = -\frac{1}{2}(1+i\sqrt{3})(\zeta - \omega_{-\frac{m-n+4}{2},\frac{3m+n+4}{2}} - 2),$$

where if $k = -\frac{m-n+4}{2}$, $l = -\frac{3m+n+4}{2}$, then $m = -\frac{k-l+4}{2}$, $n = \frac{3k+l+4}{2}$, $m+n \in 2\mathbb{Z}$, $k+l \in 2\mathbb{Z}$ and

$$(\overline{\zeta - \omega_{mn} - 2})^2 d\overline{\zeta} = (\zeta - \omega_{kl} - 2)^2 d\zeta.$$

Thus

$$\begin{aligned} b_3''' &= \frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]d\zeta \\ & - \sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{m,2-n}}-2)^2}{(\overline{\zeta-\omega_{m,2-n}}-2)^3-(\overline{z-2})^3} - \frac{3(\overline{\zeta-\omega_{m,2-n}-2})^2}{(\overline{\zeta-\omega_{m,2-n}}-2)^3+8}\right]d\overline{\zeta} = \\ & \frac{1}{2\pi i}\int\limits_{\partial_3 P^+} \gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2[(z-2)^3-(z_4-2)^3]}{[(\zeta-\omega_{mn}-2)^3-(z-2)^3][(\zeta-\omega_{mn}-2)^3-(z_4-2)^3]}\right]d\zeta, \end{aligned}$$

letting again $z \to \zeta_0 \in \partial_4 P^+$, $z_4 \to \zeta_0$, one gets

$$\lim_{z\to\zeta_0} b_3''' = 0.$$

Finally, for the boundary part $\partial_4 P^+$, $\zeta = \overline{\zeta}$, the integral b_4''' is studied. Since

$$(\overline{\zeta - \omega_{m,-n} - 2})^2 d\overline{\zeta} = (\zeta - \omega_{mn} - 2)^2 d\zeta,$$

then

$$\begin{aligned}
b_4''' &= \frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\Bigg\{\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]d\zeta \\
&-\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{m,-n}-2})^2}{(\overline{\zeta-\omega_{m,-n}-2})^3-(\overline{z}-2)^3}-\frac{3(\overline{\zeta-\omega_{m,-n}-2})^2}{(\overline{\zeta-\omega_{m,-n}-2})^3+8}\right]d\overline{\zeta}\Bigg\} = \\
&\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3}\right]d\zeta.
\end{aligned}$$

For the case $m = n = 0$

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\left[\frac{3(\zeta-2)^2}{(\zeta-2)^3-(z-2)^3}-\frac{3(\overline{\zeta-2})^2}{(\overline{\zeta-2})^3-(\overline{z-2})^3}\right]d\zeta = \\
&\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\left[\frac{3(\zeta-2)^2[(z-2)^3-(\overline{z}-2)^3]}{|(\zeta-2)^3-(z-2)^3|^2}\right]d\zeta = \\
&\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\left(\frac{3(\zeta-2)^2(z-\overline{z})[(z-2)^2+(z-2)(\overline{z}-2)+(\overline{z}-2)^2]}{|\zeta-z|^2|(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2|^2}\right)d\zeta.
\end{aligned}$$

Denote

$$\tilde{\rho}(\zeta, z) = 3(\zeta-2)^2\frac{(z-2)^2+(z-2)(\overline{z}-2)+(\overline{z}-2)^2}{|(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2|^2},$$

then letting $z \to \zeta_0 \in \partial_4 P^+$, $\overline{z} \to \zeta_0$

$$\tilde{\rho}(\zeta_0, \zeta_0) = \lim_{z\to\zeta_0}\tilde{\rho}(\zeta, z) = \frac{(\zeta_0-2)^4}{|\zeta_0-2|^4} = 1.$$

Therefore

$$\lim_{z\to\zeta_0}\left[\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\tilde{\rho}(\zeta,z)\frac{z-\overline{z}}{|\zeta-z|^2}d\zeta\right] = \lim_{z\to\zeta_0}\left[\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\tilde{\rho}(\zeta,z)\frac{z-\overline{z}}{|\zeta-z|^2}ds_\zeta\right] = \tilde{\rho}(\zeta_0,\zeta_0)\gamma(\zeta_0) = \gamma(\zeta_0).$$

This holds as well for $\zeta_0 = \pm 2$ if $\gamma(\pm 2) = 0$. Also for the other terms of b_4'''

$$\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\Bigg\{\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2\neq 0}}\frac{3(\zeta-\omega_{mn}-2)^2[(z-2)^3-(\overline{z}-2)^3]}{[(\zeta-\omega_{mn}-2)^3-(z-2)^3][(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3]}\Bigg\}d\zeta = 0.$$

Thus, the boundary condition (4.50) holds for $\zeta_0 \in \partial_4 P^+$ □

Theorem 4.3.1. *The Schwarz problem*

$$\begin{aligned}
&w_{\overline{z}} = f \text{ in } P^+,\ f \in L_p(P^+;\mathbb{C}),\ p > 2, \\
&\operatorname{Re} w = \gamma \text{ on } \partial P^+,\ \gamma \in C(\partial P^+;\mathbb{C}),\ \gamma(\zeta) = 0 \text{ for } \zeta \in \{\pm 2, \pm 1 + i\sqrt{3}\},
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
&-\frac{1}{\pi i}\int\limits_{\partial_1 P^+}\operatorname{Im} w(\zeta)\frac{\sqrt{3}}{(2\xi-3)^2+3}ds_\zeta + \frac{1}{\pi i}\int\limits_{\partial_2 P^+}\operatorname{Im} w(\zeta)\frac{\sqrt{3}}{\xi^2+3}ds_\zeta \\
&-\frac{1}{\pi i}\int\limits_{\partial_3 P^+}\operatorname{Im} w(\zeta)\frac{\sqrt{3}}{(2\xi+3)^2+3}ds_\zeta = c \text{ for } c \in \mathbb{R}
\end{aligned} \tag{4.52}$$

is uniquely solvable in the space of functions with generalized derivatives with respect to $\overline{z}$ *by*

$$\begin{aligned} w(z) \;=\; & \frac{1}{2\pi i}\int\limits_{\partial P^+}\gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}} 2[q^k_{mn}(\zeta,z)-q^k_{mn}(\zeta,0)]d\zeta - \frac{1}{\pi i}\int\limits_{\partial_1 P^+}\gamma(\zeta)\frac{2\xi-3}{(2\xi-3)^2+3}ds_\zeta \\ & +\frac{1}{\pi i}\int\limits_{\partial_2 P^+}\gamma(\zeta)\frac{\xi}{\xi^2+3}ds_\zeta - \frac{1}{\pi i}\int\limits_{\partial_3 P^+}\gamma(\zeta)\frac{2\xi+3}{(2\xi+3)^2+3}ds_\zeta + \frac{1}{\pi i}\int\limits_{\partial_4 P^+}\gamma(\zeta)\frac{1}{\xi}ds_\zeta + ic \\ & -\frac{1}{\pi}\int\limits_{P^+}\Bigg\{w_{\overline{\zeta}}(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^k_{mn}(\zeta,z)-q^k_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg) \\ & -\overline{w_{\overline{\zeta}}(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q_{mn}(\overline{\zeta},z)-q_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta \end{aligned} \tag{4.53}$$

for $k=1,2,3$.

Proof. By Theorem 4.2.1, if the solution of the Schwarz problem exists, it must be of the form (4.53). For verifying the differential equation in (4.51) for (4.53) we observe that the boundary integral is an analytic function. Let us study each of the representation forms of $w(z)$.

Consider first the representation formula (4.30) and denote

$$\begin{aligned} \tilde{T}_1 f(z) \;=\; & -\frac{1}{\pi}\int\limits_{P^+}\Bigg\{f(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\zeta,z)-q^1_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg) \\ & -\overline{f(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^1_{mn}(\overline{\zeta},z)-q^1_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta,\ z\in P^+. \end{aligned} \tag{4.54}$$

The first sum can be written as

$$\frac{3(\zeta-2)^2}{(\zeta-2)^3-(z-2)^3}-\frac{3(\zeta-2)^2}{(\zeta-2)^3+8}+\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right].$$

Moreover,

$$\begin{aligned} &\frac{3(\zeta-2)^2}{(\zeta-2)^3-(z-2)^3} = \frac{3(\zeta-2)^2}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} = \\ &\frac{1}{\zeta-z}+\left(\frac{3(\zeta-2)^2}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]}-\frac{1}{\zeta-z}\right) = \\ &\frac{1}{\zeta-z}+\frac{2(\zeta-2)^2-(\zeta-2)(z-2)-(z-2)^2}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} = \\ &\frac{1}{\zeta-z}+\frac{(\zeta-z)(2\zeta+z-6)}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} = \\ &\frac{1}{\zeta-z}+\frac{2(\zeta-3)+z}{(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2} \end{aligned}$$

Then

$$\begin{aligned} \tilde{T}_1 f(z) \;=\; & -\frac{1}{\pi}\int\limits_{P^+}\Bigg\{f(\zeta)\Bigg(\frac{1}{\zeta-z}+\frac{2(\zeta-3)+z}{(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2} \\ & -\frac{3(\zeta-2)^2}{(\zeta-2)^3+8}+\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\left[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\right]+\frac{1}{\zeta}\Bigg) \\ & -\overline{f(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3-(z-2)^3}-\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3+8}\right]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta. \end{aligned}$$

and this function is analytic in P^+ up to the Pompeiu operator with respect to z. Thus

$$\partial_{\overline{z}}[\tilde{T}_1 f(z)] = f(z),\, z \in P^+$$

provides the solution in a weak sense for the differential equation in (4.51).
For the boundary behavior of the area integral we take the real part of $\tilde{T}_1 f(z)$

$$\begin{aligned}
\operatorname{Re}\tilde{T}_1 f(z) &= -\frac{1}{2\pi}\int_{P^+}\Bigg\{f(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\Bigg]+\frac{1}{\zeta}\Bigg)\\
&+\overline{f(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3-(\overline{z}-2)^3}-\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3+8}\Bigg]+\frac{1}{\overline{\zeta}}\Bigg)\\
&-\overline{f(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3-(z-2)^3}-\frac{3(\overline{\zeta-\omega_{mn}-2})^2}{(\overline{\zeta-\omega_{mn}-2})^3+8}\Bigg]+\frac{1}{\overline{\zeta}}\Bigg)\\
&-f(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3}-\frac{3(\zeta-\omega_{mn}-2)^2}{(\zeta-\omega_{mn}-2)^3+8}\Bigg]+\frac{1}{\zeta}\Bigg)\Bigg\}d\xi d\eta.
\end{aligned}$$

By calculations in Lemma 4.3.1, substituting the relation $(\overline{z-2})^3=(z_1-2)^3$ and similarly taking the limit when $z\to\zeta_0$, shows that the integral becomes 0.

Similarly we study the representation formula (4.34) and denote

$$\begin{aligned}
\tilde{T}_2 f(z) &= -\frac{1}{\pi}\int_{P^+}\Bigg\{f(\zeta)\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\zeta,z)-q^2_{mn}(\zeta,0)]+\frac{1}{\zeta}\Bigg)\\
&-\overline{f(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}[q^2_{mn}(\overline{\zeta},z)-q^2_{mn}(\overline{\zeta},0)]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta,\; z\in P^+.
\end{aligned}\tag{4.55}$$

Then, rewriting the first term of the sum in the same manner,

$$\begin{aligned}
&\frac{3(\zeta+1-i\sqrt{3})}{(\zeta+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}=\\
&\frac{1}{\zeta-z}+\frac{2(\zeta+1-i\sqrt{3})^2-(\zeta+1-i\sqrt{3})(z+1-i\sqrt{3})-(z+1-i\sqrt{3})^2}{(\zeta-z)[(\zeta+1-i\sqrt{3})^2+(\zeta+1-i\sqrt{3})(z+1-i\sqrt{3})+(z+1-i\sqrt{3})^2]}=\\
&\frac{1}{\zeta-z}+\frac{(\zeta-z)[2\zeta+3-3i\sqrt{3}+z]}{(\zeta-z)[(\zeta+1-i\sqrt{3})^2+(\zeta+1-i\sqrt{3})(z+1-i\sqrt{3})+(z+1-i\sqrt{3})^2]}
\end{aligned}$$

then

$$\begin{aligned}
\tilde{T}_2 f(z) &= -\frac{1}{\pi}\int_{P^+}\Bigg\{f(\zeta)\Bigg(\frac{1}{\zeta-z}-\frac{3(\zeta+1-i\sqrt{3})^2}{(\zeta+1-i\sqrt{3})^3+8}\\
&\frac{2\zeta+3-3i\sqrt{3}+z}{(\zeta+1-i\sqrt{3})^2+(\zeta+1-i\sqrt{3})(z+1-i\sqrt{3})+(z+1-i\sqrt{3})^2}\\
&+\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\Bigg[\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3}-\frac{3(\zeta-\omega_{mn}+1-i\sqrt{3})^2}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3+8}\Bigg]+\frac{1}{\zeta}\Bigg)\\
&-\overline{f(\zeta)}\Bigg(\sum_{m+n\in 2\mathbb{Z}}\Bigg[\frac{3(\overline{\zeta-\omega_{mn}+1-i\sqrt{3}})^2}{(\overline{\zeta-\omega_{mn}+1-i\sqrt{3}})^3-(z+1-i\sqrt{3})^3}\\
&-\frac{3(\overline{\zeta-\omega_{mn}+1-i\sqrt{3}})^2}{(\overline{\zeta-\omega_{mn}+1-i\sqrt{3}})^3+8}\Bigg]+\frac{1}{\overline{\zeta}}\Bigg)\Bigg\}d\xi d\eta.
\end{aligned}$$

which is an analytic function of $z \in P^+$ up to the Pompeiu operator and $\partial_{\overline{z}}[\tilde{T}_2 f(z)] = f(z)$ is a weak solution of the differential equation in (4.51). We take the real part of $\tilde{T}_2 f(z)$ to check the boundary behavior

$$\begin{aligned}
\operatorname{Re} \tilde{T}_2 f(z) &= -\frac{1}{2\pi} \int_{P^+} \Bigg\{ f(\zeta) \Bigg(\sum_{m+n \in 2\mathbb{Z}} [q^2_{mn}(\zeta, z) - q^2_{mn}(\zeta, 0)] + \frac{1}{\zeta} \Bigg) \\
&+ \overline{f(\zeta)} \Bigg(\sum_{m+n \in 2\mathbb{Z}} [\overline{q^2_{mn}(\zeta, z)} - \overline{q^2_{mn}(\zeta, 0)}] + \frac{1}{\overline{\zeta}} \Bigg) - \overline{f(\zeta)} \Bigg(\sum_{m+n \in 2\mathbb{Z}} [q^2_{mn}(\overline{\zeta}, z) - q^2_{mn}(\overline{\zeta}, 0)] + \frac{1}{\overline{\zeta}} \Bigg) \\
&- f(\zeta) \Bigg(\sum_{m+n \in 2\mathbb{Z}} [\overline{q^2_{mn}(\overline{\zeta}, z)} - \overline{q^2_{mn}(\overline{\zeta}, 0)}] + \frac{1}{\zeta} \Bigg) \Bigg\} d\xi d\eta.
\end{aligned}$$

Calculating in the same way as in Lemma 4.3.2 shows that $\lim_{z \to \zeta_0} \tilde{T}_2 f(z)$ tends to 0.

Finally, for the representation form (4.43) we take

$$\begin{aligned}
\tilde{T}_3 f(z) &= -\frac{1}{\pi} \int_{P^+} \Bigg\{ f(\zeta) \Bigg(\sum_{m+n \in 2\mathbb{Z}} [q^3_{mn}(\zeta, z) - q^3_{mn}(\zeta, 0)] + \frac{1}{\zeta} \Bigg) \\
&- \overline{f(\zeta)} \Bigg(\sum_{m+n \in 2\mathbb{Z}} [q^3_{mn}(\overline{\zeta}, z) - q^3_{mn}(\overline{\zeta}, 0)] + \frac{1}{\overline{\zeta}} \Bigg) \Bigg\} d\xi d\eta, \; z \in P^+.
\end{aligned} \tag{4.56}$$

After similar calculations one gets

$$\frac{3(\zeta+2)^2}{(\zeta+2)^3 - (z+2)^3} = \frac{1}{\zeta - z} + \frac{2(\zeta+2)^2 - (\zeta+2)(z+2) - (z+2)^2}{(\zeta - z)[(\zeta+2)^2 + (\zeta+2)(z+2) + (z+2)^2]} = \frac{1}{\zeta - z} + \frac{(\zeta - z)[2\zeta + z + 6]}{(\zeta - z)[(\zeta+2)^2 + (\zeta+2)(z+2) + (z+2)^2]}$$

and

$$\begin{aligned}
\tilde{T}_3 f(z) &= -\frac{1}{\pi} \int_{P^+} \Bigg\{ f(\zeta) \Bigg(\frac{1}{\zeta - z} + \frac{2\zeta + z + 6}{(\zeta+2)^2 + (\zeta+2)(z+2) + (z+2)^2} \\
&- \frac{3(\zeta+2)^2}{(\zeta+2)^3 - 8} + \sum_{\substack{m+n \in 2\mathbb{Z}, \\ m^2+n^2>0}} \Bigg[\frac{3(\zeta - \omega_{mn} + 2)^2}{(\zeta - \omega_{mn} + 2)^3 - (z+2)^3} - \frac{3(\zeta - \omega_{mn} + 2)^2}{(\zeta - \omega_{mn} + 2)^3 - 8} \Bigg] + \frac{1}{\zeta} \Bigg) \\
&- \overline{f(\zeta)} \Bigg(\sum_{m+n \in 2\mathbb{Z}} \Bigg[\frac{3(\overline{\zeta - \omega_{mn} + 2})^2}{(\overline{\zeta - \omega_{mn} + 2})^3 - (z+2)^3} - \frac{3(\overline{\zeta - \omega_{mn} + 2})^2}{(\overline{\zeta - \omega_{mn} + 2})^3 - 8} \Bigg] + \frac{1}{\overline{\zeta}} \Bigg) \Bigg\} d\xi d\eta.
\end{aligned}$$

This function is also analytic with respect to $z \in P^+$ up to the Pompeiu operator and provides a weak solution to $w_{\overline{z}} = f$ in (4.51) because $\partial_{\overline{z}}[\tilde{T}_3 f(z)] = f(z)$, $z \in P^+$. To study the boundary behavior of $\tilde{T}_3 f(z)$ we take its real part

$$\begin{aligned}
\operatorname{Re} \tilde{T}_3 f(z) &= -\frac{1}{2\pi} \int_{P^+} \Bigg\{ f(\zeta) \Bigg(\sum_{m+n \in 2\mathbb{Z}} [q^3_{mn}(\zeta, z) - q^3_{mn}(\zeta, 0)] + \frac{1}{\zeta} \Bigg) \\
&+ \overline{f(\zeta)} \Bigg(\sum_{m+n \in 2\mathbb{Z}} [\overline{q^3_{mn}(\zeta, z)} - \overline{q^3_{mn}(\zeta, 0)}] + \frac{1}{\overline{\zeta}} \Bigg) - \overline{f(\zeta)} \Bigg(\sum_{m+n \in 2\mathbb{Z}} [q^3_{mn}(\overline{\zeta}, z) - q^3_{mn}(\overline{\zeta}, 0)] + \frac{1}{\overline{\zeta}} \Bigg) \\
&- f(\zeta) \Bigg(\sum_{m+n \in 2\mathbb{Z}} [\overline{q^3_{mn}(\overline{\zeta}, z)} - \overline{q^3_{mn}(\overline{\zeta}, 0)}] + \frac{1}{\zeta} \Bigg) \Bigg\} d\xi d\eta.
\end{aligned}$$

Using computations in Lemma 4.3.3 and formula (4.49), one gets

$$\lim_{z \to \zeta_0} \tilde{T}_3 f(z) = 0.$$

□

Chapter 5

Harmonic Dirichlet Problem for the Poisson equation

In this Chapter, the harmonic Green function for the half hexagon P^+ is constructed and the Dirichlet problem for the Poisson equation is solved explicitly.

5.1 Green representation formula

The method of reflections used to the Schwarz-Poisson representation formula in Chapter 4 can be also applied in order to find the harmonic Green function. As it was mentioned before, all the reflection points can be described either by those given in (4.4) or (4.5). Thus the meromorphic function mentioned in section 4.1 can be presented by

$$\begin{aligned} B_1(z,\zeta) &= \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\overline{z}-\omega_{mn})(\zeta-z_1-\omega_{mn})(\zeta-\overline{z_2}-\omega_{mn})}{(\zeta-z-\omega_{mn})(\zeta-\overline{z_1}-\omega_{mn})(\zeta-z_2-\omega_{mn})} = \\ & \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}, \end{aligned} \tag{5.1}$$

where $z_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3}$, $z_2 = -\frac{1}{2}(1+i\sqrt{3})z+3+i\sqrt{3}$, or by

$$\begin{aligned} B_2(z,\zeta) &= \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\overline{z}-\omega_{mn})(\zeta-\check{z_1}-\omega_{mn})(\zeta-\overline{\check{z_2}}-\omega_{mn})}{(\zeta-z-\omega_{mn})(\zeta-\overline{\check{z_1}}-\omega_{mn})(\zeta-\check{z_2}-\omega_{mn})} = \\ & \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}+2)^3-(\overline{z}+2)^3}{(\zeta-\omega_{mn}+2)^3-(z+2)^3}, \end{aligned} \tag{5.2}$$

where $\check{z_1} = -\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3}$, $\check{z_2} = -\frac{1}{2}(1-i\sqrt{3})z-3+i\sqrt{3}$. Here z is considered as a parameter and ζ is the variable.

Another representation will be needed. To deduce it the variable ζ and the parameter z in $B_1(z,\zeta)$ are rotated by the angle $\frac{\pi}{3}$, i.e. they are multiplied by $-\frac{1}{2}(1+i\sqrt{3})$. This gives

$$\begin{aligned} & B_1(-\frac{1}{2}(1+i\sqrt{3})z, -\frac{1}{2}(1+i\sqrt{3})\zeta) = \\ & \prod_{m+n\in 2\mathbb{Z}} \frac{(-\frac{1}{2}(1+i\sqrt{3})\zeta-\omega_{mn}-2)^3-(-\frac{1}{2}(1-i\sqrt{3})\overline{z}-2)^3}{(-\frac{1}{2}(1+i\sqrt{3})\zeta-\omega_{mn}-2)^3-(-\frac{1}{2}(1+i\sqrt{3})z-2)^3}. \end{aligned}$$

Here

$$\begin{aligned} & -\frac{1}{2}(1+i\sqrt{3})\zeta-\omega_{mn}-2 = -\frac{1}{2}(1+i\sqrt{3})[\zeta-\omega_{-\frac{m+n}{2},\frac{3m-n}{2}}+1-i\sqrt{3}], \\ & -\frac{1}{2}(1-i\sqrt{3})\overline{z}-2 = -\frac{1}{2}(1-i\sqrt{3})[\overline{z}+1+i\sqrt{3}], \\ & -\frac{1}{2}(1+i\sqrt{3})z-2 = -\frac{1}{2}(1+i\sqrt{3})[z+1-i\sqrt{3}]. \end{aligned}$$

Since $[-\frac{1}{2}(1+i\sqrt{3})]^3 = [-\frac{1}{2}(1-i\sqrt{3})]^3 = 1$ and

$$-\frac{m+n}{2} = k,\ \frac{3m-n}{2} = l,\ m = \frac{l-k}{2},\ n = -\frac{3k+l}{2},\ m+n\in 2\mathbb{Z},\ k+l\in 2\mathbb{Z}$$

$$B_1(-\frac{1}{2}(1+i\sqrt{3})z, -\frac{1}{2}(1+i\sqrt{3})\zeta) =$$
$$\prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(\overline{z}+1+i\sqrt{3})^3}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3-(z+1-i\sqrt{3})^3} = B_3(z,\zeta), \tag{5.3}$$

which becomes 1 on the boundary $\partial_2 P$, where $z - i\sqrt{3} = \overline{z} + i\sqrt{3}$.

Lemma 5.1.1. *The equality*

$$B_1(z,\zeta) = B_2(z,\zeta) = B_3(z,\zeta) \tag{5.4}$$

holds for $(z,\zeta)\in P^+\times\partial P^+$.

Proof. The functions $B_1(z,\cdot)$ and $B_2(z,\cdot)$ have the same poles and zeros in $\mathbb{C}$. Their quotient is an entire bounded function, which becomes 1 at infinity. $B_3(z,\zeta)$ in (5.3) is obtained from $B_1(z,\zeta)$.
Consider in detail the factors of the product $B_1(-\frac{1}{2}(1+i\sqrt{3})z, -\frac{1}{2}(1+i\sqrt{3})\zeta)$. The numerator is

$$\begin{aligned}
&N = (-\frac{1}{2}(1+i\sqrt{3})\zeta + \frac{1}{2}(1-i\sqrt{3})\overline{z} - \omega_{mn}) \times\\
&(-\frac{1}{2}(1+i\sqrt{3})\zeta + \frac{1}{2}(1+i\sqrt{3})(-\frac{1}{2}(1-i\sqrt{3}))\overline{z} - \omega_{mn} + 3 + i\sqrt{3}) \times\\
&(-\frac{1}{2}(1+i\sqrt{3})\zeta + \frac{1}{2}(1-i\sqrt{3})(-\frac{1}{2}(1-i\sqrt{3}))\overline{z} - \omega_{mn} + 3 - i\sqrt{3}) =\\
&(\zeta - \overline{z} - \omega_{-\frac{m+n}{2},\frac{3m-n-4}{2}})(\zeta + \frac{1}{2}(1+i\sqrt{3})\overline{z} - \omega_{\frac{m+n-2}{2},\frac{3m-n+2}{2}} - 3 - i\sqrt{3}) \times\\
&(\zeta + \frac{1}{2}(1-i\sqrt{3})\overline{z} - \omega_{-\frac{m+n-2}{2},\frac{3m-n-2}{2}} - 3 + i\sqrt{3}).
\end{aligned}$$

If $-\frac{m+n}{2} = k$, $\frac{3m-n}{2} = l$, $m+n\in 2\mathbb{Z}$, then $m = -\frac{k-l}{2}$, $n = -\frac{3k+l}{2}$, $k+l\in 2\mathbb{Z}$. Then

$$N = (\zeta - \overline{z} - \omega_{k,l-2})(\zeta + \frac{1}{2}(1+i\sqrt{})\overline{z} - 3 - i\sqrt{3} - \omega_{-k-1,l+1})(\zeta + \frac{1}{2}(1-i\sqrt{3})\overline{z} - 3 + i\sqrt{3} - \omega_{k+1,l-1}).$$

For the denominator

$$\begin{aligned}
&D = (-\frac{1}{2}(1+i\sqrt{3})\zeta + \frac{1}{2}(1-i\sqrt{3})z - \omega_{mn}) \times\\
&(-\frac{1}{2}(1+i\sqrt{3})\zeta + \frac{1}{2}(1-i\sqrt{3})(-\frac{1}{2}(1+i\sqrt{3}))z - 3 + i\sqrt{3} - \omega_{mn}) \times\\
&(-\frac{1}{2}(1+i\sqrt{3})\zeta + \frac{1}{2}(1+i\sqrt{3})(-\frac{1}{2}(1+i\sqrt{3}))z - 3 - i\sqrt{3} - \omega_{mn}) =\\
&(\zeta - z - \omega_{-\frac{m+n}{2},\frac{3m-n}{2}})(\zeta + \frac{1}{2}(1+i\sqrt{3})z - 3 - i\sqrt{3} - \omega_{-\frac{m+n}{2},\frac{3m-n+4}{2}}) \times\\
&(\zeta + \frac{1}{2}(1-i\sqrt{3})z - 3 + i\sqrt{3} - \omega_{-\frac{m+n-2}{2},\frac{3m-n+2}{2}})
\end{aligned}$$

and similarly

$$D = (\zeta - z - \omega_{kl})(\zeta + \frac{1}{2}(1+i\sqrt{3})z - 3 - i\sqrt{3} - \omega_{k,l+2})(\zeta + \frac{1}{2}(1-i\sqrt{})z - 3 + i\sqrt{3} - \omega_{k+1,l+1}).$$

Therefore $B_3(z,\zeta)$ and $B_1(z,\zeta)$ have the same zeros and poles. Their quotient is an entire function tending to 1 at ∞. In the same manner the equality $B_2(z,\zeta) = B_3(z,\zeta)$ can be shown. Thus, (5.4) is valid. □

The harmonic Green function for the half hexagon P^+ is

$$G_1(z,\zeta) = \log|B_1(z,\zeta)|^2 = \log|B_2(z,\zeta)|^2 = \log|B_3(z,\zeta)|^2. \tag{5.5}$$

The Green function must satisfy the following conditions [6]
1^0.$G_1(z,\zeta)$ is harmonic in $P^+\backslash\{ z\}$
2^0.$G_1(z,\zeta)+\log|\zeta-z|^2$ is harmonic in $\zeta\in P^+$ for any $z\in P^+$
3^0. $\lim_{\zeta\to\partial P^+} G(z,\zeta)=0$ for any $z\in P^+$
and the additional properties:
4^0.$G_1(z,\zeta)=G(\zeta,z)$, z and ζ in P^+, $z\neq\zeta$
5^0.$G_1(z,\zeta)>0$, z and ζ in P^+, $z\neq\zeta$.

Lemma 5.1.2. *The infinite product*

$$\prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} \tag{5.6}$$

converges, where $\omega_{mn}=3m+i\sqrt{3}n,\ m+n\in 2\mathbb{Z}$.

Proof. By the definition of the product convergence, the sum:

$$\sum_{m+n\in 2\mathbb{Z}}\left[\left(\frac{(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}\right)-1\right]=\sum_{m+n\in 2\mathbb{Z}}\left[\frac{(z-2)^3-(\overline{z}-2)^3}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}\right]$$

has to be investigated. Consider the sum

$$\sum_{m+n\in 2\mathbb{Z}}\frac{1}{|(\zeta-\omega_{mn}-2)^3-(z-2)^3|}, \tag{5.7}$$

where for $\omega_{mn}=3m+i\sqrt{3}n$, $m^2+n^2>0$,

$$\begin{aligned}
&|(\zeta-\omega_{mn}-2)^3-(z-2)^3|\geq|(\zeta-\omega_{mn}-2)^3|-|(z-2)^3|\geq\\
&(|\omega_{mn}|-|\zeta-2|)^3-|z-2|^3\geq(\frac{1}{2}|\omega_{mn}|)^3-|z-2|^3\geq(\frac{1}{3}|\omega_{mn}|)^3 \text{ for } m,n\geq 3,\\
&|\omega_{mn}|^2=|3m+i\sqrt{3}n|^2=(3m+i\sqrt{3}n)(3m-i\sqrt{3}n)=9m^2+3n^2\geq 3(m^2+n^2)\\
&m^2+n^2\geq 2mn,\ m^2+n^2\geq\frac{1}{2}(|m|+|n|)^2,\\
&|\omega_{mn}|^3\geq(|m|+|n|)^3,
\end{aligned}$$

therefore

$$|\omega_{mn}|\geq\sqrt{\frac{3}{2}}(|m|+|n|)>|m|+|n|,\ \frac{1}{|\omega_{mn}|^3}\leq\frac{1}{(|m|+|n|)^3}$$

and by Lemma 1.5. ([39], p.268) the series $\sum\limits_{(n,m)\neq(0,0)}\frac{1}{(|m|+|n|)^r}, r>2$ is convergent, therefore the sum (5.7) is convergent. □

By the properties 1^0-3^0 the Green function $G_1(z,\zeta)$ is uniquely defined. Obviously, $G_1(z,\zeta)$ is harmonic in $\zeta\in P^+\backslash\{z\}$ as $B_1(z,\zeta)$ is analytic in P^+ up to a single pole at z. Adding $\log|\zeta-z|^2$ gives a harmonic function of $\zeta\in P^+$.

Lemma 5.1.3. *The function* $G_1(z,\zeta)$ *has vanishing boundary values on* ∂P^+, *i.e.*

$$\lim_{\zeta\to\zeta_0\in\partial P^+} G_1(z,\zeta)=0. \tag{5.8}$$

Proof. By the symmetry property 4^0 it is sufficient to prove

$$\lim_{z\to z_0\in\partial P^+} G_1(z,\zeta)=0.$$

Consider the boundary behavior of $B_1(z,\zeta), B_2(z,\zeta), B_3(z,\zeta)$ on the boundary parts of ∂P^+. On $\partial_1 P^+$

$$\begin{aligned}
&B_1(z,\zeta) = \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}-2)^3 - (\overline{z}-2)^3}{(\zeta-\omega_{mn}-2)^3 - (z-2)^3}\\
&z = z_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{z} + 3 + i\sqrt{3},\\
&z-2 = -\frac{1}{2}(1+i\sqrt{3})(\overline{z}-2),\ \ (z-2)^2 = -\frac{1}{2}(1-i\sqrt{3})(\overline{z}-2)^2,\\
&(z-2)^3 = (\overline{z}-2)^3
\end{aligned}$$

and $B_1(z,\zeta) = 1$ on this boundary. On $\partial_2 P^+$

$$\begin{aligned}
&B_3(z,\zeta) = \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}+1-i\sqrt{3})^3 - (\overline{z}+1+i\sqrt{3})^3}{(\zeta-\omega_{mn}+1-i\sqrt{3})^3 - (z+1-i\sqrt{3})^3},\\
&\text{on}\, \partial_2 P^+, [1+i\sqrt{3}, -1+i\sqrt{3}],\ \ z = z_2 = \overline{z} + 2i\sqrt{z},\ z - i\sqrt{3} = \overline{z} + i\sqrt{3},
\end{aligned}$$

and $B_3(z,\zeta) = 1$. On $\partial_3 P^+$

$$\begin{aligned}
&B_2(z,\zeta) = \prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}+2)^3 - (\overline{z}+2)^3}{(\zeta-\omega_{mn}+2)^3 - (z+2)^3}\\
&z = z_3 = -\frac{1}{2}(1-i\sqrt{3})\overline{z} - 3 + i\sqrt{3},\\
&z+2 = -\frac{1}{2}(1-i\sqrt{3})(\overline{z}+2),\ \ (z+2)^2 = -\frac{1}{2}(1+i\sqrt{3})(z+2)^2,\\
&(z+2)^3 = (\overline{z}+2)^3,
\end{aligned}$$

then $B_2(z,\zeta) = 1$ here.

On the boundary $\partial_4 P^+$, where $z = \overline{z}$ it is obvious that each of B_1, B_2, B_3 becomes 1 there. Thus, $G_1(z,\zeta) = 0$ on the all parts of the boundary. □

The symmetry property 4^0 of the Green function is a consequence from the properties $1^0 - 3^0$.

Theorem 5.1.1. *[6] Any $w \in C^2(P^+;\mathbb{C}) \cap C^1(\overline{P^+};\mathbb{C})$ can be represented as*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial P^+} w(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta - \frac{1}{\pi}\int\limits_{P^+} w_{\zeta\overline{\zeta}}(\zeta) G_1(z,\zeta)d\xi d\eta,$$

where s_ζ is the arc length parameter on ∂P^+ with respect to the variable ζ and $G(z,\zeta) = \frac{1}{2}G_1(z,\zeta)$ is the harmonic Green function for P^+.

The Poisson kernel for the Green representation formula can be obtained as the outward normal derivative of the Green function on the boundary.
For the right-hand side $\partial_1 P^+$ consider for $\zeta \in \partial P^+$, $z \in P^+$,

$$G_1(z,\zeta) = \log\left|\prod_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3 - (\overline{\zeta}-2)^3}{(z-\omega_{mn}-2)^3 - (\zeta-2)^3}\right|^2.$$

Differentiation gives

$$\begin{aligned}
\partial_\zeta G_1(z,\zeta) &= \sum_{m+n\in 2\mathbb{Z}} \left\{ \frac{3(\zeta-2)^2}{(z-\omega_{mn}-2)^3 - (\zeta-2)^3} - \frac{3(\zeta-2)^2}{\overline{(z-\omega_{mn}-2)}^3 - (\zeta-2)^3} \right\},\\
\partial_{\overline{\zeta}} G_1(z,\zeta) &= \sum_{m+n\in 2\mathbb{Z}} \left\{ \frac{3(\overline{\zeta}-2)^2}{\overline{(z-\omega_{mn}-2)}^3 - (\overline{\zeta}-2)^3} - \frac{3(\overline{\zeta}-2)^2}{(z-\omega_{mn}-2)^3 - (\overline{\zeta}-2)^3} \right\}.
\end{aligned}$$

On the boundary $\partial_1 P^+$ the outward normal derivative is $\partial_{\nu_\zeta} = (\frac{\sqrt{3}}{2} + \frac{i}{2})\partial_\zeta + (\frac{\sqrt{3}}{2} - \frac{i}{2})\partial_{\overline{\zeta}}$, so we have:

$$\partial_{\nu_\zeta} G_1(z,\zeta) = -3(\sqrt{3}+i)(\zeta-2)^2 \sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3-(\overline{z-\omega_{mn}-2})^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2}, \tag{5.9}$$

taking into account that here $\zeta - 2 = -\frac{1}{2}(1+i\sqrt{3})(\overline{\zeta}-2)$, $(\zeta-2)^3 = (\overline{\zeta}-2)^3$.

For the boundary part $\partial_4 P^+$ the outward normal derivative is $\partial_{\nu_\zeta} = -i(\partial_\zeta - \partial_{\overline{\zeta}})$, $\zeta = \overline{\zeta}$, then

$$\partial_{\nu_\zeta} G_1(z,\zeta) = 6i(\zeta-2)^2 \sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3-(\overline{z-\omega_{mn}-2})^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2}. \tag{5.10}$$

For the left-hand side $\partial_3 P^+$ consider

$$G_1(z,\zeta) = \log\left|\prod_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+2)^3-(\overline{\zeta}+2)^3}{(z-\omega_{mn}+2)^3-(\zeta+2)^3}\right|^2.$$

Differentiation gives

$$\partial_\zeta G_1(z,\zeta) = \sum_{m+n\in 2\mathbb{Z}} \left\{ \frac{3(\zeta+2)^2}{(z-\omega_{mn}+2)^3-(\zeta+2)^3} - \frac{3(\zeta+2)^2}{(\overline{z-\omega_{mn}+2})^3-(\zeta+2)^3} \right\},$$

$$\partial_{\overline{\zeta}} G_1(z,\zeta) = \sum_{m+n\in 2\mathbb{Z}} \left\{ \frac{3(\overline{\zeta}+2)^2}{(\overline{z-\omega_{mn}+2})^3-(\overline{\zeta}+2)^3} - \frac{3(\overline{\zeta}+2)^2}{(z-\omega_{mn}+2)^3-(\overline{\zeta}+2)^3} \right\}.$$

On the boundary $\partial_3 P^+$ the outward normal derivative is $\partial_{\nu_\zeta} = (\frac{\sqrt{3}}{2}-\frac{i}{2})\partial_\zeta + (\frac{\sqrt{3}}{2}+\frac{i}{2})\partial_{\overline{\zeta}}$, also here $\zeta = \check{\zeta}_1 = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} - 3 + i\sqrt{3}$ and $(\zeta+2)^3 = (\overline{\zeta}+2)^3$, then

$$\partial_{\nu_\zeta} G_1(z,\zeta) = -3(\sqrt{3}-i)(\zeta+2)^2 \sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+2)^3-(\overline{z-\omega_{mn}+2})^3}{|(z-\omega_{mn}+2)^3-(\zeta+2)^3|^2}. \tag{5.11}$$

For the upper boundary part $\partial_2 P^+$ we take

$$G_1(z,\zeta) = \log \prod_{m+n\in 2\mathbb{Z}} \left| \frac{(z-\omega_{mn}+1-i\sqrt{3})^3-(\overline{\zeta}+1+i\sqrt{3})^3}{(z-\omega_{mn}+1-i\sqrt{3})^3-(\zeta+1-\sqrt{3})^3} \right|^2. \tag{5.12}$$

Then

$$\begin{aligned}
\partial_\zeta G_1(z,\zeta) \quad &= \quad \sum_{m+n\in 2\mathbb{Z}} \left\{ \frac{3(\zeta+1-i\sqrt{3})^2}{(z-\omega_{mn}+1-i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3} \right. \\
&\qquad \left. - \frac{3(\zeta+1-i\sqrt{3})^2}{(\overline{z-\omega_{mn}+1-i\sqrt{3}})^3-(\zeta+1-i\sqrt{3})^3} \right\} \\
\partial_{\overline{\zeta}} G_1(z,\zeta) \quad &= \quad \sum_{m+n\in 2\mathbb{Z}} \left\{ \frac{3(\overline{\zeta}+1+i\sqrt{3})^2}{(\overline{z-\omega_{mn}+1-i\sqrt{3}})^3-(\overline{\zeta}+1+i\sqrt{3})^3} \right. \\
&\qquad \left. - \frac{3(\overline{\zeta}+1+i\sqrt{3})^2}{(z-\omega_{mn}+1-i\sqrt{3})^3-(\overline{\zeta}+1+i\sqrt{3})^3} \right\}.
\end{aligned}$$

The outward normal derivative is $\partial_{\nu_\zeta} = i(\partial_\zeta - \partial_{\overline{\zeta}})$ and $\zeta - i\sqrt{3} = \overline{\zeta} + i\sqrt{3}$ on $\partial_2 P^+$, then

$$\partial_{\nu_\zeta} G_1(z,\zeta) = -6i(\zeta+1-i\sqrt{3})^2 \sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+1-i\sqrt{3})^3-(\overline{z-\omega_{mn}+1-i\sqrt{3}})^3}{|(z-\omega_{mn}+1-i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3|^2}. \tag{5.13}$$

5.2 Harmonic Dirichlet problem

The representation formula in Theorem 5.1.1 provides the solution for the related Dirichlet problem. At first the boundary behavior of the boundary integral

$$\varphi(z) = -\frac{1}{4\pi} \int\limits_{\partial P^+} \gamma(\zeta)\partial_{\nu_\zeta} G(z,\zeta) ds_\zeta,\ z \in P^+ \tag{5.14}$$

is to be studied.

Lemma 5.2.1. *For $\gamma \in C(\partial P^+; \mathbb{R})$ the function presented in (5.14) satisfies the relation*

$$\lim_{z\to\zeta_0} \varphi(z) = \gamma(\zeta_0), \tag{5.15}$$

where ζ_0 is a fixed point on $\partial P^+ \backslash \{\pm 2, \pm 1 + i\sqrt{3}\}$.

Proof. **1.** Let ζ_0 be defined on $\partial_1 P^+$ where $\zeta_0 = -\frac{1}{2}(1+i\sqrt{3})\overline{\zeta_0} + 3 + i\sqrt{3}$ and the relations

$$(\zeta_0 - 2)^2 = -\frac{1}{2}(1 - i\sqrt{3})(\zeta_0 - 2)^2,\ \ (\zeta_0 - 2)^3 = (\overline{\zeta_0} - 2)^3.$$

hold.
1.a. On $\partial_1 P^+$ with $\zeta = \zeta_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{\zeta} + 3 + i\sqrt{3}$, $(\zeta-2)^3 = (\overline{\zeta}-2)^3$.
Here when m=n=0 the term in (5.9) becomes

$$-3(\sqrt{3}+i)(\zeta-2)^2 \frac{(z-2)^3 - (\overline{z}-2)^3}{|(z-2)^3 - (\zeta-2)^3|^2},$$

$$(z-2)^3 - (\overline{z}-2)^3 = (z-\overline{z})[(z-2)^2 + |z-2|^2 + (\overline{z}-2)^2]$$
$$(z-2)^3 - (\zeta-2)^3 = (z-\zeta)[(z-2)^2 + (z-2)(\zeta-2) + (\zeta-2)^2].$$

Because $z_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{z} + 3 + i\sqrt{3}$, then $(z_1-2)^3 = (-\frac{1}{2}(1+i\sqrt{3})(\overline{z}-2))^3$, $(z_1-2)^3 = (\overline{z}-2)^3$, hence $(z-2)^3 - (\overline{z}-2)^3 = (z-2)^3 - (z_1-2)^3$

$$\frac{-3(\sqrt{3}+i)(\zeta-2)^2(z-z_1)[(z-2)^2 + (z-2)(z_1-2) + (z_1-2)^2]}{|(z-\zeta)[(z-2)^2 + (z-2)(\zeta-2) + (\zeta-2)^2]|^2} =$$
$$-3(\sqrt{3}+i)\frac{z-z_1}{|z-\zeta|^2}\left\{\frac{(\zeta-2)^2[(z-2)^2 + (z-2)(z_1-2) + (z_1-2)^2]}{|(z-2)^2 + (z-2)(\zeta-2) + (\zeta-2)^2|^2}\right\}.$$

Taking $z \to \zeta_0$ and $\zeta = \zeta_0$ in

$$\lim_{z\to\zeta_0}\left\{\frac{-3(\sqrt{3}+i)(\zeta-2)^2[(z-2)^2 + (z-2)(z_1-2) + (z_1-2)^2]}{|(z-2)^2 + (z-2)(\zeta-2) + (\zeta-2)^2|^2}\right\} = \frac{-(\sqrt{3}+i)(\zeta_0-2)^4}{|\zeta_0-2|^4} = (\sqrt{3}-i),\ \ \zeta_0 \neq 2.$$

For the other terms of the sum:

$$(z-\omega_{mn}-2)^3 = [-\frac{1}{2}(1-i\sqrt{3})(z-\omega_{mn}-2)]^3 = (\overline{z_1}-\omega_{kl}-2)^3,$$

which follows from

$$-\frac{1}{2}(1-i\sqrt{3})z + \frac{1}{2}(1-i\sqrt{3})(3m+i\sqrt{3}n) + 1 - i\sqrt{3} = (\overline{z_1} - \omega_{-\frac{m+n}{2},\frac{3m-n}{2}} - 2),$$

where if $-\frac{m+n}{2} = k$, $\frac{3m-n}{2} = l$, $m+n \in 2\mathbb{Z}$, then $m = -\frac{k-l}{2}$, $n = -\frac{3k+l}{2}$, $k+l \in 2\mathbb{Z}$.

Therefore

$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(z-\omega_{mn}-2)^3}{|(z-\omega_{mn}-2)^3 - (\zeta-2)^3|^2} = \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_1-\omega_{mn}-2})^3}{|(\overline{z_1-\omega_{mn}-2})^3 - (\zeta-2)^3|^2} =$$
$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_1-\omega_{mn}-2})^3}{|(z_1-\omega_{mn}-2)^3 - (\overline{\zeta}-2)^3|^2} = \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_1-\omega_{mn}-2})^3}{|(z_1-\omega_{mn}-2)^3 - (\zeta-2)^3|^2}.$$

That is why

$$\partial_{\nu_\zeta} G_1(z,\zeta) = \frac{(\sqrt{3}-i)(z-z_1)}{|z-\zeta|^2}(1+o(1)) \tag{5.16}$$

for $z \to \zeta_0$ on $\partial_1 P^+$.

1.b. On $\partial_2 P^+$ $\zeta = \zeta_2 = \overline{\zeta} + 2i\sqrt{3}$ and $\zeta - i\sqrt{3} = \overline{\zeta} + i\sqrt{3}$. For m=n=0 in (5.13) the term becomes

$$-6i(\zeta+1-i\sqrt{3})^2 \frac{(z+1-i\sqrt{3})^3 - (\overline{z+1-i\sqrt{3}})^3}{|(z+1-i\sqrt{3})^3 - (\zeta+1-i\sqrt{3})^3|^2}.$$

This term is not singular because

$$\begin{aligned}
&(z+1-i\sqrt{3})^3 - (\overline{z+1-i\sqrt{3}})^3 = \\
&[(z-i\sqrt{3}) - (\overline{z}+i\sqrt{3})][(z+1-i\sqrt{3})^2 + |z+1-i\sqrt{3}|^2 + (\overline{z+1-i\sqrt{3}})^2],\\
&(z+1-i\sqrt{3})^3 - (\zeta+1-i\sqrt{3})^3 = (z-\zeta)[(z+1-i\sqrt{3})^2\\
&+(z+1-i\sqrt{3})(\zeta+1-i\sqrt{3}) + (\zeta+1-i\sqrt{3})^2] \neq 0,\ z \neq \zeta.
\end{aligned}$$

All terms of the sum can be rewritten as

$$\begin{aligned}
(z-\omega_{mn}+1-i\sqrt{3})^3 &= [-\frac{1}{2}(1-i\sqrt{3})(z-\omega_{mn}+1-i\sqrt{3})]^3,\\
-\frac{1}{2}(1-i\sqrt{3})(z-\omega_{mn}+1-i\sqrt{3}) &= \overline{z_1} - \omega_{-\frac{m+n-2}{2}, \frac{3m-n-2}{2}} + 1 + i\sqrt{3},
\end{aligned}$$

where if $-\frac{m+n-2}{2} = k$, $\frac{3m-n-2}{2} = l$, $m+n \in 2\mathbb{Z}$ then $m = -\frac{k-l-2}{2}$, $n = -\frac{3k+l+2}{2}$, $k+l \in 2\mathbb{Z}$. Then $(z-\omega_{mn}+1-i\sqrt{3})^3 = (\overline{z_1} - \omega_{kl} + 1 + i\sqrt{3})^3$.

Thus

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+1-i\sqrt{3})^3}{|(z-\omega_{mn}+1-i\sqrt{3})^3 - (\zeta+1-i\sqrt{3})^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_1-\omega_{mn}+1-i\sqrt{3}})^3}{|(z-\omega_{mn}+1-i\sqrt{3})^3 - (\zeta+1-i\sqrt{3})}|^2.$$

Letting $z \to \zeta_0$, $z_1 \to \zeta_0 \in \partial_1 P^+$ the sum (5.13) tends to 0.

1.c. On $\partial_3 P^+$, where $\zeta = \zeta_3 = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} - 3 + i\sqrt{3}$, $(\zeta+2)^3 = (\overline{\zeta}+2)^3$.
For m=n=0 in (5.11) the term

$$3(-\sqrt{3}+i)(\zeta+2)^2 \frac{(z+2)^3 - (\overline{z}+2)^3}{|(z+2)^3 - (\zeta+2)^3|^2}$$

is not singular for $\zeta \in \partial_3 P^+$, $z \to \zeta_0 \in \partial_1 P^+$, $\zeta_0 \neq -2$ because

$$\begin{aligned}
(z+2)^3 - (\overline{z}+2)^3 &= (z-\overline{z})[(z+2)^2 + |z+2|^2 + (\overline{z}+2)],\\
(z+2)^3 - (\zeta+2)^3 &= (z-\zeta)[(z+2)^2 + (z+2)(\zeta+2) + (\zeta+2)^2] \neq 0,\ z \neq \zeta.
\end{aligned}$$

The terms of the sum are

$$(z-\omega_{mn}+2)^3 = [-\frac{1}{2}(1-i\sqrt{3})(z-\omega_{mn}+2)]^3 = (\overline{z_1} - \omega_{kl} + 2)^3,$$

because

$$-\frac{1}{2}(1-i\sqrt{3})z + \frac{1}{2}(1-i\sqrt{3})(3m+i\sqrt{3}n) - 1 + i\sqrt{3} = \overline{z_1} - \omega_{-\frac{m+n-4}{2}, \frac{3m-n-4}{2}} + 2,$$

where if $k = -\frac{m+n-4}{2}$, $l = \frac{3m-n-4}{2}$, $m+n \in 2\mathbb{Z}$ then $m = -\frac{k-l-4}{2}$, $n = -\frac{3k+l-4}{2}$, $k+l \in 2\mathbb{Z}$.
Therefore

$$\begin{aligned}
\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+2)^3}{|(z-\omega_{mn}+2)^3 - (\zeta+2)^3|^2} &= \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_1-\omega_{mn}+2})^3}{|(\overline{z_1-\omega_{mn}+2})^3 - (\zeta+2)^3|^2} = \\
&\sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_1-\omega_{mn}+2})^3}{|(z_1-\omega_{mn}+2)^3 - (\zeta+2)^3|^2},
\end{aligned}$$

and if $z \to \zeta_0 \in \partial_1 P$ then

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\zeta_0 - \omega_{mn} + 2)^3}{|(\zeta_0 - \omega_{mn} + 2)^3 - (\zeta + 2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{\zeta_0 - \omega_{mn} + 2})^3}{|(\zeta_0 - \omega_{mn} + 2)^3 - (\zeta + 2)^3|^2}$$

and the sum(5.11) tends to 0 when $z \to \zeta_0$ for $\zeta \in \partial_3 P^+$.

1.d. On $\partial_4 P^+$, $\zeta = \overline{\zeta}$. For m=n=0 in (5.10) the term of the sum becomes

$$6i(\zeta - 2)^2 \frac{(z-2)^3 - (\overline{z} - 2)^3}{|(z-2)^3 - (\zeta - 2)^3|^2}$$

and it is not singular if z tends to $\zeta_0 \in \partial_1 P^+$, $\zeta_0 \neq 2$. The terms of the sum (5.10) can be rewritten as

$$(z - \omega_{mn} - 2)^3 = [-\frac{1}{2}(1 - i\sqrt{3})(z - \omega_{mn} - 2)]^3,$$
$$-\frac{1}{2}(1 - i\sqrt{3})(z - \omega_{mn} - 2) = \overline{z_1} - \omega_{kl} - 2,$$

where if $k = -\frac{m+n}{2}$, $l = \frac{3m-n}{2}$, then $m = -\frac{k-l}{2}$, $n = -\frac{3k+l}{2}$, $m+n \in 2\mathbb{Z}$, $k+l \in 2\mathbb{Z}$. Then

$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(z - \omega_{mn} - 2)^3}{|(z - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2} = \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_1 - \omega_{mn} - 2})^3}{|(\overline{z_1 - \omega_{mn} - 2})^3 - (\zeta - 2)^3|^2} =$$
$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_1 - \omega_{mn} - 2})^3}{|(z_1 - \omega_{mn} - 2)^3 - (\overline{\zeta} - 2)^3|^2} \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_1 - \omega_{mn} - 2})^3}{|(z_1 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2}.$$

Letting $z \to \zeta_0 \in \partial_1 P^+$ the sum tends to 0 for $\zeta \in \partial_4 P^+$.

Therefore, as a result of item 1(a-d) one gets

$$\lim_{z\to\zeta_0\in\partial_1 P^+} \left[-\frac{1}{4\pi} \int\limits_{\partial P^+} \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta) ds_\zeta \right] = \lim_{z\to\zeta_0\in\partial_1 P^+} \left[-\frac{(\sqrt{3} - i)}{4\pi} \int\limits_{\partial_1 P^+} \gamma(\zeta) \frac{z - z_1}{|z - \zeta|^2} ds_\zeta \right] = \gamma(\zeta_0) \quad (5.17)$$

on the boundary $\partial_1 P$, see section 4.1.

Similar calculations are implemented for the other parts of the boundary ∂P^+.

2. Let ζ_0 be from $\partial_2 P^+$, where $\zeta_0 = \overline{\zeta_0} + 2i\sqrt{3}$, $\zeta_0 - i\sqrt{3} = \overline{\zeta_0} + i\sqrt{3}$.

2.a. On $\partial_1 P$ $\zeta = \zeta_1 = -\frac{1}{2}(1 + i\sqrt{3})\overline{\zeta} + 3 + i\sqrt{3}$, $(\zeta - 2)^3 = (\overline{\zeta} - 2)^3$.
For m=n=0 the term in (5.9) is

$$-3(\sqrt{3} + i)(\zeta - 2)^2 \frac{(z-2)^3 - (\overline{z} - 2)^3}{|(z-2)^3 - (\zeta - 2)^3|^2}$$

and it is not singular for $\zeta \in \partial_1 P^+$ if $z \to \zeta_0 \in \partial_2 P^+$, $\zeta_0 \neq 1 + i\sqrt{3}$ there. For all the terms of the sum observe

$$(z - \omega_{mn} - 2)^3 = (\overline{z_2} - \omega_{m,n-2} - 2)^3,\ m + n \in 2\mathbb{Z}$$

so that

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z - \omega_{mn} - 2)^3}{|(z - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_2 - \omega_{mn} - 2})^3}{|(z_2 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2} =$$
$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_2 - \omega_{mn} - 2})^3}{|(z_2 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2},$$

and if $z \to \zeta_0 \in \partial_2 P^+$ then

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\zeta_0 - \omega_{mn} - 2)^3}{|(\zeta_0 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{\zeta_0 - \omega_{mn} - 2})^3}{|(\zeta_0 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2}$$

and the sum(5.11) tends to 0 when $z \to \zeta_0$ for $\zeta \in \partial_1 P^+$, $\zeta_0 \neq 1 + i\sqrt{3}$.

2.b. On $\partial_2 P^+$, $\zeta = \overline{\zeta} + 2i\sqrt{3}$, $\zeta - i\sqrt{3} = \overline{\zeta} + i\sqrt{3}$.
For m=n=0 the term in (5.13) is

$$-6i(\zeta + 1 - i\sqrt{3})^2 \frac{(z + 1 - i\sqrt{3})^3 - (\overline{z + 1 - i\sqrt{3}})^3}{|(z + 1 - i\sqrt{3})^3 - (\zeta + 1 - i\sqrt{3})^3|^2} \tag{5.18}$$

and

$$\begin{aligned}
&(z + 1 - i\sqrt{3})^3 - (\overline{z + 1 - i\sqrt{3}})^3 = \\
&[(z - i\sqrt{3}) - (\overline{z} + i\sqrt{3})][(z + 1 - i\sqrt{3})^2 + |z + 1 - i\sqrt{3}|^2 + (\overline{z + 1 - i\sqrt{3}})^2], \\
&(z + 1 - i\sqrt{3})^3 - (\zeta + 1 - i\sqrt{3})^3 = (z - i\sqrt{3} - \zeta + i\sqrt{3})[(z + 1 - i\sqrt{3})^2 \\
&+ (z + 1 - i\sqrt{3})(\zeta + 1 - i\sqrt{3}) + (\zeta + 1 - i\sqrt{3})^2] \neq 0,\ z \neq \zeta.
\end{aligned}$$

On this boundary $z = z_2 = \overline{z} + 2i\sqrt{3}$ and $z - i\sqrt{3} = \overline{z} + i\sqrt{3}$ or $z_2 - i\sqrt{3} = \overline{z} + i\sqrt{3}$, therefore

$$(z + 1 - i\sqrt{3})^3 - (\overline{z + 1 - i\sqrt{3}})^3 = (z + 1 - i\sqrt{3})^3 - (z_2 + 1 - i\sqrt{3})^3.$$

Substituting into (5.18) and consider for $z \to \zeta_0 \in \partial_2 P^+,\ \zeta_0 \neq 1 + i\sqrt{3},\ \zeta = \zeta_0$

$$\lim_{z \to \zeta_0} \left\{ \frac{-6i(\zeta + 1 - i\sqrt{3})^2[(z + 1 - i\sqrt{3})^2 + (z + 1 - i\sqrt{3})(z_2 + 1 - i\sqrt{3}) + (z_2 + 1 - i\sqrt{3})^2]}{|(z + 1 - i\sqrt{3})^2 + (z + 1 - i\sqrt{3})(\zeta + 1 - i\sqrt{3}) + (\zeta + 1 - i\sqrt{3})^2|^2} \right\} =$$
$$-2i\frac{(\zeta_0 + 1 - i\sqrt{3})^4}{|\zeta + 1 - i\sqrt{3}|^4} = -2i, \zeta_0 \neq -1 + i\sqrt{3}.$$

Since the other terms of the sum (5.13) can be rewritten as $(z - \omega_{mn} + 1 - i\sqrt{3}) = (\overline{z_2} - \omega_{mn} + 1 + i\sqrt{3})$, then

$$\sum_{\substack{m+n \in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(z - \omega_{mn} + 1 - i\sqrt{3})^3}{|(z - \omega_{mn} + 1 - i\sqrt{3})^3 - (\zeta + 1 - \sqrt{3})^3|^2} = \sum_{\substack{m+n \in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_2 - \omega_{mn} + 1 + i\sqrt{3}})^3}{|(\overline{z_2 - \omega_{mn} + 1 + i\sqrt{3}})^3 - (\overline{\zeta + 1 + i\sqrt{3}})^3|^2} =$$
$$\sum_{\substack{m+n \in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_2 - \omega_{mn} + 1 + i\sqrt{3}})^3}{|(z_2 - \omega_{mn} + 1 - i\sqrt{3})^3 - (\zeta + 1 - i\sqrt{3})^3|^2}.$$

Letting $z \to \zeta_0 \in \partial_2 P^+$ and since $z_2 \to \zeta_0$, the sum tends to 0. Then

$$\partial_{\nu_\zeta} G_1(z, \zeta) = -2i\frac{z - z_2}{|z - \zeta|^2}(1 + o(1)) \tag{5.19}$$

on $\partial_2 P^+$.

2.c. On $\partial_3 P^+$, $\zeta = \zeta_3 = -\frac{1}{2}(1 - i\sqrt{3})\overline{\zeta} - 3 + i\sqrt{3}$ and $(\zeta + 2)^3 = (\overline{\zeta} + 2)^3$
and, as before, the term for m=n=0 in(5.11) is

$$3(-\sqrt{3} + i)(\zeta + 2)^2 \frac{(z + 2)^3 - (\overline{z} + 2)^3}{|(z + 2)^3 - (\zeta + 2)^3|^2}$$

and if $z \to \zeta_0 \in \partial_2 P^+$ it has no singularity for $\zeta \in \partial_3 P^+,\ \zeta_0 \neq -1 + i\sqrt{3}$. Rewriting the terms of the sum as $(z - \omega_{mn} + 2)^3 = (\overline{z_2} - \omega_{ml} + 2)^3$, where $m + n \in 2\mathbb{Z}$, $n - 2 = l$, $m + l \in 2\mathbb{Z}$, then

$$\sum_{m+n \in 2\mathbb{Z}} \frac{(z - \omega_{mn} + 2)^3}{|(z - \omega_{mn} + 2)^3 - (\zeta + 2)^3|^2} = \sum_{m+n \in 2\mathbb{Z}} \frac{(\overline{z_2 - \omega_{mn} + 2})^3}{|(\overline{z_2 - \omega_{mn} + 2})^3 - (\zeta + 2)^3|^2} =$$
$$\sum_{m+n \in 2\mathbb{Z}} \frac{(\overline{z_2 - \omega_{mn} + 2})^3}{|(z_2 - \omega_{mn} + 2)^3 - (\zeta + 2)^3|^2},$$

and if $z \to \zeta_0 \in \partial_2 P^+$, then

$$\sum_{m+n \in 2\mathbb{Z}} \frac{(\zeta_0 - \omega_{mn} + 2)^3}{|(\zeta_0 - \omega_{mn} + 2)^3 - (\zeta + 2)^3|^2} = \sum_{m+n \in 2\mathbb{Z}} \frac{(\overline{\zeta_0 - \omega_{mn} + 2})^3}{|(\zeta_0 - \omega_{mn} + 2)^3 - (\zeta + 2)^3|^2}$$

and the sum(5.11) tends to 0 when $z \to \zeta_0$ for $\zeta \in \partial_3 P^+$.

2.d. On the boundary part $\partial_4 P^+$, where $\zeta = \overline{\zeta}$, for m=n=0 in (5.10) the term of the sum becomes

$$6i(\zeta-2)^2 \frac{(z-2)^3-(\overline{z}-2)^3}{|(z-2)^3-(\zeta-2)^3|^2}$$

and it is not singular since $z \to \zeta_0 \in \partial_2 P^+$. Also on $\partial_4 P^+$ $z = z_4 = \overline{z}$ and $(z-2)^3 = (\overline{z}-2)^3$. The terms of the sum (5.10) are rewritten as

$$(z-\omega_{mn}-2)^3 = [z-2i\sqrt{3}-\omega_{mn}-2+2i\sqrt{3}]^3 = (\overline{z_2}-\omega_{m,n-2}-2)^3,\ m+n \in 2\mathbb{Z}$$

then

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_2-\omega_{mn}-2})^3}{|(\overline{z_2-\omega_{mn}-2})^3-(\zeta-2)^3|^2} =$$
$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_2-\omega_{mn}-2})^3}{|(z_2-\omega_{mn}-2)^3-(\zeta-2)^3|^2},$$

if $z \to \zeta_0 \in \partial_2 P^+$, $z_2 \to \zeta_0 \in \partial_2 P^+$

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\zeta_0-\omega_{mn}-2)^3}{|(\zeta_0-\omega_{mn}-2)^3-(\zeta-2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{\zeta_0-\omega_{mn}-2})^3}{|(\zeta_0-\omega_{mn}-2)^3-(\zeta-2)^3|^2}$$

and the sum (5.10) tends to 0 for $\zeta \in \partial_4 P^+$.

Therefore, taking the calculations of the item 2(a-d) into account, on the boundary $\partial_2 P^+$ one gets that

$$\lim_{z\to\zeta_0\in\partial_2 P^+} \left[-\frac{1}{4\pi} \int\limits_{\partial P^+} \gamma(\zeta) \partial_{\nu_\zeta} G_1(z,\zeta) ds_\zeta \right] = \lim_{z\to\zeta_0\in\partial_2 P^+} \left[-\frac{1}{2\pi i} \int\limits_{\partial_2 P^+} \gamma(\zeta) \frac{z-z_2}{|z-\zeta|^2} ds_\zeta \right] = \gamma(\zeta_0). \quad (5.20)$$

3. Let ζ_0 be defined on $\partial_3 P^+$, where $\zeta_0 = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta_0} - 3 + i\sqrt{3}$ and

$$(\zeta_0+2)^2 = -\frac{1}{2}(1+i\sqrt{3})(\overline{\zeta_0}+2)^2,\ (\zeta_0+2)^3 = (\overline{\zeta_0}+2)^3.$$

3.a. On $\partial_1 P^+$, where $\zeta = \zeta_1 = -\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} + 3 + i\sqrt{3}$, $(\zeta-2)^3 = (\overline{\zeta}-2)^3$.
For m=n=0 the term in (5.9) becomes

$$-3(\sqrt{3}+i)(\zeta-2)^2 \frac{(z-2)^3-(\overline{z}-2)^3}{|(z-2)^3-(\zeta-2)^3|^2}$$

and

$$(z-2)^3-(\overline{z}-2)^3 = (z-\overline{z})[(z-2)^2+|z-2|^2+(\overline{z}-2)],$$
$$(z-2)^3-(\zeta-2)^3 = (z-\zeta)[(z-2)^2+(z-2)(\zeta-2)+(\zeta-2)^2] \neq 0,\ z \neq \zeta.$$

For the terms of (5.9) we observe

$$(z-\omega_{mn}-2)^3 = [-\frac{1}{2}(1+i\sqrt{3})(z-\omega_{mn}-2)]^3 = (\overline{z_1}-\omega_{kl}-2)^3,$$

since

$$-\frac{1}{2}(1+i\sqrt{3})z + \frac{1}{2}(1+i\sqrt{3})(3m+i\sqrt{3}n) + 1 + i\sqrt{3} = (\overline{z_1} - \omega_{-\frac{m-n+4}{2},-\frac{3m+n+4}{2}} - 2),$$

if $k = -\frac{m-n+4}{2}$, $l = -\frac{3m+n+4}{2}$, $m+n \in 2\mathbb{Z}$ then $m = -\frac{k+l+4}{2}, n = \frac{3k-l+4}{2}$, $k+l \in 2\mathbb{Z}$. Therefore

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{\tilde{z}_1-\omega_{mn}-2})^3}{|(\overline{\tilde{z}_1-\omega_{mn}-2})^3-(\zeta-2)^3|^2} =$$
$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{\tilde{z}_1-\omega_{mn}-2})^3}{|(\tilde{z}_1-\omega_{mn}-2)^3-(\zeta-2)^3|^2},$$

and letting $z \to \zeta_0 \in \partial_3 P$ for $\zeta \in \partial_1 P^+$

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\zeta_0 - \omega_{mn} - 2)^3}{|(\zeta_0 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{\overline{(\zeta_0 - \omega_{mn} - 2)}^3}{|(\zeta_0 - \omega_{mn} - 2)^3 - (\zeta - 2)^3|^2}$$

then the sum (5.9) tends to 0 for $\zeta \in \partial_1 P^+$.

3.b. On $\partial_2 P^+$ $\zeta = \zeta_2 = \overline{\zeta} + 2i\sqrt{3}$, $\zeta + i\sqrt{3} = \overline{\zeta} - i\sqrt{3}$.
For m=n=0 in (5.13) the term becomes

$$-6i(\zeta + 1 - i\sqrt{3})^2 \frac{(z + 1 - i\sqrt{3})^3 - (\overline{z + 1 - i\sqrt{3}})^3}{|(z + 1 - i\sqrt{3})^3 - (\zeta + 1 - i\sqrt{3})^3|^2} \tag{5.21}$$

and it is again not singular for $\zeta \in \partial_2 P^+$ if $z \to \zeta_0 \in \partial_3 P^+$. All the terms of the sum are observed as

$$(z - \omega_{mn} + 1 - i\sqrt{3})^3 = (\overline{z_3 - \omega_{k,-l} + 1 - i\sqrt{3}})^3,$$

because

$$\begin{aligned}
&-\frac{1}{2}(1 + i\sqrt{3})(z - \omega_{mn} + 1 - i\sqrt{3}) = \\
&-\frac{1}{2}(1 + i\sqrt{3})z - 3 - i\sqrt{3} + \frac{1}{2}(3(m - n) + i\sqrt{3}(3m + n)) + 1 + i\sqrt{3} = \\
&\overline{z_3} - \omega_{-\frac{m-n}{2}, -\frac{3m+n}{2}} + 1 + i\sqrt{3},
\end{aligned}$$

if $k = -\frac{m-n}{2}$, $l = -\frac{3m+n}{2}$, $m + n \in 2\mathbb{Z}$ then $m = -\frac{k+l}{2}$, $n = \frac{3k-l}{2}$, $k + l \in 2\mathbb{Z}$. Then, taking a limit for $z \to \zeta_0 \in \partial_3 P^+$ gives

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(\zeta_0 - \omega_{mn} + 1 - i\sqrt{3})^3}{|(\zeta_0 - \omega_{mn} + 1 - i\sqrt{3})^3 - (\zeta + 1 - i\sqrt{3})^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{\overline{(\zeta_0 - \omega_{mn} + 1 - i\sqrt{3})}^3}{|(\zeta_0 - \omega_{mn} + 1 - i\sqrt{3})^3 - (\zeta + 1 - i\sqrt{3})^3|^2}$$

because $z_3 \to \zeta_0 \in \partial_3 P^+$ and therefore, the sum (5.13) tends to 0 here.

3.c. On the boundary part $\partial_3 P^+$ with $\zeta = \check{\zeta}_1 = -\frac{1}{2}(1 - i\sqrt{3})\overline{\zeta} - 3 + i\sqrt{3}$, $(\zeta + 2)^3 = (\overline{\zeta} + 2)^3$.
For m=n=0 in (5.11) the term becomes

$$-3(-\sqrt{3} + i)(\zeta + 2)^2 \frac{(z + 2)^3 - (\overline{z} + 2)^3}{|(z + 2)^3 - (\zeta + 2)^3|^2}.$$

Because

$$\begin{aligned}
&(z + 2)^3 - (\overline{z} + 2)^3 = (z - \overline{z})[(z + 2)^2 + |z + 2|^2 + (\overline{z} + 2)^2], \\
&(z + 2)^3 - (\zeta + 2)^3 = (z - \zeta)[(z + 2)^2 + (z + 2)(\zeta + 2) + (\zeta + 2)^2] \neq 0,\ z \neq \zeta
\end{aligned}$$

the term is

$$\begin{aligned}
3(-\sqrt{3} + i)&\frac{(z - \overline{z})(\zeta + 2)^2[(z + 2)^2 + |z + 2|^2 + (\overline{z} + 2)^2]}{|z - \zeta|^2[(z + 2)^2 + (z + 2)(\zeta + 2) + (\zeta + 2)^2]}, \\
\check{z}_1 = -\frac{1}{2}&(1 - i\sqrt{3})\overline{z} - 3 + i\sqrt{3},\ \check{z}_1 + 2 = -\frac{1}{2}(1 - i\sqrt{3})(\overline{z} + 2), \\
(\check{z}_1 + 2)^3 = (\overline{z} &+ 2)^3, \text{then } (z + 2)^3 - (\overline{z} + 2)^3 = (z + 2)^3 - (\check{z}_1 + 2)^3
\end{aligned}$$

so that it becomes

$$3(-\sqrt{3} + i)\frac{z - \check{z}_1}{|z - \zeta|^2} \frac{(\zeta + 2)^2[(z + 2)^2 + (z + 2)(\check{z}_1 + 2) + (\check{z}_1 + 2)^2]}{|(z + 2)^2 + (z + 2)(\zeta + 2) + (\zeta + 2)^2|^2}.$$

Letting $z \to \zeta_0$, $\check{z}_1 \to \zeta_0$, $\zeta = \zeta_0$ for the fraction

$$\lim_{z\to\zeta_0} \frac{3(-\sqrt{3} + i)(\zeta + 2)^2[(z + 2)^2 + (z + 2)(\check{z}_1 + 2) + (\check{z}_1 + 2)^2]}{|(z + 2)^2 + (z + 2)(\zeta + 2) + (\zeta + 2)^2|^2} = \frac{(-\sqrt{3} + i)(\zeta_0 + 2)^4}{|\zeta_0 + 2|^4} = (\sqrt{3} + i).$$

For the other terms of (5.11)

$$(z-\omega_{mn}+2)^3=[-\frac{1}{2}(1+i\sqrt{3})(z-\omega_{mn}+2)]^3=(\overline{\check{z_1}}-\omega_{kl}+2)^3,$$

as

$$-\frac{1}{2}(1+i\sqrt{3})z+\frac{1}{2}(1+i\sqrt{3})(3m+i\sqrt{3}n)-1-i\sqrt{3}=$$
$$\overline{\check{z_1}}-\omega_{-\frac{m-n}{2},-\frac{3m+n}{2}}+2,$$

if $k=-\frac{m-n}{2}$, $l=-\frac{3m+n}{2}$, $m+n\in 2\mathbb{Z}$, then $m=k+l$, $n=\frac{3k-l}{2}$, $k+l\in 2\mathbb{Z}$ and

$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\frac{(z-\omega_{mn}+2)^3}{|(z-\omega_{mn}+2)^3-(\zeta+2)^3|^2}=\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\frac{(\overline{\check{z_1}-\omega_{mn}+2})^3}{|(\overline{\check{z_1}-\omega_{mn}+2})^3-(\zeta+2)^3|^2}=$$
$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\frac{(\overline{\check{z_1}-\omega_{mn}+2})^3}{|(\check{z_1}-\omega_{mn}+2)^3-(\zeta+2)^3|^2}.$$

Therefore

$$\partial_{\nu_\zeta}G_1(z,\zeta)=(-\sqrt{3}-i)\frac{z-z_3}{|z-\zeta|^2}(1+o(1))$$

for $z\to\zeta_0\in\partial_3P^+$.

3.d. On ∂_4P^+ with $\zeta=\overline{\zeta}$ the term of the sum for m=n=0 in (5.10)becomes

$$6i(\zeta-2)^2\frac{(z-2)^3-(\overline{z}-2)^3}{|(z-2)^3-(\zeta-2)^3|^2},$$

and it has no singularity for $\zeta\in\partial_1P^+$, $z\to\zeta_4P^+$. Similarly, as for the previous items, the terms of the sum (5.10) can be rewritten as

$$(z-\omega_{mn}-2)^3=[-\frac{1}{2}(1+i\sqrt{3})(z-\omega_{mn}-2)]^3,$$
$$-\frac{1}{2}(1+i\sqrt{3})(z-\omega_{mn}-2)=-\frac{1}{2}(1+i\sqrt{3})z+\frac{1}{2}(1+i\sqrt{3})(3m+i\sqrt{3}n)+1+i\sqrt{3}=$$
$$=(\overline{\check{z_1}}-\omega_{kl}-2),$$

where if $k=-\frac{m-n+4}{2}$, $l=-\frac{3m+n+4}{2}$, $m+n\in 2\mathbb{Z}$, then $m=-\frac{k+l+4}{2}$, $n=\frac{3k-l+4}{2}$, $k+l\in 2\mathbb{Z}$. Therefore

$$\sum_{m+n\in 2\mathbb{Z}}\frac{(z-\omega_{mn}-2)^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2}=\sum_{m+n\in 2\mathbb{Z}}\frac{(\overline{\check{z_1}-\omega_{mn}-2})^3}{|(\overline{\check{z_1}-\omega_{mn}-2})^3-(\zeta-2)^3|^2}=$$
$$\sum_{m+n\in 2\mathbb{Z}}\frac{(\overline{\check{z_1}-\omega_{mn}-2})^3}{|(\check{z_1}-\omega_{mn}-2)^3-(\zeta-2)^3|^2},$$

if $z\to\zeta_0\in\partial_3P^+$ the sum (5.10) tends to 0 for $\zeta\in\partial_4P^+$.

Thus for case 3(a-d)

$$\lim_{z\to\zeta_0\in\partial_3P^+}\left[-\frac{1}{4\pi}\int_{\partial P^+}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta\right]=\lim_{z\to\zeta_0\in\partial_3P^+}\left[-\frac{(\sqrt{3}+i)}{4\pi}\int_{\partial_3P^+}\gamma(\zeta)\frac{z-\check{z}_1}{|z-\zeta|^2}ds_\zeta\right]=\gamma(\zeta_0)\quad(5.22)$$

on the boundary part ∂_3P

4. Let ζ_0 be from ∂_4P^+, where $\zeta_0=\zeta_4=\overline{\zeta}$.

4.a. On ∂_1P^+ $\zeta=\zeta_1=-\frac{1}{2}(1-i\sqrt{3})\overline{\zeta}+3+i\sqrt{3}$, $(\zeta-2)^3=(\overline{\zeta}-2)^3$. For m=n=0 the term in (5.9) becomes

$$-3(\sqrt{3}+i)(\zeta-2)^2\frac{(z-2)^3-(\overline{z}-2)^3}{|(z-2)^3-(\zeta-2)^3|^2}$$

and it is not singular for $\zeta\in\partial_1P^+$ if $z\to\zeta_0,\ z\neq 2$. All the terms in (5.9) because of

$$(z-\omega_{mn}-2)=(\overline{z_4-\omega_{m,-n}-2})$$

become

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_4-\omega_{mn}-2})^3}{|(\overline{z_4-\omega_{mn}-2})^3-(\zeta-2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_4-\omega_{mn}-2})^3}{|(z_4-\omega_{mn}-2)^3-(\zeta-2)^3|^2},$$

and if $z\to\zeta_0\in\partial_4P^+$, $z_4\to\zeta_0$ the sum

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}-2)^3-(\overline{z-\omega_{mn}-2})^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2}$$

tends to 0 for $z\in\partial_1P^+$.

4.b. On ∂_2P^+ $\zeta=\overline{\zeta}+2i\sqrt{3}$. For m=n=0 in (5.13) the term

$$-6i(\zeta+1-i\sqrt{3})^2\frac{(z+1-i\sqrt{3})^3-(\overline{z+1-i\sqrt{3}})^3}{|(z+1-i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3|^2}$$

is not singular if $z\to\zeta_0$ for $\zeta\in\partial_2P^+$. Similarly, the terms of (5.13) can be rewritten as

$$(z-\omega_{mn}+1-i\sqrt{3})^3=(\overline{z_4}-\omega_{mn}+1-i\sqrt{3})^3=(\overline{z_4-\omega_{m,-n-2}+1-i\sqrt{3}})^3,$$
$$m+n\in 2\mathbb{Z}, -n-2=k,\ m+k\in 2\mathbb{Z}.$$

Then

$$\lim_{z\to\zeta_0}\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+1-i\sqrt{3})^3}{|(z-\omega_{mn}+1-i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3|^2} = \lim_{z\to\zeta_0}\sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_4-\omega_{mn}+1-i\sqrt{3}})^3}{|(z_4-\omega_{mn}+1-i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3|^2}$$

and the sum (5.13) tends to 0 for $\zeta\in\partial_2P^+$ if $z\to\zeta_0$.

4.c. On ∂_3P^+, where $\zeta=\zeta_3=-\frac{1}{2}(1-i\sqrt{3})\overline{\zeta}-3+i\sqrt{3}$ and $(\zeta+2)^3=(\overline{\zeta}+2)^3$. The term for m=n=0 in (5.11) becomes

$$-3(-\sqrt{3}+i)(\zeta+2)^2\frac{(z+2)^3-(\overline{z}+2)^3}{|(z+2)^3-(\zeta+2)^3|^2}$$

and it is not singular for $\zeta\in\partial_3P^+$ if $z\to\zeta_0$. All the terms of the sum can be presented using

$$(z-\omega_{mn}+2)^3=(\overline{z_4-\omega_{m,-n}+2})^3,\ m+n\in 2\mathbb{Z},$$

then

$$\sum_{m+n\in 2\mathbb{Z}} \frac{(z-\omega_{mn}+2)^3}{|(z-\omega_{mn}+2)^3-(\zeta+2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_4-\omega_{mn}+2})^3}{|(\overline{z_4-\omega_{mn}+2})^3-(\zeta+2)^3|^2} = \sum_{m+n\in 2\mathbb{Z}} \frac{(\overline{z_4-\omega_{mn}+2})^3}{|(z_4-\omega_{mn}+2)^3-(\zeta+2)^3|^2}.$$

If $z\to\zeta_0\in\partial_4P^+$, $z_4\to\zeta_0$, then the sum (5.11) tends to 0 for $\zeta\in\partial_3P^+$.

4.d. On ∂_4P^+ with $\zeta=\overline{\zeta}$ for m=n=0 in (5.10) the term becomes

$$6i(\zeta-2)^2\frac{(z-2)^3-(\overline{z}-2)^3}{|(z-2)^3-(\zeta-2)^3|^2}, \tag{5.23}$$

where

$$(z-2)^3-(\overline{z}-2)^3=(z-\overline{z})[(z-2)^3+|z-2|^2+(\overline{z}-2)^2],$$
$$(z-2)^3-(\zeta-2)^3=(z-\zeta)[(z-2)^2+(z-2)(\zeta-2)+(\zeta-2)^2].$$

Here $z = z_4 = \overline{z}$, then the term (5.23) is

$$6i(\zeta-2)^2\frac{(z-z_4)}{|z-\zeta|^2}\frac{(z-2)^2+(z-2)(z_4-2)+(z_4-2)^2}{|(z-2)^2+(z-2)(\zeta-2)+(\zeta-2)^2|^2}$$

and taking the limit in the second fraction for $z \to \zeta_0 \in \partial_4 P^+$ and since $\zeta = \zeta_0$

$$\lim_{z\to\zeta_0} 6i(\zeta-2)^2\frac{(z-2)^2+(z-2)(z_4-2)+(z_4-2)^2}{|(z-2)^2+(z-2)(\zeta-2)+(\zeta-2)^2|^2} = 2i\frac{(\zeta_0-2)^4}{|\zeta_0-2|^4} = 2i.$$

The other terms of the sum (5.10) are rewritten because of

$$(z-\omega_{mn}-2)^3 = (\overline{z-\omega_{m,-n}-2})^3,\ m+n\in 2\mathbb{Z}$$

as

$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(z-\omega_{mn}-2)^3}{|(z-\omega_{mn}-2)^3-(\zeta-2)^3|^2} = \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_4-\omega_{mn}-2})^3}{|(\overline{z_4-\omega_{mn}-2})^3-(\zeta-2)^3|^2} =$$
$$\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_4-\omega_{mn}-2})^3}{|(z_4-\omega_{mn}-2)^3-(\overline{\zeta}-2)^3|^2}\ \sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}} \frac{(\overline{z_4-\omega_{mn}-2})^3}{|(z_4-\omega_{mn}-2)^3-(\zeta-2)^3|^2}.$$

If $z \to \zeta_0$, $z_4 \to \zeta_0 \in \partial_4 P^+$ this sum (5.10) tends to 0 for $\zeta \in \partial_4 P^+$. Thus on this boundary part

$$\partial_{\nu_\zeta} G_1(z,\zeta) = 2i\frac{z-z_4}{|z-\zeta|^2}. \tag{5.24}$$

Composing the results of item 4(a-d), we have

$$\begin{aligned}&\lim_{z\to\zeta_0\in\partial_4P^+}\left[-\frac{1}{4\pi}\int\limits_{\partial P^+}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta\right] = \lim_{z\to\zeta_0\in\partial_4P^+}\left[\frac{-2i}{4\pi}\int\limits_{\partial_4 P^+}\gamma(\zeta)\frac{z-z_4}{|z-\zeta|^2}ds_\zeta\right] = \\ &\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{\partial_4P^+}\gamma(\zeta)\frac{z-\overline{z}}{|z-\zeta|^2}ds_\zeta = \gamma(\zeta_0)\end{aligned} \tag{5.25}$$

on the boundary $\partial_4 P^+$. Thus, the equality (5.15) for the function $\varphi(z)$ is valid. □

In the next lemma the boundary behavior of the function $\varphi(\zeta)$ in the corner points ± 2, $\pm 1 + i\sqrt{3}$ is studied.

Lemma 5.2.2. *If $\gamma \in C(\partial P^+;\mathbb{C})$, then*

$$\lim_{z\to\zeta_0\in\partial P^+}\left\{-\frac{1}{4\pi}\int\limits_{\partial P^+}[\gamma(\zeta)-\gamma(\zeta_0)]\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta\right\} = 0,\ \zeta_0\in\{\pm 2, \pm 1 + i\sqrt{3}\},\ z\in P^+.$$

Proof. Consider first the presentation formula from Theorem 5.1.1 $w(z) \equiv 1$:

$$1 = -\frac{1}{4\pi}\int\limits_{\partial P^+}\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta,$$

then for $\varphi(z)$ from (5.14)

$$\varphi(z)-\gamma(2) = -\frac{1}{4\pi}\int\limits_{\partial P^+}[\gamma(\zeta)-\gamma(2)]\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta \tag{5.26}$$

holds.

Let $\gamma(\zeta) - \gamma(2) = \gamma_1(\zeta)$, then consider the integral on $\partial_1 P$

$$-\frac{1}{4\pi}\int\limits_{\partial_1 P^+} \gamma_1(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta = -\frac{1}{4\pi}(\sqrt{3}-i)\int\limits_{\partial_1 P^+} \gamma_1(\zeta)\frac{z-z_1}{|\zeta - z|^2}ds_\zeta =$$
$$-\frac{(\sqrt{3}-i)}{4\pi}\int\limits_{\partial H_1^+} \Gamma_1(\zeta)\frac{z-z_1}{|\zeta - z|^2}ds_\zeta,$$

where ∂H_1^+ is the line through the points 2 and $1+i\sqrt{3}$ in direction from 2 to $1+i\sqrt{3}$ and

$$\Gamma_1(\zeta) = \begin{cases} 0, & \zeta \in \partial H_1^+ \backslash \partial_1 P^+, \\ \gamma_1(\zeta), & \zeta \in \partial_1 P^+. \end{cases} \tag{5.27}$$

Similarly for $\partial_4 P^+$

$$-\frac{1}{4\pi}\int\limits_{\partial_4 P^+} \gamma_1(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta = \frac{1}{2\pi i}\int\limits_{\partial_4 P^+} \gamma_1(\zeta)\frac{z-z_4}{|\zeta - z|^2}ds_\zeta = \frac{1}{2\pi i}\int\limits_{\partial H_4^+} \hat{\Gamma_1}(\zeta)\frac{z-\overline{z}}{|\zeta - z|^2}ds_\zeta,$$

where H_4^+ is the line through the points -2 and 2 and

$$\hat{\Gamma_1}(\zeta) = \begin{cases} 0, & \zeta \in \partial H_4^+ \backslash \partial_4 P^+, \\ \gamma_1(\zeta), & \zeta \in \partial_4 P^+. \end{cases} \tag{5.28}$$

It is seen that $\Gamma_1(\zeta)$, $\hat{\Gamma_1}(\zeta)$ are both continuous at $\zeta = 2$ and the integrals tend to 0 if $z \in P^+$ tends to 2. From the results of Lemma 5.2.1 the equalities

$$\lim_{z\to 2}[-\frac{1}{4\pi}\int\limits_{\partial_2 P^+} \gamma_1(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta] = 0,$$
$$\lim_{z\to 2}[-\frac{1}{4\pi}\int\limits_{\partial_3 P^+} \gamma_1(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta] = 0,$$

follow. Thus,

$$\lim_{z\to 2}[-\frac{1}{4\pi}\int\limits_{\partial P^+} [\gamma(\zeta)-\gamma(2)]\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta] = 0$$

and, consequently,

$$\lim_{z\to 2}[-\frac{1}{4\pi}\int\limits_{\partial P^+} \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta = \gamma(2).$$

Consider now the corner point $1+i\sqrt{3}$ and take the difference:

$$\varphi(z) - \gamma(1+i\sqrt{3}) = -\frac{1}{4\pi}\int\limits_{\partial P^+} [\gamma(\zeta)-\gamma(1+i\sqrt{3})]\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta. \tag{5.29}$$

For $\partial_1 P^+$ the integral with $\gamma_2(\zeta) = \gamma(\zeta) - \gamma(1+i\sqrt{3})$

$$-\frac{1}{4\pi}\int\limits_{\partial_1 P^+} \gamma_2(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta = -\frac{1}{4\pi}(\sqrt{3}-i)\int\limits_{\partial_1 P^+} \gamma_2(\zeta)\frac{z-z_1}{|\zeta - z|^2}ds_\zeta =$$
$$-\frac{\sqrt{3}-i}{4\pi}\int\limits_{\partial H_1^+} \Gamma_2(\zeta)\frac{z-z_1}{|\zeta - z|^2}ds_\zeta,$$

where

$$\Gamma_2(\zeta) = \begin{cases} 0, & \zeta \in \partial H_1^+ \backslash \partial_1 P^+, \\ \gamma_2(\zeta), & \zeta \in \partial_1 P^+. \end{cases} \tag{5.30}$$

For $\partial_2 P^+$, the line from $1 + i\sqrt{3}$ to $-1 + i\sqrt{3}$

$$-\frac{1}{4\pi} \int\limits_{\partial_2 P^+} \gamma_2(\zeta) \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta = -\frac{1}{2\pi i} \int\limits_{\partial_2 P^+} \gamma_2(\zeta) \frac{z - z_2}{|\zeta - z|^2} ds_\zeta = \frac{1}{2\pi i} \int\limits_{\partial H_2^+} \hat{\Gamma_2}(\zeta) \frac{z - z_2}{|\zeta - z|^2} ds_\zeta,$$

where ∂H_2^+ is the line through the points $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$ in direction from $1 + i\sqrt{3}$ to $-1 + i\sqrt{3}$ and

$$\hat{\Gamma_2}(\zeta) = \begin{cases} 0, & \zeta \in \partial H_2^+ \backslash \partial_2 P^+, \\ \gamma_2(\zeta), & \zeta \in \partial_2 P^+. \end{cases} \tag{5.31}$$

Similarly, $\Gamma_2(\zeta)$, $\hat{\Gamma_2}(\zeta)$ are continuous at $\zeta = 1 + i\sqrt{3}$ and the integrals tend to 0 when z approaches $1 + i\sqrt{3}$. On the other parts of the boundary due to Lemma 5.2.1

$$\lim_{z \to 1 + i\sqrt{3}} [-\frac{1}{4\pi} \int\limits_{\partial_3 P^+} \gamma_2(\zeta) \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta] = 0,$$
$$\lim_{z \to 1 + i\sqrt{3}} [-\frac{1}{4\pi} \int\limits_{\partial_4 P^+} \gamma_2(\zeta) \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta] = 0,$$

and from (5.29)

$$\lim_{z \to 1 + i\sqrt{3}} [-\frac{1}{4\pi} \int\limits_{\partial P^+} \gamma(\zeta) \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta = \gamma(1 + i\sqrt{3})$$

follows.

For the third corner point $-1 + i\sqrt{3}$ we compute the difference

$$\varphi(z) - \gamma(-1 + i\sqrt{3}) = -\frac{1}{4\pi} \int\limits_{\partial P^+} [\gamma(\zeta) - \gamma(-1 + i\sqrt{3})] \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta. \tag{5.32}$$

Consider $\gamma_3(\zeta) = \gamma(\zeta) - \gamma(-1 + i\sqrt{3})$ and the integrals on the boundaries $\partial_2 P^+$ and $\partial_3 P^+$ and the part of the line ∂H_3^+ with the orientation from $-1 + i\sqrt{3}$ to -2

$$-\frac{1}{4\pi} \int\limits_{\partial_2 P^+} \gamma_3(\zeta) \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta = -\frac{1}{2\pi i} \int\limits_{\partial_2 P^+} \gamma_3(\zeta) \frac{z - z_2}{|\zeta - z|^2} ds_\zeta = \frac{1}{2\pi i} \int\limits_{\partial H_3^+} \Gamma_3(\zeta) \frac{z - z_2}{|\zeta - z|^2} ds_\zeta,$$

where

$$\Gamma_3(\zeta) = \begin{cases} 0, & \zeta \in \partial H_3^+ \backslash \partial_2 P^+, \\ \gamma_3(\zeta), & \zeta \in \partial_2 P^+. \end{cases} \tag{5.33}$$

And

$$-\frac{1}{4\pi} \int\limits_{\partial_3 P^+} \gamma_3(\zeta) \partial_{\nu_\zeta} G_1(z, \zeta) ds_\zeta = -\frac{(\sqrt{3} + i)}{4\pi} \int\limits_{\partial_3 P^+} \gamma_2(\zeta) \frac{z - z_3}{|\zeta - z|^2} ds_\zeta =$$
$$-\frac{(\sqrt{3} + i)}{\pi} \int\limits_{\partial H_3^+} \hat{\Gamma_3}(\zeta) \frac{z - z_3}{|\zeta - z|^2} ds_\zeta,$$

where

$$\hat{\Gamma_3}(\zeta) = \begin{cases} 0, & \zeta \in \partial H_3^+ \backslash \partial_3 P^+, \\ \gamma_3(\zeta), & \zeta \in \partial_3 P^+. \end{cases} \tag{5.34}$$

It is seen that $\Gamma_3(\zeta)$, $\hat{\Gamma_3}(\zeta)$ are continuous at $\zeta = -1+i\sqrt{3}$ and both integrals tend to 0 when z tends to $-1+i\sqrt{3}$ and by Lemma 5.2.1

$$\lim_{z\to -1+i\sqrt{3}}[-\frac{1}{4\pi}\int\limits_{\partial_1 P^+}\gamma_3(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta] = 0,$$

$$\lim_{z\to -1+i\sqrt{3}}[-\frac{1}{4\pi}\int\limits_{\partial_4 P^+}\gamma_3(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta] = 0$$

Then, from (5.32)

$$\lim_{z\to -1+i\sqrt{3}}[-\frac{1}{4\pi}\int\limits_{\partial P^+}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta = \gamma(-1+i\sqrt{3})$$

follows.

The next corner point -2 is an intersection point of $\partial_3 P^+$ and $\partial_4 P^+$. Consider the difference

$$\varphi(z) - \gamma(-2) = -\frac{1}{4\pi}\int\limits_{\partial P^+}[\gamma(\zeta)-\gamma(-2)]\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta. \tag{5.35}$$

Taking $\gamma_4(\zeta) = \gamma(\zeta) - \gamma(-2)$

$$-\frac{1}{4\pi}\int\limits_{\partial_3 P^+}\gamma_4(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta = -\frac{(\sqrt{3}+i)}{4\pi}\int\limits_{\partial_3 P^+}\gamma_4(\zeta)\frac{z-z_3}{|\zeta-z|^2}ds_\zeta =$$
$$-\frac{\sqrt{3}+i}{\pi}\int\limits_{\partial H_3^+}\Gamma_4(\zeta)\frac{z-z_3}{|\zeta-z|^2}ds_\zeta,$$

where

$$\Gamma_4(\zeta) = \begin{cases} 0, & \zeta\in\partial H_3^+\backslash\partial_3 P^+, \\ \gamma_4(\zeta), & \zeta\in\partial_3 P^+. \end{cases} \tag{5.36}$$

On the boundary part $\partial_4 P$

$$-\frac{1}{4\pi}\int\limits_{\partial_4 P^+}\gamma_4(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta = \frac{1}{2\pi i}\int\limits_{\partial_4 P^+}\gamma_4(\zeta)\frac{z-z_4}{|\zeta-z|^2}ds_\zeta = \frac{1}{2\pi i}\int\limits_{\partial H_4^+}\hat{\Gamma_4}(\zeta)\frac{z-\overline{z}}{|\zeta-z|^2}ds_\zeta,$$

where

$$\hat{\Gamma_4}(\zeta) = \begin{cases} 0, & \zeta\in\partial H_4^+\backslash\partial_4 P^+, \\ \gamma_4(\zeta), & \zeta\in\partial_4 P^+. \end{cases} \tag{5.37}$$

The functions $\Gamma_4(\zeta)$, $\hat{\Gamma_4}(\zeta)$ are continuous at $\zeta = -2$ and both integrals tend to 0 when z tends to -2. On the other parts of the boundary, due to Lemma 5.2.1

$$\lim_{z\to -2}[-\frac{1}{4\pi}\int\limits_{\partial_1 P^+}\gamma_4(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta] = 0,$$

$$\lim_{z\to -2}[-\frac{1}{4\pi}\int\limits_{\partial_2 P^+}\gamma_4(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta] = 0.$$

Then, from (5.35) the equality

$$\lim_{z\to -2}[-\frac{1}{4\pi}\int\limits_{\partial P^+}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta] = \gamma(-2)$$

follows. □

Theorem 5.2.1. *The Dirichlet problem*

$$w_{z\overline{z}} = f \text{ in } P^+,\ w = \gamma \text{ on } \partial P^+ \text{ for } f \in L_p(P^+;\mathbb{C}),\ 2 < p,\ \gamma \in C(\partial P^+;\mathbb{C}) \tag{5.38}$$

is uniquely solvable in the space $W^{2,p}(P^+;\mathbb{C}) \cap C(\overline{P^+};\mathbb{C})$ *by*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial P^+} \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta - \frac{1}{\pi}\int\limits_{P^+} f(\zeta)G_1(z,\zeta)d\xi d\eta \tag{5.39}$$

Proof. We need to prove that (5.39) is a solution to the Poisson equation in problem (5.38). Similarly, as it was done for the Schwarz problem (4.51), we use the property of the Pompeiu operator $Tf(z)$ to obtain a weak solution of the differential equation $w_{z\overline{z}} = f$.

Let us consider the Green function

$$G_1(z,\zeta) = \log\left|\prod_{m+n\in 2\mathbb{Z}} \frac{(\zeta-\omega_{mn}-2)^3-(\overline{z}-2)^3}{(\zeta-\omega_{mn}-2)^3-(z-2)^3}\right|^2 \tag{5.40}$$

and the derivative

$$\partial_z G_1(z,\zeta) = \sum_{m+n\in 2\mathbb{Z}}\left[\frac{3(z-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(z-2)^2}{(\overline{\zeta-\omega_{mn}}-2)^3-(z-2)^3}\right] =$$
$$\frac{3(z-2)^2}{(\zeta-2)^3-(z-2)^3} - \frac{3(z-2)^2}{(\overline{\zeta}-2)^3-(z-2)^3}$$
$$+\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\left[\frac{3(z-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(z-2)^2}{(\overline{\zeta-\omega_{mn}}-2)^3-(z-2)^3}\right].$$

Consider the term

$$\frac{3(z-2)^2}{(\zeta-2)^3-(z-2)^3} = \frac{3(z-2)^2}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} =$$
$$\frac{1}{\zeta-z} + \left(\frac{3(z-2)^2}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} - \frac{1}{\zeta-z}\right) =$$
$$\frac{1}{\zeta-z} + \frac{2(z-2)^2-(\zeta-2)^2-(\zeta-2)(z-2)}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} =$$
$$\frac{1}{\zeta-z} + \frac{(z-\zeta)(2z+\zeta-6)}{(\zeta-z)[(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2]} =$$
$$\frac{1}{\zeta-z} + \frac{2(3-z)-\zeta}{(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2}.$$

Denote the function

$$\begin{aligned}\tilde{g}(\zeta,z) &= \frac{2(3-z)-\zeta}{(\zeta-2)^2+(\zeta-2)(z-2)+(z-2)^2} - \frac{3(z-2)^2}{(\overline{\zeta}-2)^3-(z-2)^3}\\ &+\sum_{\substack{m+n\in 2\mathbb{Z},\\ m^2+n^2>0}}\left(\frac{3(z-2)^2}{(\zeta-\omega_{mn}-2)^3-(z-2)^3} - \frac{3(z-2)^2}{(\overline{\zeta-\omega_{mn}}-2)^3-(z-2)^3}\right)\end{aligned} \tag{5.41}$$

which is analytic with respect to $z \in P^+$, then

$$\partial_{z\overline{z}}w(z) = \partial_{z\overline{z}}\left\{-\frac{1}{\pi}\int\limits_{P^+} f(\zeta)G_1(z,\zeta)d\xi d\eta\right\} = \partial_{\overline{z}}\left\{-\frac{1}{\pi}\int\limits_{P^+} f(\zeta)\left[\frac{1}{\zeta-z}+\tilde{g}(\zeta,z)\right]d\xi d\eta\right\} = f(z)$$

provides the solution to the differential equation in the problem (5.38) in a weak sense. The boundary condition $w = \gamma$ on the boundary parts of ∂P^+ holds because of the Lemmas (5.2.1) and (5.2.2). $\square$

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Zusammenfassung

Diese Doktorarbeit ist der Untersuchung von einigen Grenzwertproblemen für komplexen partielle Differenzgleichungen im Viertelkreis und im Halbsechseck gewidmet. Unter den Hauptwerkzeugen wird die Methode der Reflexion verwendet, um die Schwarz-Poissonsche Darstellungsformel und die harmonische Green Funktion für beide Gebiete zu erhalten. Für den Viertelkreis werden die entsprechenden Schwarz, Dirichlet und Neumann Probleme für die Cauchy-Riemannsche Gleichung ausführlich gelöst. Von der harmonischen Green Funktion für dieses Gebiet wird die Neumann Funktion, die bestimmte vorgeschriebene Eigenschaften erfüllt, abgeleitet. Durch den Gebrauch der Green und Neumann Funktionen werden die ensprechenden Dirichlet und Neumann Probleme für die Poissonsche Gleichung gelöst.

Analog, wird durch den Gebrauch der Reflexionpunkte die Schwarz-Poissonsche Darstellungsformel für das Halbsechseck gefunden und die Lösung des Schwarz Problems für die Cauchy-Riemannsche Gleichung gestellt. Für dieses Gebiet wird die harmonische Green Funktion erhaltet. Diese Funktion ermöglicht, das entsprechende harmonische Dirichlet Problem zu lösen.

Weil beide Gebiete nichtregelmäßig sind, wird das Grenzverhalten in den Eckpunkten spezielle Aufmerksamkeit geschenkt.