# Chapter 5

# The Limited Potency of Simple Examples and Weak Requirements for Locally Finite Measure

## 5.1 Limitations on Approximately $j$-dimensional Sets Entering and Exiting on the Same Side

As we have already mentioned, several of the questions we are asking must be answered in the negative. To show this, clearly we need counter examples. Some of the counter examples, such as $\mathcal{N}$, $\Lambda_\delta$ and $\Lambda^2$ are relatively simple in that they are countable collections of graphs of nicely behaved functions whose relevant properties are clear. $\Gamma_\varepsilon$ is not as transparent as the sets already mentioned. It is, however, relatively clear that a set of dimension greater than $j$ satisfying a reasonable approximation of a $j$-dimensional plane (as $\Gamma_\varepsilon$ does) must be complex.

$A_\varepsilon$ and $\mathcal{A}_\varepsilon$ are the counter examples to (iv) (2) that we will use ($A_\varepsilon$ and $\mathcal{A}_\varepsilon$ are shown to satisfy defition (iv) in the next chapter). The obvious question is to ask if it is possible to find a clever way of assembling graphs of nicely behaved functions to provide a simpler counter example to (iv) (2). Definition (iv) is important because it is known to be related to singularity sets (see Lemma 3.2.1). For this reason the question regarding the simplicity of possible counter examples to definition (iv) becomes important. It is the purpose of this chapter to answer this question.

We answer the question regarding the possibility of simple counter examples to (iv) (2) with "no". Since it follows that there is no application of simple counter examples to (iv) (2) that the potency of simple counter examples is limited. The answer "no" to the above outlined question is one that we find encouraging. It is encouraging because it means that the problem of showing that sets satisfying (iv) have locally finite $\mathcal{H}^j$ measure is simplified to showing that certain (yet to be outlined) vaguely 'nice' properties are satisfied. The fact that the only known counter examples to (iv) (2), $A_\varepsilon$ and $\mathcal{A}_\varepsilon$, are not countably $j$-rectifiable (proven Chapter 9) further supports the assertion that such sets must be poorly behaved.

Mathematically speaking, this chapter is centered around answering the above discussed question in showing the following:

Any counter example, $A$, to (iv) (2) must have points, $y$, around which $A \cap B_r(y)$ has infinite $\mathcal{H}^j$ measure for any $r$. These points, $y$, are the critical points of the counter example $A$. In the case $j = 1$, we show that should there exist some set of points, $B$, with $A \supset B \ni y$ (for some critical point $y$) such that $B$ can be essentially described as a Lipschitz graph, then $A$ cannot possibly satisfy definition (iv). That is, if $A$ does satisfy definition (iv), there can be no such set $B$ in $A$. In other words, any set that is a counter example to (iv) (2) must have critical points, $y$, around which $A$ has infinite density but is a broken 'non-graph' in any neighbourhood.

The key idea in the proof is that for a graph of a function to have infinite measure in a small neighbourhood it must at some point be sharply folded on itself (that is, it must rapidly oscillate) at all levels of magnification which will prevent the set from satisfying property (iv). Should not all of the measure be contained in the graph, then there must be a graph with additional points nearby, which will again prevent the set from satisfying definition (iv).

In this section we make some necessary definitions and then prove a Lemma proving an important special case which will be used in the proof of the main Theorem proving our claim which is presented in the following section.

**Definition 5.1.1.**
*We denote the projection of a space onto a subset, $S$, whenever the concept of projection makes sense for $S$ by $\pi_S$. An exception to this rule is the projection of $\mathbb{R}^n$ onto the axis of one of the variables, in this case the projection onto the axis of the $j$-th variable will be denoted by $\pi_{x_j}$ (or $\pi_x = \pi_{x_1}$ and $\pi_y = \pi_{x_2}$ in the special case of $n = 2$.)*

**Definition 5.1.2.**
*Let $u : \mathbb{R} \to \mathbb{R}$ be a function and let*

$$\textit{graph}(u) \cap B_\rho(y) \subset L_y^\delta$$

*for some affine space $L_y \ni y$ and some $\delta \in (0, 1/4)$. Then $u$ is said to* **enter and exit the same side of $B_\rho(y)$ with respect to $L_y^\delta$** *if there is a $w \in L_y \cap \partial B_\rho(y)$ such that*

$$\max\{|w - x| : x \in \textit{graph}(u) \cap \partial B_\rho(y)\} < \frac{\pi\rho}{4}.$$

We note that for a ball $B_\rho(y)$ and an affine space $L_y \ni y$

$$L_y^\delta \cap \partial B_\rho(y) = \Psi_1 \cup \Psi_2$$

for some arcs $\Psi_1$ and $\Psi_2$ in $\mathbb{R}^2$. We can therefore make the following definition.

**Definition 5.1.3.**
*Write*

$$L_y^\delta \cap \partial B_\rho(y) = \Psi_1 \cup \Psi_2$$

*for some arcs $\Psi_1$ and $\Psi_2$ in $\mathbb{R}^2$. Now suppose a function $u$ enters and exits $B_{\rho(y)}$ on the same side with respect to $L_y^\delta$, then $\textit{graph}(u) \cap \Psi_i \neq \emptyset$ for exactly one $i = i(u) \in \{1, 2\}$. We denote this $\Psi_{i(u)}$ by $\Psi_u$ and the other by $\Psi^u$.*

**Lemma 5.1.1.**
*Suppose $u : \mathbb{R} \to \mathbb{R}$ is continuous and graph($u$) $\subset A \subset \mathbb{R}^2$. Suppose that $A$ has property (iv) and that for some $y \in A$ and $\delta \in (0, 1/16)$ $\rho_y$ is an appropriate radius at $y$ with respect to $\delta$. If $u$ enters and exits $B_{\rho_y}(y)$ on the same side, then*

$$\max\{d(\Psi_u, w) : w \in \textit{graph}(u) \cap B_{\rho_y}(y)\} < 4\delta\rho_y.$$

**Proof:**
We first show that $\mathrm{graph}(u) \cap B_{\rho_y}(y) \subset \mathrm{graph}(u(I_{u,y,\rho_y}))$ where

$$I_{u,y,\rho_y} := \left[ \inf_x \{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\}, \sup_x \{\pi_x(\mathrm{graph} u \cap \partial B_{\rho_y}(y))\} \right].$$

Suppose that this is not the case, then there is a $z \in B_{\rho_y}(y) \subset \mathbb{R}^2$ with $z \in \mathrm{graph}(u)$ (and thus $u(\pi_x(z)) = z$) and such that either

$$\pi_x(z) > \sup\{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\} \text{ or } \pi_x(z) < \inf\{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\}.$$

Without loss of generality we consider the case $\pi_x(z) > \sup\{\pi_x(\mathrm{graph} u \cap \partial B_{\rho_y}(y))\}$ the other case follows similarly. Since $u$ is a continuous function graph$(u)$ is connected and by the choice of $z$

$$\sup\{\pi_x(B_{\rho_y}(y))\} > \max\{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y)\}$$

Thus the path

$$P := u([\max\{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\}, \sup\{\pi_x(B_{\rho_y}(y))\} + 1])$$

intersects $B_{\rho_y}(y)$ only at its starting point on the boundary of $B_{\rho_y}(y)$. That is

$$P \cap B_{\rho_y}(y) = u(\max\{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y)\})$$

(otherwise $u(x) \cap \partial B_{\rho_y}(y) \neq \emptyset$ for some $x > \max\{\pi_x(\mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\}$ (in order for the connected path, $P$, to leave the ball) contradicting the choice of $\max\{\pi_x(\ \mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\}$). Thus

$$\pi_x(z) \in \left[\max\{\pi_x(\ \mathrm{graph}(u) \cap \partial B_{\rho_y}(y))\}, \sup\{\pi_x(B_{\rho_y}(y))\} + 1\right]$$

which implies $u(\pi_x(z)) \notin B_{\rho_y}(y)$. This contradiction means that $z \notin$ graph$(u)$. Therefore $\mathrm{graph}(u) \cap B_{\rho_y}(y) \subset \mathrm{graph}(u(I_{u,y,\rho_y}))$.

For any $z \in$ graph$u \cap B_{\rho_y}(y)$, let $z_\partial := \pi_x^{-1}(\pi_x(z)) \cap \Psi_u$ which will be a unique point.

Now assume

$$\sup\{d(\Psi_u, z) : z \in \mathrm{graph}(u) \cap B_{\rho_y}(y)\} \geq 4\delta\rho_y.$$

Then there is a $z \in \mathrm{graph}(u) \cap B_{\rho_y}(y)$ such that

$$|\pi_y(z) - \pi_y(z_\partial)| \geq d(z, \Psi_u) \geq 4\delta\rho_y.$$

Since for all $a \in \Psi_u$, $|a - z_\partial| < 2\delta\rho_y$ and thus $|\pi_y(a) - \pi_y(z_\partial)| < 2\delta\rho_y$, the above statement implies

$$\inf\{|\pi_y(z) - \pi_y(a)| : a \in \Psi_u\} > 2\delta\rho_y.$$

Without loss of generality assume that $\pi_y(z) > \sup\{\pi_y(a) : a \in \Psi_u\}$.

Then, as $u$ is continuous, there exist two connected paths $P_1$, $P_2$ such that

$$\pi_x(P_1) \leq \pi_x(z), \quad \pi_x(P_2) \geq \pi_x(z) \text{ and } P_1 \text{ and } P_2 \text{ are connected to } \Psi_u.$$

Thus

$$P_1 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y) \neq \emptyset \text{ and } P_2 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y) \neq \emptyset.$$

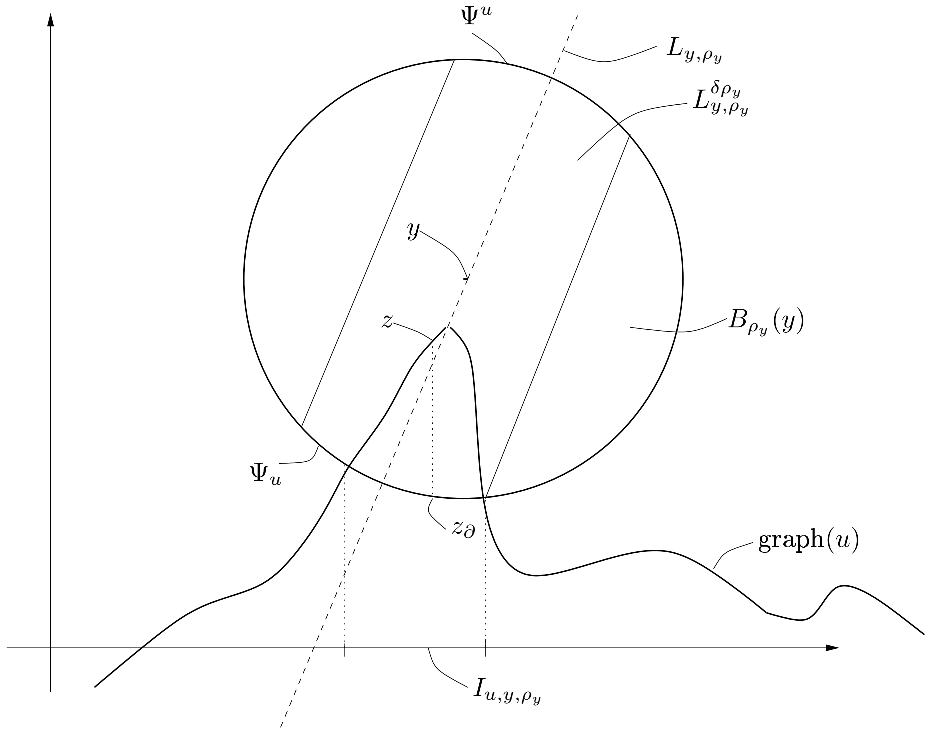


Figure 5.1: Graph entering and exiting the same side

Let

$$z_1 \in P_1 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y) \text{ and } z_2 \in P_2 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y).$$

Without loss of generality assume $|\pi_x(z_1) - \pi_x(z)| \leq |\pi_x(z_2) - \pi_x(z)|$. This choice implies that

$$|\pi_x(z_1) - \pi_x(z)| \leq 1/2 \sup\{|\pi_x(a_1) - \pi_x(a_2)| : a_1, a_2 \in \Psi_u\} \leq \delta\rho_y.$$

Then notice

$$\rho_z := |z_2 - z_1| \leq \sup\{|\pi_x(a_1) - \pi_x(a_2)| : a_1, a_2 \in \Psi_u\} = 2\delta\rho_y \leq 1/2\rho_y$$

so we consider $B_{5\rho_z/4}(z_1)$.

Notice also that $|\pi_x(z) - \pi_x(z_1)| < |\pi_x(z_2) - \pi_x(z)|$ implies

$$|\pi_x(z) - \pi_x(z_1)| \leq \frac{1}{2}\rho_z.$$

Now call the subpath of $P_1 \subset \text{graph}(u)$ connecting $z_1$ to $z$ $P_{z_1}$. Note

$$\pi_x(P_{z_1}) \subset [\pi_x(z_1), \pi_x(z)] \text{ and } z \notin B_{\rho_z}(z_1),$$

which implies

$$P_{z_1} \cap \partial B_{\rho_z}(z_1) \neq \emptyset$$

and for all $w \in \partial B_{\rho_z}(z_1)$

$$|\pi_x(w) - \pi_x(z_1)| < \frac{1}{2}\rho_z \text{ and } d(w, z_1) = \rho_z$$

which implies $|\pi_y(w) - \pi_y(z_1)| > \frac{\sqrt{3}}{4}\rho_z$. However, for any choice of $L^{\delta\rho_z}_{z_1,\rho_z}$ we must have

$$\sup\{|\pi_y(l) - \pi_y(z_1)| : l \in L^{\delta\rho_z}_{z_1,\rho_z}\} < \frac{9}{4}\delta\rho_z.$$

Since $\delta < \frac{1}{16}$ we note $\frac{\sqrt{3}}{4}\rho_z > \frac{\rho_z}{4} > \frac{9\rho_z}{64} > \frac{9}{4}\delta\rho_z$. Thus it is impossible to choose a $L_{z,\rho_z}$ such that $A \cap B_{\rho_z}(z_1) \subset L^{\delta\rho_z}_{z,\rho_z}$. This would imply $A$ does not have property (iv). This contradiction proves the Lemma. ♢

## 5.2 Constraints on Sets both Approximately $j$-dimensional and Not of Locally Finite Measure

We now prove the main theorem of this chapter. We show that any set satisfying definition (iv) that does not have locally finite measure must be poorly behaved. The proof is actually structured as a proof by contradiction. We take a set, $A$, not of locally finite measure (we denote the set of points around which the local finitness of measure of $A$ fails by $\mathcal{Y}$). We go on to show that should there exist any well behaved subset of $A$ (well behaved in the sense of being essentially described by part of a Lipschitz graph) containing a $y \in \mathcal{Y}$ then $A$ cannot satisfy definition (iv). In other words, should $A$ satisfy definition (iv) then around all such $y \in \mathcal{Y}$ $A$ all subsets of $A$ must be purely poorly behaved.

We prove the theorem by reducing the Theorem to an application of Lemma 5.1.1.

**Theorem 5.2.1.**
*Suppose $A \subset \mathbb{R}^2$ and that there exists a $y \in A$ such that*

$$\mathcal{H}^1(A \cap B_\rho(y)) = \infty \textit{ for all } \rho > 0$$

*and for some $\rho_1 > 0$,*

$$y \in G_y^{-1}(\textit{graph}(u)) \cap B_{\rho_1}(y) \textit{ and}$$

$$\overline{B_{\rho_y}(y) \cap A \cap G_y^{-1}(\textit{graph}(u))} = G_y^{-1}(\textit{graph}(u)) \cap \overline{B_{\rho_y}(y)}$$

*where $u$ is Lipschitz, $G_y \in G(1,2)$ and $G_y(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the rotation such that $G_y(G_y) = \mathbb{R}$.*

*Then $A$ does not have property (iv) for $j = 1$.*

**Proof:**
By the invariance of the relevant quantities under orthogonal transformations we can assume that $y = (0,0)$ and $G_y = \mathbb{R}$.

Assume that $A$ does satisfy property (iv).

Then for a given $\delta < 1/8$ there is a $\rho_y = \rho_y(y) \in (0, \rho_1)$ such that there exists an affine space $L_{y,\rho_y}$ such that

$$A \cap B_{\rho_y}(y) \subset L^{\delta\rho_y}_{y,\rho_y}$$

and furthermore, for each $x \in A \cap B_{\rho_y}(y)$ and $\rho \in (0, \rho_y]$ there is an affine space $L_{x,\rho}$ such that

$$A \cap B_\rho(x) \subset L_{x,\rho}^{\delta\rho}.$$

Noting that $y \in \text{graph}(u)$ and that clearly $d(y, \partial B_{\rho_y}(y)) = \rho_y > 4\delta\rho_y$ it follows that

$$\max\{d(\Psi_u, y) : y \in \text{graph}(u) \cap B_{\rho_y}(y)\} < 4\delta\rho_y$$

and thus by Lemma 5.1.1 $u$ cannot enter and exit $B_{\rho_y}$ on the same side with respect to any affine space. In particular for each $w \in L_{y,\rho_y}$

$$\text{graph}(u) \cap \pi^{-1}_{L_{y,\rho_y}}(w) \cap L^{\delta\rho_y}_{y,\rho_y} \neq \emptyset.$$

Also, if $A \cap B_{\rho_y/2}(y) \subset \text{graph}(u)$ then

$$\mathcal{H}^1(A \cap B_{\rho_y/2}(y)) \leq \frac{\rho_y}{2} \cdot \omega_1 \cdot \text{Lip}u < \infty,$$

a contradiction to our assumptions on the measure of balls around $y$. It follows that there exists an $x \in A \cap B_{\rho_y/2}(y)$ such that $x \notin \text{graph}u$. Note that $\pi^{-1}_{L_{y,\rho_y}}(x) \cap \text{graph}u \neq \emptyset$ which implies

$$d(x, \text{graph}(u)) \leq 2\delta\rho_y < \frac{1}{2}\rho_y.$$

Now select $z \in \text{graph}(u)$ such that

$$d(z, x) < \frac{9}{8} \inf\{d(w, x) : w \in \text{graph}(u)\} =: \frac{9}{8} d < \rho_y.$$

By the hypotheses there is an $z_1 \in \text{graph}(u) \cap A \cap B_{(1/16)d}(z)$. We now consider $B_{\rho_x}(z_1) \ni x$. Note that for any choice of $L_{z_1,\rho_x}$

$$L^{\delta\rho_x}_{z_1,\rho_x} \cap \partial B_{\rho_x}(z_1) = \Psi_1 \cup \Psi_2,$$

a union of two arcs as considered in Definition 5.1.3, and that $d(x, \partial B_{\rho_x}(z_1)) < \frac{1}{4}d$. This implies that for some $i = i(x) \in \{1, 2\}$

$$\Psi_i \subset B_{(1/4)d + 2\delta\rho_x}(x) = B_{(1/4)d + 2\delta(9/8)d}(x).$$

Since $\delta$ was chosen such that $\delta < 1/8$

$$\frac{1}{4}d + 2\delta\frac{5}{4}d < \frac{4}{16} + \frac{5}{16}d < \frac{15}{16}d$$

which implies $\text{graph}(u) \cap \Psi_{i(x)} = \emptyset$.

This in turn implies that $u$ enters and exits $B_{\rho_x}(z_1)$ on the same side with respect to any affine space possibly allowing property (iv) to hold. Since $z_1 \in \text{graph}(u)$

$$\max\{d(w, \partial B_{\rho_x}(z_1)) : w \in \text{graph}(u)\}) = \rho_x > 4\delta\rho_x.$$

This implies, by Lemma 5.1.1, that $A$ does not have property (iv). This contradiction completes the proof of the Theorem. $\diamondsuit$

In order to more definitely relate what has previously been discussed to this result, I observe the following immediate corollaries.

**Corollary 5.2.1.**
*Suppose $A \subset \mathbb{R}^2$ and that there exists a $y \in A$ such that*

$$\mathcal{H}^1(A \cap B_\rho(y)) = \infty \text{ for all } \rho > 0$$

*and for some $\rho_1 > 0$,*

$$A \cap B_{\rho_1}(y) = G_y^{-1}\left(\bigcup_{n=1}^{Q} \text{graph}(u_n)\right) \cap B_{\rho_1}(y)$$

*for some $Q \in \mathbb{N} \cup \{\infty\}$ where $u_n$ is Lipschitz for each $n$, $G_y \in G(1,2)$ and $G_y(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the rotation such that $G_y(G_y) = \mathbb{R}$.*

*Then $A$ does not have property (iv).*

**Proof:**
Since

$$y \in A \cap B_{\rho_1}(y) = G_y^{-1}\left(\bigcup_{n=1}^{Q} \text{graph}(u_n)\right) \cap B_{\rho_1}(y)$$

$y \in G_y^{-1}(\text{graph}(u_{n_0}))$ for some $1 \le n_0 \le Q$. With $u = u_{n_0}$ the conditions of Theorem 5.2.1 are then satisfied from which the conclusion follows. ♢

**Corollary 5.2.2.**
*$\mathcal{N}$, $\Lambda_\delta$ and $\Lambda^2$ are not counter examples to (iv) (2).*

**Proof:**
Let $\Xi = \mathcal{N}$ or $\Lambda_\delta$. Then since $\Xi$ is a countable union of Lipschitz graphs, any point of infinite density in $\Xi$ satisfies Theorem 5.2.1.

For $\Lambda^2$ we note that the only point of infinite density is $(0,0)$. Note that restricted to $[-1,1]$ the functions making up $\Lambda^2$, ($u_n = x^2/n$) are Lipschitz. Thus taking $\rho_1 = 1/2$ and $y = (0,0)$ in Theorem 5.2.1 the conditions of Theorem 5.2.1 are satisfied so that $\Lambda^2$ does not satisfy property (iv). ♢

**Remark:**
We note that in Lemma 5.1.1 and Theorem 5.2.1 we only used $\delta < 1/8$. Thus the full power of property (iv) has not been used. It is therefore possible and in fact likely that we could force any potential counter examples to (iv) (2) to be even stranger than what we have forced here. Even without using the $\delta$-fine property I believe that an improvement to Theorem 5.2.1 could be made in the form of the following conjecture.

**Conjecture 5.2.1.**
*Suppose $A \subset \mathbb{R}^2$ and that there exists a $y \in A$ such that*

$$\mathcal{H}^1(y \cap B_\rho(A)) = \infty \text{ for all } \rho > 0$$

*and for some $\rho_1 > 0$,*

$$y \in G_y^{-1}(\text{graph}(u)) \cap B_{\rho_1}(y) \text{ and}$$

$$\overline{A \cap G_y^{-1}(\text{graph}(u))} = G_y^{-1}(\text{graph}(u)) \subset A$$

*where* $u \in C^0(\mathbb{R};\mathbb{R})$*,* $G_y \in G(1,2)$ *and* $G_y(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ *is defined as the rotation such that* $G_y(G_y) = \mathbb{R}$.

*Then* $A$ *does not have property (iv).*

The difference between Conjecture 5.2.1 and Theorem 5.2.1 is that in Conjecture 5.2.1 the function $u$ is only continuous and not Lipschitz. For this reason it is possible that the graph of $u$ itself produces a point of infinite density. Unlike in Theorem 5.2.1 we are therefore unable to insist that there exists other points from the set $A$ near graph$(u)$ to prevent definition (iv) from being satisfied.

The attack idea for Conjecture 5.2.1 is that either Theorem 5.2.1 does apply or each point of infinite density in $A$ is completely represented by the graph of a function $u$ in small enough neighbourhoods. For graph$(u)$ to have so much measure it must oscillate dramatically. We should then be able to find two 'almost vertical' lines in this oscillating graph that are close enough to one another to prevent definition (iv) from holding.

More quantitatively, we note that there are several methods of attacking the proof and "almost getting there". One method, using Lemma 5.1.1, reduces the proof to the following conjecture which also shows the dependence on the high osciallation (or total variation) of the graph $u$.

**Conjecture 5.2.2.**
*Suppose* $I_1, I_2$ *are compact subintervals of* $\mathbb{R}$ *and*

$$u : I_1 \to I_2.$$

*Suppose further that for all* $x_1, x_2 \in I_1$ *such that* $u(x_1) = u(x_2)$

$$\sup\{|u(y) - u(x_1)| : y \in [x_1, x_2]\} < |x_1 - x_2|.$$

*Then, for any* $\delta > 0$ *there exists a partition* $P = \{p_1, ..., p_Q\}$ *of* $I_1$ *with*

$$\max\{|p_i - p_{i-1}| : 2 \leq i \leq Q\} < \delta\}$$

*and*

$$\sum_{i=2}^{Q} |u(p_i) - u(p_{i-1})| < C < \infty.$$

To emphasise what we have shown in this chapter we restate the result reworded. We have shown that a set $A \subset \mathbb{R}^2$ satisfying definition (iv) cannot posses any point that is both an element of an approximate Lipschitz graph (in the sense of the following definition) and possessing of a neighbourhood in which $A$ has infinite measure.

**Definition 5.2.1.**
*A subset* $A \subset \mathbb{R}^{n+k}$ *is said to possess a piece of Lipschitz* $n$*-graph at* $x \in A$ *if there exists an* $r > 0$*,* $G \in G(n, n+k)$ *and a Lipschitz function* $u : G \to G^{\perp}$ *such that*

$$x \in graph(u) \text{ and } \mathcal{H}^n((graph(u) \sim A) \cap B_r(x)) = 0.$$

*In this case* $x$ *is said to be an element of a Lipschitz graph.*

## 5.3 Notes

The entirety of the material presented here is all original and our own.