# Chapter 18

# Global Regularity

We now finally address global regularity and show that the limit surface is almost everywhere regular with respect to the induced measure on the limit surface. We show this by considering the points well behaved with respect to both the flow and with respect to the rectifiable set assumed to be the limit set. The points well behaved with respect to the flow will be called 'good points'. We show that almost all points in $\mathbb{R}^{n+1}$ are good points. It is known that almost all points in $\mathbb{R}^{n+1}$ are well behaved points with respect to the rectifiable set, where the well behaved points in this sense are taken as those that are either measure theoretically not part of the limit surface or those that have an approximate tangent space with respect to the limit surface.

We will have then proven that almost all points are well behaved both with respect to flow and structure. We show that such points provide good control over the measure convergence of the surfaces. We then use this convergence to show that all well behaved points are regular. The difference between mean curvature flow without boundary and mean curvature flow with Neumann free boundary conditions cases is that we show that under either of our regularity assumption frameworks the limit boundary, $\partial M_T$, is quite simply not there with respect to $\mathcal{H}^n$ measure. It then follows that the singular set on the boundary is also, measure theoretically speaking, not there.

We will now have to be careful about which radius we use around a central point. The radius around a central point in which all results will hold is dependent on the point itself but always greater than zero. Presently we can make the following definitions concerning the radius around which everything holds.

**Proposition 18.0.1.**
*Let $x_0 \notin \Sigma$. Then there exists a radius $l_{x_0} > 0$ around which, when applicable, the interior versions of the Upper Area Ratio Proposition (Proposition* 15.2.1*) and the Clearing out Lemma (Lemma* 16.2.1*) hold for all radii smaller than or equal to $l_{x_0}$.*

**Definition 18.0.1.**
*Let $x_0 \in \mathbb{R}^{n+1}$. We then define for $x_0 \in \Sigma$*

$$d_{x_0} := \min\left\{\frac{1}{2\kappa_\Sigma}, \left(\frac{\tau_0}{2}\right)^2\right\}$$

*and for $x_0 \notin \Sigma$*

$$d_{x_0} := \frac{1}{2}\min\{l_{x_0}, d_\Sigma(x_0)\},$$

*where $l_{x_0}$ is that mentioned in Proposition* 18.0.1.

**Remark:** As mentioned in Chapter 12, the results for mean curvature flow without boundary hold on balls centered at points not on the boundary provided that the radius is small enough to ensure that the ball does not intersect the support surface. It follows that the boundaryless results can be taken to hold on $B_{d_{x_0}}(x_0)$ for each $x_0 \notin \Sigma$.

## 18.1 Good Points

We begin by considering the area ratio behaviour of our limit sets, which will lead to the definition and properties of the 'good points'.

**Lemma 18.1.1. (Finite Total Energy/Uniform Bound on Area Ratio)**
*Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ such that for some $x_0 \in \Sigma$ $\mathcal{M}$ is a solution in*

$$B_{d_{x_0}}(x_0) \times (t_0 - d_{x_0}^2, t_0)$$

*for some $t_0 \in [0,T)$ which satisfies the area continuity and unit density hypothesis at time $t_0$. Suppose also that for some constant $A_0 < \infty$ the uniform area bound*

$$\mathcal{H}^n(M_t \cap B_{d_{x_0}}(x_0)) \leq A_0$$

*holds for all $t \in [t_0 - d_{x_0}^2, t_0]$. Then, there exists $c_0 = c_0(n) > 1$ such that*

$$(i) \qquad \int_{t_0-\rho^2}^{t_0}\int_{M_t\cap B_d(x_0)} |\vec{H}|^2 < \infty$$

*where $d = d_{x_0}/c_0$. Furthermore, there exists $A = A(n, d_{x_0}, \kappa_\Sigma, A_0) > 0$ such that*

$$(ii) \qquad \frac{\mathcal{H}^n(M_t \cap B_\rho(x))}{\rho^n} \leq A$$

*for all $x \in B_d(x_0) \cap \Sigma$, $\rho \in (0,d]$ and $t \in [t_0 - \rho^2, t_0]$.*

**Proof:**
Without loss of generality we assume that $x_0 = 0$. Then by taking $c_0 = 3\sqrt{128n}$ we see that $d = d_{x_0}/c_0 \leq d_0/3\sqrt{40}$ and $d^2 \leq d_{x_0}^2/(9(128n))$. We can therefore apply Lemma 15.1.1 to get

$$\begin{aligned}\int_{t_0-d^2}^{t_0}\int_{M_t\cap B_d} |\vec{H}|^2 d\mathcal{H}^n dt &= \int_{t_0-d^2}^{t_0}\int_{M_t\cap B_{d_{x_0}/3\sqrt{40}}} |\vec{H}|^2 d\mathcal{H}^n dt + \mathcal{H}^n(M_t \cap B_{d_{x_0}/3\sqrt{40}}) \\ &\leq 16\mathcal{H}^n(M_{t_0-d^2} \cap B_{d_{x_0}}) \\ &\leq 16A_0 \\ &< \infty,\end{aligned}$$

for any $t \in [t_0 - d^2, t_0]$ in the second term, proving part (i).

For part (ii) we use Proposition 15.2.1 as follows. For $x \in B_d \cap \Sigma$ we have for any $\rho \in (0,d]$

$$\sup_{[t_0-\rho^2,t_0)} \frac{\mathcal{H}^n(M_t \cap B_\rho(x))}{\rho^n} \leq C(n,\kappa_\Sigma)\frac{\mathcal{H}^n(M_{t_0-d^2} \cap B_{\sqrt{2+0.4n}d}(x_0))}{(\sqrt{2+0.4n}d)^n},$$

where we have used $\rho_0 \equiv d_{x_0} = \sqrt{2+0.4n}d$. Now, if $d \leq d_0(1+\sqrt{2+0.4n})^{-1}$ then $B_{\sqrt{2+0.4n}d}(x_0) \subset B_{d_{x_0}}$ for all $x_0 \in B_d$, so that

$$\begin{aligned}\sup_{[t_0-\rho^2,t_0)} \frac{\mathcal{H}^n(M_t \cap B_\rho(x))}{\rho^n} &\leq C(n,\kappa_\Sigma)\frac{\mathcal{H}^n(M_{t_0-d^2} \cap B_{d_{x_0}}(x_0))}{(\sqrt{2+0.4n}d)^n} \\ &= C(n,\kappa_\Sigma,d)\mathcal{H}^n(M_{t_0-\rho^2} \cap B_{d_{x_0}}(x_0)) \\ &\leq C(n,\kappa_\Sigma,d)A_0.\end{aligned}$$

Thus setting $c_0 = 1+\sqrt{2+0.4n}$ gives us

$$\frac{\mathcal{H}^n(M_t \cap B_\rho(x))}{\rho^n} \leq A = A(n,d,\kappa_\Sigma,A_0)$$

for all $x \in B_d \cap \Sigma$, $\rho \in (0,d]$ and $t \in [t_0-\rho^2,t_0)$.

This does not yet give the estimate for $t = t_0$. However, for any $x_0 \in B_d$ and $\rho \in (0,d]$ the Area continuity hypothesis provides

$$\mathcal{H}^n(M_{t_0} \cap B_\rho(x)) = \lim_{t \nearrow t_0} \mathcal{H}^n(M_t \cap B_\rho(x)) \leq A\rho^n$$

so that the estimate holds for $t \in [t_0-\rho^2,t_0]$.

We therefore choose

$$c_0 = \max\{1+\sqrt{2+0.4n}, 3\sqrt{128n}\} = 3\sqrt{128n}$$

so that both $(i)$ and $(ii)$ hold simultaneously for the same $c_0$. $\diamondsuit$

It is the first part of the preceeding lemma that gives the inspiration for the definition of a good point. The smaller the bounding constant less than infinity, the better the estimate. We therefore make the following definition of good points.

**Definition 18.1.1.**
*Let $\mathcal{M} = (M_t)_{t\in[t_1,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions and let $x_0 \in \mathbb{R}^{n+1}$. For $\alpha \geq 0$ we define*

$$G^\alpha_{t_0} = \left\{x \in B_{d_{x_0}/c_0}(x_0) : \limsup_{\rho \searrow 0} \frac{1}{\rho^n}\int_{T-\rho^2}^T \int_{M_t \cap B_\rho(x_0)} |\vec{H}|^2 d\mathcal{H}^n \leq \alpha^2\right\}.$$

*In the case of regularity assumptions I we define points $x \in \Sigma \cap G^\alpha_{t_0}$ to be good points only if they also satisfy $\Theta^n(\mathcal{H}^n, M_{t_0}, x) \in \{0, 1/2\}$. In this case we continue to denote the whole set as $G^\alpha_{t_0}$.*

As mentioned above, we wish to consider only the behaviour of good points. For this to help with regularity theory we first need to show that working only with good points is justified which we do by showing that almost all points are good points. To do this, we mention the interior version of the Finite Total Energy/Uniform Bound on Area Ratio, Lemma 18.1.1 above, which by the discussion in Chapter 12 also holds in our setting.

**Lemma 18.1.2.**
*Consider a smooth, properly embedded solution $\mathcal{M} = (M_t)_{t\in[0,T)}$ of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ in an open set $U \subset \mathbb{R}^{n+1}$ which satisfies the area-continuiuty and unit density hypothesis at time $T$ as well as the boundary approaches boundary assumption at time $T$. Let $x_0 \notin \Sigma$ and $d_{x_0}$ be the radius around $x_0$ such that*

1. $B_{d_{x_0}}(x_0) \times (T - d_{x_0}^2, T) \subset U \times [0, T)$, *and*

2. *all of the 'previous' non-boundary case results hold inside of* $B_{d_{x_0}}(x_0) \times (T - d_{x_0}^2, T)$.

*Suppose also that for some constant* $A_0 < \infty$ *the uniform area bound*

$$\mathcal{H}^n(M_t \cap B_{d_{x_0}}(x_0)) \leq A_0$$

*holds for all* $t \in [T - d_{x_0}^2, T]$. *Then there exists a* $c_0 = c_0(n)$ *such that*

$$\int_{T-d^2}^{T} \int_{M_t \cap B_d(x_0)} |\vec{H}|^2 d\mathcal{H}^n < \infty$$

*for* $d = d_{x_0}/c_0$. *Furthermore, there exists a constant* $A > 0$ *which depends only on* $n, d_{x_0}$ *and* $A_0$ *such that*

$$\frac{\mathcal{H}^n(M_t \cap B_\rho(x_0))}{\rho^n} \leq A$$

*holds for all* $x \in B_d(x_0)$, $\rho \in (0, d]$ *and* $t \in [T - \rho^2, T]$.

**Remark:** By observing the proof from Ecker [7] Lemma 5.10 we see that in the interior case a sufficient $c_0$ is $c_0 = (8(1 + 2n))^{-1/2}$.

For the sake of uniformity we now choose $c_0$ so that both the boundary and interior results hold.

**Definition 18.1.2.**
*We define*

$$c_0 := \min\{3(128n)^{-1/2}, (8(1 + 2n))^{-1/2}\} = 3(128n)^{-1/2}.$$

In order to prove that almost all points are good we will also use the Vitali Covering Theorem, a standard measure theoretic result which can be stated as follows.

**Theorem 18.1.1. (Vitali's Covering Theorem)**
*Let* $\mathcal{F}$ *be any collection of non-degenerate closed balls in* $\mathbb{R}^n$ *with*

$$\sup\{diam(B) : B \in \mathcal{F}\} < \infty.$$

*Then there exists a countable family* $\mathcal{G}$ *of disjoint balls in* $\mathcal{F}$ *such that*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \hat{B},$$

*where* $\hat{B}$ *denotes the ball with the same center as* $B$ *with five times the radius.*

We are now able to show that almost all points are good points. We show this in the form of two results as the proof for the case $\alpha > 0$ does not work for $\alpha = 0$. We therefore prove the result first for the case where $\alpha > 0$ and then as a corollary show that the result is also true for the case $\alpha = 0$.

**Lemma 18.1.3.**
*Let* $\mathcal{M} = (M_t)_{t \in [0,T)}$ *be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface* $\Sigma$ *in* $U \times [t_1, T)$ *for some open set* $U \subset \mathbb{R}^{n+1}$. *Let* $x_0 \in \mathbb{R}^{n+1}$ *and* $t_0 \in (0, T]$. *Then for every* $\alpha > 0$

$$\mathcal{H}^n(B_{d_{x_0}}(x_0) \sim G_{t_0}^{\alpha}) = 0.$$

**Proof:**
Regularity assumptions I (Definition 13.2.6) require that the points $x \in \Sigma$ with $\Theta^n(\mathcal{H}^n, M_T, x) \notin \{0, 1/2\}$ (which are also in the compliment of $G_T^\alpha$) have $\mathcal{H}^n$ measure zero. Thus they do not affect the result.

Fix $\delta \in (0, d_{x_0}]$. By the definition of $G_{t_0}^\alpha$, for every $x \in B_d(x_0) \sim G_{t_0}^\alpha$ there exists a radius $\rho_x \in (0, \delta/10)$ such that $B_{\rho_x}(x) \subset B_d(x_0)$ and

$$\int_{T-\rho_x^2}^{T} \int_{M_t \cap B_{\rho_x}(x)} |\vec{H}|^2 d\mathcal{H}^n > \alpha^2 \rho_x^n.$$

The Vitali covering Theorem allows us to select a disjoint family of balls $\{B_{\rho_j}(x_j)\}_{j\in\mathbb{N}}$ with $x_j \in B_{d_{x_0}} \sim G_{t_0}^\alpha$ such that

1. $\rho_j \in (0, \delta/10)$,
2. $B_{\rho_j}(x_j) \subset B_{d_x}$, and
3. $B_{d_{x_0}}(x_0) \sim G_{t_0}^\alpha \subset \bigcup_{j=1}^\infty B_{5\rho_j}(x_j)$, and

$$\int_{T-\rho_j^2}^{T} \int_{M_t \cap B_{\rho_j}(x_j)} |\vec{H}|^2 d\mathcal{H}^n > \alpha^2 \rho_j^n.$$

By noting that the $B_{\rho_j}(x_j)$'s are disjoint we can therefore estimate the $\mathcal{H}_\delta^n$-measure of $B_{d_{x_0}}(x_0) \sim G_{t_0}^\alpha$ by

$$\begin{aligned}
\mathcal{H}_\delta^n(B_{d_{x_0}}(x_0) \sim G_{t_0}^\alpha) &= \inf\left\{\sum_{j=1}^\infty \omega_n \left(\frac{diam(C_j)}{2}\right)^n : B_{d_{x_0}}(x_0) \sim G_{t_0}^\alpha \subset \bigcup_{j=1}^\infty C_j, diam(C_j) \le \delta\right\}, \\
&\le \sum_{j=1}^\infty \omega(5\rho_j)^n \\
&= c(n)\omega_n \sum_{j=1}^\infty \rho_j^n \\
&< \frac{c(n)}{\alpha^2} \sum_{j=1}^\infty \int_{T-\rho_j^2}^{T} \int_{M_t \cap B_{\rho_j}(x_j)} |\vec{H}|^2 d\mathcal{H}^n dt \\
&\le \frac{c(n)}{\alpha^2} \sum_{j=1}^\infty \int_{T-\delta^2}^{T} \int_{M_t \cap B_{\rho_j}(x_j)} |\vec{H}|^2 d\mathcal{H}^n dt \\
&= \frac{c(n)}{\alpha^2} \int_{T-\delta^2}^{T} \int_{M_t \cap \cup_{j=1}^\infty B_{\rho_j}(x_j)} |\vec{H}|^2 d\mathcal{H}^n dt \\
&\le \frac{c(n)}{\alpha^2} \int_{T-\delta^2}^{T} \int_{M_t \cap B_{d_{x_0}}(x_0)} |\vec{H}|^2 d\mathcal{H}^n dt.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\mathcal{H}^n(B_{d_{x_0}}(x_0) \sim G_{t_0}^\alpha) &= \lim_{\delta\to 0} \mathcal{H}_\delta^n(B_{d_{x_0}}(x_0) \sim G_{t_0}^\alpha) \\
&\le \frac{c(n)}{\alpha^2} \lim_{\delta\to 0} \int_{T-\delta^2}^{T} \int_{M_t \cap B_{d_{x_0}}(x_0)} |\vec{H}|^2 d\mathcal{H}^n dt \\
&= 0,
\end{aligned}$$

which completes the proof. $\diamondsuit$

By proving the corollary that the same result also holds for $\alpha = 0$ we end our discussion on good points and thus the section. We then consider the properties of good points that lead to important results for regularity theory; results on the characterisation of the structure of the boundary of the limit.

**Corollary 18.1.1.**
*Let $\mathcal{M} = (M_t)_{t\in[t_1,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$. Then*

$$\mathcal{H}^n(B_{d_{x_0}/c_0}(x_0) \sim G_T^0) = 0.$$

**Proof:**
From Lemma 18.1.3 it is known that for each $n \in \mathbb{N}$ $\mathcal{H}^n(B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/n}) = 0$. We now claim $B_{d_{x_0}/c_0}(x_0) \sim G_T^0 \subset \bigcup_{n=1}^{\infty} B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/n}$. Indeed, let $x \in B_{d_{x_0}/c_0}(x_0) \sim G_T^0$, then

$$\limsup_{\rho\searrow 0} \rho^{-n} \int_{T-\rho^2}^{T} \int_{M_t\cap B_\rho(x_0)} |\vec{H}|^2 d\mathcal{H}^n =: \beta > 0$$

($\beta = \infty$ is possible). Then there exists $N \in \mathbb{N}$ such that $1/N < \beta$. Thus

$$\limsup_{\rho\searrow 0} \rho^{-n} \int_{T-\rho^2}^{T} \int_{M_t\cap B_\rho(x_0)} |\vec{H}|^2 d\mathcal{H}^n > 1/N$$

which implies that $x \notin G_T^{1/N}$ and hence $x \in B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/N}$ so that

$$x \in \bigcup_{n=1}^{\infty} B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/n}$$

from which the claim that

$$B_{d_{x_0}/c_0}(x_0) \sim G_T^0 \subset \bigcup_{n=1}^{\infty} B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/n}$$

follows.

We can now caluculate

$$\mathcal{H}^n(B_{d_{x_0}/c_0}(x_0) \sim G_T^0) \leq \mathcal{H}^n\left(\bigcup_{n=1}^{\infty} B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/n}\right) \leq \sum_{n=1}^{\infty} \mathcal{H}^n(B_{d_{x_0}/c_0}(x_0) \sim G_T^{1/n}) = 0.$$

$\diamondsuit$

## 18.2 Non-existence of Boundary Approximate Tangent Spaces

In this section we prove the final preliminary lemmas for global regularity. Having shown that inside of appropriate balls, almost all points are good, we now show that under either set of assumptions almost all good points are regular. In the following final section we then appropriatly assemble the balls and zero sets to provide the final full global regularity result. As this section is the one where the teo forms of regularity assumptions are most vital we recall them by restating them below:

**Definition 13.2.6. (Regularity Assumptions I)**
*Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions supported on a Neumann free boundary support surface $\Sigma$. Let $t_0 \in (0,T]$. Then $\mathcal{M}$ is then said to satisfy the* **regularity assumptions I** *at time $t_0$ if $\mathcal{M}$ satisfies the area continuity and unit density hypothesis at time $t_0$ as well as the boundary approaches boundary and unit multiplicity assumptions.*

**Definition 13.2.8. (Regularity Assumptions II)**
*Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions supported on a Neumann free boundary support surface $\Sigma$. Let $t_0 \in (0,T]$. $\mathcal{M}$ is then said to satisfy the* **regularity assumptions II** *at time $t_0$ if $\mathcal{M}$ satisfies the area continuity and unit density hypothesis at time $t_0$, the boundary area continuity hypothesis at time $t_0$, the boundary approaches boundary assumption at time $t_0$ and the Type I assumption at time $t_0$.*

The main result in this section is that under either set of regularity assumptions we can ensure that there is $\mathcal{H}^n$-almost nowhere an approximate tangent space to the limit surface on the support surface. Under regularity assumptions I this follows directly from the definition of an approximate tangent space. Under regularity assumptions II we need to work harder, first proving another technical lemma before proving the result. We begin by noting that almost all of $\mathbb{R}^{n+1}$ can be separated into two sets, each satisfying a useful measure theoretic result.

**Theorem 18.2.1.**
*Let $A$ be a measurable countably $n$-rectifiable subset of $\mathbb{R}^{n+1}$ then for $\mathcal{H}^n$-almost all $x_0 \in \mathbb{R}^{n+1}$ either*

1. $\Theta^n(\mathcal{H}^n, A, x_0) = \lim_{\rho\searrow 0} \frac{\mathcal{H}^n(A\cap B_\rho(x_0))}{\omega_n \rho^n} = 0$, *or*

2. *the approximate tangent space $T_xA$ of $A$ at $x$ exists. That is,*

$$\lim_{\lambda\searrow 0} \int_{\eta_{x_0,\lambda}(A)} \phi d\mathcal{H}^n = \int_{T_xA} \phi d\mathcal{H}^n$$

   *for all $\phi \in C_0^0(\mathbb{R}^{n+1})$.*

For our purposes, the above theorem can be translated as in the following corollary.

**Corollary 18.2.1.**
*Let $\mathcal{M} = (M_t)_{[t_1,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions satisfying the $M_T$ rectifiability condition. Then for any $x_0 \in \mathbb{R}^{n+1}$ and any $x \in B_{d_{x_0}}(x_0)$, either*

1. $\Theta^n(\mathcal{H}^n, M_T, x) = 0$, *or*

2. *the approximate tangent space $T_xM_T$ of $M_T$ at $x$ exists. That is,*

$$\lim_{\lambda\searrow 0} \int_{M_T^{x_0,\lambda}} \phi d\mathcal{H}^n = \int_{T_xM_T} \phi d\mathcal{H}^n$$

   *for all $\phi \in C_0^0(\mathbb{R}^{n+1})$.*

*Here $M_T^{x_0,\lambda} = \lambda^{-1}(M_T - x_0)$, $\lambda > 0$.*

We show firstly that the above theorem allows us to prove that under regularity assumptions I $\Theta^n(\mathcal{H}^n, M_T, x) = 0$ $\mathcal{H}^n$-almost everywhere on $\Sigma$.

**Theorem 18.2.2.**
*Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions satisfying the regularity assumptions I, then for $\mathcal{H}^n$-almost all $x \in \Sigma$ we have $\Theta^n(\mathcal{H}^n, M_T, x) = 0$.*

**Proof:**
From Corollary 18.2.1 we need only show that for $\mathcal{H}^n$ almost all points $x \in \Sigma$ with $\Theta^n(\mathcal{H}^n, M_T, x) \neq 0$ there is no approximate tangent space. Regularity assumptions I tells us that for $\mathcal{H}^n$-almost all points $x \in \Sigma$, $\Theta^n(\mathcal{H}^n, M_T, x) \in \{0, 1/2\}$ so that we need only show that $\Theta^n(\mathcal{H}^n, M_T, x) = 1/2$ prevents the existence of an approximate tangent space. Assume this is not the case, then there is a point $x$ such that $\Theta^n(\mathcal{H}^n, M_T, x) = 1/2$ and such that $T_xM_T$ exists.

Take any $\phi \in C^1_C(\mathbb{R}^{n+1}, \mathbb{R})$ with $\chi_{B_{(\frac{3}{4})^{1/n}}(0)} \leq \phi \leq B_1(0)$. Since $T_xM_T$ exists

$$\lim_{\lambda\searrow 0}\int_{\eta_{x,\lambda}(M_T)} \phi d\mathcal{H}^n = \int_{T_xM_T} \phi d\mathcal{H}^n > \frac{3}{4}\omega_n.$$

Since also $\Theta^n(\mathcal{H}^n, M_T, x) = 1/2$ we have

$$\frac{1}{2} = \lim_{\lambda\searrow 0}\omega_n^{-1}\lambda^{-n}\mathcal{H}^n(M_T \cap B_\lambda(x)) = \omega_n^{-1}\lim_{\lambda\searrow 0}\int_{\eta_{x,\lambda}(M_t)}\chi_{B_1(0)}d\mathcal{H}^n > \omega_n^{-1}\lim_{\lambda\searrow 0}\int_{\eta_{x,\lambda}(M_t)}\phi d\mathcal{H}^n > \frac{3}{4}.$$

This contradiction proves the result. ♢

We can now show that for almost all points, that is at least good points in $G^\alpha_T$, we can control the convergence of the measures to the limit measure in terms of sufficiently small $\alpha$. Since we can allow $\alpha$ to go to zero, this provides very good, and as we shall see sufficient, control over the convergence of the measures.

**Lemma 18.2.1.**
*Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions satisfying either regularity assumptions I or II supported on the support surface $\Sigma$ and $\alpha \in (0, 1/2]$. Then for every $x_0 \in G^\alpha_T$ there exists a radius $\rho_0 \in (0, d_{x_0}/c_0]$ such that*

$$\sup_{t\in[T-\rho_0^2,T]}\left|\int_{M_t}\phi d\mu_t - \int_{M_T}\phi d\mu_t\right| \leq 2\alpha(\sup|\phi| + \sqrt{A}\rho\sup|D\phi|)\rho^n$$

*holds for all $\rho \in (0, \rho_0]$ and $\phi \in C^1_0(B_\rho(x_0))$.*

**Proof:**
By the definition of good points, we can find, for every $x_0 \in G^\alpha_T$, a $\rho_0 \in (0, d]$ ($d = d_{x_0}/c_0$) such that

$$\int_{T-\rho^2}^T\int_{M_t\cap B_\rho(x_0)}|\vec{H}|^2 d\mu_t dt \leq 4\alpha^2\rho^n \tag{18.1}$$

for each $\rho \in (0, \rho_0]$.

We then note that for any $\phi \in C^1_0(B_d(x_0))$

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t}\phi d\mu_t &= \int_{M_t}\frac{\partial\phi}{\partial t}d\mu_t + \int_{M_t}\phi\left(\frac{\partial}{\partial t}d\mu_t\right)\\ &= \int_{M_t}D\phi\cdot\frac{\partial}{\partial t}x - \phi|\vec{H}|^2 d\mu_t\\ &= \int_{M_t}D\phi\cdot\vec{H} - \phi|\vec{H}|^2 d\mu_t \qquad (18.2)\end{aligned}$$

as $(\partial/\partial t)x = \vec{H}$ for all $x \in M_t$. Since $\mathcal{M}$ satisfies the area continuity hypothesis we also know that

$$\lim_{t\to t_0} \int_{M_t} \phi d\mu_t = \int_{M_{t_0}} \phi d\mu_{t_0}.$$

Integrating now (18.2) with respect to $t$ over $[t_2, T]$ for any $t_2 \in [T-\rho^2, T]$ we have

$$\int_{t_2}^{T} \frac{d}{dt} \int_{M_t} \phi d\mu_t = \lim_{t\to T} \int_{M_t} \phi d\mu_t - \int_{M_{t_2}} \phi d\mu_{t_2} = \int_{M_T} \phi d\mu_T - \int_{M_{t_2}} \phi d\mu_{t_2}.$$

Thus

$$\begin{aligned}
\left| \int_{M_T} \phi d\mu_T - \int_{M_{t_2}} \phi d\mu_{t_2} \right| &= \left| \int_{t_2}^{T} \frac{d}{dt} \int_{M_t} \phi d\mu_t \right| \\
&= \left| \int_{t_2}^{T} \int_{M_t} D\phi \cdot \vec{H} - \phi |\vec{H}|^2 d\mu_t \right| \\
&\leq \int_{T-\rho^2}^{T} \int_{M_t} |D\phi||\vec{H}| + |\phi||\vec{H}|^2 d\mu_t,
\end{aligned}$$

and therefore

$$\sup_{t\in[T-\rho^2,T]} \left| \int_{M_{t_2}} \phi d\mu_{t_2} - \int_{M_T} \phi d\mu_T \right| = \left| \int_{M_T} \phi d\mu_T - \int_{M_{t_2}} \phi d\mu_{t_2} \right| \leq \int_{T-\rho^2}^{T} \int_{M_t} |D\phi||\vec{H}| + |\phi||\vec{H}|^2 d\mu_t dt.$$

for all $\phi \in C_0^1(B_\rho(x_0))$ and $\rho \in (0, \rho_0]$. We rewrite this as

$$\begin{aligned}
\int_{T-\rho^2}^{T} \int_{M_t} |D\phi||\vec{H}| + |\phi||\vec{H}|^2 d\mu_t dt &= \int_{T-\rho^2}^{T} \int_{M_t} |D\phi||\vec{H}| d\mu_t dt + \int_{T-\rho^2}^{T} \int_{M_t} |\phi||\vec{H}|^2 d\mu_t dt \\
&=: I_1 + I_2.
\end{aligned}$$

Using (18.2) it firstly follows that

$$I_2 :\leq \sup |\phi| \int_{T-\rho^2}^{T} \int_{M_t} |\vec{H}|^2 d\mu_t dt \leq \sup |\phi| 4\alpha^2 \rho^n.$$

Then using the Cauchy-Schwarz inequality, $\left( \left| \int fg d\mu \right|^2 \leq \int |f|^2 d\mu \int |g|^2 d\mu \right)$, (18.2), and Lemma 18.1.1 we have

$$\begin{aligned}
I_1 &= \left( \left| \int_{T-\rho^2}^{T} \int_{M_t} |D\phi||\vec{H}| d\mu_t dt \right|^2 \right)^{1/2} \\
&\leq \left( \int_{T-\rho^2}^{T} \int_{M_t} |D\phi|^2 d\mu_t dt \int_{T-\rho^2}^{T} \int_{M_t} |\vec{H}|^2 d\mu_t dt \right)^{1/2} \\
&\leq \left( 4\alpha^2 \rho^n \int_{T-\rho^2}^{T} \int_{M_t} |D\phi|^2 d\mu_t dt \right)^{1/2} \\
&= 2\alpha \left( \rho^n \sup |D\phi|^2 \int_{T-\rho^2}^{T} \int_{M_t \cap B_\rho(x_0)} 1 d\mu_t dt \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= 2\alpha\left(\rho^n \sup|D\phi|^2\rho^n\int_{T-\rho^2}^{T}\frac{\mathcal{H}^n(M_t\cap B_\rho(x_0))}{\rho^n}dt\right)^{1/2}\\
&\leq 2\alpha\left(\rho^n \sup|D\phi|^2\rho^n\int_{T-\rho^2}^{T}Adt\right)^{1/2}\\
&= 2\alpha\rho^n\sqrt{A}\rho\sup|D\phi|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sup_{t\in[T-\rho^2,T]}\left|\int_{M_{t_2}}\phi d\mu_{t_2}-\int_{M_T}\phi d\mu_T\right| &\leq \int_{T-\rho^2}^{T}\int_{M_t}|D\phi||\vec{H}|+|\phi||\vec{H}|^2 d\mu_t dt\\
&= I_1+I_2\\
&\leq 4\alpha^2\rho^n\sup|\phi|+2\alpha\rho^n\sqrt{A}\rho\sup|D\phi|\\
&\leq 2\alpha(\sup|\phi|+\sqrt{A}\rho\sup|D\phi|)\rho^n
\end{aligned}
$$

for all $\phi\in C_0^1(B_\rho(x_0))$ and all $\rho\in(0,\rho_0]$. ♢

As we have mentioned, the question of whether the points in $\Sigma\mathcal{M}_T$ are regular or not can be completely avoided in proving global regularity results. We show that our results, especially Lemma 18.2.1 can be combined with the rectifiable limit surface and boundary approaches boundary assumption to show that there will be $\mathcal{H}^n$ almost no points in $\Sigma\cap M_T$ at all. In particular we show that for any $x\in\Sigma\cap M_T$, there cannot exist an approximate tangent space to $M_T$ at $x_0$. We have actually already shown that this is the case under regularity assumptions I in Theorem 18.2.2. We now show in the following Lemma the equivalent result under regularity assumptions II. The proof is much more involved and depends heavily on the assumptions. The corollary following this lemma is the main result of the section that will be used in the next section when assembling the proof of the global regularity theory.

**Lemma 18.2.2.**
*Let $\mathcal{M}=(M_t)_{t\in[t_1,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ satisfying the regularity assumptions II. Then there exists an $\alpha>0$ depending only on the type I curvature constant $C_H$, $n$, and the $A$ found in Lemma 18.1.1 such that if $x_0\in G_T^\alpha\cap\Sigma$ for any $\alpha\leq\alpha_0$, then either*

1. *$x_0$ is not reached by $\mathcal{M}$, or*
2. *$T_{x_0}M_T$ does not exist.*

**Proof:**
If $x_0$ is not reached by $\mathcal{M}$ then we are done.

Now suppose $\mathcal{M}\to_T x_0$. We first show that should $T_{x_0}M_T$ exist then we must have $T_{x_0}M_T=T_{x_0}\Sigma$.

To prove this we can firstly, without loss of generality, assume that $x_0=0$ and $T_{x_0}\Sigma(=T_0\Sigma)=\mathbb{R}^n$.

Since $\Sigma$ is smooth and satisfies the rolling ball condition for balls up to a maximum radius of $1/\kappa_\Sigma$ we see that for a sufficiently small $\rho<1/2\kappa_\Sigma$, $\Sigma$ can be expressed as the smooth graph of a function $f_\Sigma$, over $\mathbb{R}^n\cap B_\rho^n(0)$ and that $\Sigma$ divides $B_\rho(0)$ into two parts expressible as the part 'above' $\Sigma$

$$\{x\in\mathbb{R}^{n+1}:x_{n+1}>f_\Sigma(x_1,...,x_n)\}$$

and the part 'below' $\Sigma$

$$\{x \in \mathbb{R}^{n+1} : x_{n+1} < f_\Sigma(x_1, ..., x_n)\}$$

and such that one of these parts is empty and $M_T \cap B_\rho(0)$ is a subset of $\Sigma$ in union with the other. Without loss of generality we will assume that $M_T$ is contained in the latter. That is

$$M_T \cap B_\rho(0) \subset \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq f_\Sigma(x_1, ..., x_n)\}.$$

Also, since $\Sigma$ is smooth with $Df_\Sigma(0) = 0$, for every $m \in \mathbb{N}$ there exists a $\lambda_m \leq \rho$ such that for all $\lambda < \lambda_m$ and $x \in \Sigma \cap B_\lambda(0)$ $x_{n+1} < \lambda/m$ and thus

$$x_{n+1} < \lambda/m, \text{ for all } x \in M_{t_0} \cap B_\lambda(0) \tag{18.3}$$

We now choose a sequence of functions $\{\phi_m\}_{m\in\mathbb{N}} \subset C^1_C(\mathbb{R}^{n+1}, \mathbb{R})$ such that

$$\phi_m = 1 \text{ on } B_{1/2}(0) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq 2/m\}$$

and

$$\phi_m = 0 \text{ outside of } B_1(0) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} > 1/m\}.$$

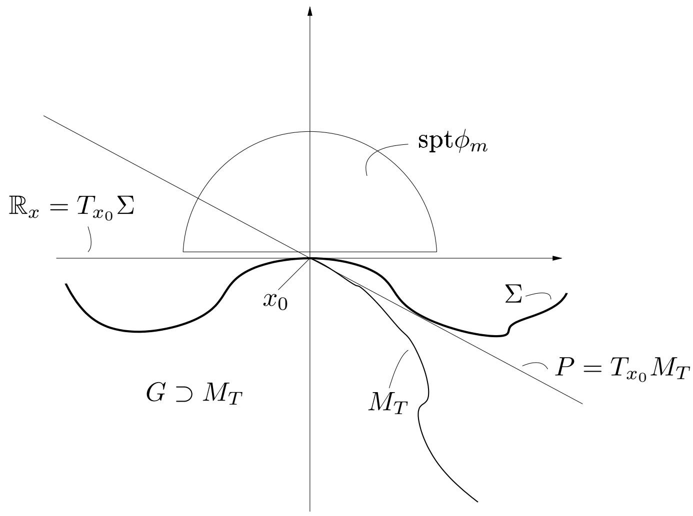


Figure 18.1: $T_{x_0}M_T$ exists $\Rightarrow T_{x_0}M_T = T_{x_0}\Sigma$

By (18.3) we see that for each $\lambda < \lambda_m$

$$\begin{aligned}
\lambda^{-1}M_T \cap \text{spt } \phi_m &= \lambda^{-1}(M_T \cap \text{spt } \phi_m) \\
&= \lambda^{-1}(M_T \cap B_\lambda(0) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} > \lambda/m\}) \\
&= \lambda^{-1}(\emptyset) \\
&= \emptyset.
\end{aligned}$$

Thus $\int_{M_T^\lambda} \phi_m d\mathcal{H}^n = 0$ for all $\lambda \leq \lambda_m$ (where $M_T^\lambda := \lambda^{-1}M_T$) and thus

$$\lim_{\lambda\to 0} \int_{M_T^\lambda} \phi_m d\mathcal{H}^n = 0 \text{ for all } m \in \mathbb{N}.$$

Since also for any hyperplane $P \neq \mathbb{R}^n (= T_{x_0}\Sigma)$ there exists a $m_P \in \mathbb{N}$ such that $P \cap B_{1/2}(0) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq 2/m_P\} \neq \emptyset$, it follows that $\int_P \phi_{m_P} d\mathcal{H}^n \neq 0$ and thus $\lim_{\lambda\to 0} \int_{M_T^\lambda} \phi_{m_P} d\mathcal{H}^n \neq \int_P \phi_{m_P}$. Consequently $T_{x_0}\Sigma$ is the only possible approximate tangent space. We complete the proof by showing that $T_{x_0}\Sigma$ is also not possible.

Since $\mathcal{M} \to_T x_0$, by the boundary approaches boundary assumption $\partial\mathcal{M} \to_T x_0$ and thus from Proposition 13.1.2 (which uses the type I assumption) there exists a $p \in \partial M^n$ such that $\lim_{t\to T} F(p,t) = x_0$. Thus by Lemma 13.1.1 we have

$$|F(p,t) - x_0| \leq 2C_H\sqrt{T-t}$$

and thus under parabolic rescaling

$$|\lambda^{-1}F(p, \lambda^2 s + T) - x_0| \leq \lambda^{-1} 2C_H \sqrt{T - \lambda^2 s - T} = \lambda^{-1} s C_H \sqrt{|s|}\lambda = 2C_H\sqrt{|s|}.$$

Thus for any chosen $s$ and sequence $\lambda_j \searrow 0$, $\lambda_j^{-1}F(p, \lambda_j^2 + T) - x_0$ is an infinite sequence in a compact set and thus there exists a subsequence $\lambda_{j_k}$ which we relabel $\lambda_j$ satisfying

$$\lim_{j\to\infty} \lambda_j^{-1} F(p, \lambda_j^2 s + T) - x_0 = x_1 \in \overline{B_{2C_H\sqrt{|s|}}(0)}.$$

Further, since $|A|^2 \leq C_H^2(T-t)^{-1}$ we have $|A_{M_s^\lambda}| \leq \lambda C_H(\lambda^2|s|)^{-1/2} = C_H(|s|)^{-1/2}$. We now select a particular $s$ which we will denote $\hat{s}$. The particular $s$ we choose is $\hat{s} := -16C_H^2$. We also select a sequence $\lambda_j \searrow 0$. Together these selctions give us $|A_{M_{\hat{s}}^{\lambda_j}}| \leq 4$ for all $j \in \mathbb{N}$ and

$$\lim_{j\to\infty} \lambda_j^{-1}(F(p, \lambda_j^2 s + T) - x_0) = x_1 \in \overline{B_{8C_H^2}(0)} \cap \Sigma' = \overline{B_{8C_H^2}(0)} \cap T_{x_0}\Sigma.$$

We now need to select test functions $\phi_j \in C_C^1(\mathbb{R}^{n+1}, \mathbb{R})$. In order to do so we need the following definitions:

$$\hat{x_j} := x_j - \nu_{\Sigma_{\lambda_j}}(x_j)$$

where $\nu_{\Sigma_{\lambda_j}}(x_j)$ is the vector unit normal to $\Sigma_{\lambda_j} = \lambda_j^{-1}(\Sigma - x_0)$ at $x_j \in \Sigma_{\lambda_j}$ with base point $x_j$,

$$B_j := B_{1/2}(\hat{x_j}) \cap T_{x_j} M_{\hat{s}}^{\lambda_j},$$

$$I_j := \{\hat{x_j} + s\nu_j : s \in [-1,1] \text{ and } \nu_j = \text{unit normal to } T_{x_j}M_{\hat{s}}^{\lambda_j} \text{ at } \hat{x_j}\}$$

$$S_j := B_j \times I_j$$

and for $S \subset \mathbb{R}^{n+1}$ and $r > 0$ we define

$$N^r(S) := \{x \in \mathbb{R}^{n+1} : d(x, S) < r\}.$$

Before defining the family $\phi_j$ we make the following observations. Firstly, for all sufficiently large $j$, say $j \geq j_0$, $\lambda_j \kappa_\Sigma < 1/5$ so that for each $j \geq j_0$

$$N^{1/4}(S_j) \cap \Sigma_{\lambda_j} = \emptyset.$$

It then follows from the fact that $|A_{M_{\hat{s}}^{\lambda_j}}| \leq 1/4$ for each $j \in \mathbb{N}$ that

$$M_{\hat{s}}^{\lambda_j} \cap S_j \neq \emptyset, \quad \partial M_{\hat{s}}^{\lambda_j} \cap S_j = \emptyset \text{ and}$$

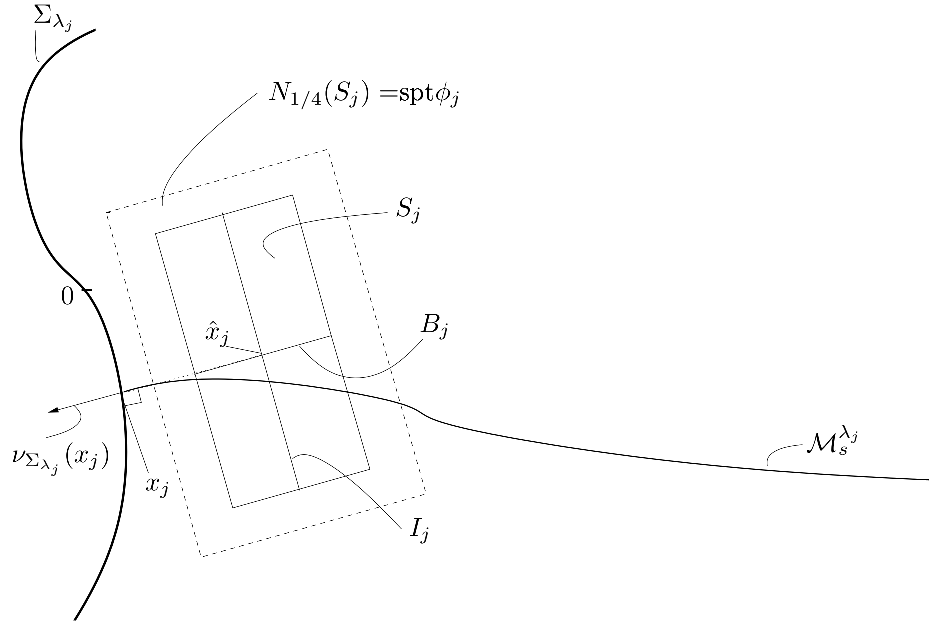


Figure 18.2: Support of a $\phi_j$

$$M_{\hat{s}}^{\lambda_j} \cap ((B_j \times (\hat{x_j} + \nu_j)) \cup (B_j \times (\hat{x_j} - \nu_j))) = \emptyset$$

and thus indeed that $M_s^{\lambda_j} \cap S_j$ is a boundaryless $n$-surface transversing the cylinder $S_j$. Thus $M_s^{\lambda_j} \cap S_j$ has measure at least that of an $n$-disc of radius 1/2. That is

$$\mathcal{H}^n(M_{\hat{s}}^{\lambda_j} \cap S_j) \geq \left(\frac{1}{2}\right)^n \omega_n =: c_n > 0. \tag{18.4}$$

We now define $\{\phi_j\}_{j=1}^{\infty} \subset C_C^1(\mathbb{R}^{n+1}, \mathbb{R})$ such that for each $j \in \mathbb{N}$, $\phi_j = 1$ on $S_j$ and $\phi_j = 0$ outside of $N^{1/4}(S_j)$. It follows from (18.4) that

$$\int_{M_{\hat{s}}^{\lambda_j}} \phi_j d\mu_{\hat{s}}^{\lambda_j} \geq c_n,$$

for each $j \geq j_0$.

Since from the Arzela-Ascoli Theorem we know that there is a subsequence of $\lambda_j$ which we again relabel $\lambda_j$ such that $M_{\hat{s}}^{\lambda_j} \to M'_{\hat{s}}$ smoothly for some smooth limit surface $M'_{\hat{s}}$, we see that $\nu_{\Sigma_{\lambda_j}}(x_j) \to \nu'(x_1)$ with $\nu(x_1)$ a unit normal to $T_{x_1}\Sigma' = T_{x_0}\Sigma'$. Define $\hat{x_i} = x_1 - \nu'(x_1)$. Also $T_{x_j} M_s^{\lambda_j} \to T'$, an $n$-plane with $\nu'(x_1) \subset T'$. Define

$$B' = B_{1/2}(\hat{x_1}) \cap T' \text{ and}$$

$$I' = \{x = \hat{x_1} + s\nu_{T'}(\hat{x_1}) : s \in [-1, 1] \text{ and } \nu_{T'} \text{ is a unit normal to } T'\}.$$

We write $S' = B' \times I'$.

We see that since $T_{x_0}\Sigma = T_{x_1}\Sigma'$ is a hyperplane parallel to the axis $\nu_{T'}(\hat{x_1})$ of the limit cylinder $S'$, and thus

$$N^\alpha(I') \cap \Sigma' = \emptyset \text{ for all } \alpha \in [0, 1/2). \tag{18.5}$$

From the convergence as sets of $M_{\hat{s}}^{\lambda_j}$ we see that for all sufficiently large $j$, say $j \geq j_1 \geq j_0$,

$$S_j \subset N^{1/4}(I'). \tag{18.6}$$

We now take any $\psi \in C_C^1(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying $\sup|\psi| = 1$, $\sup|D\psi| \leq 16$, $\psi = 1$ on $N^{1/4}(S')$ and spt $\psi \subset N^{3/8}(S')$. From (18.6) it follows that for all $j \geq j_2$

$$\int_{M_{\hat{s}}^{\lambda_j}} \psi d\mathcal{H}^n \geq \int_{M_{\hat{s}}^{\lambda_j}|_{\{x\in\mathbb{R}^{n+1}:\phi_j(x)=1\}}} \phi_j d\mathcal{H}^n \geq c_n.$$

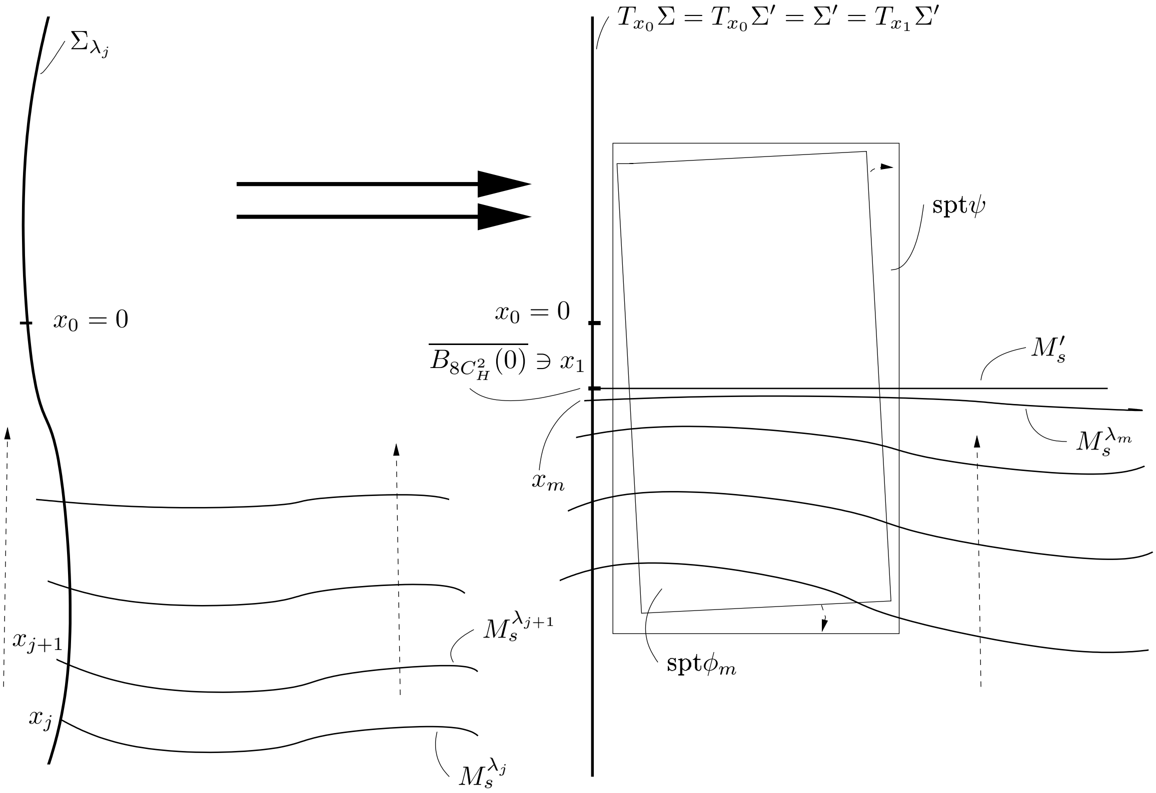


Figure 18.3: Support of the limiting test function

Since $T_{x_0}M_T = T_{x_0}\Sigma = T_{x_0}\Sigma' = T_{x_1}\Sigma'$ we see from (18.5) that

$$\lim_{\lambda\to 0} \int_{M_0^\lambda} \psi d\mathcal{H}^n = \int_{T_{x_1}\Sigma'} \psi d\mathcal{H}^n = 0.$$

It follows that for all sufficiently large $j$, say $j \geq j_3 \geq j_2$

$$\int_{M_{\hat{s}}^{\lambda_j}} \psi d\mathcal{H}^n \geq c_n \quad \text{and} \quad \int_{M_0^{\lambda_j}} \psi d\mathcal{H}^n < c_n/2$$

so that

$$\left|\int_{M_0^{\lambda_j}} \psi d\mathcal{H}^n - \int_{M_{\hat{s}}^{\lambda_j}} \psi d\mathcal{H}^n\right| > c_n/2. \tag{18.7}$$

We now note that since $x_0 \in G_T^0 \subset G_T^\alpha$ for all $\alpha > 0$ and in particular for

$$\alpha \leq \alpha_0 := \frac{c_n}{2(2(1+\sqrt{A}192C_H^2)(12C_H^2)^n)}.$$

Lemma 18.2.1 shows that there exists a $\rho_\alpha > 0$ such that

$$\sup_{t\in[T-\rho^2,T]}\left|\int_{M_t} \phi - \int_{M_T} \phi\right| \leq 2\alpha(\sup|\phi| + \sqrt{A}\rho\sup|D\phi|)\rho^n$$

holds for all $\rho \in (0,\rho_\alpha]$ and $\phi \in C_C^1(B_\rho(x_0))$. We rescale this to

$$\left|\int_{M_{\hat{s}}^\lambda} \phi - \int_{M_0^\lambda} \phi\right| \leq 2\alpha(\sup|\phi| + \sqrt{A}R\sup|D\phi|)R^n$$

for $R \in (0, \lambda^{-1}\rho_\alpha]$, $s \in [-R^2, 0]$ and $\phi \in C_C^1(B_R(0))$. Choosing $R = 12C_H^2$ we see that $\hat{s} = -16C_H^2 \in [-R^2, 0]$ and that since $x_1 \in B_{8C_H^2}(0)$

$$\psi \in C_C^1(B_{8C_H^2+4}(0)) \subset C_C^1(B_R(0)).$$

Now taking a $\mathbb{N} \ni j_4 \geq j_3$ such that for all $j \geq j_4$ $R \in (0, \lambda_j^{-1}\rho_\alpha]$ it follows that for all $j \geq j_4$

$$\left|\int_{M_{\hat{s}}^{\lambda_j}} \psi - \int_{M_0^{\lambda_j}} \psi\right| \leq 2\alpha(\sup|\psi| + \sqrt{A}R\sup|D\psi|)R^n \leq 2\alpha(1+\sqrt{A}192C_H^2)(12C_H^2)^n \leq \frac{c_n}{2}.$$

This contradiction to (18.7) shows that

$$\lim_{\lambda\to 0}\int_{M_0^\lambda} \psi d\mathcal{H}^n \neq \int_{T_{x_0}\Sigma} \psi d\mathcal{H}^n.$$

Thus $T_{x_0}\Sigma$ cannot be an approximate tangent space for $M_T$ at $x_0$ and consequently there cannot exist and approximate tangent space for $M_T$ at $x_0$. $\diamondsuit$

As was mentioned previously we now combine the above lemma, Theorem 18.2.2 and the interior results of Ecker [7] to give a corollary that holds for all good points which will be the center of the final global regularity result. The complete proof of the corollary is by no means trivial, but is already proven in Ecker [7] and will therefore not be presented here.

**Corollary 18.2.2.**
*Let $\mathcal{M} = (M_t)_{t\in[t_1,T)}$ be a smooth, properly embedded, solution of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ satisfying either the regularity assumptions I or II. Then*

*(i) For sufficiently small $\alpha = \alpha(n, A)$ the points $x_0 \in G_T^\alpha$ satisfying* 1. *in Corollary* 18.2.1 *cannot be reached at time $T$ by the solution $\mathcal{M}$. Therefore these points are regular.*

*(ii) If we choose $\alpha > 0$ with $\alpha = \alpha(n, A, \alpha_0, \varepsilon_0)$ sufficiently small then for points in $G_T^\alpha$ possessing an approximate tangent space the conditions of the Interior Local Regularity Theorem are satisfied and thus such points are also regular.*

**Proof:**
To prove (i), we note firstly that for interior points application of Ecker's ([7]) Lemma 15.5 gives the proof. Secondly, we note that for points on $\Sigma$ by using Lemma 16.2.1 instead of the Clearing out Lemma of Ecker [7], the proof of (i) follows exactly as in Ecker's ([7]) Lemma 5.15.

In case (ii), by taking $\alpha \leq \alpha_0$ for $\alpha_0$ as found in Lemma 18.2.2 we know that for each $x_0 \in G_T^{\alpha}$, $x_0 \notin \Sigma$ as for all $x_0 \in \Sigma$ points possess no approximate tangent space. For the interior points, by selecting $\rho_0$ sufficiently small to avoid the boundary (which is possible as discussed in Chapter 12) in the proof of Lemma 5.15 of [7], the proof follows in an otherwise identical manner to that of Lemma 5.15 in [7]. $\diamondsuit$

## 18.3 Global Regularity

We now assemble the proof of the global regularity theorem. We will use Corollary 18.2.2 as well as the fact that almost all points are well behaved to show that around every point there exists a neighbourhood in which there is at most an $\mathcal{H}^n$-zero measure set of singularities. The proof of the final global regularity theorem is then a question of covering theorems to show that we can appropriately cover the entire space with such balls, or compact sets covered by such balls to conclude that the measure of the entire singularity set has zero measure. In assembling this proposed proof of global regularity the following technical covering result becomes important.

**Lemma 18.3.1.**
*Let $U$ be open in $\mathbb{R}^{n+1}$, then there exists a countable collection of compact sets $\{W_j\}_{j\in\mathbb{N}}$ such that*

$$\bigcup_{j=1}^{\infty} W_j = U.$$

**Proof:**
Let $D_j$ be the set of $(n+1)$-dimensional dyadic rationals of order $j$. Define

$$D_{j_U} := \{x \in D_j : B_{1/2^{j+1}}(x) \subset U\}$$

and

$$W_j = \bigcup_{x\in D_{j_U}} B_{1/2^{j+1}}(x).$$

Let $y \in U$. Then as $U$ is open there exists an $r > 0$ such that $B_r(y) \subset U$ and an $j \in \mathbb{N}$ such that $r > (\sqrt{n}/4 + 2)2^{j-1}$. Also there must be an $x \in D_j$ such that $|y - x| < \sqrt{n}2^{-j-1}$. For this $x$ $B_{1/2^{j+1}}(x) \ni y$ and $B_{1/2^{j+1}}(x) \subset B_r(y) \subset U$ so that $B_{1/2^{j+1}}(x) \subset W_j$ and therefore $y \in W_j$.

Thus for all $y \in U$ $y \in W_j$ for some $j \in \mathbb{N}$ and therefore

$$U \subset \bigcup_{j\in\mathbb{N}} W_j. \tag{18.8}$$

Since, by construction, $W_j \subset U$ for each $j \in \mathbb{N}$,

$$\bigcup_{j\in\mathbb{N}} W_j \subset U.$$

Combining this with (18.8) gives

$$U = \bigcup_{j \in \mathbb{N}} W_j$$

which completes the proof. $\diamondsuit$

We now show that far each $x_0 \in \mathbb{R}^{n+1}$ there exists a neighbourhood with only a $\mathcal{H}^n$ neglible set of singularities and then as the final global regularity theorem show that this is sufficient to prove the full global regularity theorem.

**Theorem 18.3.1.**
*Let $\mathcal{M} = (M_t)_{t \in [t_1, T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ satisfying either the regularity assumptions I or II. Suppose that $x_0 \in \mathbb{R}^{n+1}$ and that*

$$\mathcal{H}^n(M_t \cap B_{d_{x_1}}(x_0)) < A_0$$

*holds for all $[T - d_{x_1}^2, T]$ for some $d_{x_1} \leq d_{x_0}$. Then*

$$\mathcal{H}^n(sing_T \mathcal{M} \cap B_d(x_0)) = 0$$

*where $d = d_{x_1}/c_0$ and $c_0 > 1$ is the constant dependent only on $n$ and $\kappa_\Sigma$ found in Lemma* 18.1.1.

**Proof:**
By Lemma 18.1.1, the results of Lemma 18.1.1 hold on $B_d(x_0)$ and therefore so do Lemmas 18.1.3 and 18.2.1 and thus additionally Lemma 18.2.2 and Corollary 18.2.2.

We now choose an $\alpha$ such that Lemma 18.2.2 holds. Define

$$D_1 := B_d(x_0) \cap \{x \in \mathbb{R}^{n+1} : \Theta^n(M_T, x, \mathcal{H}^n) = 0\}$$

and

$$D_2 := B_d(x_0) \cap \{x \in \mathbb{R}^{n+1} : T_x M_T \text{ exists}\}.$$

Then

$$B_d(x_0) = (B_d(x_0) \sim G_T^\alpha) \cup (G_T^\alpha \cap D_1) \cup (G_T^\alpha \cap D_2) \cup (G_T^\alpha \sim (D_1 \cup D_2))$$

and thus

$$\begin{aligned} sing_T \mathcal{M} \cap B_d(x_0) &= (sing_T \mathcal{M} \cap B_d(x_0) \sim G_T^\alpha) \cup (sing_T \mathcal{M} \cap G_T^\alpha \cap D_1) \\ &\quad \cup (sing_T \mathcal{M} \cap G_T^\alpha \cap D_2) \cup (sing_T \mathcal{M} \cap G_T^\alpha \sim (D_1 \cup D_2)). \end{aligned}$$

Now, from Lemma 18.1.3

$$\mathcal{H}^n(B_d \sim G_T^\alpha) \leq \mathcal{H}^n(B_{d_{x_0}}(x_1) \sim G_T^\alpha) \leq \mathcal{H}^n(B_{d_{x_0}}(x_0) \sim G_T^\alpha) = 0$$

and thus

$$\mathcal{H}^n(sing_T \mathcal{M} \cap B_d(x_0) \sim G_T^\alpha) = 0.$$

Similarly it follows from Corollary 18.2.1 that

$$\mathcal{H}^n(G_T^\alpha \sim (D_1 \cup D_2)) = 0$$

and thus

$$\mathcal{H}^n(sing_T \mathcal{M} \cap G_T^\alpha \sim (D_1 \cup D_2)) = 0.$$

Moreover, it follows from Lemma 18.2.2 that $sing_T\mathcal{M} \cap D_i = \emptyset$ for each $i \in \{1,2\}$ and thus

$$\mathcal{H}^n(sing_T\mathcal{M} \cap G_T^\alpha \cap D_i) = \mathcal{H}^n(\emptyset) = 0 \quad \text{for each } i \in \{1,2\}.$$

Combining the above shows

$$\begin{aligned}\mathcal{H}^n(sing_T\mathcal{M} \cap B_d(x_0)) &= \mathcal{H}^n(sing_T\mathcal{M} \cap B_d(x_0) \sim G_T^\alpha) + \mathcal{H}^n(sing_T\mathcal{M} \cap G_T^\alpha \cap D_1) \\ &\quad + \mathcal{H}^n(sing_T\mathcal{M} \cap G_T^\alpha \cap D_2) + \mathcal{H}^n(sing_T\mathcal{M} \cap G_T^\alpha \sim (D_1 \cup D_2)). \\ &= 0.\end{aligned}$$

$\diamondsuit$

We now show that we can indeed satisfy the $\mathcal{H}^n(M_t \cap B_{d_{x_1}}(x_0)) < A_0$ requirement around any $x \in \mathbb{R}^{n+1}$ and then appropriately cover all of $\mathbb{R}^{n+1}$ in order to infer that the $\mathcal{H}^n$-measure of $sing_T\mathcal{M}$ is zero. This result concludes the main body of Part II of the thesis, and therefore, apart from appendices, the thesis in general.

**Theorem 18.3.2. (Main Regularity Theorem)**
*Let $\mathcal{M} = (M_t)_{t\in[t_1,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ in $U \times [t_1, T)$, where $U$ is an open subset of $\mathbb{R}^{n+1}$, which satisfies either the regularity assumptions I or II at time $T$. Then*

$$\mathcal{H}^n(sing_T\mathcal{M}) = 0.$$

**Proof:**
Let $x_0 \in \Sigma$. Then for

$$\rho_{x_1} := \rho_{x_1}(x_0) = \min\{d_{x_0}(20^{253\kappa_\Sigma\sqrt{128n}})^{-1}, \sqrt{T}\} < d_{x_0}$$

we can apply Corollary 15.1.2 to find

$$\mathcal{H}^n(B_{\rho_{x_1}}(x_0) \cap M_t) \leq \mathcal{H}^n(B_{d_{x_0}(20^{253\kappa_\Sigma\sqrt{128n}})^{-1}}(x_0) \cap M_t) \leq 16^{2T3\kappa_\Sigma\sqrt{128n}}\mathcal{H}^n(B_{d_{x_0}}(x_0) \cap M_0)$$

for all $t \in [0,T) \supset [T - \rho_{x_1}^2, T)$.

By applying Corollary 15.2.1 we therefore have

$$\mathcal{H}^n(B_{\rho_{x_1}}(x_0) \cap M_t) \leq C(n, \kappa_\Sigma, x_0)16^{2T3\kappa_\Sigma\sqrt{128n}}\rho_{x_1}^n =: A_{x_0}(x_0) =: A_{x_0} < \infty$$

for all $t \in [T - \rho_{x_1}^2, T)$.

Similary, for $x_0 \notin \Sigma$ we can apply Proposition 4.9 in [7] finitely many times as in [7] to show that there exists a $\rho_{x_1} := \rho_{x_1}(x_0) \leq d_{x_0}$ such that

$$\mathcal{H}^n(B_{\rho_{x_1}}(x_0) \cap M_t) \leq A_{x_0}(x_0) =: A_{x_0}$$

for all $t \in [T - \rho_{x_1}^2, T)$.

Thus, for all $x_0 \in \mathbb{R}^{n+1}$ there exists a $\rho_{x_1} = \rho_{x_1}(x_0)$ and $A_{x_0} < \infty$ such that

$$\mathcal{H}^n(B_{\rho_{x_1}}(x_0) \cap M_t) \leq A_{x_0}$$

for all $t \in [T - \rho_{x_1}^2, T)$ and thus by the area continuity hypothesis (Definition 13.2.3)

$$\mathcal{H}^n(B_{\rho_{x_1}}(x_0) \cap M_t) \leq A_{x_0}(x_0) =: A_{x_0}$$

for all $t \in [T - \rho_{x_1}^2, T]$.

From Theorem 18.3.1 it then follows that for each $x_0 \in \mathbb{R}^{n+1}$ there exists a $\rho_{x_0}(x_0) = \rho_{x_1}(x_0)/c_0 > 0$ such that

$$\mathcal{H}^n(sing_T\mathcal{M} \cap B_{\rho_{x_0}}(x_0)) = 0.$$

Now, let $W$ be a compact subset of $U$. We see that

$$W \subset \bigcup_{x \in W} B_{\rho_x}(x).$$

Since $W$ is compact it follows that we can take some finite subcover of $W$, $\{B_{\rho_i}(x_i)\}_{i=1}^Q$, from the cover $\{B_{\rho_x}(x)\}_{x \in W}$ to get

$$W \subset \bigcup_{i=1}^{Q} B_{\rho_i}(x_i)$$

with $\mathcal{H}^n(sing_T\mathcal{M} \cap B_{\rho_i}(x_i)) = 0$ for each $i \in \{1, ..., Q\}$. It follows that

$$0 \leq \mathcal{H}^n(sing_T\mathcal{M} \cap W) \leq \sum_{i=1}^{Q} \mathcal{H}^n(sing_T\mathcal{M} \cap B_{\rho_i}(x_i)) = 0.$$

Since $W$ was an arbitrary compact subset of $U$, we see that by taking a countable collection of compact subsets of $U$, $\{W_i\}_{i=1}^\infty$ with

$$U = \bigcup_{i=1}^{\infty} W_i,$$

whose existence is guaranteed by Lemma 18.3.1, we get

$$0 \leq \mathcal{H}^n(sing_T\mathcal{M} \cap U) \leq \sum_{i=1}^{\infty} \mathcal{H}^n(sing_T\mathcal{M} \cap W_i) = 0$$

and therefore

$$\mathcal{H}^n(sing_T\mathcal{M}) = 0.$$

$\diamondsuit$

## 18.4 Notes

Proposition 18.0.1 is due to Ecker [7]. The concept of good points was introduced by Ecker [7] and the idea for Lemma 18.1.1 follows that of Ecker [7], though the adjustments to the Neumann free boundary conditions case are our own. Lemma 18.1.2 is the interior version, due to Ecker [7] of Lemma 18.1.1. Vitali's Covering Theorem, Theorem 18.1.1, is in principle due to Vitali [31] who proved the first theorem of this type. A good discussion of The Vitali Covering Theorem can be found in Bartle [4] or Evans and Gariepy [10]. Lemma 18.1.3, showing almost all points are $G_T^\alpha$ follows that of Ecker [7]. The generalisation of Lemma 18.1.3, Corollary 18.1.1 is our own. Theorem 18.2.1 is a standard geometric measure theoretic result, for a good discussion one can see, for example, Simon [25]. Lemma 18.2.1 is our own though draws ideas and inspiration from the non-boundary case due to Ecker [7]. Lemma 18.2.2 is our own original work. Corollary 18.2.2 is our own though, as obvious from the proof, depends heavily on Ecker's non boundary version in [7]. We understand the result of Lemma 18.3.1 to be a standard result and lay no claim to it. We have, however, no source, and the proof given is our own. The ideas in Theorems 18.3.1 and 18.3.2 are inspired by the non boundary versions to be found in Ecker [7]. The theorems, as presented here, however, are our own.