Chapter 9

Dimension, Rectifiability and Measure of Generalised Koch type Sets

We now consider the main results for Koch type sets. That is, under what conditions do we have finite, or weakly locally finite measure. Under what conditions are Koch type sets rectifiable, or not rectifiable, and under what conditions can we determine the dimension of a set in $\mathcal{K}$. The results are all determined by the construction parameters. All of the relevant parameters can be expressed in terms of the angles $\theta^A_{n,i}$. In the case of A_ε type sets we can exactly categorise the sets with respect to the above questions, for the Koch type sets it is not possible. The difference being that in the case of Koch type sets we could be generating measure from a pre-image set of measure zero in an otherwise well behaved set. The question of whether or not measure can indeed be generated remains presently unanswered. The important point for us, is that it cannot be ruled out.

For this reason some of the results will continue to be stated separately. In the general case we find, with respect to rectifiability, that

$$A \in \mathcal{K} \text{ is countably 1-rectifiable } \Leftrightarrow \mathcal{H}^1(\{x : \tilde{\Pi}^A(x) = \infty\}) = 0.$$

With respect to measure, we find that for each $A \in \mathcal{K}$

$$\mathcal{H}^1(A) = \int_{A_0 \sim \Lambda_\infty^{-1}} \tilde{\Pi}^A d\mathcal{H}^1 + \mathcal{H}^1(\Lambda_\infty).$$

and that $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty) = \infty$, In general we would also expect $\mathcal{H}^1(\Lambda_\infty^{-1}) = 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty) = 0$ (that is the non-generation of measure condition) so that we would then have

$$\mathcal{H}^1(A) = \int_{A_0} \tilde{\Pi}^A d\mathcal{H}^1.$$

While in certain cases (e.g. Λ_∞^{-1} is countable) it is certainly true, it may not be true in general. Note that this result holds also for $A \in \mathcal{K}$ with $dimA > 1$, in which case we get the uninformative result $\mathcal{H}^1(A) = \infty$.

Finally, with respect to dimension we define

$$\gamma_1^A := \sup\{a : \mathcal{H}^1(\{x : \tilde{\theta}_x^A \geq a\}) > 0\},$$

$$\gamma_2^A := \sup_{x \in A_0} \tilde{\theta}_x^A$$

and find

$$dim\Gamma_{f(\gamma_1^A)} = f_1(\gamma_1^A) \leq dimA \leq f_1(\gamma_2^A) = dim\Gamma_{f(\gamma_2^A)}$$

where

$$f(\gamma) := (1/2)(tan\gamma)$$

and therefore

$$f_1(\gamma) = -\frac{ln2}{ln((1/2)(1+(tan\gamma)^2)^{1/2})}.$$

Again, this simplifies under the hypothesis that for $B \subset A_0$ $\mathcal{H}^1(B) = 0 \Rightarrow \mathcal{H}^1(\mathcal{F}(B)) = 0$ in that we can then state

$$dimA \equiv f_1(\gamma_1^A).$$

It is in the A_ε type set case that we can ignore the possibility of generalisation of measure and thus the "nicer" results can be stated for these sets.

9.1 Lipschitz Representation and Rectifiability

We start by showing that in some cases an A_ε type set is actually a Lipschitz graph, where $\mathcal{F}$ would pass as a Lipschitz function.

Lemma 9.1.1.
Suppose $A \in A^0$ (see Definition 8.3.4) and $\sum_{n=0}^{\infty} \theta_n^A < \infty$. Then for each $l > 0$ there is an $n_0 \in \mathbb{N}$ such that $A \cap T_{n_0,i}^A$ can be expressed as the graph of a Lipschitz function with Lipschitz constant less than or equal to l over $A_{n_0,i}^A$ for each $i \in \{1, ..., 2^{n_0}\}$.

Proof:
Let n_0 be such that

$$\sum_{n=n_0}^{\infty} \theta_n^A < \frac{tan^{-1}(l)}{5}.$$

Then let $x, y \in A \cap T_{n_0,i}^A$ for some $i \in \{1, ..., 2^{n_0}\}$ with $x \neq y$. We then know that there exists a $n_1 > n_0$ such that for each $n_0 \leq n < n_1$ $x, y \in T_{n,k}^A$ for some k and that $x \in T_{n_1,j}^A$ and $y \in T_{n_1,j\pm1}^A$ for some integer j. Without loss of generality let $x \in T_{n_1,j}^A$ and $y \in T_{n_1,j+1}^A$.

By choice of n_0 we know that

$$\psi_{A_{n_0,i}^A}^{A_{n_1,j}^A} < \frac{tan^{-1}(l)}{5}$$

and by Lemma 6.2.1

$$\psi_{T_{n_1,j+1}^A}^{T_{n_1,j}^A} < 2\theta_{n_1,j}^A < 2\frac{tan^{-1}(l)}{5}$$

so that when writing $X = \{z \in \mathbb{R}^2 : z = x + ty, t \in \mathbb{R}\}$

$$\psi_{A_{n_1,j}^A}^X < 2\psi_{T_{n_1,j+1}^A}^{T_{n_1,j}^A} < 4\frac{tan^{-1}(l)}{5}.$$

Thus

$$\psi_{A_{n_0,i}^A}^X < \frac{tan^{-1}(l)}{5} + 4\frac{tan^{-1}(l)}{5} = tan^{-1}(l)$$

and hence

$$\frac{|\pi_{(A^A_{n_0,i})^\perp}(x) - \pi_{(A^A_{n_0,i})^\perp}(y)|}{|\pi_{A^A_{n_0,i}}(x) - \pi_{A^A_{n_0,i}}(y)|} < tan(tan^{-1}(l)) = l.$$

Noting that (x, y) was an arbitrarily chosen pair of distinct points completes the proof. $\diamondsuit$

Combining this Lipschitz result with Lemma 8.4.1 we are now able to present the rectifiability results. We first prove, both by Lipschitz graphs and the existence of approximate tangent spaces, the rectifiability under particular conditions of A_ε type sets. We present both in concuring with the philosophy that multiple proof methods allow further insight and understanding of the objects involved and are in any case interesting in their own right, as well as for comparative purposes.

We first prove the rectifiability using the Lipschitz lemmas to show that certain A_ε type sets can then be expressed as $\mathcal{H}^1$-almost everywhere subsets of a countable union of Lipschitz graphs.

Theorem 9.1.1.
Whenever $A \in A^0$ satisfies $\sum_{n=0}^{\infty} \theta_n^A < \infty$, A is countably 1 rectifiable.

Proof:
Since $\sum_{n=0}^{\infty} \theta_n^A < \infty$ there is, by Lemma 9.1.1, an $n_0 \in \mathbb{N}$ such that for each $i \in \{1, ..., 2^{n_0}\}$ $A \cap T^A_{n_0,i}$ can be expressed as the graph of a Lipschitz graph over $A^A_{n_0,i}$. That is there is a Lipschitz function $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A \cap T^A_{n_0,i} \subset f_i(A^A_{n_0,i})$. Then when $S^A_{n_0,i} : \mathbb{R} \to \mathbb{R}^2$ is a transformation satisfying $S^A_{n_0,i}([0, \mathcal{H}^1(A^A_{n_0,i})]) = A^A_{n_0,i}$ we can define $F_i : \mathbb{R} \to \mathbb{R}^2$ as $F_i = f_i \circ S^A_{n_o,i}$ to write

$$A = \bigcup_{i=1}^{2^{n_0}} A \cap T^A_{n_0,i} \subseteq \bigcup_{i=1}^{2^{n_0}} f_i(A^A_{n_0,i}) \subset \bigcup_{i=1}^{2^{n_0}} F_i(\mathbb{R}).$$

Since this is a subset of a form of expression of a set that is defined as being countably 1-rectifiable, the proof is complete. $\diamondsuit$

The second proof applies to sets with converging sums of base angles. In this case "potential" approximate tangent spaces eventually stop rotating and we can then use the approximate j-dimensionality to say that the set will be arbitrarily close to the limit of the rotating bases of the triangular caps containing a point and will thus have an approximate tangent space there.

Theorem 9.1.2.
Any $A \in A^0$ satisfying $\sum_{n=0}^{\infty} \theta_n^A < \infty$ has an approximate tangent space with multiplicity one almost everywhere and is thus countably 1-rectifiable.

Proof:
We first prove that $A - E$ is countably 1-rectifiable. Let $y \in A - E$, write $H := \mathcal{H}^1(A)$ and let $f \in C^0_C(\mathbb{R}^2)$. It follows in particular that f is Lipschitz with Lipschitz constant F_1 and that there is an M such that

$$sptf \subset B_M(0).$$

Let $F = max\{1, F_1\}$. Since the other case is trivial we assume $M > 0$.

Let $\varepsilon > 0$ and define $\delta = \varepsilon/(MF)$. Since $A \in A^0$ we know that $A - E$ satisfies property (iv), we therefore know that there is a $\rho_y > 0$ such that for all $\rho \in (0, \rho_y]$ there is a $L_{y,\rho}$ such that $A \cap B_\rho(y) \subset L^{\delta\rho/2}_{\rho,y}$ and we know in fact from the proof that $A - E$ satisfies (iv) that we may take $L_{y,\rho} || A^A_{n_\rho,i(y,n_\rho)}$ where $A^A_{n_\rho,i(y,n_\rho)}$ is taken such that $\mathcal{H}^1(A^A_{n_\rho,i(y,n_\rho)}) \in (\rho/2, \rho]$ and $y \in T^A_{n_\rho,i(y,n_\rho)}$.

Since $\sum_{n=0}^{\infty}\theta_n^A < \infty$ we know that $\{\psi_{\mathbb{R}}^{A_{n,i(n,y)}^A}\}$ is a convergent sequence and thus there is an affine space L such that

$$\psi_{\mathbb{R}}^L = \lim_{n\to\infty}\psi_{\mathbb{R}}^{A_{n,i(n,y)}^A}.$$

We then choose ρ_1 such that $\rho_1 < \rho_y$, so that for all $\rho < \rho_1$ the $A_{n_\rho,i(y,n_\rho)}^A$ taken as described above is such that

$$tan^{-1}(\psi_{A_{n_\rho,i(y,n_\rho)}^A}^L) < \frac{\delta}{2} \tag{9.1}$$

with n_ρ large enough for Lemma 9.1.1 to gaurantee that $A\cap T_{n_\rho,i(y,n_\rho)}^A$ can be expressed as the graph of a Lipschitz function with Lipschitz constant δ, and since $\sum_{n=0}^{\infty}\theta_n^A < \infty \Rightarrow \prod_{n=0}^{\infty}(cos\theta_n^A)^{-1} < \infty$ we take ρ_1 such that $n_{\rho_1}$is such that $\prod_{n=n_{\rho_1}}^{\infty}(cos\theta_n^A)^{-1} < 1+\varepsilon$.

Now let $\lambda < \frac{\rho_1}{M}$. Then we have that $A\cap B_{\lambda M}(y)\subset (A_{n_\lambda,i(y,n_\lambda)}^A)^{\delta\lambda M/2}$ so that by (9.1)

$$tan(\psi_L^{A_{n_\lambda,i(y,n_\lambda)}^A}) < \frac{\delta}{2}$$

so that $A\cap B_{M\lambda}(y)\subset L^{M\delta\lambda}$ and thus $\eta_{y,\lambda}A\cap B_M(0)\subset (L-y)^{M\delta}$.

On this set we also have

$$|f(x) - f(\pi_L(x))| < Lipf\cdot\delta M \leq \frac{MF\varepsilon}{MF} = \varepsilon$$

for all $x\in\eta_{y,\lambda}(A-E)$.

By otherwise considering the positive and negative parts of f we may assume that $f\geq 0$. We then note

$$\int_{\eta_{y,\lambda}(A-E)} f(y)d\mathcal{H}^1(y) \leq \int_{\eta_{y,\lambda}(A-E)}\varepsilon d\mathcal{H}^1 + \int_{\eta_{y,\lambda}(A-E)} f(\pi_L(y))d\mathcal{H}^1(y).$$

Then by Lemma 9.1.1 and Lemma 6.3.1 we know that we can apply the area formula with Jacobian calculated by taking the maximal vertical variation per unit along L as δ plus $2(2\theta_{n_\lambda}^A)$. That is, with the Jacobian factor bounded above by $(1+9\delta^2)^{1/2}$ so that we have

$$\begin{aligned}\int_{\eta_{y,\lambda}(A-E)} f(y)d\mathcal{H}^1(y) &\leq \int_{\eta_{y,\lambda}(A-E)}\varepsilon d\mathcal{H}^1 + (1+9\delta^2)^{1/2}\int_L f(y)d\mathcal{H}^1(y)\\ &< \varepsilon\mathcal{H}^1(\eta_{n,\lambda}(A-E)) + (1+9\varepsilon)\int_L f(y)d\mathcal{H}^1(y)\end{aligned}$$

which implies

$$\begin{aligned}\left|\int_{\eta_{y,\lambda}(A-E)} fd\mathcal{H}^1 - \int_L fd\mathcal{H}^1\right| &\leq \varepsilon(1+\varepsilon)2M + (1+9\varepsilon-1)\left|\int_L fd\mathcal{H}^1\right|\\ &= \varepsilon(1+\varepsilon)2M + (9\varepsilon)\int_L fd\mathcal{H}^1.\end{aligned}$$

Since this is true for all $\varepsilon > 0$ it follows that

$$\lim_{\lambda\to 0}\left|\int_{\eta_{y,\lambda}(A-E)} fd\mathcal{H}^1 - \int_L fd\mathcal{H}^1\right| = 0$$

so that

$$\lim_{\lambda\to 0}\int_{\eta_{y,\lambda}(A-E)} f d\mathcal{H}^1 = \int_L f d\mathcal{H}^1.$$

That is there is an approximate tangent space for y. Since this is true for all $y \in A - E$ and $\mathcal{H}^1(E) = 0$ we have

$$\begin{aligned}\lim_{\lambda\to 0}\int_{\eta_{y,\lambda}A} f d\mathcal{H}^1 &= \lim_{\lambda\to 0}\int_{\eta_{y,\lambda}(A-E)} f d\mathcal{H}^1 \\ &= \int_L f d\mathcal{H}^1\end{aligned}$$

for all $y \in A - E$. That is, A has an approximate tangent space for all $y \in A - E$, and therefore $\mathcal{H}^1$-almost everywhere $y \in A$ which implies that A is countably 1-rectifiable. ♢

Although these results are not for the entirety of A_ε type sets, the completion of the proofs of rectifiability fall under the proof for general $\mathcal{K}$ sets. We thus prove the more general result, stating the cleaner result for A_ε type sets as a corollary.

Theorem 9.1.3.
Let $A \in \mathcal{K}$.
If $\mathcal{H}^1(\Lambda_\infty) = 0$ then A is countably 1-rectifiable.

Remark:
It would clearly be desireable to be able to show that

$$\mathcal{H}^1(\Lambda_\infty^{-1}) = 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty)$$

which would be an a better situation since we have better understanding, perception and control of sets in A_0 than sets in A. It is, however, not necessarily in general true (though it may be). We do in some limited cases have control from A_0. For example if Λ_∞^{-1} is countable then $\mathcal{H}^1(\Lambda_\infty) = 0$ and so the above Theorem would then state that with such a Λ_∞^{-1}, A is countably 1-rectifiable.

Proof:
We note that

$$A = \Lambda_\infty \cup \bigcup_{m=1}^{\infty} \Lambda_m = \Lambda_\infty \cup \bigcup_{m=1}^{\infty} \mathcal{F}(\Lambda_m^{-1}) = \Lambda_\infty \cup \bigcup_{m=1}^{\infty} \mathcal{F}|_{\Lambda_m^{-1}}(\Lambda_m^{-1}).$$

Since, from Lemma 8.4.1, we know that $\mathcal{F}_{\Lambda_m^{-1}}$ is Lipschitz for each $m \in \mathbb{N}$, it follows that A is countably 1-rectifiable should $\mathcal{H}^1(\Lambda_\infty) = 0$. ♢

Before stating the corollary of rectifiability for A_ε sets, we prove the non-rectifiability result. In this way we will be able to demonstrate necessary and sufficient, that is, an equivalence of conditions for sets in A_ε to countable 1-rectifiability.

Theorem 9.1.4.
Let $A \in \mathcal{K}$ and $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0$. Then A is not countably 1-rectifiable.

Proof:
Let θ be any potential multiplicity function for A. Then $\theta \in L^1(\mathcal{H}^1, A, \mathbb{R})$ and thus θ is $\mathcal{H}^1$-measurable. We then claim that there is an $r > 0$ such that

$$\mathcal{H}^1(\mathcal{F}^{-1}(\{x \in A : \theta(x) > r\}) \cap \Lambda_\infty^{-1}) > 0.$$

This is true for otherwise $\mathcal{H}^1(\{x \in A_{0,0} : \theta \circ \mathcal{F}(x) = 0\}) > 0$ and thus $\mathcal{H}^1(\{x \in A : \theta(x) = 0\}) > 0$ contradicting θ being a positive function on A. Set

$$B := \mathcal{F}^{-1}(\{x \in A : \theta(x) > r\}).$$

Since θ is measurable, $\{x \in A : \theta(x) > r\}$ is measurable and thus, since from Proposition 8.4.1 we know $\mathcal{F}$ is measurable, B is $\mathcal{H}^1$-measurable in $A_{0,0}$.

It then follows from Lemma 8.5.2 that there exists a $B_1 \subset B$ with $\mathcal{H}^1(B_1) = \mathcal{H}^1(B) > 0$ such that $\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), \mathcal{F}(x)) = \infty$ for each $x \in B_1$.

Consider now $f \in C^0_C(\mathbb{R}^2, \mathbb{R})$ such that $\chi_{B_1(0)} \leq f \leq \chi_{B_2(0)}$ where χ is the characteristic function. Then for any tangent space, P, to A that may exist with respect to θ at $\mathcal{F}(x)$ for some $x \in B_1$, it follws, as in Proposition 8.4.3 that

$$\infty > \theta(\mathcal{F}(x)) \int_P f(y) d\mathcal{H}^1(y) = \lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda}(A)} f(y)\theta(x + \lambda y) d\mathcal{H}^1(y) = \infty.$$

Thus

$$\lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda} A} f(y)\theta(x + \lambda y) d\mathcal{H}^1(y) \neq \theta(\mathcal{F}(x)) \int_P f(y) d\mathcal{H}^1(y).$$

Since this is true for any $x \in \mathcal{F}(B_1)$ and $\mathcal{H}^1(\mathcal{F}(B_1)) \geq \mathcal{H}^1(B_1) > 0$ it follows that A does not have an approximate tangent space with respect to θ at x on a set of x of positive measure.

Since this holds for any allowed selection of θ it follows from the definition of rectifiable sets and Theorem A.0.1 that A is not countably 1-rectifiable. ♢

We can now state the cleaner result for A_ε type sets from which the particular results for A_ε and $\mathcal{A}_\varepsilon$ follow.

Corollary 9.1.1.
For an A_ε type set A, A is countably 1-rectifiable if and only if

$$\mathcal{H}^1((\Lambda_\infty^{-1})^A) = 0.$$

Proof:
We note that A being an A_ε type set implies $A \in \mathcal{K}$. Thus from Theorem 9.1.4, if $\mathcal{H}^1((\Lambda_\infty^{-1})^A) > 0$ then A is not countably 1-rectifiable.

Conversely, should $\mathcal{H}^1((\Lambda_\infty^{-1})^A) = 0$ then there must exist at least one point, x, for which $\tilde{\Pi}_x^A \neq \infty$. Since $\tilde{\Pi}_x^A$ is constant for all $x \in A$ for an A_ε type set it follows that $\tilde{\Pi}_y^A \neq \infty$ for each $y \in A_{0,0}$ and thus for each $y \in A$. It follows that $\Lambda_\infty^A = \emptyset$ and therefore that $\mathcal{H}^1(\Lambda_\infty^A) = 0$. It thus follows from Theorem 9.1.3 that A is countably 1-rectifiable. ♢

Theorem 9.1.5.
Let $1/100 > \varepsilon > 0$ and A be constructed as in Construction 4.2.1 with this ε. Then

$$\tilde{\Pi}_x^A \equiv \infty$$

and thus A is not 1-countably 1-rectifiable.

Proof:
From Lemma 8.3.1 we know that for any A_ε type set A_1,

$$\mathcal{H}^1(\tilde{A_n}^{A_1}) = \mathcal{H}^1(A_{0,0}^{A_1})\prod_{j=0}^{n}(cos\theta_{j,\cdot}^{A_1})^{-1} = \prod_{j=0}^{n}(cos\theta_{j,\cdot}^{A_1})^{-1}.$$

Since from Lemma 4.3.1 $\mathcal{H}^1(\tilde{A_n}^{A}) = (1 + n16\varepsilon^2)^{1/2}$ it follows that

$$\tilde{\Pi}^A = \lim_{n\to\infty}\prod_{j=0}^{n}(cos\theta_{j,\cdot}^{A})^{-1} = \lim_{n\to\infty}\mathcal{H}^1(\tilde{A_n}^{A}) = \lim_{n\to\infty}(1 + n16\varepsilon^2)^{1/2} = \infty.$$

Thus $x \in (\Lambda_\infty^{-1})^A$ for each $x \in A_{0,0}$. This completes the first part of the proof.

It thus follows that $\mathcal{H}^1((\Lambda_\infty^{-1})^A) > 0$. From Proposition 8.4.1 (3) it then follows that $\mathcal{H}^1(\Lambda_\infty^A) > 0$. Therefore, from Corollary 9.1.1, A is not countably 1-rectifiable. ♢

The proof then that $\mathcal{A}_\varepsilon$ is not countably rectifiable that we present is an indirect proof, assuming that $\mathcal{A}_\varepsilon$ is countably 1-rectifiable, which then implies that A_ε is countably 1-rectifiable. This contradiction completes the proof and the rectifiability results.

Theorem 9.1.6.
$\mathcal{A}_\varepsilon$ is not countably 1-rectifiable.

Proof:
We prove the Theorem by contradiction. So, suppose that $\mathcal{A}_\varepsilon$ is countably 1-rectifiable and so can be written in the form

$$\mathcal{A}_\varepsilon \subset A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R})$$

where $\mathcal{H}^1(A_0) = 0$ and $F_n : \mathbb{R} \to \mathbb{R}^2$ is a Lipschitz function for each $n \in \mathbb{N}$.

We now consider that by the construction of A_ε we know that $A_\varepsilon \cap T_{i,j}$ is $A_{2^{1-i}\varepsilon}$ constructed on a base of length $\mathcal{H}^1(A_{i,\cdot})$ (which we note importantly is greater than 2^{1-i} so that should A_ε be well defined, then so too is the new A_ε).

It thus follows that by contracting A_ε by 2^{1-i} in the vertical direction and by $\mathcal{H}^1(A_{i,\cdot})$ in the horizontal direction we have that the result $C(A_\varepsilon)$ is a copy of any $A_\varepsilon \cap T_{i,j}$ (where C is the contraction map satisfying the said conditions).

We thus know that there exists contraction maps for each $i \in \mathbb{N}$ and $j \in \{1, ..., 2^i\}$, $O_{ij} : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$O_{ij}(A_\varepsilon) = A_\varepsilon \cap T_{i,j}$$

which implies $O_{ij}(\mathcal{A}_\varepsilon) \subset A_\varepsilon \cap T_{i,j}$ and also that $O_{ij}(E) = E \cap T_{i,j}$.

Define

$$M_{A_\varepsilon} := A_\varepsilon \cap \bigcup_{i\in\mathbb{N}}\bigcup_{j=1}^{2^i} O_{ij}(\mathcal{A}_\varepsilon),$$

and

$$R_{A_\varepsilon} := A_\varepsilon \sim \left(\mathcal{A}_\varepsilon \cup \bigcup_{i\in\mathbb{N}}\bigcup_{j=1}^{2^i} O_{ij}(A_\varepsilon)\right) = A_\varepsilon \sim M_{A_\varepsilon}.$$

It follows that

$$L_{ijn} := O_{ij}(F(x)) \qquad i,n \in \mathbb{N}, j \in \{1,...,2^i\}$$

are Lipschitz functions $L_{ijn} : \mathbb{R} \to \mathbb{R}^2$. We note that $\{\{L_{ijn}\}_{i,n\in\mathbb{N}}\}_{j=1}^{2^i}$ is countable. Also that R_{A_ε} ia a subset of the union of balls (or deformed balls) around points in E. Also that by taking the further addition to $\mathcal{A}_\varepsilon$, $O_{ij}(\mathcal{A}_\varepsilon)$, we infinitely reduce this area by continually refining the deformed ball around each e_n, that is

$$R_{A_\varepsilon} \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\{i,j:O_{ij}((0,0))=e_n\}} O_{ij}(B_{r_1}((0,0))).$$

With this set up we can then attack the proof.

We first note that

$$\begin{aligned}
M_{A_\varepsilon} &= \mathcal{A}_\varepsilon \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(\mathcal{A}_\varepsilon) \\
&\subset A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}\left(A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R})\right) \\
&= A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i}\bigcup_{n=1}^{\infty} L_{ijn}(\mathbb{R}) \\
&= A_0 \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0) \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i}\bigcup_{n=1}^{\infty} L_{ijn}(\mathbb{R}),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}^1\left(A_0 \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0)\right) &\leq \mathcal{H}^1(A_0) + \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \mathcal{H}^1(O_{ij}(A_0)) \\
&\leq \mathcal{H}^1(A_0) + \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \mathcal{H}^1(A_0) \\
&= 0 + \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 0 \\
&= 0
\end{aligned}$$

and $\bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i}\bigcup_{n=1}^{\infty} L_{ijn}(\mathbb{R})$ is a countable collection of Lipschitz images.

It thus follows that M_{A_ε} is countably 1-rectifiable. That is

$$M_{A_\varepsilon} = M_0 \cup \bigcup_{n=1}^{\infty} M_n(\mathbb{R})$$

where $M_0 = A_0 \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0)$ is a set of measure zero and $\{M_n\}_{n=1}^{\infty}$ is a reordering of $\{F_n\}_{n=1}^{\infty} \cup \{\{L_{ijn}\}_{i,n=1}^{\infty}\}_{j=1}^{2^i}$. We now show that $\mathcal{H}^1(R_{A_\varepsilon}) = 0$.

Let $\eta > 0$. For each $i, n \in \mathbb{N}$ there exists $j_n = j_n(i,n) \in \{1, ..., 2^i\}$ such that $O_{ij}((0,0)) = e_n$. That is, $O_{ij}(B_{r_1}((0,0)))$ covers the part of R_{A_ε} centered on e_n, so that since

$$\lim_{i\to\infty} \mathcal{H}^1(A_{i,\cdot}) = 0$$

for each n we can choose an $i_n \in \mathbb{N}$ such that $diam(O_{ij}(B_{r_1}((0,0)))) < \eta 2^{-n}$. Then, since

$$R_{A_\varepsilon} \subset \bigcup_{n=1}^{\infty} O_{i_n j_n}(B_{r_1}((0,0)))$$

and since $diam(O_{ij}(B_{r_1}((0,0)))) < \eta 2^{-n} < \eta$ for each $n \in \mathbb{N}$ we then have that $\{O_{i_n j_n}(B_{r_1}((0,0)))\}_{n=1}^{\infty}$ is an appropriate covering set to estimate $\mathcal{H}^1_\eta$ and in fact we have

$$\mathcal{H}^1_\eta(R_{A_\varepsilon}) \le \mathcal{H}^1_\eta\left(\bigcup_{n=1}^{\infty} O_{i_n j_n}(B_{r_1}((0,0)))\right) \le \sum_{n=1}^{\infty} diam(O_{i_n j_n}(B_{r_1}((0,0)))) < \sum_{n=1}^{\infty} \eta 2^{-n} = \eta.$$

Thus

$$\mathcal{H}^1(R_{A_\varepsilon}) = \lim_{\eta\to 0} \mathcal{H}^1_\eta(R_{A_\varepsilon}) < \lim_{\eta\to 0} \eta = 0.$$

Now since $\mathcal{A}_\varepsilon = M_{A_\varepsilon} \cup R_{A_\varepsilon}$ we have

$$\mathcal{A}_\varepsilon = R_{A_\varepsilon} \cup M_0 \cup \bigcup_{n=1}^{\infty} M_n(\mathbb{R}).$$

Since $\mathcal{H}^1(R_{A_\varepsilon}) = 0$,

$$\mathcal{H}^1(R_{A_\varepsilon} \cup M_0) = 0$$

and it follows that A_ε is countably 1-rectifiable. This contradicts Theorem 9.1.5, thus $\mathcal{A}_\varepsilon$ is not countably 1-rectifiable. $\diamondsuit$

This completes our study of rectifiability, we move on to the measure results before finally considering the dimension of Koch type sets.

9.2 Measure Formulae for Koch Type Sets

For our measure result we present, as previously seen, a formula that resembles the Area Formula. We could also have applied the Area Formula (for more information on the Area Formula see for example Simon [25]) but not without some difficulty. We therefore present a self contained direct proof of the result.

Theorem 9.2.1.
Let $A \in \mathcal{K}$. Then, for all measurable $B \subset A_{0,0}$ the following holds

$$\mathcal{H}^1(\mathcal{F}(B)) = \int_{B\sim\Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 + \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty).$$

Remark:
As with the rectifiability theorem, the statement of this theorem would be simplified should it be true that

$$\mathcal{H}^1(\Lambda_\infty^{-1}) = 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty) = 0$$

in which case we could write

$$\mathcal{H}^1(\mathcal{F}(B)) = \int_B \tilde{\Pi} d\mathcal{H}^1,$$

since, should $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0$, both sides would then be ∞ so that they could in this case also be reconciled with one another.

It seems as though an application of the Area Formula for rectifiable sets is all that is necessary, which is likely to be true, however, since the convergence of $\tilde{\Pi}_n(x)$ is equivalent to the convergence of $\sum_n \theta_{n,i}^A(x)^2$ and thus not necessarily of $\sum_n \theta_{n,i}^A(x)$, the Jacobian is by no means a trivial quantity to calculate or show that it is equal to $\tilde{\Pi}$ on $A_{0,0} \sim \Lambda_\infty^{-1}$.

Proof:
We note that for any measurable $D \subset A_{0,0}$

$$\mathcal{F}(D) = \bigcap_{n=1}^{\infty} \bigcup_{i \in X_n} T_{n,i}^A$$

where

$$X_n := \{i \in \{1, ..., 2^n\} : i = i(n, x) \text{ for some } x \in D\}$$

and so can be constructed from countable unions and intersections of $\mathcal{H}^1$-measurable subsets of $\mathbb{R}^2$ and is therefore measurable. Also, since from Lemma 8.4.1 F_n is Lipschitz for each $n \in \mathbb{N}$ these sets are also measurable.

Further, since F_n is a Lipschitz map for each $n \in \mathbb{N}$, if $D \subset A_{0,0}$ so to is $F_n(D)$ for each $n \in \mathbb{N}$.

It follows then that

$$\mathcal{H}^1(\mathcal{F}(B)) = \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty) + \mathcal{H}^1(\mathcal{F}(B) \sim \Lambda_\infty).$$

We consider the second term.

Let $q \in \mathbb{N}$ and define

$$H_{nq} := \left\{ x \in A_0 : \frac{n-1}{q} < \tilde{\Pi}(x) \leq \frac{n}{q} \right\}.$$

We see

$$B \sim \Lambda_\infty^{-1} = \bigcup_{n=1}^{\infty} H_{nq}.$$

we now estimate $\mathcal{H}^1(\mathcal{F}(B \cap H_{nq}))$. Firstly $H_{nq} \subset \Lambda_{n/q}$ so that $\mathcal{F}(B \cap H_{nq}) = \mathcal{F}|_{\Lambda_{n/q}}(B \cap H_{nq})$ is a Lipschitz graph with $\text{Lip}\mathcal{F}|_{\Lambda_{n/q}} \leq n/q$ and so that $\mathcal{H}^1(\mathcal{F}(B \cap H_{nq})) \leq \frac{n}{q}\mathcal{H}^1(H_{nq})$.

It is now necessary to establish a lower estimate. To do this we define

$$H_{nqj} := \{x \in H_{nq} : \tilde{\Pi}_j(x) > (n-1)q^{-1} \geq \tilde{\Pi}_{j-1}(x)\}$$

and note that $H_{nqi} \cap H_{nqj} = \emptyset$ whenever $i \neq j$. We also define

$$J_{nq} := \{i \in \{1, ..., 2^j\} : i = i(n, x) \text{ for some } x \in H_{nqj}\}.$$

We note that $\mathcal{F}|_{\Lambda_{n/q}} \circ F_j^{-1}$ is a Lipschitz expansion map on $F_j(\Lambda_{n/q})$. It follows that

$$\begin{aligned}
\mathcal{H}^1(\mathcal{F}(H_{nqj})) &= \mathcal{H}^1(\mathcal{F} \circ F_j^{-1} \circ F_j(H_{nqj})) \\
&= \mathcal{H}^1\left(\bigcup_{i \in J_{nq}} \mathcal{F}|_{\Lambda_{n/q}} \circ F_j^{-1}(F_j(H_{nqj}) \cap A_{j,i})\right) \\
&= \sum_{i \in J_{nq}} \mathcal{H}^1(\mathcal{F}|_{\Lambda_{n/q}} \circ F_j^{-1}(F_j(H_{nqj}) \cap A_{j,i})) \\
&\geq \sum_{i \in J_{nq}} \mathcal{H}^1(F_j(H_{nqj}) \cap A_{j,i}) \\
&= \sum_{i \in J_{nq}} \tilde{\Pi}_{j,i} \mathcal{H}^1(H_{nqj} \cap [(i-2)2^{-j}, i2^{-j}]) \\
&> \sum_{i \in J_{nq}} \frac{n-1}{q} \mathcal{H}^1(H_{nqj} \cap [(i-1)2^{-j}, i2^{-j}]) \\
&= \frac{n-1}{q} \mathcal{H}^1(H_{nqj}).
\end{aligned}$$

Since H_{nq} is the disjoint union of $\{H_{nqj}\}_{j=1}^{\infty}$ it follows that

$$\mathcal{H}^1(\mathcal{F}(H_{nq})) = \sum_{j=1}^{\infty} \mathcal{H}^1(\mathcal{F}(H_{nqj})) > \frac{n-1}{q} \sum_{j=1}^{\infty} \mathcal{H}^1(H_{nqj}) = \frac{n-1}{q} \mathcal{H}^1(H_{nq}).$$

It then follows that

$$\frac{n-1}{q} \mathcal{H}^1(H_{nq}) \leq \mathcal{H}^1(\mathcal{F}(H_{nq})) \leq \frac{n}{q} \mathcal{H}^1(H_{nq}).$$

Correspondingly we have direct from the definition of H_{nq} that

$$\frac{n-1}{q} \mathcal{H}^1(H_{nq}) < \int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 \leq \frac{n}{q} \mathcal{H}^1(H_{nq})$$

so that

$$\left| \int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(H_{nq})) \right| < \frac{1}{q} \mathcal{H}^1(H_{nq})$$

and therefore

$$\begin{aligned}
\left| \int_{B \sim \Lambda_{\infty}^{-1}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(B \sim \Lambda_{\infty}^{-1})) \right| &= \left| \sum_{n=1}^{\infty} \int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(H_{nq})) \right| \\
&\leq \sum_{n=1}^{\infty} \left| \int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(H_{nq})) \right| \\
&< \sum_{n=1}^{\infty} \frac{1}{q} \mathcal{H}^1(H_{nq}) \\
&= \frac{1}{q} \mathcal{H}^1(B \sim \Lambda_{\infty}^{-1}) \\
&\leq \frac{1}{q}.
\end{aligned}$$

Since this is true for all $q \in \mathbb{N}$ it follows that

$$\left| \int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(B \sim \Lambda_\infty^{-1})) \right| = 0$$

and thus that

$$\int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 = \mathcal{H}^1(\mathcal{F}(B \sim \Lambda_\infty^{-1})).$$

This gives us

$$\begin{aligned} \mathcal{H}^1(\mathcal{F}(B)) &= \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty) + \mathcal{H}^1(\mathcal{F}(B) \sim \Lambda_\infty) \\ &= \int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 + \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty) \end{aligned}$$

completing the proof. ♢ As we mentioned at the beginning of this chapter, we present the simplified result for A_ε type sets. In this case, however, the result does not simplify. This is because, should $\tilde{\Pi}^A_{(\cdot)} \equiv \infty$ for some A_ε type set A then it could be this very A that allows for creation of measure. Then for any set $B \subset A_{0,0}$ with $\mathcal{H}^1(B) > 0$ we get

$$\int_B \tilde{\Pi} d\mathcal{H}^1 = \mathcal{H}^1(\mathcal{F}(B)) = \infty.$$

However, for a measurable set $B \subset A_{0,0}$ with $\mathcal{H}^1(B) = 0$ from which measure is created we would have $\int_B \tilde{\Pi} d\mathcal{H}^1 = 0$ but $\mathcal{H}^1(\mathcal{F}(B)) > 0$ preventing the simplified version of Theorem 9.2.1

$$\int_B \tilde{\Pi} d\mathcal{H}^1 = \mathcal{H}^1(\mathcal{F}(B))$$

holding as desired.

This, therefore, concludes our discussion of measure formulae and we now conclude with the results on dimension.

9.3 A Full Spectrum of Dimension

We complete Part I with a discussion of the Dimension of A_ε and Koch type sets. As we discussed earlier in this Chapter, in order to gather results about dimension we essentially want to place sets either inside of or around sets that we know the dimension of. Unfortunately, different A_ε type sets do not generally stay neatly inside of one another. We therefore need to use our centralisation results to rearrange each stage of construction to ensure that strict containment is retained by the necessary sets.

As with the rectifiability results, the A_ε type sets allow for a more cleanly stated result than the Koch type sets. Unlike some of the previous result, we shall not prove the aesthetically more pleasing results for the A_ε type sets as a corollary of the more general Koch type sets but shall rather prove the result directly. This is mainly because the proof attached to the A_ε type sets is much cleaner allowing the essential ingredients to be more clearly seen. The proof associated with the Koch type sets is then presented afterwards where the difficulties of allowing full variation of base angles require a much more technical proof.

As we will see from the results, a complete closed interval in $\mathbb{R}$ represents the possible dimensions of sets in $\mathcal{K}$. This shows the rich variation of the sets, which could otherwise perhaps have been of a dimension from a finite set of values.

Following the proof of the dimension of the A_ε type sets, we present a Corollary showing how the dimension of A_ε (which we directly proved to be 1 in Theorem A.0.1) follows easily from the more general result.

Theorem 9.3.1.
For $r \geq 0$ and $A \in A^r$ (see Definition 8.3.4)

$$dimA = -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})}.$$

Proof:
The proof is dependent on the dimension of Γ_ε. We thererfore first note that for any scaling $\lambda \in \mathbb{R}$

$$dim\lambda\Gamma_\varepsilon = dim\Gamma_\varepsilon.$$

We also note that

$$\Gamma_{1/2(tanr)} \in A^r$$

and finally, recalling $dim\Gamma_\varepsilon = -ln2/(lnl)$ where l is the shrinking factor associated with approximation stage, we calculate that l_r, the approriate l for $\Gamma_\varepsilon \in A^r$ is

$$l_r = -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})}.$$

Now, since for $A \in A^r$ $\theta_n^A \searrow r$ we have $\theta_n^A \geq r$ for all $n \in \mathbb{N}$. Thus, since $\theta_n^{\Gamma_{1/2(tanr)}} \equiv r$ for all $n \in \mathbb{N}$ and thus also $\mathcal{H}^1(A_{0,1}^A)T_{0,1}^{\Gamma_{1/2(tanr)}} \subset T_{0,1}^A$ Proposition 8.5.1 then gives us that

$$\mathcal{H}^1(A_{0,1}^A)\Gamma_{1/2(tanr)} \hookrightarrow^c A.$$

Since then

$$A = \bigcap_{n=0}^{\infty}\bigcup_{j=1}^{2^n} T_{n,j}^A, \quad \mathcal{H}^1(A_{0,1}^A)\Gamma_{1/2(tanr)} = \bigcap_{n=0}^{\infty}\bigcup_{j=1}^{2^n} T_{n,j}^{\mathcal{H}^1(A_{0,1}^A)\Gamma_{1/2(tanr)}}$$

and $|\{1,...,2^n\}| = 2^n \geq 2^{n-2n_0}$ for each $n_0 \in \mathbb{N} \cup \{0\}$ and each $n \in \mathbb{N} \cup \{0\}$ we can apply Lemma 8.5.2 to get

$$dimA \geq dim\mathcal{H}^1(A_{0,1}^A)\Gamma_{1/2(tanr)} = dim\Gamma_{1/2(tanr)}. \tag{9.2}$$

Then, for any $r_1 > r$ there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ $\theta_n^A \leq r_1$. It follows that by choosing arbitrarily and $j \in \{1,...,2^{n_0}\}$

$$\mathcal{H}^1(A_{n_0,j}^A)T_{0,1}^{\Gamma_{1/2(tanr_1)}} \supset T_{n_0,j}^A.$$

Now taking $T_j \in A^r$ to be the set generated by starting with $T_{n_0,j}^A$ and $\theta_n^{T_j} \equiv \theta_{n+n_0}^A$, we have by Proposition 8.5.1 that $T_j \hookrightarrow^c \mathcal{H}^1(A_{n_0,j}^A)\Gamma_{1/2(tanr_1)}$.

It then follows from Lemma 8.5.2 that

$$dimT_j \leq dim\mathcal{H}^1(A_{n_0,j}^A)\Gamma_{1/2(tanr_1)} = dim\Gamma_{1/2(tanr_1)}.$$

Taking a finite union of such sets will not alter the dimension, thus

$$dimA = dim \bigcup_{j=1}^{2^{n_0}} T_j = dimT_j \leq dim\Gamma_{1/2(tanr_1)} = -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})}.$$

Since this is true for all $r_1 > r$ it follows that

$$dimA \leq -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})} = dim\Gamma_{1/2(tanr)}.$$

Combining this with (9.2) gives the result. ♢

Corollary 9.3.1.
$dimA_\varepsilon = 1$.

Proof:
Since from Proposition 8.3.1 we know $A_\varepsilon \in A^0$ for any given ε, we can directly apply Theorem 9.3.1 to calculate

$$dimA_\varepsilon = -\frac{ln2}{ln(\frac{1}{2}(1+(tan(0))^2)^{1/2})} = -\frac{ln2}{ln(1/2)} = 1.$$

♢

Our final result is then the characterisation of dimension for the more general Koch type sets. As we see, the basic principle is the same as that used for A_ε type sets, the difference being the need to adjust for individually varying rates of change of base angle in the more general set up. We slowly eliminate those more rapidly decreasing, leaving those with a base measure enough to make a difference that reduce base angle slowly and would then, in the sense of Theorem 9.3.1 have higher dimension. It is these sets that dictate the dimension of the general whole set.

Theorem 9.3.2.
Let $A \in \mathcal{K}$ and

$$\gamma_1^A = \sup\{a : \mathcal{H}^1(\{x \in A_0 : \lim_{n\to\infty} \theta^A_{n,i(n,x)} \geq a\}) > 0\}$$

and

$$\gamma_2^A = \sup_{x\in A_0} \tilde{\theta}^A_x.$$

Then

$$dim\Gamma_{f(\gamma_1^A)} = f_1(\gamma_1^A) \leq dimA \leq f_1(\gamma_2^A) = dim\Gamma_{f(\gamma_2^A)}$$

where

$$f(\gamma) := (1/2)(tan\gamma)$$

and therefore

$$f_1(\gamma) := -\frac{ln2}{ln((1/2)(1+(tan\gamma)^2)^{1/2})}.$$

Should the hypothesis that for $B \subset A_0$ $\mathcal{H}^1(B) = 0 \Rightarrow \mathcal{H}^1(\mathcal{F}(B)) = 0$ hold, or should for a given $A \in \mathcal{K}$ we have $\mathcal{H}^1(\Upsilon_{\gamma_1^A+}) = 0$ then

$$dimA \equiv f_1(\gamma_1^A).$$

Proof:
We start by proving that $dimA \leq f_1(\gamma_1^A)$. Let $\xi < \gamma_1^A$. Then $\mathcal{H}^1(\Upsilon_{\xi+}^{-1}) > 0$. There is therefore an $n_0 \in \mathbb{N}$ such that $\mathcal{H}^1(\Upsilon_{\xi+}^{-1}) > 2^{-n_0}$. It follows that $\Upsilon_{\xi+}^{-1} \cap [(i-1)2^{-n_0}, i2^{-n_0}] \neq \emptyset$ for at least 2^{n-n_0} $i \in \{1,...,2^n\}$. From which we have $T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset$ for at least 2^{n-n_0} $i \in \{1,...,2^n\}$. In particular, this is true for all $n \geq n_0$. Since $\theta_{n,i}^A$ is decreasing in the sense that $\theta_{n,i}^A \geq \theta_{n+1,2i-k}^A$ for $k \in \{0,1\}$

$$\cup\{T_{n+1,i}^A : \theta_{n+1,i}^A > \xi\} \subset \cup\{T_{n,i}^A : \theta_{n,i}^A > \xi\}.$$

For each $T_{n_0,i}^A$, if there exists an $n > n_0$ such that there are less than $2^{n-2n_0} T_{n,i}^A \subset T_{n_0,i}^A$ satisfying $T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset$, then there is a minimum such n, say n_{0_i}.

Note that if $T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) = \emptyset$ for all $T_{n+k,j}^A \subset T_{n,i}^A$.

Thus for all $n > n_{0i}$ there are less than 2^{n-2n_0} $T_{n,i}^A \subset T_{n_0,i}^A$ satisfying $T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset$.

Suppose that there exists such an n_{0_i} for each $i \in \{1,...,2^{n_0}\}$ then let $n_{0_m} := \max_{i\in\{1,...,2^{n_0}\}} n_{0_i}$. We then see that for each $n > n_{0_m}$

$$\begin{aligned}
|\{T_{n,i}^A : T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset\} &\leq \sum_{j=1}^{2^{n_0}} |\{T_{n,i}^A : T_{n,i}^A \subset T_{n_0,j}^A \text{ and } T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset\}| \\
&< \sum_{j=1}^{2^{n_0}} 2^{n-2n_0} \\
&= 2^{n-n_0}.
\end{aligned}$$

This contradiction shows that there exists a trianglular cap $T_{n_0,i}^A$ such that for all $n \geq n_0$

$$|\{T_{n,j}^A : T_{n,j}^A \subset T_{n_0,i}^A \text{ and } T_{n,j}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset\}| \geq 2^{2-2n_0}.$$

Define

$$J_n := \{j \in \{1,...,2^{n-n_0}\} : T_{n,j}^A \subset T_{n_0,i}^A \text{ and } T_{n,j}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset\},$$

$$A' := \bigcap_{n=0}^{\infty} \bigcup_{j\in I_n} T_{n+n_0,j}^A \quad \text{and} \quad \tilde{\Gamma_\varepsilon} := \bigcap_{n=0}^{\infty} \bigcup_{j\in I_n} \mathcal{H}^1(A_{n_0,i}^A) T_{n,j}^{\Gamma_{f(\xi)}}$$

where $I_n := J_{n+n_0}$. We note that since $A' \subset A$ $\dim A' \leq \dim A$. Note also that $A_1 := \tilde{\Gamma_\varepsilon}$ and $A_2 := A'$ are in the form of A_1 and A_2 in the definition of centralisation (Definition 8.5.1). By observing that $|I_n| > 2^{n-n_0}$ for each $n \in \mathbb{N} \cup \{0\}$ and that $n_0 \in \mathbb{N}$, we see that in order to show that $\dim A \geq \dim \Gamma_{f(\xi)}$ it suffices, by Lemma 8.5.2, to show that $\tilde{\Gamma_\varepsilon} \hookrightarrow^c A'$.

Clearly, for any $T_{n_0+m+1,i}^A \subset A_{2(m+1)}$, $T_{n_0+m+1,i}^A \cap F_{n_0+m+1}(\Upsilon_{\xi+}^{-1}) \neq \emptyset$, also $T_{n_0+m+1}^A \subset T_{n_0+m,int(i/2)+1}^A$ so that

$$T_{n_0+m,int(i/2)+1}^A \cap F_{n_0+m+1}(\Upsilon_{\xi+}^{-1}) \neq \emptyset$$

and thus

$$T_{n_0+m,int(i/2)+1}^A \cap F_{n_0+m}(\Upsilon_{\xi+}^{-1}) \neq \emptyset,$$

so that $T_{n_0+m,\int(i/2)+1}^A \subset A_{2m}$ and hence we have $A_{2(m+1)} \subset A_{2m}$ for any $m \in \mathbb{N}$.

We see that in putting $\tilde{\Gamma}$ into the required form for Definition 8.5.1

$$\tilde{\Gamma} = A_1, T_n^{\tilde{\Gamma}} := \bigcup_{i=1}^{2^n} T_{n,i}^{\tilde{\Gamma}} = A_{1n}, T_{n,i}^{\tilde{\Gamma}} = A_{1ni}, \text{ and } n_1(m) = 2^m.$$

So that A_1 and A_2 individually satisfy the requirements of A_1 and A_2. Also, $n_1(m) = n_2(m)$. We therefore only need to show the existence of the transformations $\mathcal{T}_{n,i}^{A_1,A_2}$.

We note that each $A_{1mi} = T_{m,i}^{\tilde{\Gamma}}$ is a triangular cap of base length

$$2^{-m}\left(\prod_{i=0}^{m-1}(cos\xi)^{-1}\right) \times (\text{base length } T_{0,1}^{\tilde{\Gamma}}) = 2^{-m-n_0}\prod_{i=0}^{m+n_0-1}(cos\xi)^{-1}$$

and of base angle ξ.

We also note that for each $i \in \{1,...,n_1(m)\}$, $i \in \{1,...,n_2(m)\}$ so that A_{2mi} exists and is a triangular cap $T_{n_0+m,k}^A$ for some $k \in \{1,...,2^{n_0+m}\}$ with base angle $\theta_{n_0+m,k}^A \geq \xi$ and base length

$$2^{-n_0-m}\tilde{\Pi}_{n_0+m,k}^A.$$

Since a sequence $\{\theta_{n,i(n)}\}$ of angles in the construction of A is decreasing and $\theta_{n_0+m,k}^A \geq \xi$ it follows that

$$2^{-n_0-m}\tilde{\Pi}_{n_0+m,k}^A \geq 2^{-m-n_0}\prod_{i=0}^{m+n_0-1}(cos\xi)^{-1} = \mathcal{H}^1(A_{1mi}).$$

It follows, since A_{1mi} and A_{2mi} are isosceles triangles where A_{2mi} has a longer base and larger base angles that A_{2mi} is strictly larger than A_{1mi} in the sense that A_{1mi} could be placed inside of A_{2mi} and thus there must exist an orthogonal transformation $\mathcal{T}_{m,i}^{\tilde{\Gamma},A_2}$ such that

$$\mathcal{T}_{m,i}^{\tilde{\Gamma},A_2}(A_{1mi}) \subset A_{2mi}.$$

Since this is true for any $m \in \mathbb{N}$ and $i \in \{1,...,n_1(m)\}$ it follows that $\tilde{\Gamma} \hookrightarrow^c A_2 = A'$.

Since this is true for each $\xi < \gamma_1^A$ it follows that

$$dimA \geq dim\Gamma_{f(\gamma_1^A)} = f_1(\gamma_1^A).$$

For the $\leq$ inequalities, we let $B \subset A_0$ be $\mathcal{H}^a$-measurable for each $a \in \mathbb{R}$ and show that for $\gamma := \sup_{x\in B}\tilde{\theta}_x^A$

$$dim\mathcal{F}(B) \leq dim\Gamma_{f(\gamma)} = f_1(\gamma).$$

Let $\xi > \gamma$ and for each $n \in \mathbb{N}$ define

$$\chi_n := \cup\{T_{n,i}^A : \theta_{n,i}^A \geq \xi\}.$$

Then $\Psi_n := T_n - \chi_n$ is the finite union of triangular caps $T_{n,j}^A$ with $\theta_{n,j}^A \leq \xi$. We see that for each such triangular cap $T_{n,j}^A \subset \Phi_n$,

$$\mathcal{H}^1(A_{n,j}^A) \leq \mathcal{H}^1(A_{0,1}^A) = \mathcal{H}^1(A_{0,1}^{\Gamma_{f(\xi)}})$$

and that for each later triangular cap $T^A_{n,m,k} \subset T^A_{n,j}$

$$\theta^A_{n+m,k} \leq \theta^A_{n,j} \leq \xi = \theta^{\Gamma_{f(\xi)}}_{n+m,\cdot}.$$

It therefore follows from Proposition 8.5.1 that for each $T^A_{n,j} \subset \Psi_n$ $T^A_{n,j} \hookrightarrow^c \Gamma_{f(\xi)}$. Therefore, since $A \cap T^A_{n,j}$ equals the final set resulting from the Koch set construction starting from $T^A_{n,j}$, Lemma 8.5.2 gives

$$dim(A \cap T^A_{n,j}) \leq dim\Gamma_{f(\xi)}$$

and hence, since this is true for any such triangular cap, that

$$dim(A \cap \Psi_n) = dim(A \cap T^A_{n,j}) \leq dim\Gamma_{f(\xi)}.$$

Now, suppose that there exists a $y \in \mathcal{F}(B)$ with

$$y \notin \bigcup_{n=1}^{\infty} \Psi_n.$$

Then for each $n \in \mathbb{N}$ $\theta^A_{n,i(n,y)} \geq \xi$ and therefore $\tilde{\theta}^A_y = \lim_{n\to\infty} \theta^A_{n,i(n,y)} \geq \xi > \gamma$.

Since this is impossible it follows that $\mathcal{F}(B) \subset \cup_{n=1}^{\infty}(\Psi_n \cap A)$ and therefore that $dim\mathcal{F}(B) \leq dim\Gamma_{f(\xi)}$. Since this is true for each $\xi > \gamma$ we have $dim\mathcal{F}(B) \leq dim\Gamma_{f(B)} = f_1(B)$.

To finish the proof we note that $f_1(\gamma) \geq 1$ for each $\gamma \geq 0$, and consider firstly that for each $x \in A_0$, $\tilde{\theta}^A_x \leq \gamma^A_2$ so that immediately from the above we have

$$dimaA \leq dim\Gamma_{f(\gamma^A_2)} = f_1(\gamma^A_2).$$

For the second conclusion we consider $B = \Upsilon^{-1}_{\gamma^A_1}$. It follows that

$$dim\Upsilon_{\gamma^A_1+} \leq dim\Gamma_{f(\gamma^A_1)} = f_1(\gamma^A_1).$$

Should the hypothesis hold that for all $D \subset A_0$, $\mathcal{H}^1(D) = 0 \Rightarrow \mathcal{H}^1(\mathcal{F}(D)) = 0$, or should we directly have $\mathcal{H}^1(\Upsilon_{\gamma^A_1+}) = 0$, then we have $\mathcal{H}^1(\Upsilon_{\gamma^A_1+}) = 0$ and therefore

$$dim\Upsilon_{\gamma^A_1+} \leq dim\Gamma_{f(\gamma^A_1)} = f_1(\gamma^A_1).$$

We therefore have

$$dimA \leq max\{dim\Upsilon_{\gamma^A_1}, \Upsilon_{\gamma^A_1+}\} \leq dim\Gamma_{f(\gamma^A_1)} = f_1(\gamma^A_1),$$

which completes the proof. $\diamondsuit$

9.4 Notes

All of the results in this chapter are our own. We do however, note that Theorem 9.2.1 is similar in concept and proof idea to the well known Area Formula. A detailed discussion and proof of the Area Formula can be found in Simon [25]. Also, Theorem 9.3.1, although completely our own, is implicitly dependent on the work of Hutchinson [15].

Glossary

Symbol/Term	Definition/Result	Page
$\mathcal{H}^r$ Hausdorff r measure	Definition 3.1.1	12
$dimA$ Hausdorff dimension (of A)	Definition 3.1.1	12
$\Theta^{*n}(\mu, A, x)$ Upper n-dimensional density	Definition 3.1.2	12
$\Theta^n_*(\mu, A, x)$ Lower n-dimensional density	Definition 3.1.2	12
$\Theta^n(\mu, A, x)$ n-dimensional density	Definition 3.1.2	12
$\eta_{y,\rho}$ Blow up function	Definition 3.1.4	14
L^ρ neighbourhood of a subspace	Definition 3.1.4	14
Multiplicity one class	Definition 3.2.1	15
sing$M$ Interior singular set	Definition 3.2.2	16
$sing_{t_0}\mathcal{M}$ Singular set	Definition 3.2.4	16
reg$M$ Interior regular set	Definition 3.2.2	16
$reg_{t_0}\mathcal{M}$ Regular set	Definition 3.2.4	16
$S_+(z)$	Lemma 3.2.1	16
Definitions (i) to (viii)	Definition A	17
Questions (1) to (3)	Questions 3.2.1	19
(weak) locally finite $\mathcal{H}^j$ measure	Definition 3.2.5	19
$\mathcal{N}$		25
Λ_{δ_0}		25
Λ^2		25
ε-triangular cap	Definition 4.1.1	26
Γ_ε	Construction 4.1.1	27
$A_{i,j}$	Construction 4.1.1 and 4.2.1	27, 31
$T_{i,j}$	Construction 4.1.1 and 4.2.1	27, 31
Γ^E_ε	Definition 4.1.3	28
$E(\cdot)$	Definition 4.1.2	28
A_ε	Construction 4.2.1	31
e_i	Construction 4.2.2	32
$\mathcal{A}_\varepsilon$	Construction 4.2.2	33
π_S Projection onto a set	Definition 5.1.1	38
π_x	Definition 5.1.1	38
Ψ_1, Ψ_2	Definition 5.1.2	38
Ψ_u, Ψ^u	Definition 5.1.2	38
enter and exit the same side	Definition 5.1.2	38
$I_{u,y,\rho}$	Lemma 5.1.1	39
$G(n, m)$ Grassman manifold	Theorem 5.2.1	41
$C_\delta(x)$ δ-cone	Definition 6.0.1	46
$C_{\delta,L}$	Definition 6.0.1	46
ψ^A_B angle between A and B	Definition 6.2.1	48
$\theta^A_{n,i}$	Definition 6.2.2	49
$\psi(n, \varepsilon)$	Definition 6.2.2	49
$O_L(\cdot)$	Definition 6.2.2	49
$R_{n,i}$	Lemma 6.2.1	50
$\delta_1(\cdot)$	Proposition 7.1.1	65

Symbol/Term	Definition/Result	Page
$E_{\phi,\rho,x}$	Proposition 7.1.2	67
$l_{m,i}, r_{m,i}$	Proposition 7.1.2 and Definition 8.2.1	67
$a_{m,i}$	Proposition 7.1.2	67
A_ε type set	Definitions 8.1.1 and 8.1.4	71, 72
$\mathcal{K}$ Koch type set	Definitions 8.1.2 and 8.1.5	71, 73
$\tilde{A}_n^A$	Definition 8.1.3	72
D_n dyadic interval of order n	Definition 8.2.2	76
$D_{n,i}$	Definition 8.2.2	76
F_n	Proposition 8.2.1	76
$\mathcal{F}$	Proposition 8.2.1	76
f_i	Proposition 8.2.2	79
$i(n,x)$	Definition 8.3.1	81
$\tilde{\theta}_x^A$	Definition 8.3.1	81
$\tilde{\Pi}^A$, $\tilde{\Pi}_n^A$, $\tilde{\Pi}_{n,i}^A$	Definition 8.3.1	81
Λ_m, Λ_m^{-1}, Λ_{m+}, Λ_{m+}^{-1}, Λ_∞, Λ_∞^{-1}	Definition 8.3.1	81
Υ_a^{-1}, Υ_a, Υ_{a+}^{-1}, Υ_{a+}	Definition 8.3.1	81
$\psi_{L_2}^{L_1}$	Definition 8.3.2	81
$i(n,B)$	Definition 8.3.3	82
A^r	Definition 8.3.4	82
$\mathcal{F}_m$	Definition 8.4.1	84
$B_1^{\mathcal{A}_\varepsilon}$, $B_2^{\mathcal{A}_\varepsilon}$, $\mathcal{B}^{\mathcal{A}_\varepsilon}$	Definition 8.4.2	90
$A \hookrightarrow_C B$ A centered in B	Definition 8.5.1	95, 96
A_{im}, A_{imj}	Definition 8.5.1	95, 96
$\mathcal{T}_{mj}^{A,B}$	Definition 8.5.1	99, 96
$C_n^{A,B}$	Definition 8.5.1	95, 96
γ_1^A	Theorem 9.3.2	119
γ_2^A	Theorem 9.3.2	119
$f(\gamma)$	Theorem 9.3.2	119
$f_1(\gamma)$	Theorem 9.3.2	119