Chapter 1

General Arrangements

For a topological space X and a set $\mathcal{A}$ of subspaces of X (an *arrangement* in X) we will be interested in the homotopy type of $\bigcup \mathcal{A}$ and in the calculation of the homology groups of this space. We will also be interested in the space $X \setminus \bigcup \mathcal{A}$, especially in its cohomology ring. If X is a manifold, we will describe the latter, using Poincaré duality, via the homology of the pair $(X, \bigcup \mathcal{A})$ and intersection products therein.

The arrangement $\mathcal{A}$ defines a partially ordered set (*poset*), the intersection poset $Q := \{\bigcap S \colon S \subset Q\}$ ordered by inclusion. This is a special case of a *diagram of spaces*. We will more generally investigate diagrams over a small category $\mathfrak{C}$ (instead of the poset Q) where maps are not required to be inclusions. This additional generality will not complicate the proofs, and we hope that it will clarify the relevant concepts by allowing the reader the comparison with other special cases, for example that of the category $\mathfrak{C}$ being a group. We also envision applications of these tools to arrangements with group actions where $\mathfrak{C}$ would be the orbit category of the group, but this will not be explained in the current work.

1.1 Diagrams of spaces

In this section we will denote by $\mathfrak{T}op$ a suitable category of topological spaces, such as the category presented in [Vog71].

Let $\mathfrak{C}$ be a (discrete) small category. We call an $X \in \mathfrak{T}op^{\mathfrak{C}}$, i.e. a functor $X \colon \mathfrak{C} \to \mathfrak{T}op$, a $\mathfrak{C}$-diagram of spaces. If the category $\mathfrak{C}$ is a group G, a G-diagram is a space with a G-operation defined on it. We will choose similar notation in such a way that an $X \in \mathfrak{T}op^{\mathfrak{C}}$ corresponds to a left operation and an $X \in \mathfrak{T}op^{\mathfrak{C}^o}$ corresponds to a right operation.

Diagrams of spaces have been used as a tool for studying homotopy types of arrangements in [ZŽ93] and [WZŽ99]. Since our main interest lies in computing homology groups and products, not in determining homotopy types, we summarize the needed results only briefly, giving proofs only where they illustrate concepts that will be useful later on, and using [HV92] and [FF89] as references. This is also hoped to motivate the material on diagrams of chain complexes in the next section, which proceeds analogously in some respects. In particular, we want to draw attention to the notions of free diagrams and ZŽ–maps.

The setting in which these results will be applied to arrangements is explained in Section 1.3.

1.1.1 Definition. Let $X \in \mathfrak{T}op^{\mathfrak{C}^o}$, $Y \in \mathfrak{T}op^{\mathfrak{C}}$. We define

$$X \times_{\mathfrak{C}} Y := \coprod_{q \in \operatorname{Obj} \mathfrak{C}} X(q) \times Y(q) \Big/ \sim, \tag{1.1}$$

where $\sim$ is the relation generated by $(x, Y(f)y) \sim (X(f)x, y)$ for $x \in X(q)$, $y \in Y(p)$, $f \in \mathfrak{C}(p,q)$.

1.1.2 Definition. For $S \in \mathfrak{T}op^{\operatorname{Obj} \mathfrak{C}}$ we define $i_\# S \in \mathfrak{T}op^{\mathfrak{C}}$ by

$$(i_\# S)(q) := \coprod_{p \in \operatorname{Obj} \mathfrak{C}} \coprod_{f \in \mathfrak{C}(p,q)} S(p) \tag{1.2}$$

for $q \in \operatorname{Obj} \mathfrak{C}$, and for $f \in \mathfrak{C}(q,q')$ letting $(i_\# S)(f) \colon (i_\# S)(q) \to (i_\# S)(q')$ map the copy of $S(p)$ indexed by f identically to that indexed by gf. $i_\# \colon \mathfrak{T}op^{\operatorname{Obj} \mathfrak{C}} \to \mathfrak{T}op^{\mathfrak{C}}$ is made into a functor in the obvious way.

1.1.3 Definition. Let $X \in \mathfrak{T}op^{\mathfrak{C}}$. We call X a *free* $\mathfrak{C}$*-diagram*, if there exists a filtration $X = \bigcup_{n \geq 0} X_n$, $X_n \in \mathfrak{T}op^{\mathfrak{C}}$, such that X carries the final topology with respect to the inclusion maps $X_n \to X$, $X_0(q) = \emptyset$ for all $q \in \operatorname{Obj} \mathfrak{C}$, and for each $n \geq 1$ exist $A_n, B_n \in \mathfrak{T}op^{\operatorname{Obj} \mathfrak{C}}$, a map $j \colon B_n \to A_n$ consisting of closed cofibrations, and a diagram

$$\begin{array}{ccc} i_\# B_n & \longrightarrow & X_{n-1} \\ {\scriptstyle i_\# j}\downarrow & & \downarrow{\scriptstyle \text{incl.}} \\ i_\# A_n & \longrightarrow & X_n \end{array}$$

which is a pushout diagram.

1.1.4 Remark. What we call a free $\mathfrak{C}$-diagram is called a principal $\mathfrak{C}$-space in [FF89]. Indeed it can be argued that the term *free diagram* would better be reserved to those in the image of the functor $i_\#$. However, a chain complex is also called free, if it is free in every degree, and similarly free diagrams are made up of pieces in the image of $i_\#$. Also compare with Definition 1.2.2.

1.1.5 Definition. We define a $(\mathfrak{C} \times \mathfrak{C}^o)$-diagram of simplicial sets by

$$\begin{aligned} E\mathfrak{C}(q', q)_n &:= \left\{ (f_0, f_1, \dots, f_{n+1}) \colon f_i \in \mathfrak{C}, f_0 f_1 \cdots f_{n+1} \in \mathfrak{C}(q, q') \right\} \\ E\mathfrak{C}(g', g)_n \left((f_0, f_1, \dots, f_{n+1}) \right) &:= (g' f_0, f_1, \dots, f_{n+1} g) \end{aligned}$$

with boundaries

$$d^i(f_0, f_1, \dots, f_{n+1}) := (f_0, \dots, f_i f_{i+1}, \dots, f_{n+1})$$

and degeneracies

$$s^i(f_0, f_1, \dots, f_{n+1}) := (f_0, \dots, f_i, \mathrm{id}, f_{i+1}, \dots, f_{n+1}).$$

We define $E\mathfrak{C} \in \mathfrak{T}op^{\mathfrak{C} \times \mathfrak{C}^o}$ to be the simplicial realization of this diagram.

1.1.6 Remark. We think of a $X \in \mathfrak{Top}^{\mathfrak{C}\times\mathfrak{C}^o}$ as being equipped with commuting right and left operations of $\mathfrak{C}$. For example, for $Y \in \mathfrak{Top}^{\mathfrak{C}^o}$, $X \times_{\mathfrak{C}} Y$ is defined and inherits the left operation, i.e. $X \times_{\mathfrak{C}} Y \in \mathfrak{Top}^{\mathfrak{C}}$.

1.1.7 Proposition. *Let $X \in \mathfrak{Top}^{\mathfrak{C}}$. $E\mathfrak{C} \times_{\mathfrak{C}} X \in \mathfrak{Top}^{\mathfrak{C}}$ is a free $\mathfrak{C}$-diagram.*

Sketch of proof. The idea is to choose the pairs $(A_n(q), B_n(q))$ of Definition 1.1.3 as $(\Delta^n, \partial\Delta^n) \times \coprod_{q' \in \operatorname{Obj}\mathfrak{C}} (\{(f_1, \dots, f_n)\colon f_1 \cdots f_n \in \mathfrak{C}(q', q)\} \times X(q'))$. □

1.1.8 Proposition. *Let $X \in \mathfrak{Top}^{\mathfrak{C}}$. A point of $(E\mathfrak{C} \times_{\mathfrak{C}} X)(q)$ is determined by $f_0, \dots, f_{n+1}$ with $f_0 f_1 \cdots f_{n+1} \in \mathfrak{C}(q', q)$, $s \in \Delta^n$ and $x \in X(q')$. Sending this point to $X(f_0 f_1 \cdots f_{n+1})(x) \in X(q)$ yields a well-defined map of $\mathfrak{C}$-diagrams $E\mathfrak{C} \times_{\mathfrak{C}} X \to X$.*

1.1.9 Definition. For $X \in \mathfrak{Top}^{\mathfrak{C}}$ we define the *colimit of X* as (or identify it with) $\operatorname{colim} X := * \times_{\mathfrak{C}} X$ and we define the *homotopy colimit of X* as $\operatorname{hcolim} X := * \times_{\mathfrak{C}} E\mathfrak{C} \times_{\mathfrak{C}} X$, where $*$ denoted the constant $\mathfrak{C}^o$-diagram of spaces consisting of a single point.

The map from Proposition 1.1.8 induces a map $\operatorname{hcolim} X \to \operatorname{colim} X$.

1.1.10 Remark. For $X \in \mathfrak{Top}^{\mathfrak{C}^o}$ we have $\operatorname{hcolim} X = * \times_{\mathfrak{C}^o} E\mathfrak{C}^o \times_{\mathfrak{C}^o} X \approx X \times_{\mathfrak{C}} E\mathfrak{C} \times_{\mathfrak{C}} *$, and we will switch between both versions ad libitum.

1.1.11 Definition. Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}}$, $f\colon X \to Y$ a map of $\mathfrak{C}$-diagrams. The map f is called a *homotopy equivalence*, if there exist $g\colon Y \to X$, $F\colon I \times X \to X$, $G\colon I \times Y \to Y$, all of them maps of $\mathfrak{C}$-diagrams, such that F is a homotopy from $g \circ f$ to id_X and G a homotopy from $f \circ g$ to id_Y.

1.1.12 Definition. Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}}$, $f\colon X \to Y$ a map of $\mathfrak{C}$-diagrams. The map f is called a *weak homotopy equivalence*, if $f(q)\colon X(q) \to Y(q)$ is a homotopy equivalence for all $q \in \operatorname{Obj}\mathfrak{C}$.

1.1.13 Proposition. *The map $E\mathfrak{C} \times_{\mathfrak{C}} X \to X$ from Proposition 1.1.8 is a weak homotopy equivalence.*

See Lemma 1.2.9 for a proof of an algebraic analogue.

The following Proposition is our main technical tool. A proof can be found in [FF89, Thm 4.3] where it is attributed to [BV73].

1.1.14 Proposition. *Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}}$, $f\colon X \to Y$ a weak homotopy equivalence. If X and Y are free diagrams, then f is a homotopy equivalence.*

1.1.15 Definition. Let $X, Y \in \mathfrak{Top}^{\mathfrak{C}^o}$. We will call a weak equivalence $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ a *topological ZŽ–map*.

1.1.16 Remark. A ZŽ–map $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ yields homotopy equivalences $X(q) \simeq (X \times_{\mathfrak{C}} E\mathfrak{C})(q) \simeq Y(q)$ and should be seen as a collection of homotopy equivalences $f_q\colon X(q) \xrightarrow{\simeq} Y(q)$ which not necessarily satisfy the equations $f_p \circ X(g) = Y(g) \circ f_q$ that would make these maps into a map of $\mathfrak{C}$-diagrams, but which satisfy these equations up to homotopies which fit together up to higher homotopies and so on. This will be made clearer for the algebraic analogue in Definition 1.2.12 and the calculations following it.

1.1.17 Proposition. *Let* $X, Y \in \mathfrak{T}op^{\mathfrak{C}^o}$ *and* $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ *be a ZŽ–map. If* Y *is a free* $\mathfrak{C}^o$*-diagram, then the induced map* $\operatorname{hcolim} X \approx X \times_{\mathfrak{C}} E\mathfrak{C} \times_{\mathfrak{C}} * \to Y \times_{\mathfrak{C}} * \approx \operatorname{colim} Y$ *is a homotopy equivalence.*

Proof. $X \times_{\mathfrak{C}} E\mathfrak{C}$ is a free diagram by Proposition 1.1.7, Y by assumption. Therefore f is a homotopy equivalence by Proposition 1.1.14. It follows that the induced map is a homotopy equivalence. □

1.1.18 Remark. Our usage of the term *ZŽ–map* is motivated by the fact that the homotopy equivalence to the link of a linear arrangement from its combinatorially defined homotopy model given by Ziegler and Živaljević [ZŽ93] can be seen to arise in this way. We present their homotopy model in Proposition 2.1.10.

Because of Proposition 1.1.13 we can note a special case.

1.1.19 Corollary. *Let* $X \in \mathfrak{T}op^{\mathfrak{C}}$*. If* X *is a free* $\mathfrak{C}$*-diagram, then the canonical map* $\operatorname{hcolim} X \to \operatorname{colim} X$ *is a homotopy equivalence.* □

We conlude this short overview with a simple proposition that allows us to compose ZŽ–maps.

1.1.20 Proposition. *Let* $f\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ *be a ZŽ–map. The map*

$$\begin{aligned} L(f)\colon X \times_{\mathfrak{C}} E\mathfrak{C} &\to Y \times_{\mathfrak{C}} E\mathfrak{C} \\ [(x,s)] &\mapsto [(f(x,s),s)] \end{aligned}$$

is a well defined weak homotopy equivalence (and hence homotopy equivalence) of $\mathfrak{C}^o$*-diagrams which makes the diagram*

$$\begin{array}{ccc} X \times_{\mathfrak{C}} E\mathfrak{C} & \xrightarrow{L(f)} & Y \times_{\mathfrak{C}} E\mathfrak{C} \\ & \searrow^{f} & \downarrow \\ & & Y \end{array} \tag{1.3}$$

commute. Consequently, if $g\colon Y \times_{\mathfrak{C}} E\mathfrak{C} \to Z$ *is another ZŽ–map, then so is* $g \circ L(f)\colon X \times_{\mathfrak{C}} E\mathfrak{C} \to Z$*.*

Proof. It is easily checked by computation that (1.3) is a well defined commutative diagram of $\mathfrak{C}^o$-diagrams. Since f and the natural map $Y \times_{\mathfrak{C}} E\mathfrak{C} \to Y$ are weak homotopy equivalences, so is $L(f)$. By Proposition 1.1.7 and Proposition 1.1.14 $L(f)$ is even a homotopy equivalence. $g \circ L(f)$ is a composition of weak homotopy equivalences and therefore a weak homotopy equivalence. □

1.2 Diagrams of chain complexes

We introduce notation and terminology for diagrams of chain complexes similar to those introduced in the previous section for diagrams of spaces. In particular, free diagrams and algebraic ZŽ–maps will be defined. These tools will be used later on to study homology groups of arrangements, starting in Section 1.3.

The main results of this section will be proved by use of the spectral sequences of a certain double complex. It is one of these spectral sequences that will also be the basis for the proof of the graded formula for intersection products in an arrangement in a manifold, Proposition 1.3.20.

Homology

Let $\mathfrak{C}$ be a small category, R a hereditary ring.

1.2.1 Definition. Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $N \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$. We define $M \otimes_{\mathfrak{C}} N \in R\text{-}\mathfrak{Mod}$,

$$M \otimes_{\mathfrak{C}} N := \bigoplus_{q \in \operatorname{Obj} \mathfrak{C}} M(q) \otimes N(q) \Big/ K,$$

where K is the submodule generated by the elements $a \otimes N(f)(b) - M(f)(a) \otimes b$ for $a \in M(q)$, $b \in N(p)$, $f \in \mathfrak{C}(p,q)$.

1.2.2 Definition. For $S \in \mathfrak{Set}^{\operatorname{Obj} \mathfrak{C}}$, we define $F^{\mathfrak{C}} S \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ by

$$F^{\mathfrak{C}} S(q) := \mathrm{F}\left\{(f,s) \colon f \in \mathfrak{C}(p,q), s \in S(p)\right\},$$

and for $g \in \mathfrak{C}(q,q')$ letting $F^{\mathfrak{C}} S(g)$ be the morphism sending (f,s) to (gf,s). $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ is called a *free $\mathfrak{C}$-diagram of abelian groups* if M is isomorphic to $F^{\mathfrak{C}} S$ for some S, and $X \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ is called a *free $\mathfrak{C}$-diagram of chain complexes*, if X_n is a free diagram of abelian groups for every $n \in \mathbb{Z}$.

1.2.3 Lemma. *Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $S \in \mathfrak{Set}^{\operatorname{Obj} \mathfrak{C}}$. The map*

$$\bigoplus_{q \in \operatorname{Obj} \mathfrak{C}} \bigoplus_{s \in S(q)} M(q) \to M \otimes_{\mathfrak{C}} F^{\mathfrak{C}} S$$

which sends an element m of the summand $M(q)$ indexed by $s \in S(q)$ to $m \otimes (\mathrm{id}_q, s)$ is an isomorphism. □

1.2.4 Proposition. *If $N \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ is a free $\mathfrak{C}$–diagram, then the functor $\bullet \otimes_{\mathfrak{C}} N \colon R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o} \to R\text{-}\mathfrak{Mod}$ is exact.*

Proof. This follows from Lemma 1.2.3. □

1.2.5 Definition. We define $B(\mathfrak{C}) \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}\times\mathfrak{C}^o}$ by

$$B(\mathfrak{C})(q',q)_n := F\left\{q' \xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}} q\right\},$$

$$B(\mathfrak{C})(g',g)(q' \xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}} q) := p' \xleftarrow{g'f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}g} p,$$

for $g \in \mathfrak{C}(p,q)$, $g' \in \mathfrak{C}(q',p')$,

$$\mathfrak{d}(q' \xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_n} \cdot \xleftarrow{f_{n+1}} q) := \sum_{k=0}^{n}(-1)^k\, q' \xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{k-1}} \cdot \xleftarrow{f_k f_{k+1}} \cdot \xleftarrow{f_{k+2}} \cdot \cdots \cdot \xleftarrow{f_{n+1}} q$$

1.2.6 Remark. $B(\mathfrak{C})(q',q)$ is the chain complex associated to the simplicial set $E\mathfrak{C}_\bullet$ of Definition 1.1.5.

1.2.7 Lemma. *$B(\mathfrak{C})$ is a free $\mathfrak{C} \times \mathfrak{C}^o$–diagram of chain complexes, $B(\mathfrak{C})_n \cong F^{\mathfrak{C}\times\mathfrak{C}^o} S_n$ with $S_n(q',q) := \left\{q' \xleftarrow{f_1} \cdots \xleftarrow{f_n} q\right\}$.* □

1.2.8 Definition and Proposition. *For $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ we define a map $\varepsilon \in \mathrm{Hom}_{\mathfrak{C}}(B(\mathfrak{C}) \otimes_{\mathfrak{C}} K, K)$ by*

$$\varepsilon\colon B(\mathfrak{C}) \otimes_{\mathfrak{C}} K \to K$$
$$\left(\xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{n+1}}\right) \otimes k \mapsto \begin{cases} 0, & n > 0, \\ K(f_0 f_1)(k), & n = 0. \end{cases}$$

This is a chain map if the K in second position is regarded as a chain complex concentrated in degree zero.

Proof. Since

$$\varepsilon\left(\mathfrak{d}\left(\left(\xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \xleftarrow{f_2}\right) \otimes k\right)\right) = \varepsilon\left(\left(\xleftarrow{f_0 f_1} \cdot \xleftarrow{f_2}\right) \otimes k - \left(\xleftarrow{f_0} \cdot \xleftarrow{f_1 f_2}\right) \otimes k\right)$$
$$= K(f_0 f_1 f_2)(x) - K(f_0 f_1 f_2)(x) = 0,$$

ε is a chain map. □

1.2.9 Lemma. *The map ε from the preceding proposition is a chain homotopy equivalence. In other words, it is an acyclic resolution of K.*

Proof. We define

$$L\colon (B(\mathfrak{C})_r \otimes_{\mathfrak{C}} K)(q) \to (B(\mathfrak{C})_{r+1} \otimes_{\mathfrak{C}} K)(q)$$
$$\left(\xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{r+1}}\right) \otimes k \mapsto \left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{r+1}}\right) \otimes k$$

and calculate for $x = \left(\xleftarrow{f_0} \cdot \cdots \cdot \xleftarrow{f_{r+1}}\right) \otimes k$, that

$$(\mathfrak{d}L + L\mathfrak{d})x = \begin{cases} x, & r > 0, \\ x - \left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{\mathrm{id}}\right) \otimes \varepsilon(x), & r = 0, \end{cases}$$

proving that $k \mapsto \left(\xleftarrow{\mathrm{id}} \cdot \xleftarrow{\mathrm{id}}\right) \otimes k$ is a homotopy inverse of ε. □

1.2.10 Proposition. *Let $X \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$, $X_p = 0$ for $p < 0$. If X is a free $\mathfrak{C}$-diagram, then the map*

$$H(X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K) \xrightarrow{H(\mathrm{id}_X \otimes \varepsilon)} H(X \otimes_{\mathfrak{C}} K)$$

is an isomorphism.

Proof. Again viewing the right hand side K as a complex concentrated in degree zero, the map $\mathrm{id}_X \otimes \varepsilon$ is a map of double complexes. Since X is free, the induced map $''H(X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K) \to {''H}(X \otimes_{\mathfrak{C}} K) = X \otimes_{\mathfrak{C}} K$, where $''H$ denotes homology with respect to the second differential of a double complex, is isomorphic to $\mathrm{id}_X \otimes H(\varepsilon)\colon X \otimes_{\mathfrak{C}} H(B(\mathfrak{C}) \otimes_{\mathfrak{C}} K) \to X \otimes_{\mathfrak{C}} H(K) = X \otimes_{\mathfrak{C}} K$ by Proposition 1.2.4, and the latter map is an isomorphism by Lemma 1.2.9. By [God58, Thm I.4.3.1] it follows that $H(\mathrm{id}_X \otimes \varepsilon)$ is also an isomorphism. □

1.2.11 Definition and Proposition. *We define the diagonal chain map $\Delta \in \mathrm{Hom}_{\mathfrak{C} \times \mathfrak{C}^o}(B(\mathfrak{C}), B(\mathfrak{C}) \otimes_{\mathfrak{C}} B(\mathfrak{C}))$ by*

$$\begin{aligned} \Delta(q' \xleftarrow{f_0} p_0 \xleftarrow{f_1} p_1 \cdots p_{n-1} \xleftarrow{f_n} p_n \xleftarrow{f_{n+1}} q) := \\ \sum_{k=0}^{n} q' \xleftarrow{f_0} p_0 \cdots p_{k-1} \xleftarrow{f_k} p_k \xleftarrow{id_{p_k}} p_k \otimes \\ p_k \xleftarrow{id_{p_k}} p_k \xleftarrow{f_{k+1}} p_{k+1} \cdots p_n \xleftarrow{f_{n+1}} q. \end{aligned}$$

Proof. This is a chain map by the usual calculation, additionally using

$$\begin{aligned} q' \xleftarrow{f_0} p_0 \cdots p_{k-1} \xleftarrow{f_k} p_k \otimes p_k \xleftarrow{id_{p_k}} p_k \xleftarrow{f_{k+1}} p_{k+1} \cdots p_n \xleftarrow{f_{n+1}} q = \\ = q' \xleftarrow{f_0} p_0 \cdots p_{k-2} \xleftarrow{f_{k-1}} p_{k-1} \xleftarrow{id_{p_{k-1}}} p_{k-1} \otimes p_{k-1} \xleftarrow{f_k} p_k \cdots p_n \xleftarrow{f_{n+1}} q \end{aligned}$$

here. □

1.2.12 Definition and Proposition. *Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$. A chain map between diagrams $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$ induces chain maps*

$$\begin{aligned} f_q\colon X(q) &\to Y(q) \\ x &\mapsto f(x \otimes q \xleftarrow{\mathrm{id}_q} q \xleftarrow{\mathrm{id}_q} q). \end{aligned}$$

The maps $H(f_q)$ form a homomorphism $H(X) \xrightarrow{\cong} H(Y)$, where $H(X), H(Y) \in \left(R\text{-}\mathfrak{Mod}^{\mathbb{Z}}\right)^{\mathfrak{C}^o}$. We call the map f an algebraic ZŽ–map, *if for every $q \in \mathfrak{C}$ and every R-module M the map $H(f_q)\colon H(X(q); M) \to H(Y(q); M)$ is an isomorphism.*

Proof. Using the isomorphism

$$\eta\colon \operatorname{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y) \xrightarrow{\cong} \operatorname{Hom}_{\mathfrak{C}\times\mathfrak{C}^o}(B(\mathfrak{C}), \operatorname{Hom}(X, Y))$$

we can describe f_q by $f_q = \eta(f)\left(q \xleftarrow{\mathrm{id}_q} q \xleftarrow{\mathrm{id}_q} q\right)$. Now

$$\mathfrak{d} f_q = \eta(f)\left(\mathfrak{d}\left(q \xleftarrow{\mathrm{id}_q} q \xleftarrow{\mathrm{id}_q} q\right)\right) = 0,$$

i.e. f_q is a chain map. For $k \in \mathfrak{C}(p, q)$,

$$\begin{aligned} \mathfrak{d}\left(\eta(f)\left(\xleftarrow{\mathrm{id}_q} \cdot \xleftarrow{k} \cdot \xleftarrow{\mathrm{id}_p}\right)\right) &= \eta(f)\left(\xleftarrow{k} \cdot \xleftarrow{\mathrm{id}_p} - \xleftarrow{\mathrm{id}_q} \cdot \xleftarrow{k}\right) \\ &= f_p \circ X(k) - Y(k) \circ f_q, \end{aligned}$$

i.e. $f_p \circ X(k) \simeq Y(k) \circ f_q$. □

Looking at the preceding calculation from the other side, we get:

1.2.13 Proposition. *Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$. Assume we are given chain maps $f_q\colon X(q) \to Y(q)$ for all $q \in \mathfrak{C}$ and for every $k \in \mathfrak{C}(p, q)$ a chain homotopy from $Y(k) \circ f_q$ to $f_p \circ X(k)$. If $K_{kl} = Y(l) \circ K_k + K_l \circ X(k)$ whenever kl is defined, then the map*

$$\begin{aligned} f\colon X \otimes_{\mathfrak{C}} B(\mathfrak{C}) &\to Y \\ x \otimes \left(\xleftarrow{k_1} p_0 \xleftarrow{k_0}\right) &\mapsto f_{p_0}(x \cdot k_1) \cdot k_0, \\ x \otimes \left(\xleftarrow{k_2} p_1 \xleftarrow{k_1} p_0 \xleftarrow{k_0}\right) &\mapsto (-1)^{|x|} K_{k_1}(x \cdot k_2) \cdot k_0, \\ x \otimes \left(\xleftarrow{k_0} \cdots \xleftarrow{k_{n+1}}\right) &\mapsto 0, \qquad n > 1 \end{aligned} \tag{1.4}$$

is a chain map of $\mathfrak{C}^o$–diagrams as in Definition 1.2.12.

Proof. f is a well-defined map of $\mathfrak{C}^o$–diagrams, and to check that it is a chain map, it will suffice to show that $\eta(f)$ is. We have

$$\begin{aligned}
\mathfrak{d}\left(\eta(f)\left(\xleftarrow{id_q} q \xleftarrow{id_q}\right)\right) &= \mathfrak{d}f_q = 0,\\
\mathfrak{d}\left(\eta(f)\left(\xleftarrow{id_q} q \xleftarrow{k} p \xleftarrow{id_p}\right)\right) &= \mathfrak{d}K_k = k\cdot k\cdot f_p - f_q\cdot k\\
&= \eta(f)\left(\xleftarrow{k} p \xleftarrow{\mathrm{id}_p} - \xleftarrow{id_q} q \xleftarrow{k}\right)\\
&= \eta(f)\left(\mathfrak{d}\left(\xleftarrow{id_q} q \xleftarrow{k} p \xleftarrow{id_p}\right)\right),\\
\eta(f)\left(\mathfrak{d}\left(\xleftarrow{\mathrm{id}}\cdot\xleftarrow{k}\cdot\xleftarrow{l}\cdot\xleftarrow{\mathrm{id}}\right)\right) &= \eta(f)\left(\xleftarrow{k}\cdot\xleftarrow{l}\cdot\xleftarrow{\mathrm{id}}\right) - \eta(f)\left(\xleftarrow{\mathrm{id}}\cdot\xleftarrow{kl}\cdot\xleftarrow{\mathrm{id}}\right)\\
&\quad + \eta(f)\left(\xleftarrow{\mathrm{id}}\cdot\xleftarrow{k}\cdot\xleftarrow{l}\right)\\
&= k\cdot K_l - K_{kl} + K_k\cdot l = 0,
\end{aligned}$$

proving this. □

We come to an algebraic analogue of Proposition 1.1.20.

1.2.14 Definition. Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$ and $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$. We define $L(f) \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y \otimes_{\mathfrak{C}} B(\mathfrak{C}))$ as the composition

$$L(f)\colon X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \xrightarrow{\mathrm{id}\otimes\Delta} X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} B(\mathfrak{C}) \xrightarrow{f\otimes\mathrm{id}} Y \otimes_{\mathfrak{C}} B(\mathfrak{C}).$$

1.2.15 Proposition. *Let $X, Y, Z \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$, $g \in \mathrm{Hom}_{\mathfrak{C}^o}(Y \otimes_{\mathfrak{C}} B(\mathfrak{C}), Z)$. Then $h := g \circ L(f) \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Z)$, $L(h) = L(g) \circ L(f)$, and if f and g are ZŽ–maps, then so is h.*

Proof. $L(h) = L(g) \circ L(f)$ and $h_q = g_q \circ f_q$ are verified by calculation, using $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ for the first formula. □

1.2.16 Remark. Just in case that the equation $L(g) \circ L(f) = L(g \circ L(f))$ or the following diagram might seem oddly familiar to some readers, we remark that the functor $T\colon X \mapsto X \otimes_{\mathfrak{C}} B(\mathfrak{C})$ together with the natural transformations $\varepsilon'\colon T \to 1$ defined as in Definition 1.2.8 and $\mathrm{id} \otimes \Delta\colon T \to T^2$ is a comonad.

We now obtain an algebraic analogue of Proposition 1.1.17.

1.2.17 Proposition. *Let $X, Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $X_r = 0$, $Y_r = 0$ for $r < 0$, $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$, $f \in \mathrm{Hom}_{\mathfrak{C}^o}(X \otimes_{\mathfrak{C}} B(\mathfrak{C}), Y)$. Then the diagram*

$$\begin{array}{ccc}
X \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K & \xrightarrow{L(f)\otimes\mathrm{id}_K} & Y \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K \\
& \searrow^{f\otimes\mathrm{id}_K} & \Big\downarrow{\scriptstyle \mathrm{id}_Y\otimes\varepsilon} \\
& & Y \otimes_{\mathfrak{C}} K
\end{array}$$

commutes. If f is a ZŽ–map, then $H(L(f) \otimes \mathrm{id}_K)$ is an isomorphism. Therefore, $H(f \otimes \mathrm{id}_K)$ is also an isomorphism, if additionally Y is a free diagram.

Proof. The commutativity of the diagram follows by calculation, the last sentence from Proposition 1.2.10. We have to check that $H(L(f)\otimes \mathrm{id}_K)$ is an isomorphism for a ZŽ–map f.

Let $x \in X(q)_r$, $c = \xleftarrow{\mathrm{id}_q} \cdot \xleftarrow{f_1} \cdots \xleftarrow{f_n} \cdot \xleftarrow{id_q} \in B(\mathfrak{C})(q,q')_s$, $k \in K(q')$. Then $(L(f)\otimes \mathrm{id}_K)(x\otimes c\otimes k) \in \bigoplus_{i=0}^{s} Y_{p+i}\otimes_{\mathfrak{C}} B(\mathfrak{C})_{s-i}\otimes_{\mathfrak{C}} K$. Since $L(f)\otimes \mathrm{id}_K$ respects the filtration by s, it induces a homomorphism between the corresponding spectral sequences. The term of $(L(f)\otimes \mathrm{id}_K)(x\otimes c\otimes k)$ in $X_r\otimes_{\mathfrak{C}} B(\mathfrak{C})_s\otimes_{\mathfrak{C}} K$ is $f_q(x)\otimes c\otimes k$. This describes the induced homomorphism between the E^1–terms $'H(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \cong H(X\otimes K)\otimes_{\mathfrak{C}\times\mathfrak{C}^o} B(\mathfrak{C})$ and $'H(Y\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \cong H(Y\otimes K)\otimes_{\mathfrak{C}\times\mathfrak{C}^o} B(\mathfrak{C})$ of the spectral sequences which is an isomorphism, because f is a ZŽ–map. By [God58, Thm I.4.3.1] it follows that $H(L(f)\otimes \mathrm{id}_K)$ is also an isomorphism. □

Proposition 1.2.17 will later be used to substitute for a diagram Y a simpler diagram X. We now examine conditions on X which make this especially worthwhile.

1.2.18 Proposition. *Let $X \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, $K \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$, $K(q)$ a free R–module for all $q \in \mathfrak{C}$. If X is isomorphic to a direct sum $\bigoplus_{i\in I} X_i$, $X_i \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$, such that there exist $n_i \in \mathbb{Z}$ with $H_r(X_i(q)) = 0$ for all $q \in \mathfrak{C}$, $r \neq n_i$, then $H(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \cong H(H(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$. This isomorphism is natural with respect to K and such that the diagram*

$$\begin{array}{ccc} H_l(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})_0\otimes_{\mathfrak{C}} K & \longrightarrow & H_l(X\otimes_{\mathfrak{C}} B(\mathfrak{C})_0\otimes_{\mathfrak{C}} K) \\ \downarrow & & \\ H_0(H_l(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) & & \downarrow \\ \downarrow & & \\ \bigoplus_r H_r(H_{l-r}(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) & \xrightarrow{\cong} & H_l(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \end{array} \tag{1.5}$$

commutes.

1.2.19 Remark. Most of the time we will actually be able to choose an X with zero differentials, in which case the proposition becomes trivial.

Proof. Because of the additivity of all of the involved constructions, we may assume that there exists an $n \in \mathbb{Z}$ such that $H_r(X) = 0$ for all $r \neq n$.

We will consider the spectral sequence of the double complex $X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K$ associated to the filtration by the degree of $B(\mathfrak{C})$. The E^1–term of this spectral sequence is $'H(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$. Since $B(\mathfrak{C})\otimes_{\mathfrak{C}} K$ is a free $\mathfrak{C}$–diagram, it follows that $E^1_{r,s} \cong H_r(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})_s\otimes_{\mathfrak{C}} K$ and $E^2_{r,s} \cong H_s(H_r(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$. Since $E^2_{r,s} = 0$ for $r \neq n$, $E^\infty = E^2$ and $H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \cong H_s(H_n(X)\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)$, proving the existence of the isomorphism.

The isomorphism is given by the maps

$$E^\infty_{n,s} \cong \frac{F_s H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)}{F_{s-1}H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K)} \xleftarrow{\cong} F_s H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K) \xrightarrow{\cong} H_{n+s}(X\otimes_{\mathfrak{C}} B(\mathfrak{C})\otimes_{\mathfrak{C}} K),$$

where F denotes the induced filtration on the homology of the double complex. All of the maps are natural with respect to K.

The commutativity of the diagram is easily checked. $\square$

1.2.20 Remark. If K does not consist of free modules, then the result will still hold, if we have that $H_r(X_i(q); K(q')) = 0$ for all $q, q' \in \mathfrak{C}$, $r \neq n_i$ and if we replace $H(H(X) \otimes_{\mathfrak{C}} B(\mathfrak{C}) \otimes_{\mathfrak{C}} K)$ by $H(H(X \otimes K) \otimes_{\mathfrak{C} \times \mathfrak{C}^o} B(\mathfrak{C}))$.

Products

When investigating intersection products in manifolds, many of the occurring chain complexes will carry products. We introduce some terminology to ease the description of these products.

We fix a functor $\lambda \colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$.

1.2.21 Definition. Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$. A *$\lambda$-product* on M is a natural transformation from the functor

$$\begin{aligned} \mathfrak{C} \times \mathfrak{C} &\to R\text{-}\mathfrak{Mod} \\ (p, q) &\mapsto M(p) \otimes M(q) \end{aligned}$$

to the functor $M \circ \lambda$.

1.2.22 Definition and Proposition. *Let $M \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}^o}$ and $N \in R\text{-}\mathfrak{Mod}^{\mathfrak{C}}$ be equipped with λ-products, written by juxtaposition. Then*

$$\begin{aligned} (M \otimes_{\mathfrak{C}} N) \otimes (M \otimes_{\mathfrak{C}} N) &\to (M \otimes_{\mathfrak{C}} N) \\ (m \otimes n) \otimes (m' \otimes n') &\mapsto mm' \otimes nn' \end{aligned}$$

is well-defined, and we will usually equip $M \otimes_{\mathfrak{C}} N$ with this product. $\square$

1.2.23 Definition and Proposition. *We define the cross product $\times$ on $B(\mathfrak{C})$, $\times \in \operatorname{Hom}_{(\mathfrak{C}\times\mathfrak{C})\times(\mathfrak{C}\times\mathfrak{C})^o}(B(\mathfrak{C}) \otimes B(\mathfrak{C}), B(\mathfrak{C} \times \mathfrak{C}))$, by*

$$\Big(\xleftarrow{f_0} \cdot \xleftarrow{f_1} \cdot \cdots \cdot \xleftarrow{f_k} \cdot \xleftarrow{f_{k+1}}\Big) \otimes \Big(\xleftarrow{g_0} \cdot \xleftarrow{g_1} \cdot \cdots \cdot \xleftarrow{g_l} \cdot \xleftarrow{g_{l+1}}\Big) \mapsto$$

$$\sum_{\substack{(i_0,j_0)<\cdots<(i_{k+l},j_{k+l}) \\ (i_0,j_0)=(0,0) \\ (i_{k+l},j_{k+l})=(k,l) \\ i_{r+1}\leq i_r+1 \\ j_{r+1}\leq j_r+1}} \varepsilon^{i_0,\ldots,i_{k+l}}_{j_0,\ldots,j_{k+l}} \left(\xleftarrow{(f_0,g_0)} \cdot \xleftarrow{h^{i_0,i_1}_{j_0,j_1}} \cdot \cdots \cdot \xleftarrow{h^{i_{k+l-1},i_{k+l}}_{j_{k+l-1},j_{k+l}}} \cdot \xleftarrow{(f_{k+1},g_{k+1})} \right)$$

with $h^{i,i}_{j,j+1} = (\mathrm{id}, g_{j+1})$, $h^{i,i+1}_{j,j} = (f_{i+1}, \mathrm{id})$, $\varepsilon^{0,0,\ldots,0}_{0,1,\ldots,l} = \varepsilon^{0,1,\ldots,k}_{0,0,\ldots,0} = 1$, and the remaining $\varepsilon^{i_0,\ldots,i_{k+l}}_{j_0,\ldots,j_{k+l}} \in \{+1, -1\}$ determined by the requirement that $\times$ be a chain map.

The functor λ induces a natural transformation λ_* from $B(\mathfrak{C}\times\mathfrak{C})$ to $B(\mathfrak{C})\circ(\lambda\times\lambda^o)$ and the composition $x \otimes y \mapsto \lambda_*(x \times y)$ is a λ-product (more precisely a $(\lambda \times \lambda^o)$-product) on $B(\mathfrak{C})$ that we will sometimes denote by $\overset{\lambda}{\times}$. $\square$

1.3 Arrangements

Let X be a topological space and $\mathcal{A}$ a set of subspaces of X. We set $Q := \{\bigcap S\colon S \subset \mathcal{A}\}$ and order Q by inclusion. The resulting partially ordered set Q will be considered a small category with a single arrow from p to q if $p \geq q$. We define $D \in \mathfrak{T}op^{Q^o}$ by $D(p) := p$ and letting $D(q \leftarrow p)$ be the inclusion from q to p.

1.3.1 Notation. The minimum map $\wedge\colon Q \times Q \to Q$, $p \wedge q = p \cap q$ is order preserving, hence a functor. Q has a minimum $\bigcap \mathcal{A}$ and a maximum $X = \bigcap \emptyset$. These will be denoted by $\perp$ and $\top$ respectively. For $p, q \in Q$, we denote by $[p, q]$ the *closed interval* $\{x\colon p \leq x \leq q\}$ and similar for open and half-open intervals. For a partially ordered set P, ΔP denotes the *order complex* of P, i.e. the simplicial complex with vertex set P and simplices all chains (totally ordered subsets) in P. By $C(\Delta P)$ we denote the ordered simplicial chain complex of the simplicial complex ΔP, for example $B(Q)(p, q) = C_*(\Delta[p, q])$.

Homotopy

When discussing homotopy properties of the arrangement $\mathcal{A}$ and the diagram D, we will assume that for all $q \in Q$ the inclusion map $\bigcup_{p<q} D(p) \to D(q)$ is a closed cofibration. The union on the left hand side can also be formulated as the colimit of the restrition of D to the poset $\{p\colon p < q\}$.

Under this hypothesis, we get the following proposition.

1.3.2 Proposition. *Let Q be finite. Then $D \in \mathfrak{T}op^{Q^o}$ is a free Q^o-diagram.*

Proof. We enumerate Q as $Q = \{q_1, \ldots, q_m\}$ with $i > j$ whenever $q_i > q_j$ and define diagrams $D^k \in \mathfrak{T}op^{Q^o}$ by

$$D^k(p) := \bigcup \{D(q_i)\colon i \leq k, q_i \leq p\} = \bigcup_{i \leq k} D(p \wedge q_i).$$

These D^k form a filtration of $D^m = D$, $D^0(p) = \emptyset$ for all $p \in Q$, and we will show that this filtration satisfies the conditions of Definition 1.1.3.

Let $0 < k \leq m$. For $p \geq q_k$, we have $D^k(p) = D^{k-1}(p) \cup D(q_k)$ and

$$D^{k-1}(p) \cap D(q_k) = \bigcup_{i<k} D(p \wedge q_i) \cap D(q_k) = \bigcup_{i<k} D(q_k \wedge q_i) = \bigcup_{q'<q_k} D(q').$$

The equation in the middle is the one that is special to a diagram derived from an arrangement and would not hold for an arbitrary diagram of inclusion maps. The key point is that the right hand side is independent of p. It follows that

$$\begin{array}{ccc} \bigcup_{q'<q_k} D(q') & \longrightarrow & D^{k-1}(p) \\ \downarrow & & \downarrow \\ D(q_k) & \longrightarrow & D^k(p) \end{array}$$

is a pushout diagram. For $p \not\geq q_k$, $D^k(p) = D^{k-1}(p)$ and therefore

$$\begin{CD} \emptyset @>>> D^{k-1}(p) \\ @VVV @VVV \\ \emptyset @>>> D^k(p) \end{CD}$$

is a pushout diagram. Defining $A_k, B_k \in \mathfrak{Top}^{\operatorname{Obj} Q}$ by

$$A_k(p) := \begin{cases} D(q_k) & p = q_k, \\ \emptyset, & p \neq q_k, \end{cases} \qquad B_k(p) := \begin{cases} \bigcup_{q' < q_k} D(q') & p = q_k, \\ \emptyset, & p \neq q_k, \end{cases}$$

these combine to give a pushout diagram

$$\begin{CD} i_\# B_k @>>> D^{k-1} \\ @VVV @VVV \\ i_\# A_k @>>> D^k \end{CD}$$

of Q^o-diagrams. □

1.3.3 Remark. From the construction in the proof it follows immediately that a diagram obtained from D by restriction to a sub-poset $\{q_1, \ldots, q_{m'}\}$ with $m' < m$ and q_i as in the proof is also free.

1.3.4 Remark. If X allows a triangulation such that every $A \in \mathcal{A}$ is a subcomplex, a condition that we will assume later when considering intersection products but that is less natural when considering homotopy theory, then a proof more along the lines of Proposition 1.3.9 is available to show that the diagram D is free. This would involve defining $A_k(p)$ to be the disjoint union of all k-simplices in $D(p)$ not contained in any $D(q)$ with $q < p$ and $B_k(p)$ the boundaries of those simplices. Finiteness of Q would not be needed.

Let us assume we are given the following data: A Q^o-diagram of spaces $E \in \mathfrak{Top}^{Q^o}$ (in applications this will be easier to describe than D and possibly carry less information), and for all $p \in Q$ maps

$$f^p \colon E(p) \times \Delta[p, \top] \to X$$

with $\operatorname{im} f^p|_{E(p)\times\Delta[p,q]} \subset D(q) = q$, $f^p(\cdot, \langle p \rangle) \colon E(p) \to D(p) = p$ a homotopy equivalence, and such that for $q \leq p$ the diagram

$$\begin{array}{ccccc} & & E(p) \times \Delta[p, \top] & & \\ & \nearrow^{\mathrm{id}\times\mathrm{incl}} & & \searrow^{f^p} & \\ E(p) \times \Delta[q, \top] & & & & X \\ & \searrow_{E(p\leftarrow q)\times\mathrm{id}} & & \nearrow_{f^q} & \\ & & E(q) \times \Delta[q, \top] & & \end{array} \tag{1.6}$$

commutes.

1.3.5 Proposition. *In the above situation and with Q finite, the maps f^p induce a homotopy equivalence* $\operatorname{hcolim} E' \xrightarrow{\simeq} \bigcup \mathcal{A}$*, where E' is the diagram obtained by restricting E to $Q \setminus \{\top\}$.*

Proof. We set $Q' := Q \setminus \{\top\}$ and denote the restriction of D to Q' by D'. D is free by Proposition 1.3.2, and so is D'. The inclusion $D'(q) \to \bigcup \mathcal{A}$ induce a map $\operatorname{colim} D' \to \bigcup \mathcal{A}$. Since $\bigcup \mathcal{A} = \bigcup_{q \in Q'} D'(q)$ and all of the $D'(q)$ are closed in $\bigcup \mathcal{A}$, this map is a homeomorphism.

Since in Q viewed as a category the composition of two morphisms is never an identity unless one of the original morphisms was (indeed both of them), degenerate simplices may be omitted in the construction of EQ, and $EQ(p,q) \approx \Delta[p,q]$. Hence the maps f^p are exactly what it takes to define a map of Q^o-diagrams $E \times_Q EQ \to D$, and they also define a map of $Q^{o\prime}$-diagrams $E' \times_Q' EQ' \to D'$. The assumption that $f^p(\cdot, \langle p \rangle) \colon E(p) \to p$ is a homotopy equivalence means that this map is a ZŽ–map. The diagram D' is free as shown in Proposition 1.3.2 respectively Remark 1.3.3. By Proposition 1.1.17 it follows that this ZŽ–map induces a homotopy equivalence $\operatorname{hcolim} E' \xrightarrow{\simeq} \operatorname{colim} D' \approx \bigcup \mathcal{A}$. □

1.3.6 Remark. For Proposition 1.3.5 it would not have been necessary to include $\bigcap \emptyset = X$ in Q when defining the maps f^p. Indeed, it may seem like a nuissance that we have included $\bigcap \emptyset = X$ in the definition of the intersection poset. However, when dealing with linear arrangements in Chapter 2 the constructed maps will naturally include the top element of the intersection poset and, more importantly, when turning to homology in the next section, we will also consider the relative case of $(X, \bigcup \mathcal{A})$ and for this the top element will be needed.

1.3.7 Example. Let us assume that every intersection of elements of $\mathcal{A}$ is either empty or contractible. As discussed in the preceding remark, we allow us to ignore the empty intersection. We set

$$E(q) := \begin{cases} \emptyset, & D(q) = \emptyset \\ *, & D(q) \neq \emptyset \end{cases}$$

and make E into a Q^o-diagram in the obvious and unique way. In this case $\operatorname{hcolim} E' \approx \Delta N$ where $N := \{q \in Q \colon D(q) \neq \emptyset,\ q < \top\}$. Maps f^p as above will automatically satisfy that $f^p(\cdot, \langle p \rangle)$ is a homotopy equivalence. Fulfilling the commutativity of (1.6) amounts to constructing a map $h \colon \Delta N \to \bigcup \mathcal{A}$ with $h[\langle q_0, \ldots, q_r \rangle] \subset D(q_r)$ for all chains $q_0 < \cdots < q_r$ with $q_0 \in N$. Since $D(q)$ is contractible for every $q \in P$, such a map is easily defined by recursion over the skeleton of ΔP. By Proposition 1.3.5 the map h is a homotopy equivalence.

1.3.8 Remark. The fact that the diagram D is free can be seen as the reason for the appearance of the intersection poset in descriptions of the homotopy type of $\bigcup \mathcal{A}$ or the homology formulas in the next section.

An alternative construction of a diagram D for which Proposition 1.3.5 holds is to define the poset Q to be the power set of $\mathcal{A}$ ordered by reverse inclusion and to set $D(q) := \bigcap q$. This is again a free diagram, which is proven by repeating the proof of Proposition 1.3.2 verbatim. Vassiliev calls the diagram defined via the power set of $\mathcal{A}$ the *naive resolution* and the diagram defined via the intersection poset the *economical resolution* [Vas01].

When using the naive resolution, the space ΔN in Example 1.3.7 becomes the *nerve of* $\mathcal{A}$ and the result of the example the Nerve Theorem.

We will only consider the economical resultion in the following. Some of the applications to linear arrangements would work equally well for both kinds of resolutions, while for some the economical resolution is more practical. We will look at this again in Remark 2.1.26.

We will meet the naive resolution briefly once again in Section 2.4 in the guise of the atomic complex.

Homology

We will be interested in describing $H(\bigcup\mathcal{A})$ and $H(X,\bigcup\mathcal{A})$. For simplicity, we assume $\bigcup\mathcal{A} \neq X$. We assume the inclusion $\sum_{A\in\mathcal{A}} S(A) \to S(\bigcup\mathcal{A})$, where S denotes the singular chain complex, to induce an isomorphism in homology. Again we will write $S(D)$ instead of $S \circ D$ for the diagram of chain complexes arising from D by applying the singular chain functor.

1.3.9 Proposition. *$S(D) \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{Q^o}$ is a free Q^o–diagram.*

Proof. Let $p \in Q$ and $\sigma\colon \Delta^r \to D(p)$ be a singular simplex. We set $q_\sigma := \bigcap\{q' \in Q\colon \operatorname{im}\sigma \subset q'\}$. Then $q_\sigma \leq p$ and $\sigma \in S_r(q_\sigma)$. It follows that $S_r(D)$ is freely generated by the system $(\{\sigma\colon \Delta^r \to D(p)\colon q_\sigma = p\})_{p\in Q}$. □

1.3.10 Definition. We define $K^u, K^p \in R\text{-}\mathfrak{Mod}^Q$ by

$$K^u(q) := \begin{cases} 0, & q = \top, \\ R, & q < \top, \end{cases} \qquad K^u(q' \to q) := \begin{cases} 0, & q' = \top, \\ \mathrm{id}_R, & q' < \top, \end{cases}$$

$$K^p(q) := \begin{cases} R, & q = \top, \\ 0, & q < \top, \end{cases} \qquad K^p(q' \to q) := \begin{cases} \mathrm{id}_R, & q' = q = \top, \\ 0, & q < \top. \end{cases}$$

The notation is chosen because of the following connection of K^u and K^p with the singular chain complexes of the union $\bigcup\mathcal{A}$ and the pair $(X,\bigcup\mathcal{A})$, respectively.

1.3.11 Proposition. *The chain maps*

$$\begin{aligned} S(D) \otimes_Q K^u &\to S\left(\bigcup\mathcal{A}\right), & S(D) \otimes_Q K^p &\to S\left(X,\bigcup\mathcal{A}\right) \\ c \otimes k &\mapsto kc & c \otimes k &\mapsto kc \end{aligned}$$

are well defined and they induce isomorphisms $H(S(D) \otimes_Q K^u) \cong H(\bigcup\mathcal{A})$ and $H(S(D) \otimes_Q K^p) \cong H(X,\bigcup\mathcal{A})$.

Proof. By the proof of Proposition 1.3.9 and Lemma 1.2.3,

$$S_r(D) \otimes_Q K^u \cong \bigoplus_{p<\top} \bigoplus_{\substack{\sigma \in S_r(X) \\ q_\sigma = p}} R = \bigoplus_{\substack{\sigma \in S_r(X) \\ q_\sigma \in Q \setminus \{\top\}}} R = \sum_{A \in \mathcal{A}} S(A) =: S\mathcal{A},$$

and the first induced map factorizes as $H(S_r(D) \otimes_Q K^u) \xrightarrow{\cong} H(S\mathcal{A}) \xrightarrow{\cong} H(\bigcup \mathcal{A})$. Similarly, $S_r(D) \otimes_Q K^p$ is free on the r–simplices σ with $q_\sigma = \top$, and the second induced map factorizes as $H(S_r(D) \otimes_Q K^p) \xrightarrow{\cong} H(SX/S\mathcal{A}) \xrightarrow{\cong} H(X, \bigcup \mathcal{A})$. □

1.3.12 Proposition. *In the situation described in* (1.6)*, with the condition on* $f^p(\cdot, \langle p \rangle)\colon E(p) \to p$ *weakened to induce an isomorphism in homology,*

$$\begin{gathered} g\colon S(E) \otimes_Q B(Q) \to S(D) \\ c \otimes p \leftarrow q_0 \leftarrow \cdots \leftarrow q_n \leftarrow p' \mapsto f^p_*(c \times \langle q_0, \dots, q_n \rangle), \end{gathered}$$

defines a ZŽ–map. For $K \in R\text{-}\mathfrak{Mod}^Q$ *the map* $g_*\colon H(S(E) \otimes_Q B(Q) \otimes_Q K) \to H(S(D) \otimes_Q K)$ *is an isomorphism.*

Proof. Because $\operatorname{im} f^p|_{E(p) \times \Delta[p,p']} \subset p'$, $f^p_*(c \times \langle q_0, \dots, q_n \rangle)$ is in $S(D)(p) = S(p)$. It is well-defined because of the commutativity of (1.6). The map is a map of Q^o–diagrams, because the right hand side is independent of p'. That it is a chain map is now easily checked.

The map $S(E)(p) \to S(D)(p) = S(p)$, $c \mapsto g(c \otimes p \leftarrow p \leftarrow p) = f^p_*(c \times \langle p \rangle)$ induces an isomorphism in homology by assumption, so g is a ZŽ–map. The map g_* is an isomorphism by Proposition 1.2.17 and Proposition 1.3.9. □

In this situation, one may be lucky and able to prove that $H(S(E) \otimes_Q B(Q) \otimes_Q K)$ is isomorphic to $H(H(E) \otimes_Q B(Q) \otimes_Q K)$, e.g. by Proposition 1.2.18. It then follows that $H(S(D) \otimes_Q K) \cong H(H(D) \otimes_Q B(Q) \otimes_Q K)$.

The preceding results, and those in the section to come, are easily extended to the relative case. We will formulate and prove only the key step.

1.3.13 Proposition. *Let* $Y \subset X$ *and assume that the inclusion maps*

$$\begin{gathered} S(Y) + S\left(\bigcup \mathcal{A}\right) \to S\left(Y \cup \bigcup \mathcal{A}\right), \\ \sum_{A \in \mathcal{A}} S(A \cap Y) \to S\left(Y \cap \bigcup \mathcal{A}\right) \end{gathered}$$

also induce isomorphisms in homology. Let D' *be the* Q^o*–diagram of pairs of spaces defined by* $D'(q) := (q, q \cap Y)$*. Then* $S(D')$ *is a free* Q^o*–diagram, and the chain maps*

$$\begin{array}{cc} S(D') \otimes_Q K^u \to S\left(\bigcup \mathcal{A}, Y \cap \bigcup \mathcal{A}\right), & S(D') \otimes_Q K^p \to S\left(X, Y \cup \bigcup \mathcal{A}\right) \\ c \otimes k \mapsto kc & c \otimes k \mapsto kc \end{array}$$

are well defined and induce isomorphisms $H(\bigcup\mathcal{A}, Y\cap\bigcup\mathcal{A}) \cong H(S(D')\otimes_Q K^u)$ *and* $H(X, Y\cup\bigcup\mathcal{A}) \cong H(S(D')\otimes_Q K^p)$.

Let $\bar{Q} := \{p\in Q\colon D(p)\not\subset Y\}$ *and* $\bar{D}'$ *the* $\bar{Q}^o$*–diagram obtained by restricting* D'. *Then* $S(\bar{D}')$ *is a free diagram and for any* $K \in \mathfrak{Ab}^Q$ *the obvious map* $S(\bar{D}')\otimes_{\bar{Q}}\bar{K} \to S(D')\otimes_Q K$, *where* $\bar{K}$ *is the restriction of* K, *is an isomorphism. In particular,* $H(\bigcup\mathcal{A}, Y\cap\bigcup\mathcal{A}) \cong H(S(\bar{D}')\otimes_{\bar{Q}} K^u)$ *and* $H(X, Y\cup\bigcup\mathcal{A}) \cong H(S(\bar{D}')\otimes_{\bar{Q}} K^p)$.

Proof. Taking up the notation of the proof of Proposition 1.3.9, $S_r(D')$ is free on the system $(\{\sigma\colon\Delta^r\to D(p)\colon q_\sigma = p, \operatorname{im}\sigma\not\subset Y\})_{p\in Q}$, and as in Proposition 1.3.11 the first induced map factorizes as $H(S(D')\otimes_Q K^u) \xrightarrow{\cong} H(S\mathcal{A}/(S\mathcal{A}\cap SY)) \xrightarrow{\cong} H(\bigcup\mathcal{A}, Y\cap\bigcup\mathcal{A})$. $S(D')\otimes_Q K^p$ is free on the singular simplices neither in Y nor in any of the $A\in\mathcal{A}$, and the second induced map factorizes as $H(S(D')\otimes_Q K^p) \xrightarrow{\cong} H(SX/(SY\cup S\mathcal{A})) \xrightarrow{\cong} H(X, Y\cup\bigcup\mathcal{A})$.

To justify the claims regarding $\bar{Q}$, it suffices to remark that the free diagram $S_r(D')$ has no generators for $p\in Q\setminus\bar{Q}$. □

Intersection products in manifolds

If X is a compact n-dimensional manifold oriented over R, we are interested in the intersection products $\bullet$ defined by commutativity of

$$\begin{array}{ccc} H_k(X,\bigcup\mathcal{A})\otimes H_l(X,\bigcup\mathcal{A}) & \xrightarrow{\quad\bullet\quad} & H_{k+l-n}(X,\bigcup\mathcal{A}) \\ {\scriptstyle\frown[X]}\uparrow{\scriptstyle\cong} & & {\scriptstyle\frown[X]}\uparrow{\scriptstyle\cong} \\ H^{n-k}(X\setminus\bigcup\mathcal{A})\otimes H^{n-l}(X\setminus\bigcup\mathcal{A}) & \xrightarrow{\quad\smile\quad} & H^{2n-k-l}(X\setminus\bigcup\mathcal{A}) \end{array}$$

and

$$\begin{array}{ccc} H_k(\bigcup\mathcal{A})\otimes H_l(\bigcup\mathcal{A}) & \xrightarrow{\quad\bullet\quad} & H_{k+l-n}(\bigcup\mathcal{A}) \\ {\scriptstyle\frown[X]}\uparrow{\scriptstyle\cong} & & {\scriptstyle\frown[X]}\uparrow{\scriptstyle\cong} \\ H^{n-k}(X,X\setminus\bigcup\mathcal{A})\otimes H^{n-l}(X,X\setminus\bigcup\mathcal{A}) & \xrightarrow{\quad\smile\quad} & H^{2n-k-l}(X,X\setminus\bigcup\mathcal{A}). \end{array}$$

For Poincaré duality to hold and for technical reasons, we assume X to allow a triangulation such that all $A\in\mathcal{A}$ are subcomplexes.

In this section we will see what information about the intersection products can obtained algebraically without special geometric knowledge of the class of arrangements at hand. For linear arrangements, this will yield the graded formulas of Section 2.2.

For the description of these products it will be important that there is a product on $C_*(\Delta Q)$.

1.3.14 Definition and Remark. *Let* $c \in C_r(\Delta Q)$, $d \in C_s(\Delta Q)$. *Then we have* $c \times d \in C_{r+s}(\Delta Q \times \Delta Q) = C_{r+s}(\Delta(Q \times Q))$ *and* $\wedge_*(c \times d) \in C_{r+s}(Q)$, *since* $\wedge\colon Q \times Q \to Q$ *is order preserving and hence a simplicial map* $\Delta(Q \times Q) \to \Delta Q$. *If* $c = \langle p_0, \dots, p_r\rangle$ *and* $d = \langle q_0, \dots, q_s\rangle$, *then* $\wedge_*(c \times d)$ *is a linear combination of simplices with first vertex* $p_0 \wedge q_0$ *and last vertex* $p_r \wedge q_s$. *This specializes Definition 1.2.23 with* $\wedge$ *for* λ, *and as there we will set* $c \mathbin{\hat{\times}} d := \wedge_*(c \times d)$. *The multiplication in* R *defines* $\wedge$*-products (see Definition 1.2.21) on* K^u *and* K^p *in the obvious way. These products and Definition 1.2.22 will be used to define products on several chain complexes.*

1.3.15 Proposition. *Let* K *be equipped with a* $\wedge$*-product in the above situation. The spectral sequence of the filtration of* $S(D) \otimes_Q B(Q) \otimes_Q K$ *by the grading of* $B(Q)$ *can be made into a multiplicative* E^1*-spectral sequence with the multiplication on* E^1 *isomorphic to the multiplication on* $H(D) \otimes_Q B(Q) \otimes_Q K$ *given by*

$$\begin{aligned}(a \otimes \langle p_0, \dots, p_r\rangle \otimes m) &\otimes (b \otimes \langle q_0, \dots, q_s\rangle \otimes m')\\ &\mapsto (-1)^{r(n-l)}[(a \bullet b) \otimes (\langle p_0, \dots, p_r\rangle \mathbin{\hat{\times}} \langle q_0, \dots, q_s\rangle) \otimes (m \cdot m'), \qquad (1.7)\end{aligned}$$

where $a \in H_k(D(p))$, $b \in H_l(D(q))$, $a \bullet b \in H_{k+l-n}(D(p \wedge q))$, $m \in K(p')$, $m' \in K(q')$, $m \cdot m' \in K(p' \wedge q')$. *The multiplication is a chain map of degree* $(-n, 0)$. *If* $K = K^u$ *or* $K = K^p$ *the multiplication on* E^∞ *is induced, via the isomorphism* $H(S(D) \otimes_Q B(Q) \otimes_Q K^u) \cong H(\bigcup \mathcal{A})$ *or* $H(S(D) \otimes_Q B(Q) \otimes_Q K^p) \cong H(X, \bigcup \mathcal{A})$ *respectively, by the intersection product on* $H(\bigcup \mathcal{A})$ *or* $H(X, \bigcup \mathcal{A})$.

1.3.16 Remark. The intersection product $a \bullet b$ in the above proposition is defined by commutativity of

$$\begin{array}{ccc} H_k(A) \otimes H_l(B) & \xrightarrow{\quad\bullet\quad} & H_{k+l-n}(A \cap B) \\ {\scriptstyle \frown[X]}\big\uparrow{\scriptstyle \cong} & & {\scriptstyle \frown[X]}\big\uparrow{\scriptstyle \cong} \\ H^{n-k}(X, X \setminus A) \otimes H^{n-l}(X, X \setminus A) & \xrightarrow{\ \smile\ } & H^{2n-k-l}(X, (X \setminus A) \cup (X \setminus B)). \end{array}$$

Proof of Proposition 1.3.15

We will from now on consider X to be triangulated by a barycentric subdivision of a triangulation of which all $A \in \mathcal{A}$ are subcomplexes. This will make all $p \in Q$ full subcomplexes of X. We will denote the face poset of this triangulation by FX and by $C(FX)$ the chain complex of ascending (from 0-simplices to n-simplices) chains in FX.

1.3.17 Definition and Proposition. *For a subcomplex* A *of* X, *cap products*

$$\begin{aligned} C^r(FX, FX \setminus FA) \otimes C_s(FX) &\xrightarrow{\frown} C_{s-r}(FA)\\ C^r(FX \setminus FA) \otimes C_s(FX) &\xrightarrow{\frown} C_{s-r}(FX, FA)\\ h \otimes \langle f_0, \dots, f_s\rangle &\mapsto (-1)^{r(s-r)} h(\langle f_{s-r}, \dots, f_s\rangle)\langle f_0, \dots, f_{s-r}\rangle \end{aligned}$$

are defined, where

$$\begin{aligned} C^r(FX, FX \setminus FA) &= \ker\left(\mathrm{Hom}(C_r(FX), R) \to \mathrm{Hom}(C_r(FX \setminus FA), R)\right), \\ C_{s-r}(FX, FA) &= \operatorname{coker}\left(C_{s-r}(FA) \to C_{s-r}(FX)\right). \end{aligned}$$

Proof. If $\langle f_0, \ldots, f_{s-r}\rangle$ is not in $C_*(FA)$, then f_{s-r} is not in A and therefore $\langle f_{s-r}, \ldots, f_s\rangle$ is in $C_*(FX \setminus FA)$, so that $h(\langle f_{s-r}, \ldots, f_s\rangle) = 0$ for the first kind of product or $h(\langle f_{s-r}, \ldots, f_s\rangle)$ is defined for the second kind of product. □

Since $\Delta(FX)$ is just the barycentric subdivison of X, we have $H(C(FX)) \cong H(X)$. Let $o \in C_n(FX)$ represent the orientation class $[X] \in H_n(X)$. Regarding $C(FA)$ as a subcomplex of the singular chain complex $S(A)$, this yields a map $C(FX, FX \setminus FA) \xrightarrow{\frown o} S(A)$ which induces an isomorphism in homology, if A is a full subcomplex. $\Delta(FX \setminus FA)$ is the subcomplex of the barycentric subdivision of X that consists of all simplices which do not meet A. It is the complement of an open normal neighbourhood of A.

1.3.18 Definition and Proposition. *If A, B are subcomplexes of X, a cup product*

$$\begin{aligned} C^r(FX, FX \setminus FA) \otimes C^s(FX, FX \setminus FB) &\to C^{r+s}(FX, FX \setminus F(A \cap B)) \\ g \otimes h &\mapsto g \smile h, \end{aligned}$$

$(g \smile h)(\langle f_0, \ldots, f_{r+s}\rangle) := (-1)^{rs} g(\langle f_0, \ldots, f_r\rangle) h(\langle f_r, \ldots, f_{r+s}\rangle)$, *is defined.*

Proof. If $\langle f_0, \ldots, f_{r+s}\rangle$ is in $C_*(FX \setminus F(A \cap B))$, then $f_0 \notin F(A \cap B)$ and therefore either $f_0 \notin FA$ and $\langle f_0, \ldots, f_r\rangle \in C(FX \setminus FA)$ or $f_0 \notin FB$ and $\langle f_r, \ldots, f_{r+s}\rangle \in C(FX \setminus FB)$. In either case $(g \smile h)(\langle f_0, \ldots, f_{r+s}\rangle) = 0$. □

We now define $Y \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{Q^o}$ by $Y(p)_r := C^{-r}(FX, FX \setminus Fp)$. We equip this with the $\wedge$–product given by the cup product just defined. This also defines products on $Y \otimes_Q K$ and $Y \otimes_Q B(Q) \otimes_Q K$ by Definition 1.2.22 and Definition 1.2.23.

The product on the double complex $Y \otimes_Q B(Q) \otimes_Q K$ makes its second spectral sequence into a multiplicative spectral sequence with the multiplication on the E^1-term $H(Y) \otimes_Q B(Q) \otimes_Q K$ isomorphic to

$$\begin{aligned} &[\alpha \otimes \langle p_0, \ldots, p_r\rangle \otimes m] \otimes [\beta \otimes \langle q_0, \ldots, q_s\rangle \otimes m'] \\ &\qquad \mapsto (-1)^{rl}[(\alpha \smile \beta) \otimes (\langle p_0, \ldots, p_r\rangle \mathbin{\hat{\times}} \langle q_0, \ldots, q_s\rangle) \otimes (m \cdot m'), \end{aligned}$$

where $\alpha \in H^k(X, X \setminus p)$, $\beta \in H^l(X, X \setminus q)$, $\alpha \smile \beta \in H^{k+l}(X, X \setminus (p \cap q))$, $m \in K(p')$, $m' \in K(q')$, $m \cdot m' \in K(p' \wedge q')$.

Now $\frown o$ is a Q^o-chain-map (of degree n) from Y to $S(D)$, inducing isomorphisms

$$H^*(X, X \setminus p) \cong H(Y(p)) \xrightarrow[\cong]{\frown[X]} H(S(D(p))) = H_*(p)$$

for all p. It therefore induces an isomorphism between the second spectral sequences of the double complexes $Y \otimes_Q B(Q) \otimes_Q K$ and $S(D) \otimes_Q B(Q) \otimes_Q K$ from the E^1-terms on. We use this isomorphism to make the spectral sequence of $S(D) \otimes_Q B(Q) \otimes_Q K$ multiplicative. Since $\frown [X]$ takes cup products into intersection products, this already proves the first part of the proposition.

1.3.19 Proposition. *The maps*

$$\begin{aligned} Y \otimes_Q K^u &\to C^*(FX, FX \setminus F \bigcup \mathcal{A}), \\ Y \otimes_Q K^p &\to C^*(FX \setminus F \bigcup \mathcal{A}), \\ [f \otimes k] &\mapsto kf \end{aligned}$$

are well defined and respect products.

Proof. $Y(\top) \otimes K^u(\top) = 0$ and for $q < \top$ the complex $Y(q) = C^*(FX, FX \setminus Fq)$ is a subcomplex of $C^*(FX, FX \setminus F \bigcup \mathcal{A})$, since $q \subset \bigcup \mathcal{A}$. Therefore the map $Y \otimes_Q K^u \to C^*(FX, FX \setminus F \bigcup \mathcal{A})$ is well defined.

$Y(q) \otimes K^p(q) = 0$ for $q < \top$ and $Y(\top) = C^*(FX)$ restricts to $C^*(FX \setminus F \bigcup \mathcal{A})$. Let $q < \top$, $k \in K^p(\top)$, $f \in Y(q) = C^*(FX, FX \setminus Fq)$. Then $Y(q \leftarrow \top) f \otimes k = f \otimes K^p(q \leftarrow \top) k = 0$ and f restricts to 0 in $C^*(FX \setminus F \bigcup \mathcal{A})$. Therefore the map $Y \otimes_Q K^p \to C^*(FX \setminus F \bigcup \mathcal{A})$ is well defined.

Both maps respect products because of the naturality of cup products. □

The multiplication on E^∞ is induced by the multiplication on the homology of the double complex. In the commutative diagram

$$\begin{array}{ccc}
H(Y \otimes_Q B(Q) \otimes_Q K^p) & \xrightarrow[\cong]{H((\frown o)\otimes \mathrm{id} \otimes \mathrm{id})} & H(S(D) \otimes_Q B(Q) \otimes_Q K^p) \\
\downarrow{\scriptstyle H(\mathrm{id}_Y \otimes \varepsilon)} & & {\scriptstyle \cong}\downarrow{\scriptstyle H(\mathrm{id}_{S(D)} \otimes \varepsilon)} \\
H(Y \otimes_Q K^p) & \xrightarrow[\cong]{H((\frown o)\otimes \mathrm{id})} & H(S(D) \otimes_Q K^p) \\
\downarrow & & {\scriptstyle \cong}\downarrow \\
H^*(X \setminus \bigcup \mathcal{A}) & \xrightarrow{\frown [X]} & H_*(X, \bigcup \mathcal{A})
\end{array}$$

the maps on the left are ring homomorphisms, while the map at the bottom takes cup products into cap products. This proves the second part of the proposition for K^p and $H_*(X, \bigcup \mathcal{A})$. The corresponding diagram with K^u, $H^*(X, X \setminus \bigcup \mathcal{A})$, and $H_*(\bigcup \mathcal{A})$ completes the proof of Proposition 1.3.15 □

Product formulas

We apply Proposition 1.3.15 to a class of arrangements for which the E^2-term of the spectral sequence is isomorphic to the homology of the arrangement.

1.3.20 Proposition. *Assume that there is* $Z \in \mathfrak{d}R\text{-}\mathfrak{Mod}^{Q^o}$ *satisfying the condition of Proposition 1.2.18 and a* $Z\check{Z}$*–map* $\zeta\colon Z \otimes_Q B(Q) \to S(D)$*. Then*

$$\begin{aligned} H(H(D) \otimes_Q B(Q) \otimes_Q K) &\xleftarrow{\cong} H(H(Z) \otimes_Q B(Q) \otimes_Q K) \\ &\xrightarrow[\alpha]{\cong} H(Z \otimes_Q B(Q) \otimes_Q K) \qquad (1.8) \\ &\xrightarrow{\cong} H(S(D) \otimes_Q B(Q) \otimes_Q K), \end{aligned}$$

the isomorphism α *being that from Proposition 1.2.18 (and trivial, if the boundary map in* Z *equals zero) and the other two induced by* ζ*: The first arrow by the isomorphism* $H(Z) \cong H(D)$ *described in Definition 1.2.12 and the last one by Proposition 1.2.17. We denote the composition of these isomorphisms by* $\tilde{\phi}$ *and decompose the resulting isomorphism* $\phi\colon H(H(D) \otimes_Q \otimes B(Q) \otimes_Q K^p) \to H_*(X, \bigcup \mathcal{A})$ *as* $\phi = \sum_k \phi_k$ *with*

$$\phi_k\colon H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^p) \to H_{k+r}\left(X, \bigcup \mathcal{A}\right). \qquad (1.9)$$

Then

$$\phi_k(x) \bullet \phi_l(y) - \phi_{k+l-n}(x \cdot y) \in \bigoplus_{i>k+l-n} \operatorname{im} \phi_i, \qquad (1.10)$$

where $x \cdot y$ *denotes the product induced by* (1.7)*.*

Proof. The filtration defining the spectral sequence in Proposition 1.3.15 induces a filtration on $H(S(D) \otimes_Q B(Q) \otimes_Q K^p)$ and we will first identify its image in $H_*(X, \bigcup \mathcal{A})$. On $H(Z \otimes_Q B(Q) \otimes_Q K^p)$ the filtration given by the degree of $B(Q)$ is

$$F_k(H_t(Z \otimes_Q B(Q) \otimes_Q K^p)) = \bigoplus_{i=0}^{k} \alpha\left[H_i(H_{t-i}(Z) \otimes_Q B(Q) \otimes_Q K^p)\right].$$

The map $L(\zeta) \otimes \mathrm{id}_{K^p}$ induces an isomorphism respecting filtrations, therefore the filtration on $H_*(X, \bigcup \mathcal{A})$ induced by that on $H(S(D) \otimes_Q B(Q) \otimes_Q K^p)$ is

$$F_k\left(H_t\left(X, \bigcup \mathcal{A}\right)\right) = \bigoplus_{i=0}^{k} \phi_{t-i}\left[H_i(H_{t-i}(D) \otimes_Q B(Q) \otimes_Q K^p)\right].$$

For $x \in H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^p)$, $\tilde{\phi}(x) \in F_r(H_{k+r}(S(D) \otimes_Q B(Q) \otimes_Q K^p))$. The E^∞-term $F_r(H_{k+r}(S(D) \otimes_Q B(Q) \otimes_Q K^p))/F_{r-1}(H_{k+r}(S(D) \otimes_Q B(Q) \otimes_Q K^p))$ equals the E^2-term $H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^p)$, and the class that $\tilde{\phi}(x)$ represents in the E^∞-term is again x. From this and Proposition 1.3.15 it follows that for $y \in H_s(H_l(D) \otimes_Q B(Q) \otimes_Q K^p)$ we have

$$\begin{aligned} \phi_k(x) \bullet \phi_l(y) - \phi_{k+l-n}(x \cdot y) &\in F_{r+s-1}\left(H_{k+l-n+r+s}\left(X, \bigcup \mathcal{A}\right)\right) \\ &\subset \bigoplus_{i=0}^{r+s-1} \operatorname{im} \phi_{k+l-n+r+s-i} = \bigoplus_{i=1}^{r+s} \operatorname{im} \phi_{k+l-n+i} \subset \bigoplus_{i>k+l-n} \operatorname{im} \phi_i \end{aligned}$$

as stated. □

In Proposition 1.3.15 the case of $K = K^u$, i.e. of intersection products in $H_*(\bigcup\mathcal{A})$, is the less interesting one.

1.3.21 Proposition. *The product*

$$H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^u) \otimes H_s(H_l(D) \otimes_Q B(Q) \otimes_Q K^u) \\ \to H_{r+s}(H_{k+l-n}(D) \otimes_Q B(Q) \otimes_Q K^u)$$

is zero except for $r = s = 0$.

Proof. Let $x \in H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^u)$, $y \in H_s(H_l(D) \otimes_Q B(Q) \otimes_Q K^u)$. The sequence

$$H_{r+1}(H_k(D) \otimes_Q B(Q) \otimes_Q R^p) \xrightarrow{\mathfrak{d}} H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K^u) \\ \to H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K)$$

is exact. If $r > 0$, then $H_r(H_k(D) \otimes_Q B(Q) \otimes_Q K) = 0$ and therefore $z \in H_{r+1}(H_k(D) \otimes_Q B(Q) \otimes_Q R^p)$ exists with $\mathfrak{d}z = x$. It is easily checked that with the obvious definition of the $\wedge$–product $R^p \otimes R^u \to R^u$ it follows that $xy = (\mathfrak{d}z)y = \mathfrak{d}(zy) = 0$. ☐

We will, however, often be able to get a better result on intersection products in $H_*(\bigcup\mathcal{A})$ without resorting to Proposition 1.3.15.

1.3.22 Proposition. *We assume the situation of Proposition 1.3.20 and define* $\phi_k \colon H_r(H_k(D) \otimes_Q B(Q) \otimes K^u) \to H_{k+r}(\bigcup\mathcal{A})$ *analogously. Then we have* $\phi_k(x) \bullet \phi_l(y) = 0$, *unless* $|x| = |y| = 0$, *in which case the product is determined by*

$$\phi_k([a \otimes \langle p\rangle \otimes m]) \bullet \phi_l([b \otimes \langle q\rangle \otimes m']) = \phi_{k+l-n}([(a \bullet b) \otimes \langle p \wedge q\rangle \otimes mm']).$$

Proof. Assume that $|x| > 0$. As in the preceding proof, x maps to zero in $H(H(D) \otimes_Q B(Q) \otimes_Q R)$. Since the isomorphisms in (1.8) are natural with respect to K, the diagram

$$\begin{array}{ccc} H(H(D) \otimes_Q B(Q) \otimes_Q R^u) & \longrightarrow & H(H(D) \otimes_Q B(Q) \otimes_Q R) \\ \downarrow \phi & & \downarrow \\ H_*(\bigcup\mathcal{A}) & \xrightarrow{i_*} & H_*(X) \end{array}$$

commutes. It follows that $i_*(\phi(x)) = 0$. From this it follows that $\phi(x) \in \operatorname{im}\Big(H_*(X, \bigcup\mathcal{A}) \xrightarrow{\mathfrak{d}} H_*(\bigcup\mathcal{A})\Big)$ and therefore $\phi(x) \bullet \phi(y) = 0$, since $\phi(x) \bullet \phi(y) \in \operatorname{im}\Big(H_*(\bigcup\mathcal{A}, \bigcup\mathcal{A}) \xrightarrow{\mathfrak{d}} H_*(\bigcup\mathcal{A})\Big)$.

From the commutativity of (1.5), it follows that

$$\phi_k([a \otimes \langle p\rangle \otimes m]) \bullet \phi_l([b \otimes \langle q\rangle \otimes m']) = mi_*^p(a) \bullet m' i_*^q(b) = \\ = mm' i_*^{p\wedge q}(a \bullet b) = \phi_{k+l-n}([(a \bullet b) \otimes \langle p \wedge q\rangle \otimes mm']),$$

where $i^p \colon D(p) \to X$ is the inclusion map. ☐

The intersection product in $H_*(X, \bigcup\mathcal{A})$ is much more interesting and will be our object of study in concrete classes of arrangements.