Introduction

This work is concerned with homotopy and homology properties of arrangements. An *arrangement* in a topological space X is a finite set $\mathcal{A}$ of subspaces of X. One goal in the study of arrangements is the description of the *union* $\bigcup \mathcal{A}$ and the *complement* $X \setminus \bigcup \mathcal{A}$. For a linear subspace arrangement in real or projective space the intersection of the union of the arrangement with the unit sphere is also of interest. It is called the *link* of the arrangement. The link can be regarded as the union of an arrangement of spheres. By Alexander duality, the homology groups of the link determine the cohomology groups of the complement of the arrangement. They do not determine the multiplication in the cohomology ring.

Main result

This work treats the general theory of arrangements as well as properties of linear arrangements. Its main contribution however is the description of the cohomology ring of the complement of a complex projective arrangement. We will describe this result here in some detail, which will give us the opportunity to introduce some terminology.

Let V be a finite dimensional vector space over $\mathbb{C}$ and let $\mathcal{A}$ be a finite set of linear subspaces of V. We denote the complex dimension of the projective space PV by n and set $P\mathcal{A} := \{PA \colon A \in \mathcal{A}\}$. We define $Q := \{\bigcap M \colon M \subset \mathcal{A}\}$ and order this set by inclusion. The partially ordered set (poset) defined in this way is called the *intersection poset* of $\mathcal{A}$. On it the *dimension function* d is defined by $d(q) := \dim Pq$. In particular $d(\top) = d(\bigcap \emptyset) = d(V) = n$. The result will be an explicit description of the cohomology ring of the complement of the projective arrangement $P\mathcal{A}$ in terms of the intersection poset and the dimension function.

Additively the cohomology of the complement is given by

$$H^{2n-i}\left(PV \setminus \bigcup P\mathcal{A}\right) \cong H_i\left(PV, \bigcup P\mathcal{A}\right) \cong \bigoplus_{k=0}^{n} H_{i-2k}(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}). \quad (1)$$

Here $Q_{[k,n]} := \{q \in Q \colon k \le d(q) \le n\}$, $Q_{[k,n)} := \{q \in Q \colon k \le d(q) < n\}$ and for a poset P the *order complex of* P, i.e. the simplicial complex with simplices all chains in P, is denoted by ΔP. The first isomorphism above is Poincaré-Lefschetz-duality, and the second isomorphism will be described explicitly by maps $h_k\colon H_{i-2k}(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to H_i\,(PV, \bigcup P\mathcal{A})$.

On the ordered simplicial chain complex C_* of ΔQ a product $\hat{\times}$ is defined as the composition

$$\hat{\times}\colon C_r(\Delta Q)\otimes C_s(\Delta Q) \xrightarrow{\times} C_{r+s}(\Delta Q\times\Delta Q) = C_{r+s}(\Delta(Q\times Q)) \xrightarrow{\wedge_*} C_{r+s}(\Delta Q), \quad (2)$$

where $\wedge\colon Q\times Q \to Q$ is the map taking (u,v) to the minimum $u\cap v$. The map $\wedge$ is order preserving and hence a simplicial map. The product $\hat{\times}$ induces products in homology. Denoting the intersection product on the homology of $(PV, \bigcup P\mathcal{A})$ that corresponds via duality to the cup product on the cohomology of the complement $PV\setminus\bigcup P\mathcal{A}$ by $\bullet$ the cohomology ring will be fully described by (1) together with the following formula.

Theorem. *For* $c\in H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})$ *and* $d\in H_*(\Delta Q_{[l,n]}, \Delta Q_{[l,n)})$

$$h_k(c)\bullet h_l(d) = \begin{cases} h_{k+l-n}(c\,\hat{\times}\,d), & k+l\geq n,\\ 0, & k+l<n. \end{cases} \quad (3)$$

This will appear as Theorem 2.2.1 in this work. Since the intersection product is a composition of the cross product and the transfer of the diagonal map, the connection between it and the product $\hat{\times}$ seems very natural in the light of (2).

History

We mention parts of the history of this subject that build a suitable context for the description of the content of this work.

Arnol'd has given a simple presentation in terms of generators and relations for the cohomology ring of the classifying space of the coloured braid group [Arn69]. This classifying space is the complement of the arrangement $\{\{z\colon z_i = z_j\} : i\neq j\}$ in $\mathbb{C}^n$, i.e. it is the complement of a linear complex hyperplane arrangement. This result has been extended to arbitrary linear complex hyperplane arrangements, where the cohomology is described by the Orlik-Solomon algebra of the intersection poset [OS80]. Several generalizations for other classes of complex linear subspace arrangements have been obtained afterwards.

For an arbitrary complex linear and projective subspace arrangement Goresky and MacPherson have given descriptions of the cohomology groups of the complement in terms of the intersection poset and the dimension function as an application of their stratified Morse theory [GM88]. A formula equivalent to (1) appears in that work. Ziegler and Živaljević have given a concrete homotopy equivalence between a space determined by the intersection poset and the dimension function and the link of a linear arrangement from which the homology formula can be read of [ZŽ93]. Their approach is to view an arrangement as a diagram of spaces (and inclusion maps). They have given an overview of homotopy theoretic tools which are useful in this setting and further applications to problems in combinatorics in [WZŽ99].

De Concini and Procesi produced rational models to show that the cohomology rings with rational coefficients of the complements of complex linear arrangements are determined by the intersection poset and the dimension function [DCP95]. Yuzvinsky used these models to endow the homology formulas given by Goresky and MacPherson for these arrangements with a combinatorially defined product which describes the cohomology ring of the complement of a complex linear arrangement [Yuz02]. This product is equivalent to the product $\hat{\times}$ defined above under an isomorphism shifting dimensions by two, although the connection with the cross product was not made explicit. Starting from this description of the cohomology ring he was in a position to attack the problem of giving presentations in terms of generators and relations for special classes of arrangements in a purely combinatorial way and he generalized previous results [Yuz99]. His results were however, by the nature of the rational models which were at the foundations of this, confined to complex arrangements and cohomology with rational coefficients. The generalization of the product formula to integral coefficients and a class of real linear arrangements containing all complex linear arrangements was done independently by Deligne, Goresky and MacPherson [DGM00] and by de Longueville and the current author [dLS01]. The latter work uses quite explicit geometrical constructions for which it is important that the homology isomorphism are induced by the topological maps of Ziegler and Živaljević. It also introduces the product $\hat{\times}$ in the form above.

Leitfaden

In Chapter 1 we deal with general arrangements in topological spaces. We first give a minimal overview over homotopy properties of diagrams of spaces. We then develop a corresponding theory of diagrams of chain complexes suitable to the study of homology properties of arrangements. This section features a spectral sequence that will be crucial in studying products later on. In the third section we show how to apply the results presented so far to the study of arrangements. Possibly new is the proof that a product formula like Yuzvinsky's holds quite generally for the cohomology ring of the complement of an almost arbitrary arrangement in a manifold, albeit only in the graded object defined by the filtration of the cohomology ring induced by the spectral sequence. This graded formula will be be the basis for proofs of exact formulas in the second chapter. In the case of projective arrangements discussed above, such a graded formula would describe $h_k(c) \bullet h_l(d)$ only up to elements $h_i(r_i)$ with $i > k + l - n$.

In Chapter 2 we enter the more concrete realm of linear arrangements. In Section 2.1 we prove several homology and also a few homotopy formulas for central linear, projective and affine arrangements, among them (1). While the isomorphisms and homotopy equivalences are constructed in a uniform manner. Still some redundancy is to be expected, as the aim is to demonstrate the use of the tools from Chapter 1 and connections between the different isomorphisms. There is probably nothing really new in that section, except perhaps that some things may not have been made quite so explicit before, including the connections between affine and projective arrangements and the homotopy equivalence for projective arrangements in Proposition 2.1.17. The topological maps forming this homotopy equivalance also induce the isomorphisms h_k defined above. To have such explicit descriptions of them makes our approach to the calculations of products possible.

In Section 2.2 we turn to determining the products in the cohomology rings of complements of linear arrangements. We first state the results for affine and projective arrangements. The product formula in Theorem 2.2.3 is the main result of [dLS01]. The corresponding formula for projective arrangements in Theorem 2.2.1 is the one discussed above. We then prove graded versions of these formulas by identifying them as special cases of a result from the first chapter. We show how the exact formulas can be derived from these by an inductive argument, if the vanishing of certain products can be guaranteed. An easy geometric argument then proves this vanishing for affine arrangements. For projective arrangements, this necessary vanishing is just the case $k + l < n$ in (3) above. Its proof costs considerably more effort than in the affine case and takes up Section 2.3.

In Section 2.4 we derive from the product formula for projective arrangements thus proved a presentation of the cohomology ring of a projective c-arrangement. This is done by methods employed by Yuzvinsky in proving presentations for similar classes of linear arrangements.

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