Chapter 15

Area and Ratio Estimates

Global regularity is about the measure of the singularity set and thus it makes sense that we wish to develop area estimates. In particular it will come in useful as one of the problems that 'could have' arisen is that the measure in a ball grows uncontrollably quickly thus accumulating very rapidly in a small space forcing a collapse of the surface's structure. To show that this can't happen we need area bounds. Further, ratio bounds are necessary to establish the characterisation of Gaussian density, a key tool in our research, which we consider in the next chapter. In this chapter we provide those area and area ratio bounds that are necessary for the main results.

We begin with both an area bound and a boundary area bound, proving first properties of yet another test function of the same form as $\varphi_{(x_0,t_0),\sigma}$, and then directly attacking the area bounds.

We then prove area ratio bounds ending with a result of the form

$$\mathcal{H}^n(M_t \cap B_{R/C}(x_0)) \leq C(n, \kappa_\Sigma)R^n.$$

15.1 Surface and Boundary Area Estimates

We start with a Corollary of Proposition 11.3.1 that we will use in conjunction with the new test function in proving the area bounds.

Corollary 15.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free-boundary conditions and U an open subset of $\mathbb{R}^{n+1}$ containing $\mathcal{M}$. Then, for any function $\phi : U \times [0,T) \to \mathbb{R}$ which satisfies $\phi \in C_0^2(U)$ and $\frac{\partial\phi}{\partial t} \in C_0^0(U)$ we have

$$\frac{d}{dt}\int_{\mathcal{M}_t} \phi d\mu_t = \int_{M_t} \left(\left(\frac{d}{dt} - \triangle_{M_t}\right)\phi - |H|^2\phi\right) d\mu_t + \int_{\partial M_t} < D\phi, \nu_\Sigma > d\sigma_t.$$

Furthermore, if ϕ satisfies $(\frac{d}{dt} - \triangle_{M_t})\phi \leq 0$ then

$$\frac{d}{dt}\int_{\mathcal{M}_t} \phi d\mu_t \leq -\int_{M_t} |H|^2\phi d\mu_t + \int_{\partial M_t} < D\phi, \nu_\Sigma > d\sigma_t.$$

Proof:
From (11.5) we have with $f = \phi$ and $g = 1$

$$\frac{d}{dt}\int_{M_t} \phi d\mu_t \quad = \quad -\int_{M_t} \phi|\vec{H}|^2 d\mu_t + \int_{M_t} \left(\frac{d}{dt} - \triangle_{M_t}\right)\phi d\mu_t$$

$$+\int_{\partial M_t} < D\phi, \nu_\Sigma > \sigma_t,$$

which gives the first part of the result. In the case $\left(\frac{d}{dt} - \triangle_{M_t}\right)\phi \leq 0$ the second part follows trivially. $\diamondsuit$

The general form of ϕ in which we are presently interested in association with the above corollary is a generalised class of functions based on the localisation function $\varphi_{(x_0,t_0),\sigma}$

Definition 15.1.1.
Let Σ be a Neumann free boundary support surface, let $c, c_2 \in \mathbb{R}$ and $R > 0$. We define the class of **boundary localisation functions** $\phi_{R,c,c_2} : \mathbb{R}^{n+1} \times R \to \mathbb{R}$ *by*

$$\phi_{R,c,c_2}(x,t) = \left(1 - \frac{r_{x_0} + c(t - c_2)}{R^2}\right)^4_+ .$$

Remark: We suppress mention of Σ in the definition of ϕ_{R,c,c_2} since its use will always be in conjunction with an understood previously fixed support surface.

We need to show some simple properties of the boundary localisation functions before proving the first area bound. To do this we state the following general result for mean curvature flows.

Proposition 15.1.1.
Let $M_t \equiv F(M^n, t)$ be a family of hypersurfaces evolving by mean curvature flow and let $f = f(x,t)$ for $x = F(p,t)$, $p \in M^n$. Then

$$\left(\frac{d}{dt} - \triangle_{M_t}\right) f = \left(\frac{\partial}{\partial t} - div_{M_t} D\right) f.$$

Proposition 15.1.2.
Let Σ be a Neumann free boundary support surface. Then, for ϕ_{R,c,c_2} as defined in Definition 15.1.1 *for any $c \in \mathbb{R}$ we have*

$$\left(\frac{d}{dt} - \triangle_{M_t}\right)\phi_{R,c,c_2} \leq \phi'_{R,c,c_2}\left(\frac{\partial}{\partial t} - div_{M_t} D\right) z,$$

where $z(x,t) = r_{x_0} + c(t - c_2)$ and $< D\phi_{R,c,c_2}, \nu_\Sigma > \equiv 0$ on Σ.

Proof:
Since $\phi_{R,c,c_2}(x,t) = \phi_{R,c,c_2}(z(x,t))$ and $\phi''_{R,c,c_2}(z(x,t)) = 12R^{-4}(1 - ZR^{-2})^2_+$ it follows, using the chain rule, that

$$\begin{aligned}\left(\frac{d}{dt} - \triangle_{M_t}\right)\phi_{R,c,c_2} &= \frac{d}{dt}\phi_{R,c,c_2} - \triangle_{M_t}\phi_{R,c,c_2} \\ &= \phi'_{R,c,c_2}\frac{d}{dt}z - \phi'_{R,c,c_2}\triangle_{M_t} z - \phi''_{R,c,c_2}|\nabla^{M_t} z|^2 \\ &\leq \phi'_{R,c,c_2}\left(\frac{d}{dt} - \triangle_{M_t}\right) z,\end{aligned}$$

where ϕ'_{R,c,c_2} is the first derivative of ϕ_{R,c,c_2} with respect to z. Applying Proposition 15.1.1 to $\left(\frac{d}{dt} - \triangle_{M_t}\right) z$ gives

$$\left(\frac{d}{dt} - \triangle_{M_t}\right)\phi_{R,c,c_2} = \phi'_{R,c,c_2}\left(\frac{\partial}{\partial t} - div_{M_t} D\right) z$$

as required.

For the second property in the statement of the proposition we calculate, using Proposition 11.3.4,

$$< D\phi_{R,c,c_2}, \nu_\Sigma >= -\frac{\phi^{3/4}_{R,c,c_2}}{R^2} < Dr_{x_0}, \nu_\Sigma >= 0.$$

$\diamondsuit$

A particular naturally self selecting member of the class of boundary localisation functions will be used to prove our fundamental area bound.

Theorem 15.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface Σ in $B_{d_0}(x_0)$ for some $x_0 \in \Sigma$, for all $t \in [0,T) \supset [T-d_0^2, T)$ where $d_0 \leq R/(3\sqrt{128n})$ and $R \leq 3/(2\kappa_\Sigma)$. Then for any $t_0 \in [t_1 + d_0^2, T]$, $\rho \in (0, d_0]$ we have for all $t \in [t_0 - \rho^2, t_0)$ and $i, j, k \in \{1, 2\}$

$$\begin{aligned}
\mathcal{H}^n(M_t \cap B_{\rho/3\sqrt{40},j}) &+ \int_{t_0-\rho^2}^{t_0}\int_{M_s\cap B_{\rho/3\sqrt{40},i}} |H|^2 d\mu_s ds \\
&\leq \mathcal{H}^n(M_t \cap (B_{\rho/3\sqrt{40},1} \cup B_{\rho/3\sqrt{40},2})) \\
&\quad + \int_{t_0-\rho^2}^{t_0}\int_{M_s\cap(B_{\rho/3\sqrt{40},1}\cup B_{\rho/3\sqrt{40},2})} |H|^2 d\mu_s ds \\
&\leq 16\mathcal{H}^n(M_{t_0-\rho^2} \cap (B_{\rho/3,1} \cup B_{\rho/3,2})) \\
&\leq 16\mathcal{H}^n(M_{t_0-\rho^2} \cap B_{\rho,k})
\end{aligned}$$

where $B_{R,1} := \{x \in \mathbb{R}^{n+1} : |x - x_0| \leq R\}$ and $B_{R,2} := \{x \in \mathbb{R}^{n+1} : |\widetilde{x - x_0}| \leq R\}$.

Proof:
Let $\phi_{\rho/3,c,c_2}$ be the test function as defined above with $c_2 = t_0$ and c yet to be chosen. By Proposition 11.3.4 and Proposition 15.1.1 we see (with $z(x,t) := r_{x_0} + c(t - t_0)$) that

$$\begin{aligned}
\frac{d}{dt}\int_{M_t}\phi_{\rho/3,c,c_2} d\mu_t &= \int_{M_t}\left(\frac{d}{dt} - \triangle_{M_t}\right)\phi_{\rho/3,c,c_2} d\mu_t + \int_{\partial M_t} < D\phi_{\rho/3,c,c_2}, \nu_\Sigma > d\sigma_t \\
&= \int_{M_t}\phi'_{\rho/3,c,c_2}\left(\frac{d}{dt}z - div_{M_t}Dz\right) - \phi''_{\rho/3,c,c_2}|\nabla^{M_t}z|^2 - |H|^2\phi_{\rho/3,c,c_2} d\mu_t. \\
&\leq \int_{M_t}\phi'_{\rho/3,c,c_2}\left(\frac{d}{dt}z - div_{M_t}Dz\right) - |H|^2\phi_{\rho/3,c,c_2} d\mu_t.
\end{aligned}$$

Then, since $\phi'_{\rho/3,c,c_2} = -(4/(\rho/3)^2)(1 - z/(\rho/3)^4)^3_+$, $\partial z/\partial t = c$ and using Proposition 11.3.4 we have

$$\begin{aligned}
\phi'_{\rho/3,c,c_2}\left(\frac{\partial z}{\partial t} - div_{M_t}Dz\right) &= \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)^3_+ (div_{M_t}Dz - c) \\
&\leq \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)^3_+ (|div_{M_t}Dr - 4n| + 4n - c) \\
&\leq \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)^3_+ \left(4n + \frac{20n\kappa|x - x_0|}{1 - d\kappa} + \frac{4n\kappa^2|x - x_0|^2}{(1 - d\kappa)^2} - c\right)
\end{aligned}$$

Moreover, since for any $t \geq t_0 - \rho^2$

$$spt\left(1 - \frac{z}{(\rho/3)^2}\right)_+^3 = \{x \in \mathbb{R}^{n+1} : r_{x_0} + ct \leq (\rho/3)^2\} \subset \{x \in \mathbb{R}^{n+1} : |x - x_0| \leq \rho/3\}$$

with $\rho < R \leq 3/2\kappa$ and $x_0 \in \Sigma$ (for which $d(x) \leq |x - x_0|$) we can therefore firstly estimate $d\kappa \leq |x - x_0|\kappa \leq (\rho/3)\kappa \leq (R/3)\kappa \leq (3/6\kappa)\kappa = 1/2$ so that $1 - d\kappa \geq 1 - 1/2 = 1/2$ thus $(1 - d\kappa)^{-1} \leq 2$ and thusly also

$$\begin{aligned}
\phi'_{\rho/3,c,c_2}\left(\frac{\partial z}{\partial t} - div_{M_t} Dz\right) &\leq \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)_+^3 \left(4n + \frac{20n\kappa|x - x_0|}{1 - d\kappa} + \frac{4n\kappa^2|x - x_0|^2}{(1 - d\kappa)^2} - c\right) \\
&\leq \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)_+^3 (4n + 40n\kappa|x - x_0| + 32n\kappa^2|x - x_0|^2 - c) \\
&\leq \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)_+^3 (4n + 40n(2R/3)^{-1}\rho + 32n(2\rho/3)^{-2}(\rho/3)^2 - c) \\
&= \frac{4}{(\rho/3)^2}\left(1 - \frac{z}{(\rho/3)^2}\right)_+^3 (32n - c).
\end{aligned}$$

Therefore, setting $c = 32n$ in $\phi_{\rho/3,c,c_2}$ gives us

$$\frac{d}{dt}\int_{M_t} \phi d\mu_t \leq \int_{M_t} \phi'\left(\frac{\partial z}{\partial t} - div_{M_t} Dz\right) - \phi|H|^2 d\mu_t \leq -\int_{M_t} \phi|H|^2 d\mu_t$$

where

$$\phi := \phi_{\rho/3,c,c_2} = \left(1 - \frac{r_{x_0} + 32n(t - t_0)}{(\rho/3)^2}\right)_+^4.$$

Integrating this equation in time and adding $\lim_{t\nearrow t_0}\int_{M_t} \phi d\mu_t$ (where we take the limit for the case $t_0 = T$ since it is not necessarily clear what $\int_{M_T} \rho d\mu_T$ is) yields

$$\begin{aligned}
\int_{t_0-\rho^2}^{t_0}\int_{M_t} \phi|H|^2 d\mu_t dt + \lim_{t\to t_0}\int_{M_t} \phi d\mu_t &\leq -\int_{t_0-\rho^2}^{t_0} \frac{d}{ds}\int_{M_t} \phi d\mu_t + \lim_{t\to t_0}\int_{M_t} \phi d\mu_t \\
&= -\left[\lim_{t\to t_0}\int_{M_t} \phi d\mu_t - \int_{M_{t_0-\rho^2}} \phi d\mu_{t_0-\rho^2}\right] + \lim_{t\to t_0}\int_{M_t} \phi d\mu_t \\
&= \int_{M_{t_0-\rho^2}} \phi d\mu_{t_0-\rho^2}
\end{aligned}$$

where since $\phi \leq 1$ and

$$spt\phi(\cdot, 0) = \{x \in \mathbb{R}^{n+1} : r_{x_0} \leq (\rho/3)^2\} \subset (B_{\rho/3,1} \cup B_{\rho/3,2}) \subset B_{\rho,i}$$

for $i \in \{1, 2\}$ we have

$$\int_{M_{t_0-\rho^2}} \phi d\mu_{t_0-\rho^2} \leq \int_{M_{t_0-\rho^2}\cap spt\phi} 1 d\mu_{t_0-\rho^2} \leq \mathcal{H}^n(M_{t_0-\rho^2} \cap (B_{\rho/3,1} \cup B_{\rho/3,2})) \leq \mathcal{H}^n(M_{t_0-\rho^2} \cap B_{\rho,i})$$

for $i \in \{1, 2\}$. For each $r_{x_0} \leq \rho^2/36$ and $t \in [t_0 - \rho^2, t_0]$ we have

$$\phi(x, t) = \left(1 - \frac{r_{x_0} + 32n(t - (t_0 - \rho^2))}{(\rho/3)^2}\right)_+^4 \geq \left(1 - \frac{r_{x_0}}{(\rho/3)^2}\right)_+^4 \geq \left(1 - \frac{\rho^2/4}{\rho^2}\right)_+^4 = \frac{1}{16}.$$

So that since, applying Proposition 11.3.5 to give us $|\widetilde{x - x_0}| \leq 3|x - x_0|$, $|x - x_0| \leq 3|\widetilde{x - x_0}|$ and

$$\begin{aligned}\{x : r_{x_0} \leq \rho^2/36\} &= \{x : |x - x_0|^2 + |\widetilde{x - x_0}|^2 \leq \rho^2/36\} \\ &\supset \{x : 10\min\{|x - x_0|^2, |\widetilde{x - x_0}|^2\} \leq \rho^2/36\} \\ &= \{x : |x - x_0|^2 \leq \rho^2/9 \cdot 40\} \cup \{x : |\widetilde{x - x_0}|^2 \leq \rho^2/9 \cdot 40\} \\ &= B_{\rho/3\sqrt{40},1} \cup B_{\rho/3\sqrt{40},2} \\ &\supset B_{\rho/3\sqrt{40},i}\end{aligned}$$

for each $i \in \{1, 2\}$, we have

$$\int_{M_t} \phi d\mu_t \geq \frac{1}{16}\mathcal{H}^n(M_t \cap (B_{\rho/3\sqrt{40},1} \cup B_{\rho/3\sqrt{40},2})) \geq \frac{1}{16}\mathcal{H}^n(M_t \cap B_{\rho/3\sqrt{40},i})$$

for each $i \in \{1, 2\}$ and

$$\begin{aligned}\int_{t_0-\rho^2}^{t_0}\int_{M_t} \phi|H|^2 d\mu_t dt &\geq \frac{1}{16}\int_{t_0-\rho^2}^{t_0}\int_{M_t\cap(B_{\rho/3\sqrt{40},1}\cup B_{\rho/3\sqrt{40},2})} |H|^2 d\mu_t dt \\ &\geq \frac{1}{16}\int_{t_0-\rho^2}^{t_0}\int_{M_t\cap B_{\rho/3\sqrt{40},i}} |H|^2 d\mu_t dt\end{aligned}$$

for each $i \in \{1, 2\}$ and all $t \leq \rho^2/128n$. Multiplying through by 16 we now have for each $i, j \in \{1, 2\}$ and each $t \in [t_0 - \rho^2, t_0]$

$$\begin{aligned}\mathcal{H}^n(M_t \cap B_{\rho/3\sqrt{40},j}) &+ \int_{t_0-\rho^2}^{t_0}\int_{M_t\cap B_{\rho/3\sqrt{40},i}} \phi_{\rho/3,c}|H|^2 d\mu_t dt \\ &\leq \mathcal{H}^n(M_t \cap (B_{\rho/3\sqrt{40},1} \cup B_{\rho/3\sqrt{40},2})) \\ &\quad + \int_{t_0-\rho^2}^{t_0}\int_{M_t\cap(B_{\rho/3\sqrt{40},1}\cup B_{\rho/3\sqrt{40},2})} \phi_{\rho/3,c}|H|^2 d\mu_t dt \\ &\leq 16\mathcal{H}^n(M_{t_0-\rho^2} \cap (B_{\rho/3,1} \cup B_{\rho/3,2})) \\ &\leq 16\mathcal{H}^n(M_{t_0-\rho^2} \cap B_{\rho,j}).\end{aligned}$$

◇

In the proof of Global regularity we will need to apply Theorem 15.1.1 finitely many times repeatedly. In the following Corollary we make the appropriate multiple applications to give a bound on balls that now have a relationship between their radius and the total time T. This relationship is made explicable through the function defined below.

Definition 15.1.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions. We define $C_T : \mathbb{R}^+ \to \mathbb{R}$ by

$$C_T(R) := \frac{2T}{R}.$$

We now show the area bound following from multiple applications of Theorem 15.1.1.

Corollary 15.1.2.
Let $(M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions in $B_R(x_0)$, for all $t \in [0, T)$ with $R \leq (3\kappa_\Sigma\sqrt{128n})^{-1}$. Then for all $t \in [0, T)$

$$\mathcal{H}^n(B_{R/(20^{C_T(R)})}(x_0) \cap M_t) \leq 16^{C_T(R)}\mathcal{H}^n(B_R(x_0) \cap M_0).$$

Proof:
We note that for $t_1 = R/2$, $t_1 \in [0, R/2\kappa_\Sigma\sqrt{128n})$ so that Theorem 15.1.1 applies for all $t \in [0, 2t_1)$ and thus for all $t \in [0, 2t_1)$

$$\mathcal{H}^n(M_t \cap B_{R/20}(x_0)) \leq 16\mathcal{H}^n(M_0 \cap B_{\frac{3\sqrt{40}}{20}R}(x_0)) \leq 16\mathcal{H}^n(M_0 \cap B_R(x_0)).$$

In particular this holds for $t = t_1$. If $T \leq t_1$ we are finished. Otherwise, by relabelling the flow as $M_t^1 := M_{t-t_1}$ then $(M_t^1)_{t\in[0,T-t_1)}$ satisfies the requirements to apply Theorem 15.1.1 so that for all $t \in [0, 2t_1)$, we have, identically to the above,

$$\mathcal{H}^n(M_t^1 \cap B_{R/20^2}(x_0)) \leq 16\mathcal{H}^n(M_0^1 \cap B_{R/20}(x_0)).$$

Relabelling back to the original gives

$$\mathcal{H}^n(M_t \cap B_{R/20^2}(x_0)) \leq 16\mathcal{H}^n(M_{t_1} \cap B_{R/20}(x_0)) \leq 16^2\mathcal{H}^n(M_0 \cap B_R(x_0))$$

for all $t \in [t_1, 3t_1)$. In particular this holds for $t = 2t_1$. Repeating this step inductively gives

$$\mathcal{H}^n(M_t \cap B_{R/20^p}(x_0)) \leq 16\mathcal{H}^n(M_{(p-1)t_1} \cap B_{R/20^{p-1}}(x_0)) \leq ... \leq 16^p\mathcal{H}^n(M_0 \cap B_R(x_0))$$

whenever $t \in [(p-1)t_1, pt_1)$, $1 \leq p \leq C_T(R)$.

Now, for any $t \in [0, T)$ we are able to select a $p_t \in \{1, ..., C_T(R)\}$ such that $t \in [(p_t - 1)t_1, p_t t_1)$. Thus, since $B_{R/(20^{C_T(R)})}(x_0) \subset B_{R/20^{p_t}}(x_0)$ for such a p_t

$$\begin{aligned}\mathcal{H}^n(M_t \cap B_{R/(20^{C_T(R)})}(x_0)) &\leq \mathcal{H}^n(M_t \cap B_{R/20^p}(x_0)) \\ &\leq 16^p\mathcal{H}^n(M_0 \cap B_R(x_0)) \\ &\leq 16^{C_T(R)}\mathcal{H}^n(M_0 \cap B_R(x_0)).\end{aligned}$$

♢

We will, in the next section, further apply Theorem 15.1.1 to get upper area ratio bounds. We continue, for now by looking at boundary area. In order to examine the boundary area we find it necessary to apply the well known Divergence Theorem, which we state below.

Theorem 15.1.2.
Let M be a smooth, orientable hypersurface with boundary embedded in $\mathbb{R}^{n+1}$. Then for any $X \in C_C^1(M, \mathbb{R}^{n+1})$ we have

$$\int_M div_M X d\mathcal{H}^n = -\int_M < X, \vec{H} > d\mathcal{H}^n + \int_{\partial M} < X, \nu_{\partial M} > d\mathcal{H}^{n-1}, \tag{15.1}$$

where $\vec{H}$ denotes the mean curvature vector of M and $\nu_{\partial M}$ is the outer unit conormal to ∂M.

We apply the Divergence Theorem to bound the boundary measure in terms of the area and the integral of the mean curvature over the surface in some slightly larger ball than that in which we are estimating the boundary area.

Lemma 15.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions supported on the support surface Σ and let $R_1 = \frac{1}{2\kappa_\Sigma}$. Then for any $t \in [0, T)$, $R \in [0, R_1]$ and $x_0 \in \mathbb{R}^{n+1}$

$$\mathcal{H}^{n-1}(\partial M_t \cap B_{R/2}(x_0)) \leq C(n, \kappa_\Sigma)\left[\int_{M_t \cap B_R(x_0)} |\vec{H}| d\mu_t + \mathcal{H}^n(M_t \cap B_R(x_0))\right].$$

Proof:
Note that if $x \notin \Sigma_{1/2\kappa_\Sigma}$ then $B_R(x_0) \cap \partial M_t \subset B_R(x_0) \cap \Sigma = \emptyset$, and thus $\mathcal{H}^{n-1}(B_R(x_0) \cap \partial M_t) = 0$. We can therefore assume that $x_0 \in \Sigma_{1/2\kappa_\Sigma}$ and thus that $B_R(x_0) \subset \Sigma_{1/\kappa_\Sigma}$ on which $d(\cdot, \Sigma)$ is well defined.

Choose now any $R \leq R_1$ and define $C_1 = (R\kappa_\Sigma)^{-1}$ We define two test functions that we will superimpose on one another in our composition of the vector field that we will choose to apply the Divergence Theorem to. Choose firstly $\phi_1 \in C_C^1(\mathbb{R}^{n+1})$ such that $\chi_{B_{R/2}(x_0)} \leq \phi_1 \leq \chi_{B_R(x_0)}$ and $|D\phi_1| \leq 2$. Then define $\phi_2 \in C_C^1(\mathbb{R}^{n+1})$ by

$$\phi_2(x) = (1 - C_1 d\kappa_\Sigma)_+^3.$$

We consider now the vector field $X \in C_C^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ defined by

$$X := \phi_1\phi_2 Dd.$$

We note firstly that $|\phi_1\phi_2| \leq 1$ and that, on $B_{R/2}(x_0) \subset B_{1/2C_1\kappa_\Sigma}(x_0)$

$$|X| = |\phi_1\phi_2 Dd| = |(1 - c_1 R\kappa_\Sigma)_+^3| \geq |(1 - c_1|x - x_0|\kappa_\Sigma)^3| \geq 2^{-3}.$$

We observe that $|X| = |\phi_1||\phi_2||Dd| \leq 1$ and that

$$< X, \nu_{\partial M} >= \phi_1\phi_2 < Dd, \nu_{\partial M} >= \phi_1\phi_2 = |X|.$$

Then, using the divergence Theorem we have

$$\begin{aligned} \mathcal{H}^{n-1}(\partial M_t \cap B_{R/2}(x_0)) &\leq 8\int_{\partial M_t} < X, \nu_{\partial M_t} > d\mathcal{H}^{n-1} \\ &= 8\int_{M_t} < X, \vec{H} > d\mathcal{H}^n + 8\int_{M_t \cap \text{spt } X} div_{M_t} X d\mathcal{H}^n. \end{aligned}$$

We observe that spt $X \subset$ spt $\phi_1 \cap$ spt $\phi_2 \subset B_R(x_0)$ and that on spt $X \subset$ spt ϕ_2 $1 - C_1 d\kappa_\Sigma \geq 0 \Rightarrow d\kappa_\Sigma \leq C_1^{-1} = R\kappa_\Sigma < \frac{1}{2}$ and thus $\frac{1}{1-d\kappa_\Sigma} \leq 2$. It follows, using Proposition 11.3.3, that

$$|div_{M_t} Dd| \leq \frac{n\kappa_\Sigma}{1 - d\kappa_\Sigma} \leq 2n\kappa_\Sigma \tag{15.2}$$

on spt X. We now expand $div_{M_t} X$ as

$$div_{M_t} X = \phi_1\phi_2 div_{M_t} Dd + \phi_1 < \nabla^{M_t}\phi_2, Dd > +\phi_2 < \nabla^{M_t}\phi_1, Dd > .$$

From (15.2) it follows that $|\phi_1\phi_2 div_{M_t} Dd| \leq 2n\kappa_\Sigma$ and from the properties of $|D\phi_2|, |\phi_2|$ and $|X|$ it follows that $|\phi_2 < \nabla^{M_t}\phi_1, Dd > | \leq 2$. We can then calculate for some orthonormal basis to $T_x M_t$, $\{x_i\}_{i=1}^n$ for any x that

$$\phi_1 < \nabla^{M_t}\phi_2, Dd >= \phi_1 \sum_{i=1}^n \frac{\partial}{\partial x_i}\phi_2 (Dd)^i = -C_1\kappa_\Sigma\phi_1(1 - C_1 d\kappa_\Sigma)_+^2 \sum_{i=1}^n \left(\frac{\partial d}{\partial x_i}\right)^2 \leq 0.$$

Finally noting that since $|X| \leq 1$, $| < X, \vec{H} > | \leq |\vec{H}|$ and thus we can calculate

$$\mathcal{H}^{n-1}(\partial M_t \cap B_{R/2}(x_0)) \leq 8\left(\int_{M_t \cap \text{spt } X} < X, \vec{H} > d\mathcal{H}^n + \int_{M_t \cap \text{spt } X} div_{M_t} X d\mathcal{H}^n\right)$$

$$
\begin{aligned}
&= 8\int_{M_t\cap \text{spt } X} <X,\vec{H}> d\mathcal{H}^n + 8\int_{M_t\cap\text{spt } X} \phi_1\phi_2 div_{M_t} Dd \;\; d\mathcal{H}^n \\
&\quad +8\int_{M_t\cap\text{spt } X} \phi_1 <\nabla^{M_t}\phi_2, Dd> d\mathcal{H}^n \\
&\quad +8\int_{M_t\cap\text{spt } X} \phi_2 <\nabla^{M_t}\phi_1, Dd> d\mathcal{H}^n \\
&\leq 8\int_{M_t\cap\text{spt } X} |<X,\vec{H}>| d\mathcal{H}^n + 8\int_{M_t\cap\text{spt } X} |\phi_1\phi_2 div_{M_t} Dd| d\mathcal{H}^n \\
&\quad + \int_{M_t\cap\text{spt } X} |\phi_2 <\nabla^{M_t}\phi_1, Dd>| d\mathcal{H}^n \\
&\leq 8\left[\int_{M_t\cap B_R(x_0)} |\vec{H}| d\mathcal{H}^n + \int_{M_t\cap B_R(x_0)} 2n\kappa_\Sigma + 2 d\mathcal{H}^n\right] \\
&\leq 16n\kappa_\Sigma\left[\int_{M_t\cap B_R(x_0)} |\vec{H}| d\mathcal{H}^n + \mathcal{H}^n(M_t\cap B_R(x_0))\right].
\end{aligned}
$$

$\diamondsuit$

15.2 Upper Area Ratio Bounds

In this section instead of considering absolute bounds on measure we look at the measure of the surface inside of a ball as compared to the usual $\mathcal{H}^n$ measure of a ball of the same radius. This then also acts as a measure of how crumpled up inside the ball the surface could be. In regularity theory (particularly local regularity) it is then the aim to show that this ratio approaches 1 (or a half on the boundary) as the radius becomes very small and the time goes to the singular time (in a way that will be later formalised). Should the ratio approach 1 (or a half on the boundary) then the surface is approaching a flat (half) hyperplane in a small ball around the point being considered and would thus be regular. If the ratio is significantly larger than 3/2 then there exists significant non-flatness arbitrarily close in space and time, which would lead to non-smoothness and thus singularity.

We first present the main technical lemma forming the basis of area ratio bounds. This is followed by a fixed time ratio bound based on the assumption that (M_t) is smooth and properly embedded for $t < T$. These two results are followed by a subsidiary result making the presentation of the ratio results neater.

Proposition 15.2.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions on the support surface Σ. Let $\rho_0 := (\tau_0/2)^{1/2}$. Then for any $x_0 \in \Sigma$ and any $\rho \in (0, (2+0.4n)^{-1/2}\rho_0)$

$$
\sup_{t\in(T-\rho^2,T)} \frac{\mathcal{H}^n(M_t\cap B_\rho(x_0))}{\rho^n} \leq C(n,\kappa_\Sigma)\frac{\mathcal{H}^n\left(M_{T-\frac{\rho_0^2}{2+0.4n}}\cap B_{\rho_0}(x_0)\right)}{\rho_0^n}.
$$

where we recall that $\tau_0 = (3/160n)^{2/\delta}/\kappa_\Sigma^2$.

Proof:
Let $\rho_1 = (2+0.4n)^{-1/2}\rho_0$. Let $\rho \in (0,\rho_1)$. Set $\sigma = (20+4n)\rho_1^2$, $t_1 = T+\rho^2$, $t_2 = T+\rho_0^2$

and $t_3 = T - \rho_1^2$. We can then apply the monotonicity formula, Theorem 14.2.1, to get for each $t \in (T - \rho^2, T)$

$$e^{C\kappa_\Sigma^{2\delta}(t_1-t)^\delta} \int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t \leq e^{C\kappa_\Sigma^{2\delta}(t_1-t_3)^\delta} \int_{M_{t_3}} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t.$$

Clearly, $e^{C\kappa_\Sigma^{2\delta}(t_1-t)^\delta - (t_1-t_3)^\delta} < C(n, \kappa_\Sigma)$ so that we can write

$$\int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t \leq C(n, \kappa_\Sigma) \int_{M_{t_3}} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t. \tag{15.3}$$

Next we observe, from Proposition 11.3.6, that $\eta_2 \leq 256$. We also calculate directly that $\varphi_3(\cdot, t_3) \leq 1$, that

$$\rho^1_{\kappa_\Sigma}(\cdot, t_3) \leq \frac{1}{(4\pi(T + \rho^2 - T + \rho_1^2)^{n/2}} \leq \frac{1}{(4\pi)^{n/2}} \rho_1^{-n},$$

and that, since $|\widetilde{x - x_0}| \leq 3|x - x_0|$ and $|x - x_0| \leq 3|\widetilde{x - x_0}|$

$$\text{spt } \varphi_3(\cdot, t_3) = \{x : 1 - r_{x_0}^2/\sigma^2 \geq 0\} \subseteq \{x : 10|x - x_0|^2 \leq (20 + 4n)\rho_1^2\} = B_{\sqrt{2+0.4n}\rho_1}(x_0).$$

It follows that

$$\begin{aligned} \int_{M_{t_3}} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t &\leq \frac{256}{(4\pi)^{n/2}} \rho_1^{-n} \mathcal{H}^n(M_{t_3} \cap B_{\sqrt{2+0.4n}\rho_1}(x_0)) \\ &= C(n) \frac{\mathcal{H}^n(M_{t_3} \cap B_{\sqrt{2+0.4n}\rho_1}(x_0))}{\rho_1^n}. \end{aligned} \tag{15.4}$$

We then note that for $\rho < \rho_1 < \rho_0$ and $t \in (T - \rho^2, T) \subset (T - \tau_0/2, T)$, we have, from Proposition 11.3.6, that $\eta_2 \geq 1/256$ on $B_{\rho_0}(x_0) \supset B_\rho(x_0)$. Then calculating directly that

$$\varphi_3 = \left(1 - \frac{r_{x_0} - 2n(t_3 - t)}{(20 + 4n)\rho_1^2}\right)_+^3 \geq \left(1 - \frac{10\rho_1^2 - 2n\rho_1^2}{(20 + 4n)\rho_1^2}\right)_+^3 \geq \left(\frac{1}{12}\right)^3,$$

and that

$$\rho^1_{\kappa_\Sigma} = \frac{1}{(4\pi(T + \rho^2 - t))^{n/2}} e^{-\frac{r_{x_0}}{8(16(\kappa_\Sigma^2(t_1-t))^\delta + 1)(t_1-t)}} \geq \frac{1}{(8\pi\rho^2)^{n/2}} e^{-\frac{10\rho^2}{8\rho^2}} = \frac{e^{-\frac{10}{8}}}{(8\pi)^{n/2}} \rho^{-n},$$

so that for all $t \in (T - \rho^2, T)$

$$\int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t \geq \frac{1}{1728} \frac{1}{256} \frac{e^{-\frac{10}{8}}}{(8\pi)^{n/2}} \rho^{-n} \mathcal{H}^n(M_t \cap B_\rho(x_0)) = C(n) \frac{\mathcal{H}^n(M_t \cap B_\rho(x_0))}{\rho^n}. \tag{15.5}$$

Combining (15.3), (15.4), and (15.5) gives

$$\sup_{t \in (T-\rho^2, T)} \frac{\mathcal{H}^n(M_t \cap B_\rho(x_0))}{\rho^n} \leq C(n, \kappa_\Sigma) \frac{\mathcal{H}^n(M_{t_3} \cap B_{\sqrt{2+0.4n}\rho_1}(x_0))}{\rho_1^n}.$$

Since $\rho_1 = (2 + 0.4n)^{-1/2}\rho_0$ this gives the result. ♢

We now show that for any given time, we can bound the area in a ball in terms of only the radius for sufficiently small radii. We will combine a particular choice of time for this lemma with the above proposition to give us our final upper area ratio result. Although the following result, at first, does not appear to improve results, we have shown above that we can bound all area ratios up to (but not yet including) the first singular time by the area ratio of some given time but not that this given time is appropriately bounded (that is, finite). We show below that the area ratio for any given time and any small enough (not time dependent) radius is indeed finite.

Proposition 15.2.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions supported on the support surface Σ, $t_0 \in [0,T)$ and $x_0 \in \mathbb{R}^{n+1}$. Then there exists a constant depending only on x_0, $C(x_0)$, such that for all $R \in [0, 1/2\kappa_\Sigma]$

$$\frac{\mathcal{H}^n(M_{t_0} \cap B_R(x_0))}{\omega_n R^n} \leq c(x_0).$$

Proof:
Since $\mathcal{M}$ is a smooth solution we know at time t_0 that $\Theta^n(\mathcal{H}^n, M_{t_0}, x_0) =: \Theta^n_{x_0} < \infty$, (and will infact be smaller than or equal to 1). It follows that there exists an $\varepsilon > 0$ such that $\omega_n^{-1}R^{-n}\mathcal{H}^n(B_R(x_0) \cap M_{t_0}) < 2\Theta^n_{x_0}$ for all $R \leq \varepsilon$.

Now suppose that the claim is not true. Then for all $m \in \mathbb{N}$ with $m > 2\Theta^n_{x_0}$ there exists $R_m \in [\varepsilon, 1/2\kappa_\Sigma]$ such that $\mathcal{H}^n(B_{R_m}(x_0) \cap M_{t_0}) > m\omega_n R^n_m$. Note that $\lim_{m\to\infty} R_m =: R_0 \in [\varepsilon, 1/2\kappa_\Sigma]$ and that there exists a subsequence (which we continue to label $\{R_m\}$) such that either

1) $R_m \nearrow R_0$, or
2) $R_m \searrow R_0$.

In either case it is true that

$$\lim_{m\to\infty} \frac{\mathcal{H}^n(B_{R_m}(x_0) \cap M_{t_0})}{\omega_n R^n_m} = \infty,$$

but also that since M_{t_0} is properly embedded $\mathcal{H}^n(M_{t_0} \cap K) < \infty$ for each compact $K \subset \mathbb{R}^{n+1}$ so that

$$\frac{\mathcal{H}^n(B_{R_0}(x_0) \cap M_{t_0})}{\omega_n R^n_0} < \infty.$$

We wish to show that these two facts lead to a contradiction. Note also in particular that $R_0 \neq 0$.

Suppose first that $\mathbb{R}_m \searrow R_0$. Then

$$\begin{aligned}
\infty &= \lim_{m\to\infty} \frac{\mathcal{H}^n(B_{R_m}(x_0) \cap M_{t_0})}{\omega_n R^n_m} \\
&= \lim_{m\to\infty} \left[\frac{\mathcal{H}^n(B_{R_0}(x_0) \cap M_{t_0})}{\omega_n R^n_m} + \frac{\mathcal{H}^n((B_{R_m}(x_0) \sim B_{R_0}(x_0)) \cap M_{t_0})}{\omega_n R^n_m}\right] \\
&= \frac{\mathcal{H}^n(B_{R_0}(x_0) \cap M_{t_0})}{\omega_n R^n_0} \lim_{m\to\infty} \frac{R^n_0}{R^n_m} + \lim_{m\to\infty} \frac{1}{\omega_n R^n_m} \lim_{m\to\infty} \mathcal{H}^n((B_{R_m}(x_0) \sim B_{R_0}(x_0)) \cap M_{t_0}) \\
&= \frac{\mathcal{H}^n(B_{R_0}(x_0) \cap M_{t_0})}{\omega_n R^n_0} + \frac{1}{\omega_n R^n_0}\mathcal{H}^n\left(\bigcap_{m=1}^{\infty} (B_{R_m}(x_0) \sim B_{R_0}(x_0)) \cap M_{t_0}\right) \\
&= \frac{\mathcal{H}^n(B_{R_0}(x_0) \cap M_{t_0})}{\omega_n R^n_0} + \frac{\mathcal{H}^n(\emptyset)}{\omega_n R^n_0} \\
&= \frac{\mathcal{H}^n(B_{R_0}(x_0) \cap M_{t_0})}{\omega_n R^n_0},
\end{aligned}$$

where the third inequality follows since each part is finite. This contradiction shows that $R_m \searrow R_0$ is impossible.

Alternatively, if $R_m \nearrow R_0$, then

$$\begin{aligned}
\frac{\mathcal{H}^n(B_{R_0}(x_0)\cap M_{t_0})}{\omega_n R_0^n} &= \lim_{m\to\infty} \frac{\mathcal{H}^n((B_{R_0}(x_0)\sim B_{R_m}(x_0))\cap M_{t_0}) + \mathcal{H}^n(B_{R_m}(x_0)\cap M_{t_0})}{\omega_n R_0^n} \\
&\geq \lim_{m\to\infty} \frac{\mathcal{H}^n(B_{R_m}(x_0)\cap M_{t_0})}{\omega_n R_0^n} \\
&= \lim_{m\to\infty} \frac{R_m^n}{R_0^n}\frac{\mathcal{H}^n(B_{R_m}(x_0)\cap M_{t_0})}{\omega_n R_m^n} \\
&\geq \left(\frac{1}{2}\right)^n \lim_{m\to\infty} \frac{\mathcal{H}^n(B_{R_m}(x_0)\cap M_{t_0})}{\omega_n R_m^n} \\
&= \infty.
\end{aligned}$$

This contradiction shows that $R_m \searrow R_0$ is impossible. Thus there is a $N \in \mathbb{N}$ such that

$$\frac{\mathcal{H}^n(B_R(x_0)\cap M_{t_0})}{\omega_n R^n} \leq N$$

for all $R \in [\varepsilon, 1/2\kappa_\Sigma]$.

Defining $C(x_0) := \sup\{N, 2\Theta^n_{x_0}\}$ completes the proof. ♢

We conclude the chapter with a final upper area ratio. The final result combines Propositions 15.2.1 and 15.2.2 to be able to state united upper area ratio result which will later prove useful in its phrasing. In the following chapter we go on to the crucial concept to our research, Gaussian density, which we also use immediately in the same chapter to consider lower density ratio results.

Corollary 15.2.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution of mean curvature flow with Neumann free boundary conditions on the support surface Σ. Let $\rho_0 = (\tau_0/2)^{1/2}$ and $\delta \in (1/3, 2/5]$. Then for any $x_0 \in \Sigma$ there is a constant $C(n, \kappa_\Sigma, x_0)$ depending only on n, κ_Σ and x_0 such that for all $R \in (0, \rho_0]$ and $t \in (T - R^2, T)$

$$\mathcal{H}^n(M_t \cap B_R(x_0)) \leq C(n, \kappa_\Sigma, x_0) R^n.$$

Proof:
Since $\rho_0 = (\tau_0)^{1/2}$ we can apply Proposition 15.2.1 to obtain, for all $R \in (0, \rho_0]$

$$\sup_{t\in(T-R^2,T)} \frac{\mathcal{H}^n(M_t\cap B_\rho(x_0))}{\rho^n} \leq C(n,\kappa_\Sigma)\frac{\mathcal{H}^n\left(M_{T-\frac{\rho_0^2}{2+0.4n}} \cap B_{\rho_0}(x_0)\right)}{\rho_0^n}.$$

We note then that for any $\delta \in (1/3, 2/5]$ and any $n \geq 1$ $(\tau_0/2)^{1/2} < 1/2\kappa_\Sigma$ so that we can apply Proposition 15.2.2 to the right hand side with $t_0 = T - \frac{\rho_0^2}{2+0.4n}$ to get

$$\sup_{t\in(T-R^2,T)} \frac{\mathcal{H}^n(M_t\cap B_\rho(x_0))}{\rho^n} \leq C(n,\kappa_\Sigma)C(x_0)\rho_0^n = C(n,\kappa_\Sigma,x_0)$$

from which the result follows. ♢

15.3 Notes

Corollary 15.1.1 is actually a simpler version of Proposition 11.3.1, for an alternative proof see Proposition 2.2.1 in Buckland [6]. Proposition 15.1.1 is similarly standard and, again, a proof can be found in [6]. The definition, Definition 15.1.1, of the class of boundary localisation functions, as well as their properties, presented in Proposition 15.1.2 are our own. A version of Theorem 15.1.1 was attempted in [6] which in turn was meant as the Neumann free boundary version of an area bound for boundaryless mean curvature flow presented as Proposition 4.9 in Ecker [7]. The result presented here is a generalisation and (we believe) correction of the result attempted in [6]. Corollary 15.1.2 is our own. The Divergence Theorem, Theorem 15.1.2, is a standard result. A proof can be found in Simon [25]. Lemma 15.1.1 as well as Proposition 15.2.2 and Corollary 15.2.1 are our own. The result Proposition 15.2.1 is also our own though draws inspiration and ideas from the boundaryless version due to Ecker [7].