Chapter 3

Background, Definition and Existing Results

3.1 Preliminary Geometric Measure Theory

We start with some relevant measure theoretic background. The standard references are [25] and [11]. We assume basic familiarity with general measure theory and we use the usual symbol for r-dimensional Hausdorff measure $\mathcal{H}^r$ for $r \in \mathbb{R}$. Also, we denote the Hausdorff volume of the unit n-ball by ω_n.

As mentioned, a major part of our investigation regards dimension. We use and are interested in dimension in the sense of Hausdorff dimension which we define as follows.

Definition 3.1.1.
*Set $A \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Then the **Hausdorff dimension** of A is defined as*

$$
\begin{aligned}
dim A \quad &:= \quad \inf\{r \in \mathbb{R} : \mathcal{H}^r(A) = 0\} \\
&= \quad \sup\{r \in \mathbb{R} : \mathcal{H}^r(A) = +\infty\}.
\end{aligned}
$$

Another important quantity that we will be using is density, and indeed n-dimensional density.

Definition 3.1.2.
Let $(X, \mathcal{B}, \mu)$ be a measure space. Then for any subset A of X, and any point $x \in X$, we define the n-dimensional upper and lower densities $\Theta^{*n}(\mu, A, x)$, $\Theta^n_*(\mu, A, x)$ respectively by

$$
\Theta^{*n}(\mu, A, x) = \limsup_{\rho \to 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}
$$

and

$$
\Theta^n_*(\mu, A, x) = \liminf_{\rho \to 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}.
$$

In the case that the two quantities are equal we call the common quantity the n-dimensional μ-density of A at x denoted by $\Theta^n(\mu, A, x)$.

Remark:
Depending on which quantities are understood from the context, we will also use the terms density of A at x or simply the density at x.

The σ-algebra $\mathcal{B}$ here is mentioned for formality, we will use the usual σ-algebra of all measurable sets in the ambient space.

Also fundamental to our considerations is the concept of rectifiability. We will need several forms of the definition of rectifiability. Their equivalences are well presented in [25]. We shall not here be interested in general rectifiable sets, so we restrict ourselves immediately to countably rectifiable sets. Firstly and most basically we have the following definition.

Definition 3.1.3.
A set $M \subset \mathbb{R}^{n+k}$ is said to be countably n-rectifiable if

$$M \subset M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n)$$

where $F_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ are Lipschitz functions and $\mathcal{H}^n(M_0) = 0$.

Remark: By standard Lipschitz extension results we know that we can also write

$$M = M_0 \cup \bigcup_{j=1}^{\infty} F_j(A_j)$$

for subsets $A_j \subset \mathbb{R}^n$.

Notice that we have not required that the sets be measurable, which is occasionally required in definitions of rectifiable sets. It is however not necessary since, as we will see, all of the relevant sets we will be considering are in any case measurable since they can be shown to be expressable as countable unions and intersections of Borel sets in the appropriate Euclidean space.

From this basic definition it is known that the following expression for rectifiable sets holds.

Lemma 3.1.1.
M is countably n-rectifiable if and only if

$$M \subset \bigcup_{j=0}^{\infty} N_j,$$

where $\mathcal{H}^N(N_0) = 0$ and where each N_j, $j \geq 1$, is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+k}$.

We will need one more representation of rectifiability, which will be given below. However, we now present a result concerning rectifiability that, although intuitively clear, needs to be formally stated for the purposes of our classification.

Proposition 3.1.1.
Let $j, n \in \mathbb{N}$ satisfy $j < n$ and let $A \subset \mathbb{R}^n$ satisfy $dim(A) > j$. Then A is not countably j-rectifiable.

Proof:
Since $dim A > j$, $dim A = j + \varepsilon$ for some $\varepsilon > 0$ and thus

$$\mathcal{H}^{j+\varepsilon/2}(A) = \infty. \tag{3.1}$$

Now, should A be countably j-rectifiable, we could write

$$A = M_0 \cup \bigcup_{i=1}^{\infty} F_i(\mathbb{R}^j)$$

where $\mathcal{H}^j(M_0) = 0$ (and thus $\mathcal{H}^{j+\varepsilon/2}(M_0) = 0$) and $F_i : \mathbb{R}^j \to \mathbb{R}^n$ are Lipschitz functions. As $\mathcal{H}^{j+\varepsilon/2}(\mathbb{R}^j) = 0$, $\mathcal{H}^{j+\varepsilon/2}(F_i(\mathbb{R}^j)) = 0$ for each $i \in \mathbb{N}$ and thus

$$\mathcal{H}^{j+\varepsilon/2} \left(\bigcup_{i=1}^{\infty} F_i(\mathbb{R}^j) \right) \leq \sum_{i=1}^{\infty} \mathcal{H}^{j+\varepsilon/2}(F_i(\mathbb{R}^j)) = 0.$$

It follows that $\mathcal{H}^{j+\varepsilon/2}(A) \leq \mathcal{H}^{j+\varepsilon/2}(M_0) + \mathcal{H}^{j+\varepsilon/2}(\cup_{i=1}^{\infty} F_i(\mathbb{R}^j)) = 0$. This contradiction to (3.1) proves the result. $\diamond$

To introduce the final representation of rectifiability that we need we first need the following two definitions.

Definition 3.1.4.
We let the blow-up function be denoted by η, that is for any subset $A \subset \mathbb{R}^n$

$$\eta_{y,\rho}(A) = \rho^{-1}(A - y).$$

Let L be a subspace of $\mathbb{R}^n$ and $\rho \in \mathbb{R}$, $\rho > 0$, then

$$L^\rho = \{x \in \mathbb{R}^n : |x - y| < \rho \text{ for some } y \in L\}.$$

Definition 3.1.5.
If M is an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ and θ is a positive locally $\mathcal{H}^n$-integrable function on M, then we say that a given n-dimensional subspace P of $\mathbb{R}^{n+k}$ is the approximate tangent space for M with respect to θ if

$$
\begin{aligned}
\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}M} f(y)\theta(x + \lambda y)d\mathcal{H}^n(y) &= \lim_{\lambda \to 0} \lambda^{-n} \int_M f(\lambda^{-1}(z - x))\theta(z)d\mathcal{H}^n(z) \\
&= \theta(x) \int_P f(y)d\mathcal{H}^n(y)
\end{aligned}
$$

for all $f \in C_C^0(\mathbb{R}^{n+k})$. The function θ is called the multiplicity function of M.

We will in general consider sets with the multiplicity function set to 1.

Our final definition of countably n-rectifiable sets is now stated in the form of the following theorem.

Theorem 3.1.1.
Suppose M is $\mathcal{H}^n$-measurable. Then M is countably n-rectifiable if and only if there is a positive locally $\mathcal{H}^n$-integrable function θ on M with respect to which the approximate tangent space $T_x M$ exists for $\mathcal{H}^n$-a.e. $x \in M$.

Remark: We note that, for example in [26], it is often required that the total or $\mathcal{H}^n$ measure of a set M be finite or at least that $\mathcal{H}^n(M \cap K)$ be finite for each compact set K. We do not, a priori, make this assumption.

Rectifiability can be seen as the weakest form of structure that a set can possess. However, we can explore parts of even unrectifiable sets in the case that they contain rectifiable parts. This fact will be useful to us, particularly in chapter 4. For this reason we also define purely unrectifiable sets.

Definition 3.1.6.
A set P is said to be purely n-unrectifiable if it contains no countably n-rectifiable subsets of $\mathcal{H}^n$ positive measure.

We note to this definition that for any set in $\mathbb{R}^{n+k}$, A, A can always be decomposed into the disjoint union of two sets $A = R \cup P$ where R is countably R rectifiable and P is purely n-unrectifiable.

3.2   Motivation of the Classification

We now give the motivation and construction of the problem at hand, previous results and results that follow more or less trivially from the literature.

An additional motivation to that mentioned in the introduction to this work was to perhaps uncover a way to attack the local $\mathcal{H}^j$-finality of singularity sets for minimal surfaces or surfaces moving by their mean curvature. This is supported by the mentioned results in Leon Simon's [26] paper on the rectifiability of minimal surfaces, and recent work by Huisken and Sinistrari that shows that estimates on the shape of singularity sets is heading in the direction of satisfying the properties of the definitions under consideration. In particular, in Simon [26] a Lemma (the same one as has been previously discussed) shows that at least parts of the singularity sets of particular types of minimal surfaces exactly satisfy one of the approximation properties.

We state this Lemma (after appropriate definitions) as a motivational starting point and also as it highlights some of the interesting points of results. We then state the definitions intended for classification mentioned in the introduction and provide more fully a discussion of the intended classification of these definitions. We also provide here a summary of the classification central to our work.

Definition 3.2.1.
*By a **multiplicity one class** of minimal surfaces, $\mathcal{M}$, we will mean a set of smooth (i.e. infinitely differentiable) n-dimensional minimal submanifolds. Each $M \in \mathcal{M}$ is assumed to be properly embedded in $\mathbb{R}^{n+k}$ in the sense that for each $x \in M$ there is a $\sigma > 0$ such that $M \cap \overline{B}_\sigma(x)$ is a compact connected embedded smooth manifold with boundary contained in $\partial B_\sigma(x)$. We also assume that for each $M \in \mathcal{M}$ there is a corresponding open set $U_M \supset M$ such that $\mathcal{H}^n(M \cap K) < \infty$ for each compact $K \subset U_M$, and such that M is stationary in U_M in the sense that*

$$\int_M div_M \Phi d\mu = 0,$$

whenever $\Phi = (\Phi^1, ..., \Phi^{n+k}) : U_M \to \mathbb{R}^{n+k}$ is a C^∞ vector field with compact support in U_M. Here we have used $\mu = \mathcal{H}^n|_M$. We also require that the multiplicity one class of submanifolds are closed with respect to sequential compactness, orthogonal transformations and homotheties, that is:

1. *$M \in \mathcal{M} \Rightarrow q \circ \eta_{x,\rho} M \in \mathcal{M}$ and $q \circ \eta_{x,\rho} U_M = U_{q \circ \eta_{x,\rho} M}$ for each $\rho \in (0, 1]$, and for each orthogonal transformation q of $\mathbb{R}^{n+k}$.*

2. *If $\{M_j\}_j \subset \mathcal{M}$, $U \subset \mathbb{R}^{n+k}$ with $U \subset U_M$, for all sufficiently large j, and $\sup_{j \geq 1} \mathcal{H}^n(M_j \cap K) < \infty$ for each compact $K \subset U$, then there is a subsequence $M_{j'}$ and an $M \in \mathcal{M}$ such that $U_M \supset U$ and $M_{j'} \to M$ in U in the sense that*

$$\int_{M_{j'}} f d\mathcal{H}^n \to \int_M f d\mathcal{H}^n$$

   for any $f \in C^0_C(U, \mathbb{R})$.

We assume here that the $M \in \mathcal{M}$ have no removable singularities: thus if $x \in \overline{M} \cap U_M$ and there is a $\sigma > 0$ such that $\overline{M} \cap \overline{B}_\sigma(z)$ is a smooth connected embedded n-dimensional submanifold with boundary contained in $\partial B_\sigma(z)$, then $z \in M$. Subject to this agreement we can make the following definition:

Definition 3.2.2.
Suppose that $\mathcal{M}$ is as above and that $M \in \mathcal{M}$, then the (**interior**) **singular set of** M *(relative to U_M) is defined by*
$$sing M := U_M \cap \overline{M} \sim M$$
and the regular set of M is simply M itself, that is

$$reg M := M.$$

With these definitions we can now state the motivating Lemma due to Simon [26]:

Lemma 3.2.1.
If $\mathcal{M}$ is a multiplicity one class of minimal surfaces, $M \in \mathcal{M}$,

$$m := \max\{dim \ sing M : M \in \mathcal{M}\},$$

$$z_0 \in sing M$$

and
$$S_+(z_0) := \{z \in M : \Theta^m(M, z) \geq \Theta^m(M, z_0)\}.$$

Then for each $\varepsilon > 0$ there is a $p = p(\varepsilon, z_0, M) > 0$ such that $S_+(z_0)$ has the following approximation property in $\overline{B_p(z_0)}$:

For each $\sigma \in (0, p]$ and $z \in S_+(z_0) \cap \overline{B_p(z_0)}$ there is an m-dimensional affine subspace $L_{z,\sigma}$ containing z with
$$S_+ \cap B_\sigma(z) \subset \ the \ (\varepsilon\sigma) - nhood \ of \ L_{z,\sigma}.$$

We note that in the case of Mean Curvature Flows, the singularity set can also be defined as in the following two definitions:

Definition 3.2.3.
We say that a solution of Mean Curvature Flow $(M_t)_{t < t_0}$ reaches $x_0 \in \mathbb{R}^{n+1}$ at time t_0 if there exists a sequence (x_j, t_j) with $t_j \nearrow t_0$ so that $x_j \in M_j$ and $x_j \to x_0$.

Definition 3.2.4.
Let $\mathcal{M} = (M_t)$ be a smooth solution of mean curvature flow in $U \times (t_1, t_0)$. We say that $x_0 \in U$ is a **singular point** *of the solution at time t_0 if $\mathcal{M}$ reaches x_0 at time t_0 and has no smooth extension beyond time t_0 in any neighbourhood of x_0. All other points are called* **regular points**. *The* **singular set** *at time t_0 will be denoted by $sing_{t_0}\mathcal{M}$ and the regular set by $reg_{t_0}\mathcal{M}$.*

As singularity sets are the motivation rather than the subject of our investigation, the properties of singular sets are used very little. However, in determining how applicable our results may be to singular sets we find that it is important to note that singular sets (from either definintion) are closed.

Proposition 3.2.1.
Singular sets as defined in either Definition 3.2.2 or Definition 3.2.4 are closed.

16

Proof:
Suppose that the statement is not true, then there is a point $x \in regM$ such that for all $r > 0$ $B_r(x) \cap singM \neq \emptyset$. In particular since $x \in regM$ there is a radius $\rho_x > 0$ such that $\bar{M} \cap \overline{B_{\rho_x}(x)}$ is "smooth" (either in the infinitely differentiable in space time sense for mean curvature flow, or the sense outlined in Definition 3.2.1, depending on whether we are proving the result for Definition 3.2.2 or 3.2.4) and such that $B_{\rho_x} \cap singM \neq \emptyset$. Thus there is a $z \in singM$ and $\rho_z > 0$ such that $B_{\rho_z}(z) \subset B_{\rho_x}(x)$. It follows that $\bar{M} \cap \overline{B_{\rho_z}(z)}$ is "smooth" and thus $z \in regM$. This contradiction shows such a point x cannot be found which completes the proof. $\diamondsuit$

We now construct the properties that we will be investigating. We will always be considering sets being approximated by j-dimensional affine spaces that are subspaces of $\mathbb{R}^n$. We will identify $\mathbb{R} \times \{0\}$ with $\mathbb{R}$ and denote the projection onto $\mathbb{R}$ by π_x. Further, if L is a j-dimensional affine space in $\mathbb{R}^2$ we will denote the projection onto L by π_L.

Definition A.
Let $A \subset \mathbb{R}^n$ be an arbitrary set and $1 > \delta > 0$; then

(i) A has the weak j-dimensional δ-approximation property if for all $y \in A$ there is $\rho_y > 0$ such that for all $\rho \in (0, \rho_y]$, there exists an affine space $L_{y,\rho}$ such that $B_\rho(y) \cap A \subset L_{y,\rho}^{\delta\rho} \ni y$.

(ii) A has the weak j-dimensional δ-approximation property with local ρ_y-uniformity if for all $y \in A$ there is a $\rho_y > 0$ such that for all $\rho \in (0, \rho_y]$ and all $x \in B_{\rho_y}(y) \cap A$, there exists an affine space $L_{x,\rho}$ such that $B_\rho(x) \cap A \subset L_{x,\rho}^{\delta\rho} \ni x$.

(iii) A is said to have the fine weak j-dimensional approximation property if A satisfies (i) for each $\delta > 0$.

(iv) A has the fine weak j-dimensional approximation property with local ρ_y-uniformity if A satisfies (ii) for all $\delta > 0$.

(v) The property (i) is said to be ρ_0-uniform, if A is contained in some ball of radius ρ_0 and if, for every $y \in A$ and every $\rho \in (0, \rho_0]$, there exists an affine space $L_{y,\rho}$ such that $B_\rho(y) \cap A \subset L_{y,\rho}^{\delta\rho} \ni y$.

(vi) A has the strong j-dimensional δ-approximation property if for each $y \in A$ there is a j-dimensional affine space L_y containing y such that definition (i) holds with $L_{y,\rho} = L_y$ for every $\rho \in (0, \rho_y]$.

(vii) A has the strong j-dimensional δ-approximation property with local ρ_y-uniformity if for all $y \in A$ there exists a $\rho_y > 0$ and an affine space L_y such that for all $x \in B_{\rho_y}(y)$ and all $\rho \in (0, \rho_y]$ $B_\rho(x) \cap A \subset (L_y + y - x)^{\delta\rho} \ni x$.

(viii) The property in (vi) is said to be ρ_0-uniform if A is contained in some ball of radius ρ_0 and if for each $y \in A$ and $rhoin(0, \rho_0$ there is a j-dimensional affine space L_y containing y such that $B_\rho(y) \cap A \subset L_y^{\delta\rho} \ni y$.

Due to the long names of the properties, they will be henceforth referred to only by their number.

Since the definitions are not easy to properly understand or to properly distinguish from one another at first glance, we present a table below summarising the construction of the definitions. Apart from the basic approximating principle of fitting a set in a $\delta\rho$ neighbourhood of an affine space within a ρ-ball, the definitions are all made up of four basic elements.
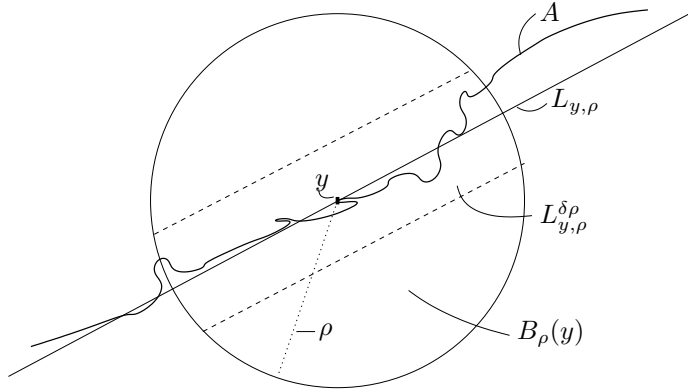
17

Figure 3.1: δ-approximation

1. δ-**approximation type**: Each definition will be either of strong or weak δ-approximation type. To be of weak type allows the approximating affine space to alter for each point being approximated. To be of stong type insists that given a point, y, in the set being approximated, A, and radius ρ_y there is an affine space that may not be changed when making the subsequent approximations required by the definition in question after the initial choice of a point and radius.

2. ρ_0-**uniformity**: Each definition either possesses this property or not. It requires that the entire set being approximated be contained in a ball of a fixed radius $\rho_0 > 0$. Further, the set must be appropriately approximated by an affine space around each point in the set in a neighbourhood of any radius up to and including ρ_0.

3. **Local ρ_y-uniformity**: Each definition either possesses this property or not. It requires simply that for each y in the set being approximated, A, there exists a $\rho_y > 0$ such that $A \cap B_{\rho_y}(y)$ have the ρ_0 uniformity property with $\rho_0 = \rho_y$.

4. δ-**fine**: Each definition either possesses this property or not. It requires that the definitions other properties hold simultaneously for all $\delta > 0$ and not just some pre-chosen $\delta > 0$.

We can now present the definitions in Definition A in table form showing the elements possessed by each definition.

Property	δ-approx.	ρ_0-uniform	ρ_y-uniform	δ-fine
(i)	Weak	No	No	No
(ii)	Weak	No	Yes	No
(iii)	Weak	No	No	Yes
(iv)	Weak	No	Yes	Yes
(v)	Weak	Yes	Yes	No
(vi)	Strong	No	No	No
(vii)	Strong	No	Yes	No
(viii)	Strong	Yes	Yes	No

(3.2)

The classification we make is to get a simple yes or no answer for each of the eight definitions with respect to the following three questions.

Question 3.2.1.
We wish to classify the definitions in Definition 1 with respect to the following questions:

1. *If the set will be of dimension j (or rather $\leq j$),*

2. *if the set will have some locally finite Hausdorff measure property and*

3. *if the set will be countably j-rectifiable.*

With these questions in mind we will concern ourselves with asking about the answer to (1), (2) or (3) with respect to a certain definition. For example, in saying that property (i) does not ensure that a set have Hausdorff dimension less than or equal to j, we are answering the first question in the negative for property (i). In this case we say that the answer to (i) (1) is no.

As we are generally probing here for 'free information' about singularity sets, and the use of more than one definition of the terms about which we are asking in the literature we remain open as to which definition it is that we are making classifications with respect to. We therefore allow for two strengths of locally finite $\mathcal{H}^j$ measure. In only one case do we find that the answer as to possessing locally finite $\mathcal{H}^j$ measure is affected by the choice of strength of definition, that is for (vii) where the definition ensures satisfaction of the weaker but not the stronger definition. The definitions are:

Definition 3.2.5.
A subset $A \subset \mathbb{R}^n$ is said to have **locally finite $\mathcal{H}^j$ measure** *(or local $\mathcal{H}^j$-finality) if for all compact subsets $K \subset \mathbb{R}^n$,*

$$\mathcal{H}^j(K \cap A) < \infty,$$

or equivalently, if for all $y \in \mathbb{R}^n$ there exists a radius $\rho_y > 0$ such that

$$\mathcal{H}^j(B_{\rho_y}(y) \cap A) < \infty.$$

A subset is said to have **weakly locally finite $\mathcal{H}^j$ measure** *(or weak local $\mathcal{H}^j$-finality) if for each $y \in A$ there exists a radius $\rho_y > 0$ such that*

$$\mathcal{H}^j(B_{\rho_y}(y) \cap A) < \infty.$$

An example of the difference is that

$$\mathcal{N} := \bigcup_{n=1}^{\infty} \mathbb{R} \times \left\{\frac{1}{n}\right\}$$

has weak local $\mathcal{H}^j$-finality but not local $\mathcal{H}^j$-finality. The use of allowing the weak definition is that in some cases, such as $\overline{\mathcal{N}}$, a set without weak local $\mathcal{H}^j$-finality will be the finite union of a collection of sets with local $\mathcal{H}^j$-finality. Which still could be understood as having reasonably behaved local measure when the structure giving the locally infinite measure is known.

Also, the definition of rectifiability often requires that the set in question be weakly locally finite in addition to satisfying the structural requirements (see for example [26]). Since the question of locally finite measure is addressed in question (2) we do not add this consideration. It is for this reason that we restrict ourselves as much as possible to the structural requirements of rectifiable sets.

The classification we get (which shall be proved in the following chapters) can be tabulated as follows:

<div align="center">

Classification Table

</div>

Property	(1)	(2) (weak, strong)	(3)
(i)	No	No, No	No
(ii)	No	No, No	No
(iii)	Yes	No, No	No
(iv)	Yes	No, No	No
(v)	No	No, No	No
(vi)	Yes	No, No	Yes
(vii)	Yes	Yes, No	Yes
(viii)	Yes	Yes, Yes	Yes

$$(3.3)$$

In answering a "No" we always give a counter example. They have not been given in the above table as they have not yet been constructed. For reference, a table of counterexamples is given at the end of Chapter 4 in (**??**) and a complete table of all classifications with the corresponding counterexamples is provided at the end of Chapter 6 in (6.4).

As we see, and was hinted at, we do not necessarily get very much information for free. Most notably we find that it is not true that the definition relating to Simon's Lemma (property (iv)) does not guarantee locally finite measure or rectifiability. However, as mentioned in the introduction, in this case we do show that in order for something to go wrong the set does have to be truly badly behaved which should be helpful. We now note formally that the condition in Simon's Lemma is definition (iv).

Proposition 3.2.2.
The $S_+(z_0)$ sets introduced in Lemma 3.2.1 are (iv).

Proof:
Direct comparison between the property shown in Lemma 3.2.1 and (iv) shows that this is exactly what is shown in Lemma 3.2.1. $\diamondsuit$

3.3 Results Following from the Literature

Although the problem we are looking at has not previously been systematically investigated, a few of the results follow easily from results already in the literature for which proofs can be found, for example in Simon [27]. The relevant results can be stated in the form of one counter example (discussed later) and the following lemma (which is a summary of results to be found in Simon [27]):

Lemma 3.3.1.
(i) There is a function $\beta : [0,\infty) \to [0,\infty)$ with $\lim_{\delta \searrow 0} \beta(\delta) = 0$ such that if $A \subset \mathbb{R}^n$ has the j-dimensional weak δ-approximation property for some given $\delta \in (0,1]$, then $\mathcal{H}^{j+\beta(\delta)}(A) = 0$. (In particular if A has the j-dimensional weak δ- approximation property for each $\delta > 0$, then $\dim A \leq j$.)

(ii) If $A \subset \mathbb{R}^n$ has the strong j-dimensional δ-approximation property for some $\delta \in (0,1]$, then

$A \subset \cup_{k=1}^{\infty} G_k$, where each G_k is the graph of some Lipschitz function over some j-dimensional subspace of $\mathbb{R}^n$.

(iii) If $A \subset \mathbb{R}^n$ has the ρ_0 uniform strong j-dimensional δ-approximation property for some $\delta \in (0, 1]$, then $A \subset \cup_{k=1}^{Q} G_k$, where G_k is the graph of some Lipschitz function over some j-dimensional subspace of $\mathbb{R}^n$, L.

We show in the following Corollary that the above Lemma allows us to answer yes to properties (vi) (1), (viii) (1) and (2), (iii) (1), (iv) (1) and (vii) (1) and (2), although we answer yes to (vii) (2) only with weak local $\mathcal{H}^j$-finality, to local $\mathcal{H}^j$-finality we answer no.

Corollary 3.3.1.
The answer to the following Definitions is yes:
(1): (iii), (iv), (vi), (vii) and (viii).
(2): (vii) and (viii).
(3): (vi), (vii) and (viii).

Proof:
(iii) (1) follows from Lemma 3.3.1 (i) since

"In particular if A has the j-dimensional weak δ-approximation property for each $\delta > 0$, then $dim A \leq j$."

means that should A satisfy (iii), then $dim A \leq j$ which proves that the answer to (iii) (1) is yes. Further, since (iv) (1) is a strengthening of (iii), sets satisfying the properties of (iv) must further satisfy any properties following from sets satisfying (iii), thus the answer to (iv) (1) must also be yes.

Any graph of a Lipschitz function over a j-dimensional affine space clearly has dimension less than or equal to j. It follows then that any countable union of such graphs will also have dimension bounded above by j. It thus follows from Lemma 3.3.1 (ii) and (iii) that the answers to (vi) (1) and (viii) (1) are yes. Similarly to the preceeding paragraph, the fact that (vii) is a strengthening of (vi) that the answer to (vii) (1) is yes.

Further concerning (viii), suppose that we have a set A satisfying the conditions of property (viii). Suppose also that $x \in \mathbb{R}^n$ and $\rho > 0$. Then we know that

$$A \cap B_\rho(x) \subset \bigcup_{k=1}^{Q} g_k(\pi_{L_k}(B_\rho(x)))$$

where L_k are the j-dimensional affine spaces that Lemma 3.3.1 ensures exist and the g_k are the Lipschitz functions over the L_k that combined contain A. Thus

$$\mathcal{H}^j(A \cap B_\rho(x)) \leq \sum_{k=1}^{Q} \mathcal{H}^j(g_k(\pi_{L_k}(B_\rho(x)))).$$

Since $card(\{g_k\}_{k=1}^{Q}) = Q < \infty$ there exists an $M = \max_k \text{ Lip} g_k < \infty$ so that by the Area Formula

$$\mathcal{H}^j(A \cap B_\rho(x)) \leq \sum_{k=1}^{Q} M \mathcal{H}^j(\pi_{L_k}(B_\rho(x))) = QM\omega_j\rho_j < \infty.$$

We thus have that property (viii) does ensure locally finite measure, and hence that the answer to (viii) (2) is yes.

We now note that should we have a set satisfying (vii), then, by definition, for each $y \in A$ there is a $\rho_y > 0$ and an affine space L_y such that for all $x \in B_{\rho_y}(y)$ and all $\rho \in (0, \rho_y]$ $B_\rho(x) \cap A \subset L_y^{\delta\rho}$.

It follows that $B_{\rho_y}(y) \cap A$ satisfies (viii), thus $\mathcal{H}^j(K \cap A) < \infty$ for each compact $K \subset \mathbb{R}^n$, that is

$$\mathcal{H}^j(B_{\rho_y}(y) \cap A) \leq \mathcal{H}^j(\overline{B_{\rho_y}(y)} \cap A) < \infty.$$

giving weak local $\mathcal{H}^j$-finality, and the answer to (vii) (2) as yes.

For the answers to (3) we observe that Lemma $reflem2$ (ii) states that any set A satisfying definition (vi) can be written as a countable union of Lipschitz graphs. By the definition of rectifiability (Definition **??**) we see that the answer to (vi) (3) is yes.

By observing the structure of the definitions (vii) and (vii) (see in particular Table 3.2) we see that definitions (vii) and (viii) are strictly stronger than (vi). That is, for any set, A, satisfying either definition (vii) or (viii), A also satisfies definition (vi) and so by the preceeding paragraph is countably j-rectifiable. This shows that the answers to (vii) (3) and (viii) (3) are both yes and thus concludes the proof. $\diamond$

Remark: We note that the proof as written is also optimal in that we cannot get better than weak local $\mathcal{H}^j$-finality for (vii) as seen in the already given example of $\mathcal{N}$. For each $y \in \mathcal{N}$ we can find a $\rho_y > 0$ such that $B_{\rho_y}(y) \cap \mathcal{N} \subset \mathbb{R} \times \{1/n\}$ for some $n \in \mathbb{N}$, and by setting L_y as this affine space for each y it is clear that $\mathcal{N}$ satisfies (vii). However, for each $r > 0$

$$\mathcal{H}^j(B_r((0,0)) \cap \mathcal{N}) = \infty$$

so that $\mathcal{N}$ does not have locally finite $\mathcal{H}^j$-measure.

Another contribution that comes from Simon [27] is a set that is similar in form to the main and most interesting counter example that is presented here. Its actual construction and properties will be discussed in the following section, however, in noting results that have already been essentially shown, we acknowledge its existence and that it is known to satisfy one of the definitions.

Lemma 3.3.2.
There is a set, Γ_ε, that satisfies (i) for $j = 1$ that has dimension greater than 1.

In latter chapters the dimension of Γ_ε and related sets will be discussed. The original proofs that we present will be based on the knowledge of how to calculate the dimension of Γ_ε. The proof of the relevant formula will, however, not be presented, as it also already exists in the literature. The proof can be found in [15].

Corollary 3.3.2.
The answers to (i) (1), (2) and (3) are no.

Proof:
The set Γ_ε of Lemma 3 constructed in the following section provides a counter example to the answer to (i) (1) being yes. It also follows that Γ_ε is not weakly $\mathcal{H}^1$-finite. Thus the answer to (i) (2) is also no. Similarly, Proposition 3.1.1 shows that the fact that $dim\Gamma_\varepsilon > 1$ prevents Γ_ε being countably 1-rectifiable showing that the answer to (i) (3) is no. $\diamond$

All of the results following easily from the literature have now been shown. Further all of those questions in our classification to be answered with yes have now been answered. What has already been shown can be summarised in the following table:

Property	Question			Counter Example		
	(1)	(2) (weak, strong)	(3)	(1)	(2)	(3)
(i)	No	No, No	No	Γ_ε	Γ_ε	Γ_ε
(ii)						
(iii)	Yes					
(iv)	Yes					
(v)						
(vi)	Yes		Yes			
(vii)	Yes	Yes, No	Yes			
(viii)	Yes	Yes, Yes	Yes			

$$(3.4)$$

It remains only to show that the answers to all of the remaining classification questions are no.

3.4 Notes

Definitions 3.1.1, 3.1.2, 13.2.1, 3.1.4, 3.1.5 and 3.1.6 are all standard geometric measure theory as are the results Lemma 3.1.1 and Theorem 3.1.1. We understand Proposition 3.1.1 to be standard and lay no claim to it. We have, however, no source and the proof given is our own. Good sources on geometric measaure theory are Federer [11], Simon [25] or Evans and Gariepy [10]. The motivating Definitions 3.2.1 and 3.2.2 as well as the motivating Lemma for this work, Lemma 3.2.1, are due to Simon [26]. Definitions 3.2.3 and 3.2.4 come from Ecker [7]. Although the result proven in Proposition 3.2.1 is well known, see Simon [27], the proof given is our own. Definition A and Question 3.2.1 is our own fundamental set up of this research. The two definitions of locally finite measure in Definition 3.2.5 both occur in the literature, both generally referred to as simply locally finite measure. Strong locally finite measure can be found in, for example Ecker [7] or Evans and Gariepy [10]. Weak locally finite measure can be found in, for example, Simon [26], [27] or Brakke [5]. Proposition 3.2.2 is, as stated in the proof, simply a reforming of the statement of Lemma 3.2.1 which is due to Simon [26]. Lemma 3.3.1 is due to Simon and was the motivation for looking at variations of definition in Definition A. Corollary 3.3.1 is, however, our own. Lemma 3.3.2 is proven later in this thesis and its origin will be discussed in the appropriate chapter. Corollary 3.3.2 is our own.