

# On the Singularity Sets of Minimal Surfaces and a Mean Curvature Flow

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## Certification

I, Amos Nathan Koeller, declare that this Dissertation, submitted in (partial) fulfilment of the requirements for the award of doctor rerum naturalium, in the Mathematischen Institut at the Freie Universität Berlin, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other institution.

## Erklärung

Ich versichere, dass ich alle Hilfsmittel und Hilfen zur Erstellung der Dissertation in der vorliegenden Arbeit angegeben habe. Ich versichere, dass ich die vorliegende Dissertation auf Grundlage der angegebenen Hilfsmittel und Hilfen selbständig angefertigt habe.

Amos Nathan Koeller

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