Chapter 16

Gaussian Density and the Clearing Out Lemma

In this chapter we introduce the theory of Gaussian density for mean curvature flow with Neumann free boundary conditions. Gaussian density has been used to great effect by White (see, for e.g. [32]) and in Ecker [7] in proving regularity theory for the boundaryless case. Since we are following, conceptually, the same path to regularity as in [7] we need to have a concept of Gaussian density. As with the monotonicities and area bounds, the boundaryless case does not directly translate. We show however, that we can define analogies that provide sufficiently functional results.

We first define the usual Gaussian density and state its well known existence. We then define what we will use as Gaussian density in this Thesis and prove its existence. Following this we prove the technical crux of the theory, a characterisation of the Gaussian density for times previous to the first singularity time. To prove this technical proposition we will need the upper area ratio results. We follow the characterisation result with further technical lemmas leading up to the central theorem of this chapter; the well known Clearing Out Lemma. We will then be ready to consider regularity theory directly, which we do in the following two chapters.

16.1 Gaussian Density

The usual Gaussian density is the parabolic limit of the integral of the backward heat kernel. The increasing concentration over time of the backward heat kernel around its center point shows that the Gaussian density is somewhat like a time dependent asymptote of the Lebesgue density around a point. Its usefulness is dependent on the monotonicity with respect to the variable in which we take the limit. The usual Gaussian density for mean curvature flows is defined formally below.

Definition 16.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)} \subset U \times [0,T)$ be a smooth, properly embedded solution of mean curvature flow. Let $x_0 \in U$ and $t_0 \in (0,T]$ for $U \subset \mathbb{R}^{n+1}$. Then the usual Gaussian density at (x_0,t_0), denoted by $\Theta^u(\mathcal{M},x_0,t_0)$ is defined by

$$\Theta^u(\mathcal{M},x_0,t_0) := \lim_{t\to t_0}\int_{M_t}\rho d\mathcal{H}^n$$

With respect to the usual Gaussian density for mean curvature flows without boundary we have the following result. The existence of the usual Gaussian density is guaranteed by Huisken's monotonicity formula. We state this along with the interior value of the Gaussian density formally in the following proposition.

Proposition 16.1.1.
Let $\mathcal{M} = (M_t)_{t\in[t_1,T)} \subset U \times [0,T)$ be a smooth properly embedded solution of mean curvature flow. Then for all $x_0 \in \mathbb{R}^{n+1}$ and $t_0 \in (0,T]$ the usual Gaussian density exists.

Moreover, should $x_0 \in M_{t_0}$ for some time $t_0 \in (t_1, T)$ then

$$\Theta^u(\mathcal{M}, x_0, t_0) = 1.$$

In analogy to the usual Gaussian density we define a Gaussian density for mean curvature flow with Neumann free boundary conditions that we will call simply the Gaussian density. Due to its necessity in the local regularity results we define the Gaussian Density in this case by way of the approximating Gaussian density ratio.

Definition 16.1.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions on the support surface Σ. Then for any $x_0 \in \mathbb{R}^{n+1}$, $t_0 \in (0,T]$ and $\sigma \in (0, \tau_0^{1/2}/2)$ the Gaussian density ratio at time $t \in (x_0, t_0)$ is defined by

$$\Theta(\mathcal{M}, x_0, t_0, t) := e^{C\kappa_\Sigma^{2\delta}\tau_{t_0}^\delta} \int_{M_t} \varphi_{(x_0,t_0)\sigma}\eta_{(x_0,t_0)}\rho_{\kappa_\Sigma,x_0,t_0} d\mu_t.$$

The Gaussian density at (x_0, t_0) is then defined by

$$\Theta(\mathcal{M}, x_0, t_0) := \lim_{t\nearrow t_0} \Theta(\mathcal{M}, x_0, t_0, t).$$

Remark: We note that the appearance of the $\delta(\in (1/3, 2/5]$ does not effect the Gaussian density ratio or Gaussian density itself. It is only present since it is necessary in the monotonicity formulas on which the existence of the Gaussian density is dependent. We may take it that for the remainder of the thesis δ is an arbitrary but fixed element of $(1/3, 2/5]$.

As in the case of the usual Gaussian density, the existence of the Gaussian density is ensured by the monotonicity formula associated to the integral defining the density.

Proposition 16.1.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions on the support surface Σ. Then for any $x_0 \in \Sigma$ and $t_0 \in (0,T]$ the Gaussian density at (x_0, t_0) exists.

Proof:
By the monotonicity formula in Theorem 14.2.1 we see that

$$e^{C\kappa_\Sigma^{2\delta}\tau_{t_0}^\delta} \int_{M_t} \varphi_{(x_0,t_0)\sigma}\eta_{(x_0,t_0)}\rho_{\kappa_\Sigma,x_0,t_0} d\mu_t$$

is decreasing in time for $t < T$. It is also clear that the same expression is non-negative. It follows that the limit $t \nearrow t_0$ exists for any $t_0 \in (0,T)$. $\diamondsuit$

Before proving the characterisation of Gaussian density we need a standard result concerning blow up limit surfaces.

Proposition 16.1.3.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow (with or without boundary) and $x_0 \in M_{t_0}$ for some $t_0 \in (0,T)$. Then the flow $(M_t)_{t\in[0,t_0)}$ is smooth up to and including t_0 and therefore has a limiting flow under parabolic rescaling (M'_s). In this case, for all $s < 0$, M'_s also satisfies

$$M'_s = \lim_{\lambda\searrow 0} M_s^{(x_0,t_0),\lambda} = \begin{cases} T_{x_0}M_{t_0} & x_0 \notin \partial M_{t_0} \\ \Pi \cap T_{x_0}M_{t_0} & x_0 \in \partial M_{t_0} \end{cases}$$

where $T_{x_0}M_{t_0}$ is the tangent plane to M_{t_0} at x_0 and $\Pi \cap T_{x_0}M_{t_0}$ is the half tangent plane corresponding to the smooth M_{t_0} at $x_0 \in \partial M_{t_0}$. (That is, Π is the $n+1$ dimensional half space with boundary $T_{x_0}\Sigma$ for which $M_{t_0} \cap \Pi \cap B_r(x_0) \neq \emptyset$ for all $r > 0$.)

Remark: The proof exploits only the uniformly bounded second fundamental form over the entirety of the flow $(M_t)_{t\in[0,t_0)}$ so that whether or not $\mathcal{M}$ has a boundary does not affect the result at all.

We now present the characterisation of Gaussian density central to our discussion of Gaussian density in general. It says that the value of the Gaussian density for $t_0 \in [0,T)$ can be written as the integral of the usual backward heat kernel over the limiting blowup flow (M'_s).

Proposition 16.1.4.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth properly embedded solution to mean curvature flow with Neumann free boundary conditions on the support surface Σ. Let $\mathcal{M}$ reach $x_0 \in \Sigma$ at the time $t_0 \in [0,T)$. Then for any $s < 0$ and any $\delta \in (1/3, 2/5]$

$$\Theta(\mathcal{M}, x_0, t_0) = \int_{M'_s} \rho \, d\mu'_S.$$

Proof:
In the definition of Gaussian density we have implicitly chosen $\sigma \in (0, \tau_0/2)$. Select and fix any $s < 0$. We note firstly that

$$\lim_{\lambda\to 0} e^{C(-\lambda)\kappa_\Sigma^{2\delta} s^\delta} = 1$$

so that should

$$\lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho} \, d\hat{\mu}_s \tag{16.1}$$

be finite then

$$\lim_{\lambda\to 0} e^{C(-\lambda)\kappa_\Sigma^{2\delta} s^\delta} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho} \, d\hat{\mu}_s = \lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho} \, d\hat{\mu}_s,$$

and should (16.1) not be finite, then the entire limit is infinite and the proof is complete. We therefore concentrate firstly on this quantity and assume the finiteness of (16.1).

We note that since $M_s^{(x_0,t_0),\lambda} \to M'_s$ smoothly, for each function $\phi \in C_C^1(\mathbb{R}^{n+1}, \mathbb{R})$

$$\lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \phi \, d\tilde{\mu}_s = \int_{M'_s} \phi \, d\mu_s.$$

We note also that for $R_0 = (\tau_0/2)^{1/2} < 1/2\kappa_\Sigma$ (since $\delta \in (1/3, 2/5]$) the unscaled flow $\mathcal{M} = (M_t)_{t\in[0,T)}$ exists on $B_{R_0}(x_0)$ for $t \in [0,T)$. We then note that for all $t \in [t_0 - R_0^2/4n, t_0)$ and x satisfying $|x - x_0| > R_0$, $r_{x_0} > |x - x_0|^2 > R_0^2$ and thus

$$\varphi_{(x_0,t_0),\sigma}(x,t) = \left(1 - \frac{r_{x_0} - 2n(t_0 - t)}{\sigma^2}\right)_+^3 \leq \left(1 - \frac{R_0^2 - 2nR_0^2/4n}{R_0^2/2}\right)_+^3 = 0,$$

so that spt $\varphi_{(x_0,t_0),\sigma}(\cdot,t) \subset B_{R_0}(x_0)$ for each $t \in [t_0 - R_0^2/4n, t_0)$ and thus, by (2) in Proposition 14.1.2 spt $\varphi_{(x_0,t_0),\sigma}(\cdot,s) \subset B_{\lambda^{-1}R}(0)$ for all $s \in [\lambda^{-2}(R_0^2/4n), 0)$.

We also note that for any $\lambda > 0$, $M_s^{(x_0,t_0),\lambda}$ is a mean curvature flow with Neumann free boundary conditions on $B_{\lambda^{-1}R_0}(0) \times [-T/\lambda^2, 0)$ and that $\kappa_{\Sigma_\lambda} = \lambda\kappa_\Sigma$ so that

$$\lambda^{-1}R_0 < \lambda^{-1}(\tau_0/2)^{1/2} = \frac{(3/160n)^{1/\delta}}{\sqrt{2}\lambda\kappa_\Sigma}.$$

Since $(3/160n)^{1/\delta}/(\sqrt{2}\lambda\kappa_\Sigma)$ is the '$(\tau_0/2)^{1/2}$' of the rescaled flow we can apply Corollary 15.2.1 to show that there exists a constant depending only on n, κ_Σ and x_0 such that for all $R \in [0, \lambda^{-1}R_0)$ and $\tau \in [-\lambda^{-2}R_0^2/4n, 0) \subset [-R^2, 0)$

$$\mathcal{H}^n(M_\tau^{(x_0,t_0),\lambda} \cap B_R(0)) \leq C(n, 0, \kappa_\Sigma)R^n.$$

In particular, we can choose a λ_s such that $(\lambda^2 4n)^{-1}R_0^2 > |s|$ for all $\lambda < \lambda_s$ and thus for each such λ we have spt $\tilde{\varphi}_{(x_0,t_0),\sigma} \subset B_{\lambda^{-1}R_0}(0)$ and that for all $R \in (0, \lambda^{-1}R_0)$

$$\mathcal{H}^n(M_s^{(x_0,t_0),\lambda} \cap B_R(0)) \leq C(n, 0, \kappa_\Sigma)R^n.$$

We now choose an $R > 1$ and take any $\lambda < \lambda_0$ where $\lambda_0 \leq \lambda_s$ is chosen such that $\lambda_0^{-1}R_0 > R$. We now define (why we make this definition will be become apparent shortly)

$$Q := \text{integer part}(ln(\lambda^{-1}R_0 - R)) + 1.$$

Then, using Proposition 14.1.2 (1), we calculate

$$\begin{aligned}
I &:= \int_{M_s^{(x_0,t_0),\lambda} \sim B_R(0)} \hat{\varphi_\sigma}\hat{\eta}\hat{\rho}_{\kappa_\Sigma} d\hat{\mu}_s \\
&\leq \sup\hat{\varphi_\sigma} \sup\hat{\eta} \int_{(M_s^{(x_0,t_0),\lambda} \cap \text{spt } \hat{\varphi_\sigma}) \sim B_R(0)} \hat{\rho_{\kappa_\Sigma}} d\hat{\mu}_s \\
&< \left(1 + \frac{\lambda^2 4n|s|}{\sigma}\right)^3 256 \int_{(M_s^{(x_0,t_0),\lambda} \cap B_{\lambda^{-1}R_0}(0)) \sim B_R(0)} \hat{\rho_{\kappa_\Sigma}} d\hat{\mu}_s \\
&= \frac{C(n,s,\sigma)}{(-s)^{n/2}} \int_{(M_s^{(x_0,t_0),\lambda} \cap B_{\lambda^{-1}R_0}(0)) \sim B_R(0)} e^{-\frac{r_y^2}{8(1+16(-(\lambda\kappa_\Sigma)^2 s)^\delta s}} d\hat{\mu}_s(y) \\
&\leq C(n,s,\sigma) \sum_{j\geq 1} \int_{(M_s^{(x_0,t_0),\lambda} \cap B_{\lambda^{-1}R_0}(0)) \cap (B_{R^{j+1}}(0) \sim B_{R^j}(0))} e^{-\frac{R^{2j}}{8|s|}} d\hat{\mu}_s \\
&= C(n,s,\sigma) \sum_{j=1}^{Q} e^{-\frac{R^{2j}}{8|s|}} \int_{(M_s^{(x_0,t_0),\lambda} \cap B_{\lambda^{-1}R_0}(0)) \cap (B_{R^{j+1}}(0) \sim B_{R^j}(0))} 1 d\hat{\mu}_s \\
&\leq C(n,s,\sigma) \left(\sum_{j=1}^{Q-1} e^{-\frac{R^{2j}}{8|s|}} \mathcal{H}^n(M_s^{(x_0,t_0),\lambda} \cap B_{R^{j+1}}(0)) \right) + \mathcal{H}^n(M_s^{(x_0,t_0),\lambda} \cap B_{\lambda^{-1}R_0}(0)) \\
&\leq C(n,s,\sigma) C(n,\kappa_\Sigma,x_0) \sum_{j=1}^{Q} R^{n(j+1)} e^{-\frac{R^{2j}}{8|s|}} \\
&=: f(R)
\end{aligned}$$

where (as one can directly calculate) $\lim_{R\to\infty} f(R) = 0$. Although for any given λ we cannot let $R \to \infty$ this is not necessary since for any $R \in \mathbb{R}$ the above calculation holds, independently of λ provided λ is small enough.

Note that the Q gives an appropriate number of terms in the summation since after $j = Q$

$$(B_{R^{j+1}}(0) \sim B_{R^j}(0)) \cap B_{\lambda^{-1}R_0}(0) = \emptyset,$$

and needed as we cannot apply Corollary 15.2.1 for $R > \lambda^{-1}R_0$.

We also need an estimate on the 'outer integral' for the limit surface. To do this we apply Proposition 16.1.3. From Proposition 16.1.3 we know that M_s' is a subset of a hyperplane through the origin. It then follows from standard integration theory (or direct calculation) that there exists, for each $\varepsilon > 0$ a radius R^ε such that for all $R \geq R^\varepsilon$

$$\int_{M_s' \sim \overline{B_R(0)}} \rho d\mu_s' < \varepsilon.$$

We now take any small $\varepsilon > 0$ and an $R^\varepsilon \leq R_\varepsilon \in \mathbb{R}$ so that (for sufficiently small λ, say $\lambda \leq \lambda_{R_\varepsilon}$)

$$\int_{M_s^{(x_0,t_0),\lambda} \sim B_{R_\varepsilon}(0)} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho}_{\kappa\Sigma} d\hat{\mu}_s < \varepsilon,$$

noting again that $\hat{\varphi_\sigma} \to 1$, $\hat{\eta} \to 1$ and $\hat{\rho_{\kappa\Sigma}} \to \rho$ locally uniformly as $\lambda \searrow 0$. We can thus choose a λ_ε so that for all $\lambda < \lambda_\varepsilon$

$$\int_{M_s^{(x_0,t_0),\lambda} \sim \overline{B_{R_\varepsilon}(0)}} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho}_{\kappa\Sigma} d\hat{\mu}_s < \varepsilon, \quad \int_{M_s' \sim \overline{B_{R_\varepsilon}(0)}} \rho d\mu_s' < \varepsilon,$$

$$|\hat{\varphi_\sigma} - 1|, |\hat{\eta} - 1| < \varepsilon \quad \text{on} \quad B_{2R_\varepsilon}, \text{ and } |\hat{\rho_{\kappa\Sigma}} - \rho| < min\{\varepsilon, \varepsilon\rho(y)\} \quad \text{on} \quad B_{2R_\varepsilon}$$

for any $|y| = 2R_\varepsilon$.

We now also take any test function $\psi \in C_C^1(\mathbb{R}^{n+1}, \mathbb{R})$ such that $\psi = 1$ on B_{R_ε} and $\psi = 0$ outside of B_{2r_ε}. Then

$$\int_{M_s^{(x_0,t_0),\lambda} \sim B_{R_\varepsilon}(0)} \psi \rho d\hat{\mu} s \leq 2 \int_{M_s^{(x_0,t_0),\lambda} \sim B_{R_\varepsilon}(0)} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho_{\kappa\Sigma}} d\hat{\mu}_s < 2\varepsilon \quad \text{and}$$

$$\int_{M_s' \sim \overline{B_{R_\varepsilon}}} \psi \rho d\mu_s \leq \int_{M_s' \sim \overline{B_{R_\varepsilon}}} \rho d\mu_s < \varepsilon.$$

Since $t_0 < T$, the flow $(M_t)_{t\in[0,t_0)}$ is smooth so that we can use Theorem 11.4.2 to calculate

$$\begin{aligned}
J &:= \lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho}_{\kappa\Sigma} d\hat{\mu}_s \\
&= \lim_{\lambda\to 0} \left(\int_{M_s^{(x_0,t_0),\lambda} \sim B_{R_\varepsilon}(0)} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho}_{\kappa\Sigma} d\hat{\mu}_s + \int_{M_s^{(x_0,t_0),\lambda} \cap B_{R_\varepsilon}(0)} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho}_{\kappa\Sigma} d\hat{\mu}_s \right) \\
&= O(\varepsilon) + \lim_{\lambda\to 0} \left(\int_{M_s^{(x_0,t_0),\lambda} \sim B_{R_\varepsilon}(0)} \psi \rho d\hat{\mu}_s + \int_{M_s^{(x_0,t_0),\lambda} \cap B_{R_\varepsilon}(0)} \hat{\varphi_\sigma} \hat{\eta} \hat{\rho}_{\kappa\Sigma} d\hat{\mu}_s \right) \\
&= O(\varepsilon) + \lim_{\lambda\to 0} \left(\int_{M_s^{(x_0,t_0),\lambda} \sim B_{R_\varepsilon}(0)} \psi \rho d\hat{\mu}_s + (1 + O(\varepsilon)) \int_{M_s^{(x_0,t_0),\lambda} \cap B_{R_\varepsilon}(0)} \psi \rho d\hat{\mu}_s \right)
\end{aligned}$$

$$
\begin{aligned}
&= O(\varepsilon) + (1+O(\varepsilon)) \lim_{\lambda\to 0} \int \psi \rho d\mathcal{H}^n|_{M_s^{(x_0,t_0),\lambda}} \\
&= O(\varepsilon) + (1+O(\varepsilon)) \int \psi \rho d\mu_s \\
&= O(\varepsilon) + (1+O(\varepsilon)) \left(\int_{M_s' \cap \overline{B_{R_\varepsilon}(0)}} \rho d\mathcal{H}^n + O(\varepsilon) - O(\varepsilon) + \int_{M_s' \sim \overline{B_{R_\varepsilon}(0)}} \rho d\mathcal{H}^n \right) \\
&= O(\varepsilon) + \int_{M_s'} \rho d\mathcal{H}^n,
\end{aligned}
$$

Since this is true for any $\varepsilon > 0$ we have

$$
\lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho}_{\kappa_\Sigma} d\hat{\mu}_s = \int_{M_s'} \rho d\mathcal{H}^n.
$$

Thus

$$
\begin{aligned}
\Theta(\mathcal{M}, x_0, t_0) &= \lim_{t\to t_0} e^{C\kappa_\Sigma^{2\delta}\tau_0^\delta} \int_{M_t} \varphi_\sigma \eta \rho_{\kappa_\Sigma} d\mu_t \\
&= \lim_{\lambda\to 0} e^{C(-\lambda)\kappa_\Sigma^{2\delta}s^\delta} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho}_{\kappa_\Sigma} d\hat{\mu}_s \\
&= \lim_{\lambda\to 0} e^{C(-\lambda)\kappa_\Sigma^{2\delta}s^\delta} \lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho}_{\kappa_\Sigma} d\hat{\mu}_s \\
&= \lim_{\lambda\to 0} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\varphi}_\sigma \hat{\eta} \hat{\rho}_{\kappa_\Sigma} d\hat{\mu}_s \\
&= \int_{M_s'} \rho d\mathcal{H}^n.
\end{aligned}
$$

$\diamondsuit$

In proving the Clearing Out Lemma we will need an absolute lower bound on the Gaussian density for points in the flow. The following three technical results are leading up to this result. Following the lower bound on Gaussian density we can then prove the Clearing out Lemma. The first of these three results can be seen as corollaries of Proposition 16.1.4.

Corollary 16.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions on the support surface Σ. Then $\Theta(\mathcal{M}, x_0, t_0)$ exists for each $t_0 \in (0, T]$ and $x_0 \in \Sigma$ independently of the $\sigma \in (0, \tau_0^{1/2}/2)$ chosen for φ_σ and for each $t \in [0, t_0)$

$$
\Theta(\mathcal{M}, x_0, t_0) \le e^{C\kappa_\Sigma^{2\delta}\tau_{t_0}^\delta} \int_{M_t} \varphi_\sigma \eta \rho_{\kappa_\Sigma} d\mu_t.
$$

Proof:
The existence follows directly from Proposition 16.1.2. The fact that it is independent of the choice of σ follows from Proposition 16.1.4. From Theorem 14.2.1 we know that the quantity

$$
e^{C\kappa_\Sigma^{2\delta}\tau_{t_0}^\delta} \int_{M_t} \varphi_\sigma \eta \rho_{\kappa_\Sigma} d\mu_t
$$

is reducing in t so that

$$
e^{C\kappa_\Sigma^{2\delta}\tau_{t_0}^\delta} \int_{M_t} \varphi_\sigma \eta \rho_{\kappa_\Sigma} d\mu_t \ge \lim_{t\nearrow t_0} e^{C\kappa_\Sigma^{2\delta}\tau_{t_0}^\delta} \int_{M_t} \varphi_\sigma \eta \rho_{\kappa_\Sigma} d\mu_t = \Theta(\mathcal{M}, x_0, t_0).
$$

$\diamondsuit$

Corollary 16.1.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions on the support surface Σ. Then for any $t_0 \in (0,T)$ and $x_0 \in \partial M_{t_0}$

$$\Theta(\mathcal{M}, x_0, t_0) = \frac{1}{2}.$$

Proof:
For any $t_0 < T$, the limiting blow up flow (M_s') exists. Since the surface M_{t_0} is a smooth manifold with smooth boundary we have for all $s < 0$

$$M_s' = \begin{cases} \mathbb{R}^n & x_0 \in M_{t_0} \sim \partial M_{t_0} \\ \mathbb{R}^n \cap \Pi & x_0 \in \partial M_{t_0}, \end{cases}$$

where Π is an n-dimensional halfspace with boundary intersecting 0, it follows from Proposition 16.1.4 and standard integration results that if $x_0 \in \partial M_{t_0}$

$$\Theta(\mathcal{M}, x_0, t_0) = \int_{M_s'} \rho d\mu_s' = \int_{\mathbb{R}^n \cap \Pi} \rho d\mathcal{H}^n = \frac{1}{2}\int_{\mathbb{R}^n} \rho d\mathcal{H}^n = \frac{1}{2}.$$

$\diamondsuit$

We now wish to show the upper semi continuity of the Gaussian density from which, together with Corollaries 16.1.1 and 16.1.2 will provide the necessary lower bound to prove the Clearing out Lemma. To prove the upper semi continuity of the Gaussian Density we will need to apply the following standard measure theoretic result.

Theorem 16.1.1. (Lebesgue Dominated Convergence Theorem)
Let g be a $\mathcal{H}^n$-integrable function on $\mathbb{R}^{n+1}$ and let f and the sequence of functions $\{f_k\}_{k=1}^{\infty}$ be $\mathcal{H}^n$ measurable on $\mathbb{R}^{n+1}$. Suppose $|f_k| \leq g$ and $f_k \to f$ $\mathcal{H}^n$ almost everywhere as $k \to \infty$. Then

$$\lim_{k\to\infty} \int |f_k - f| d\mathcal{H}^n = 0.$$

We can now prove the upper semi continuity of the Gaussian density.

Proposition 16.1.5.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions on support surface Σ and let (x_j, t_j) be a sequence in $\Sigma \times (t_0 - \tau_0, t_0)$ satisfying $x_j \to x_0$ and $t_j \nearrow t_0 \leq T$. Then

$$\limsup_{j\to\infty} \Theta(\mathcal{M}, x_j, t_j) \leq \Theta(\mathcal{M}, x_0, t_0).$$

Proof:
We take a fixed but arbitrary $t \in (t_0 - \tau_0, t_0)$ and take $j_0 \in \mathbb{N}$ so that for all $j \geq j_0$, $t_j > t$. Then, for these j, we have from the Monotonicity Formula, Theorem 14.2.1 applied at the space-time point (x_j, t_j)

$$e^{C\kappa_\Sigma^{2\delta}\tau_j^\delta} \int_{M_t} (\varphi_\sigma \eta \rho_{\kappa_\Sigma})_j d\mu_t \geq \lim_{t\nearrow t_j} e^{C\kappa_\Sigma^{2\delta}\tau_j^\delta} \int_{M_t} (\varphi_\sigma \eta \rho_{\kappa_\Sigma})_j d\mu_t \equiv \Theta(\mathcal{M}, x_j, t_j),$$

where $\tau_j(t) = t - t_j$ and $(\varphi_\sigma \eta \rho_{\kappa_\Sigma})_j(x,t) := \varphi_{(x_j,t_j),\sigma}\eta_{(x_j,t_j)}(x,t)\rho_{\kappa_\Sigma,x_j,t_j}(x,t)$.

We then want to take the limiting supremum with respect to j of the far left and far right terms.

On the left hand side we note that we have

$$\limsup_{j\to\infty} e^{C\kappa_\Sigma^{2\delta}\tau_j^\delta} = e^{C\kappa_\Sigma^{2\delta}(t-t_0)} < \infty.$$

From Proposition 11.3.6, we have $\eta_{(x_j,t_j)}(x,t)\rho_{\kappa_\Sigma,x_j,t_j}(x,t) \leq 256(4\pi(t-t_{j_0}))^{-n/2}\chi_{spt\eta_{(x_{j_0},t_{j_0})}}$ for each $j \geq j_0$. Additionally we have $\varphi_{(x_j,t_j),\sigma} \leq (1-2n(t-t_{j_0})\sigma^{-2})^3$ for each $j \geq j_0$ so that

$$(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_j(x,t) \leq \frac{256(1-2n(t-t_{j_0})\sigma^{-2})^3}{(4\pi(t-t_{j_0}))^{n/2}}\chi_{spt\eta_{(x_{j_0},t_{j_0})}}$$

for each $j \leq j_0$, noting also that

$$\int_{M_t} \frac{256(1-2n(t-t_{j_0})\sigma^{-2})^3}{(4\pi(t-t_{j_0}))^{n/2}}\chi_{spt\eta_{(x_{j_0},t_{j_0})}} d\mu_t < \infty,$$

and that $(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_j \to \varphi_{\sigma,x_0,t_0}\eta_{(x_0,t_0)}\rho_{\kappa_\Sigma,x_0,t_0} =: (\varphi_\sigma\eta\rho_{\kappa_\Sigma})_{t_0}$ it follows from the Lebesgue Dominated Convergence Theorem that

$$\limsup_{j\to\infty}\int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_j d\mu_t = \int_{M_t}\limsup_{j\to\infty}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_j d\mu_t = \int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_{t_0} d\mu_t < \infty.$$

We can therefore separate the two multiplying factors of the left limiting supremum into the product of two limiting suprema as

$$\begin{aligned}
\limsup_{j\to\infty} e^{C\kappa_\Sigma^{2\delta}\tau_j^\delta}\int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_j d\mu_t &= \limsup_{j\to\infty} e^{C\kappa_\Sigma^{2\delta}\tau_j^\delta}\limsup_{j\to\infty}\int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_j d\mu_t \\
&= e^{C\kappa_\Sigma^{2\delta}(t-t_0)}\int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_{t_0} d\mu_t.
\end{aligned}$$

Thus

$$e^{C\kappa_\Sigma^{2\delta}(t-t_0)}\int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_{t_0} d\mu_t \geq \limsup_{j\to\infty}\Theta(\mathcal{M},x_j,t_j).$$

Since this is true for any $t \in (t_0-\tau_0, t_0)$ we have

$$\Theta(\mathcal{M},x_0,t_0) = \lim_{t\nearrow t_0} e^{C\kappa_\Sigma^{2\delta}(t-t_0)}\int_{M_t}(\varphi_\sigma\eta\rho_{\kappa_\Sigma})_{t_0} d\mu_t \geq \limsup_{j\to\infty}\Theta(\mathcal{M},x_j,t_j).$$

◇

16.2 The Clearing Out Lemma

The Clearing Out Lemma is important in regularity theory in deciding when points are not in the limiting surface where the first singularity occurs at all. It gives a lower ratio bound below which we know that there is not enough of the surface in the tested ball to retain a presence there. That is, if the area ratio is too low, we know that a short time later there is no surface at all in a ball half the size. The first version, applicable to mean curvature flows without boundary is due to Brakke [5]. The proof however, has been simplified by the emergence of Gaussian density. A proof using Gaussian density is given in Ecker [7]. Our proof, adjusted for Neumann free boundary conditions follows the ideas of Ecker.

We first combine the results of the previous section to provide an absolute lower bound for Gaussian density for points reached by the flow at time $t_0 \leq T$.

Proposition 16.2.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions on the support surface Σ satisfying the boundary approaches boundary assumption, let $t_0 \in (0,T]$ and suppose that $\mathcal{M}$ reaches $x_0 \in \Sigma$ at time t_0. Then

$$\Theta(\mathcal{M}, x_0, t_0) \geq \frac{1}{2}.$$

Proof:
If $t_0 < T$ then this follows immediately from Corollary 16.1.2. Otherwise, since $\mathcal{M}$ reaches x_0 at time t_0, then from the boundary approaches boundary assumption $\partial\mathcal{M}$ reaches x_0 at time t_0 and thus there exists a sequence $(x_j, t_j)_{j\in\mathbb{N}}$ with $t_j \nearrow t_0$ and $x_j \in \partial M_{t_j}$ for each $j \in \mathbb{N}$. From Corollary 16.1.2 it then follows that

$$\Theta(\mathcal{M}, x_j, t_j) \geq \frac{1}{2}$$

for each $j \in \mathbb{N}$. From Proposition 16.1.5 we then have

$$\Theta(\mathcal{M}, x_0, t_0) \geq \limsup_{j\to\infty} \Theta(\mathcal{M}, x_j, t_j) \geq \frac{1}{2}.$$

$\diamondsuit$

With this result we are now able to prove the Clearing Out Lemma.

Lemma 16.2.1. (Clearing Out Lemma)
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions supported on the support surface Σ satisfying the boundary approaches boundary assumption. Let $\rho_0 \leq (\tau_0/2)^{1/2}$. If $\mathcal{M}$ reaches $x_0 \in \Sigma$ at time t_0 for some $t_0 \in (0,T]$ then for any $\beta \in (0, 1/2n)$ there exists $\theta = \theta(n,\beta) \in (0,1/2)$ such that for all $\rho \in (0,\rho_0)$

$$\rho^{-n}\mathcal{H}^n(M_{t_0-\beta\rho^2} \cap B_\rho(x_0)) \geq \theta.$$

Equivalently, if for some $\rho \in (0,\rho_0)$ and $\beta \in (0,1/2n)$

$$\rho^{-n}\mathcal{H}^n(M_{t_0-\beta\rho^2} \cap B_\rho(x_0)) < \theta$$

then there exists $\varepsilon > 0$ such that

$$M_t \cap B_\varepsilon(x_0) = \emptyset$$

for all $t \in (t_0 - \varepsilon^2, t_0)$. That is $\mathcal{M}$ does not reach x_0 at time t_0.

Proof:
Since M_t reaches $x_0 \in \Sigma$ at time t_0 we know from Corollary 16.1.1 and Proposition 16.2.1 that the inequality

$$\frac{1}{2} \leq e^{C\kappa_\Sigma^{2\delta}\tau^\delta} \int_{M_t} \varphi_\sigma \eta \rho_{\kappa_\Sigma} d\mu_t$$

holds for all $t \in (t_0 - \tau_0, t_0)$ and $\sigma \in (0, \tau_0^{1/2}/2)$. We can thus, for $\rho_0 \leq (\tau_0/2)^{1/2}$, note that the same inequality holds for all $\sigma \in (0, \rho_0/\sqrt{2})$ and $t \in (t_0 - \sigma^2, t_0)$. Take $\sigma < \rho_0(1+2n)^{-1/2}$, $\alpha \in (0,1)$ and $t = t_0 - \alpha\sigma^2$. In this case we note that from Proposition 11.3.6 $\eta \leq 256$, that we can calculate

$$\varphi_\sigma = \left(1 - \frac{r_{x_0} + 2n(t_0 - \alpha\sigma^2 - t_0)}{\sigma^2}\right)_+^3 \leq \left(1 + \frac{2n\alpha\sigma^2}{\sigma^2}\right)^3 = (1+2n\alpha)^3$$

and

$$\rho_{\kappa_\Sigma} = \frac{1}{4\pi(t_0 - t_0 + \alpha\sigma^2)^{n/2}} e^{-\frac{r_{x_0}}{8(16(\kappa_\Sigma^2\tau)^\delta+1)\tau}} \leq \frac{1}{(4\pi\alpha)^{n/2}}\sigma^{-n}.$$

We now note that for $|x - x_0| > \sqrt{1-2n\alpha}\sigma$, $r_{x_0} \geq (1+2n\alpha)\sigma^2$ so that

$$\begin{aligned} 1 - \frac{r_{x_0} - 2n(t_0 - (t_0 - \alpha\sigma^2))}{\sigma^2} &\leq 1 - \frac{(1+2n\alpha)\sigma^2 - 2n\alpha\sigma^2}{\sigma^2} \\ &= 0, \end{aligned}$$

and thus that $spt\varphi_\sigma \subset B_{\sqrt{1+2n\alpha}\sigma}$. We therefore obtain

$$\begin{aligned} \frac{1}{2} &\leq e^{C\kappa_\Sigma^{2\delta}\alpha^\delta\sigma^{2\delta}} \int_{M_{t_0-\alpha\sigma^2}\cap spt\varphi_\sigma} \sup\varphi_\sigma \sup\eta \sup\rho_{\kappa_\Sigma} d\mu_t \\ &\leq e^{C\kappa_\Sigma^{2\delta}\alpha^\delta\sigma^{2\delta}} \frac{(1+2n\alpha)^3 256}{(4\pi\alpha)^{n/2}}\sigma^{-n} \int_{M_{t_0-\alpha\sigma^2}\cap B_{\sqrt{1+2n\alpha}\sigma}} 1 d\mathcal{H}^n \\ &= e^{C\kappa_\Sigma^{2\delta}\alpha^\delta\sigma^{2\delta}} \frac{(1+2n\alpha)^3 256}{(4\pi\alpha)^{n/2}}\sigma^{-n}\mathcal{H}^n(M_{t_0-\alpha\sigma^2}\cap B_{\sqrt{1+2n\alpha}\sigma}). \end{aligned}$$

This implies

$$\mathcal{H}^n(M_{t_0-\alpha\sigma^2}\cap B_{\sqrt{1+2n\alpha}\sigma}) \geq e^{-C\kappa_\Sigma^{2\delta}\alpha^\delta\sigma^{2\delta}} \frac{(4\pi\alpha)^{n/2}}{(1+2n\alpha)^3 512}\sigma^n.$$

Setting $\rho = \sqrt{1+2n\alpha}\sigma$ and $\alpha\sigma^2 = \beta\rho^2$ (which implies $\beta \in (0, 1/2n)$), we then have

$$\begin{aligned} \mathcal{H}^n(M_{t_0-\beta\rho^2}\cap B_\rho(x_0) &\geq e^{-C\kappa_\Sigma^{2\delta}(\beta\rho^2)^\delta} \frac{(4\pi\beta)^{n/2}\rho^n\sigma^{-n}}{512(1+2n\beta\rho^2\sigma^{-2})^3}\sigma^n \\ &\geq e^{-C\kappa_\Sigma^{2\delta}\beta^\delta\tau_0^\delta} \frac{(4\pi\beta)^{n/2}\rho^n}{512(1+2n\beta)^3} \\ &\geq e^{-C(3/160n)^2} \frac{(4\pi\beta)^{n/2}\rho^n}{512(1+2n\beta)^3} \\ &=: \theta(n,\beta)\rho^n. \end{aligned}$$

It is then easily checked that $\theta(n, \beta) \in (0, 1/2)$. ◇

Remark: We note that the constant $\theta(n, \beta) = e^{-C(3/160n)^2}\frac{(4\pi\beta)^{n/2}}{512(1+2n\beta)^3}$ can be left as a larger constant also depending on δ. That is, the clearing out lemma also holds for the constant

$$\theta = \theta(n, \beta, \delta) := e^{-C\beta^\delta(3/160n)^2}\frac{(4\pi\beta)^{n/2}}{512(1+2n\beta)^3} \tag{16.2}$$

16.3 Notes

Gaussian density is a standard tool in the study of mean curvature flows, used to great effect by White (see, for e.g. [32]) and Ilmanen (see for e.g. [17]) it has also been used by Ecker [7] and Buckland [6]. The usual Gaussian density, Definition 16.1.1, in particular can be found in Ecker [7]. Proposition 16.1.2 is our own. Proposition 16.1.3 can be found with further discussion in Ecker [7]. Proposition 16.1.4 was mentioned in Buckland [6] though the proof is our own. The Neumann free boundary versions of Corollaries 16.1.1 and 16.1.2, as well as Propositions 16.1.5 and 16.2.1 are our own though they follow closely the boundaryless versions which can be found in Ecker [7]. The Lebesgue Dominated Convergence Theorem, originally due to Lebesgue [21] is a standard measure theoretic result. A good discussion can be found in Bartle [4] and a good background in the general measure theory required can be found in Bartle [4], Evans and Gariepy [10] or Rudin [24]. The first version of the Clearing out Lemma was due to Brakke [5], a proof using Gaussian density can be found in Ecker [7]. We follow the proof in Ecker [7], though the adjustments for Neumann free boundary conditions are our own.