Chapter 4

Construction of the Counter Examples

Having answered all the questions that will be answered with yes, we now turn our attention to providing counter examples for the remaining questions so as to answer no to each of these.

The sets being considered are not all trivial sets to construct or to understand. We therefore leave the proofs that they actually satisfy the definitions that they are respectively intended to be counterexamples to until later. For the more complicated sets, particularly A_ε, there is more than one method to construct the set. Some of these will be discussed further in Chapters 7 and 8. For now, however, we satisfy ourselves with the definitions most easily used to fit the constructed sets to the relevant definitions and thus complete the classification.

In this chapter we construct 3 sets and 3 1-parameter families of sets. Of the latter three the first is our own construction of a known set, the same that appears in Lemma 3, which we provide since the necessary properties for our purposes are more easily proven with our construction method. The latter two are then variations of the same set allowing for important extra properties by adding another point of variation. For the sets with a variable there is a range of values of the parameter (independent of which set) for which each resultant specific example is appropriate for our purposes. We will, however, calculate with the parameter left arbitrary since it provides more generality and makes no difference to the proofs of the results that we want to prove with these sets.

The three simpler sets are of little interest apart from the fact that they are appropriate counter examples to particular definitions. The other three are of independent interest. As well as allowing us to show that some good behaviour is ensured by the approximate j-dimensionality of the sets if not as definite as we had hoped, they provide a range of interesting results on dimension, rectifiability and measure density. General proofs concerning properties of these sets are included in the discussion of generalised Koch Type sets in Chapters 7 and 8. We include in any case the direct proofs of the properties that we are interested that are relevant to the classification work.

We construct firstly the three simpler sets. We then construct Γ_ε which will be a counter example to (v) (1) followed by a property of Γ_ε important to our study. We then construct the second more complicated set A_ε which is a counter example to (iv) (2) and (3). Since A_ε is not closed and is therefore not possibly a singularity set we make the third construction $\mathcal{A}_\varepsilon$, which is a subset of the second, constructed to be closed but retain the necessary properties. We then prove some necessary

proprties of A_ε and $\mathcal{A}_\varepsilon$.

4.1   Simple and Known Sets

The first set has already been defined, and is:

$$\mathcal{N} := \bigcup_{n=1}^{\infty} \mathbb{R} \times \left\{ \frac{1}{n} \right\}.$$

Note that we will henceforth identify $\mathbb{R}^n \times [0]^{N-n}$ with $\mathbb{R}^n$ in $\mathbb{R}^N$ for each choice of $n, N \in \mathbb{N}$ with
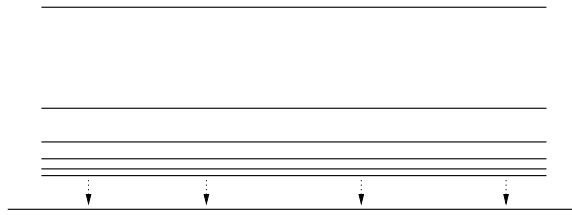

Figure 4.1: $\mathcal{N}$

$n < N$. The other simple sets, are used in a similar way to $\mathcal{N}$ but need differing levels of fineness approximation with bad properties at one point. Being a collection of flat sheets, $\mathcal{N}$ does not have this property, we therefore define the subset of $\mathbb{R}^2$ defined for each $\delta > 0$ as

$$\Lambda_{\delta_0} = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2} \text{graph} \left\{ \frac{(-1)^i \delta_0 x}{n} \right\}$$
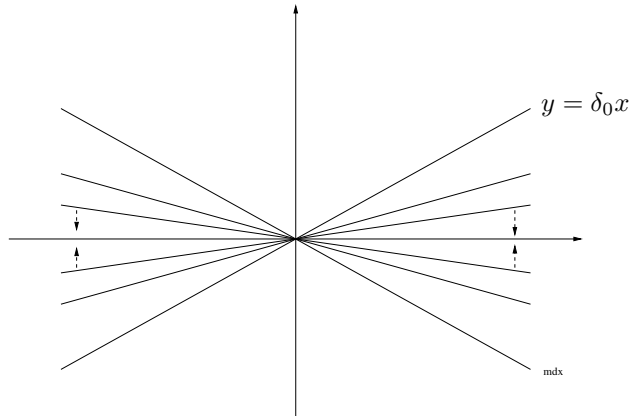

Figure 4.2: Λ_{δ_0}

and the subset of $\mathbb{R}^2$ defined as

$$\Lambda^2 = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2} \text{graph} \left\{ \frac{(-1)^i x^2}{n} \right\}$$
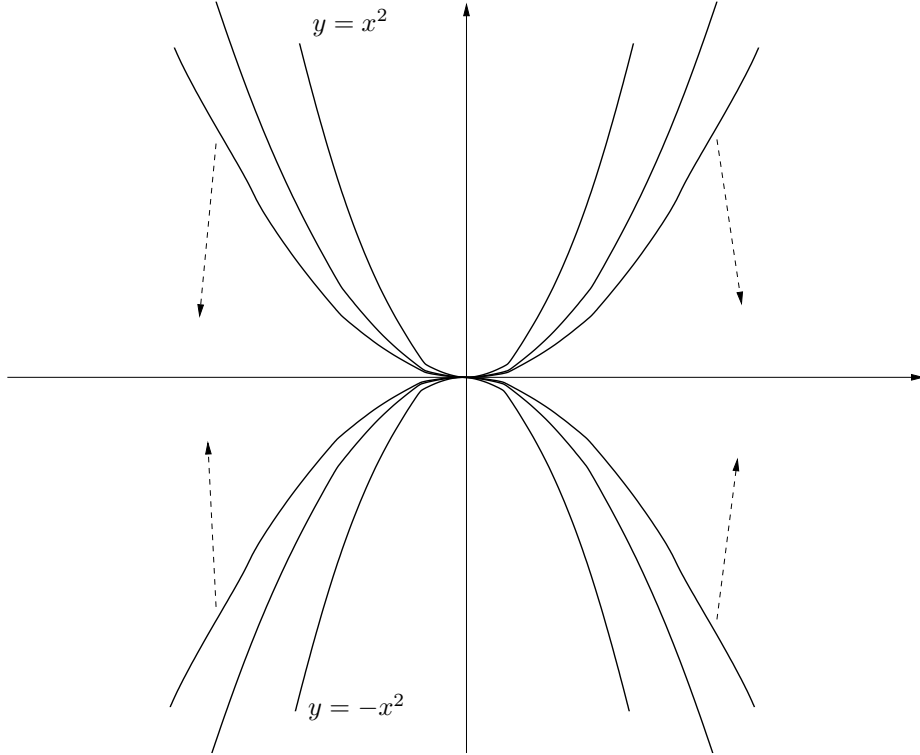
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Figure 4.3: Λ^2

We now construct the more complicated examples. They are both based on the "Koch Curve" which was originally constructed as a fractal set being of dimension between 1 and 2. The first we construct is the set Γ given by Simon in [27], on which the remaining sets are based. The second set, which is actually a function from $\mathbb{R}^+$ into $2^{\mathbb{R}^n}$ (that is, the set is constructed with respect to a variable $\varepsilon \in \mathbb{R}^+$) will be denoted A_ε, and is used as a counter example to (iv) (2) and (3). Although Γ was actually constructed as a fixed set in [27], we will allow the set to be constructed with respect to a variable ε, which will later allow us to find appropriate counter examples with respect to (v) (1) for any given δ. The variable set will then be denoted Γ_ε

These constructions rely heavily on the use of triangles so we first make the following definition.

Definition 4.1.1.
*Let $L = (a, b) = ((a_1, a_2), (b_1, b_2))$ be a line in $\mathbb{R}^2$. An ε-**triangular cap** or, when the context is clear, simply a **cap** will be the isosceles triangle, T, with vertices a, b and $c + (a + b)/2$ (we write c also as (c_1, c_2)), where c is chosen such that*

$$|c| = \varepsilon \ and \ < c, b - a > = 0.$$

The above conditions on c allow for two possible points. Should L be a side edge of a previously constructed triangular cap, T_0, we choose c from the two possible points so that $\mathcal{H}^2(T \cap T_0) > 0$. Otherwise we choose c to satisfy $c_1 \geq (a_1 + b_1)/2$ and $c_2 \geq (a_2 + b_2)/2$.
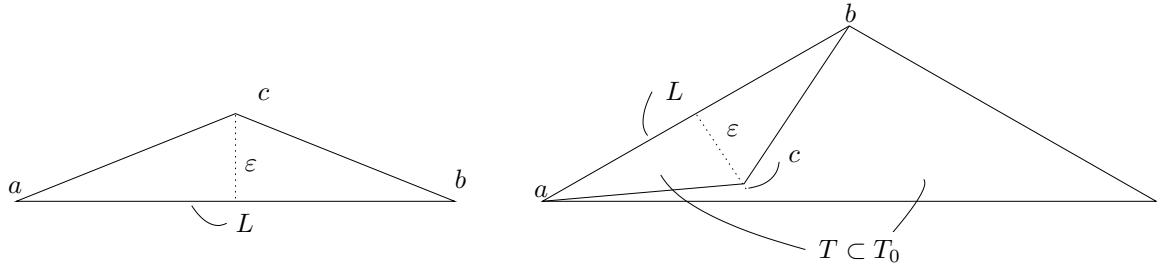
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Figure 4.4: ε-triangular caps

Construction 4.1.1. .
We construct the set Γ_ε as follows.

Let $1/100 > \varepsilon > 0$. We begin with an ε-triangular cap, T_0, constructed over the line $A_{0,1} := ((0,0),(1,0))$. We then name the two new edges $A_{1,j}$, $j = 1,2$. We write $A_0 := T_0$. We note that $l := \mathcal{H}^1(A_{1,j}) < \mathcal{H}^1(A_{0,1})$, $j = 1,2$. We then construct $l\varepsilon$-triangular caps $T_{1,j}$ on $A_{1,j}$. We name the four new edges $A_{2,j}$, $j \in \{1,2,3,4\}$. We write $A_1 := \cup_{j=1}^2 T_{1,j}$. We note that $A_{2,j}$, $j = 1,...,4$ are the 2^2 shortest edges of length l^2. We note also that A_1 can also be constructed by the appropriately rotated and scaled union of two copies of A_0
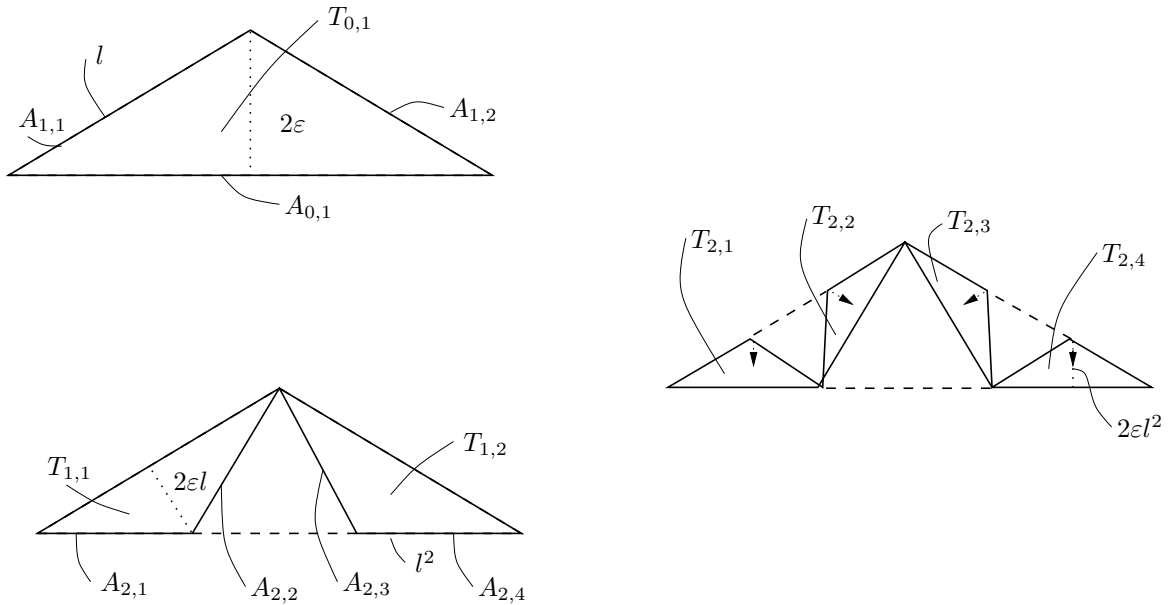

Figure 4.5: Construction of Γ_ε

We now continue inductively. Suppose that we have a set A_n consisting of 2^n triangular caps,

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$T_{n,j}$ with base length l^n and altogether 2^{n+1} "shortest sides", $A_{n+1,j}$ of length l^{n+1}. On each $A_{n+1,j}$ we construct an $l^{n+1}\varepsilon$-triangular cap $T_{n+1,j}$. We set

$$A_{n+1} := \bigcup_{j=1}^{2^{n+1}} T_{n+1,j}.$$

This A_{n+1} will then have all of the same properties as A_n with n replaced by $n+1$. We note also that with the numbering of the caps, we always count from "left" to "right" so that $T_{n+1,2j-1} \cup T_{n+1,2j} \subset T_{n,j}$.

We then define

$$\Gamma_\varepsilon := \bigcap_{n=0}^{\infty} A_n$$

where the dependence on ε comes from the initial choice of ε. $\diamondsuit$

Remark: The selection of $\varepsilon < 1/100$ is important. For further comment on the reasoning see the remarks following Construction 4.2.1.

One property of Γ_ε that should be noted now, as it is particularly intrinsic to the construction is that Γ_ε is essentially the union of two scaled copies of itself. We show this after the following definitions.

Definition 4.1.2.
*We denote the end points of a line of finite length, A, by $E(A)$, and call them the **edge points of** A. Let $T_{n,i}$ be a triangular cap. $T_{n,i}$ will then have 3 vertices which will be called the **edge points of** T. Let A_n be a stage of construction in the construction of Γ_ε (Construction 4.1.1), we then define the edge points of A_n by*

$$E(A_n) := \bigcup_{i=1}^{2^n} E(T_{n,i})$$

*and the **edge points of** Γ_ε are*

$$E(\Gamma_\varepsilon) := \bigcup_{n=1}^{\infty} E(A_n).$$

We see that the edge points are all of the corners that appear in the constructions of Γ_ε.

Definition 4.1.3.
*We define the **edgepointless** Γ_ε as*

$$\Gamma_\varepsilon^E := \Gamma_\varepsilon \sim E(\Gamma_\varepsilon).$$

Proposition 4.1.1.
There are contraction mappings, S_1 and S_2, and an open set, O, such that

$$\Gamma_\varepsilon^E \subset O, \quad S_1(O) \cup S_2(O) \subset O, \quad S_1(\Gamma_\varepsilon^E) \cup S_2(\Gamma_\varepsilon^E) = \Gamma_\varepsilon^E \text{ and } S_1(O) \cap S_2(O) = \emptyset.$$

Further

$$LipS_1 = LipS_2 = l := (1/4 + \varepsilon^2)^{1/2}$$

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Proof:
It is not too difficult to check that the contraction mappings with Lipschitz constants l defined by

$$S_i(x,y) = \begin{pmatrix} cos((-1)^i tan^{-1}(\varepsilon) - \pi) & -sin((-1)^i tan^{-1}(\varepsilon) - \pi) \\ sin((-1)^i tan^{-1}(\varepsilon) - \pi) & cos((-1)^i tan^{-1}(\varepsilon) - \pi) \end{pmatrix} v(x,y)$$

where

$$v(x,y) = \left(l \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1/2 \\ \varepsilon/2 \end{pmatrix} \right) + \begin{pmatrix} 1/2 \\ \varepsilon/2 \end{pmatrix} \right)$$

are such that $S_1(T_0) = T_{1,2}$, $S_2(T_0) = T_{1,1}$ and thus

$$S_1(A_0) \cup S_2(A_0) = S_1(T_0) \cup S_2(T_0) = T_{1,1} \cup T_{1,2} = A_1.$$

Further, by setting O to be the open quadrilateral with vertices $\{(0,0),(1/2,3\varepsilon/2),(1,0),(1/2,-\varepsilon/2\}$ we see

$$\Gamma_\varepsilon^E \subset T_0 = A_0 \subset O,$$

that $S_1(O)$ is the quadrilateral with vertices

$$\{(1/2,\varepsilon),(1,0),l((1,0) - (1/2,3\varepsilon/2)) + (1/2,3\varepsilon/2),l((1,0) - (1/2,-\varepsilon/2)) + (1/2,-\varepsilon/2)\}$$

and $S_2(O)$ is the quadrilateral of vertices $\{(0,0),l(1/2,3\varepsilon/2),(1/2,\varepsilon),l(1/2,-\varepsilon/2)\}$.

It follows that $S_1(O) \cup S_2(O) \subset O$ and that $S_1(O) \cap S_2(O) = \emptyset$.

By the procedure P we will mean a procedure of replacing a triangular cap with two smaller triangular caps that are subsets of the first. Indeed, for a triangular cap T with base length b and side length l we apply proceedure P to T to get $P(T) = T_1 \cup T_2$ where $T_1, T_2 \subset T$, T_1 and T_2 are triangular caps with base length lb, side length l^2 and base coinciding with one of the sides of T. Further the side of T coinciding with the base of T_2 is the side not coinciding with the base of T_1.
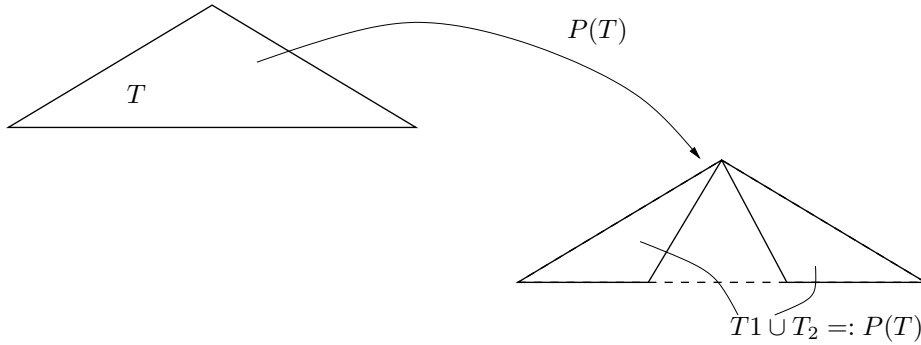

Figure 4.6: The proceedure P

Note that $S(T_{n,i}) \cup S_2(T_{n,i}) = P(T_{n,i})$ for any $T_{n,i}$ in the construction of Γ_ε.

Note that since the proceedure, P, of taking two triangular caps on the shorter sides of a union of isosceles triangle is clearly invariant under orthogonal transformation (since chosing the new cap

to be within the previous triangle is independent of orientation) and homothety, that is $P(R(T)) = R(P(T))$ where T is an isosceles triangles and R is any orthogonal transformation on $\mathbb{R}^2$, and if $l \in \mathbb{R}$, $P(lT) = lP(T)$. Since S_1 and S_2 are indeed just combinations of homothety and orthogonal transformation we have $P(S_i(T)) = S_i(P(T))$ for $i = 1, 2$.

We claim that $A_n = S_1(A_{n-1}) \cup S_2(A_{n-1})$ for each $n \in \mathbb{N}$. We already have a starting point $(n = 1)$. Now, supposing that $A_n = S_1(A_{n-1}) \cup S_2(A_{n-1})$ for some $n \in \mathbb{N}$, we then have

$$A_{n+1} = P(A_n) = P(S(A_{n-1}) \cup S_2(A_{n-1})) = S_1(PA_{n-1}) \cup S_2(PA_{n-1}) = S_1(A_n) \cup S_2(A_n),$$

completing the inductive sets. Then, since $A_1 \subset A_0$, we have

$$
\begin{aligned}
\Gamma_\varepsilon^E &= \bigcap_{n=0}^\infty A_n \sim E \\
&= \bigcap_{n=1}^\infty S_1(A_{n-1} \sim E) \cup S_2(A_{n-1} \sim E) \\
&= \bigcap_{n=0}^\infty S_1(A_n \sim E) \cup S_2(A_n \sim E) \\
&= S_1\left(\bigcap_{n=1}^\infty A_n \sim E\right) \cup S_2\left(\bigcap_{n=1}^\infty A_n \sim E\right) \\
&= S_1(\Gamma_\varepsilon^E \sim E) \cup S_2(\Gamma_\varepsilon^E \sim E).
\end{aligned}
$$

$\diamondsuit$

4.2 Pseudo-Fractal Sets

We now construct the "strangest" sets. These are similar to Γ_ε in construction, however, as we noted in Proposition 4.1.1, the construction for Γ_ε retains the basic shape of the triangular caps. This will not be sufficient for the cases when we want to prove properties for the case where approximations should hold for all $\delta > 0$. We therefore allow the relative height of the triangular caps to shrink, so that the "angles" involved in the triangles approach zero as we look at smaller and smaller sections of the triangles. As we will see later, even this adjustment is not sufficient. We therefore remove all of the interior at each stage, take, in a sense, a limit and remove the approximating sets and the edges. We make the specific constructions below in Constructions 4.2.1 and 4.2.2. As has been mentioned, the third set is then a carefully selected subset of this chosen in such a way as to ensure that it is closed. Incorporated into the first of the constructions is the definition of edge points relevant to the construction. The need for the concept of edge points is bypassed in the second construction as shall be seen.

Construction 4.2.1. .
Let $1/200 > \varepsilon > 0$. We then construct the set, as previously, as a subset of $\mathbb{R}^2$. We start with

$$A_{0,1} := [(0,0),(1,0)].$$

We then denote by $T_{0,1}$ the 2ε-triangular cap on $A_{0,1}$.

We now set

$$A_1 := \overline{(\partial T_{0,1} \sim A_{0,1})},$$

which is the union of two lines (namely the two shorter edge lines of $T_{0,1}$), we name the two lines $A_{1,i}$, $i = 1, 2$. To continue, we denote by $T_{1,i}$ the ε-triangular cap constructed on $A_{1,i}$, the $A_{1,i}$ being considered as an edge of $T_{0,1}$ for each i.

We then set

$$A_2 := \overline{\left(\partial \left(\bigcup_{i=1}^{2} T_{1,i} \right) \sim A_1 \right)},$$

which will be a union of 4 lines $A_{2,i}$, $i = 1, 2, 3, 4$, each $A_{2,i}$ being an edge of a triangular cap $T_{1,i}$.
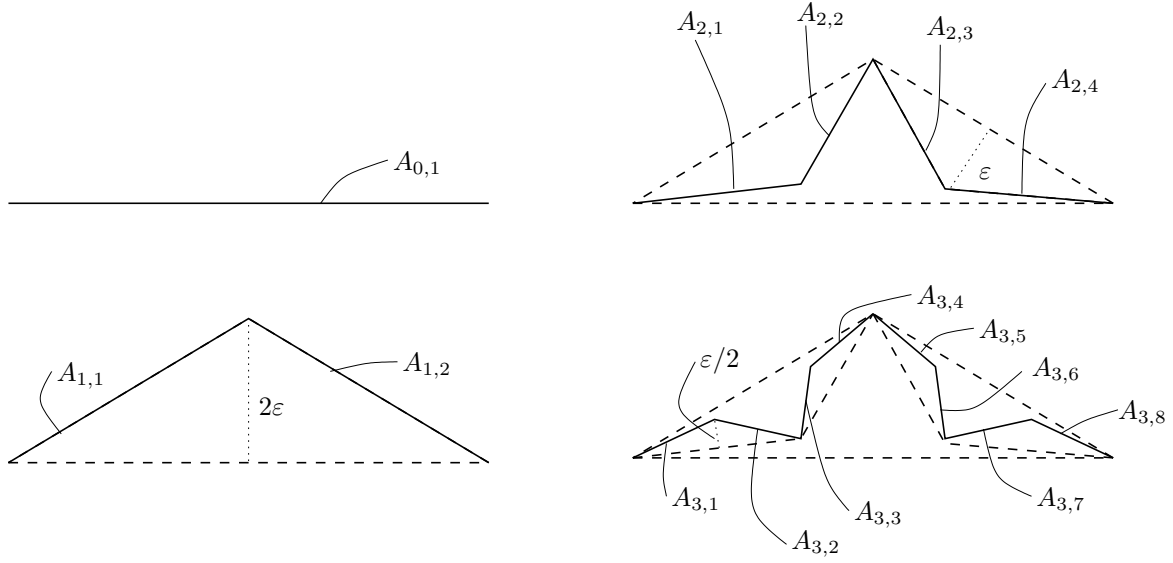

Figure 4.7: Construction of A_ε

We continue the construction inductively. Assuming we have A_n, a union of 2^n lines, $\{A_{n,i}\}_{i=1}^{2^n}$ that lie on the boundary of 2^{n-1} triangular caps $\{T_{n-1,i}\}_{i=1}^{2^{n-1}}$, (and A_n a union of 2^n triangular caps), we construct 2^n $(2^{1-n}\varepsilon)$-triangular caps, $\{T_{n,i}\}_{i=1}^{2^n}$, on each of the 2^n lines. As previously we number from "left" to "right" so that $T_{n+1,2j-1} \cup T_{n+1,2j} \subset T_{n,j}$. We then set

$$A_{n+1} := \overline{\left(\partial \left(\bigcup_{i=1}^{2^n} T_{n,i} \right) \sim A_n \right)}.$$

Finally, we define

$$A_\varepsilon := \bigcap_{i=1}^{\infty} \overline{\left(\bigcup_{n=1}^{\infty} A_n \sim \bigcup_{n=1}^{i} (A_n \sim E) \right)} \sim E = \overline{\bigcup_{n=1}^{\infty} A_n} \sim \bigcup_{n=1}^{\infty} A_n,$$

where

$$E := \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} E(A_{n,i}),$$

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and $E(A_{n,i})$ denotes the endpoints of the line $A_{n,i}$. As previously, the ε refers to the arbitrarily chosen $\varepsilon > 0$ at the beginning of the construction, which may, of course, be chosen, as small as is necessary. $\diamond$

Remarks:

(1) The removal of the endpoints is very important for the example. Including the endpoints leaves points in the set that are vertices. It would then follow that the set A_ε would not be able to be appropriately approximated by a plane simultaneously for all $\delta > 0$. By removing the endpoints we ensure that no element of A_ε is an endpoint. In this way, for any $x \in A_\varepsilon$ and $\delta > 0$ we can choose a radius, $r_x > 0$, such that $B_{r_x}(x)$ contains no vertex too sharp for the appropriate δ-approximation.

(2) We are asking measure theoretic questions. For this reason it is important to note that the edgepoints

$$E = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2^n} E(A_{n,i})$$

are countable and therefore are of zero $\mathcal{H}^1$ measure. E therefore has no effect on any $\mathcal{H}^1$-measure properties that we are looking at.

(3) As stated in the remark following Contruction 4.1.1 the choice of $\varepsilon < 1/100$ is important. The reason that it be so small comes from properties required later on the base angles of the triangular caps. This will be further explained when the requirements on the angles becomes clear. For now, however, we note that it is important that $\varepsilon < \mathcal{H}^1(A_{0,1})$ (which for Γ_ε and A_ε means $\varepsilon < 1/4$). In the constructions of Γ_ε and A_ε it is important that each stage of triangular caps is a subset of the previous (that is $\cup_i T_{n,i} \subset \cup_i T_{n-1,i}$). It is also important that although clearly one triangular cap in some stage of construction will have a common point with a neighbouring triangular cap (that is $T_{n,i}$ and $T_{n,i\pm 1}$ will have a common point) we need the triangular caps of a given stage of construction to be otherwise disjoint. Despite the construction ensuring that the triangular caps have non-increasing base angle with increasing stage of approximation (that is (base angle)$T_{n+k,i} \le$ (base angle)$T_{n,i}$), should the vertical height (that is ε) of $T_{0,1}$ be greater than the base length, then $T_{1,1}$ will not be a subset of $T_{0,1}$. Also, we note that if $T_{0,1}$ is an equilateral triangle (which would be the case if $\varepsilon = \sqrt{3/4}$) then it is a triangular cap. In this case, however, $T_{n,i} = T_{0,1}$ for each $n \in \mathbb{N}$ and $i \in \{1, 2, ..., 2^n\}$. It can be calculated that should $\varepsilon < 1/4$ then

$$l \le (\sqrt{b^2/4 + b^2/4})^2/b = 5b/16 < b/2$$

where l is the side length of $T_{n,i}$ (for some n and any i) and b is the base length of $T_{n-1,j}$ (for any j). Taking $\varepsilon < 1/100$ clearly satisfies $\varepsilon < 1/4$ so that the desired structural properties mentioned above are indeed satisfied under this assumption on ε.

Definition 4.2.1.
For each $n \in \mathbb{N} \cup \{0\}$ and each $i \in \{1, ..., 2^n\}$ there is a triangular cap $T_{n,i}$ constructed on $A_{n,i}$. We denote the vertiex of $T_{n,i}$ that is not in $A_{n,i}$ (that is, the new vertiex created) by $a_{n,i}$.

Construction 4.2.2. .
As previously mentioned we will be looking at a subset of A_ε. We have already noted that the edge points of A_ε are countable, we now give them an ordering. We take

$$e_1 = (0,0), \quad e_2 = (1,0), \quad e_3 = a_{0,1}$$

and then in general

$$e_{2+i+\sum_{j=0}^{n-1} 2^j} = a_{n,i}.$$

To give a closed version of A_ε we define simply

$$\mathcal{A}_\varepsilon := A_\varepsilon \sim \mathcal{B}^{\mathcal{A}_\varepsilon}$$

where $\mathcal{B}^{\mathcal{A}_\varepsilon}$ is the union of a countable collection of open balls forming an open cover of $E(A_\varepsilon)$ such that

$$\mathcal{H}^1(\pi_x(\mathcal{B}^{\mathcal{A}_\varepsilon})) < \frac{1}{2}$$

and thus

$$\mathcal{H}^1(\pi_x(\mathcal{A}_\varepsilon)) > \frac{1}{2}.$$

That $\mathcal{A}_\varepsilon$ is closed is proven in the following Lemma. Since $E(A_\varepsilon)$ is countable it is clear that such a cover exists. We need, however, that $\mathcal{B}^{\mathcal{A}_\varepsilon}$ be chosen to allow $\mathcal{A}_\varepsilon$ to satisfy further measure properties. In particular, we need that $\mathcal{H}^1(\mathcal{A}_\varepsilon) = \infty$. This result is claimed in Lemma 6.3.2 and proven following Proposition 8.4.1. How $\mathcal{B}^{\mathcal{A}_\varepsilon}$ should be chosen is described in Definition 8.4.2 after the necessary preliminary concepts have been established. $\diamondsuit$

Remark: There are four points concerning A_ε and $\mathcal{A}_\varepsilon$ that are important that should be noted. Firstly, the entire purpose of altering A_ε to $\mathcal{A}_\varepsilon$ was that $\mathcal{A}_\varepsilon$ should be closed. We therefore prove that this important property indeed holds. Secondly, although we will show that A_ε and $\mathcal{A}_\varepsilon$ have property (iv) with respect to $j = 1$ and thus have dimension 1, the sets have some interesting properties in themselves. For this reason and as support for the consistency of the results here we provide, in Appendix A, a direct proof that the dimension of A_ε and $\mathcal{A}_\varepsilon$ is 1. Thirdly, as we will show in chapter 5, the exotic counterexamples of A_ε and $\mathcal{A}_\varepsilon$ are necessary. Finally, as a support to the idea that counter examples to (iv) (2) need necessarily be badly behaved but more importantly to answer (iii) and (iv) (3), we note that A_ε, $\mathcal{A}_\varepsilon$ are not rectifiable. In these first six chapters describing the classification the result will be formally stated but without proof, the proof is provided later in the Thesis. This is because the proof of the fact that A_ε and $\mathcal{A}_\varepsilon$ are not rectifiable is very technical and requires substantial preparation making it much more convenient to prove the result as a corollary of the same result for the generalised sets later in the Theorem.

Lemma 4.2.1.
$\mathcal{A}_\varepsilon$ *is closed.*

Proof:
We first show that $A_\varepsilon \cup E$ is closed.

Consider a convergent sequence of points $\{x_n\} \subset A_\varepsilon \cup E$. We must show that

$$x := \lim_{n \to \infty} x_n \in A_\varepsilon \cup E.$$

If $x \in E$ we are finished, so assume that this is not the case. We will for each n define an appropriate sequence, $\{x_{n,j}\}$ converging to x_n. We will then apply a diagonal argument to the resulting sequnce of converging sequences.

Now, for each x_n, either $x_n \in E$ or $x_n \in A_\varepsilon$.

In the first case $x_n = e_i \in E$ for some i and there is an $n_0 \in \mathbb{N}$ such that $e_i \in A_m$ for each $m > n_0$. It follows that by taking $\{x_{n,j}\}_{j=1}^\infty$ such that $x_{n,j} = x_n$ for each j, we can see that for each j there is an $m \geq j$ such that $x_{n,j} \in A_m$. With the sequence $x_{n,j} = x_n$ for each j we also have $\lim_{j \to \infty} x_{n,j} = x_n$.

In the second case

$$x_n \in A_\varepsilon = \overline{\left(\bigcup_{m=1}^{\infty} A_m \right)} \sim \bigcup_{m=1}^{\infty} A_m.$$

Thus there exists a sequence $x_{n,j}$ such that $|x_{n,j} - x_n| < 1/j$ so that $\lim_{j \to \infty} x_{n,j} = x_n$ and $\{x_{n,j}\}_{j=1}^{\infty} \subset \bigcup_{m=1}^{\infty} A_m$. We now assume that there is a finite number q such that $\{x_{n,j}\}_j \subset \cup_{m=1}^{q} A_m$ and show that the assumption leads to a contradiction. Under this assumption, since $\cup_{m=1}^{q} A_m$ is a finite union of closed lines it is closed so that $\lim_j x_{n,j} \in \cup_{m=1}^{q} A_m$ and thus $x_n \in \cup_{m=1}^{q} A_m$. However, since $x \notin E$ and $\cup_{m=1}^{q} E(A_m)$ is finite, $d(x, \cup_{m=1}^{q} E(A_m)) > 0$. Thus in this case $x_n \in \cup_{m=1}^{q} A_m \sim E$. It follows then that we would have

$$x_n \notin \overline{\left(\bigcup_{m=1}^{\infty} A_m \right)} \sim \bigcup_{m=1}^{\infty} A_m \cup E = A_\varepsilon \cup E,$$

giving us the required contradiction. There does not exist, therefore, a $q \in \mathbb{N}$ such that $\{x_{n,j}\}_j \subset \cup_{m=1}^{q} A_m$. We can thusly take a subsequence and relabel to assume that $x_{n,j} \in A_m$ for some $m \geq j$ for each $j \in \mathbb{N}$.

Combining the two cases, we now take the sequence $\{x_m\}_{m=1}^{\infty}$ given by

$$x_m = x_{m,m},$$

and note that $\{x_m\} \subset \bigcup_{n=1}^{\infty} A_n$ so that $\lim_{m \to \infty} x_m \in \overline{\cup_{n=1}^{\infty} A_n}$. By the condition that $|x_{n,j} - x_n| < 1/j$, this diagonal selection gives us

$$x = \lim_{m \to \infty} x_m \in \overline{\bigcup_{n=1}^{\infty} A_n}.$$

Since, following from construction 2, for each $n \in \mathbb{N}$ and each $y \in A_n \sim E$ there is a radius $r > 0$ such that $d(y, \cup_{m=n+1}^{\infty} A_m) > r$ it follows that for each $n \in \mathbb{N}$ $x \notin A_n \sim E$. Thus

$$x \in \overline{\left(\bigcup_{m=1}^{\infty} A_m \right)} \sim \bigcup_{m=1}^{\infty} (A_m \sim E) = A_\varepsilon \cup E.$$

We therefore have that A_ε is closed.

Now since $\mathcal{B}^{\mathcal{A}_\varepsilon}$ is the countable union of open balls it is also open. Since $E \subset \mathcal{B}^{\mathcal{A}_\varepsilon}$ we can write

$$\mathcal{A}_\varepsilon = A_\varepsilon \sim \mathcal{B}^{\mathcal{A}_\varepsilon} = A_\varepsilon \cup E \sim \mathcal{B}^{\mathcal{A}_\varepsilon}$$

which is a closed set with an open set removed and thus is closed, proving the Lemma. $\diamond$

4.3   Properties of A_ε and $\mathcal{A}_\varepsilon$

We now mention two properties of A_ε and $\mathcal{A}_\varepsilon$.

The first is that $dim A_\varepsilon = dim \mathcal{A}_\varepsilon = 1$, which can be proved directly without too much difficulty. A direct proof is often mathematically instructive and is therefore given in Appendix A. The reason that the direct proof is not given here is that the result is a direct consequence of two of the more

general results presented later. Firstly, since A_ε and $\mathcal{A}_\varepsilon$ satisfy (iv) with $j = 1$ (proven in Chapter 6) it follows that they both have dimension 1. The second general result from which the dimension of A_ε and $\mathcal{A}_\varepsilon$ follow is Theorem 9.3.1 which would take too long to introduce and discuss here.

The second property we mention here (which we also prove below) is that the $\mathcal{H}^1$ measure of the approximating lines $A_{n,i}$ (and therefore also the $\mathcal{H}^1$ measure of the approximating sets A_n) can be explicitly calculated. The resulting formula is very important to many results concerning A_ε and $\mathcal{A}_\varepsilon$ and is regularly applied. We also point out here that the form of the resulting formula already provides some indication of the motivation to the selection of the ρ_n's in the construction of $\mathcal{A}_\varepsilon$.

Lemma 4.3.1.
For each $n \in \mathbb{N}$ and each $j \in \{1, 2, ..., 2^n\}$ the base length of a triangular cap $T_{n,j}$ in the construction of A_ε has length

$$\mathcal{H}^1(A_{n,j}) = \frac{(1 + n16\varepsilon^2)^{1/2}}{2^n},$$

and thus

$$\mathcal{H}^1(A_n) = (1 + n16\varepsilon^2)^{1/2}$$

for each $n \in \mathbb{N}$.

Proof:
Clearly $\mathcal{H}^1(A_0) = 1$. Then $\mathcal{H}^1(A_1)$ is the sum of two hypothenuses of triangles of $(1/2)\mathcal{H}^1(A_0)$ base length and 2ε height. That is

$$\mathcal{H}^1(A_1) = 2 \left(\left(\frac{1}{2} \right)^2 + (2\varepsilon) \right)^{1/2} = (1 + 16\varepsilon^2)^{1/2}.$$

Having that it is true for $n = 0, 1$ I now claim that $\mathcal{H}^1(A_n) = (1 + n16\varepsilon^2)^{1/2}$. Assuming it is true for n we note that $\mathcal{H}^1(A_{n+1})$ is the sum of 2^{n+1} hypothenuses of triangles of base length $\mathcal{H}^1(A_n)/2^{n+1}$ and height $2^{2-n}\varepsilon$. That is

$$
\begin{aligned}
\mathcal{H}^1(A_{n+1}) &= 2^{n+1} \left(\left(\frac{\mathcal{H}^1(A_n)}{2^{n+1}} \right)^2 + (2^{2-(n+1)}\varepsilon)^2 \right)^{1/2} \\
&= ((\mathcal{H}^1(A_n))^2 + 16\varepsilon^2)^{1/2} \\
&= (1 + (n+1)16\varepsilon^2)^{1/2},
\end{aligned}
$$

proving the inductive claim. Finally for each $n \in \mathbb{N}$ and $j \in \{1, 2, ..., 2^n\}$ the base length of a triangular cap $T_{n,i}$, $\mathcal{H}^1(A_{n,i})$, is one equal 2^n-th part of the length of A_n, $\mathcal{H}^1(A_n)$ which completes the proof. $\diamond$

To conclude the Chapter, having introduced all of the counter examples, we present a table showing the questions the constructed sets can be used as counter examples for.

Couner Examples			
Property	Counter Example		
	(1)	(2)	(3)
(i)	Γ_ε	Γ_ε or $\mathcal{N}$	Γ_ε
(ii)	Γ_ε	Γ_ε	Γ_ε
(iii)		Λ^2, A_ε or $\mathcal{A}_\varepsilon$	A_ε or $\mathcal{A}_\varepsilon$
(iv)		A_ε or $\mathcal{A}_\varepsilon$	A_ε or $\mathcal{A}_\varepsilon$
(v)	Γ_ε	Γ_ε	Γ_ε
(vi)		Λ_{δ_0}	
(vii)		$\mathcal{N}$	
(viii)			

$$(4.1)$$

4.4   Notes

The construction of Γ_ε is based on a fractal presented by Koch [18], the general form we use comes from Simon [27] and the specific example we use is our own. The remaining definitions, constructions and results in this chapter are our own.