# Chapter 17

# Local Regularity

In this chapter we show local regularity, which, as mentioned in the introduction(s), is understood here to be referring to conditions by which we can determine whether or not a point is regular. The condition that we will use to determine the regularity of a point is the boundedness of the difference between the Gaussian density ratio and its expected value in a neighbourhood of the point.

In order to prove local regularity we will show that having a Gaussian density ratio close enough to the value of the Gaussian density for a point around which the flow is smoothly expressible as a graph, implies that the second fundamental form of the surface is bounded from above, in some small ball, for all small enough $t$. This then leads to $x_0$ being a regular point.

## 17.1 Bounding the Second Fundamental Form

We show in this section that the appropriately bounded Gaussian density ratio inside of a neighbourhood around the observed point leads to a uniform bound on the second fundamental form for all times up to the observed time in some neighbourhood of the observed point. Before attacking that proof directly we find the following technical lemma useful.

**Lemma 17.1.1.**
*Suppose a sequence of smooth subsets of space-time $\{(M_s^j)_{s\leq 0}\}_{j\in\mathbb{N}}$ converges smoothly to a limiting family $(M_s')$ and that $\{\rho_j\}_{j\in\mathbb{N}}$ and $\{f_j\}_{j\in\mathbb{N}}$ are sequences of positive functions in $C^2(\mathbb{R}^{n+1}\times(-\infty,0],\mathbb{R})$ such that $\rho_j\to\rho$ and $f_j\to 1$ uniformly on compact subsets of $\mathbb{R}^{n+1}\times(-\infty,0]$ where $\rho$ is as defined in Definition 11.3.4. Suppose also that there exists a constant $C>0$ such that*

$$C\leq\int_{M_s^j} f_j\rho_j d\mathcal{H}^n\leq C+\frac{1}{j}$$

*for all $j\in\mathbb{N}$. Then*

$$\int_{M_s'}\rho d\mathcal{H}^n=C$$

*for all $s\leq 0$.*

**Proof:**
We recall that $(M_s^j)\to(M_s')$ smoothly implies

$$\lim_{j\to\infty}\int_{M_s^j}\psi d\mathcal{H}^n=\int_{M_s'}\psi d\mathcal{H}^n$$

for each $\psi \in C^1_C(\mathbb{R}^{n+1}, \mathbb{R})$. For each $\delta > 0$ and $k \in \mathbb{N}$ we see, by considering $\psi \in C^1_C(\mathbb{R}^{n+1}, \mathbb{R})$ with $\chi_{B_k} \leq \psi \leq \chi_{B_{k+1}}$, so that $\psi\rho \in C^1_C(\mathbb{R}^{n+1}, \mathbb{R})$ and using that the $f_j$ and $\rho_j$ converge locally uniformly, that

$$\limsup_{j\to\infty} \int_{M^j_s} f_j \rho_j d\mathcal{H}^n \geq \limsup_{j\to\infty} \int_{M^j_s \cap B_{k+1}} f_j \rho_j \psi d\mathcal{H}^n \geq (1-\delta) \limsup_{j\to\infty} \int_{M^j_s} \rho\psi d\mu_s \geq (1-\delta) \int_{M'_s \cap B_k} \rho d\mu'_s$$

for all $s \leq 0$ and similarly that

$$\liminf_{j\to\infty} \int_{M^j_s \cap B_k} f_j \rho_j d\mathcal{H}^n \leq (1+\delta) \liminf_{j\to\infty} \int_{M^j_s} \rho\psi d\mathcal{H}^n = (1+\delta) \int_{M'_s} \rho\psi d\mu_s.$$

for all $s \leq 0$.

Since this is true for all $k \in \mathbb{N}$ we see

$$(1-\delta) \int_{M'_s} \rho d\mathcal{H}^n = (1-\delta) \int_{\cup_{k\in\mathbb{N}}(M'_s \cap B_k)} \rho d\mu_s = \liminf_{k\to\infty} (1-\delta) \int_{M'_s \cap B_k} \rho d\mu_s \leq \liminf_{j\to\infty} \int_{M^j_s} f_j \rho_j d\mathcal{H}^n$$

for all $s \leq 0$, and

$$\limsup_{j\to\infty} \int_{M^j_s \cap B_k} f_j \rho_j d\mathcal{H}^n$$

is an increasing sequence in $k$ bounded above by $(1+\delta) \int_{M'_s} \rho d\mathcal{H}^n$ so that

$$\limsup_{j\to\infty} \int_{M^j_s \cap B_k} f_j \rho_j d\mathcal{H}^n = \limsup_{j\to\infty} \int_{\cup_{k\in\mathbb{N}}(M^j_s \cap B_k)} f_j \rho_j d\mathcal{H}^n \leq (1+\delta) \int_{M'_s} \rho d\mathcal{H}^n$$

for all $s \leq 0$. Therefore

$$(1-\delta) \int_{M'_s} \rho d\mu_s \quad \leq \quad \liminf_{j\to\infty} \int_{M^j_s} f_j \rho_j d\mathcal{H}^n \leq \limsup_{j\to\infty} \int_{M^j_s} f_j \rho_j d\mathcal{H}^n \leq (1+\delta) \int_{M'_s} \rho d\mu_s$$

for all $s \leq 0$.

Since this is true for each $\delta > 0$ it follows that

$$\lim_{j\to\infty} \int_{M^j_s} f_j \rho_j d\mathcal{H}^n = \int_{M'_s} \rho d\mathcal{H}^n$$

for all $s \leq 0$ and thus, since

$$C \leq \int_{M^j_s} f_j \rho_j d\mathcal{H}^n \leq C + \frac{1}{j},$$

we have

$$\lim_{j\to\infty} C \leq \lim_{j\to\infty} \int_{M^j_s} f_j \rho_j d\mathcal{H}^n \leq \lim_{j\to\infty} C + \frac{1}{j}$$

for all $s \leq 0$, and thus

$$\int_{M'_s} \rho d\mu_s = C$$

for all $s \leq 0$. ◇

In the main technical lemma of the local regularity result we will find it necessary to directly use Huisken's [14] monotonicity results for boundaryless flow and thus state his result directly for reference.

**Theorem 17.1.1. (Huisken's Monotonicity Formula)**
*Let $\mathcal{M} = (M_t)_{t\in I}$ be a mean curvature flow (without boundary) over the time interval $I$. Let $\rho$ denote the usual backward heat kernel as defined in Definition* 11.3.4. *Then*

$$\frac{d}{dt}\left(\int_{M_t} \rho d\mu_t\right) = -\int_{M_t} \left|\vec{H} - \frac{D^{\perp}\rho}{\rho}\right|^2 d\mu_t.$$

We can now prove the main technical lemma, bounds on the second fundamental form, which we then use in the next section to prove the main local regularity theorem.

**Lemma 17.1.2.**
*Let $\mathcal{M} = (M_t)_{[t_1,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions supported on the support surface $\Sigma$ with $\mathcal{M} \to_T x_0$. Let $\rho_1 \in (0, (2\kappa_\Sigma)^{-1})$, $\rho_2 > 0$. Then there exist constants $1/2 > \varepsilon_0 > 0$ and $c_0 > 0$ such that whenever*

$$\Theta(\mathcal{M}, y, \tau, t) \le \Xi(\Theta, y, \tau) + \varepsilon_0$$

*for all $(y, \tau) \in B_{\rho_1}(x_0) \times (T - \rho_1^2, T]$ and $t \in [\tau - \rho_2^2, \tau]$ then*

$$|A(x,t)|^2 \le c_0 \rho^{-2}$$

*for some $\rho > 0$ and for all $x \in M_t \cap B_\rho(x_0)$ and $t \in (T - \rho^2, T)$. Here*

$$\Theta(\mathcal{M}, y, \tau, t) := \begin{cases} e^{C\kappa_\Sigma^{2\delta}(\tau - t)^\delta} \int_{M_t} \varphi_{\sigma,y,\tau}\eta_{y,\tau}\rho_{\kappa_\Sigma,y,\tau} d\mu_t & y \in \Sigma, \\ \int_{M_t} \psi_{\sigma(y),y,\tau}\rho_{(t,\tau)} d\mu_t & y \notin \Sigma \end{cases},$$

*where $\sigma \in (0, \tau_0^{1/2}/2)$, $\sigma(y) := \frac{1}{2}\sqrt{d(y,\Sigma)^2 + 8n}$, $\psi_{\sigma(y),y,\tau}$ is as defined in Definition* 12.0.1 *and*

$$\Xi(\Theta, y, \tau) := \begin{cases} 1 & y \in M_\tau \sim \Sigma \\ \frac{1}{2} & y \in M_\tau \cap \Sigma \\ 0 & \textit{otherwise} \end{cases}$$

**Proof:**
We prove firstly that the result holds with $\varepsilon_0$ only being required to be greater than zero. Since making $\varepsilon_0$ smaller only improves the behaviour of $\Theta(\mathcal{M}, y, \tau, t)$ it follows that should the result hold for some $\varepsilon_0 > 1/2$ then it also holds for $\varepsilon_1 := 1/4$ in the place of $\varepsilon_0$. In this case, by redefining $\varepsilon_0$ to equal $\varepsilon_1 = 1/4$ the result holds for an $\varepsilon_0 < 1/2$ as required. Thus proving the result with the only requirement on $\varepsilon_0$ being strict positivity is sufficient.

We first claim that we can assume without loss of generality that $\mathcal{M} = (M_t)_{t\in[t_1,t_0]}$ is smooth up to and including time $t_0$. We can do this as otherwise we would apply the following proof to the flow up to time $t_0 - \delta$ for some $\delta > 0$. By considering $\mathcal{M}$ as the images of the one-parameter family of functions $F_t$ there is a $p_{x_0} \in M_0$ such that $F_{t_0}(p_{x_0}) = x_0$, we then see that

$$|A(x)|^2 \le c_0 \rho^{-2}$$

for some $\rho > 0$ and for all $x \in M_t \cap B_\rho(F_{t_0-\delta}(p_{x_0})$ and $t \in (t_0 - \delta - \rho^2, t_0 - \delta)$. Since this is true for each $\delta > 0$ ($c_0$, $\rho$ not being dependent on $t$ or $x_0$) we see that for all $x \in B_\rho(x_0)$ and $t \in (t_0 - \rho^2, t_0)$ there exists a $\delta_0 > 0$ such that $(x,t) \in B_\rho(F_{t_0-\delta}(p_{x_0})) \times (t_0 - \delta - \rho^2, t_0 - \delta)$ for all $\delta < \delta_0$ and thus $|A(x)|^2 \le c_0\rho^{-2}$ for all $(x,t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$.

We further assume without loss of generality (since it is just a reorientation of coordinates) that

$(x_0, t_0) = (0, 0)$.

We next claim that by the hypotheses, and by scaling the solution if necessary, that it is sufficient to prove the following statement (Statement A):

There exists constants $\varepsilon_0$ and $c_0$ such that whenever a smooth, properly embedded solution $\mathcal{M} = (M_t)$ of mean curvature flow with Neumann free boundary conditions supported on a support surface $\Sigma$ with $\kappa_\Sigma \leq 1$ reaches $0 \in \mathbb{R}^{n+1} \cap \Sigma$ at time 0 and satisfies

$$\Theta(\mathcal{M}, y, \tau, t) \leq \Xi(\Theta, y, \tau) + \varepsilon_0$$

for all $(y, \tau) \in B_1 \times (-1, 0)$ and $t \in [\tau - 1, \tau]$ then

$$\sigma^2 \sup_{(-(1-\sigma^2),0)} \sup_{M_t \cap B_{1-\sigma}} |A|^2 \leq c_0$$

for some $\sigma \in (0, 1)$. We now show that we can make this assertion. From the hypotheses we have

$$\Theta(\mathcal{M}, y, \tau, t) \leq \Xi(\Theta, y, \tau) + \varepsilon_0$$

for all $(y, \tau) \in B_{\rho_1}(x_0) \times (T - \rho_1^2, T)$ and $t \in [\tau - \rho_2, \tau]$. If $\rho_1, \rho_2 > 1$ and $\kappa_\Sigma < 1$ we are done, otherwise rescaling by a factor of $\lambda = \min\{\rho_1, \rho_2, \kappa_\Sigma^{-1}\}$ with the change of variables $y = \lambda x + x_0$ and $\tau = \lambda^2 s + T$ gives $\Sigma' = \lambda^{-1}\Sigma$ and $\kappa_{\Sigma'} = \lambda\kappa_\Sigma \leq \kappa_\Sigma$. Also $\kappa_\Sigma \leq 1$. Notationally, after rescaling, we write $M_t \to M_s$, $d\mu_t = \lambda^n d\mu_s$, and $\rho_1, \rho_2 \geq 1$. In the case $y \in \Sigma$ we also have $\eta_{y,\tau} \to \tilde{\eta}_{x,s}$, $\varphi_{\sigma,y,\tau} \to \tilde{\varphi}_{(x,s),\lambda^{-1}\sigma}$, $\rho_{\kappa_\Sigma,y,\tau} \to \tilde{\rho}_{(x,s),\lambda\kappa_\Sigma}$ and $e^{C\kappa_\Sigma^{2\delta}(-t)^\delta} \to e^{C(\lambda\kappa_\Sigma)^{2\delta}(-s)^\delta}$ as described in previous chapters. We see in this case that then

$$\Xi(\Theta, x, s) \leq e^{C(\lambda\kappa_\Sigma)^{2\delta}(-s)^\delta} \int_{M_s} \tilde{\varphi}_{\lambda^{-1}\sigma,x,s}\tilde{\eta}_{x,s}\tilde{\rho}_{\lambda\kappa_\Sigma,x,s} d\mu_s \leq \Xi(\Theta, x, s) + \varepsilon_0$$

for all $(x, s) \in B_1 \times [-1, 0)$ and $t \in [s-1, s]$. Similarly, in case $y \notin \Sigma$ we have $\psi_{\sigma(y),y,\tau} \to \psi_{\lambda^{-1}\sigma(y),x,s}$ and in this case

$$\Xi(\Theta, x, s) \leq \int_{M_s} \psi_{\lambda^{-1}\sigma(y),x,s}\rho_{x,s} d\mu_s \leq \Xi(\Theta, x, s) + \varepsilon_0$$

for all $(x, s) \in B_1 \times [-1, 0)$ and $t \in [s - 1, s]$. That is, in all cases after rescaling we have $\Xi(\Theta, x, s) \leq \Theta((M_s), x, s, t) \leq \Xi(\Theta, x, s) + \varepsilon_0$ for all $(x, s) \in B_1 \times [-1, 0)$ and $t \in [s - 1, s]$.

Should this then imply that there exists a $\sigma \in (0, 1)$ such that

$$\sigma^2 \sup_{t\in(-(1-\sigma)^2,0)} \sup_{M_s \cap B_{1-\sigma}} |A|^2 \leq c_0$$

then if $\sigma \geq 1/2$, then $1 - \sigma \leq \sigma$ giving

$$(1 - \sigma)^2 \sup_{t\in(-(1-\sigma)^2,0)} \sup_{M_s \cap B_{1-\sigma}} |A|^2 \leq c_0$$

and thus

$$\sigma'^2 \sup_{t\in(-(\sigma')^2,0)} \sup_{M_s \cap B_{\sigma'}} |A|^2 \leq c_0,$$

where $\sigma' := 1 - \sigma$ and if $\sigma \leq 1/2$ then $\sigma \leq 1 - \sigma$ so that

$$\sigma^2 \sup_{t\in(-\sigma^2,0)} \sup_{M_s \cap B_\sigma} |A|^2 \leq c_0.$$

That is, there exists a $\sigma \in (0,1)$ such that

$$\sigma^2 \sup_{t\in(-\sigma^2,0)} \sup_{M_s\cap B_\sigma} |A|^2 \leq c_0.$$

Rescaling back to the flow $(M_t)$ supported on $\Sigma$ gives

$$\sup_{\tau\in(-\rho^2,0)} \sup_{M_t\cap B_\rho} |A|^2 \leq c_0\rho^{-2}$$

as required.

Suppose now that Statement A is not correct. Then for every $j \in \mathbb{N}$ one can find a smooth, properly embedded solution to mean curvature flow with Neumann free boundary conditions supported on a support surface $\Sigma^j$ with $\kappa_{\Sigma^j} \leq 1$, $\mathcal{M}^j = (M^j_t)_{t\in[-1,0)}$, which reaches $0 \in \mathbb{R}^{n+1} \cap \Sigma$ at time 0, and such that for some $\rho_j > 1$ and all $(y,\tau) \in B_1 \times [-1,0)$ and $t \in [\tau-1,\tau]$

$$\Theta(\mathcal{M}^j, y, \tau, t) \leq \Xi(\Theta, y, \tau) + \frac{1}{j}$$

holds, but that

$$\lim_{j\to\infty} \gamma_j^2 := \sup_{\sigma\in(0,1)} \left(\sigma^2 \sup_{(-(1-\sigma^2),0)} \sup_{M^j_t\cap B_{1-\sigma}} |A|^2\right) = \infty.$$

(Note that since $\mathcal{M}^j$ is smooth up to and including $t=0$ by assumption we have $\gamma_j^2 < \infty$ for each $j \in \mathbb{N}$.)

Then since $[0,1]$ is compact and $\gamma_j^2$ is continuous in $\sigma$ there is a $\sigma_j \in [0,1]$ for each $j \in \mathbb{N}$ such that

$$\gamma_j^2 = \sigma_j^2 \sup_{(-(1-\sigma_j^2),0)} \sup_{M^j_t\cap B_{1-\sigma_j}} |A|^2.$$

Since for $\sigma_j \in \{0,1\}$, $\gamma_j^2 = 0$ and in general $\gamma_j^2 \geq 0$, it follows that in this case $\gamma_j^2$ is constantly 0 and we hence we can choose $\sigma_j \in (0,1)$. Thus there exists, for each $j \in \mathbb{N}$, a $\sigma_j \in (0,1)$ such that

$$\gamma_j^2 = \sigma_j^2 \sup_{(-(1-\sigma_j^2),0)} \sup_{M^j_t\cap B_{1-\sigma_j}} |A|^2.$$

Further, it similarly follows that for each $j \in \mathbb{N}$ there exists $\tau_j \in [-(1-\sigma_j)^2, 0]$ and $y_j \in M^j_{\tau_j} \cap \overline{B_{1-\sigma_j}}$ so that

$$\gamma_j^2 = \sigma_j^2 |A(y_j)|^2.$$

Since

$$\left(\frac{\sigma_j}{2}\right)^2 \sup_{(-(1-(\sigma_j^2/2)),0)} \sup_{M^j_t\cap B_{1-(\sigma_j/2)}} |A|^2 \leq \gamma_j^2,$$

$$\sup_{(-(1-(\sigma_j^2/2)),0)} \sup_{M^j_t\cap B_{1-(\sigma_j/2)}} |A|^2 \leq 4\gamma_j^2\sigma_j^2 = 4|A(y_j)|^2.$$

Also, since $\sigma_j/2 < 1/2$, $1-(\sigma_j/2) \geq \sigma_j/2$ and thus $B_{\sigma_j/2} \subset B_{1-\sigma_j/2}$.

As $\tau_j \leq 0$, $\tau_j \in [-(1-\sigma_j)^2, 0]$ and $\sigma_j < 1$ imply

$$(1-(\sigma_j/2))^2 = -1+\sigma_j-(\sigma_j/2)^2 < -1+2\sigma_j-\sigma_j^2-(\sigma_j/2)^2 = -(1-\sigma_j)^2-(\sigma_j/2)^2 \leq \tau_j-(\sigma_j/2)^2,$$

we have

$$\sup_{(\tau_j-(\sigma_j/2)^2,\tau_j)}\ \sup_{M_t^j\cap B_{\sigma_j/2}} |A|^2 \leq 4\gamma_j^2.$$

We now want to rescale this sequence. Rescaling each term differently to give an appropriately convergent sequence. How we do so, however, is dependent on the sequence that we already have. We need to consider two cases:

1. There exists a subsequence $\{j_k\}$ of $\{j\}$ such that

$$d(y_{j_k}, \Sigma^{j_k})|A(y_{j_k}, \tau_{j_k})| \leq C$$

   for some fixed $C < \infty$.

2. There exists a subsequence $\{j_k\}$ of $\{j\}$ such that

$$\lim_{j\nearrow\infty} d(y_{j_k}, \Sigma^{j_k})|A(y_{j_k}, \tau_{j_k})| = \infty.$$

We first consider case 1. We note that in this case, since $|A(y_{j_k}, \tau_{j_k})| \to \infty$ we must have $d(y_{j_k}, \Sigma^{j_k}) \to 0$. We relabel the subsequence $\{j_k\}$ as simply $\{j\}$ and assume without loss of generality that $d(y_j, \Sigma^j) < 1/2$ for each $j \in \mathbb{N}$. We then define $\lambda_j = |A(y_j, \tau_j)|^{-1}$ and take parabolic blow-ups around $(P_{\Sigma^j}(y_j), \tau_j)$. (Where we recall that $P_\Sigma$ denotes the perpendicular projection onto the support surface $\Sigma$.)

Note that since $y_j \in B_{1/2}(0) \subset \Sigma^j_{1/\kappa_{\Sigma^j}}$, $P_{\Sigma^j}(y_j)$ is well defined. We see that $P_{\Sigma^j}(y_j) \in B_1(0) \cap \Sigma^j$ and thus

$$\Theta(\mathcal{M}, P_{\Sigma^j}(y_j), \tau_j, t) \leq \frac{1}{2} + \frac{1}{j} \tag{17.1}$$

for all $t \in (\tau_j - 1, \tau_j]$ for each $j \in \mathbb{N}$.

Under the parabolic rescaling by a factor of $\lambda_j$ for each $j$ we get a sequence of mean curvature flow with Neumann free boundary conditions,

$$\tilde{M}_s^j := \lambda_j^{-1}(M^j_{\lambda_j^2 s+\tau_j} - P_{\Sigma^j}(y_j)),$$

supported on the Neumann free boundary support surface

$$\Sigma_j := \lambda_j^{-1}(\Sigma^j - P_{\Sigma^j}(y_j))$$

for $s \in [-\lambda_j^2\sigma_j^2/4, 0]$ which is a rescaling with the change of variables $x = \lambda_j y + P_{\Sigma^j}(y_j)$ and $t = \lambda_j^2 s + \tau_j$.

$\tilde{\mathcal{M}}^j = (\tilde{M}_s^j)$ is a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions satisfying

$$0 \in \tilde{M}_0^j, \quad |A(z_j, 0)| = 1$$

and

$$\sup_{(-\lambda_j^2\sigma_j^2/4,0)}\ \sup_{(\tilde{M}_s^j\cap B_{\lambda_j^{-1}\sigma_j/2}(z_j)} |A|^2 \leq 4$$

where $z_j = \lambda_j^{-1}(y_j - P_{\Sigma^j}(y_j)) \in B_C(0)$ (since 1. holds).

Further $0 \in \Sigma_j$ and $\kappa_{\Sigma_j} = \lambda_j \kappa_{\Sigma^j}$. Also, from (17.1) we see

$$\Theta(\tilde{\mathcal{M}}^j, 0, 0, s) \leq \frac{1}{2} + \frac{1}{j}$$

for each $j \in \mathbb{N}$.

Since $\{z_j\}_{j\in\mathbb{N}} \subset \overline{B_C(0)}$ there exists a convergent subsequence of $\{z_j\}$ converging to some $z \in \overline{B_C(0)}$ which we relabel as simply $\{z_j\}$ again. We note that since $\lambda_j^{-2}\sigma_j^2 = \gamma_j^2 \to \infty$ we can conclude that for every $R > 0$ and for all sufficiently large $j$ dependent on $R$ we have

$$\sup_{(-(R+2C)^2,0)} \sup_{\tilde{M}_s^j \cap B_{R+2C}(z_j)} |A|^2 \leq 4$$

and thus

$$\sup_{(-R^2,0)} \sup_{\tilde{M}_s^j \cap B_R(0)} |A|^2 \leq 4.$$

Also $\kappa_{\Sigma_j} = \lambda_j \kappa_{\Sigma^j} \leq \lambda_j \to 0$ as $j \to \infty$. Moreover by the interior estimates of Theorem 11.2.2

$$\sup_{(-R^2,0)} \sup_{\tilde{M}_s^j \cap B_R(0)} |\nabla^k A|^2 \leq C_k$$

for each $k \in \mathbb{N}$ for all sufficiently large $j \in \mathbb{N}$ depending on $R$. We can therefore apply the Arzela-Ascoli Theorem to take a subsequence of $\{\tilde{M}_s^j\}$, which we immediately relabel $\{\tilde{M}_s^j\}$, which converges locally smoothly to a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions, $\mathcal{M}' = (M'_s)_{s\leq 0}$, supported on the support surface $\Sigma'$. Passing to the limit for $j \to \infty$ we get

$$0, z \in M'_0, \quad |A(z,0)| = 1 \quad \text{and } |A(y,s)|^2 \leq 4 \quad \text{for all } y \in M'_s, \quad s \leq 0$$

and $\kappa_{\Sigma'} = 0$. From $\kappa_{\Sigma'} = 0$ it follows that $\Sigma'$ is a hyperplane and since $0 \in \Sigma_j$ for all $j \in \mathbb{N}$ we also have $0 \in \Sigma'$.

We now note that $\{\tilde{M}_s^j\}$ is a smoothly convergent sequence of subsets of space-time, that $e^{C(\lambda_j^{-1}\kappa_\Sigma^j)^{2\delta}(-s)^\delta}\tilde{\varphi}_{\lambda_j^{-1},0,0}\tilde{\eta}_{0,0} \to 1$ locally uniformly, that $\tilde{\rho}_{\lambda_j\kappa_{\Sigma^j},0,0} \to \rho$ locally uniformly and that for any $R > 0$

$$\Theta(\mathcal{M}^j, 0, 0, s) \leq \frac{1}{2} + \frac{1}{j}$$

for $s \in (-R^2, 0]$ for sufficiently large $j$. It therefore follows from Lemma 17.1.1 that

$$\int_{M'_s} \rho d\mu' \leq \frac{1}{2}$$

for all $s \leq 0$. Since also $\mathcal{M}^j \to_0 0$ it follows from Proposition 16.2.1 and the monotonicity formula, since $\mathcal{M}^j$ is smooth up to and including time 0 and $0 \in \Sigma'$, that

$$\frac{1}{2} \leq \Theta(\mathcal{M}^j, 0, 0) = \int_{M'_s} \rho d\mu'$$

for all $s \leq 0$. It therefore also follows that

$$\int_{M'_s} \rho d\mu' = \frac{1}{2}$$

for all $s \leq 0$.

We can therefore, since $\Sigma'$ is a hyperplane, reflect $M'_s$ across $\Sigma'$ in the usual way (as opposed to the tilde reflection function of Definition 11.3.2) to obtain a solution of mean curvature flow without boundary $M'^*_s$. By then taking $M''_s := M'^*_s - z$ we obtain a mean curvature flow satisfying $0 \in M''_s$, $|A(0,0)| = 1$, and $|A(y,s)|^2 \leq 4$ for all $y \in M''_S$ and $s \leq 0$, and

$$\int_{M''_s} \rho d\mu_s = 1$$

for all $s \leq 0$.

We now consider case 2. In this case we take a subsequence of $\{M^j_t\}$ which we relabel $\{M^j_t\}$ such that

$$\lim_{j \to \infty} d(y_j, \Sigma^j)|A(y_j, \tau_j)| = \infty.$$

We then define $\lambda_j := |A(y_j, \tau_j)|^{-1}$ and rescale parabolically around $(y_j, \tau_j)$. Since $d(y_j, \Sigma^j)|A(y_j, \tau_j)| \to \infty$ we can assume $y_j \notin \Sigma$ for each $j \in \mathbb{N}$. Thus

$$\Theta(\mathcal{M}, y_j, \tau_j, t) \leq 1 + \frac{1}{j}$$

for all $t \in (\tau_j - 1, \tau_j)$ for each $j \in \mathbb{N}$.

Under the parabolic rescaling by a factor of $\lambda_j$ for each $j$ we get a sequence of mean curvature flows with Neumann free boundary conditions $\tilde{M}^j_s := \lambda_j^{-1}(M^j_{\lambda_j^2 s + \tau_j} - y_j)$ supported on the support surface $\Sigma_j := \lambda_j^{-1}(\Sigma_j^{-1} - y_j)$ for $s \in [-\lambda_j^2 \sigma_j^2/4, 0]$ which is a rescaling with the change of variables $x = \lambda_j y + y_j$ and $t = \lambda_j^2 s + \tau_j$. $\tilde{\mathcal{M}}^j = (\tilde{M}^j_s)$ is a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions satisfying

$$0 \in \tilde{M}^j_0, \quad |A(0,0)| = 1 \text{ and } \sup_{(-\lambda_j^2\sigma_j^2/4, 0)} \sup_{\tilde{M}^j_s \cap B_{\lambda_j^{-1}\sigma_j/2}(0)} |A|^2 \leq 4.$$

Further $d(0, \Sigma_j) = \lambda_j^{-1} d(y_j, \Sigma^j) = |A(y_j, \tau_j)| d(y_j, \Sigma^j) \to \infty$ Also

$$\Theta(\tilde{\mathcal{M}}^j, 0, 0, s) \leq 1 + \frac{1}{j}.$$

We note that since $\lambda_j^{-2}\sigma_j^2 = \gamma_j^2 \to \infty$ we conclude that for every $R > 0$ and for all sufficiently large $j$

$$\sup_{(-R^2, 0)} \sup_{\tilde{M}^j_s \cap B_R} |A|^2 \leq 4$$

and similarly since $d(0, \Sigma_j) \to \infty$ we see that for any $R > 0$ we have $B_R \cap \Sigma_j = \emptyset$ for sufficiently large $j$ dependent on $R$. Moreover, by the interior estimates of Theorem 11.2.2 of the thesis

$$\sup_{(-R^2, 0)} \sup_{\tilde{M}^j_s \cap B_R} |\nabla^k A|^2 \leq C_k$$

for each $k \in \mathbb{N}$ and for all sufficiently large $j \in \mathbb{N}$ depending on $R$.

We can therefore apply the Arzela-Ascoli Theorem and take a subsequence of $\{\tilde{M}_s^j\}$, which we immediately relabel $\{\tilde{M}_s^j\}$, which converges smoothly to a smooth, properly embedded solution $\mathcal{M}' = (M_s')_{s\leq 0}$ of mean curvature flow with Neumann free boundary conditions supported on some support surface $\Sigma'$.

Passing to the limit for $j \to \infty$ we get

$$0 \in M_S', \quad |A(0,0)| = 1 \text{ and } |A(y,s)|^2 \leq 4$$

for all $y \in M_s'$ and $s \leq 0$. Also $\Sigma' \cap B_R = \emptyset$ for all $R > 0$ and thus $\Sigma' = \emptyset$ so that $\mathcal{M}'$ is actually a mean curvature flow without boundary. We then note that

$$\varphi_{\lambda_j \sigma(y_j)}(y,s) = \left(1 - \lambda_j^2 \sigma(y_j)^{-2}(|y|^2 - 2ns)\right)_+^3 = \left(1 - \frac{4|A(y_j,\tau_j)|^{-2}}{d(y_j,\Sigma^j)^2 + 8n}(|y|^2 - 2ns)\right)_+^3$$

which, since $d(y_j,\Sigma^j)|A(y_j,\tau_j)| \to \infty$, we see converges locally uniformly to 1 on compact sets. Since also $\tilde{M}_s^j$ is a smoothly convergent sequence in space-time, for each $s \leq 0$

$$\Theta(\tilde{\mathcal{M}}^j, 0, 0, s) \leq 1 + \frac{1}{j}$$

for sufficiently large $j$ and clearly $\rho \to \rho$ we can therefore apply Lemma 17.1.1 to get

$$\int_{M_s'} \rho d\mathcal{H}^n \leq 1$$

for all $s \leq 0$. Using then Corollary 4.20 from [7] in place of Proposition 16.2.1 (Noting that we can use the result as justified in Chapter 12) we see that in the same way as in case 1 it follows that

$$\int_{M_s'} \rho d\mathcal{H}^n = 1$$

for all $s \leq 0$.

It follows that in either case we come to a smooth limit mean curvature flow $\mathcal{N} = (N_t)_{t\leq 0}$ with

$$0 \in N_0,$$

$$|A(0,0)| = 1, \tag{17.2}$$

$$|A(y,t)|^2 \leq 4 \tag{17.3}$$

for each $y \in N_t$, $t \leq 0$ and

$$\int_{N_t} \rho d\mathcal{H}^n = 1 \tag{17.4}$$

for all $t \leq 0$.

These conditions are the same as those found in the Local Regularity Theorem of Ecker in [7]. As in [7] it then follows from (17.4) and Huisken's Monotonicity Formula that $N_t$ is a homothetic solution to mean curvature flow with $|\vec{H}(y)| = D^{\perp}\rho(y)/\rho(y)$ for each $y \in N_t$ and each $t \leq 0$. It then follows as in [7], that $N_t$ is a cone with vertex at 0. Again, as in [7], it then follows by using (17.3) as in [7] that the cone is smooth, is thus a plane and therefore satisfies $|A(y,t)| = 0$ for all $y \in N_t$ and $t \leq 0$. This contradiction to (17.2) proves the Lemma. ♢

## 17.2 Local Regularity

We are now able to prove the full local regularity theorem.

**Theorem 17.2.1.**
*Suppose $\mathcal{M} = (M_t)_{t\in[0,T)}$ is a smooth, properly embedded mean curvature flow with Neumann free boundary conditions satisfying supported on the support surface $\Sigma$ satisfying the conditions of Lemma* 17.1.2 *at a point $x_0 \in \Sigma$. Then $x_0$ is a regular point.*

**Proof:**
From Lemma 17.1.2 we know that under the given conditions there exists $c_0 > 0$ and $\rho > 0$ such that

$$|A(x)|^2 \leq \frac{c_0}{\rho^2} \tag{17.5}$$

for all $x \in M_t \cap B_\rho(x_0)$ and $t \in (T - \rho^2, T)$. The interior estimates of Stahl (see [29]) then imply that for each $k \in \mathbb{N}$

$$|\nabla^k A(x)|^2 \leq \frac{c_1}{\rho^{2(k+1)}} \tag{17.6}$$

for all $x \in M_t \cap B_{\rho/(2(1+C))}(x_0)$ and $t \in (T - \rho^2/4, T)$ where $C$ is a constant depending only on $c_0$ and $\rho$.

The estimates (17.5) and (17.6) imply that there exists a $\rho_1 \leq \rho/2(1 + C)$ such that $M_t \cap B_{\rho_1}(x_0)$ can be described locally graphically. This graphical representation may be multilayered.

Should there be at least two sheets, say $M_t^1$ and $M_t^2$, converging to $x_0$ at time $T$, then by the estimates (17.5) and (17.6) each sheet can be described locally graphicaly with bounded derivatives so that we can apply the Arzela-Ascoli Theorem to give smooth limit surfaces $M_T^1$ and $M_T^2$ supported on the support surface $\Sigma \ni x_0$ so that from Proposition 16.1.4 it follows that by denoting $(M_t^i \cap B_{\rho_1}(x_0))_{t\in[T-\rho_1^2,T]}$ by $\mathcal{M}^i$ for each $i = 1, 2$ we have

$$\Theta(\mathcal{M}^i, x_0, T) = \int_{T_{x-0}M_T^i} \rho d\mathcal{H}^n = \frac{1}{2}$$

and hence

$$\Theta(\mathcal{M}, x_0, T) \geq \sum_{i=1}^{2} \Theta(\mathcal{M}^i, x_0, T) = 1.$$

Since this contradicts the hypothesis that $\Theta(\mathcal{M}, x_0, T) < 1/2 + \varepsilon < 1$ it follows that there is at most one sheet converging to $x_0$.

It thus follows that our local graphical representation of $M_t \cap B_{\rho_1}(x_0) \times (T - \rho_1^2, T)$ is single valued and since it is a mean curvature flow with Neumann free boundary conditions satisfies

$$F_t(\partial M^n) \cap B_{\rho_1}(x_0) = \Sigma \cap B_{\rho_1}(x_0) \cap M_t \quad \text{for all} \quad t \in [T - \rho_1^2, T), \text{ and}$$

$$< \nu_\Sigma, \nu > (F_t(p)) = 0 \quad \text{for all} \quad (p, t) \in \partial M^n \times (T - \rho_1^2, T)$$

for some choice of unit normal $\nu$ of $M_t$. We can therefore apply the Arzela-Ascoli Theorem to give us a limit surface $M_T$ that is smooth in $B_{\rho_2}(x_0)$ for some $\rho_2 \leq \rho_1/2$ satisfying the same curvature bounds as in (17.5) and (17.6) that is

$$|A(x, T)|^2 \leq \frac{c_0}{\rho^2} \tag{17.7}$$

for all $x \in M_t \cap B_{\rho_2}(x_0)$ and for each $k \in \mathbb{N}$

$$|\nabla^k A(x,T)|^2 \leq \frac{c_1}{\rho^{2(k+1)}} \tag{17.8}$$

for all $x \in M_t \cap B_{\rho_2}(x_0)$. Also, in passing to the limit we have

$$F_T(\partial M^n) \cap B_{\rho_2}(x_0) \subset \Sigma \cap B_{\rho_1}(x_0) \cap M_t \text{ and}$$

$$< \nu_\Sigma, \nu > (F_T(p)) = 0 \text{ for all } p \in \{p \in \partial M^n : F_T(p) \in B_{\rho_2}(x_0))\} \tag{17.9}$$

and choice of unit normal $\nu$ of $M_T$. In particular

$$< \nu_\Sigma, \nu > (x_0) = 0. \tag{17.10}$$

We now need to show that $F_T(\partial M^n) = \Sigma \cap B_{\rho_2}(x_0) \cap M_T$.

Note that since $\Sigma$ satisfies the rolling ball condition there exists some radius $\rho_3 \leq \min\{\rho_2, 1/\kappa_\Sigma\}$ such that $M_T$ is a single continuous surface. (That is we rule out the possibility of some part of $M_T$ being relatively disconnected to $x_0$ in $B_{\rho_3}(x_0)$ that would then be supported on a part of $\Sigma$ relatively disconnected to $x_0$ in $B_{\rho_3}(x_0)$.)

Now, the curvature bounds (17.7) and (17.8) together with (17.10) imply that there exists a $\rho_4 \leq \rho_3$ such that we can describe $M_T \cap B_{\rho_4}(x_0)$ as a subset of a graph over some hyperplane which satisfies the following conditions: for each $x \in M_T \cap B_{\rho_4}(x_0)$ and $\tau \in T_x M_T$

$$| < \tau, \nu_\Sigma > | < \frac{1}{2} \tag{17.11}$$

and for each $y \in \Sigma \cap B_{\rho_4}(x_0)$ and each $\eta \in T_y \Sigma$

$$| < \eta, \nu_\Sigma > | > \frac{1}{2}. \tag{17.12}$$

Now, should there exist $p \in M^n \sim \partial M^n$ with $F_T(p) \in \Sigma \cap B_{\rho_4}(x_0)$ then, as $F_T$ is a smooth embedding $F_T(p) \notin \partial F_T(M^n)$ and thus $T_{F_T(p)}\Sigma = T_{F_T(p)} M_T$. Using (17.11) and (17.12) this implies that for each unit vector $\tau \in T_{F_T(p)} M_T$

$$| < \tau, \nu_\Sigma > | \in [0, 1/2) \cap (1/2, 1] = \emptyset.$$

This contradiction shows that $F_T(M^n \sim \partial M^n) \cap \Sigma = \emptyset$ and thus using 17.9

$$F_T(\partial M^n) \cap B_{\rho_4}(x_0) = M_T \cap \Sigma \cap B_{\rho_4}(x_0) \quad \text{and} \quad < \nu_\Sigma, \nu > (F_T(p))) = 0$$

for any choice $\nu$ of unit normal to $F_T$ and all $p \in \{p \in \partial M^n : F_T(p) \in B_{\rho_4}(x_0)\}$.

Thus the smooth, orientable $n$-dimensional manifold with smooth compact boundary $M_2^n := \{p \in M^n : F_{t_0}(p) \in B_{\rho_4}(x_0)\}$ is a Neumann free boundary conditions initial surface supported on the support surface $\Sigma$. We can therefore apply Theorem 11.2.1 to deduce the existence of a $T_1 > 0$ for which there is a unique solution to mean curvature flow with Neumann free boundary conditions $(M_t')_{t \in [t_0, t_0+T_1)}$. This is a smooth extension of $\mathcal{M}$ in a neighbourhood of $x_0$. Thus, by definition, $x_0$ is a regular point. ♢

## 17.3 Notes

The results presented here are analogous to those presented without boundary in Ecker [7] which in turn draw from the results of White [32]. However, the proofs here have needed to be significantly altered from those in Ecker and White to allow for the boundary conditions. The proofs given here are our own.