Chapter 6

Fitting the Counter Examples

We mentioned in Chapter 3 that only questions with the answer "no" remain to be shown. In this section we show these results by appropriately fitting counter examples. For us this means showing two things for each of the sets constructed in chapter 4. Firstly, that the sets constructed in Chapter 4 do in fact correspond to the definitions with respect to which they were constructed. Secondly, we show that the a given constructed set has the property required to make it a counter example (one of not being j-dimensional, not having locally finite $\mathcal{H}^n$ measure or not being j-dimensional depending on what a given set is supposed to be a counter example to). As mentioned in the introduction, the higher dimensional cases will be discussed the following chapter. The reason the general dimension is not dealt with here is that they in any case reduce to the 1-dimensional case as we shall see.

There is in fact, in terms of completing the classification being presented here, little that remains to be shown. What remains, however, is technical and involved. So much so that, as mentioned in Chapter 4, we do not present everything here. We state but do not prove that A_ε and $\mathcal{A}_\varepsilon$ are not countably j-rectifiable. The proof follows from and is presented after the more general results presented after all of the necessary preparatory results in Chapters 8 and 9.

Fitting the counter example to (iv) (2) in particular shows that a non-rectifiable set (working on the later proved understanding that A_ε is not rectifiable) spiralling at all points and magnifications does not spiral too tightly around any given point.

The structure of the Chapter is that we show that Λ_{δ_0} satisfies (vi) which will answer (vi) (2) in the negative. We do the same with Λ^2 for (iii). $\mathcal{A}_\varepsilon$ is then shown to satisfy (iv) (actually via first showing that A_ε satisfies (iv)), from which (iv) (2) is answered in the negative, and as a corollary therefore (iii) (2) is also answered in the negative. Γ_ε is then shown to satisfy (v), from which it follows from Lemmas 3.3.2 and 6.5.2 that (v) (1) is answered with a no, and therefore as a corollary, the remaining questions: (v) (2), (ii) (1) and (ii) (2) are also answered with no. We then finally state formally that A_ε and $\mathcal{A}_\varepsilon$ are not rectifiable. Through Proposition 3.1.1 we can then use A_ε and Γ_ε to show that the answers to (i), (ii), (iii), (iv) and (v) (3) are no.

The proofs that the sets satisfy the definitions are mainly geometric and will actually mostly involve fitting sets in cones and then considering an appropriate neighbourhood of the center point. For this we need to develop notation to describe the cones we are using. As we will also find sets that should be covered by two cones meeting at their vertex, notation and theory also need to be developed for angles between sets. The appropriate definitions will be made as (or shortly before) they are used.

Definition 6.0.1.
*Let A be a 1-dimensional affine subspace of $\mathbb{R}^2$, $\delta > 0$ and $x \in \mathbb{R}^2$, then A is said to be a subset of the δ-**cone** at x, $C_\delta(x)$, if*

$$A \subset \{y = (y_1, y_2) \in \mathbb{R}^2 : |y_2| < \delta|y_1|\} + x =: C_\delta(x).$$

More generally, if L is a 1-dimensional affine space in $\mathbb{R}^2$, $x \in A \cap L$ and ϕ is the orthogonal transformation such that

$$\phi(L) = \mathbb{R} \text{ and } \phi(x) = 0$$

*then we say that A is a subset of the δ-**cone around** L **at** x, $C_{\delta,L}(x)$ if*

$$A \subset \phi^{-1}\left(\{y = (y_1, y_2) \in \mathbb{R}^n : |y_2| < \delta|y_1|\}\right) =: C_{\delta,L}(x).$$

6.1 Simple Counter Examples

We now present the classification results that follow from the use of the simpler counter examples.

Proposition 6.1.1.
Let $\delta > 0$. Then Λ_δ satisfies the definition (vi) with respect to δ, and further does not have weak locally finite $\mathcal{H}^1$ measure so that the answer to (vi) (2) (weakly locally finite measure) is no.

Proof:
There are two types of points to consider. If $x = (x_1, x_2) \in \Lambda$ with $x \neq (0,0)$, then

$$x \in \text{ graph}\left(\frac{sgn(x_1)sgn(x_2)\delta x}{n}\right)$$

for some $n \in \mathbb{N}$. Then for $r_x = \frac{|x|\delta}{4(n+1)}$,

$$B_{r_x}(x) \cap \Lambda_\delta \subset \text{ graph}\left(\frac{sgn(x_1)sgn(x_2)\delta x}{n}\right) \subset G^{\delta r}_{\delta/n,x},$$

where $G_{\delta/n,x} \in G(1,2)$ is the affine space defined by graph$((sgn(x_1)sgn(x_2)\delta x)/n)$, for each $r \in (0, r_x]$. Thus, by setting $L_x = G_{\delta/n,x}$, x is an acceptable point with respect to (vi).

If $x = (0,0)$, then by construction, we may choose $L_x = \mathbb{R}$ and note that $G_{\delta/n,x} \subset C_\delta(x)$ for each $n \in \mathbb{N}$, so that $\Lambda_\delta \subset C_\delta(x)$.

It follows that $\Lambda_\delta \subset \mathbb{R}^{\delta\rho} = L_x^{\delta\rho}$ for each $\rho > 0$. Thus choosing an $r_x > 0$ at random we have $\Lambda_\delta \subset L_x^{\delta r}$ for each $r \in (0, r_x]$. It follows that Λ_δ satisfies (vi).

Note, however, that due to the fact that there are countably infinitely many lines of length $2r$ going through any ball of radius r around $(0,0)$,

$$\mathcal{H}^1(\Lambda_\delta \cap B_r((0,0))) = \infty \text{ for all } r > 0$$

so that Λ_δ is not weak locally $\mathcal{H}^1$ finite. It follows that the answer to (vi) (2) is no. $\diamondsuit$

Remark: It is in fact true that should definition (vi) be made to be a δ-approximation for some $\delta > 0$, then Λ_{δ_0} satisfies definition (vi) for all $\delta_0 \leq \delta$. It is, however, not necessary to prove this generalisation here.

Proposition 6.1.2.
Λ^2 satisfies (iii), and further does not have weak locally finite $\mathcal{H}^1$ measure so that the answer to (iii) (2) is no.

Proof:
There are two types of points to consider. If $x = (x_1, x_2) \in \Lambda^2$, $x \neq (0,0)$, then

$$x \in \text{ graph} \left(\frac{sgn(x_1)sgn(x_2)x^2}{n} \right)$$

for some $n \in \mathbb{N}$. Then for $r_x = \frac{|x|^2 \delta}{4(n+1)}$,

$$B_{r_x}(x) \cap \Lambda^2 \subset \text{ graph} \left(\frac{sgn(x_1)sgn(x_2)x^2}{n} \right)$$

Since also x^2 is differentiable there is a tangent line L_x to $sgn(x_1)sgn(x_2)x^2/n$ at x and a radius that can be chosen to be smaller than r_x, $r_{x_1} = r_{x_1}(\delta) > 0$, such that for all

$$y \in \text{ graph} \frac{sgn(x_1)sgn(x_2)x^2}{n} \cap B_{r_{x_1}}(x)$$

$$|\pi_{L_x^\perp}(y) - \pi_{L_x^\perp}(x)| < \delta |\pi_{L_x}(y) - \pi_{L_x}(x)|$$

so that $B_r(x) \cap \Lambda^2 \subset L_x^{\delta r}$ for each $r \in (0, r_{x_1}]$. Thus x is an acceptable point with respect to (vi).

If $x = (0,0)$, then by construction, we may choose $L_x = \mathbb{R}$ and note that for $|x| < \delta$

$$\frac{|x^2|}{n} = \frac{|x||x|}{n} < |x|\delta$$

for each $n \in \mathbb{N}$. Thus it follows that for each $r \in (0, r_x = \delta]$ $\Lambda^2 \cap B_r((0,0)) \subset L_x^{r\delta}$. It follows that Λ^2 satisfies (vi).

As in Proposition 6.1.1 the fact that there are countably infinitly many lines in Λ^2 of length greater than or equal to $2r$ going through any ball of radius r around $(0,0)$, shows that Λ^2 is not weak locally $\mathcal{H}^1$ finite. It follows that the answer to (vi) (2) is no. $\diamondsuit$

6.2 Spiralling

For A_ε and $\mathcal{A}_\varepsilon$ we first show that the required measure properties hold. That is that both of the sets are not weakly locally $\mathcal{H}^1$-finite. We then demonstrate that the sets A_ε and $\mathcal{A}_\varepsilon$ indeed satisfy (iv). We have to work quite hard to get the necessary classification results for Γ_ε and A_ε. This arises from the fact, as has been mentioned and as will be shown in the next chapter, that Γ_ε and A_ε develop spirals. In order to show the required properties we need to show that these spirals are not too tight. We now prove a technical lemma showing that we can find a "spiral free" view of our sets Γ_ε and A_ε. We can then discuss the measure properties of A_ε and $\mathcal{A}_\varepsilon$.

In order to discuss spiralling, we clearly need to discuss angles. For us, most essential will be the angle between two sets, particularly the angle between two triangular caps. As what is meant by simply saying 'the angle between the sets A and B' is unclear, we make a definition that will be sufficient for our needs.

Definition 6.2.1.
Let A and B be two sets that can be divided by some $G \in G(1,2)$ (in a sense that is explained below) and which have a single common point z. Then the angle between the two sets ψ_B^A is defined by

$$\psi_B^A := \min\{\theta : C_\theta(z) \supset G(A \cup B) \text{ for some } G \in G(1,2) \text{ dividing } A \text{ and } B\}$$

where as usual $G(1,2)$ is the Grassman manifold, $G(\cdot)$ denotes the rotation that takes $G \in G(1,2)$ to $\mathbb{R}_x$, and G divides A and B if for all $X \in A$, $\pi_x(G(X)) \leq 0$ and for all $Y \in B$, $\pi_x(G(Y)) \geq 0$.

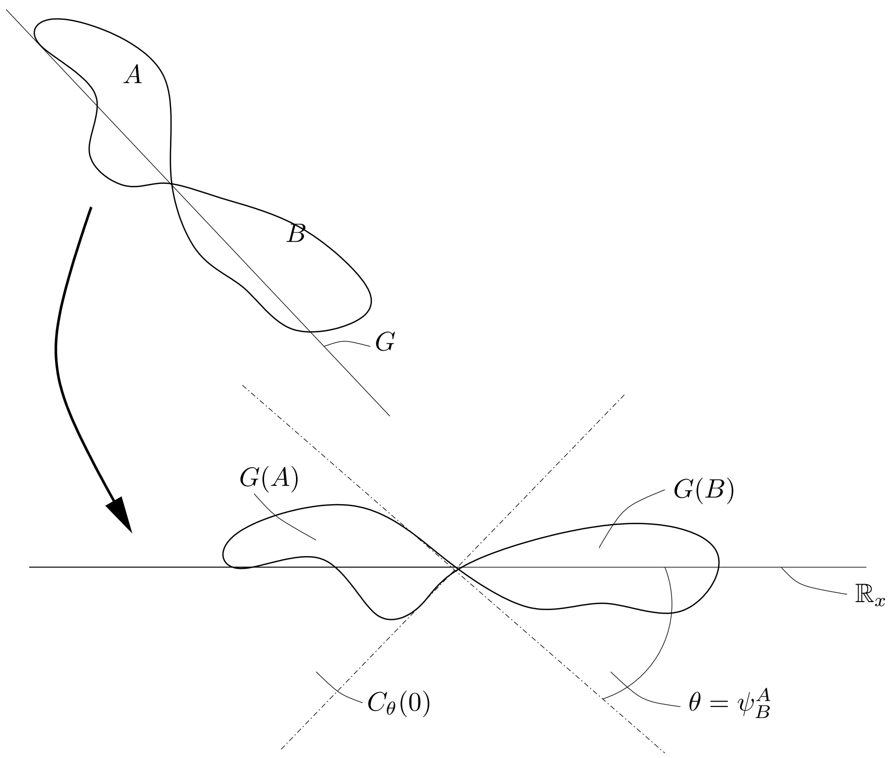

Figure 6.1: Angle between sets

Remarks: Clearly if $A_1 \subset A$, and $B_1 \subset B$ are such that $A_1 \cap B_1 = A \cap B = \{z\}$ then $\psi_{B_1}^{A_1} \leq \psi_B^A$. Note that the order is important due to the dividing of A and B. The notation ψ_B^A will always denote that A is in the "left cone half" (i.e. $\pi_x(G(A)) \subset \mathbb{R}_x^-$) and B is in the "right cone half" (i.e. $\pi_x(G(B)) \subset \mathbb{R}_x^+$) for the G giving the minimum. We note that $\psi_{(\cdot)}^{(\cdot)}$ is subadditive in the sense that, if A, B and C are sets for which the definition makes sense for the pairings $\{A, B\}$ and $\{B, C\}$ with $z_1 = A \cap B$ and $z_2 \in B \cap C$, then

$$\psi_{C-\{z_2-z_1\}}^A \leq \psi_B^A + \psi_C^B,$$

provided that such a value is less than $\pi/2$ (to ensure the dividing of the sets continues to make sense). Note that $\psi_{(\cdot)}^{(\cdot)}$ is translation and rotation invariant. We note also particularly that in considering the angle between sets A and B, if there is an affine space L such that $A \cap L = \{z, z_a\}$ (i.e.

contains the point common with B, z, and another point), then $\psi_B^L \leq \psi_B^A$ otherwise it would be impossible to contain z_a and B in a cone of angle ψ_B^A around z.

We also need to consider the angles that are actually intrinsic to the triangular caps.

Definition 6.2.2.
Let $n \in \mathbb{N} \cup \{0\}$, $j \in \{1, 2, ..., 2^n\}$ and A be one of Γ_ε, A_ε or $\mathcal{A}_\varepsilon$, then we see from the definition of triangular caps and Constructions 4.1.1, 4.2.1 and 4.2.2 that the triangular cap $T_{n,j}$ is an isosceles triangle. We denote the angles of $T_{n,j}$ as $\theta_{n,j}^A$ and $\pi - 2\theta_{n,j}^A$ where

$$\theta_{n,j}^A = \tan^{-1}(\psi(n, \varepsilon)),$$

where

$$\psi(n, \varepsilon) := \frac{2^{2-n}\varepsilon}{\frac{(1+n16\varepsilon^2)^{1/2}}{2^{n+1}}}$$

and where the ε is that associated with the construction of A_ε. Should the set A be understood we will simply write $\theta_{n,j}$. Further, should the $\theta_{n,j}$ be independent of j for the understood set A; $\theta_{n,j}$ will be written $\theta n, \cdot$.

Also, suppose that L is an 1-dimensional affine subspace (i.e. a line) of $\mathbb{R}^2$ of finite length (so that it has a middle point l), then we use O_L to denote the orthogonal transformation such that

$$O_L : L \to \mathbb{R}$$

and

$$O_L(l) = (0, 0).$$

Remark: In this chapter the angles $\theta_{n,j}^A$ are independent of the index j. However, in Chapters 8 and 9 when we look at general forms of the construction of A_ε, the angles will be allowed to vary in both n and j. For uniformity and simplicity later in the work, we introduce the symbol for the more general needs immediately.

Note: We are now in a position to comment further on the selection of $\varepsilon < 1/100$ in Constructions 4.1.1, 4.2.1 and 4.2.2 (we have previouisly commented on the selection in the remarks following Constructions 4.1.1 and 4.2.1. The selection of such a small ε is actually to ensure that we have $\psi(0, \varepsilon) < \pi/32$. To ensure that this requirement on $\psi(0, \varepsilon)$ is satisfied we need ε selected so that

$$tan^{-1}\left(\frac{8\varepsilon}{(1 + 16\varepsilon^2)^{1/2}}\right) < \frac{\pi}{32}$$

(coming from the definition of $\psi(n, \varepsilon)$.) That is

$$\frac{8\varepsilon}{(1 + 16\varepsilon^2)^{1/2}} < 0.09$$

so that taking $0 < \varepsilon < \frac{1}{100}$, as we have done, is sufficient. Since we in any case want to look at very small ε and eventually will also be looking at $\varepsilon \to 0$, this presents us with no problems. We will therefore henceforth assume the ε used to construct Γ_ε, A_ε, $\mathcal{A}_\varepsilon$ and other similar sets is less than 0.01. The reason for this assumption is that it is required for the spiralling lemmas to work.

Lemma 6.2.1.
Suppose that A_ε, $\mathcal{A}_\varepsilon$ and Γ_ε are as defined in Constructions 1,2 and 3. Then

(1) should two neighbouring trianglular caps, $T_{n,i}$ and $T_{n,i+1}$, be contained in another (necessarily earlier) triangular cap $T_{m,j(i)}$ $(m \leq n)$ then

$$\psi_{T_{n,i+1}}^{T_{n,i}} \leq 2\theta_{m,j(i)} \leq 2\theta_{0,1} \text{ and}$$

(2) the rectangle

$$R_{n,i} = \pi_x \left(O_{A_{n,i}} \left(\cup_{j:|i-j|\leq 1} A_{n,j} \right) \right) \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]$$

satisfies

$$O_{A_{n,i}}^{-1}(R_{n,i}) \cap A \subset \bigcup_{j:|i-j|\leq 1} T_{n,j},$$

for each $A \in \{A_\varepsilon, \mathcal{A}_\varepsilon, \Gamma_\varepsilon\}$.

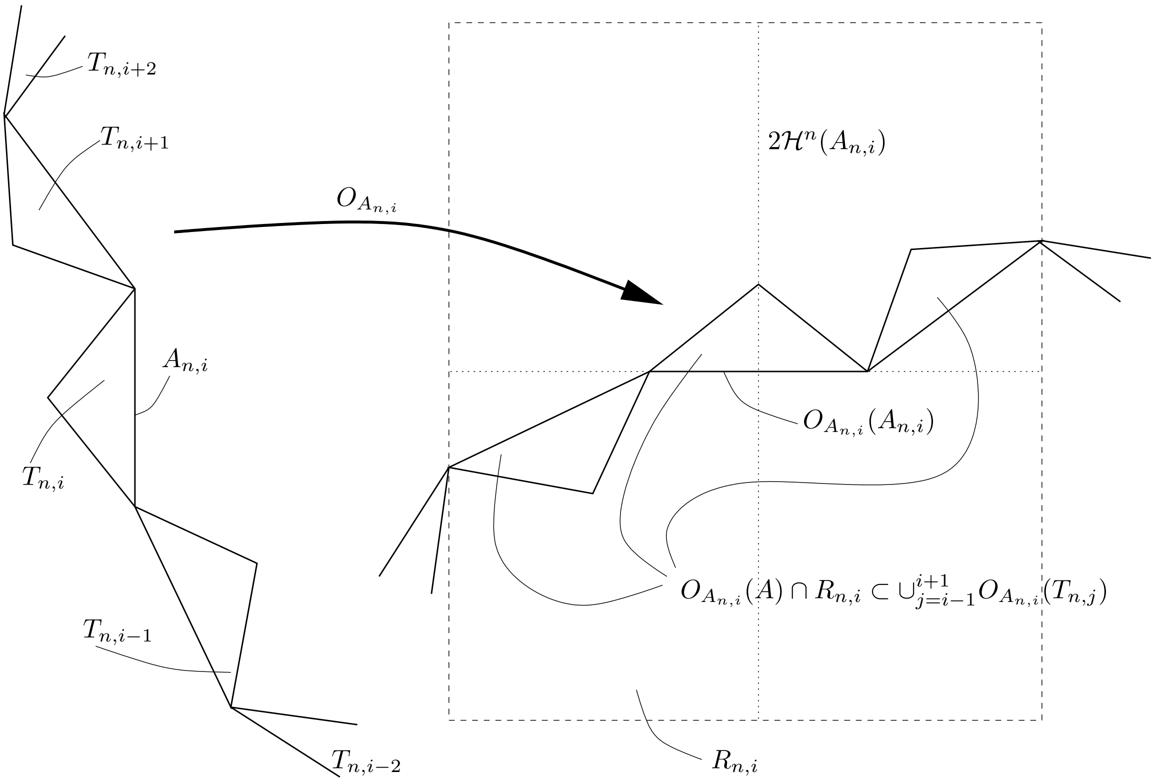

Figure 6.2: Restricted spiralling

Proof:
We give the proof for A_ε, from which the proofs for $\mathcal{A}_\varepsilon$ and Γ_ε follow. This is true for $\mathcal{A}_\varepsilon$ since $\mathcal{A}_\varepsilon \subset A_\varepsilon$ and it is true for Γ_ε since we make all claims with respect to the triangular caps. The

only additional tool used is properties of $\theta_{n,j}$. However, since the only property of $\theta_{n,j}$ from the construction of A_ε that is used is that $\theta_{n,j} \leq \theta_{m,i}$ for $m \leq n$ and since $\theta_{n,j} \equiv \theta_{0,1}$ for all $n \in \mathbb{N}$, $j \in \{1, ..., 2^n\}$ in the construction of Γ_ε, all arguments involving $\theta_{\cdot,\cdot}$ also translate directly to Γ_ε.

For (1), let $T_{n,i}$ and $T_{n,i+1}$ be two neighbouring triangular caps with common point z. Then, by the construction of A_ε, $z = z_{n_1+1,2i_1}$ is the vertex of a triangular cap T_{n_1,i_1} for some $n_1 < n$ and some appropriate i_1. Further, since $z \in T_{m,j(i)}$ and $T_{n,i}, T_{n,i+1} \subset T_{m,j(i)}$ so that $z \notin E(A_{m,j(i)})$ $m < n_1$ as otherwise the vertex a_{n_1,i_1} cannot be in $T_{m,j(i)}$.

Then by considering $G_{n_1,i_1} \in G(1,2)$ chosen such that $G_{n_1,i_1} || A_{n_1,i_1}$ we see that we can choose two "halves" (divided at $z_{n_1+1,2i_1}$) of G_{n_1,i_1}, $G^-_{n_1,i_1}$ and $G^+_{n_1,i_1}$, such that

$$\psi^{A_{n_1+1,2i_1}}_{G_{n_1,i_1^+}+z} \leq \theta_{n_1,\cdot} \text{ and } \psi^{G_{n_1,i_1^-}+z}_{A_{n_1+1,2i_1-1}} \leq \theta_{n_1,\cdot}$$

so that, since in both cases in finding the minimum over cones, from which the definition of $\psi^{A_{n_1+1,2i_1}}_{G_{n_1,i_1^+}+z}$ and $\psi^{G_{n_1,i_1^-}+z}_{A_{n_1+1,2i_1-1}}$ comes, we used the cone with respect to $G_{n_1,i-1}$, we have

$$\psi^{A_{n_1+1,2i_1}}_{A_{n_1+1,2i_1-1}} \leq \theta_{n_1,\cdot}.$$

Since then $T_{n_1+1,2i_1-1}$ and $T_{n_1+1,2i_1}$ are constructed on the interior of T_{n_1,i_1} with a base angle of $\theta_{n_1+1,\cdot}$, it follows similarly that

$$\psi^{T_{n_1+1,2i_1}}_{G_{n_1,i_1^+}+z} \leq \theta_{n_1,\cdot} + \theta_{n_1+1,\cdot} \text{ and } \psi^{G_{n_1,i_1^-}+z}_{T_{n_1+1,2i_1-1}} \leq \theta_{n_1,\cdot} + \theta_{n_1+1,\cdot}$$

so that, since we have, as above, in both cases again made the statements about $\psi^{\cdot}_{\cdot}$ with respect to a cone around G_{n_1,i_1}

$$\psi^{T_{n_1+1,2i_1-1}}_{T_{n_1+1,2i_1}} \leq \theta_{n_1,\cdot} + \theta_{n_1+1,\cdot}.$$

Now, since $\theta_{n,\cdot} > \theta_{m,\cdot}$ for all $n < m$ it follows that $\theta_{n_1,\cdot} \leq \theta_{m,\cdot} \leq \theta_{0,\cdot}$ and that $\theta_{n_1+1,\cdot} \leq \theta_{m,\cdot} \leq \psi(0,\varepsilon)$ so that

$$\psi^{T_{n_1+1,2i_1-1}}_{T_{n_1+1,2i_1}} \leq 2\psi(m,\varepsilon) \leq 2\psi(0,\varepsilon).$$

Finally, we note that now, by construction (in that A_ε is defined through intersection of the constructing levels) that $T_{n,i} \subset T_{n_1,i_1}$ and $T_{n,i+1} \subset T_{n_1,i_1+1}$ so that $\psi^{T_{n,i}}_{T_{n,i+1}} \leq 2\psi(m,\varepsilon) \leq 2\psi(0,\varepsilon)$ proving (1).

For (2), note that since $\varepsilon < 1/100$, $\psi(0,\varepsilon) < \pi/32$. We first need to make a subclaim.

The claim is that if $T_{n,i}$ and $T_{n,j}$ are triangular caps with $2 \leq |i-j| \leq 3$ then

$$\pi_x\left(O_{A_{n,i}}\left(\bigcup_{j:|i-j|<2} T_{n,j}\right)\right) \cap \pi_x(O_{A_{n,i}}(T_{n,j}) - \{z_{n,i-2}, z_{n,i+1}\}) = \emptyset.$$

From this claim we will prove (2). As claimed above, we note that since

$$\bigcup_{j:|i-j|<2} A_{n,j} = A \cap \bigcup_{j:|i-j|<2} T_{n,j}$$

it is sufficient to prove that for any $T_{n,i}, T_{n,i+1}, T_{n,i+2}$ we have

$$A \cap \pi_x \left(O_{A_{n,i+1}} \left(\bigcup_{j:|i+1-j|\leq 1} T_{n,j} \right) \right) \times [-2\mathcal{H}^1(A_{n,\cdot}), 2\mathcal{H}^1(A_{n,\cdot})] \subset \bigcup_{j:|i+1-j|\leq 1} T_{n,j}.$$

We now consider our claim.

We prove the case for $j - i > 0$, the other case following symmetrically. Note that we know from (1) that

$$\psi^{T_{n,i}}_{T_{n,i+1}} \leq 2\psi(0, \varepsilon)$$

and that

$$\psi^{T_{n,i+1}}_{T_{n,i+2}} \leq 2\psi(0, \varepsilon)$$

so that

$$\psi^{T_{n,i}}_{T_{n,i+2}-(z_{n,i+1}-z_{n,i})} \leq 4\psi(0, \varepsilon).$$

Indeed, since

$$\psi^{T_{n,i+2}}_{T_{n,i+3}} \leq 2\psi(0, \varepsilon),$$

$$\psi^{T_{n,i}}_{T_{n,i+2}-(z_{n,i+2}-z_{n,i})} = \psi^{T_{n,i}}_{T_{n,i+3}-(z_{n,i+2}-z_{n,i+1})-(z_{n,i+1}-z_{n,i})} \leq \psi^{T_{n,i}}_{T_{n,i+1}} + \psi^{T_{n,i+1}}_{T_{n,i+2}} + \psi^{T_{n,i+2}}_{T_{n,i+3}} \leq 6\psi(0, \varepsilon).$$

It thus follows that $\psi^{A_{n,i}}_{T_{n,i+3}-(z_{n,i+2}-z_{n,i})} \leq 6\psi(0, \varepsilon)$. Since $A_{n,i}$ is a line meeting the center of the cone

$$C_{6\psi(0,\varepsilon)}(G(z_{n,i})) \supset G(A_{n,i} \cup (T_{n,i+3} - (z_{n,i+2} - z_{n,i})))$$

it follows that

$$O_{A_{n,i}}(G^{-1}(C_{6\psi(0,\varepsilon)}(G(z_{n,i})))) \subset C_{12\psi(0,\varepsilon)}((0, \mathcal{H}^1(A_{n,i})/2))$$

and thus that

$$O_{A_{n,i}}(T_{n,i+3} - (z_{n,i+2} - z_{n,i})) \subset C^+_{12\psi(0,\varepsilon)}((0, \mathcal{H}^1(A_{n,i})/2))$$

(where C^+ denotes the right hand side of the cone), and therefore from the translation invariance of the cone containing a set

$$O_{A_{n,i}}(T_{n,i+3}) \subset C^+_{12\psi(0,\varepsilon),\mathbb{R}_x+z_{n,i+2}}(z_{n,i+2}).$$

This being the worse of the two possible j cases ($j = i + 1$ and $j = i + 2$), an identical proceedure can be used to show that $O_{A_{n,i}}(T_{n,i+2}) \subset C^+_{8\psi(0,\varepsilon),\mathbb{R}_x+z_{n,i+1}}(z_{n,i+1})$.

We note that $8\psi(0, \varepsilon) < 12\psi(0, \varepsilon) < \frac{12\pi}{32} < \frac{\pi}{2}$. Thus

$$\pi_x(O_{A_{n,i}}(T_{n,i+2} \cup T_{n,i+3})) \subset [\pi_x(O_{A_{n,i}}(z_{n,i+1})), \infty)$$

and

$$\pi_x(O_{A_{n,i}}(T_{n,i+2} \cup T_{n,i+3}) - \{z_{n,i+1}, z_{n,i-2}\}) \subset (\pi_x(O_{A_{n,i}}(z_{n,i+1})), \infty).$$

We find that a similar argument to the above produces $O_{A_{n,i}}(T_{n,i+1}) \subset C^+_{4\psi(0,\varepsilon),\mathbb{R}_x+z_{n,i}}(z_{n,i})$, so that since $4\psi(0, \varepsilon) < \pi/2 - \psi(0, \varepsilon)$

$$\begin{aligned}
\max\{\pi_x(y) : y \in O_{A_{n,i}}(T_{n,i+1})\} &= \pi_x(O_{A_{n,i}}(z_{n,i+1})) \\
&> \pi_x(O_{A_{n,i}}(z_{n,i})) \\
&= \max\{\pi_x(y) : y \in O_{A_{n,i}}(T_{n,i})\} \\
&= \pi_x(O_{A_{n,i}}(z_{n,i-1})) + \mathcal{H}^1(A_{n,\cdot}) \\
&\geq \max\{\pi_x(y) : y \in O_{A_{n,i}}(T_{n,i-1})\}.
\end{aligned}$$

Thus clearly

$$\pi_x\left(\bigcup_{j:|i-j|<2} O_{A_{n,i}}(T_{n,j})\right) \subset (-\infty, \pi_x(O_{A_{2,1}}(z_{2,3}))],$$

so that

$$\pi_x\left(\bigcup_{j:|i-j|<2} O_{A_{n,i}}(T_{n,j})\right) \cap \pi_x(O_{A_{n,i}}(T_{n,i+2}\cup T_{n,i+3}) - \{z_{n,i+1}, z_{n,i-2}\}) = \emptyset$$

proving the claim.

We now prove (2) by induction. We first note that for A_0 and A_1 it is obvious, as there are 1 and 2 triangular caps respectively, meaning that A is clearly a subset of any "triple" (using " " as it is actually impossible to choose a triple) of the form required. For A_2 there are four triangular caps, so that there is something to prove. However, we note that for any chosen i every triangle is either in the "triple" around i or has an index j such that $2 \leq |i-j| \leq 3$. Since A is a subset of the four triangles, the required result follows directly from the above proved claim.

We now prove the inductive step. We suppose that the inductive hypothesis (i.e. (2)) holds for all triples $\{T_{p,i-1}, T_{p,i}, T_{p,i+1}\}$ for a given $p \in \mathbb{N}$ and show that it holds for an arbitrary triple $\{T_{p+1,i-1}, T_{p+1,i}, T_{p+1,i+1}\}$. We set

$$\mathcal{T} = \cup\{T_{p+1,i-1}, T_{p+1,i}, T_{p+1,i+1}\}.$$

Note first that

$$\bigcup_{j:|i-j|<2} T_{p+1,j} \subset \bigcup_{j:|i_1-j|<2} T_{p,j}$$

where $i_1 = (i/2)^{\sqcap} - 1$ ($x^{\sqcap}$ is the smallest integer $q \geq x$), so that the triple is in fact a subset of a triple in the pth construction level. This triple in the pth construction level, by the induction hypothesis contains exactly 6 trianglular caps in the $(p+1)$th construction level, namely $\{T_{p+1,j}\}_{j=2i_1-3}^{2i_1+2}$ with $T_{p+1,i} \in \{T_{p+1,2i_1-1}, T_{p+1,2i_1}\}$. We also have by the inductive hypothesis that

$$A \cap R_{p,i_1} \subset \bigcup_{j=2i_1-3}^{2i_1+2} T_{p+1,j}.$$

It follows that

$$A \cap R_{p+1,i} \cap R_{p,i_1} \subset \bigcup_{j=2i_1-3}^{2i_1+2} T_{p+1,j}.$$

Now, since $i \in \{2i_1-1, 2i_1\}$ we see that for all $j \in \{2i_1-3, ..., 2i_1+2\}$, either $|i-j| < 2$ or $2 \leq |i-j| \leq 3$. From the above proven claim it follows that for each j such that $2 \leq |i-j| \leq 3$, $(T_{p+1,j} \sim \mathcal{T}) \cap R_{p+1,i} = \emptyset$. Thus

$$A \cap R_{p+1,i} \cap R_{p,i_1} \subset \mathcal{T}.$$

The induction then follows in the case that $R_{p+1,i} \subset R_{p,i_1}$, as in this case

$$A \cap R_{p+1,i} = A \cap R_{p+1,i} \cap R_{p,i_1} \subset \mathcal{T}.$$

We therefore prove that this is the case. It is clearly sufficient to show that

$$O_{A_{p,i_1}}(R_{p+1,i}) \subset O_{A_{p,i_1}}(R_{p,i_1})$$

as in this case

$$R_{p+1,i} = O_{A_{p,i_1}} \circ O^{-1}_{A_{p,i_1}}(R_{p+1,i}) \subset O_{A_{p,i_1}} \circ O^{-1}_{A_{p,i_1}}(R_{p,i_1}) = R_{p,i_1},$$

which is what we need.

Without loss of generality we may assume that

$$\begin{aligned} O_{A_{p,i_1}}(T_{p+1,i}) &\subset \triangle((0,0),(-\mathcal{H}^1(A_{p,j}/2,0),(0,\varepsilon\mathcal{H}^1(A_{p,j})))) \\ &\subset \triangle((0,0),(-\mathcal{H}^1(A_{p,j}/2,0),(0,\mathcal{H}^1(A_{p,j})/100))) \end{aligned}$$

where $\triangle(a,b,c)$ denotes the triangle in $\mathbb{R}^2$ with vertices a, b and c. The other cases follow with symmetric arguments.

We have

$$\pi_{O_{A_{p,i_1}}}\left(\bigcup_{j:|i-j|<2} O_{A_{p,i_1}}(T_p+1,j)\right) \subset$$
$$\left\{t\left(-\mathcal{H}^1(A_{p,j}),-\frac{\mathcal{H}^1(A_{p+1,j})}{100}\right)+(1-t)\left(\frac{\mathcal{H}^1(A_{p,j})}{2},\frac{2\mathcal{H}^1(A_{p+1,j})}{100}\right) : t\in[0,1]\right\};$$

so that

$$O_{A_{p,i_1}}\left(\bigcup_{j:|i-j|<2} O_{A_{p,i_1}}(T_p+1,j)\right) \subset \{x = y+z\}$$

where

$$y \in \left\{t\left(-\mathcal{H}^1(A_{p,j}),-\frac{\mathcal{H}^1(A_{p+1,j})}{100}\right)+(1-t)\left(\frac{\mathcal{H}^1(A_{p,j})}{2},\frac{2\mathcal{H}^1(A_{p+1,j})}{100}\right) : t\in[0,1]\right\}$$

and

$$z \in \left\{2s\mathcal{H}^1(A_{p+1,j})\left(\frac{-4}{100},2\right) : s\in[-1,1]\right\}.$$

That is $O_{A_{p,i_1}}(R_{p,i})$ is a subset of the quadrilateral with vertices

$$V_1 := (-1.54\mathcal{H}^1(A_{p,j}), 2\mathcal{H}^1(A_{p+1,j})),\quad V_2 := (0.96\mathcal{H}^1(A_{p,j}), 2.04\mathcal{H}^1(A_{p+1,j}))$$
$$V_3 := (1.04\mathcal{H}^1(A_{p,j}), -2\mathcal{H}^1(A_{p+1,j})) \text{ and } V_4 := (-1.46\mathcal{H}^1(A_{p,j}), -2.04\mathcal{H}^1(A_{p+1,j})).$$

Noting then that, due to the fact that $\psi(0,\varepsilon) < \pi/32$ and the general fact that $\psi^{T_{p,j}}_{T_{p,j+1}} < 2\psi(0,\varepsilon)$ (from (1)) we get

$$\mathcal{H}^1(\pi_x(O_{A_{p,i_1}}(T_{p,j}))) > \cos\left(\frac{\pi}{8}\right)\mathcal{H}^1(A_{p,j}) > 0.9\mathcal{H}^1(A_{p,j})$$

for all j such that $|j-i_1|<2$, and since

$$\mathcal{H}^1(A_{p+1,j}) = \frac{1}{2}(1+16\varepsilon^2)^{1/2}\mathcal{H}^1(A_{p,k}) < 0.6\mathcal{H}^1(A_{p,k})$$

we have

$$\begin{aligned} R_{p,i_1} &= O_{A_{p,i_1}}(R_{p,i_1}) \\ &\supset [-1.9\mathcal{H}^1(A_{p,j}), 1.9\mathcal{H}^1(A_{p,j})] \times [-2\mathcal{H}^1(A_{p,j}), 2\mathcal{H}^1(A_{p,j})] \\ &\supset [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})] \times [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})]. \end{aligned}$$

Since clearly

$$V_1, V_2, V_3, V_4 \in [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})] \times [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})]$$

it follows that $O_{A_{p,i_1}}(R_{p+1,i}) \subset O_{A_{p,i_1}}(R_{p,i_1})$ and thus that $R_{p+1,i} \subset R_{p,i_1}$ completing the proof of (2). $\diamondsuit$

6.3 Measure Properties of A_ε and $\mathcal{A}_\varepsilon$

To complete the classification of Definition A we need to establish some measure properties of A_ε and $\mathcal{A}_\varepsilon$. The complete proofs of the properties, however, involve further construction and preliminary results that apply to more general sets than to just A_ε and $\mathcal{A}_\varepsilon$. So as not to doubly present material, we provide the necessary proofs for the properties (stated below) that we need for A_ε and $\mathcal{A}_\varepsilon$ in Chapter 8. They will be presented as corollaries to the necessary technical results proven after all of the necessary definitions estabilshing the generalisations of the set A_ε have been made.

In essence, we need to show that A_ε and $\mathcal{A}_\varepsilon$ are not weakly locally $\mathcal{H}^1$-finite. In particular we present the following results:

Lemma 6.3.1.
A_ε is not weakly locally $\mathcal{H}^1$-finite.

Corollary 6.3.1.
For each $y \in \overline{A_\varepsilon}$, $\Theta^1(\mathcal{H}^1, A_\varepsilon, y) = \infty$.

Lemma 6.3.2.
$\mathcal{H}^1(\mathcal{A}_\varepsilon) = \infty$.

Corollary 6.3.2.
A_ε and $\mathcal{A}_\varepsilon$ are not weakly locally $\mathcal{H}^1$-finite.

In order to apply the more general results proved later to the above listed lemmata and corollaries we do need to prove a technical proposition specifically applicable to A_ε and $\mathcal{A}_\varepsilon$ concerning the angles $\theta^A_{n,i}$ for $A \in \{A_\varepsilon, \mathcal{A}_\varepsilon\}$.

Proposition 6.3.1.
For each sequence $\{i(n)\}_{n=0}^{\infty}$ satisfying $i(n) \in \{1, ..., 2^n\}$ for each $n \in \mathbb{N} \cup \{0\}$, and for each $A \in \{A_\varepsilon, \mathcal{A}_\varepsilon\}$

$$\prod_{n=0}^{\infty} (cos\theta^A_{n,i(n)})^{-1} = \infty.$$

Proof:
Since $\mathcal{A}_\varepsilon$ is constructed as simply A_ε with an open set removed, the construction has not been altered. It follows that $\theta^{\mathcal{A}_\varepsilon}_{n,i(n)} = \theta^{A_\varepsilon}_{n,i(n)}$ for each $n \in \mathbb{N} \cup \{0\}$ and $i \in \{1, ..., 2^n\}$. For this reason, it is sufficient to prove the proposition for $\mathcal{A}_\varepsilon$.

From the definitions of *cos* and $\theta^{\mathcal{A}_\varepsilon}_{n,i}$ (Definition 6.2.2) we see that for any defined choice of n and i

$$(cos\theta^{\mathcal{A}_\varepsilon}_{n,i})^{-1} = \frac{\mathcal{H}^1(A^{\mathcal{A}_\varepsilon}_{n+1,2i-1})}{\frac{1}{2}\mathcal{H}^1(A^{\mathcal{A}_\varepsilon}_{n,i})}.$$

Since $\mathcal{H}^1(A_{n,i}^{\mathcal{A}_\varepsilon}) = \mathcal{H}^1(A_{n,j}^{\mathcal{A}_\varepsilon}$ for each $n \in \mathbb{N} \cup \{0\}$ and each $i, j \in \{1, ..., 2^n\}$ we have

$$(cos\theta_{n,i}^{\mathcal{A}_\varepsilon})^{-1} = \frac{\mathcal{H}^1(A_{n+1,2i-1}^{\mathcal{A}_\varepsilon}) + \mathcal{H}^1(A_{n+1,2i}^{\mathcal{A}_\varepsilon})}{\mathcal{H}^1(A_{n,i}^{\mathcal{A}_\varepsilon})}$$

and thus

$$(cos\theta_{n,i}^{\mathcal{A}_\varepsilon})^{-1} = \frac{\sum_{j=1}^{2^{n+1}} \mathcal{H}^1(A_{n+1,j}^{\mathcal{A}_\varepsilon})}{\sum_{i=1}^{2^n} \mathcal{H}^1(A_{n,i}^{\mathcal{A}_\varepsilon})} = \frac{\mathcal{H}^1(A_{n+1}^{\mathcal{A}_\varepsilon})}{\mathcal{H}^1(A_n^{\mathcal{A}_\varepsilon})}.$$

By using Lemma 4.3.1 and the fact that the i can be arbitrarily chosen, we see that for any sequence $\{i(m)\}_{m=0}^{\infty}$

$$\prod_{m=0}^{n} (cos\theta_{m,i(m)}^{\mathcal{A}_\varepsilon})^{-1} = \prod_{m=0}^{n} \frac{\mathcal{H}^1(A_{m+1}^{\mathcal{A}_\varepsilon})}{\mathcal{H}^1(A_m^{\mathcal{A}_\varepsilon})} = \frac{\mathcal{H}^1(A_n^{\mathcal{A}_\varepsilon})}{\mathcal{H}^1(A_0^{\mathcal{A}_\varepsilon})} = \mathcal{H}^1(A_n^{\mathcal{A}_\varepsilon}) = (1 + n16\varepsilon^2)^{1/2}.$$

Thus

$$\prod_{m=0}^{\infty} (cos\theta_{m,i(m)}^{\mathcal{A}_\varepsilon})^{-1} = \lim_{n\to\infty} (1 + n16\varepsilon^2)^{1/2} = \infty.$$

◇

The Proofs to Lemma 6.3.1 and Lemma 6.3.2 will be presented following Proposition 8.4.1 in Chapter 8. However, under the understanding that lemmata 6.3.1 and 6.3.2 hold, we can prove now directly corollaries 6.3.1 and 6.3.2.

Proof of Corollary 6.3.1

Let $y \in \overline{A_\varepsilon}$ and $\rho > 0$, then there is a $y_1 \in A_\varepsilon \cap B_{\rho/2}(y)$ such that $B_{\rho/2}(y_1) \subset B_\rho(y)$. Since $y_1 \in A_\varepsilon$ there is a triangular cap $T_{n,i(n,y)} \ni y$ for each $n \in \mathbb{N}$. Also, there is an $n_0 \in \mathbb{N}$ such that $\mathcal{H}^1(A_{n,i}) < \rho/4$ for each $n > n_0$ and $i \in \{1, ..., 2^n\}$ so that $T_{n,i(n,y)} \subset B_{\rho/2}(y_1)$ for each $n > n_0$.

Now, from the symmetry of the construction we see that $T_{n_0+1,i(n_0+1,y)}$ is a $\mathcal{H}^1(A_{n_0+1,i})$ scale copy of $A_{2^{-n_0}\varepsilon}$. However, from Lemma 6.3.1 we know
$\mathcal{H}^1(A_{2^{-n_0}\varepsilon}) = \infty$, thus

$$\mathcal{H}^1(A_\varepsilon \cap B_\rho(y)) \geq \mathcal{H}^1(A_\varepsilon \cap T_{n_0+1,i(n_0+1,y)}) = \mathcal{H}^1(A_{n_0+1,\cdot}) \cdot \mathcal{H}^1(A_{2^{-n_0}\varepsilon}) = \infty.$$

It follows that

$$\Theta(A_\varepsilon, y) = \lim_{\rho\to 0} \frac{\mathcal{H}^1(B_\rho(y) \cap A_\varepsilon)}{\omega_1 \rho} = \infty.$$

◇

Proof of Corollary 6.3.2

For A_ε this follows directly from Corollary 6.3.1.

Now, suppose that $\mathcal{A}_\varepsilon$ is weakly locally $\mathcal{H}^1$-finite. Then for each $y \in \mathcal{A}_\varepsilon$ there is a radius $\rho_y > 0$ such that $\mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_{\rho_y}(y)) < \infty$. $\{B_{\rho_y}(y)\}_{y\in\mathcal{A}_\varepsilon}$ is an open cover of $\mathcal{A}_\varepsilon$ so that since $\mathcal{A}_\varepsilon$ is compact there must exist a finite subcover $\{B_{\rho_{y_n}}(y_n)\}_{n=1}^{Q}$ of $\mathcal{A}_\varepsilon$ and further we know that

$$M := \max\{\mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_{\rho_{y_n}}(y_n)) : 1 \leq n \leq Q\} < \infty.$$

It follows that

$$\mathcal{H}^1(\mathcal{A}_\varepsilon) \le \mathcal{H}^1\left(\mathcal{A}_\varepsilon \cap \bigcup_{n=1}^{Q} B_{\rho_{y_n}}(y_n)\right) \le \sum_{n=1}^{Q} \mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_{\rho_{y_n}}(y_n)) < \infty.$$

This contradiction implies that there must exist a $y \in \mathcal{A}_\varepsilon$ such that for each $\rho > 0$ $\mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_\rho(y)) = \infty$ and therefore that $\mathcal{A}_\varepsilon$ is not weakly locally $\mathcal{H}^1$-finite. $\diamondsuit$

Remarks:
1) Although having an infinitely dense point is not that uncommon, and in fact having a set of $\mathcal{H}^1$ positive measure of points of $\mathcal{H}^1$ infinite density is not uncommon, that A_ε is a set of Hausdorff dimension 1 of positive $\mathcal{H}^1$ measure that has infinite $\mathcal{H}^1$ density at all points of its closure is less common, which makes A_ε a set of peculiar interest in its own right without association to the properties that we are currently discussing. This interesting feature is one of the motivations for the generalisations of A_ε in the later Chapters of Part I of this Thesis.

2) Althouh we have only shown that one such point exists for $\mathcal{A}_\varepsilon$, we can in fact show (and will later show) that there is a subset of $\mathcal{A}_\varepsilon$ of infinite $\mathcal{H}^1$ measure for which each element, x satisfies $\Theta^1(\mathcal{H}^1, \mathcal{A}_\varepsilon, x) = \infty$). It may be, but is not necessarily true that the peculiar property of A_ε (that each element of A_ε has infinite density) holds for $\mathcal{A}_\varepsilon$.

3) The fact that there are infinite points of density in A_ε and $\mathcal{A}_\varepsilon$ becomes important for a second reason (that is, for a reason other than showing non-weak local $\mathcal{H}^1$-finality) in showing the non-rectifiability of A_ε and $\mathcal{A}_\varepsilon$.

6.4 Approximate j-dimensionality of A_ε and $\mathcal{A}_\varepsilon$

Having discussed the measure theoretic properties of A_ε and $\mathcal{A}_\varepsilon$ that are required for them to be appropriate counter examples to (iv) (2), we now go on to show that A_ε and $\mathcal{A}_\varepsilon$ actually do satisfy the requirements of the Definition of (iv).

Lemma 6.4.1.
A_ε and $\mathcal{A}_\varepsilon$ satisfy property (iv).

Proof:
Since $\mathcal{A}_\varepsilon \subset A_\varepsilon$, proving that A_ε satisfies (iv) is sufficient to prove the Lemma. We therefore proceed to prove that A_ε satisfies (iv).

We first consider an arbitrary triangular cap, $T_{n,i}$ from somewhere in our construction. From the construction it is clear that it must be isosceles. From Lemma 6.2.1 and Construction 4.2.1 (particularly the constructed vertical heights, and Lemma 6.2.1 (1)) we see that it must have the two sorts of angles, $\psi(n, \varepsilon)$ and $\pi - 2\psi(n, \varepsilon)$, where, as in Definition 6.2.2

$$\psi(n, \varepsilon) = \tan^{-1}\left(\frac{2^{2-n}\varepsilon}{\frac{(1+n16\varepsilon^2)^{1/2}}{2^{n+1}}}\right)$$

so that we have

$$\lim_{n\to\infty} \psi(n, \varepsilon) = \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{2-n}\varepsilon}{\frac{(1+n16\varepsilon^2)^{1/2}}{2^{n+1}}}\right) = \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{2-n+n+1}\varepsilon}{(1+n16\varepsilon^2)^{1/2}}\right) = 0. \tag{6.1}$$

We now choose arbitrarily some $\delta > 0$ and $x \in A_\varepsilon$. We show that there is a r_x such that for each $r \in (0, r_x]$ $A_\varepsilon \cap B_r(x) \subset L^{\delta r}_{x,r}$.

Since the endpoints of $A_{n,i}$ for each n, i are not in A_ε, x is not an endpoint so that we know from (6.1) that we can choose an $r_x > 0$ such that $B_{r_x}(x) \cap A \subset T_{n_0,j_0}$ for some choice of $n_0 \in \mathbb{N}$ and $j_0 \in \{1, ..., 2^{n_0}\}$ and such that for all $n > n_0$

$$\theta_\delta := tan^{-1}(\delta) > 3\psi(n-1, \varepsilon) + 2\psi(n-2, \varepsilon) > \psi(n, \varepsilon).$$

Since $x \in A_\varepsilon \cap T_{n_0,j_0}$, for each $n > n_0$, $x \in T_{n,j(n)}$ for some $j(n) \in \{1, ..., 2^n\}$. For each $r \in (0, r_x]$ we can therefore choose an $n_1 > n_0$ and $j_1 = j(n_1)$ such that $\mathcal{H}^1(A_{n_1,j_1}) \in [r/2, r)$.

We now consider x as simply being some element of T_{n_1,j_1} and set $L_{r,x}$ to be the affine space parallel to A_{n_1,j_1} containing x.

We now check that $2^{2-n_1}\varepsilon > \frac{\delta r}{2}$. First, we note that

$$\delta > tan(\psi(n_1, \varepsilon)) = \frac{2^{2-n_1+n_1+1}\varepsilon}{(1+n16\varepsilon^2)^{1/2}} = \frac{8\varepsilon}{(1+n_1 16\varepsilon^2)^{1/2}}$$

which we get from the selection of n_1. Also, from the selection of n_1 with respect to r that we have

$$r > \mathcal{H}^1(A_{n_1}, j_1) = \frac{(1+(n_1)16\varepsilon^2)^{1/2}}{2^{n_1}}$$

so that

$$\delta r > \frac{8\varepsilon(1+(n_1)16\varepsilon^2)^{1/2}}{(1+n_1 16\varepsilon^2)^{1/2} 2^{n_1}} \geq 2^{3-n_1}$$

giving the desired inequality. This gives us that the vertical height of T_{n_1,j_1} is less than half the diameter of the neighbourhood that we need around $L_{r,x}$ (that is $L^{\delta r}_{r,x}$). Thus

$$B_r(x) \cap T_{n_1,j_1} \subset L^{\delta R}_{r,x}.$$

It only remains to show that the remainder of $A_\varepsilon \cap B_r(x)$ is inside of an appropriate cone around $L^{\delta r}_{r,x}$. Since from the choice of n_1 with respect to r we have that

$$B_r(x) \subset \pi_x \left(O_{A_{n,i}} \left(\cup_{j:|i-j|\leq 1} A_{n,j} \right) \right) \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})].$$

Thus from Lemma 6.2.1 (2) it follows that the remainder of A is contained in

$$\bigcup_{i:0<|i-j|<3} T_{n_1,i}$$

so that it suffices to prove that these four caps are in the appropriate cone around $L^{\delta r}_{r,x}$. We note that the union of these four caps is the subset of three $T_{n_1-1,k}$ caps,

$$T_{n_1,j_1-2} \cap T_{n_1,j_1-1} \cap T_{n_1,j_1+1} \cap T_{n_1,j_1+2} \subset T_{n_1-1,j_1-1} \cap T_{n_1-1,j_1} \cap T_{n_1-1,n_1+1}.$$

By construction, the maximal angle divergence from $L^{\delta r}_{r,x}$ that an edge on a neighbouring triangular cap of order n_1 can have is $2\psi(n-1, \varepsilon)$ and similarly for a triangular cap of order $n_1 - 1$, the maximal angular divergence is $2\psi(n-2, \varepsilon)$. Adding these together (which is actually worse than could possibly occur) we find that the maximal angle *requirement* for a cone around $L^{\delta r}_{r,x}$ is

$$\begin{aligned} 2\psi(n-1, \varepsilon) + 2\psi(n-2, \varepsilon) &< 3\psi(n-1, \varepsilon) + 2\psi(n-2, \varepsilon) \\ &< \theta_\delta. \end{aligned}$$

It follows that we now have

$$B_r(x) \cap A \subset L_{r,x}^{\delta r}.$$

Since x and δ were arbitrary, this shows that A_ε has the fine weak 1-dimensional ε-approximation property with local r_x uniformity, (that is, it satisfies (iv)) and thus completes the proof. $\diamondsuit$

Corollary 6.3.1 and Lemma 6.4.1 allow us to provide the answer to question (iv) (2). We present this result formally in the following Theorem.

Theorem 6.4.1.
The answer to (iv) (2) is no.

Proof:
From Lemma 6.4.1 A_ε is a set that satisfies (iv) (2). Since, from Corollary 6.3.1 we know that A_ε is not weakly locally $\mathcal{H}^1$-finite it follows that A_ε is a counter example to the answer to (iv) (2) being yes. $\diamondsuit$

6.5 Approximate j-dimensionality of Γ_ε

As previously discussed, the remainder of the answers to our definitions are completely dependent on showing that Γ_ε satisfies (v). We show that this is true, or at least sufficiently true in the following Lemma. Sufficiently true here means that for any $\delta > 0$ we can find an appropriate ε such that Γ_ε constructed with this ε satisfies (v) for the chosen δ. This is sufficient since definition (v) is dependent on some arbitrary but fixed δ unlike (iv) which requires δ to be able to be chosen arbitrarily for any set satisfying (iv). We show first that Γ_ε satisfies (v) and then how the remaining classification for questions (1) and (2) follows.

Lemma 6.5.1.
For all $\delta > 0$ there exists an $\varepsilon_\delta = \varepsilon_\delta(\delta) > 0$ such that $\Gamma_{\varepsilon_\delta}$ satisfies property (v) with respect to δ.

Proof:
Let $0 < \varepsilon < 1/100$. We show, in fact, that there exists a function

$$\delta(\varepsilon) : \mathbb{R} \to \mathbb{R}$$

such that

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$$

such that Γ_ε satisfies (v) with respect to $\delta(\varepsilon)$. It then follows that for all $\delta > 0$ there is an $\varepsilon_\delta > 0$ such that $\delta(\varepsilon_\delta) < \delta$; Γ_ε then satisfies (v) with respect to $\delta(\varepsilon_\delta)$ and therefore with respect to δ.

Let $w \in \Gamma_\varepsilon$ and $\rho \in (0, \rho_0](= (0,1])$. Then, as in Lemma 6.4.1, we know that there exists an $n \in \mathbb{N}$ such that $w \in T_{n,i}$ for some i with $\mathcal{H}^1(T_{n,i}) \in [\rho, 2\rho)$.

Now from Lemma 6.2.1 (1) $\{\psi_{T_{n,j+1}}^{T_{n,j}}\}_{j=i-1}^{j=i} < 2\psi(0,\varepsilon) < \frac{\pi}{16}$ so that

$$\begin{aligned} O_{A_{n,i}}^{-1}(R_{n,i}) &= O_{A_{n,i}}^{-1}(\pi_x(O_{A_{n,i}}(\cup_{j:|i-j|\le 1} A_{n,j})) \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]) \\ &\supset O_{A_{n,i}}^{-1}([-(0.5\mathcal{H}^1(T_{n,\cdot}) + 0.9\mathcal{H}^1(T_{n,\cdot})), 0.5\mathcal{H}^1(T_{n,\cdot}) + 0.9\mathcal{H}^1(T_{n,\cdot})] \\ &\quad \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]) \\ &\supset O_{A_{n,i}}^{-1}([-\rho, \rho] \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]). \end{aligned}$$

This implies that

$$B_\rho(w) \subset O^{-1}_{A_{n,i}}(R_{n,i}). \tag{6.2}$$

From Lemma 6.2.1 (2) it follows that

$$\Gamma_\varepsilon \cap B_\rho(w) \subset \bigcup_{j:|i-j|<2} T_{n,j} \cup O^{-1}_{A_{n,i}}(R_{n,i})^c.$$

Since, from (6.2)

$$B_\rho(w) \cap O^{-1}_{A_{n,i}}(R_{n,i})^c = \emptyset,$$

$$\Gamma_\varepsilon \cap B_\rho(w) \subset \bigcup_{j:|i-j|<2} T_{n,j} \tag{6.3}$$

and more importantly, that

$$\Gamma_\varepsilon \cap \left(B_{\rho(w)} \sim \bigcup_{j:|i-j|<2} T_{n,j} \right) = \emptyset.$$

Since

$$\sup\{\pi_y(x) : x \in O_{A_{n,i}}(T_{n,i})\} \leq \varepsilon \mathcal{H}^1(A_{n,i}) \leq \varepsilon 2\rho$$

and since from Lemma 6.2.1 (2)

$$O_{A_{n,i}} \left(\bigcup_{j:|i-j|=2} T_{n,j} \right) \subset C_{4\psi(0,\varepsilon)}((0,0))$$

and hence

$$\sup \left\{ |\pi_y(z)| : z \in O_{A_{n,i}} \left(\bigcup_{j:|i-j|=1} T_{n,j} \cap B_\rho(w) \right) \right\} \quad \leq \quad \sin(4\psi(0,\varepsilon))\rho$$

it follows that

$$\sup\{|\pi_y(z)| : z \in O_{A_{n,i}}(\Gamma_\varepsilon \cap B_\rho(w))\} < \sup\{2\varepsilon, sin(4\psi(0,\varepsilon))\}\rho = sin(4\psi(0,\varepsilon))\rho$$

and thus by choosing $L_{w,\rho}||A_{n,i}$ we have

$$\sup\{|\pi^{\perp}_{L_{w,\rho}}(z)| : z \in \Gamma_\varepsilon \cap B_\rho(w)\} < sin(4\psi(0,\varepsilon))\rho,$$

that is $\Gamma_\varepsilon \cap B_\rho(w) \subset L^{sin(4\psi(0,\varepsilon))\rho}_{w,\rho}$. Thus Γ_ε satisfies (v) for $\delta > sin(4\psi(0,\varepsilon))$. Which, since $\lim_{\varepsilon\to 0} sin(4\psi(0,\varepsilon)) = 0$, by setting $\delta(\varepsilon) = sin(4\psi(0,\varepsilon))$, proves the lemma. ♢

The dimension of Γ_ε follows from the work of Hutchinson [15]. The proof is quite involved and so we do not present it here. We will however apply Hutchinsons proof regularly as a fundamental theorem of dimension to which we can reduce all of our investigations into the dimension of the generalised Koch Sets considered in Chapters 7 and 8. It is therefore important to state the Theorem and to show that Γ_ε satisfies the conditions required for the Theorem to be applied.

We first mention a result of Mandelbrot [22] required to make sense of the result in [15] that we use.

Proposition 6.5.1.
Let $\{r_i\}_{i=1}^N$ be a sequence of positive real numbers, then there exists a unique $D \in \mathbb{R}$ such that

$$\sum_{i=1}^{N} r_i^D = 1.$$

With this D we can now consider the appropriate result about dimension from [15].

Theorem 6.5.1.
If

$$K = \bigcup_{i=1}^{N} S_i(K)$$

where S_i are contraction mappings and if there exists an open set O such that

1. $0 \neq \emptyset$
2. $\bigcup_{i=1}^{N} S_i(O) \subset O$
3. $S_i(O) \cap S_j(O) = \emptyset$ *whenever* $i \neq j$.

Then if $LipS_i =: r_i$ for each $1 \leq i \leq N$ and D is the unique real number for which

$$\sum_{i=1}^{N} r_i^D = 1$$

$$dimK = D.$$

We can apply this Theorem directly to our case with Γ_ε by appealing to Proposition 4.1.1 as follows.

Lemma 6.5.2.
For each $\varepsilon > 0$, $dim\Gamma_\varepsilon > 1$.

Proof:
By Proposition 4.1.1 there exist, for each $\varepsilon > 0$ contraction maps S_1, S_2 with $\mathrm{Lip}S_i = l(\varepsilon) > 1/2$ for each $i = 1, 2$ and an open set O such that the requirements of Theorem 6.5.1 are satisfied for $K = \Gamma_\varepsilon$.

It follows that

$$\sum_{i=1}^{2} (\mathrm{Lip}S_i)^{\mathrm{dim}\Gamma_\varepsilon} = 1.$$

That is $2l^{\mathrm{dim}\Gamma_\varepsilon} = 1$, or $\mathrm{dim}\Gamma_\varepsilon = -\frac{\ln 2}{\ln l} > 1$. ♢

We now have the tools to, and do in the following Theorem and Corollary, give the answers to (1) and (2) for our remaining definitions.

Theorem 6.5.2.
The answer to (v) (1) is no.

Proof:
From Lemma 6.5.1 we know that Γ_ε satisfies (v). Lemma 6.5.2 shows that $\mathrm{dim}\Gamma_\varepsilon > 1$ and therefore that Γ_ε is a counter example to the answer to (v) (1) being yes. ♢

Corollary 6.5.1.
The answer to the following definitions is no.

$$(v)(2), \quad (ii)(1), \ \textit{and}\ (ii)(2).$$

Proof:
Since from Lemma 6.5.2 we know that the dimension of Γ_ε is greater than 1, it follows that Γ_ε cannot be weakly locally $\mathcal{H}^1$-finite. Since Lemma 6.5.1 then shows that Γ_ε satisfies (v), it follows that the answer to (v) (2) must be no.

Since Property (v) is strictly stronger than Property (ii), any set that satisfies (v) must also satisfy (ii). It then follows that Γ_ε satisfies (ii) and thus in the same way that the answer to (v) (1) and (2) is no it follows that the answers to (ii) (1) and (2) is no. ♢

6.6 The Non-rectifiability of Γ_ε, A_ε and $\mathcal{A}_\varepsilon$

In this section we complete the classification results by answering question (3) for definitions (i) to (v). We see below that only two counter examples are necessary to achieve this aim. We first make the formal statement of the non-rectifiability of A_ε and $\mathcal{A}_\varepsilon$.

Lemma 6.6.1.
The sets A_ε and $\mathcal{A}_\varepsilon$ are not countably 1-rectifiable in the sense of Definition 13.2.1.

Proof:
As in previous comments the proof will be deferred until after the necessary preparation has been made. The actual proofs can be found in Theorems 9.1.5 and 9.1.6. ♢

We now complete the classification results by answering question (i)-(v) (3).

Corollary 6.6.1.
The answer to the questions (i) (3), (ii) (3), (iii) (3), (iv) (3) and (v) (3) is no.

Proof:
From Definition A, made easire through the observation of Table 3.2 we see that definition (v) is strictly stronger than definitions (i) and (ii) and that definition (iv) is strictly stronger than definition (iii). It follows that should there exist a non-rectifiable set satisfying definition (v) then this same set serves as an example of a non-rectifiable set satisfying definitions (iii) and (iv). Similarly, should there exist a non-rectifiable set satisfying definition (iv) then there is also one satisfying definition (iii). Thus it is sufficient to show that there exist non-rectifiable sets satisfying definitions (iv) and (v).

From Lemma 6.4.1 we know that A_ε satisfies definition (iv) and from Lemma 6.6.1 we know that A_ε is not countably 1-rectifiable. Thus the answers to questions (iii) (3) and (iv) (3) are no.

From Lemma 6.5.1 we know that Γ_ε satisfies definition (v). From Lemma 6.5.2 we know that $dim\Gamma_\varepsilon > 1$ and thus from Proposition 3.1.1 that Γ_ε is not countably 1-rectifiable. It follows that the answers to the questions (i) (3), (ii) (3) and (v) (3) are no. ♢

This completes the classification results that were the initial motivating aim for this work. We present here a summary of the classification results:

Theorem 6.6.1. *The classification of the definitions in Definition 1 with respect to the questions presented in Questions 1 is as follows:*

Property	*Question*		
	(1)	*(2)* *(weak, strong)*	*(3)*
(i)	*No*	*No, No*	*No*
(ii)	*No*	*No, No*	*No*
(iii)	*Yes*	*No, No*	*No*
(iv)	*Yes*	*No, No*	*No*
(v)	*No*	*No, No*	*No*
(vi)	*Yes*	*No, No*	*Yes*
(vii)	*Yes*	*Yes, No*	*Yes*
(viii)	*Yes*	*Yes, Yes*	*Yes*

Proof:
These results are a summary of those stated in Corollaries 3.3.1, 3.3.2, 8.4.1, 6.6.1, Theorems 6.4.1, 6.5.2 and Proposition 6.1.2 ♢

Having now completed the classification we present the complete classification results (with counter examples) in a summarising table.

Complete Classification Table

Property	Question			Counter Example		
	(1)	(2) (weak, strong)	(3)	(1)	(2)	(3)
(i)	No	No, No	No	Γ_ε	Γ_ε or $\mathcal{N}$	Γ_ε
(ii)	No	No, No	No	Γ_ε	Γ_ε	Γ_ε
(iii)	Yes	No, No	No		Λ^2, A_ε or $\mathcal{A}_\varepsilon$	A_ε or $\mathcal{A}_\varepsilon$
(iv)	Yes	No, No	No		A_ε or $\mathcal{A}_\varepsilon$	A_ε or $\mathcal{A}_\varepsilon$
(v)	No	No, No	No	Γ_ε	Γ_ε	Γ_ε
(vi)	Yes	No, No	Yes		Λ_{δ_0}	
(vii)	Yes	Yes, No	Yes		$\mathcal{N}$	
(viii)	Yes	Yes, Yes	Yes			

(6.4)

We next continue with results related to the fitting of the counter examples to the eight properties. In particular we show that A_ε does indeed spiral in a sense that will be defined and we show that the counter examples can be extended to higher dimensions.

We have already seen that a rich tapestry of results follows from these more complicated examples. In the interest of finding as much interesting mathematics as possible that could arise from these sets we then in Chapters 7 and 8 allow for generalisation of these sets and show various measure theoretic properties of the resulting sets.

6.7 Notes

The cone, Definition 6.0.1, is a standard concept in both geometry and set theory. The definition given is however, our own. The general result concerning dimension for Fractals, Theorem 6.5.1 is due to the work of Hutchinson [15] whose work, in this case, depended on the work of Mandelbrot [22]. The relevant part of Mandelbrots work presented here as Proposition 6.5.1. The remainder of the results in this chapter are our own.