Chapter 14

Localised Monotonicity with Boundary Conditions

Just as Buckland translated Huisken's Monotonicity Formula to the mean curvature flow with Neumann free boundary conditions case, we find it necessary to translate the boundaryless localisation function (see Definition 12.0.1) to the Neumann free boundary situation. In this chapter we develop this analog and prove the monotonicity formulas that we will need.

We begin by defining the localisation function that we need. We prove that it possesses the necessary properties and then prove the appropriate monotonicity formulas.

14.1 An Additional Localisation Function

Although Bucklands η does indeed function as a localisation function, we cannot sufficiently control how local it is. That is, we cannot make the support arbitrarily small as we can with Ecker's ψ by chossing a smaller σ. Ecker's ψ however, does not allow for the nullification of the boundary terms of (11.5). We thus combine the properties of the two in our monotonicity formula. We accomplish this by multiplying a variation of the boundaryless localisation function ψ by η to take the place of f in (11.5).

Definition 14.1.1.
Let Σ be a Neumann free boundary support surface, $x_0 \in \Sigma$, $t_0 \in \mathbb{R}$ and $\sigma > 0$. We then define the Localisation function $\varphi_{(x_0,t_0),\sigma}(x,t) : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{(x_0,t_0),\sigma}(x,t) := \left(1 - \frac{r_{x_0} - 2n\tau_{t_0}}{\sigma^2}\right)_+^3,$$

where $\tau_{t_0} := t_0 - t$. Should (x_0, t_0) be understood we write simply φ_σ.

We prove immediately an important property of this localisation function below.

Lemma 14.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions on the support surface Σ, let $t_0 \in \mathbb{R}$, $x_0 \in \Sigma$ and $\sigma > 0$. Then for any $t \in [0, t_0)$ we have, on spt $\eta_{(x_0,t_0)}$,

$$\left(\frac{d}{dt} - \triangle_{M_t}\right)\varphi \leq 0.$$

Proof:
We define $Z := (r_{x_0} - 2n\tau_{t_0})\sigma^{-2}$ We can then consider φ as a function of Z. We note

$$\varphi'(Z) = -3(1-Z)_+^2 \leq 0 \text{ and } \varphi''(Z) = 6(1-Z)_+ \geq 0. \tag{14.1}$$

We also note from Proposition 11.3.4 (3) that

$$|div_{M_t} Dr - 4n| \leq \frac{20n\kappa_\Sigma |x|}{1-d\kappa_\Sigma} + \frac{4n\kappa_\Sigma^2 |x|^2}{(1-d\kappa_\Sigma)}^3$$

and that, from Proposition 11.3.6, we have on $\text{spt}\eta_{(x_0,t_0)}$ $(1-d\kappa_\Sigma)^{-1} < 2$ and $\kappa_\Sigma |x-x_0| \leq (3/160n)^2$. Then using standard differentiation theory, (14.1) and the above three inequalities we calculate

$$\begin{aligned}
\left(\frac{d}{dt} - \triangle_{M_t}\right)\varphi &= \varphi'\left(\frac{d}{dt} - \triangle_{M_t}\right) Z - \varphi''|\nabla Z|^2 \\
&\leq \varphi'\left(\frac{d}{dt} - \triangle_{M_t}\right) Z \\
&= (-\varphi')\sigma^{-2}(-2n + div_{M_t} Dr_{x_0}) \\
&\leq (-\varphi')\sigma^{-2}\left(-2n - 4n + \frac{20n\kappa_\Sigma |x-x_0|}{(1-d\kappa_\Sigma)} + \frac{4n\kappa_\Sigma^2 |x-x_0|^2}{(1-d\kappa_\Sigma)^3}\right) \\
&\leq 0.
\end{aligned}$$

♢

Together with properties mentioned in Chapter 11 this completes the necessary preparation for our Monotonicity formulas. We are, however, not finished with the study of the new localisation function. Although not needed for the monotonicity formulas, we will later need to observe properties of the rescaled version of $\varphi_{(x_0,t_0),\sigma}$. We first state the rescaled version of the function.

Proposition 14.1.1.
Let Σ be a Neumann free boundary support surface $(x_0, t_0) \in \Sigma \times \mathbb{R}$ and $\varphi_{(x_0,t_0),\sigma}$ be as defined in Definition 14.1.1. *Then under parabolic rescaling by a factor of λ we get*

$$\hat{\varphi}_\sigma = \left(1 - \lambda^2 \frac{r(y) - 2n|s|}{\sigma^2}\right)_+^3$$

where $r(y)$ is the tilde reflection radius function of y with respect to the rescaled support surface $\Sigma_{x_0}^\lambda$.

Proof:
As rescaling is simply a change of variables, one need only carry out the parabolic rescale change of variables to obtain the identity. ♢

We also need the following relatively trivial properties of $\varphi_{(x_0,t_0),\sigma}$ and $\hat{\varphi}_{(x_0,t_0),\sigma}$.

Proposition 14.1.2.
Let Σ be a Neumann free boundary support surface, $(x_0, t_0) \in \Sigma \times \mathbb{R}$ and $\varphi_{(x_0,t_0),\sigma}$ be the localisation function as defined in Definition 14.1.1. *Then the following properties hold:*

1. *for $\lambda > 0$ and $s \leq 0$, $\hat{\varphi}_{(x_0,t_0),\sigma} \leq \left(1 - \frac{\lambda^2 2n|s|}{\sigma^2}\right)_+^3$,*
2. *for $\lambda, R,$ and $t \in [0, t_0)$*

$$\textit{spt } \varphi_{(x_0,t_0),\sigma)}(\cdot, t) \subset B_R(x_0) \Rightarrow \textit{spt } \hat{\varphi}_{(x_0,t_0),\sigma}(\cdot, \lambda^{-2}(t-t_0)) \subset B_{\lambda^{-1}R}(0), \textit{ and}$$

3. for any fixed $s \leq 0$, $\hat{\varphi}_{(x_0,t_0),\sigma} \to 1$ locally uniformly as $\lambda \to 0$.

Proof:
That (1) holds follows from the definition of the rescaled test function since for $\tilde{y} = y - 2(< y, D\tilde{d} > -\tilde{d})D\tilde{d}$ (where $\tilde{d}(y)$ =signed dist$(y, \Sigma_\lambda^{x_0})$) $|y|^2 + |\tilde{y}|^2 \geq 0$ and therefore

$$\tilde{\varphi}_{(x_0,t_0),\sigma}(y,s) = \left(1 - \lambda^2 \frac{|y|^2 + |\tilde{y}|^2 - 2n(-s)}{\sigma^2}\right)_+^3 \leq \left(1 - \lambda^2 \frac{\lambda^2 2n|s|}{\sigma^2}\right)_+^3$$

from which the result follows.

(2) follows similarly easily from the general principle that the parabolic rescale $\tilde{f}$ of any function $f : \mathbb{R}^{n+1} \times [0,T)$ satisfies $\tilde{f}(y,s) = f(\lambda y + x_0, \lambda^2 s + t_0)$ as follows:

If $y \notin B_{\lambda^{-1}R}(0)$ then $\lambda y + x_0 \notin B_R(x_0)$ and thus $\tilde{\varphi}_{(x_0),\sigma}(y,s) = \varphi_{(x_0,t_0),\sigma}(\lambda y + x_0, \lambda^2 s + t_0) = 0$ from which the result follows.

Finally, set any $s \leq 0$ and $\sigma > 0$. Then for any compact set K, $K \subset B_R(0)$ for some $R > 0$ so that using Proposition 11.3.5 $r(y) \leq 10R^2$ for $y \in K$. Thus for any small $\varepsilon > 0$ we can take

$$\lambda_0 := \varepsilon \sqrt{\frac{\sigma^2}{2n|s| + 10R^2}}$$

so that for each $\lambda < \lambda_0$ we have, using (2)

$$1 \geq \hat{\varphi}_{(x_0,t_0),\sigma} = \left(1 - \lambda^2 \frac{r(y) + 2n|s|}{\sigma^2}\right)_+^3 \geq (1-\varepsilon)_+^3 > 1 - 3\varepsilon - \varepsilon^3,$$

which clearly suffices to show locally uniform convergence, proving (5). ♢

Having proven the properties of $\varphi_{(x_0,t_0),\sigma}$ necessary for this thesis, we move on to the first application. That is, the localised monotonicity formula for mean curvature flow with Neumann free boundary conditions.

14.2 Localised Monotonicity Formulas

We prove two monotonicity formulas in this section. One is a special case of the second. The first gives us a particular formula that we will be using. The second is the obvious generalised form of the first. An additional generalisation of the monotonicity formula of Buckland is that we allow the time centres of the test functions to vary from each other. This is necessary for later ratio bounds. The spatial centres must, however, remain the same.

Theorem 14.2.1. (Local Monotonicity)
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions on the support surface Σ. Let $\sigma > 0$, $\delta \in (0, 2/5]$, let $t_1, t_2, t_3 \in (0, T + \tau_0/2]$, $x_0 \in \Sigma$ and let $\varphi_{(x_0,t_0),\sigma}$ be as defined in Definition 14.1.1. *Then for any $t \in [t_1 - \tau_0/2, \min\{t_1, T\})$*

$$\frac{d}{dt}\left(e^{C\kappa_\Sigma^{2\delta}\tau^\delta} \int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t\right) \leq -e^{C\kappa_\Sigma^{2\delta}\tau_0^\delta} \int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} \left|\vec{H} - \frac{D^\perp \rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}}\right|^2 d\mu_t,$$

where $C = 17n$, $\varphi_3 := \varphi_{(x_0,t_3),\sigma}$, $\eta_2 := \eta_{x_0,t_2}$, and $\rho^1_{\kappa_\Sigma} := \rho_{\kappa_\Sigma,x_0,t_1}$.

Proof:
By choosing $g = \rho^1_{\kappa_\Sigma}$ and $f = \varphi_3\eta_2$ in (11.5) we see that (11.5) reduces to

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\varphi_3\eta_2\rho^1_{\kappa_\Sigma}\mu_t &\leq -\int_{M_t}\varphi_3\eta_2\left|\vec{H}-\frac{D^\perp\rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}}\right|^2 d\mu_t+\int_{M_t}\varphi_3\eta_2 Q(\rho^1_{\kappa_\Sigma})d\mu_t\\ &+\int_{M_t}\rho^1_{\kappa_\Sigma}\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\eta_2 d\mu_t\\ &+\int_{\partial M_t}\rho^1_{\kappa_\Sigma}<D\varphi_3\eta_2,\nu_\Sigma>d\sigma_t,\end{aligned}$$

so that applying Theorem 11.3.1 gives

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\varphi_3\eta_2\rho^1_{\kappa_\Sigma}d\mu_t &\leq -\int_{M_t}\varphi_3\eta_2\left|\vec{H}-\frac{D^\perp\rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}}\right|^2 d\mu_t+\frac{17n(\kappa^2_\Sigma(t_1-t))^\delta}{t_1-t}\int_{M_t}\varphi_3\eta_2\rho^1_{\kappa_\Sigma}d\mu_t\\ &+\int_{M_t}\rho^1_{\kappa_\Sigma}\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\eta_2 d\mu_t+\int_{\partial M_t}\rho^1_{\kappa_\Sigma}<D\varphi_3\eta_2,\nu_\Sigma>d\sigma_t, \quad (14.2)\end{aligned}$$

We note that $D\varphi_3 = Dr_{x_0}(3/\sigma^2)\varphi^{2/3}$ so that using $< Dr_{x_0},\nu_\Sigma >= 0$ and thus $< D\eta_2,\nu_\Sigma >= 0$ from Proposition 11.3.4 and Proposition 11.3.6 we have

$$<D\varphi_3\eta_2,\nu_\Sigma>=\varphi_3<D\eta_2,\nu_\Sigma>+\eta_2<D\varphi_3,\nu_\Sigma>=\eta_2\frac{3}{\sigma^2}\varphi_3^{3/2}<Dr_{x_0},\nu_\Sigma>=0.$$

Thus

$$\int_{\partial M_t}\rho^1_{\kappa_\Sigma}<D\varphi_3\eta_2,\nu_\Sigma>d\sigma_t=0.$$

Secondly we consider $\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\eta_2$. For $x\notin \text{spt}\eta_2$, $\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\eta_2=0$. Otherwise we choose for $x\in M_t$ an orthonormal basis $\tau_1,...,\tau_n$ for T_xM_t and calculate using the product rule

$$\begin{aligned}\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\eta_2 &= \varphi_3\frac{d}{dt}\eta_2+\eta_2\frac{d}{dt}\varphi_3-2\nabla^{M_t}\varphi_3\cdot\nabla^{M_t}\eta_2\\ &= \varphi_3\left(\frac{d}{dt}-\triangle_{M_t}\right)\eta_2+\eta_2\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3-2\sum_{i=1}^n D_{\tau_i}\varphi_3 D_{\tau_j}\eta_2.\end{aligned}$$

Since φ_3 and η_2 are radially symmetric functions monotonically decreasing in x as x moves away from a common spatial center point x_0 it follows that for any vector $\tau\in\mathbb{R}^{n+1}$

$$sign(D_\tau\varphi_3)=sign(D_\tau\eta_2)$$

so that

$$\begin{aligned}\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\eta_2 &= \varphi_3\left(\frac{d}{dt}-\triangle_{M_t}\right)\eta_2+\eta_2\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3-2\sum_{i=1}^n D_{\tau_i}\varphi_3 D_{\tau_j}\eta_2\\ &= \varphi_3\left(\frac{d}{dt}-\triangle_{M_t}\right)\eta_2+\eta_2\left(\frac{d}{dt}-\triangle_{M_t}\right)\varphi_3\\ &\quad -2\sum_{i=1}^n sign(D_\tau\varphi_3)sign(D_\tau\eta_2)|D_{\tau_i}\varphi_3||D_{\tau_j}\eta_2|\end{aligned}$$

$$
\begin{aligned}
&= \varphi_3 \left(\frac{d}{dt} - \triangle_{M_t}\right) \eta_2 + \eta_2 \left(\frac{d}{dt} - \triangle_{M_t}\right) \varphi_3 \\
&\quad -2 \sum_{i=1}^{n} sign(D_\tau \varphi_3)^2 |D_{\tau_i} \varphi_3| |D_{\tau_j} \eta_2| \\
&\leq \varphi_3 \left(\frac{d}{dt} - \triangle_{M_t}\right) \eta_2 + \eta_2 \left(\frac{d}{dt} - \triangle_{M_t}\right) \varphi_3.
\end{aligned}
$$

Since from Proposition 11.3.6, $\left(\frac{d}{dt} - \triangle_{M_t}\right) \eta_2 \leq 0$ and Lemma 14.1.1, $\left(\frac{d}{dt} - \triangle_{M_t}\right) \varphi_3 \leq 0$ we thus have $\left(\frac{d}{dt} - \triangle_{M_t}\right) \varphi_3 \eta_2 \leq 0$. Returning to (14.2) we therefore have

$$
\begin{aligned}
\frac{d}{dt} \int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t &\leq -\int_{M_t} \varphi_3 \eta_2 \left| \vec{H} - \frac{D^\perp \rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}} \right|^2 d\mu_t + \frac{17n(\kappa^2_\Sigma (t_1 - t))^\delta}{t_1 - t} \int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t \\
&\quad + \int_{M_t} \rho^1_{\kappa_\Sigma} \left(\frac{d}{dt} - \triangle_{M_t}\right) \varphi_3 \eta_2 d\mu_t + \int_{\partial M_t} \rho^1_{\kappa_\Sigma} < D\varphi_3 \eta_2, \nu_\Sigma > d\sigma_t \\
&\leq -\int_{M_t} \varphi_3 \eta_2 \left| \vec{H} - \frac{D^{\perp^1} \rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}} \right|^2 d\mu_t + \frac{17n(\kappa^2_\Sigma (t_1 - t))^\delta}{t_1 - t} \int_{M_t} \varphi_3 \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t.
\end{aligned}
$$

Introducing the integrating factor $e^{C\kappa_\Sigma^{2\delta} \tau^\delta}$ then gives the result. ♢

We write the generalisation of the above immediately. It follows with the same calculations, since we take exactly those test functions for which all necessary properties continue to hold.

Theorem 14.2.2. (general local monotonicity)
Let $\mathcal{M} = (M_t)_{t \in [0,T)}$ be a mean curvature flow with Neumann free boundary condition on the support surface Σ. Let $\delta \in (0, 2/5]$, $t_1, t_2 \in (0, T + \tau_0/2]$, $x_0 \in \Sigma$. Then for any f such that $f \in C^{2,1}(U)$

$$
\left(\frac{d}{dt} - \triangle_{M_t}\right) f \leq 0, f \geq 0,
$$

f has a single local maximum (and global maximum) at x_0 for all $t \in [0, T)$ and $< Df, \nu_\Sigma > (x, t) = 0$ for all $x \in \partial M_t$, $t \in [0, T)$,

and for any $t \in [t_1 - \tau_0, \min\{t_1, T\})$

$$
\frac{d}{dt} \left(e^{C\kappa_\Sigma^{2\delta} \tau^\delta} \int_{M_t} f \eta_2 \rho^1_{\kappa_\Sigma} d\mu_t \right) \leq -e^{C\kappa_\Sigma^{2\delta} \tau^\delta} \int_{M_t} f \eta_2 \rho^1_{\kappa_\Sigma} \left| \vec{H} - \frac{D^\perp \rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}} \right|^2 d\mu_t
$$

where $C = 17n$, $\eta_2 := \eta_{x_0, t_2}$ and $\rho^1_{\kappa_\Sigma} := \rho_{\kappa_\Sigma, x_0, t_1}$.

Proof:
As in Theorem 14.2.1, we see

$$
\begin{aligned}
\frac{d}{dt} \int_{M_t} f \eta_2 \rho^1_{\kappa_\Sigma} \mu_t &\leq -\int_{M_t} f \eta_2 \left| \vec{H} - \frac{D^\perp \rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}} \right|^2 d\mu_t + \int_{M_t} f \eta_2 Q(\rho^1_{\kappa_\Sigma}) d\mu_t \\
&\quad + \int_{M_t} \rho^1_{\kappa_\Sigma} \left(\frac{d}{dt} - \triangle_{M_t}\right) f \eta_2 d\mu_t + \int_{\partial M_t} \rho^1_{\kappa_\Sigma} < Df \eta_2, \nu_\Sigma > d\sigma_t,
\end{aligned}
$$

thus, from the hypothesis that $< Df, \nu_\Sigma > (x,t) = 0$ and Proposition 11.3.1 we have

$$\begin{aligned}\frac{d}{dt}\int_{M_t} f\eta_2\rho^1_{\kappa_\Sigma}\mu_t &\leq -\int_{M_t} f\eta_2\left|\vec{H} - \frac{D^\perp\rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}}\right|^2 d\mu_t + \frac{17n(\kappa^2_\Sigma(t_1-t))^\delta}{t_1-t}\int_{M_t} f\eta_2\rho^1_{\kappa_\Sigma}d\mu_t \\ &\quad + \int_{M_t}\rho^1_{\kappa_\Sigma}\left(\frac{d}{dt} - \triangle_{M_t}\right) f\eta_2 d\mu_t\end{aligned}$$

Using that f and τ_2 have the same unique spatial maximum so that $signD_\tau f = signD_\tau\eta_2$ we calculate as in Theorem 14.2.1 for some orthonormal basis for T_xM_t around an arbitrary point $x \in M_t$

$$\begin{aligned}\left(\frac{d}{dt} - \triangle_{M_t}\right) f\eta_2 &= f\left(\frac{d}{dt} - \triangle_{M_t}\right)\eta_2 + \eta_2\left(\frac{d}{dt} - \triangle_{M_t}\right) f - 2\sum_{i=1}^n D_{\tau_i} f D_{\tau_i}\eta_2 \\ &= f\left(\frac{d}{dt} - \triangle_{M_t}\right)\eta_2 + \eta_2\left(\frac{d}{dt} - \triangle_{M_t}\right) f - 2\sum_{i=1}^n sign(D_{\tau_i} f)^2 |D_{\tau_i} f||D_{\tau_i}\eta_2| \\ &\leq 0,\end{aligned}$$

since by hypothesis $\left(\frac{d}{dt} - \triangle_{M_t}\right) f \leq 0$, and it is known that $\left(\frac{d}{dt} - \triangle_{M_t}\right)\eta_2 \leq 0$. It follows that

$$\frac{d}{dt}\int_{M_t} f\eta_2\rho^1_{\kappa_\Sigma}\mu_t \leq -\int_{M_t} f\eta_2\left|\vec{H} - \frac{D^\perp\rho^1_{\kappa_\Sigma}}{\rho^1_{\kappa_\Sigma}}\right|^2 d\mu_t + \frac{17n(\kappa^2_\Sigma(t_1-t))^\delta}{t_1-t}\int_{M_t} f\eta_2\rho^1_{\kappa_\Sigma}d\mu_t$$

Introducing the integrating factor $e^{C\kappa^{2\delta}_\Sigma\tau^\delta}$, as in Theorem 14.2.1, completes the proof. ♢

14.3 Notes

The definition and results presented in this chapter are all original.