New Applications of Topological Methods in Discrete Geometry

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Contents

Chapter 1

Introduction

One of the first applications of topological methods or, more precisely, *equivariant* topological methods to a problem in discrete geometry was Lovász's 1978 proof [55] of the Kneser conjecture, which was simplified a few months later by Bárány [7]. The Kneser conjecture, which was posed as an exercise/problem by Kneser in 1956 [51], states that one needs at least $\ell + 2$ colors for a vertex coloring of the Kneser graph $\mathrm{KG}_{k,\ell}$. The Kneser graph $\mathrm{KG}_{k,\ell}$ is the graph with vertices given by the k-element subsets of the set $\{1, \ldots, 2k + \ell\}$ and edges given by pairs of vertices whose corresponding k-element subsets are disjoint. Both proofs of the Kneser conjecture rely on the Borsuk–Ulam theorem, which states that there is no $\mathbb{Z}/2$-equivariant map $S^\ell \to S^{\ell-1}$ from a sphere of dimension ℓ to a sphere of dimension $\ell - 1$, when the group $\mathbb{Z}/2$ acts antipodally.

Let us consider the proof of the Kneser conjecture by Bárány. We need to show that there is no vertex coloring of $\mathrm{KG}_{k,\ell}$ with $\ell + 1$ colors. A result due to Gale [9, Thm. 1] implies that for any two non-negative integers k and ℓ there is a set X of $2k + \ell$ points on the sphere S^ℓ such that every open hemisphere contains at least k points from X. Identify the vertices of $\mathrm{KG}_{k,\ell}$ with the k-element subsets of X. Assume that the vertices of $\mathrm{KG}_{k,\ell}$ can be colored by $\ell + 1$ colors. Define a covering of S^ℓ by open sets $U_i \subset S^\ell$ for $i = 1, \ldots, \ell + 1$ by letting U_i consist of the normal vectors of open hemispheres that contain a vertex of $\mathrm{KG}_{k,\ell}$ of color i. Take a closed covering $V_1, \ldots, V_{\ell+1}$ of S^ℓ with $V_i \subset U_i$ for all i and define a $\mathbb{Z}/2$-equivariant map $f\colon S^\ell \to \mathbb{R}^\ell$ by $f(x) = (\mathrm{dist}(x, V_1) - \mathrm{dist}(-x, V_1), \ldots, \mathrm{dist}(x, V_\ell) - \mathrm{dist}(-x, V_\ell))$. Then $f(x) = 0$ implies that there is an index i_0 with $1 \leq i_0 \leq \ell + 1$ such that the set V_{i_0} contains both x and $-x$. This means that $\mathrm{KG}_{k,\ell}$ has two vertices of the same color in opposite open hemispheres and hence that $\mathrm{KG}_{k,\ell}$ has a monochromatic edge, leading to a contradiction. On the other hand, if f is never zero, then composing f with the radial retraction yields a $\mathbb{Z}/2$-equivariant map $S^\ell \to S^{\ell-1}$, which contradicts the Borsuk–Ulam theorem.

Other early examples of results in discrete geometry that are proved using topological methods include the topological Tverberg theorem by Bárány, Shlosman, and Szűcs 1981 [10], the later extension of this result by Özaydin 1987 [64], and the proof of the necklace splitting theorem by Alon 1987 [2]. For a survey of these and other results see Matoušek [59]. For further surveys of topological methods with applications to problems in discrete geometry and combinatorics see Björner [14] and Blagojević and Ziegler [29].

Many proofs of theorems in discrete geometry using topological methods follow a common "scheme" that we refer to as the *configuration space/test map scheme*; see [83], [84]. One assumes by way of contradiction that the theorem in question fails and defines from this "failure" an equivariant map $X \to Y$, called the *test map*, between a *configuration space* X and a *test space* Y. Both spaces X and Y are equipped with an action by a non-trivial group G. One then employs tools from equivariant topology to determine that such an equivariant map cannot exist, leading to a contradiction. (We will elaborate on some of these tools below.) We point out that in general the converse implication is false. The existence of a test map $X \to Y$ does not necessarily imply that the theorem is false. In the proof of the Kneser conjecture the spaces X and Y are spheres of dimensions ℓ and $\ell - 1$ and the question of nonexistence of a $\mathbb{Z}/2$-equivariant map $X \to Y$ is answered by the Borsuk–Ulam theorem. Of central importance to this approach are the properties of the group G and its action on the spaces X and Y. Is G finite, or at least a compact Lie group? Is its action free? Does it have fixed points?

There are several ways to approach the questions of existence and nonexistence of equivariant maps $X \to Y$ between G-spaces X and Y. We point out three frequently used approaches that are also used in this dissertation. From a theoretical standpoint they are not quite independent, but in concrete settings they each come with a different set of technical challenges. In the following, we tacitly assume that X and Y are G-spaces.

In the *connectivity-based* approach one argues with an extension of the Borsuk–Ulam theorem known as Dold's theorem [35]; see [77] for a more general version that is also applicable in this context. Dold's theorem asserts that if G is a finite non-trivial group and if X is n-connected and Y is a free G-CW complex of dimension at most n, then a continuous G-equivariant map $X \to Y$ cannot exist. This approach is taken in [10], [70], and in Chapter 4. It can be seen as a special case of the approach by equivariant obstruction theory described below. However, in the connectivity-based approach a concrete CW model for X or Y is not needed, often making the approach easier.

In the *degree-based* approach one argues with an equivariant extension of the Hopf theorem [34, Thm. II.4.11]: If G is finite and if X is a compact oriented n-dimensional free G-manifold and Y is the sphere S^n, then the degrees of any two G-equivariant maps $X \to Y$ are congruent modulo the order of G. One then calculates the degree of the test map and gives an example of an equivariant map $X \to Y$ with a different degree, leading to a contradiction. This approach can only work in the setting where both spaces X and Y are compact orientable manifolds of the same dimension. We take this approach in Chapter 2.

In the approach by *relative equivariant obstruction theory* one argues with an extension of obstruction theory to the equivariant setting; see tom Dieck [34, Sec. II.3]. Here the group G can be infinite as long as it is a compact Lie group. The space X however must be a relative G-CW complex (A, B) with G acting freely on the complement $A \setminus B$. If for some $k < \dim(X)$ the obstruction class in the $(k+1)$-th equivariant cohomology group of the space X does not vanish, then the test map defined on the k-skeleton of X cannot be extended to the $(k+1)$-skeleton of X, implying that it cannot exist as a map from $X \to Y$. The space Y should ideally be k-simple to avoid the use of local coefficients. The difficulty in calculating the obstruction class depends on the complexity of the CW model for the space X and on the action of the group on X. This approach is taken in [28], [31], and in Chapter 3.

One problem in discrete geometry that has been an active testing ground for methods from equivariant topology is the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem. It goes back to Grünbaum 1960 [46, Sec. 4.(v)], who asked if any convex body in $\mathbb{R}^d$ can be cut into 2^d pieces of equal volume by d suitably-chosen affine hyperplanes. For $d \leq 2$ this is an easy consequence of the intermediate value theorem. For $d = 3$ Grünbaum's question was answered positively by Hadwiger in 1966 [48]. In 1984 Avis [4] answered Grünbaum's question negatively for $d \geq 5$. The case $d = 4$ was left open.

Grünbaum's question was independently raised in computational geometry, motivated by the search for structures that efficiently store high-dimensional data. In this context, Willard [80] reproved the case $d = 2$, while the case $d = 3$ was reproved by Yao, Dobkin, Edelsbrunner, and Paterson [81]. In this context Grünbaum's question was extended to the setting where convex bodies are replaced with well-behaved finite Borel measures, called *masses*. Given a collection $\mathcal{M} = \{\mu_1, \ldots, \mu_j\}$ of j masses on $\mathbb{R}^d$ we say that an arrangement $\mathcal{H}$ of k affine hyperplanes in $\mathbb{R}^d$ *equiparts* $\mathcal{M}$, if for every orthant $\mathcal{O}$ defined by $\mathcal{H}$ the measure $\mu_i(\mathcal{O})$ is equal to $\mu_i(\mathbb{R}^d)/2^k$ for all i.

In 1996 Ramos [68] formulated the general version of the hyperplane mass partition problem: Determine the minimal dimension $d = \Delta(j, k)$ such that for every collection $\mathcal{M}$ of j masses on $\mathbb{R}^d$ there exists an arrangement $\mathcal{H}$ of k affine hyperplanes in $\mathbb{R}^d$ that equiparts $\mathcal{M}$.

The special case $\Delta(j, 1) = j$ of the Grünbaum–Hadwiger–Ramos problem for a single hyperplane follows from the ham sandwich theorem, which was conjectured by Steinhaus and proved by Banach in 1938; see [11]. This turns out to be an incarnation of the Borsuk–Ulam theorem. By placing one-dimensional masses along a curve in $\mathbb{R}^d$ of degree d Ramos [68] and Avis [4] obtained lower bounds: $(2^k - 1)j/k \leq \Delta(j, k)$. The best upper bounds to date were obtained by Mani-Levitska et al. [57, Thm. 39]: $\Delta(j, k) \leq j + (2^{k-1} - 1)2^{\lfloor \log_2 j \rfloor}$, where $2^{\lfloor \log_2 j \rfloor}$ is "j rounded down to the nearest power of 2." Thus far, surprisingly few exact values of $\Delta(j, k)$ are known. Section 2.1 contains a survey of exact values and bounds for $\Delta(j, k)$ that have been claimed or proved in the past.

In order to apply a configuration space/test map scheme to the Grünbaum–Hadwiger–Ramos problem we first need a suitable configuration space X. There are several possibilities. An oriented affine hyperplane $\hat{H}$ in $\mathbb{R}^d$ can be parametrized by a point H on the sphere S^d by mapping $\hat{H}$ to $\mathbb{R}^{d+1}$ via the embedding $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 1)$ and then extending its image to a linear hyperplane in $\mathbb{R}^{d+1}$, whose normal vector H lies on the sphere S^d. The north and south poles of S^d correspond to hyperplanes at infinity. One configuration space for k affine hyperplanes in $\mathbb{R}^d$ is given by the k-fold Cartesian product $(S^d)^k$, called the *product configuration space*. This space has low connectivity and low dimension. Another possibility is the k-fold join $(S^d)^{*k}$, called the *join configuration space*. This space has high connectivity and high dimension.

Both configuration spaces have symmetries arising from permuting the order of the hyperplanes and changing their orientations, which corresponds to permuting the spheres and acting antipodally on each sphere. These symmetries are realized by an action of the hyper-octahedral group $(\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$, denoted in the following by $\mathfrak{S}_k^{\pm}$, which can be described as the symmetry group of the k-dimensional cube. The action by $\mathfrak{S}_k^{\pm}$ is not free on either configuration space, since each space contains tuples of points corresponding to the same hyperplane, possibly with opposite orientations. These points are fixed by a permutation and possible orientation change. By deleting the points with non-trivial stabilizer one obtains a *free configuration space*.

By evaluating each of the j masses on each orthant of the hyperplane arrangement and by making a few technical modifications one obtains a $\mathfrak{S}_k^{\pm}$-equivariant map that is zero if and only if the corresponding hyperplane arrangement equiparts the masses. If the masses cannot be equiparted by any arrangement of k affine hyperplanes in $\mathbb{R}^d$ one obtains by radial retraction a $\mathfrak{S}_k^{\pm}$-equivariant test map $X \to Y$ to a sphere Y, the test space, whose dimension depends only on k and j. Hence the nonexistence of an $\mathfrak{S}_k^{\pm}$-equivariant map $X \to Y$ implies that any collection of j masses can be equiparted by an arrangement of k affine hyperplanes in $\mathbb{R}^d$.

In **Chapter 2** we give a critical review of the work on the Grünbaum–Hadwiger–Ramos problem. We point out which results come with valid proofs and which proofs do not hold up under critical inspection. In Sections 2.6, 2.7, and 2.8 we give counterexamples and point out essential gaps. Furthermore, we show that Hadwiger's result [48] remains true if we replace convex bodies by masses, implying that $\Delta(2,2) = 3$; see Section 2.4. Finally, we apply the product scheme in a setting where the degree based approach is possible and obtain the exact values $\Delta(2^t+1,2) = 3\cdot 2^{t-1}+2$ for $t \geq 1$; see Section 2.5.

In **Chapter 3** we apply the join scheme to the Grünbaum–Hadwiger–Ramos problem. We build an efficient equivariant CW model for the configuration space $(S^d)^{*k}$ and exploit the fact that the space is highly connected. By connectivity, all obstruction classes aside from a critical obstruction class that admits a combinatorial interpretation vanish. This allows us to take a unified approach via relative equivariant obstruction based on calculating the critical obstruction class. This yields several new as well as already known exact values of $\Delta(j,k)$; see Theorems 3.5 and 3.6: We retrieve the exact values of $\Delta(2^t-1,2)$ due to [57] and the exact values of $\Delta(2^t+1,2)$ that were obtained in Chapter 2. We recover the exact values of $\Delta(2^t,2)$ that were previously claimed by Mani-Levitska et al. [57, Prop. 25]. Finally, we calculate the exact value $\Delta(2,3) = 5$ that was previously claimed by Ramos and obtain the new exact value $\Delta(4,3) = 10$.

The relatively few known exact values of $\Delta(j,k)$ nevertheless seem to support the following conjecture due to Ramos [68]: $\Delta(j,k) = \lceil \frac{2^k-1}{k} j \rceil$ for every $j \geq 1$ and $k \geq 1$. Perhaps the most notorious case of the conjecture is the situation in dimension 4, which was already noted by Avis [4]. It is known that $4 \leq \Delta(1,4) \leq 5$, but none of the standard approaches seem capable of deciding whether $\Delta(1,4)$ is equal to 4 or 5.

The study of Tverberg-type problems has played a central role in developing topological methods for applications to problems in discrete geometry. We say that a problem or result is of *Tverberg-type* if it is related to the following theorem from 1966 due to Tverberg [75]: For any affine map $f\colon \Delta_{(k-1)(d+1)} \to \mathbb{R}^d$ from a simplex of dimension $(k-1)(d+1)$ to $\mathbb{R}^d$ there is a collection $\{\sigma_1,\dots,\sigma_k\}$ of k pairwise disjoint faces of $\Delta_{(k-1)(d+1)}$ such that $\bigcap_{i=1}^k f(\sigma_i) \neq \emptyset$. We call such a collection $\{\sigma_1,\sigma_2,\dots,\sigma_k\}$ of faces a *Tverberg k-partition.* By a codimension argument one verifies that the dimension of the simplex is minimal for the implication of the theorem to be true. In an equivalent version, Tverberg's theorem states that any set of $(k-1)(d+1)+1$ points in $\mathbb{R}^d$ can by partitioned into k sets whose convex hulls have a common point of intersection. Tverberg's theorem for $k=2$ is known as Radon's theorem [67] and already has strong implications in discrete geometry. For example, it yields that any k-neighborly d-polytope for $k > \lfloor d/2 \rfloor$ is combinatorially equivalent to a d-simplex [47, Sec. 7.1]. A more advanced result that can obtained by using Tverberg's theorem is a version of the celebrated Hadwiger–Debrunner (p,q)-problem [49] from 1957 established by Alon and Kleitman 1992 [3]: Let $p \geq q \geq d+1$ and let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that among any p sets of $\mathcal{F}$ there are q sets with a common point. Then there exists a *transversal number* $\mathrm{t}(p,q,d)$, also called the *piercing number*, such that there is a set $X \subseteq \bigcup \mathcal{F}$ of cardinality at most $\mathrm{t}(p,q,d)$, called a *transversal*, that intersects all elements of $\mathcal{F}$. Matoušek gives a modern exposition of this proof in [58, Sec. 10.5]. See [36, Sec. 9] for further implications of Tverberg's theorem and an early survey of related results.

In 1981 Bárány, Shlosman, and Szűcs extended Tverberg's theorem [10]: They showed that if k is prime, then any continuous map $f\colon \Delta_{(k-1)(d+1)} \to \mathbb{R}^d$ has a Tverberg k-partition. This result is known as the *topological Tverberg theorem* and is one of the landmark applications of topological methods to a problem in discrete geometry. It was later reproved by Sarkaria [70] and extended to the case where k is a prime power by Özaydin [64]. The question whether the topological Tverberg theorem is true for $k \geq 1$, known as the *topological Tverberg conjecture*, remained open until recently, when Frick [42], [25], using the "constraint method" [24] and building on the work by Mabillard and Wagner [56], showed that the topological Tverberg conjecture is false when k is not a prime power and $d \geq 3k+1$.

The proofs of the topological Tverberg theorem mentioned above all use configuration space/test map schemes. In the proof by Bárány, Shlosman, and Szűcs and in the proof by Özaydin the configuration space X is the k-fold deleted product $(\Delta_{(k-1)(d+1)})^{\times k}_{\Delta}$ of the simplex $\Delta_{(k-1)(d+1)}$, called the *product configuration space*. In the proof by Sarkaria the configuration space X is the k-fold deleted join $(\Delta_{(k-1)(d+1)})^{*k}_{\Delta}$ of the simplex $\Delta_{(k-1)(d+1)}$, called the *join configuration space*. The test spaces in both cases are spheres, whose dimensions grow as k and d increase. In the case where k is prime, all spaces and in particular the two spheres (which are odd-dimensional if $k > 2$) admit a free action by the group $\mathbb{Z}/k$. The topological Tverberg theorem for k prime is then obtained by showing that the connectivity of the configuration space is at least as high as the dimension of the test space. If k is a prime power but not prime, one does not have a free action on the test space Y and hence a different result is needed to show nonexistence of the test map; see [64, Lem. 4.2] or [77, Lem. 1].

There are a number of interesting Tverberg-type results that follow directly from the topological Tverberg theorem by using the constraint method and applying the theorem as a black box. With this method one obtains a weak colored Tverberg theorem [24, Thm. 5.3], which states that if we color the vertices of the simplex $\Delta_{(k-1)(2d+2)}$ with $d+1$ colors such that each color class has cardinality at most $2k-1$, then in the case where k is a prime power any continuous map $f\colon \Delta_{(k-1)(2d+2)} \to \mathbb{R}^d$ has a Tverberg k-partition $\{\sigma_1, \dots, \sigma_k\}$ such that each σ_i has at most one vertex of each color. Using the same method one can show that given a simplex of somewhat larger dimension one can impose restrictions on the dimensions of the simplices σ_i in Tverberg k-partitions when k is a prime power [24, Thm. 6.5]. This result implies the van Kampen–Flores theorem [76], [40], which states that if $d \geq 2$ is even, then for any continuous map $f\colon \Delta_{d+2} \to \mathbb{R}^d$ there are two faces $\sigma_1, \sigma_2 \subset \Delta_{d+2}$ each of dimension at most $d/2$ such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$. As a special case we obtain the non-planarity of the complete graph on 5 vertices. See [59] and [8] for recent surveys of these and other (topological) Tverberg-type results.

Our starting point in **Chapter 4** is the recent Tverberg-type result for matroids by Bárány, Kalai, and Meshulam [9, Thm. 1]. They introduced the *topological Tverberg number* $\mathrm{TT}(M,d)$ of a matroid M as the maximal integer $k \geq 1$ such that any continuous map $f\colon M \to \mathbb{R}^d$ has a Tverberg k-partition, where a matroid is viewed as the simplicial complex given by its independent sets. If k

is a prime power and M is the uniform matroid $\Delta_{(k-1)(d+1)}$ of rank $(k-1)(d+1)+1$, then the topological Tverberg theorem implies that $\mathrm{TT}(\Delta_{(k-1)(d+1)}, d) = k$. By approximating f with a general position map Schöneborn and Ziegler [71, Prop. 2.2] showed that the topological Tverberg theorem implies the stronger statement $\mathrm{TT}(\Delta^{(d)}_{(k-1)(d+1)}, d) = k$ for the d-skeleton of the simplex. Bárány, Kalai, and Meshulam showed that for an arbitrary matroid of rank $r = d+1$ with b disjoint bases the topological Tverberg number satisfies $\mathrm{TT}(M, d) \geq \sqrt{b}/4$ [9, Thm. 1]. For the d-skeleton of the simplex this result implies that $\mathrm{TT}(\Delta^{(d)}_{(k-1)(d+1)}, d) \geq \sqrt{k-1}/4$. In the proof of [9, Thm. 1] the join scheme is used, where the configuration space given by the k-fold deleted join M^{*k}_{Δ} of the matroid M. By [9, Cor. 3] the connectivity of M^{*k}_{Δ} is at least $br/(\lceil b/k \rceil + 1) - 2$ for any $k \geq 1$, where r denotes the rank of M. The result then follows by applying Dold's theorem.

The proofs of the topological Tverberg theorem show that the connectivity-based approach yields tight bounds for the topological Tverberg number of the simplex skeleton $\Delta^{(d)}_{(k-1)(d+1)}$ when k is a prime power. The questions we are concerned with regard the connectivity-based approach in the matroid case: What is the connectivity of the configuration spaces? Which results can be obtained by a connectivity-based approach and which results cannot?

In Theorem 4.2 we give an example of a family of matroids M_r of rank $r \geq 2$ with r disjoint bases such that the connectivity of the 2-fold deleted join $(M_r)^{*2}$ is $2r-3$, while its dimension is $2r-1$. This disproves a conjecture by Bárány, Kalai, and Meshulam [9, Conj. 4] and shows that the connectivity of the deleted join of a matroid is not independent of its number of disjoint bases. In Theorem 4.20 we show that the connectivity of the k-fold deleted product $M^{\times k}_{\Delta}$ of a matroid M of rank r grows as the number of disjoint bases is increased and stabilizes when it reaches the value $(r-2)$. In particular it does not increase as k increases. Since the dimension of the test space (a sphere) grows as k is increased, this shows that a connectivity-based approach involving the product scheme will not yield good results.

In Theorem 4.3 we show, using a Fadell–Husseini index argument, a sharp Radon theorem for the counterexample family of matroids M_r. This yields better bounds for the topological Tverberg number $TT(M_r, d)$ than can be obtained with a connectivity-based approach. We thus show that the connectivity-based approach does not yield the best bounds for the topological Tverberg number. A similar phenomenon can be observed in the case of the optimal colored Tverberg theorem [28], where the configuration space, a "chessboard complex", has low connectivity. Finally, in Section 4.4.1 we deduce from [9, Cor. 3] lower bounds for the topological Tverberg number of matroids of arbitrary rank and provide upper bounds in the case when the rank r is at most $d-2$.

Several open questions concerning Tverberg-type results for matroids remain. Our method of proof of Theorem 4.2 fails when $k > 2$; see Section 4.4.4. Can we show for $k > 2$ that $(M_r)^{*k}_{\Delta}$ has a similar connectivity drop? Computations seem to suggest that this may be correct. Can we perhaps find a different family for which the connectivity drop is easier to show? More generally, we can ask if the connectivity bound by Bárány, Kalai, and Meshulam [9, Cor. 3] is optimal. Ultimately, how do we obtain optimal bounds for the topological Tverberg number of a matroid? The evidence suggests that we have to take an approach that is not based on connectivity alone. We can also ask if it even makes sense to expect an optimal result for a family of simplicial complexes as general as matroids. If not, what is the right family?

Part I

The Grünbaum–Hadwiger–Ramos hyperplane mass partition problem

Chapter 2

A critical review

Abstract

In 1960 Grünbaum asked whether for any finite mass in $\mathbb{R}^d$ there are d hyperplanes that cut the mass into 2^d equal parts. This was proved by Hadwiger (1966) for $d \leq 3$, but disproved by Avis (1984) for $d \geq 5$, while the case $d = 4$ remained open. More generally, Ramos (1996) asked for the smallest dimension $d_0 = \Delta(j, k)$ such that for any j masses in $\mathbb{R}^{d_0}$ there are k hyperplanes that cut each of the masses into 2^k equal parts. At present the best lower bounds on $\Delta(j, k)$ are provided by Avis (1984) and Ramos (1996), the best upper bounds by Mani-Levitska, Vrećica and Živaljević (2006). Ramos' conjecture is that the Avis–Ramos necessary lower bound condition $\Delta(j, k) \geq j(2^k - 1)/k$ is also sufficient. The problem has been an active testing ground for advanced machinery from equivariant topology. We give a critical review of the work on the Grünbaum–Hadwiger–Ramos problem, which includes the documentation of essential gaps in the proofs for some previous claims. Furthermore, we establish that $\Delta(j, 2) = \frac{1}{2}(3j + 1)$ in the cases when $j - 1$ is a power of 2 and $j \geq 5$.

Publication Remark. The results of this chapter are joint work with Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler [22].

2.1 Introduction

Our starting point is the following problem that is due to Grünbaum [46, Sec. 4.(v)], Hadwiger [48], and Ramos [68].

The Grünbaum–Hadwiger–Ramos problem. *Determine the minimal dimension $d = \Delta(j, k)$ such that for every collection $\mathcal{M}$ of j masses on $\mathbb{R}^d$ there exists an arrangement $\mathcal{H}$ of k affine hyperplanes in $\mathbb{R}^d$ that equiparts $\mathcal{M}$.*

It turns out that the most natural configuration spaces parameterizing k-tuples of oriented affine hyperplanes are products of spheres, such as $(S^d)^k$, which do not have the high connectivity that is required for a simple application of Borsuk–Ulam-type machinery, for example via Dold's Theorem; see Matoušek [59] for an introduction to this approach. Thus more sophisticated machinery is needed in order to decide about the existence of the equivariant maps proposed by various applications of the configuration space/test map scheme as developed by Sarkaria and Živaljević; see again [59] for an introduction. Methods that have been employed to settle such existence problems include

- equivariant cohomology (the Fadell–Husseini index [38]),
- equivariant obstruction theory (see tom Dieck [34, Sec. II.3]), and
- the normal bordism approach of Koschorke [52].

In this chapter we attempt to provide a status report about the partial results obtained for the Grünbaum–Hadwiger–Ramos problem up to now. This in particular includes the lower and upper bounds

$$\left\lceil \tfrac{2^k-1}{k} j \right\rceil \;\leq\; \Delta(j,k) \;\leq\; j + (2^{k-1}-1)2^{\lfloor \log_2 j \rfloor},$$

where $j, k \geq 1$ are integers and $2^{\lfloor \log_2 j \rfloor}$ is j "rounded down to the nearest power of 2," thus $\frac{1}{2}j < 2^{\lfloor \log_2 j \rfloor} \leq j$.

The lower bound was derived by Avis [4] (for $j = 1$) and Ramos [68] from measures concentrated on the moment curve. The upper bound was obtained by Mani-Levitska, Vrećica and Živaljević [57] from a Fadell–Husseini index calculation. A table below will show that there is quite a gap between the lower and the upper bounds – they only coincide in the ham sandwich case $\Delta(j,1) = j$, and in the case of two hyperplanes if $j+1$ is a power of 2, with $\Delta(j,2) = \frac{1}{2}(3j+1)$. All the available evidence up to now is consistent with the expectation that Ramos' lower bound is tight for all j and k; we will refer to this in the following as the *Ramos conjecture*. For example, while the above bounds specialize to $3 \leq \Delta(2,2) \leq 4$, Hadwiger [48] proved that indeed $\Delta(2,2) = 3$.

In addition to the general lower and upper bounds, a number of papers have treated special cases, reductions, and relatives of the problem. As a basis for further work we will in the following provide a critical review of all the key contributions to this study, which will also include short proofs as far as feasible. In this context we have to observe, however, that quite a number of published proofs do not hold up upon critical inspection, and indeed some of the approaches employed cannot work. As some of these errors have not been pointed out in print (although they may be known to experts), we will provide detailed reviews and explanations in these cases.

We have been able to salvage one of these results, with different methods: We will prove below (Theorem 2.12) that $\Delta(j,2) = \frac{1}{2}(3j+1)$ also holds if $j-1$ is a power of 2, $j \geq 5$. So in this case again the Ramos lower bound is tight while the Mani-Levitska et al. upper bound is not. (It is tight in the case $j = 3$.)

2.1.1 Set-up and terminology

Any affine hyperplane $H = H_v(a) = \{x \in \mathbb{R}^d : \langle x, v \rangle = a\}$, given by a vector $v \in \mathbb{R}^d \backslash \{0\}$ and scalar $a \in \mathbb{R}$, determines two closed halfspaces, which we denote by

$$H^0 = \{x \in \mathbb{R}^d : \langle x, v \rangle \geq a\} \qquad \text{and} \qquad H^1 = \{x \in \mathbb{R}^d : \langle x, v \rangle \leq a\}.$$

Let $\mathcal{H}$ be an *arrangement* (ordered tuple) of $k \geq 1$ affine hyperplanes in $\mathbb{R}^d$, and $\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbb{Z}/2)^k = \{0,1\}^k$. The *orthant* determined by the arrangement $\mathcal{H}$ and an element $\alpha \in (\mathbb{Z}/2)^k$ is the intersection of halfspaces

$$\mathcal{O}_\alpha^{\mathcal{H}} = H_1^{\alpha_1} \cap \dots \cap H_k^{\alpha_k}.$$

A *mass* on $\mathbb{R}^d$ is a finite Borel measure on $\mathbb{R}^d$ that vanishes on every affine hyperplane. Without loss of generality we deal only with probability measures (that is, masses such that $\mu(\mathbb{R}^d) = 1$). Examples of masses that appear frequently include

- measures given by the d-dimensional volume of a compact convex body $K \subset \mathbb{R}^d$,
- measures induced by an interval on the moment curve in $\mathbb{R}^d$,
- measures given by a finite family of (small, disjoint) balls.

An arrangement $\mathcal{H} = (H_1, \ldots, H_k)$ *equiparts* a collection of masses $\mathcal{M} = (\mu_1, \ldots, \mu_j)$ if for every element $\alpha \in (\mathbb{Z}/2)^k$ and every $\ell \in \{1, \ldots, j\}$

$$\mu_\ell(\mathcal{O}_\alpha^{\mathcal{H}}) = \frac{1}{2^k}\,.$$

Clearly this can happen only if $k \le d$.

The Grünbaum–Hadwiger–Ramos problem thus asks for the smallest dimension $d = \Delta(j,k)$ in which any collection $\mathcal{M}$ of j masses in $\mathbb{R}^d$ admits an arrangement $\mathcal{H}$ of k affine hyperplanes that equiparts $\mathcal{M}$.

For the proofs using equivariant topology methods, we make additional assumptions on the masses to be considered, namely that the measures μ_i that we deal with have compact connected support. This assumption can be made as we can strongly approximate each mass by masses with compact connected support. (This can be done "*mit passender Grenzbetrachtung und Kompaktheitserwägung auf die übliche schulmäßige Weise*" [48, S. 275] as we learn from Hadwiger.) It guarantees that the measure captured by an affine halfspace depends continuously on the halfspace, and more generally that the measure captured by an orthant depends continuously on the hyperplanes that define the orthant. Moreover, it yields that for any mass μ and a given vector v the hyperplane $H_v(a)$ that halves the mass μ is unique, and depends continuously on v.

One could also allow for measures supported on finitely many points, as often considered in the computational geometry context; see e.g. [4] and [81]. Such point measures do *not* satisfy the assumptions above, but they can be approximated by masses that do. To accommodate for point measures, one would have to modify the definition of "equiparts" in such a way that each open orthant captures *at most* a fraction of $1/2^k$ of each measure.

2.1.2 Summary of known Results

We have noted that the ham-sandwich theorem yields $\Delta(j,1) = j$ and that trivially $k \le \Delta(j,k)$. A stronger lower bound was given by Ramos [68]:

$$\tfrac{2^k-1}{k} j \le \Delta(j,k). \tag{2.1}$$

Ramos believed that his bound is tight:

The Ramos conjecture. *$\Delta(j,k) = \lceil \tfrac{2^k-1}{k} j \rceil$ for every choice of integers $j \ge 1$ and $k \ge 1$.*

The best upper bound to date, due to Mani-Levitska et al. [57, Thm. 39], can be phrased as follows:

$$\Delta(2^t + r, k) \le 2^{t+k-1} + r \qquad \text{for } t \ge 0,\ 0 \le r \le 2^t - 1. \tag{2.2}$$

The proofs of these bounds are subject of Section 2.3 (Theorems 2.2 and 2.3). In particular, for $k = 2$ and $j = 2^{t+1} - 1$ the lower bound (2.1) and the upper bound (2.2) coincide, implying that

$$\Delta(2^{t+1} - 1, 2) = 3 \cdot 2^t - 1 \qquad \text{for } t \geq 0.$$

The first result that is not a consequence of a coincidence between the lower and upper bounds (2.1) and (2.2) is due to Hadwiger [48], who showed that two masses in $\mathbb{R}^3$ can be simultaneously cut into four equal parts by two (hyper)planes. We give a degree-based proof of a generalization of this result in Section 2.4 by showing that $\Delta(2, 2) = 3$. As Hadwiger observed, by a simple reduction (2.5) this also implies that $\Delta(1, 3) = 3$.

Despite a number of published papers in prominent journals on new cases of the Ramos conjecture, the values and bounds for $\Delta(j, k)$ just mentioned appear to be the only ones available before with correct proofs: The papers by Ramos [68] from 1996, by Mani-Levitska et al. [57] from 2006, and by Živaljević [85] from 2008 and [86] from 2011 all contain essential gaps; see Sections 2.6, 2.7 and 2.8. In Table 2.1 we summarized the situation.

Lower		$\Delta(j,k)$		Upper	Reference of upper bound
8	$\leq$	$\Delta(5,2)$	$\leq$	8	[57, Thm. 4]
$\frac{3}{2} \cdot 2^t$	$\leq$	$\Delta(2^t,2)$	$\leq$	$\frac{3}{2} \cdot 2^t$	[68, Thm. 6.3] [57, Prop. 25]
$\frac{3}{2} \cdot 2^t + 2$	$\leq$	$\Delta(2^t+1,2)$	$\leq$	$\frac{3}{2} \cdot 2^t + 2$	[86, Thm. 2.1]
$\frac{7}{3} \cdot 2^t$	$\leq$	$\Delta(2^t,3)$	$\leq$	$\frac{5}{2} \cdot 2^t$	[68, Thm. 6.3]
4	$\leq$	$\Delta(1,4)$	$\leq$	5	[68, Thm. 6.3]
$\frac{15}{4} \cdot 2^t$	$\leq$	$\Delta(2^t,4)$	$\leq$	$\frac{9}{2} \cdot 2^t$	[68, Thm. 6.3]
7	$\leq$	$\Delta(1,5)$	$\leq$	9	[68, Thm. 6.3]
$\frac{31}{5} \cdot 2^t$	$\leq$	$\Delta(2^t,5)$	$\leq$	$\frac{15}{2} \cdot 2^t$	[68, Thm. 6.3]

Table 2.1: Upper bounds claimed in the literature with incorrect/incomplete proofs, where $t \geq 1$. For comparison, we also show the Ramos lower bounds, which are conjectured to be tight.

Furthermore, in Section 2.6 we show that Živaljević's approach in [85] towards the last remaining open case $\Delta(1, 4) = 4$ of the Grünbaum problem fails in principle as well as in details.

Finally, in Section 2.5 we prove using a degree calculation that

$$\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 2 \quad \text{for } t \geq 2. \tag{2.3}$$

By this we verify an instance of the Ramos conjecture previously claimed by Živaljević in [86, Thm. 2.1].

The resulting status of the Grünbaum–Hadwiger–Ramos problem is summarized in Table 2.2.

2.2 Transition to equivariant topology

In this section we demonstrate how the Grünbaum–Hadwiger–Ramos problem induces a problem of Borsuk–Ulam type.

2.2.1 The configuration spaces

Consider a collection of j masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$ on $\mathbb{R}^d$. We would like to find an arrangement of k affine hyperplanes $\mathcal{H} = (H_1, \dots, H_k)$ in $\mathbb{R}^d$ such that $\mathcal{H}$ equiparts $\mathcal{M}$. The sphere S^d can be seen as the space of all oriented affine hyperplanes in $\mathbb{R}^d$, where the north pole e_{d+1} and the south pole $-e_{d+1}$ lead to *hyperplanes at infinity*. For this we embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ via the map $(x_1, \dots, x_d) \longmapsto (x_1, \dots, x_d, 1)$. An oriented affine hyperplane in $\mathbb{R}^d$ is mapped to an oriented affine $(d-1)$-dimensional subspace of $\mathbb{R}^{d+1}$ and is extended (uniquely) to an oriented linear hyperplane. The unit normal vector on the positive side of the linear hyperplane defines a point on the sphere S^d. There is a one-to-one correspondence between points v in $S^d \setminus \{e_{d+1}, -e_{d+1}\}$ and oriented affine hyperplanes H_v in $\mathbb{R}^d$. Let H_v^0 and H_v^1 denote the positive resp. the negative closed half-space determined by H_v. The positive side of the hyperplane at infinity is $\mathbb{R}^d$ for $v = e_d$ and $\emptyset$ for $v = -e_d$. Hence $H_{-v}^0 = H_v^1$ for every v.

There are three natural configuration spaces that parametrize arrangements of k oriented affine hyperplanes in $\mathbb{R}^d$. Note that hyperplanes at infinity cannot arise as solutions to the mass partition problem, since they produce empty orthants. Hence we do not need to worry about the fact that the following configuration spaces incorporate these.

The configuration spaces we consider are

(i) the *join configuration space* $X_{d,k} = (S^d)^{*k} \cong S^{dk+k-1}$, the k-fold join of spheres S^d,

(ii) the *product configuration space* $Y_{d,k} = (S^d)^k$, the k-fold Cartesian product of spheres S^d, and

(iii) the *free configuration space* $Z_{d,k} = \{(x_1, \dots, x_k) \in Y_{d,k} : x_i \neq \pm x_j \text{ for } i < j\}$, the largest subspace of $Y_{d,k}$ on which the group action described below is free.

2.2.2 The group

The Weyl group $\mathfrak{S}_k^{\pm} = (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$, also known as the group of signed permutations, or as the symmetry group of the k-dimensional cube, acts naturally on the configuration spaces we consider: It permutes the hyperplanes, and changes their orientations. Correspondingly it also acts on the test spaces, which record the fractions of the j measures captured in each of the 2^k orthants.

2.2.3 The action on configuration spaces

Elements in $X_{d,k}$ can be presented as formal ordered convex combinations $t_1 v_1 + \dots + t_k v_k$, where $t_i \geq 0$, $\sum t_i = 1$ and $v_i \in S^d$. The action of the group $\mathfrak{S}_k^{\pm} = (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ on the space $X_{d,k}$ is defined as follows. Each copy of $\mathbb{Z}/2$ acts antipodally on the corresponding sphere S^d while the symmetric group $\mathfrak{S}_k$ acts by permuting coordinates. More precisely, let $((\beta_1, \dots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $t_1 v_1 + \dots + t_k v_k \in X_{d,k}$, then

$$((\beta_1, \dots, \beta_k) \rtimes \tau) \cdot (t_1 v_1 + \dots + t_k v_k) = t_{\tau^{-1}(1)} (-1)^{\beta_1} v_{\tau^{-1}(1)} + \dots + t_{\tau^{-1}(k)} (-1)^{\beta_k} v_{\tau^{-1}(k)}.$$

The diagonal subspace $\{\frac{1}{k} v_1 + \dots + \frac{1}{k} v_k \in X_{d,k}\} \cong Y_{d,k}$ of $X_{d,k}$ is invariant under the $\mathfrak{S}_k^{\pm}$-action and thus has a well-defined induced $\mathfrak{S}_k^{\pm}$-action. Furthermore, there is a well-defined induced action of $\mathfrak{S}_k^{\pm}$ on $Z_{d,k}$, since the action leaves the subset $Y_{d,k}^{>1}$ of all points in $Y_{d,k}$ with non-trivial stabilizers invariant. Note that for $k \geq 2$ the $\mathfrak{S}_k^{\pm}$-action is free on $Z_{d,k}$ but not on $X_{d,k}$ or on $Y_{d,k}$.

Values of $\Delta(j,k)$ for j measures and k hyperplanes and $t \geq 1$				
j \ k	1	2	3	4
1	$1 \leq$ 1 ≤ 1	$2 \leq$ 2 ≤ 2	$3 \leq$ $\mathbf{3}$ ≤ 4	$4 \leq$ ≤ 8
2	$2 \leq$ 2 ≤ 2	$3 \leq$ $\mathbf{3}$ ≤ 4	$5 \leq$ ≤ 8	$8 \leq$ ≤ 16
3	$3 \leq$ 3 ≤ 3	$5 \leq$ 5 ≤ 5	$7 \leq$ ≤ 9	$12 \leq$ ≤ 17
$\vdots$	$\vdots$			
$2^t - 1$	$2^t - 1 \leq$ $2^t - 1$ $\leq 2^t - 1$	$3 \cdot 2^{t-1} - 1 \leq$ $3 \cdot 2^{t-1} - 1$ $\leq 3 \cdot 2^{t-1} - 1$		
2^t	$2^t \leq$ 2^t $\leq 2^t$	$3 \cdot 2^{t-1} \leq$ $\boldsymbol{\leq 3 \cdot 2^{t-1} + 1}$ $\leq 4 \cdot 2^{t-1}$		
$2^t + 1$	$2^t + 1 \leq$ $2^t + 1$ $\leq 2^t + 1$	$3 \cdot 2^{t-1} + 2 \leq$ $\boldsymbol{3 \cdot 2^{t-1} + 2}$ $\leq 4 \cdot 2^{t-1} + 1$		

Table 2.2: Each square in this table records the lower bound (2.1) in the north-west corner, the upper bound (2.2) in the south-east corner, and the exact value or improved bound in the center. The values/bounds that do *not* simply follow from the two bounds coinciding are typeset in **boldface**.

2.2.4 The test space

Consider the vector space $\mathbb{R}^{(\mathbb{Z}/2)^k}$ and the subspace of codimension one

$$U_k = \Big\{ (y_\alpha)_{\alpha \in (\mathbb{Z}/2)^k} \in \mathbb{R}^{(\mathbb{Z}/2)^k} : \sum_{\alpha \in (\mathbb{Z}/2)^k} y_\alpha = 0 \Big\}.$$

We define an action of $\mathfrak{S}_k^{\pm}$ on $\mathbb{R}^{(\mathbb{Z}/2)^k}$ as follows: $((\beta_1, \ldots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ acts on a vector

$$\big(y_{(\alpha_1,\ldots,\alpha_k)}\big)_{(\alpha_1,\ldots,\alpha_k) \in (\mathbb{Z}/2)^k} \in \mathbb{R}^{(\mathbb{Z}/2)^k}$$

by acting on its indices

$$((\beta_1, \ldots, \beta_k) \rtimes \tau) \cdot (\alpha_1, \ldots, \alpha_k) = (\beta_1 + \alpha_{\tau^{-1}(1)}, \ldots, \beta_k + \alpha_{\tau^{-1}(k)}),$$

where the addition is in $\mathbb{Z}/2$. With respect to this action of $\mathfrak{S}_k^{\pm}$ the subspace U_k is a $\mathfrak{S}_k^{\pm}$-subrepresentation. The *test space* related to both configuration spaces $Y_{d,k}$ and $Z_{d,k}$ and a family of j masses is the $\mathfrak{S}_k^{\pm}$-representation $U_k^{\oplus j}$, where the action is diagonal.

2.2.5 The test map

Consider the following map from the configuration space $Y_{d,k}$ to the test space $U_k^{\oplus j}$ associated to the collection of masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$:

$$\begin{aligned} \phi_{\mathcal{M}} \colon Y_{d,k} &\longrightarrow U_k^{\oplus j}, \\ (v_1, \dots, v_k) &\longmapsto \Big(\big(\mu_i(H_{v_1}^{\alpha_1} \cap \dots \cap H_{v_k}^{\alpha_k}) - \tfrac{1}{2^k}\big)_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{Z}/2)^k} \Big)_{i\in\{1,\dots,j\}}. \end{aligned}$$

The map $\phi_{\mathcal{M}}$ is $\mathfrak{S}_k^{\pm}$-equivariant with respect to the actions introduced in Sections 2.2.3 and 2.2.4. The essential property of the map $\phi_{\mathcal{M}}$ is that *the oriented hyperplanes $H_{v_1}, \dots, H_{v_k}$ equipart $\mathcal{M}$ if and only if $\phi_{\mathcal{M}}(v_1, \dots, v_k) = 0 \in U_k^{\oplus j}$.* Note that the space $U_k^{\oplus j}$ does not depend on the dimension d.

Finally, we define the $\mathfrak{S}_k^{\pm}$-equivariant map $\psi_{\mathcal{M}} \colon Z_{d,k} \longrightarrow U_k^{\oplus j}$ as the restriction of $\phi_{\mathcal{M}}$ to $Z_{d,k}$. Again, the essential property holds: *The oriented hyperplanes $H_{v_1}, \dots, H_{v_k}$ equipart $\mathcal{M}$ if and only if $\psi_{\mathcal{M}}(v_1, \dots, v_k) = 0 \in U_k^{\oplus j}$.*

The maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ are called *test maps.* Thus we have established the following criteria.

Proposition 2.1. *Let $d \geq 1$, $k \geq 1$, and $j \geq 1$ be integers.*

(1) *Let $\mathcal{M}$ be a collection of j masses on $\mathbb{R}^d$, and let $\phi_{\mathcal{M}} \colon Y_{d,k} \longrightarrow U_k^{\oplus j}$ and $\psi_{\mathcal{M}} \colon Z_{d,k} \longrightarrow U_k^{\oplus j}$ be the $\mathfrak{S}_k^{\pm}$-equivariant maps defined above. If $0 \in \operatorname{im} \phi_{\mathcal{M}}$, or $0 \in \operatorname{im} \psi_{\mathcal{M}}$, then there are k oriented hyperplanes that equipart $\mathcal{M}$.*

(2) *Let $S(U_k^{\oplus j})$ denote the unit sphere in the vector space $U_k^{\oplus j}$. If there is no $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$, or $Z_{d,k} \longrightarrow S(U_k^{\oplus j})$, then $\Delta(j,k) \leq d$.*

We have an equivalence $0 \in \operatorname{im} \phi_{\mathcal{M}} \iff 0 \in \operatorname{im} \psi_{\mathcal{M}}$, since on the non-free part two hyperplanes are equal or opposite, so some orthants are empty, and we do not loose any equipartitions by deleting the non-free part. However, the nonexistence of a $\mathfrak{S}_k^{\pm}$-equivariant map $Z_{d,k} \longrightarrow S(U_k^{\oplus j})$ only implies the nonexistence of a $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$, but not conversely.

The join configuration spaces $X_{d,k}$ were introduced in [30]. They will not be used here, but will be essential in Chapter 3. The construction of the corresponding $\mathfrak{S}_k^{\pm}$-equivariant test map is given in [30, Sec. 2.1]. The product configuration space $Y_{d,k}$ embeds into $X_{d,k}$ via the diagonal embedding $Y_{d,k} \hookrightarrow X_{d,k}$, $(v_1, \dots, v_k) \mapsto \frac{1}{k}v_1 + \dots + \frac{1}{k}v_k$. They play a central role for the configuration space/test map scheme that will produce all major results in the following.

The free configuration spaces $Z_{d,k}$ appear in the literature as *orbit configuration spaces*; see for example [39], where they are denoted by $F_{\mathbb{Z}/2}(S^d, k)$. We will show below that the restriction of the configuration space/test map scheme to $Z_{d,k}$ is problematic, as for this restricted scheme the equivariant maps, whose nonexistence would be needed for settling new cases of the Ramos conjecture, do exist, partially for trivial reasons; see in particular Section 2.6.

2.3 Bounds and reductions for $\Delta(j,k)$

In this section we present the general lower and upper bounds for the function $\Delta(j,k)$. For the sake of completeness we present proofs.

2.3.1 The lower bounds by Ramos

Theorem 2.2 (Ramos [68]). *For integers $j \geq 1$ and $k \geq 1$, the minimal dimension $d = \Delta(j,k)$ such that any j masses on $\mathbb{R}^d$ can be equiparted by k hyperplanes satisfies*

$$\lceil \tfrac{(2^k-1)}{k} j \rceil \leq \Delta(j,k).$$

Proof. Let $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^d$ given by $\gamma(t) = (t, t^2, \dots, t^d)$ be the moment curve in $\mathbb{R}^d$. Choose j pairwise disjoint intervals on this curve and let $\mu_1, \dots, \mu_j$ be the corresponding masses. Any equipartition of these masses by k hyperplanes must give rise to at least $(2^k - 1)j$ intersections of the hyperplanes with $\operatorname{im}\gamma$. The result now follows if we recall that the moment curve has degree d: Any hyperplane meets it in at most d distinct points, so k hyperplanes can intersect it in at most dk points. □

2.3.2 The upper bounds by Mani-Levitska et al.

Theorem 2.3 (Mani-Levitska et al. [57, Thm. 39]). *Given integers $0 \leq t$, $0 \leq r \leq 2^t - 1$ and $1 \leq k$, the minimal dimension $d = \Delta(2^t + r, k)$ such that any $j = 2^t + r$ masses on $\mathbb{R}^d$ can be equiparted by k hyperplanes satisfies*

$$\Delta(2^t + r, k) \leq 2^{t+k-1} + r.$$

Proof. Let $d = 2^{t+k-1} + r$ and $j = 2^t + r$. According to Proposition 2.1 it suffices to prove that there is no $(\mathbb{Z}/2)^k$-equivariant, and consequently no $\mathfrak{S}_k^{\pm}$-equivariant, map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$. We prove this using the Fadell–Husseini ideal-valued index theory [38], for the group $(\mathbb{Z}/2)^k$ and $\mathbb{F}_2$ coefficients.

Let $(\mathbb{Z}/2)^k = \langle \varepsilon_1, \dots, \varepsilon_k \rangle$ with ε_i acting antipodally on the i-th sphere in the product $Y_{d,k} = (S^d)^k$. The cohomology of $(\mathbb{Z}/2)^k$ is $H^*((\mathbb{Z}/2)^k; \mathbb{F}_2) = \mathbb{F}_2[u_1, \dots, u_k]$, where $\deg(u_i) = 1$ and the variable u_i corresponds to the generator ε_i, $1 \leq i \leq k$. Then according to [38, Ex. 3.3]

$$\operatorname{Index}_{(\mathbb{Z}/2)^k}(Y_{d,k}; \mathbb{F}_2) = \langle u_1^{d+1}, \dots, u_k^{d+1} \rangle.$$

According to [38, Prop. 3.7] or [30, Prop. 3.13] we have that

$$\operatorname{Index}_{(\mathbb{Z}/2)^k}(S(U_k^{\oplus j}); \mathbb{F}_2) = \Big\langle \Big(\prod_{(\alpha_1,\dots,\alpha_k) \in (\mathbb{Z}/2)^k \setminus \{0\}} (\alpha_1 u_1 + \dots + \alpha_k u_k) \Big)^j \Big\rangle.$$

Now assume that there is a $(\mathbb{Z}/2)^k$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$. Then a basic property of the Fadell–Husseini index [38, Sec. 2] implies that

$$\operatorname{Index}_{(\mathbb{Z}/2)^k}(S(U_k^{\oplus j}); \mathbb{F}_2) \subseteq \operatorname{Index}_{(\mathbb{Z}/2)^k}(Y_{d,k}; \mathbb{F}_2),$$

and consequently

$$\Big(\prod_{(\alpha_1,\dots,\alpha_k) \in (\mathbb{Z}/2)^k \setminus \{0\}} (\alpha_1 u_1 + \dots + \alpha_k u_k) \Big)^j \in \langle u_1^{d+1}, \dots, u_k^{d+1} \rangle. \tag{2.4}$$

Let us denote

$$p = \prod_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{Z}/2)^k\setminus\{0\}} (\alpha_1 u_1 + \cdots + \alpha_k u_k) \in \mathbb{F}_2[u_1,\dots,u_k].$$

As a Dickson polynomial of maximal degree [1, Sec. III.2] it can be presented as

$$p = \sum_{\pi\in\mathfrak{S}_k} u_{\pi(1)}^{2^{k-1}} u_{\pi(2)}^{2^{k-2}} \cdots u_{\pi(k)}^{2^0}.$$

Therefore,

$$\begin{aligned}
p^j &= \Big(\prod_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{Z}/2)^k\setminus\{0\}} (\alpha_1 u_1 + \cdots + \alpha_k u_k) \Big)^j \\
&= \Big(\sum_{\pi\in\mathfrak{S}_k} u_{\pi(1)}^{2^{k-1}} u_{\pi(2)}^{2^{k-2}} \cdots u_{\pi(k)}^{2^0} \Big)^{2^t+r} \\
&= \Big(\sum_{\pi\in\mathfrak{S}_k} u_{\pi(1)}^{2^{k+t-1}} u_{\pi(2)}^{2^{k+t-2}} \cdots u_{\pi(k)}^{2^t} \Big) \Big(\sum_{\pi\in\mathfrak{S}_k} u_{\pi(1)}^{2^{k-1}} u_{\pi(2)}^{2^{k-2}} \cdots u_{\pi(k)}^{2^0} \Big)^r \\
&= \big(u_1^{2^{k+t-1}} u_2^{2^{k+t-2}} \cdots u_k^{2^t} \big) \cdot \big(u_1^r u_2^{2r} \cdots u_k^{2^{k-1}r} \big) + \text{Rest} \\
&= u_1^{2^{k+t-1}+r} u_2^{2^{k+t-2}+2r} \cdots u_k^{2^t+2^{k-1}r} + \text{Rest},
\end{aligned}$$

where Rest does not contain the monomial

$$u_1^{2^{k+t-1}+r} u_2^{2^{k+t-2}+2r} \cdots u_k^{2^t+2^{k-1}r}.$$

Thus $p^j \notin \langle u_1^{d+1},\dots,u_k^{d+1}\rangle$, which contradicts (2.4). This concludes the proof of the nonexistence of a $(\mathbb{Z}/2)^k$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$. $\square$

2.3.3 Dimension reductions via constraints

In order to bound $\Delta(j,k)$ it is not always necessary to make use of advanced topological methods, as there are also reduction arguments available: Hadwiger and Ramos used the rather obvious fact that

$$\Delta(j,k) \le \Delta(2j,k-1), \tag{2.5}$$

while Matschke in [60] proved that

$$\Delta(j,k) \le \Delta(j+1,k) - 1. \tag{2.6}$$

We employ a simple combinatorial reduction argument to deduce the nonexistence of equivariant maps and, in particular, to obtain a topological analog of Matschke's result, Proposition 2.4. Recently, Blagojević, Frick, and Ziegler used this approach to give elementary proofs of old and new Tverberg-type results [24].

For $\alpha \in (\mathbb{Z}/2)^k\setminus\{0\}$ let V_α be the one-dimensional real $(\mathbb{Z}/2)^k$-representation for which $\beta \in (\mathbb{Z}/2)^k$ acts non-trivially if and only if $\sum_{i=1}^k \alpha_i\beta_i = 1 \mod 2$. Then there is an isomorphism of $(\mathbb{Z}/2)^k$-representations $U_k \cong \bigoplus_{\alpha\in(\mathbb{Z}/2)^k\setminus\{0\}} V_\alpha$. Denote by $A \subseteq (\mathbb{Z}/2)^k$ the subset of all $\alpha = (\alpha_1,\dots,\alpha_k) \in (\mathbb{Z}/2)^k$ with exactly one α_i non-zero, and let $B \subseteq (\mathbb{Z}/2)^k$ be the subset of all

$\alpha \in (\mathbb{Z}/2)^k$ with more than one α_i non-zero. The representation U_k splits into $\bigoplus_{\alpha\in A} V_\alpha \oplus \bigoplus_{\alpha\in B} V_\alpha$.

Proposition 2.4. *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$, then there is also no $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d-1,k} \longrightarrow S\big(U_k^{\oplus(j-1)} \oplus \bigoplus_{\alpha\in B} V_\alpha\big)$.*

Proof. There is a $\mathfrak{S}_k^{\pm}$-equivariant map $\Phi\colon Y_{d,k} \longrightarrow \bigoplus_{\alpha\in A} V_\alpha$ with $\Phi^{-1}(0) = Y_{d-1,k}$, where $Y_{d-1,k} \subseteq Y_{d,k}$ is naturally identified with a product of equators. In fact, the space $Y_{d,k}$ contains all real $(d+1)\times k$ matrices whose columns have norm one. Now define

$$\Phi\colon Y_{d,k} \longrightarrow \bigoplus_{\alpha\in A} V_\alpha, \qquad A \longmapsto (x_{d+1,1},\dots,x_{d+1,k})$$

as the map that evaluates the last row of a given matrix $A \in Y_{d,k}$.

Let $f\colon Y_{d-1,k} \longrightarrow U_k^{\oplus(j-1)} \oplus \bigoplus_{\alpha\in B} V_\alpha$ be an arbitrary equivariant map. We need to show that f has a zero. Extend f somehow to an equivariant map $F\colon Y_{d,k} \longrightarrow U_k^{\oplus(j-1)} \oplus \bigoplus_{\alpha\in B} V_\alpha$. The map $F\oplus\Phi\colon Y_{d,k} \longrightarrow U_k^{\oplus j}$ has a zero x_0, otherwise it would induce a $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$ by retraction. Since $\Phi(x_0) = 0$, we have $x_0 \in Y_{d-1,k}$ and it is a zero of the map f. $\square$

By induction we obtain the following criterion.

Theorem 2.5. *Suppose there is no $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \to S(U_k^{\oplus j})$, then $\Delta(j-m,k) \le d-m$ for all $m = 0,\dots,j-1$.*

Corollary 2.6. *Let $j,k \ge 1$ be integers, then we have*

$$\Delta(j-m,k) \le \Delta(j,k) - m \quad \textit{for} \quad m = 0,\dots,j-1.$$

2.4 The Ramos conjecture for $\Delta(2,2)$

The first result on the Grünbaum–Hadwiger–Ramos problem for more than one hyperplane is due to Hadwiger [48]. He proved the following result.

Theorem 2.7 (Hadwiger [48])**.** *Let $A, B \subseteq \mathbb{R}^3$ be two compact sets with positive Lebesgue measure and denote by μ_A and μ_B the restriction of the Lebesgue measure to the respective sets. Then there is an arrangement of two affine hyperplanes that equipart the measures μ_A and μ_B.*

We prove, using as a main ingredient a degree-theoretic argument, that any two masses in $\mathbb{R}^3$ can be equiparted by two affine hyperplanes, so $\Delta(2,2) \le 3$. For this we use that equivariant maps have restricted homotopy types.

Lemma 2.8 (Equivariant Hopf Theorem [34, Thm. II.4.11])**.** *Let G be a finite group that acts on S^d and acts freely on a closed oriented d-manifold M. Then for any two G-equivariant maps $\Phi,\Psi\colon M \longrightarrow S^d$*

$$\deg\Phi \equiv \deg\Psi \mod |G|.$$

First we consider measures with continuous densities that have connected support. This guarantees that the measure captured by each orthant depends continuously on the hyperplanes that define the orthant. The general result then follows by approximation; see [48, S. 275].

Lemma 2.9. *Let μ_1 and μ_2 be masses on $\mathbb{R}^3$. The space $C \subset S^3$ of all oriented affine hyperplanes that simultaneously bisect both μ_1 and μ_2 admits a $\mathbb{Z}/2$-equivariant map $S^1 \longrightarrow C$ where the action on the sphere S^1 is antipodal.*

Proof. The sphere S^3 parametrizes all oriented affine hyperplanes in $\mathbb{R}^3$ including the ones at infinity. Consider the following subspace of S^3:

$$S = \{u \in S^3 \,:\, \mu_1(H_u^0) = \tfrac{1}{2}\}.$$

The space S is homeomorphic to a 2-sphere that is invariant with respect to the antipodal action on S^3 (that is, with respect to change of orientation of the hyperplane): Any normal vector in $\mathbb{R}^3$ determines a unique bisecting affine hyperplane for μ_1. For this we need that μ_1 has connected support.

Let us define a map $\phi\colon S \longrightarrow \mathbb{R}$ by $u \longmapsto \mu_2(H_u^0) - \mu_2(H_u^1)$. The map ϕ is $\mathbb{Z}/2$-equivariant where the action on both spaces is antipodal. Set $C = \phi^{-1}(0) = \bigcup_{i\in I} C_i$ where the C_i are the path-components of C. First we prove that there exists a $\mathbb{Z}/2$-invariant path-component C_j of C.

According to the general Borsuk–Ulam–Bourgin–Yang Theorem [27, Sec. 6.1]

$$\mathrm{Index}_{\mathbb{Z}/2}(C;\mathbb{F}_2)\cdot \mathrm{Index}_{\mathbb{Z}/2}(\mathbb{R}\backslash\{0\};\mathbb{F}_2) \subseteq \mathrm{Index}_{\mathbb{Z}/2}(S;\mathbb{F}_2). \tag{2.7}$$

Let the cohomology of $\mathbb{Z}/2$ be denoted by $H^*(\mathbb{Z}/2;\mathbb{F}_2) = \mathbb{F}_2[t]$, where $\deg(t) = 1$. Using [30, Prop. 3.13] we get

$$\mathrm{Index}_{\mathbb{Z}/2}(\mathbb{R}\backslash\{0\};\mathbb{F}_2) = \mathrm{Index}_{\mathbb{Z}/2}(S^0;\mathbb{F}_2) = \langle t\rangle, \qquad \mathrm{Index}_{\mathbb{Z}/2}(S;\mathbb{F}_2) = \langle t^3\rangle.$$

If C did not have a path-component that the $\mathbb{Z}/2$-action maps to itself, then the path-components of C would come in pairs that the group action would exchange. Consequently, there exists a $\mathbb{Z}/2$-equivariant map $C \to S^0$ implying that $\mathrm{Index}_{\mathbb{Z}/2}(C;\mathbb{F}_2) = \langle t\rangle$. This contradicts (2.7), and so C contains a path-component that the $\mathbb{Z}/2$-action maps to itself.

Let C_j be a $\mathbb{Z}/2$-invariant path-component of C. We prove that there exists a $\mathbb{Z}/2$-equivariant map $S^1 \longrightarrow C_j$ where the action on S^1 is antipodal. Connect two antipodal points in C_j via an injective path and extend to S^1 via the $\mathbb{Z}/2$-symmetry. □

Theorem 2.10. $\Delta(2,2) = 3$.

Proof. Let μ_1 and μ_2 be masses on $\mathbb{R}^3$. The subspace $C \subseteq S^3$ of oriented hyperplanes that simultaneously bisect both masses admits a $\mathbb{Z}/2$-equivariant map $i\colon S^1 \longrightarrow C$, where the action on the sphere S^1 is antipodal.

Consider the composition $\Phi\colon S^1 \times S^1 \longrightarrow C \times C \longrightarrow \mathbb{R}^2$ defined by

$$(u,v) \longmapsto \left(\mu_1\big(H^0_{i(u)} \cap H^0_{i(v)}\big) - \tfrac{1}{4}, \mu_2\big(H^0_{i(u)} \cap H^0_{i(v)}\big) - \tfrac{1}{4}\right).$$

Assume that μ_1 and μ_2 do not have any equipartition by two hyperplanes in $\mathbb{R}^3$. Consequently $0 \notin \Phi(S^1 \times S^1)$, since the zeros of the map Φ are pairs of hyperplanes that equipart μ_1 and μ_2. Now Φ composed with radial retraction $\mathbb{R}^2\backslash\{0\} \longrightarrow S^1$ induces the map $\Psi\colon S^1 \times S^1 \longrightarrow S^1$. Notice that $\Psi(u,u) = \big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big)$ for each $u \in S^1$. Thus the map $\Psi|_D\colon D \longrightarrow S^1$, where $D = \{(u,u) : u \in S^1\}$ is the diagonal, is constant and so has degree 0.

Let t be a generator of $\mathbb{Z}/4$. Then $t \cdot (u, v) = (v, -u)$ defines a free $\mathbb{Z}/4$-action on $S^1 \times S^1$. The circle $\Gamma = \{(u, e^{i\frac{\pi}{2}} \cdot u) : u \in S^1\} \subseteq S^1 \times S^1$ is a $\mathbb{Z}/4$-invariant subspace that is homotopic to the diagonal D in $S^1 \times S^1$. Thus $\deg \Psi|_\Gamma = \deg \Psi|_D = 0$.

On the other hand, the map $\Psi|_\Gamma : \Gamma \to S^1$ is $\mathbb{Z}/4$-equivariant with the generator t acting antipodally on the codomain sphere S^1. All such maps have the same degree modulo 4 by Lemma 2.8 and $z \mapsto z^2$ is such a map of degree 2. This yields a contradiction, and so the map Φ has a zero. □

The reduction argument (2.5) applied to the result of the previous theorem in combination with Ramos' lower bound yields the following consequence.

Corollary 2.11 (Hadwiger [48])**.** $\Delta(1, 3) = 3$.

2.5 The Ramos conjecture for $\Delta(2^t + 1, 2)$

In this section we prove the following theorem, establishing a family of exact values for the function $\Delta(j, 2)$ in the case of two hyperplanes. It is a nontrivial instance of the Ramos conjecture that was previously claimed by Živaljević [86, Thm. 2.1], but the proof given there is not complete; see Section 2.8.

Theorem 2.12. $\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 2$ *for any integer* $t \geq 2$.

Using the reduction of (2.6) we obtain from this that

$$\Delta(2^t, 2) \leq 3 \cdot 2^{t-1} + 1 \quad \text{for any } t \geq 2.$$

as listed in Table 2.2.

The rough outline of the proof is as follows: For $d = 3 \cdot 2^{t-1} + 1$ the existence of j masses in $\mathbb{R}^d$ that do not admit an equipartition by two affine hyperplanes yields the D_8-equivariant test map $\psi\colon S^d \times S^d \longrightarrow S^{2d-2}$. The restricted map $\overline{\psi}\colon S^{d-1} \times S^{d-1} \longrightarrow S^{2d-2}$ has degree zero since it factors through $S^d \times S^d$. We then consider the test map ϕ for j specific masses and compute the degree of the restricted map $\overline{\phi}$ on $S^{d-1} \times S^{d-1}$ by counting the zeros of ϕ on $B^d \times S^{d-1}$ (where B^d is a hemisphere of S^d) with sign and multiplicity. This is done by counting equipartitions for this specific set of measures. The maps $\overline{\psi}$ and $\overline{\phi}$ need not be homotopic and so their degrees might not coincide. This is remedied by exploiting the equivariance of both maps, yielding $\deg \overline{\psi} \equiv \deg \overline{\phi}$ mod 8, which gives a contradiction if $j - 1$ is a power of two, $j \geq 5$.

2.5.1 Equipartitions restrict degrees of equivariant maps

In order to show that $\Delta(j, k) \leq d$ we use Proposition 2.1(2) and prove that there is no $\mathfrak{S}_k^\pm$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$.

Lemma 2.13. *Let* $j, k, d \geq 1$ *be integers. Assume* $\Delta(j, k) > d$ *for* $k(d-1) = (2^k - 1)j - 1$ *and assume that* $k(d-1)$ *is not divisible by* d*. Then any* $\mathfrak{S}_k^\pm$*-equivariant map* $\psi\colon Y_{d,k} \longrightarrow S(U_k^{\oplus j})$ *induces a* $\mathfrak{S}_k^\pm$*-equivariant map* $\overline{\psi}\colon Y_{d-1,k} \longrightarrow S(U_k^{\oplus j})$ *with* $\deg \overline{\psi} = 0$.

Proof. Since $\Delta(j, k) > d$ there is a $\mathfrak{S}_k^\pm$-equivariant map $\psi\colon Y_{d,k} \longrightarrow S(U_k^{\oplus j})$. This map restricts to a $\mathfrak{S}_k^\pm$-equivariant map $\overline{\psi}\colon Y_{d-1,k} \longrightarrow S(U_k^{\oplus j})$ on the product of the equators. The domain and codomain of $\overline{\psi}$ are closed orientable manifolds of the same dimension, and thus $\overline{\psi}$ has a well-defined degree up to a sign. Consider the following commutative diagram of $\mathfrak{S}_k^{\pm}$-equivariant maps

$$\begin{tikzcd} Y_{d,k}=(S^d)^k \arrow[r, "\psi"] & S(U_k^{\oplus j}) \\ Y_{d-1,k}=(S^{d-1})^k. \arrow[u, hook] \arrow[ur, "\overline{\psi}"] & \end{tikzcd}$$

After applying the $k(d-1)$-dimensional homology functor we get

$$\begin{tikzcd} H_{k(d-1)}((S^d)^k;\mathbb{Z}) \arrow[r, "\psi_*"] & H_{k(d-1)}(S(U_k^{\oplus j});\mathbb{Z}) \\ H_{k(d-1)}((S^{d-1})^k;\mathbb{Z}). \arrow[u] \arrow[ur, "\overline{\psi}_*"] & \end{tikzcd}$$

Thus the map $\overline{\psi}_*$ factors through $H_{k(d-1)}((S^d)^k;\mathbb{Z})$. Since d does not divide $k(d-1)$ we have that $H_{k(d-1)}((S^d)^k;\mathbb{Z})\cong 0$. Consequently, $\deg\overline{\psi}=0$. $\square$

The equality $k(d-1)=(2^k-1)j-1$ implies that $d=\frac{(2^k-1)}{k}j-\frac{1}{k}+1=\lceil\frac{(2^k-1)}{k}j\rceil$, which coincides with the lower bound (2.1). The space $Y_{d-1,k}=(S^{d-1})^k$ is naturally a subspace of $Y_{d,k}$ by identifying it with oriented linear hyperplanes in $\mathbb{R}^d$, that is, $(x_1,\dots,x_k)\in Y_{d,k}\subseteq(\mathbb{R}^{d+1})^k$ is in $Y_{d-1,k}$ precisely if $\langle e_{d+1},x_i\rangle=0$ for $i=1,\dots,k$.

We use the following generalized equivariant Hopf theorem.

Theorem 2.14 (Kushkuley & Balanov [54, Cor. 2.4])**.** *Let M be a compact oriented n-dimensional manifold with an action of a finite group G. Let $N\subseteq M$ be a closed G-invariant subset containing the set of all points with non-trivial stabilizers. Then any two G-equivariant maps $\phi,\psi\colon M\longrightarrow S^n$ that are equivariantly homotopic on N satisfy* $\deg\phi\equiv\deg\psi \mod |G|$.

The set $Y_{d-1,k}^{>1}$ of points in $Y_{d-1,k}$ with non-trivial stabilizers with respect to the action of $\mathfrak{S}_k^{\pm}$ is

$$\{(x_1,\dots,x_k)\in Y_{d-1,k} : x_r=x_s \text{ or } x_r=-x_s \text{ for some } r\neq s\}.$$

Observe that for $k\geq 3$ and $d\geq 2$, the space $Y_{d-1,k}^{>1}$ is path-connected, while for $k=2$ it consists of two path-components.

Corollary 2.15. *Let $k(d-1)=(2^k-1)j-1$ and let $k(d-1)$ be not divisible by d. Let $\mathcal{M}=(\mu_1,\dots,\mu_j)$ be a collection of masses on $\mathbb{R}^d$ that cannot be equiparted by k linear hyperplanes with the corresponding test map $\phi=\phi_{\mathcal{M}}\colon Y_{d,k}\longrightarrow U_k^{\oplus j}$. Denote the (normalized) test map restricted to linear hyperplanes by $\overline{\phi}\colon Y_{d-1,k}\longrightarrow S(U_k^{\oplus j})$. If $\deg\overline{\phi}\not\equiv 0 \mod 2^k k!$, then $\Delta(j,k)=d$, that is, the Ramos conjecture holds for j masses and k hyperplanes.*

Proof. Suppose $\Delta(j,k)>d$. Then from Lemma 2.13 we get a $\mathfrak{S}_k^{\pm}$-equivariant map $\overline{\psi}\colon Y_{d-1,k}\longrightarrow S(U_k^{\oplus j})$ with $\deg\overline{\psi}=0$. By assumption there is a $\mathfrak{S}_k^{\pm}$-equivariant map $\overline{\phi}\colon Y_{d-1,k}\longrightarrow S(U_k^{\oplus j})$ with $\deg\overline{\phi}\not\equiv 0 \mod |\mathfrak{S}_k^{\pm}|$. Set $N=Y_{d-1,k}^{>1}$. Once we have shown that $\overline{\phi}$ and $\overline{\psi}$ are equivariantly homotopic on N we can apply Theorem 2.14 and get that $\deg\overline{\phi}\equiv\deg\overline{\psi} \mod |\mathfrak{S}_k^{\pm}|$. This is a contradiction with $\deg\overline{\psi}=0$, and therefore $\Delta(j,k)\leq d$.

The equivariant homotopy from $\overline{\phi}|_N$ to $\overline{\psi}|_N$ is just the linear homotopy in $U_k^{\oplus j}$ normalized to the unit sphere. For this to be well-defined we need to show that the linear homotopy does not have a zero. This follows from the fact that for each point $z \in N$ the vectors $\phi(z)$ and $\psi(z)$ lie in some affine subspace of $U_k^{\oplus j}$ that is not a linear subspace. Since $z = (x_1, \ldots, x_k) \in N$ has non-trivial stabilizer there are $r \neq s$ with $x_r = \pm x_s$. Thus the corresponding affine hyperplanes H_r and H_s coincide with perhaps opposite orientations. This implies that the arrangement of hyperplanes has an empty orthant, implying that any test map has value equal to $-1/2^k$ in the coordinate corresponding to the empty orthant. This implies that $\phi(z)$ and $\psi(z)$ lie in an affine subspace not containing zero. □

2.5.2 The standard configuration along the moment curve

Now we specialize to the problem of two hyperplanes, $k = 2$. In this case the relevant group is the dihedral group $\mathfrak{S}_2^{\pm} = D_8 = (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/2 = \langle \varepsilon_1, \varepsilon_2 \rangle \rtimes \langle \omega \rangle$, and the corresponding test space is $Y_{d,2} = S^d \times S^d$. Thus the test map is a D_8-equivariant map $\phi\colon S^d \times S^d \to U_2^{\oplus j}$ whose zeros correspond to equipartitions.

Before proceeding further we recall how, in this case, $D_8 = (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/2 = \langle \varepsilon_1, \varepsilon_2 \rangle \rtimes \langle \omega \rangle$ acts on $S^d \times S^d$ and U_2. For $(u, v) \in S^d \times S^d$ we have that

$$\varepsilon_1 \cdot (u, v) = (-u, v), \quad \varepsilon_2 \cdot (u, v) = (u, -v), \quad \omega \cdot (u, v) = (v, u).$$

The real 3-dimensional D_8-representation U_2 considered as a $(\mathbb{Z}/2)^2$-representation decomposes into a direct sum of irreducible real 1-dimensional representations as $U_2 = V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$, where $V_{(1,0)} = V_{(0,1)} = V_{(1,1)} = \mathbb{R}$ and

$$\varepsilon_1 \cdot (a, b, c) = (-a, b, -c), \quad \varepsilon_2 \cdot (a, b, c) = (a, -b, -c), \quad \omega \cdot (a, b, c) = (b, a, c)$$

for $(a, b, c) \in V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$.

We will now define masses $\mu_1, \ldots, \mu_j$ for which computing the degree of the normalized test map restricted to linear hyperplanes is particularly simple. Recall that the *moment curve* $\gamma(t) = (t, \ldots, t^d)$ in $\mathbb{R}^d$ has the special property that any set of pairwise distinct points on γ is in general position. Hence every affine hyperplane intersects γ in at most d points. For the rest of this section we consider the masses $\mu_1, \ldots, \mu_j$ to be concentrated along j pairwise disjoint intervals along the moment curve that do not include the origin.

The masses $\mu_1, \ldots, \mu_j$ satisfy the hypotheses of Corollary 2.15 for $k = 2$ and $2d = 3j + 1$: Any equipartition of $\mu_1, \ldots, \mu_j$ by two affine hyperplanes intersects the moment curve in $3j$ points. Additionally requiring that both hyperplanes pass through the origin prescribes one more intersection point with γ for each hyperplane. Two hyperplanes intersect the moment curve in at most $2d$ points, that is, the space of linear hyperplanes $Y_{d-1,2}$ contains no pair of equiparting hyperplanes if $2d < 3j + 2$. Now we will compute the degree of the restricted test map by counting equipartitions.

Lemma 2.16. *Let $2d = 3j+1$ and let $i \geq 1$ be an integer. Let $\mu_1, \dots, \mu_j$ be masses concentrated on the pairwise disjoint intervals $\gamma(i, i+1)$ of length 1 along the moment curve in $\mathbb{R}^d$. Then there are $\binom{j}{\frac{j-1}{2}}$ pairs of unoriented (non-parallel) affine hyperplanes (H_1, H_2) equiparting $\mu_1, \dots, \mu_j$ such that H_2 passes through the origin.*

Proof. To equipart $\mu_1, \dots, \mu_j$ the pair (H_1, H_2) needs to have at least $3j$ intersection points with the moment curve. Moreover, H_2 is a linear hyperplane. Thus each hyperplane H_1 and H_2 intersects the moment curve in at least $3j+1$ points. Since $2d = 3j+1$ and every hyperplane can intersect in at most d points, there are exactly $3j+1$ intersection points. In particular, each μ_i has either one intersection with H_1 (in the midpoint of μ_i) and two intersections with H_2 (in the midpoint of the two halves defined by H_1) or vice versa. Consequently, the intersection points of the pair (H_1, H_2) with the interval μ_i are uniquely determined by the number of intersections of μ_i and H_1. There are $\binom{j}{2j-d}$ masses with exactly one point of intersection with H_1. Since $d = \frac{3j+1}{2}$ this is equal to $\binom{j}{\frac{j-1}{2}}$. □

2.5.3 Computing the degree of the restricted test map geometrically

Let $\phi = \phi_{\mathcal{M}} \colon S^d \times S^d \longrightarrow U_2^{\oplus j}$ be the D_8-equivariant test map associated to the standard configuration $\mathcal{M}$ of j masses along the moment curve in $\mathbb{R}^d$ where $2d = 3j+1$. By Lemma 2.16 such an equipartition exists and thus $\phi^{-1}(0)$ is non-empty. However there is no such equipartition by linear hyperplanes since this would require more than d intersection points of some hyperplane with the moment curve γ.

Denote by $\overline{\phi} \colon S^{d-1} \times S^{d-1} \to S(U_2^{\oplus j})$ the normalized restriction of ϕ to linear hyperplanes. Note that $\dim S^{d-1} \times S^{d-1} = 2d-2 = 3j-1 = \dim S(U_2^{\oplus j})$ and thus $\overline{\phi}$ has well-defined degree (up to a sign). For even d this degree modulo 8 was previously computed by Živaljević [86, Prop. 9.15].

Lemma 2.17. *For even d the map $\overline{\phi} \colon S^{d-1} \times S^{d-1} \longrightarrow S(U_2^{\oplus j})$ has degree*

$$\deg \overline{\phi} = 2\binom{j}{\frac{j-1}{2}}.$$

For odd d the degree of $\overline{\phi}$ vanishes.

We will now prove this lemma by counting zeros of ϕ with signs and multiplicities. Theorem 2.12 then follows from an application of Corollary 2.15 once we have established that $2\binom{2^t+1}{2^{t-1}}$ is not divisible by 8 for $t \geq 2$.

Proof of Lemma 2.17. Let $W \subseteq S^d \times S^d$ be the subspace of hyperplanes (H_1, H_2), where H_1 has the origin in its positive half-space and H_2 is a linear hyperplane. The subspace W is a manifold homeomorphic to $B^d \times S^{d-1}$ with boundary $S^{d-1} \times S^{d-1}$. By Lemma 2.16 ϕ has $2\binom{j}{\frac{j-1}{2}}$ zeros on W. The orientation of H_1 is prescribed by the requirement that the origin be in its positive half-space, but the orientation of H_2 is not prescribed. We will show that for d even all local degrees of ϕ on W are 1 and that $\deg \overline{\phi}$ is the sum of local degrees of ϕ on W.

Denote by $\widetilde{W} = W \setminus \phi^{-1}(B_\epsilon(0))$ for a sufficiently small $\epsilon > 0$ such that $W \setminus \phi^{-1}(0)$ deformation retracts to $\widetilde{W}$. The boundary $\partial\widetilde{W}$ consists of $Y_{d-1,2}$ and disjoint copies of $(2d-2)$-spheres $S_1, \dots, S_\ell$, one for each zero of ϕ on W. Let $\phi' \colon \widetilde{W} \longrightarrow S(U_2^{\oplus j})$ denote the composition of ϕ and radial retraction restricted to $\widetilde{W}$. The fundamental class $[Y_{d-1,2}]$ is equal to $\sum[S_i]$ in $H_{2d-2}(\widetilde{W})$ since $Y_{d-1,2}$ and $\bigcup S_i$ are cobordant in $\widetilde{W}$. Now $\sum \phi'_*([S_i]) = \phi'_*([Y_{d-1,2}]) = \deg\overline{\phi}\cdot[S(U_2^{\oplus j})]$, and hence $\deg\overline{\phi} = \sum \deg \phi'|_{S_i}$; see [62, Prop. IV.4.5].

That local degrees of ϕ are ± 1 is simple to see since in a small neighborhood U around any zero (u,v) the test map ϕ is a continuous bijection: For any sufficiently small vector $w \in \mathbb{R}^{3j}$ there is exactly one tuple $(u',v') \in U$ with $\phi(u',v') = w$. Thus $\phi|_{\partial U}$ is a continuous bijection into some $(3j-1)$-sphere around the origin and by compactness of ∂U is a homeomorphism.

The symmetry of the configuration allows us to compute the local signs of the test map. First let us describe a neighborhood of every zero of the test map in W. Let $(u,v) \in W$ with $\phi(u,v) = 0$. Denote the intersections of H_u with the moment curve by $x_1,\dots,x_d$ in the correct order along the moment curve. Similarly, let $y_1,\dots,y_d$ be the intersections of H_v with the moment curve. In particular, $y_1 = 0$. Choose an $\epsilon > 0$ such that ϵ-balls around the $x_1,\dots,x_d$ and around $y_2,\dots,y_d$ are pairwise disjoint and such that these balls intersect the moment curve only in precisely one interval μ_i.

Tuples of hyperplanes $(H_{u'},H_{v'})$ with $(u',v') \in W$ that still intersect the moment curve in the corresponding ϵ-balls parametrize a neighborhood of (u,v). The local neighborhood consisting of pairs of hyperplanes with the same orientation still intersecting the moment curve in the corresponding ϵ-balls can be naturally parametrized by $\prod_{i=2}^{2d}(-\epsilon,\epsilon)$, where the first d factors correspond to neighborhoods of the x_i and the last $d-1$ factors to ϵ-balls around $y_2,\dots,y_d$. A natural basis of the tangent space at (u,v) is obtained via the push-forward of the canonical basis of $\mathbb{R}^{2d-1}$ as tangent space at the origin.

Consider the subspace $Z \subseteq W$ that consists of pairs of hyperplanes (H_u,H_v) in W that each intersect the moment curve in d points. It has two path-components determined by the orientation of H_v. The path-components of Z are contractible as each hyperplane can be continuously moved to intersect the moment curve in d fixed points. On each part the orientation around the zeros given above derives from the same global orientation since the given bases of tangent spaces transform into one another along this contraction path. The map $\varepsilon_2\colon (H_u,H_v) \longmapsto (H_u,H_{-v})$ is orientation-preserving if and only if d is even.

Any two neighborhoods of distinct zeros of the test map ϕ can be mapped onto each other by a composition of coordinate charts since their domains coincide. This is a smooth map of degree 1: the Jacobian at the zero is the identity map. Let (u,v) and (x,y) be zeros in the same path-component of Z of the test map ϕ and let Ψ be the change of coordinate chart described above. Then ϕ and $\phi\circ\Psi$ differ in a neighborhood of (u,v) just by a permutation of coordinates. This permutation is always even by the following:

Claim 2.18. *Let A and B be finite sets of the same cardinality. Then the cardinality of the symmetric difference $A \vartriangle B$ is even.*

Up to orientation of H_u the hyperplanes H_u and H_v are completely determined by the set of measures that H_u cuts once. Let $A \subseteq \{1,\dots,j\}$ be the set of indices of measures that H_u intersects once, and let $B \subseteq \{1,\dots,j\}$ be the same set for H_v. Then Ψ is a composition of a multiple of $A \vartriangle B$ transpositions and, hence, an even permutation.

The linear map $\varepsilon_2\colon U_2^{\oplus j} \to U_2^{\oplus j}$ always has determinant equal to 1 since ε_2 is a composition of $2j$ reflections in hyperplanes on $U_2^{\oplus j}$. Thus for d even all local degrees of ϕ on W are the same since the coordinate change Ψ preserves orientation (on a path-component), and we have proved Lemma 2.17. Thus for d even $\deg \overline{\phi} = 2\binom{j}{\frac{j-1}{2}}$. ☐

To apply Corollary 2.15 it is essential to know when the binomial coefficient $\binom{j}{\frac{j-1}{2}}$ is divisible by 4. This is answered by the following lemma by Kummer.

Lemma 2.19 (Kummer [53]). *Let $n \geq m \geq 0$ be integers and let p be a prime. The maximal integer k such that p^k divides $\binom{n}{m}$ is the number of carries when m and $n-m$ are added in base p.*

Putting these statements together we obtain a proof of Theorem 2.12.

Proof of Theorem 2.12. Let $k = 2$, $j = 2^t + 1$ with $t \geq 2$, and $d = 3 \cdot 2^{t-1} + 2$. Then $2(d-1) = j(2^k - 1) - 1$ and d does not divide k. Thus we can apply Corollary 2.15 to the standard configuration $\mathcal{M}$ of j masses along the moment curve. The restriction to linear hyperplanes $\overline{\phi}$ of the corresponding test map $\phi_{\mathcal{M}}$ has degree $\binom{j}{\frac{j-1}{2}}$ by Lemma 2.17 since d is even. This degree is non-zero modulo 8 by Lemma 2.19. ☐

2.6 The failure of the free configuration space

Here we prove the following theorem about the existence of $\mathfrak{S}_k^{\pm}$-equivariant maps from the free configuration space $Z_{d,k}$. Recall that $Z_{d,k} = \{(x_1, \ldots, x_k) \in Y_{d,k} : x_s \neq \pm x_r \text{ for } s < r\}$ is the largest subspace of $Y_{d,k}$ on which the $\mathfrak{S}_k^{\pm}$-action is free.

Theorem 2.20. *Let $d \geq k \geq 3$ be integers and let $(2^k - 1)j + 2 \geq \max\{dk, dk + 4 - k\}$. Then there is a $\mathfrak{S}_k^{\pm}$-equivariant map $Z_{d,k} \longrightarrow S(U_k^{\oplus j})$.*

Theorem 2.20 will be proved in Section 2.6.2. As $\dim S(U_k^{\oplus j}) = (2^k - 1)j$ and $\dim Z_{d,k} = dk$, it exhibits a disadvantage of the free configuration spaces.

As a direct consequence of Theorem 2.20 we prove the first main result claimed in Živaljević's 2008 paper [85, Thm. 5.9].

Corollary 2.21. *There is a $\mathfrak{S}_k^{\pm}$-equivariant map $f\colon Z_{4,4} \longrightarrow S(U_4)$.*

In Section 2.6.3 we explain why the proof given in [85] for this result is invalid. To compare the results, note that $Z_{4,4}$ is there denoted by $(S^4)^4_\delta$. Furthermore, in Section 2.6.3 we exhibit a gap in the proof of the second main (positive) result of the same paper, [85, Thm. 5.1].

2.6.1 Existence of equivariant maps

Let G be a finite group, let X be a free G-CW complex and W be an orthogonal real G-representation. Let us further denote by $\operatorname{cohdim} X = \max\{i : H^i(X;\mathbb{Z}) \neq 0\}$ the *cohomological dimension* of the space X.

In this section we consider the existence of a G-equivariant map $X \longrightarrow S(W)$ under specific conditions and prove the following theorem.

Theorem 2.22. *Let G be a finite group, let X be a free G-CW complex, let W be an orthogonal real G-representation, and let $I = \{i : \dim W - 1 \leq i \leq \dim X - 1\}$. If*

(i) $2 \leq \operatorname{cohdim} X < \dim W$, *and*

(ii) $\pi_i S(W)$ *is a trivial* $\mathbb{Z}[G]$*-module for every* $i \in I$,

then there exists a G-equivariant map $X \longrightarrow S(W)$.

The proof of the theorem will be obtained via equivariant obstruction theory, as presented by tom Dieck in [34, Sec. II.3]. In the proof of the theorem we use the following special case of a result given as an exercise by Bredon [32, Exer. 9, p. 168]. It is an extension (for acyclicity above a certain dimension) of the important result from Smith theory that the quotient of a compact, acyclic space by a finite group action is still acyclic.

Lemma 2.23. *Let G be a finite group acting cellularly on the compact G-CW-complex X. If $H^i(X;\mathbb{Z}) = 0$ for all $i > n$, then $H^i(X/G;\mathbb{Z}) = 0$ for all $i > n$.*

Proof of Theorem 2.22. Let us denote by $N = \dim X$, $n = \operatorname{cohdim} X$ and $w = \dim W$. For $i \in \{0, \dots, N\}$, the i-th skeleton of X is denoted as usual by $X^{(i)}$.

Since $S(W)$ is $(w-2)$-connected, $(w-1)$-simple and X is a free G-CW complex there is no obstruction for the existence of a G-equivariant map $f\colon X^{(w-1)} \longrightarrow S(X)$. The proof continues by induction.

The first obstruction for the extension of the map f to the w-skeleton $X^{(w)}$ lives in the specially defined Bredon type equivariant cohomology [34, pp. 111–114]:

$$\mathcal{H}^w_G(X;\pi_{w-1}S(W)) \cong \mathcal{H}^w_G(X;\mathbb{Z}),$$

Now $\pi_{w-1}S(W) \cong \mathbb{Z}$ is a trivial $\mathbb{Z}[G]$-module by the assumption of the theorem. The isomorphism of [34, II, Prop. 9.7, (ii)] implies $\mathcal{H}^w_G(X;\mathbb{Z}) \cong H^w(X/G;\mathbb{Z})$, where on the right we have singular cohomology. Since $n = \operatorname{cohdim} X < w$, by the assumption of the theorem, an application of Lemma 2.23 gives $\mathcal{H}^w_G(X;\mathbb{Z}) = 0$. Thus $\mathcal{H}^w_G(X;\pi_{w-1}S(W)) = 0$, and the map f can be G-equivariantly extended to the w-skeleton of X.

The process continues in the same way until we reach the N-th skeleton of X since all the ambient groups $\mathcal{H}^i_G(X;\pi_{i-1}S(W))$, $i \in \{w, \dots N\}$, for the obstructions vanish. □

2.6.2 Proof of Theorem 2.20

Let $d \geq k \geq 3$ be integers and let

$$(2^k - 1)j + 2 \geq \max\{dk, dk + 4 - k\}.$$

We prove the existence of a $\mathfrak{S}_k^{\pm}$-equivariant map $Z_{d,k} \longrightarrow S(U_k^{\oplus j})$ by direct application of Theorem 2.22.

Let X be a dk-dimensional $\mathfrak{S}_k^{\pm}$-CW complex with the property that $X \subseteq Z_{d,k}$ is an equivariant deformation retract of $Z_{d,k}$. Then X is a dk-dimensional free $\mathfrak{S}_k^{\pm}$-CW complex and it suffices to prove that there exists a $\mathfrak{S}_k^{\pm}$-equivariant map $X \longrightarrow S(U_k^{\oplus j})$.

If $dk = \dim X \leq \dim S(U_k^{\oplus j}) = (2^k - 1)j - 1$ then a $\mathfrak{S}_k^{\pm}$-equivariant map $X \longrightarrow S(U_k^{\oplus j})$ exists since X is a free $\mathfrak{S}_k^{\pm}$-CW complex and all obstructions vanish. Thus we can in addition assume that $dk-1 \geq (2^k-1)j-1$. Now

$$I = \{i : \dim W - 1 \leq i \leq \dim X - 1\} = \{i : (2^k-1)j - 1 \leq i \leq dk - 1\}.$$

Since $(2^k-1)j+2 \geq \max\{dk, dk+4-k\}$ and $dk-1 \geq (2^k-1)j-1$ we have that $|I| \leq 3$, that is,

$$I \subseteq \{(2^k-1)j-1, (2^k-1)j, (2^k-1)j+1\}.$$

The following fact is known. For completeness we give a brief proof.

Claim. $\operatorname{cohdim} Z_{d,k} = (d-1)k+1$ *for* $d \geq k \geq 3$.

Proof. The free configuration space $Z_{d,k}$ is defined as a difference $Y_{d,k}\backslash Y_{d,k}^{>1}$ of an oriented dk-manifold $Y_{d,k} = (S^d)^k$ and the regular CW-complex $Y_{d,k}^{>1}$.

The CW-complex $Y_{d,k}^{>1}$ can be covered by a family

$$\mathcal{L} = \{L_{s,r}^+ : 1 \leq s < r \leq k\} \cup \{L_{s,r}^- : 1 \leq s < r \leq k\}$$

of subcomplexes

$$Y_{d,k}^{>1} = \bigcup_{1\leq s<r\leq k} (L_{s,r}^+ \cup L_{s,r}^-),$$

where for $1 \leq s < r \leq k$ we set

$$L_{s,r}^+ = \{(x_1,\dots,x_k) \in Y_{d,k} : x_s = x_r\}, \quad L_{s,r}^- = \{(x_1,\dots,x_k) \in Y_{d,k} : x_s = -x_r\}.$$

Every subcomplex $L_{s,r}^{\pm}$ as well as any finite non-empty intersection of them is $(d-1)$-connected. Therefore, by a version of the nerve lemma [15, Th. 6], we have that $\pi_r(Y_{d,k}^{>1}) \cong \pi_r(\Delta(P_{\mathcal{L}}))$ for all $r \leq d-1$, where $\Delta(P_{\mathcal{L}})$ denotes the order complex of the intersection poset $P_{\mathcal{L}}$ of the family $\mathcal{L}$. The intersection poset $P_{\mathcal{L}}$ can be identified as a subposet of the type B partition lattice Π_k^B, consult Wachs [79, Ex. 5.3.6]. Moreover, Π_k^B is a geometric semilattice, which implies that $\Delta(P_{\mathcal{L}}) \simeq \bigvee S^{k-2}$. Thus $Y_{d,k}^{>1}$ is $(k-3)$-connected.

The Poincaré–Lefschetz duality [33, Cor. VI.8,4] relates the homology of $Z_{d,k}$ to the cohomology of the pair $(Y_{d,k}, Y_{d,k}^{>1})$:

$$H_{dk-i}(Z_{d,k};\mathbb{Z}) \cong H^i(Y_{d,k}, Y_{d,k}^{>1};\mathbb{Z}).$$

Using the long exact sequence in cohomology for the pair $(Y_{d,k}, Y_{d,k}^{>1})$ and the facts that $Y_{d,k}$ is $(d-1)$-connected and $Y_{d,k}^{>1}$ is $(k-3)$-connected we get that $\tilde{H}^i(Y_{d,k}, Y_{d,k}^{>1};\mathbb{Z}) = 0$ for $i \leq k-2$ and $H^{k-1}(Y_{d,k}, Y_{d,k}^{>1};\mathbb{Z}) \cong H^{k-2}(Y_{d,k}^{>1};\mathbb{Z}) \neq 0$ is free abelian. Consequently, using the universal coefficient theorem [33, Cor. V.7.2], we conclude that $\operatorname{cohdim} Z_{d,k} = (d-1)k+1$. $\square$

In order to apply Theorem 2.22 and complete the proof we need to verify the conditions (i) and (ii).

(i) By assumption $(2^k-1)j+2 \geq \max\{dk, dk+4-k\}$ and $k \geq 3$. Consequently,

$$\dim U_k^{\oplus j} = (2^k-1)j > dk-1 = \dim X - 1 > (d-1)k+1 = \operatorname{cohdim} Z_{d,k}.$$

(ii) Since $S(U_k^{\oplus j}) \approx S^{(2^k-1)j-1}$ and $I \subseteq \{(2^k-1)j-1, (2^k-1)j, (2^k-1)j+1\}$ we consider

$$\pi_{(2^k-1)j-1}S(U_k^{\oplus j}) \cong \mathbb{Z}, \quad \pi_{(2^k-1)j}S(U_k^{\oplus j}) \cong \mathbb{Z}/2, \quad \pi_{(2^k-1)j+1}S(U_k^{\oplus j}) \cong \mathbb{Z}/2$$

as $\mathbb{Z}[\mathfrak{S}_k^{\pm}]$-modules. Since the second two groups are $\mathbb{Z}/2$ and therefore trivial $\mathbb{Z}[\mathfrak{S}_k^{\pm}]$-modules it remains to be shown that $\mathfrak{S}_k^{\pm}$ acts orientation preserving on $S(U_k^{\oplus j})$.
Each of the generators ε_i of $(\mathbb{Z}/2)^k$ acts on the top integral homology of the sphere $S(U_k^{\oplus j})$ by multiplication with

$$(-1)^{j\left(\binom{k-1}{0}+\binom{k-1}{1}+\cdots+\binom{k-1}{k-1}\right)} = 1.$$

Furthermore, each of the transpositions $\tau_{sr} = (sr)$ for $1 \leq s < r \leq k$, which generate $\mathfrak{S}_k$, acts on the top integral homology of the sphere $S(U_k^{\oplus j})$ by multiplication with

$$(-1)^{j\left(\binom{k-2}{0}+\binom{k-2}{1}+\cdots+\binom{k-2}{k-2}\right)} = 1.$$

Thus $\mathfrak{S}_k^{\pm}$ preserves orientation of $S(U_k^{\oplus j})$ and consequently $\pi_{(2^k-1)j-1}S(U_k^{\oplus j})$ is a trivial $\mathbb{Z}[\mathfrak{S}_k^{\pm}]$-module.

Now Theorem 2.22 implies the existence of a $\mathfrak{S}_k^{\pm}$-equivariant map $Z_{d,k} \longrightarrow S(U_k^{\oplus j})$, and we have completed the proof of Theorem 2.20.

2.6.3 Gaps in [85]

In this section we exhibit and explain essential gaps in [85] that invalidate Živaljević's proofs for both main results of that paper.

A Gap in [85, Lemma 4.3]

We note that this lemma is the starting point for the explicit calculations related to both main results of that paper and thus crucial for their validity.

First we recall some notation from [85]:

- $(S^n)^n_\delta = \{x \in (S^n)^n : x_i \neq \pm x_j \text{ for } i \neq j\}$, consult [85, (2.2)]; in the notation of this chapter, $(S^n)^n_\delta$ is equal to the free configuration space $Z_{n,n}$.
- $SP^4_\delta(\mathbb{R}\mathrm{P}^4) := (S^4)^4_\delta/\mathfrak{S}_4^{\pm} \subseteq SP^4(\mathbb{R}\mathrm{P}^4)$ where $SP^m(X) = X^m/\mathfrak{S}_m$ denotes the symmetric product of X, consult [85, Prop. 3.1].

The following statement is claimed to be "an easy consequence of Poincaré duality"; the homology is considered with coefficients in the field $\mathbb{Z}/2$.

> [85, **Lemma 4.3**] *There is an isomorphism $H_2(SP^4_\delta(\mathbb{R}\mathrm{P}^4)) \longrightarrow H_2(SP^4(\mathbb{R}\mathrm{P}^4))$ of homology groups, induced by the inclusion map $SP^4_\delta(\mathbb{R}\mathrm{P}^4) \hookrightarrow SP^4(\mathbb{R}\mathrm{P}^4)$.*

Further on, it was claimed that

$$H_2(SP^4(\mathbb{R}\mathrm{P}^4)) \cong H_2(SP^4(\mathbb{R}\mathrm{P}^\infty)) \cong \\ H_2(K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,3) \times K(\mathbb{Z}/2,4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Now we prove that $H_2(SP^4_\delta(\mathbb{R}\mathrm{P}^4))$ is not isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Indeed, there is a sequence of isomorphisms

$$\begin{aligned}
H_2(SP^4_\delta(\mathbb{R}\mathrm{P}^4)) &\cong H^2(SP^4_\delta(\mathbb{R}\mathrm{P}^4)) && \text{by the Universal Coefficient Theorem,}\\
&\cong H^2((S^4)^4_\delta/\mathfrak{S}^\pm_4) && \text{by definition of } SP^4_\delta(\mathbb{R}\mathrm{P}^4),\\
&\cong H^2(\mathrm{E}\mathfrak{S}^\pm_4 \times_{\mathfrak{S}^\pm_4} (S^4)^4_\delta) && \text{since the action of } \mathfrak{S}^\pm_4 \text{ is free,}\\
&\cong H^2(\mathfrak{S}^\pm_4) && \text{since } (S^4)^4_\delta \text{ is 2-connected [39].}
\end{aligned}$$

A result of Nakaoka [37, Thm. 5.3.1] combined with $H^2(\mathfrak{S}_4) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ [1, Ex. VI.1.13] implies that

$$\begin{aligned}
H^2(\mathfrak{S}^\pm_4) &\cong \bigoplus_{p=0}^{2} H^p(\mathfrak{S}_4, H^{2-p}((\mathbb{Z}/2)^4))\\
&\cong H^0(\mathfrak{S}_4, H^2((\mathbb{Z}/2)^4)) \oplus H^1(\mathfrak{S}_4, H^1((\mathbb{Z}/2)^4)) \oplus H^2(\mathfrak{S}_4, H^0((\mathbb{Z}/2)^4))\\
&\cong H^2((\mathbb{Z}/2)^4)^{\mathfrak{S}_4} \oplus H^1(\mathfrak{S}_4, H^1((\mathbb{Z}/2)^4)) \oplus H^2(\mathfrak{S}_4)\\
&\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus H^1(\mathfrak{S}_4, H^1((\mathbb{Z}/2)^4)) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\end{aligned}$$

Thus $H_2(SP^4_\delta(\mathbb{R}\mathrm{P}^4))$ is not isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and therefore [85, Lemma 4.3] is not true.

A Gap in the proof of [85, Thm. 5.1]

Here we discuss a gap in the proof of the following theorem, the second main result in [85].

> [85, **Theorem 5.1**] *Suppose that μ is a measure on $\mathbb{R}^4$ admitting a 2-dimensional plane of symmetry in the sense that for some 2-plane $L \subset \mathbb{R}^4$ and the associated reflection $R_L\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, for each measurable set $A \subset \mathbb{R}^4$, $\mu(A) = \mu(R_L(A))$. Then μ admits a 4-equipartition.*

The proof of the theorem is based on [85, Claim on p. 165]. For convenience we copy the claim with the first two sentences of its proof from [85].

> [85, **Claim on p. 165**] *There does not exist a G-equivariant map $f\colon (S^4)^4_\Delta \longrightarrow S(U_4 \oplus \lambda)$, where $S(U_4 \oplus \lambda)$ is the G-invariant unit sphere in $U_4 \oplus \lambda$. In other words each G-invariant map $f\colon (S^4)^4_\Delta \longrightarrow U_4 \oplus \lambda$ has a zero.*
>
> *Proof of the Claim.* The claim is equivalent to the statement that the vector bundle $\xi\colon (S^4)^4_\Delta \times_G (U_4 \oplus \lambda) \longrightarrow (S^4)^4_\Delta/G$ does not admit a non-zero continuous cross section. For this it is sufficient to show that the top Stiefel–Whitney class $w_n(\xi)$ is non-zero.

The group G is the direct sum $\mathfrak{S}^\pm_4 \oplus \mathbb{Z}/2$, and $(S^4)^4_\Delta$ is the largest subspace of $(S^4)^4$ on which the group G acts freely. The base space $(S^4)^4_\Delta/G$ of the vector bundle ξ is an open manifold of dimension 16. The real G-representation $U_4 \oplus \lambda$ is 16-dimensional and therefore ξ is a 16-dimensional vector bundle. Thus the top Stiefel–Whitney class $w_{16}(\xi)$ lives in $H^{16}((S^4)^4_\Delta/G; \mathbb{Z}/2) = 0$ and so it vanishes. This contradicts the proof of the claim.

Actually, more is true: Since ξ is a 16-dimensional vector bundle over a connected non-compact 16-dimensional manifold $(S^4)^4_\Delta/G$, an exercise from Koschorke [52, Exer. 3.11] guarantees the existence of a nowhere vanishing cross section, again contradicting the proof of the claim.

2.7 A gap in Ramos [68]

In this section we will give a counterexample to [68, Lem. 6.2], from which Ramos derives his main result [68, Thm. 6.3] by induction. Our Counterexample 2.30 exploits the fact that a certain coordinate permutation action has fixed points, a crucial fact that is missed in the proof of [68, Lem. 6.2].

The following table lists bounds for $\Delta(j,k)$ that are obtained directly from [68, Thm. 6.3]. They cannot be obtained from [68, Thm. 4.6] or any other result in his article.

$\Delta(2^m, 2)$	$\leq$	$3 \cdot 2^m/2$
$\Delta(2^m, 3)$	$\leq$	$5 \cdot 2^m/2$
$\Delta(2^m, 4)$	$\leq$	$9 \cdot 2^m/2$
$\Delta(2^m, 5)$	$\leq$	$15 \cdot 2^m/2$

Table 2.3: Here $m \geq 0$. From [68, p. 164].

In order to clarify Ramos' approach, we will describe his initial configuration space, which he modifies twice. The second modification is the basis for [68, Lem. 6.2]. Given a dimension $d \geq 1$ and a number of hyperplanes $k \geq 1$ and masses $\mu_1, \dots, \mu_j$, the initial configuration space is defined as

$$B^{d-1} \times \cdots \times B^{d-1} = B^{k(d-1)}.$$

Here B^{d-1} is regarded as the upper hemisphere of S^{d-1}, where each sphere S^{d-1} is the space of directions of normal vectors in $\mathbb{R}^d$ of hyperplanes in $\mathbb{R}^d$ that bisect the first mass μ_1. We make the assumption that each mass has a unique bisecting hyperplane, which is the case, if masses have connected support. The results for general masses then follow by approximation; see [48, S. 275]. In order for his first result [68, Thm. 4.6] to hold, Ramos makes restricts the configuration space a first time to

$$B^{n_1} \times \cdots \times B^{n_k} \subseteq B^{k(d-1)},$$

where $n_i \leq d-1$ for all $i = 1, \dots, k$ and $\sum n_i = (2^k - 1)j - k$ [68, Sec. 4]. Note that [68, Thm. 4.6] does not yield the upper bounds in Table 2.3.

Let $\mu_1, \dots, \mu_j$ be masses on $\mathbb{R}^d$. For $x = (x_1, \dots, x_k) \in B^{n_1} \times \cdots \times B^{n_k}$ and $i \in [k]$, let $H(x_i)$ be the unique hyperplane in $\mathbb{R}^d$ with normal vector x_i that bisects the first mass μ_1, where we regard the x_i in $\mathbb{R}^d$ via the inclusions $B^{n_i} \subseteq B^{d-1} \hookrightarrow S^{d-1}$. For $\alpha \in \{0,1\}$, let $H^\alpha(x_i)$ be the positive (if $\alpha = 0$) respectively negative (if $\alpha = 1$) closed half-space defined by $H(x_i)$. Observe the difference in notation to H_{x_i}, which we used to denote the affine hyperplane corresponding to a point x_i in the sphere S^d of one dimension higher.

Ramos defines the test map

$$\begin{aligned} \Phi\colon B^{n_1} \times \cdots \times B^{n_k} &\xrightarrow{\psi} (\mathbb{R}^{2^k})^{\oplus j} \xrightarrow{U \oplus \cdots \oplus U} (\mathbb{R}^{2^k})^{\oplus j} \xrightarrow{\pi} (\mathbb{R}^{2^k - 1})^{\oplus j} = U_k^{\oplus j}, \\ (x_1, \dots, x_k) &\overset{\psi}{\longmapsto} \Big(\mu_1\big(\bigcap_{i=1}^{k} H^{\alpha_i}(x_1)\big), \dots, \mu_j\big(\bigcap_{i=1}^{k} H^{\alpha_i}(x_k)\big)\Big)_{(\alpha_1, \dots, \alpha_k) \in (\mathbb{Z}/2)^k}. \end{aligned}$$

The map ψ is followed by an orthogonal coordinate transformation $U \oplus \cdots \oplus U = U^{\oplus j}$ given by the matrix

$$U = \left(\epsilon^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}\right) \quad \text{for} \quad (i_1,\ldots,i_k),(j_1,\ldots,j_k) \in (\mathbb{Z}/2)^k,$$

where

$$\epsilon^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} = (-1)^b \quad \text{and} \quad b = (i_1,\ldots,i_k)^t(j_1,\ldots,j_k).$$

The map π chops off the coordinates of $(U \oplus \cdots \oplus U) \circ \phi$ corresponding to the row of U with index $(i_1,\ldots,i_k) = (0,\ldots,0)$. In these coordinates, $(U \oplus \cdots \oplus U) \circ \phi$ is constant and equal to 1, since the value of such a coordinate is the sum, for a fixed mass, of the masses of all of the orthants. The map Φ can be viewed as a map to a $((2^k-1)j-k)$-dimensional subspace of $(\mathbb{R}^{2^k-1})^{\oplus j} \cong U_k^{\oplus j}$ since the map Φ has k zero-components due to the fact that all hyperplanes bisect the first mass by definition.

Proposition 2.24 ([68, Property 4.4]). *Let $x = (x_1,\ldots,x_k) \in \mathbb{R}^d$, then $\Phi(x) = 0$ if and only if the hyperplanes $H(x_i) \subset \mathbb{R}^d$ with normal vectors x_i that bisect the first mass μ_1 form an equipartition of the masses $\mu_1,\ldots,\mu_j$. Moreover, if $\Phi(x) = 0$, then $\Delta(j,k) \leq d$.*

In the following definition, Ramos introduces the notion of a map that is equivariant on the boundary of the domain and calls this *antipodal*. For this we let $(\mathbb{Z}/2)^k$ act antipodally on the boundary of $B^{n_1} \times \cdots \times B^{n_k}$.

Definition 2.25 ([68, p. 151]). A continuous map $f\colon B^{n_1} \times \cdots \times B^{n_k} \longrightarrow \mathbb{R}^{(2^k-1)j}$ is *antipodal in the m-th component with respect to the n-th ball with antipodality a_{pq}* $\in \{0,1\}$, for $p \in [(2^k-1)j]$ and $q \in [k]$, if

$$f(x_1,\ldots,-x_q,\ldots,x_k) = (-1)^{a_{pq}} f_p(x_1,\ldots,x_q,\ldots,x_k)$$
$$\text{for all } (x_1,\ldots,x_k) \in B^{n_1} \times \cdots \times S^{n_q-1} \times \cdots \times B^{n_k}.$$

Call f *antipodal* if f is antipodal in all components with respect to all balls. In this case we call $A = (a_{pq})_{p,q} \in \mathbb{R}^{(2^k-1)j \times k}$ the *antipodality matrix* of f.

Using the antipodality matrix A, we define an action of $(\mathbb{Z}/2)^k$ on $\mathbb{R}^{(2^k-1)j}$ by letting the generators of $(\mathbb{Z}/2)^k$ act by changing the signs of vectors in $\mathbb{R}^{(2^k-1)j}$ according to the columns of A. In this restricted sense, f is equivariant on the boundary of $B^{n_1} \times \cdots \times B^{n_k}$.

Proposition 2.26 ([68, Property 4.3]). *The test map Φ is antipodal. Its antipodality in the component with index $(i_1,\ldots,i_q,\ldots,i_k)$ with respect to the q-th ball is i_q (for any mass). Hence the rows of A are precisely all $0/1$-vectors of length k, each repeated j times, up to some re-ordering of the rows that depends only the labeling of the components of Φ. If we regard Φ as mapping into $U_k^{\oplus j}$, then A consists of all $0/1$-vectors of length k except of $(0,\ldots,0)$, each repeated j times.*

Ramos's method of proof is to show that the parity of the number of zeros of the test map Φ on the given domain is odd and hence the map has at least one zero. In [68, Thm. 4.6] he shows that if the permanent of a certain matrix is odd, then the parity of the number of zeros of Φ is also odd. However, this permanent is odd in only a few cases and in particular in none of the cases listed in Table 2.3. To prove [68, Lem. 6.2] and obtain the results in Table 2.3, Ramos restricts the configuration space a second time with the goal of obtaining more cases where the matrix permanent is odd. Instead of a product of balls, he uses a subspace of a product of balls: For $p, q \geq 1$ define

$$(B^p)^q_{\leq} = \{(x_1, \ldots, x_t) \in (\mathbb{R}^p)^q \;:\; \|x_1\| \leq \|x_2\| \leq \cdots \leq \|x_q\| \leq 1\} \subseteq (B^p)^q.$$

The space $(B^p)^q_{\leq}$ is a closed fundamental domain (meaning that boundary points may be contained in an orbit of an interior point) for the action of the symmetric group $\mathfrak{S}_q$ on $(B^p)^q$ given by permuting copies. [68, Lem. 6.2] is a result that relates the parity of the number of zeros of the test map Φ on $(B^p)^q_{\leq}$ to the parity of the number of zeros of Φ on the boundary of $(B^p)^q_{\leq}$. Ramos parametrizes the boundary as follows: For $1 \leq m < n \leq q$, define sets

$$\begin{aligned} C_{m,n} &= \{(x_1, \ldots, x_q) \in (B^p)^q_{\leq} \;:\; \|x_m\| = \|x_n\|\}, \\ C_{q,q+1} &= \{(x_1, \ldots, x_q) \in (B^p)^q_{\leq} \;:\; \|x_q\| = 1\}. \end{aligned}$$

Here $C_{q,q+1}$ can be regarded as the "lid" of $(B^p)^q_{\leq}$, where the "top lid" $X^+_{q,q+1} = X_{q,q+1} \cap \{x_q \geq 0\}$ and the "bottom lid" $X^-_{q,q+1} = X_{q,q+1} \cap \{x_q \leq 0\}$ are homeomorphic to $(B^p)^{q-1}_{\leq} \times B^{p-1}$. Hence

$$\operatorname{bd}(B^p)^q_{\leq} \;=\; \biguplus_{1 \leq m \leq q} C_{m,m+1}\;,$$

where "$\biguplus$" denotes the union of sets whose relative interiors are disjoint. On the sets $C_{m,n}$, Ramos defines a permutation action given by

$$\begin{aligned} \beta_{mn} \colon C_{m,n} &\longrightarrow C_{m,n} \\ (x_1, \ldots, x_m, \ldots, x_n, \ldots, x_q) &\mapsto (x_1, \ldots, x_n, \ldots, x_m, \ldots, x_q). \end{aligned}$$

Notice that points in the subsets $\{x \in (B^p)^q_{\leq} \;:\; x_m = x_n\} \subset C_{m,n}$ are fixed by this action. Hence the action is not fixed point free.

For the proofs, Ramos switches to a piecewise-linear (PL) approximation of the test map that maps the simplices of a "symmetric" triangulation of $(B^p)^q_{\leq}$ into general position with respect to the origin. See the following definition for these notions.

Definition 2.27 ([68, p. 149])**.** If T is a pseudomanifold, then we call a map $r \colon \|T\| \to \mathbb{R}^n$ *piecewise linear* if it is affine on every simplex of T. We call r *non-degenerate* if given any m-simplex $\sigma \in T$, any m component functions of r have at most one common zero on σ and any common zero lies in the relative interior of σ. We will say that r is *NDPL* if r is both non-degenerate and piecewise linear.

The test map or its NDPL approximation is again required to be "equivariant" in some sense. This is made precise in the following definition.

Definition 2.28 ([68, p. 162])**.** Given a map $r = (r', r'') \colon (B^p)^q_{\leq} \longrightarrow \mathbb{R}^{pq}$, where r' denotes the first $pq - 1$ components of r and r'' the last component, we call r *symmetric for the zeros in the boundary* if for all $1 \leq m < n \leq q$ and all $x \in C_{m,n}$ the following implication holds:

$$r'(x) = 0 \;\text{ implies that }\; r'(\beta_{mn}(x)) = 0 \text{ and } r''(x) = r''(\beta_{mn}(x)).$$

Lemma 2.29 ([68, Lem. 6.2]). *Let $r = (r', r'') \colon (B^p)^q_\le \longrightarrow \mathbb{R}^{pq}$ be a map where r' denotes the first $pq-1$ components and r'' the last component. Suppose r is NDPL and symmetric for the zeros in the boundary. Let r be antipodal in the last component with respect to the q-th ball and let $a = a_{pq,q} \in \{0,1\}$ be its antipodality. If $P(r', r''; (B^p)^q_\le)$ denotes the parity of the number of zeros of r in $(B^p)^q_\le$ and $P(r'; (B^p)^{q-1}_\le \times B^{p-1})$ denotes the parity of the number of zeros of r' in $X^+_{q,q+1} \approx (B^p)^{q-1}_\le \times B^{p-1}$, the "top lid" of the boundary of $(B^p)^q_\le$, then we have the following equality:*

$$P(r', r''; (B^p)^q_\le) = a \cdot P(r'; (B^p)^{q-1}_\le \times B^{p-1}).$$

Example 2.30 (Counterexample to [68, Lem. 6.2]). This example exploits the simple fact that the permutation action on the coordinates in $C_{m,n}$ has fixed points, a fact that Ramos does not account for in his proof of [68, Lem. 6.2]. Let $p = 1$ and $q = 3$. Then

$$(B^p)^q_\le = (B^1)^3_\le = \{(x, y, z) \in \mathbb{R}^3 : |x| \le |y| \le |z| \le 1\}.$$

See Figures 2.1a and 2.1b for a visualization of $(B^1)^3_\le$. Define the following sets and color them as in the Figures:

$$\begin{aligned}
F_{x,y} &= \{(x,y,z) \in (B^1)^3_\le \ : \ x = y\} \subset C_{1,2}, && \text{“red”}\\
F_{y,z} &= \{(x,y,z) \in (B^1)^3_\le \ : \ y = z\} \subset C_{2,3}, && \text{“blue”}\\
F_{x,z} &= \{(x,y,z) \in (B^1)^3_\le \ : \ x = z\} \subset C_{1,3}, && \text{“green”}\\
\mathrm{Top} &= \{(x,y,z) \in (B^1)^3_\le \ : \ z = 1\} \subset C_{3,4}, && \text{“black”}\\
\mathrm{Bot} &= \{(x,y,z) \in (B^1)^3_\le \ : \ z = -1\} \subset C_{3,4}. && \text{“black”}
\end{aligned}$$

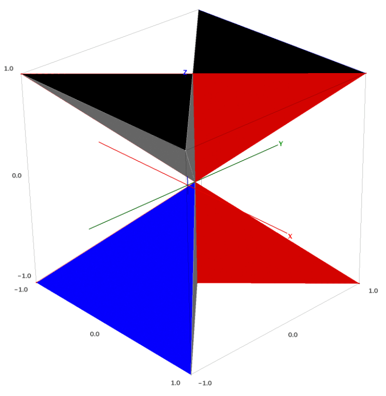
We will now construct a map $r = (r', r'') \colon (B^1)^3_\le \longrightarrow \mathbb{R}^3$ that contradicts [68, Lem. 6.2].

(i) Rotate $(B^1)^3_\le$ by 90° to the right along the y-axis. Now Top and Bot lie in the two parallel hyperplanes $\{x = 1\}$ and $\{x = -1\}$.

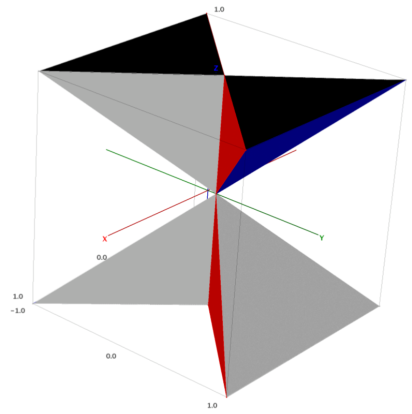
(ii) Rotate $(B^1)^3_\le$ along the x-axis and translate it such that the z-axis runs through $F_{x,y,1}$ and $F_{y,z,1}$ and the origin lies in the interior of the tetrahedron that has $F_{x,y,1}$ and $F_{y,z,1}$ as two of its faces. See Figure 2.1c.

The map r' has a zero in $F_{x,y}$ and $F_{y,z}$. By exploiting the fact that the permutation $\beta_{1,2}$ and $\beta_{2,3}$ fix these zeros, we see that the map $r = (r', r'')$ is in fact symmetric for the zeros in the boundary. Since $r''(x, y, -z) = r''(x, y, z) = (-1)^0 r''(x, y, z)$, the map r is antipodal in the last component with respect to the third ball with antipodality $a = 0$. It is easy to check that r is non-degenerate. Moreover, r has exactly one zero in $(B^1)^3_\le$. Hence

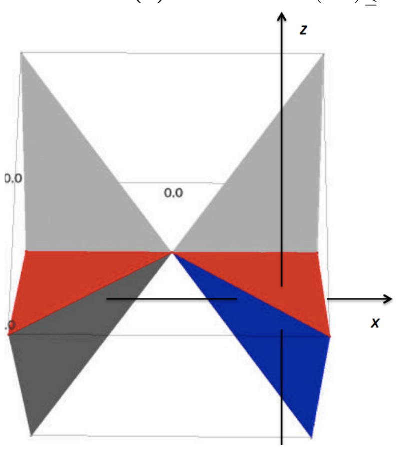
$$P(r', r''; (B^1)^3_\le) = 1 \ne 0 = 0 \cdot P(r'; (B^1)^2_\le \times B^0).$$



(a) One view of $(B^1)^3_{\leq}$



(b) Another view of $(B^1)^3_{\leq}$



(c) The r image of $(B^1)^3_{\leq}$

Figure 2.1

2.8 Further gaps in the literature

In this section we explain essential gaps in proofs of the main results in the papers of Mani-Levitska et al. [57] and Živaljević [86].

2.8.1 Gaps in [57]

Mani-Levitska et al. in their 2006 paper [57] studied the Ramos conjecture in the case of two hyperplanes, $k = 2$. One of the main result of this paper [57, Thm. 4] was a criterion under which for special values of m, in particular for $m = 1$, one would get $\Delta(4m+1, 2) \leq 6m+2$.

To get this criterion, they used the product configuration space/test map scheme and applied the equivariant obstruction theory of tom Dieck [34, Sec. II.3] in order to study the nonexistence of D_8-equivariant maps $S^d \times S^d \longrightarrow S(U_2^{\oplus j})$. Indeed, in the beginning of [57, Sec. 2.3.3] the authors supply details on the equivariant obstruction theory they apply as well as about the first isomorphism that will be used in the identification of the obstruction element:

> [57, **Section 2.3.3**] Once a problem is reduced to the question of (non) existence of equivariant map, one can use some standard topological tools for its solution. For example, one can use the cohomological index theory for this purpose [14,17,45,47]. This approach is discussed in Section 4.1. In this paper our main tool is elementary equivariant obstruction theory [13], refined by some basic equivariant bordism, and group homology calculations.
>
> Suppose that M^n is orientable, n-dimensional, **free** G-manifold and that V is a m-dimensional, real representation of G. Then the first obstruction for the existence of an equivariant map $f\colon M \longrightarrow S(V)$, is a cohomology class
>
> $$\omega \in H^m_G\big(M, \pi_{m-1}\big(S(V)\big)\big)$$
>
> in the appropriate equivariant cohomology group [13, Section II.3], where $\pi_k(S(V))$ is seen as a G-module. The action of G on M induces a G-module structure on the group $H_n(M, \mathbb{Z}) \cong \mathbb{Z}$ which is denoted by $\mathcal{O}$. The associated homomorphism $\theta\colon G \longrightarrow \{-1, +1\}$ is called the orientation character. Let A be a (left) G-module. The Poincaré duality for equivariant (co)homology is the following isomorphism [39],
>
> $$H^k_G(M, A) \xrightarrow{D} H^G_{n-k}(M, A \otimes \mathcal{O}).$$

(Boldface added for emphasis.) In [57, Sec. 2.6] they present further isomorphisms that will be used in the identification of the obstruction element:

> [57, **Section 2.6**] By equivariant Poincaré duality, Section 2.3.3, the dual $D(\omega)$ of the first obstruction cohomology class $\omega \in H^m_G(M, \pi_{m-1}S(V))$ lies in the equivariant homology group $H^G_{n-m}(M, \pi_{m-1}S(V) \otimes \mathcal{Z})$. If M is $(n-m)$-connected, then there is an isomorphism [11, Theorem II.5.2]
>
> $$H^G_{n-m}(M, \pi_{m-1}S(V) \otimes \mathcal{Z}) \xrightarrow{\cong} H_{n-m}(G, \pi_{m-1}S(V) \otimes \mathcal{Z}).$$
>
> This allows us to interpret $D(\omega)$ as an element in the latter group. Moreover, if the coefficient G-module $\pi_{m-1}S(V) \otimes \mathcal{Z}$ is trivial, then the homology group $H_{n-m}(G, \mathbb{Z}) \cong H_{n-m}(BG, \mathbb{Z})$ is for $n - m \leq 3$ isomorphic to the oriented G-bordism group $\Omega_{n-m}(G) \cong \Omega_{n-m}(BG)$, that is to the groups based on free, oriented G-manifolds [12].
>
> Our objective is to identify the relevant obstruction classes. Already the algebraically trivial case $H_0(G, M) \cong M_G$, where $M_G = \mathbb{Z} \otimes M$ is the group of coinvariants, may be combinatorially sufficiently interesting. Indeed, the parity count formulas applied in [32], see also [49, Section 14.3], may be seen as an instance of the case $M_G \cong \mathbb{Z}/2$.
>
> However, the most interesting examples explored in this paper involve the identification of 1-dimensional obstruction classes. Since these classes in practice usually arise as the fundamental classes of zero set manifolds, our first choice will be the bordism group $\Omega_1(G)$.

After presenting the method used in the paper [57] for the study of the nonexistence of D_8-equivariant maps $S^d \times S^d \longrightarrow S(U_2^{\oplus j})$ we can point out the gap. For the method to work the action of the group (in this case D_8) on the manifold (in this case $S^d \times S^d$) has to be free. The action of D_8 on $S^d \times S^d$ is **not free** and therefore the method can not be applied to the problem of the nonexistence of D_8-equivariant maps $S^d \times S^d \longrightarrow S(U_2^{\oplus j})$. Consequently, all the claims by Mani-Levitska et al. derived from the application of this method — namely [57, Thm. 4, Prop. 25, Thm. 33, Cor. 37] — are not proven.

Furthermore, we point out that

- the Poincaré duality isomorphism $H^k_G(M, A) \xrightarrow{D} H^G_{n-k}(M, A \otimes \mathcal{O})$ stands only with the assumption that M is an oriented compact manifold with a free G-action; a complete proof can be found in [21, Thm. 1.4],
- the isomorphism $H^G_{n-m}(M, A) \xrightarrow{\cong} H_{n-m}(G, A)$ holds for a trivial G-module A when M is an $(n-m)$-connected space on which the G-action is free.

Finally, let us mention that already in 1998 Živaljević [84, Proof of Prop. 4.9] has given a suggestion how to deal with the presence of non-free actions in the context of equivariant obstruction theory applied to the Ramos conjecture: There he studied the nonexistence of a $(\mathbb{Z}/2 \oplus D_8)$-equivariant map $(S^3)^3 \longrightarrow S(\mathbb{R}^9)$ with non-free action on the domain using relative equivariant obstruction theory.

2.8.2 A gap in [86]

In his 2015 paper [86], Živaljević studied the Ramos conjecture in the case of two hyperplanes, $k = 2$. The main result [86, Thm. 2.1] claims that $\Delta(4 \cdot 2^k + 1, 2) = 6 \cdot 2^k + 2$. For this claim we gave a degree-based proof, see Theorem 2.12.

In order to study the nonexistence of D_8-equivariant maps induced by the product configuration scheme $S^d \times S^d \longrightarrow S(U_2^{\oplus j})$ Živaljević in [86, App. B] introduces an "algebraic equivariant obstruction theory." We explain why the proofs for [86, Thms. 2.1 and 2.2] using this obstruction theory are not complete, as they fail to validate essential preconditions that are not automatically provided by this theory.

Following [86, App. B], suppose that X is a d-dimensional G-space with admissible filtration [86, Def. B.3]:

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n \subset X_{n+1} \subset \cdots \subset X_d = X.$$

Furthermore, let Y be a G-CW-complex with associated filtration by skeleta:

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_{n-1} \subset Y_n \subset Y_{n+1} \subset \cdots \subset Y_\nu = Y.$$

Then, according to [86, Prop. B.6], if we assume that there exists a G-equivariant map $f\colon X \longrightarrow Y$, then there exists a chain map

$$f_*\colon H_n(X_n, X_{n-1}; \mathbb{Z}) \longrightarrow H_n(Y_n, Y_{n-1}; \mathbb{Z})$$

between the associated augmented chain complexes of $\mathbb{Z}[G]$-modules:

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] & \dots \arrow[r, "\partial"] & C_1 \arrow[r, "\partial"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "="] & 0 \\
\dots \arrow[r, "\partial"] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & D_1 \arrow[r, "\partial"] & D_0 \arrow[r, "\partial"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where $C_n = H_n(X_n, X_{n-1}; \mathbb{Z})$ and $D_n = H_n(Y_n, Y_{n-1}; \mathbb{Z})$ for every n.

Now [86, Sec. B.3] studies the existence of chain maps between chain complexes of $\mathbb{Z}[G]$-modules. [86, Prop. B.7] introduces an obstruction theory as follows: For $n+1 \leq d$ we are given

- a finite chain complexes of $\mathbb{Z}[G]$-modules $C_* = \{C_k\}_{k=-1}^{d}$ and $D_* = \{D_k\}_{k=-1}^{d}$, with $C_{-1} = D_{-1} = \mathbb{Z}$, and
- a fixed partial chain map $F_{n-1} = (f_j)_{j=-1}^{n-1} \colon \{C_k\}_{k=-1}^{n-1} \longrightarrow \{D_k\}_{k=-1}^{n-1}$.

We further assume that F_{n-1} can be extended to dimension n, that is, there exists $f_n \colon C_n \longrightarrow D_n$ such that $\partial f_n = f_{n-1}\partial$. Then [86, (B.7)] defines the obstruction to the existence of a partial chain map as

$$F_{n+1} = (f_j)_{j=-1}^{n+1} \colon \{C_k\}_{k=-1}^{n+1} \longrightarrow \{D_k\}_{k=-1}^{n+1},$$

which extends the partial chain map F_{n-1}, with a possible modification of f_n, as an appropriate element θ of the cohomology group:

$$H^{n+1}(C_*; H_n(D_*)) = H_{n+1}(\mathrm{Hom}(C_*, H_n(D_*)).$$

The element θ is represented by the cocycle [86, (B.8)]:

$$\theta(f_n) \colon C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f_n} Z_n(D_*) \xrightarrow{\pi} H_n(D_*).$$

Now [86, Prop. B.7] states that vanishing of θ is not only necessary but also sufficient for the existence of the extension F_{n+1} if C_n and C_{n+1} are projective modules.

The obstruction θ highly depends on the partial chain map $F_{n-1} = (f_j)_{j=-1}^{n-1}$. The first paragraph of [86, Sec. B.4] comments on this issue as follows:

> [86, **Section B.4. Heuristics for evaluating the obstruction** θ**.**] In many cases the chain map $F_{n-1} = (f_j)_{j=-1}^{n-1}$, which in Proposition B.7 serves as an input for calculating the obstruction θ, is unique up to a chain homotopy. This happens for example when D_* is a chain complex associated to a G-sphere Y of dimension n.

The last sentence is not true: In order to guarantee that the input partial chain map $F_{n-1} = (f_j)_{j=-1}^{n-1}$ is unique up to a chain homotopy an additional condition on the chain complex C_* needs to be fulfilled, for example that $\{C_k\}_{k=-1}^{n-1}$ is a sequence of projective $\mathbb{Z}[G]$-modules.

The algebraic obstruction theory just described is applied in [86] to the problem of the nonexistence of a D_8-equivariant map $S^d \times S^d \longrightarrow S(U_2^{\oplus j})$:

- in [86, Sec. 6] an admissible filtration of $S^d \times S^d$ is defined,
- in [86, Sec. 7] the top three levels of the associated chain complex C_* of $S^d \times S^d$ are described as projective $\mathbb{Z}[D_8]$-modules,
- in [86, Prop. 9.9] evaluates the obstruction θ for particular input data $F_{2d-2} = (f_j)_{j=-1}^{2n-2}$ proving that it does not vanish.

Since the D_8-action on $S^d \times S^d$ is not free the chain complex C_* of $\mathbb{Z}[D_8]$-modules associated to $S^d \times S^d$ is not a chain complex of projective $\mathbb{Z}[D_8]$-modules. Thus different input data $F_{2d-2} = (f_j)_{j=-1}^{2n-2}$ need not define the same obstruction θ computed in [86, Prop. 9.9]. Consequently, no conclusion about the nonexistence of an extension of F_{2d-1}, and further of a D_8-equivariant map $S^d \times S^d \longrightarrow S(U_2^{\oplus j})$ can be obtained from the computation of just one obstruction. This exhibits an essential gap in the proof of the main result [86, Thm. 2.1] as well as a serious deficiency in the proposed algebraic obstruction theory.

Chapter 3

A unified approach via relative equivariant obstruction theory

Abstract

The Grünbaum–Hadwiger–Ramos hyperplane mass partition problem asks for the smallest dimension $d_0 = \Delta(j,k)$ such that for any j masses in $\mathbb{R}^{d_0}$ there are k hyperplanes that cut each of the masses into 2^k equal parts. Ramos' conjecture is that the Avis–Ramos necessary lower bound condition $\Delta(j,k) \geq j(2^k-1)/k$ is also sufficient. We develop a join scheme for this problem, such that for any $d \geq 1$ the nonexistence of an $\mathfrak{S}_k^{\pm}$-equivariant map between spheres $(S^d)^{*k} \to S(W_k \oplus U_k^{\oplus j})$ that extends a test map on the subspace of $(S^d)^{*k}$ where the hyperoctahedral group $\mathfrak{S}_k^{\pm}$ acts non-freely, implies that $\Delta(j,k) \leq d$. For the sphere $(S^d)^{*k}$ we obtain a regular equivariant CW model, whose cells get a combinatorial interpretation with respect to measures on a modified moment curve. This allows us to apply relative equivariant obstruction theory successfully, even in the case when the difference of dimensions of the spheres $(S^d)^{*k}$ and $S(W_k \oplus U_k^{\oplus j})$ is greater than one. We give a rigorous, unified treatment of the previously announced cases of the Grünbaum–Hadwiger–Ramos problem, as well as a number of new cases for Ramos' conjecture.

Publication Remark. The results of this chapter are joint work with Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler [23].

3.1 Introduction

3.1.1 Grünbaum–Hadwiger–Ramos hyperplane mass partition problem

Recall the following problem that is due to Grünbaum [46, Sec. 4.(v)], Hadwiger [48], and Ramos [68].

The Grünbaum–Hadwiger–Ramos problem. *Determine the minimal dimension $d = \Delta(j,k)$ such that for every collection $\mathcal{M}$ of j masses on $\mathbb{R}^d$ there exists an arrangement $\mathcal{H}$ of k affine hyperplanes in $\mathbb{R}^d$ that equiparts $\mathcal{M}$.*

All available evidence up to now supports the following conjecture, though it has been established rigorously only in few special cases; see Section 2.1.2 for a summary of known results, excluding the new results presented in this chapter.

The Ramos conjecture. $\Delta(j,k) = \lceil \frac{2^k-1}{k} j \rceil$ *for every choice of integers* $j \geq 1$ *and* $k \geq 1$.

3.1.2 Product scheme and join scheme

It seems natural to use $Y_{d,k} := (S^d)^k$ as a configuration space for any k oriented affine hyperplanes/halfspaces in $\mathbb{R}^d$. Indeed, this was our main approach in Chapter 2. This leads to the following *product scheme*: If there is no equivariant map

$$(S^d)^k \longrightarrow_{\mathfrak{S}_k^{\pm}} S(U_k^{\oplus j})$$

from the configuration space to the unit sphere in the space $U_k^{\oplus j}$ of values on the orthants of $\mathbb{R}^k$ that sum to 0, which is equivariant with respect to the hyperoctahedral (signed permutation) group $\mathfrak{S}_k^{\pm}$, then there is no counter-example for the given parameters, so $\Delta(j,k) \leq d$.

However, our critical review in Chapter 2 of the main papers on the Grünbaum–Hadwiger–Ramos problem since 1998 has shown that this scheme is very hard to handle. Except for the 2006 upper bounds by Mani-Levitska, Vrećica and Živaljević [57], derived from a Fadell–Husseini index calculation, it has produced very few valid results: The group action on $(S^d)^k$ is not free, the Fadell–Husseini index is rather large and thus yields weak results, and there is no efficient cell complex model at hand.

In this chapter, we provide a new unified approach. For this, we use a *join scheme*, as introduced by Blagojević and Ziegler [27], which takes the form

$$F: \quad (S^d)^{*k} \longrightarrow_{\mathfrak{S}_k^{\pm}} S(W_k \oplus U_k^{\oplus j}).$$

Here the domain $(S^d)^{*k} \subseteq \mathbb{R}^{(d+1)\times k}$ is a sphere of dimension $dk+k-1$, given by

$$X_{d,k} := \{(\lambda_1 x_1, \dots, \lambda_k x_k) : x_1, \dots, x_k \in S^d,\ \lambda_1, \dots, \lambda_k \geq 0,\ \lambda_1 + \dots + \lambda_k = 1\},$$

where we write $\lambda_1 x_1 + \dots + \lambda_k x_k$ as a short-hand for $(\lambda_1 x_1, \dots, \lambda_k x_k)$. The co-domain is a sphere of dimension $j(2^k-1)+k-2$. Both domain and co-domain are equipped with canonical $\mathfrak{S}_k^{\pm}$-actions. We observe that the map restricted to the points with non-trivial stabilizer (the "non-free part")

$$F': \quad X_{d,k}^{>1} \subset (S^d)^{*k} \longrightarrow_{\mathfrak{S}_k^{\pm}} S(W_k \oplus U_k^{\oplus j})$$

is the same up to homotopy for all test maps. If for any parameters (j,k,d) an equivariant extension F of F' does not exist, we get that $\Delta(j,k) \leq d$.

To decide the existence of this map, or at least obtain necessary criteria, we employ relative equivariant obstruction theory, as explained by tom Dieck [34, Sect. II.3]. This approach has the following aspects:

- The Fox–Neuwirth [41]/Björner–Ziegler [19] combinatorial stratification method yields a simple and efficient cone stratification for the space $\mathbb{R}^{(d+1)\times k}$, which is equivariant with respect to the action of $\mathfrak{S}_k^{\pm}$ on the columns.

- This yields a small equivariant regular CW complex model for the sphere $(S^d)^{*k} \subseteq \mathbb{R}^{(d+1)\times k}$, for which the the non-free part, given by an arrangement of k^2 subspheres of codimension $d+1$, is an invariant subcomplex. The cells $D_I^S(\sigma)$ in the complex are given by combinatorial data.
- To evaluate the obstruction cocycle, we use measures on a non-standard (binomial coefficient) moment curve. For the resulting test map, the relevant cells $D_I^S(\sigma)$ can be interpreted as k-tuples of hyperplanes such that some of the hyperplanes have to pass through prescribed points of the moment curve, or equivalently, they have to bisect some extra masses.

3.1.3 Statement of the main results

The join scheme reduces the Grünbaum–Hadwiger–Ramos problem to a combinatorial counting problem that can be solved by hand or by means of a computer: A k-bit *Gray code* is a $k \times 2^k$ binary matrix of all column vectors of length k such that two consecutive vectors differ by only one bit. Such a k-bit code can be interpreted as a Hamiltonian path in the graph of the k-cube $[0,1]^k$. The *transition count* of a row in a binary matrix A is the number of bit-changes, *not* counting a bit change from the last to the first entry. By *transition counts* of a matrix A we refer to the vector of the transition counts of the rows of the matrix A. Two binary matrices A and A' are *equivalent*, if A can be obtained from A' by a sequence of permutations of rows and/or inversion of bits in rows.

Definition 3.1. Let $d \geq 1$, $j \geq 1$, $\ell \geq 0$ and $k \geq 1$ be integers such that $dk = (2^k-1)j + \ell$ with $0 \leq \ell \leq d-1$. A binary matrix A of size $k \times j2^k$ is an *ℓ-equiparting matrix* if

(a) $A = (A_1, \ldots, A_j)$ for Gray codes $A_1, \ldots, A_j$ with the property that the last column of A_i is equal to the first column of A_{i+1} for $1 \leq i < j$; and

(b) there is one row of the matrix A with the transition count $d-\ell$, while all other rows have transition count d.

Example 3.2. If $d=5$, $j=2$, $\ell=1$ and $k=3$, then a possible 1-equiparting matrix is

$$A = (A_1, A_2) = \begin{pmatrix} 0&0&1&1&0&0&1&1 & & 1&1&1&1&0&0&0&0 \\ 0&0&0&1&1&1&1&0 & & 0&1&1&0&0&0&1&1 \\ 0&1&1&1&1&0&0&0 & & 0&0&1&1&1&0&0&1 \end{pmatrix}.$$

Here the first row of A has transition count 4 while the other two rows have transition count 5.

Theorem 3.3. *Let $d \geq 1$, $j \geq 1$, $\ell \geq 0$ and $k \geq 2$ be integers with the property that $dk = (2^k-1)j + \ell$ and $0 \leq \ell \leq d-1$. The number of non-equivalent ℓ-equiparting matrices is the number of arrangements of k affine hyperplanes $\mathcal{H}$ that equipart a given collection of j disjoint intervals on a moment curve γ in $\mathbb{R}^d$, up to renumbering and orientation change of hyperplanes in $\mathcal{H}$, when it is forced that one of the hyperplanes passes through ℓ prescribed points on γ that lie to the left of the j disjoint intervals.*

In some situations this yields a solution for the Grünbaum–Hadwiger–Ramos problem.

Theorem 3.4. *Let $j \geq 1$ and $k \geq 3$ be integers with $d := \lceil \frac{2^k-1}{k} j \rceil$ and $\ell := \lceil \frac{2^k-1}{k} j \rceil k - (2^k-1)j = dk - (2^k-1)j$, which implies $0 \leq \ell < k \leq d$. If the number of non-equivalent ℓ-equiparting matrices of size $k \times j2^k$ is odd, then*

$$\Delta(j,k) = \lceil \tfrac{2^k-1}{k} j \rceil.$$

Theorem 3.4 is also true for $k = 1$ (and thus $d = j$, $\ell = 0$), where it yields the Ham Sandwich theorem: In this case an equiparting matrix A is a row vector of length $2d$ and transition count d. Thus each A_i is either $(0, 1)$ or $(1, 0)$, where A_i uniquely determines A_{i+1}. Hence, up to inversion of bits, A is unique and so $\Delta(d, 1) \leq d$, and consequently $\Delta(d, 1) = d$.

While the situation for $k = 1$ hyperplane is fully understood, we seem to be far from a complete solution for the case of $k = 2$ hyperplanes. However, we do obtain the following instances.

Theorem 3.5. *Let $t \geq 1$ be an integer. Then*

(i) $\Delta(2^t - 1, 2) = 3 \cdot 2^{t-1} - 1$,

(ii) $\Delta(2^t, 2) = 3 \cdot 2^{t-1}$,

(iii) $\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 2$.

The statements (i) and (iii) were already known: Part (i) is the only case where the lower bound of Ramos and the upper bound of Mani-Levitska, Vrećica, and Živaljević [57, Thm. 39] coincide. Part (ii) is Hadwiger's result [48] for $t = 1$; the general case was previously claimed by Mani-Levitska et al. [57, Prop. 25]. However, the proof of the result was incorrect and not recoverable, as explained in Section 2.8.1. Here we recover this result by a different method of proof. Similarly, statement (iii) was claimed by Živaljević [86, Thm. 2.1] with a flawed proof; for an explanation of the gap see Section 2.8.2. We gave a proof of (iii) via degrees of equivariant maps in Section 2.5. Here we will prove all three cases of Theorem 3.5 in a uniform way.

In the case of $k = 3$ hyperplanes we prove using Theorem 3.4 the following instances of the Ramos conjecture.

Theorem 3.6.

(i) $\Delta(2, 3) = 5$,

(ii) $\Delta(4, 3) = 10$.

Statement (i) was previously claimed by Ramos [68, Sec. 6.1]. A gap in the method that Ramos developed and used to get this result was explained in Section 2.7. It is also claimed by Vrećica and Živaljević in the recent preprint [78] without a proof for the crucial [78, Prop. 3].

The reduction result of Hadwiger and Ramos $\Delta(j, k) \leq \Delta(2j, k - 1)$ applied to Theorem 3.6 implies the following consequences. For details on reduction results see Section 2.3.

Corollary 3.7.

(i) $4 \leq \Delta(1, 4) \leq 5$,

(ii) $8 \leq \Delta(2, 4) \leq 10$.

Note that $\Delta(1, 4)$ is the open case of Grünbaum's original conjecture.

3.2 The join configuration space test map scheme

In this section we develop the join configuration test map scheme that was introduced in [30, Sec. 2.1]. A sufficient condition for $\Delta(j, k) \leq d$ will be phrased in terms of the nonexistence of a particular equivariant map between representation spheres.

3.2.1 Arrangements of k hyperplanes

Let $\hat{H} = \{x \in \mathbb{R}^d : \langle x, v\rangle = a\}$ be an affine hyperplane determined by a vector $v \in \mathbb{R}^d\backslash\{0\}$ and a constant $a \in \mathbb{R}$. The hyperplane $\hat{H}$ determines two (closed) halfspaces

$$\hat{H}^0 = \{x \in \mathbb{R}^d : \langle x, v\rangle \geq a\} \qquad \text{and} \qquad \hat{H}^1 = \{x \in \mathbb{R}^d : \langle x, v\rangle \leq a\}.$$

Let $\mathcal{H} = (\hat{H}_1, \dots, \hat{H}_k)$ be an arrangement of k affine hyperplanes in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \dots, \alpha_k)$ be an element of $(\mathbb{Z}/2)^k$. The *orthant* determined by the arrangement $\mathcal{H}$ and $\alpha \in (\mathbb{Z}/2)^k$ is the intersection

$$\mathcal{O}_\alpha^{\mathcal{H}} = \hat{H}_1^{\alpha_1} \cap \cdots \cap \hat{H}_k^{\alpha_k}.$$

Let $\mathcal{M} = (\mu_1, \dots, \mu_j)$ be a collection of finite Borel probability measures on $\mathbb{R}^d$ such that the measure of each hyperplane is zero. Such measures will be called *masses*. The assumptions about the measures guarantee that $\mu_i(\hat{H}_s^0)$ depends continuously on $\hat{H}_s^0$.

An arrangement of affine hyperplanes $\mathcal{H} = (\hat{H}_1, \dots, \hat{H}_k)$ *equiparts* the collection of masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$ if for every element $\alpha \in (\mathbb{Z}/2)^k$ and every $\ell \in \{1, \dots, j\}$

$$\mu_\ell(\mathcal{O}_\alpha^{\mathcal{H}}) = \tfrac{1}{2^k}.$$

3.2.2 The configuration spaces

The space of all oriented affine hyperplanes (or closed affine halfspaces) in $\mathbb{R}^d$ can be parametrized by the sphere S^d, where the north pole e_{d+1} and the south pole $-e_{d+1}$ represent hyperplanes at infinity. An oriented affine hyperplane in $\mathbb{R}^d$ *at infinity* is the set $\mathbb{R}^d$ or $\emptyset$, depending on the orientation. Indeed, embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ via the map $(\xi_1, \dots, \xi_d)^t \longmapsto (1, \xi_1, \dots, \xi_d)^t$. Then an oriented affine hyperplane $\hat{H}$ in $\mathbb{R}^d$ defines an oriented affine $(d-1)$-dimensional subspace of $\mathbb{R}^{d+1}$ that extends (uniquely) to an oriented linear hyperplane H in $\mathbb{R}^{d+1}$. The outer unit normal vector that determines the oriented linear hyperplane is a point on the sphere S^d.

We consider the following configuration spaces that parametrize arrangements of k oriented affine hyperplanes in $\mathbb{R}^d$:

(1) *The join configuration space*: $X_{d,k} := (S^d)^{*k} \cong S(\mathbb{R}^{(d+1)\times k})$,

(2) *The product configuration space*: $Y_{d,k} := (S^d)^k$.

The elements of the join $X_{d,k}$ can be presented as formal convex combinations $\lambda_1 v_1 + \cdots + \lambda_k v_k$, where $\lambda_i \geq 0$, $\sum \lambda_i = 1$ and $v_i \in S^d$.

3.2.3 The group actions

The space of all ordered k-tuples of oriented affine hyperplanes in $\mathbb{R}^d$ has natural symmetries: Each hyperplane can change orientation and the hyperplanes can be permuted. Thus the group $\mathfrak{S}_k^{\pm} := (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ encodes the symmetries of both configuration spaces.

The group $\mathfrak{S}_k^{\pm}$ acts on $X_{d,k}$ as follows. Each copy of $\mathbb{Z}/2$ acts antipodally on the appropriate sphere S^d in the join while the symmetric group $\mathfrak{S}_k$ acts by permuting factors in the join product.

More precisely, for $((\beta_1, \dots, \beta_k) \rtimes \pi) \in \mathfrak{S}_k^{\pm}$ and $\lambda_1 v_1 + \cdots + \lambda_k v_k \in X_{d,k}$ the action is given by

$$((\beta_1, \dots, \beta_k) \rtimes \tau) \cdot (\lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda_{\tau^{-1}(1)}(-1)^{\beta_1} v_{\tau^{-1}(1)} + \cdots + \lambda_{\tau^{-1}(k)}(-1)^{\beta_k} v_{\tau^{-1}(k)}.$$

The product space $Y_{d,k}$ is a subspace of the join $X_{d,k}$ via the diagonal embedding

$$Y_{d,k} \longrightarrow X_{d,k}, (v_1, \dots, v_k) \longmapsto \frac{1}{k} v_1 + \cdots + \frac{1}{k} v_k.$$

The product $Y_{d,k}$ is an invariant subspace of $X_{d,k}$ with respect to the $\mathfrak{S}_k^{\pm}$-action and consequently inherits the $\mathfrak{S}_k^{\pm}$-action from $X_{d,k}$. For $k \geq 2$, the action of $\mathfrak{S}_k^{\pm}$ is *not* free on either $X_{d,k}$ or $Y_{d,k}$.

The sets of points in the configuration spaces $X_{d,k}$ and $Y_{d,k}$ that have non-trivial stabilizer with respect to the action of $\mathfrak{S}_k^{\pm}$ can be described as follows:

$$X_{d,k}^{>1} = \{\lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_1 \cdots \lambda_k = 0, \text{ or } \lambda_s = \lambda_r \text{ and } v_s = \pm v_r \text{ for some } 1 \leq s < r \leq k\},$$

and

$$Y_{d,k}^{>1} = \{(v_1, \dots, v_k) : v_s = \pm v_r \text{ for some } 1 \leq s < r \leq k\}.$$

3.2.4 Test spaces

Consider the vector space $\mathbb{R}^{(\mathbb{Z}/2)^k}$, where the group element $((\beta_1, \dots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ acts on a vector $(y_{(\alpha_1,\dots,\alpha_k)})_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{Z}/2)^k} \in \mathbb{R}^{(\mathbb{Z}/2)^k}$ by acting on its indices as

$$((\beta_1, \dots, \beta_k) \rtimes \tau) \cdot (\alpha_1, \dots, \alpha_k) = (\beta_1 + \alpha_{\tau^{-1}(1)}, \dots, \beta_k + \alpha_{\tau^{-1}(k)}). \tag{3.1}$$

The subspace of $\mathbb{R}^{(\mathbb{Z}/2)^k}$ defined by

$$U_k = \Big\{(y_\alpha)_{\alpha\in(\mathbb{Z}/2)^k} \in \mathbb{R}^{(\mathbb{Z}/2)^k} : \sum_{\alpha\in(\mathbb{Z}/2)^k} y_\alpha = 0\Big\}$$

is $\mathfrak{S}_k^{\pm}$-invariant and therefore an $\mathfrak{S}_k^{\pm}$-subrepresentation.

Next we consider the vector space $\mathbb{R}^k$ and its subspace

$$W_k = \Big\{(z_1, \dots, z_k) \in \mathbb{R}^k : \sum_{i=1}^{k} z_i = 0\Big\}.$$

The group $\mathfrak{S}_k^{\pm}$ acts on $\mathbb{R}^k$ by permuting coordinates, that is, for $((\beta_1, \dots, \beta_k) \rtimes \tau) \in \mathfrak{S}_k^{\pm}$ and $(z_1, \dots, z_k) \in \mathbb{R}^k$ we have

$$((\beta_1, \dots, \beta_k) \rtimes \tau) \cdot (z_1, \dots, z_k) = (z_{\tau^{-1}(1)}, \dots, z_{\tau^{-1}(k)}). \tag{3.2}$$

In particular, the subgroup $(\mathbb{Z}/2)^k$ of $\mathfrak{S}_k^{\pm}$ acts trivially on $\mathbb{R}^k$. The subspace $W_k \subset \mathbb{R}^k$ is $\mathfrak{S}_k^{\pm}$-invariant and consequently a $\mathfrak{S}_k^{\pm}$-subrepresentation.

3.2.5 Test maps

The product test map associated to the collection of j masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$ from the configuration space $Y_{d,k}$ to the *test space* $U_k^{\oplus j}$ is defined by

$$\begin{aligned} \phi_{\mathcal{M}}\colon \qquad & Y_{d,k} \longrightarrow U_k^{\oplus j}, \\ (v_1, \dots, v_k) & \longmapsto \Big(\big(\mu_i(H_{v_1}^{\alpha_1} \cap \dots \cap H_{v_k}^{\alpha_k}) - \tfrac{1}{2^k} \big)_{(\alpha_1,\dots,\alpha_k) \in (\mathbb{Z}/2)^k} \Big)_{i \in \{1,\dots,j\}}. \end{aligned}$$

In this chapter we mostly work with the join configuration space $X_{d,k}$. The corresponding join test map associated to a collection of j masses $\mathcal{M} = (\mu_1, \dots, \mu_j)$ maps the configuration space $X_{d,k}$ into the related *test space* $W_k \oplus U_k^{\oplus j}$. It is defined by

$$\begin{aligned} \psi_{\mathcal{M}}\colon \qquad & X_{d,k} \longrightarrow W_k \oplus U_k^{\oplus j}, \\ \lambda_1 v_1 + \dots + \lambda_k v_k & \longmapsto (\lambda_1 - \tfrac{1}{k}, \dots, \lambda_k - \tfrac{1}{k}) \oplus (\lambda_1 \cdots \lambda_k) \cdot \phi_{\mathcal{M}}(v_1, \dots, v_k). \end{aligned}$$

Both maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ are $\mathfrak{S}_k^{\pm}$-equivariant with respect to the actions defined in Sections 3.2.3 and 3.2.4. Let $S(U_k^{\oplus j})$ and $S(W_k \oplus U_k^{\oplus j})$ denote the unit spheres in the vector spaces $U_k^{\oplus j}$ and $W_k \oplus U_k^{\oplus j}$, respectively. The maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ are called test maps since we have the following criterion, which reduces finding an equipartition to finding zeros of the test map.

Proposition 3.8. *Let $d \geq 1$, $k \geq 1$, and $j \geq 1$ be integers.*

(i) *Let $\mathcal{M}$ be a collection of j masses on $\mathbb{R}^d$, and let*

$$\phi_{\mathcal{M}}\colon Y_{d,k} \longrightarrow U_k^{\oplus j} \quad \text{and} \quad \psi_{\mathcal{M}}\colon X_{d,k} \longrightarrow W_k \oplus U_k^{\oplus j}$$

be the $\mathfrak{S}_k^{\pm}$-equivariant maps defined above. If $0 \in \operatorname{im} \phi_{\mathcal{M}}$, or $0 \in \operatorname{im} \psi_{\mathcal{M}}$, then there is an arrangement of k affine hyperplanes that equiparts $\mathcal{M}$.

(ii) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map of either type*

$$Y_{d,k} \longrightarrow S(U_k^{\oplus j}) \quad \text{or} \quad X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j}),$$

then $\Delta(j,k) \leq d$.

It is worth pointing out that $0 \in \operatorname{im} \phi_{\mathcal{M}}$ if and only if $0 \in \operatorname{im} \psi_{\mathcal{M}}$, while the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map $Y_{d,k} \longrightarrow S(U_k^{\oplus j})$ implies the existence of a $\mathfrak{S}_k^{\pm}$-equivariant map $X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ but not vice versa.

The homotopy class of the restrictions of the test maps $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ on the set of points with non-trivial stabilizer (as maps avoiding the origin) is independent of the choice of the masses $\mathcal{M}$, by the following proposition.

Proposition 3.9. *Let $\mathcal{M}$ and $\mathcal{M}'$ be collections of $j \geq 1$ masses in $\mathbb{R}^d$. Then*

(i) *$0 \notin \operatorname{im} \phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$ and $0 \notin \operatorname{im} \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$,*

(ii) *$\phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$ and $\phi_{\mathcal{M}'}|_{Y_{d,k}^{>1}}$ are $\mathfrak{S}_k^{\pm}$-homotopic as maps $Y_{d,k}^{>1} \longrightarrow U_k^{\oplus j} \backslash \{0\}$, and*

(iii) *$\psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ and $\psi_{\mathcal{M}'}|_{X_{d,k}^{>1}}$ are $\mathfrak{S}_k^{\pm}$-homotopic as maps $X_{d,k}^{>1} \longrightarrow (W_k \oplus U_k^{\oplus j}) \backslash \{0\}$.*

Proof. (i) If $(v_1, \dots, v_k) \in Y_{d,k}^{>1}$, then $v_s = \pm v_r$ for some $1 \leq s < r \leq k$. Consequently, the corresponding hyperplanes H_{v_i} and H_{v_j} coincide, possibly with opposite orientations. Thus some of the orthants associated to the collection of hyperplanes $(H_{v_1},\dots,H_{v_k})$ are empty. Consequently, Proposition 3.8 implies that $0\notin \operatorname{im}\phi_{\mathcal M}|_{Y^{>1}_{d,k}}$.

In the case where $\lambda_1v_1+\dots+\lambda_kv_k\in X^{>1}_{d,k}$ the additional case $\lambda_s=0$ for some $1\le s\le k$ may occur. If $\lambda_s=0$, then the s-th coordinate of $\psi(\lambda_1v_1+\dots+\lambda_kv_k)\in W_k\oplus U_k^{\oplus j}$ is equal to $-\frac1k$, and hence $0\notin\operatorname{im}\psi_{\mathcal M}|_{X^{>1}_{d,k}}$.
(ii) The equivariant homotopy between $\phi_{\mathcal M}|_{Y^{>1}_{d,k}}$ and $\phi_{\mathcal M'}|_{Y^{>1}_{d,k}}$ is just the linear homotopy in $U_k^{\oplus j}$. For this the linear homotopy should not have zeros; compare to Corollary 2.15. It suffices to prove that for each point $(v_1,\dots,v_k)\in Y^{>1}_{d,k}$, the points $\phi_{\mathcal M}(v_1,\dots,v_k)$ and $\phi_{\mathcal M'}(v_1,\dots,v_k)$ belong to some affine subspace of the test space that is not linear.

First observe that $\mathbb R^{(\mathbb Z/2)^k}$, considered as a real $(\mathbb Z/2)^k$ representation, is the real regular representation of $(\mathbb Z/2)^k$ and therefore it decomposes into the direct sum of all real irreducible representations. For this we use the fact that all real irreducible representations of $(\mathbb Z/2)^k$ are 1-dimensional. The subspace U_k seen as a real $(\mathbb Z/2)^k$ subrepresentation of $(\mathbb Z/2)^k$ decomposes as follows:

$$U_k\cong\bigoplus_{\alpha\in(\mathbb Z/2)^k\backslash\{0\}}V_\alpha. \tag{3.3}$$

Here V_α is the 1-dimensional real representation of $(\mathbb Z/2)^k$ determined by $\beta\cdot v=-v$ for $x\in V_\alpha$ if and only if $\alpha\cdot\beta:=\sum\alpha_s\beta_s=1\in\mathbb Z/2$, for $\beta\in(\mathbb Z/2)^k$. The isomorphism (3.3) is given by the direct sum of the projections $\pi_\alpha\colon U_k\longrightarrow V_\alpha$, $\alpha\in(\mathbb Z/2)^k\backslash\{0\}$,

$$(y_\beta)_{\beta\in(\mathbb Z/2)^k\backslash\{0\}}\longmapsto\sum_{\alpha\cdot\beta=1}y_\beta-\sum_{\alpha\cdot\beta=0}y_\beta.$$

Now let $v_s=\pm v_r$. Consider $\alpha\in(\mathbb Z/2)^k$ given by $\alpha_s=1=\alpha_r$ and $\alpha_\ell=0$ for $\ell\notin\{s,r\}$, and the corresponding projection $\pi_\alpha^{\oplus j}\colon U_k^{\oplus j}\longrightarrow V_\alpha^{\oplus j}$. Then

$$\pi_\alpha^{\oplus j}\circ\phi_{\mathcal M}(v_1,\dots,v_k)=\pi_\alpha^{\oplus j}\circ\phi_{\mathcal M'}(v_1,\dots,v_k)=(\pm1,\dots,\pm1).$$

(iii) Likewise, the linear homotopy between $\psi_{\mathcal M}|_{X^{>1}_{d,k}}$ and $\psi_{\mathcal M'}|_{X^{>1}_{d,k}}$ is equivariant and avoids zero. Let $\lambda_1v_1+\dots+\lambda_kv_k\in X^{>1}_{d,k}$. If $\lambda:=\lambda_1\cdots\lambda_k\neq0$, $\lambda_s=\lambda_r$ and $v_s=\pm v_r$, then

$$(\pi_\alpha^{\oplus j}\circ\eta\circ\psi_{\mathcal M})(\lambda_1v_1+\dots+\lambda_kv_k)=(\pi_\alpha^{\oplus j}\circ\eta\circ\psi_{\mathcal M'})(\lambda_1v_1+\dots+\lambda_kv_k)=(\pm\lambda,\dots,\pm\lambda),$$

where $\eta\colon W_k\oplus U_k^{\oplus j}\longrightarrow U_k^{\oplus j}$ is the projection. Finally, in the case when $\lambda_s=0$ for some $1\le s\le k$, $\psi_{\mathcal M}(\lambda_1v_1+\dots+\lambda_kv_k)$ and $\psi_{\mathcal M'}(\lambda_1v_1+\dots+\lambda_kv_k)$ after projection to the sth coordinate of the subrepresentation W_k are equal to $-\frac1k$. □

Denote the radial projections by

$$\rho\colon U_k^{\oplus j}\backslash\{0\}\longrightarrow S(U_k^{\oplus j})\quad\text{and}\quad\nu\colon(W_k\oplus U_k^{\oplus j})\backslash\{0\}\longrightarrow S(W_k\oplus U_k^{\oplus j}).$$

Note that ρ and ν are $\mathfrak S_k^{\pm}$-equivariant maps. Now the criterion stated in Proposition 3.8 (ii) can be strengthened as follows.

Theorem 3.10. *Let $d \geq 1$, $k \geq 1$ and $j \geq 1$ be integers and let $\mathcal{M}$ be a collection of j masses in $\mathbb{R}^d$. We have the following two criteria:*

(i) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map*

$$Y_{d,k} \longrightarrow S(U_k^{\oplus j})$$

whose restriction to $Y_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\rho \circ \phi_{\mathcal{M}}|_{Y_{d,k}^{>1}}$, then $\Delta(j,k) \leq d$.

(ii) *If there is no $\mathfrak{S}_k^{\pm}$-equivariant map*

$$X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j})$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$, then $\Delta(j,k) \leq d$.

3.3 Applying relative equivariant obstruction theory

In order to prove Theorems 3.4, 3.5, and 3.6 via Theorem 3.10(ii), we study the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map

$$X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j}), \tag{3.4}$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ for some fixed collection $\mathcal{M}$ of j masses in $\mathbb{R}^d$. If we prove that such a map cannot exist, Theorems 3.4, 3.5, and 3.6 follow.

Denote by

$$N_1 := (d+1)k - 1$$

the dimension of the sphere $X_{d,k} = (S^d)^{*k}$, and by

$$N_2 := (2^k - 1)j + k - 2$$

the dimension of the sphere $S(W_k \oplus U_k^{\oplus j})$. If $N_1 \leq N_2$, then

$$\dim X_{d,k} = N_1 \leq \operatorname{conn}\big(S(W_k \oplus U_k^{\oplus j})\big) + 1 = N_2.$$

Consequently, all obstructions to the existence of an $\mathfrak{S}_k^{\pm}$-equivariant map (3.4) vanish and so the map exists. Here $\operatorname{conn}(\cdot)$ denotes the connectivity of a space.

Therefore, we assume that $N_1 > N_2$, which is equivalent to the Ramos lower bound $d \geq \frac{2^k-1}{k} j$. Furthermore, the following prerequisites for applying equivariant obstruction theory are satisfied:

- The N_1-sphere $X_{d,k}$ can be given the structure of a relative $\mathfrak{S}_k^{\pm}$-CW complex $X := (X_{d,k}, X_{d,k}^{>1})$ with a free $\mathfrak{S}_k^{\pm}$-action on $X_{d,k} \backslash X_{d,k}^{>1}$: In Section 3.4 we construct an explicit relative $\mathfrak{S}_k^{\pm}$-CW complex that models $X_{d,k}$.
- The sphere $S(W_k \oplus U_k^{\oplus j})$ is path connected and N_2-simple, except in the trivial case of $k = j = 1$ when $N_2 = 0$. Indeed, the group $\pi_1(S(W_k \oplus U_k^{\oplus j}))$ is abelian for $N_2 = 1$ and trivial for $N_2 > 1$ and therefore its action on $\pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))$ is trivial.
- The $\mathfrak{S}_k^{\pm}$-equivariant map $h \colon X_{d,k}^{>1} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ given by the composition $h := \nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$, for a fixed collection of j masses $\mathcal{M}$, serves as the base map for extension.

Since the sphere $S(W_k \oplus U_k^{\oplus j})$ is $(N_2 - 1)$-connected, the map h can be extended to a $\mathfrak{S}_k^{\pm}$-

equivariant map from the N_2-skeleton $X^{(N_2)} \longrightarrow S(W_k \oplus U_k^{\oplus j})$. A necessary criterion for the existence of the $\mathfrak{S}_k^{\pm}$-equivariant map (3.4) extending h is that the $\mathfrak{S}_k^{\pm}$-equivariant map $h = \nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ can be extended to a map from the (N_2+1)-skeleton $X^{(N_2+1)} \longrightarrow S(W_k \oplus U_k^{\oplus j})$.

Given the above hypotheses, we can apply relative equivariant obstruction theory, as presented by tom Dieck [34, Sec. II.3], to decide the existence of such an extension.

If g is an equivariant extension of h to the N_2-skeleton $X^{(N_2)}$, then the obstruction to extending g to the (N_2+1)-skeleton is encoded by the equivariant cocycle

$$\mathfrak{o}(g) \in \mathcal{C}^{N_2+1}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big).$$

The $\mathfrak{S}_k^{\pm}$-equivariant map $g\colon X^{(N_2)} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ extends to $X^{(N_2+1)}$ if and only if $\mathfrak{o}(g) = 0$. Furthermore, the cohomology class

$$[\mathfrak{o}(g)] \in \mathcal{H}^{N_2+1}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big),$$

vanishes if and only if the restriction $g|_{X^{(N_2-1)}}$ to the (N_2-1)-skeleton can be extended to the (N_2+1)-skeleton $X^{(N_2+1)}$. Any two extensions g and g' of h to the N_2-skeleton are equivariantly homotopic on the (N_2-1)-skeleton and therefore the cohomology classes coincide: $[\mathfrak{o}(g)] = [\mathfrak{o}(g')]$. Hence it suffices to compute the cohomology class $[\mathfrak{o}(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}})]$ for a fixed collection of j masses $\mathcal{M}$ with the property that $0 \notin \mathrm{im}(\psi_{\mathcal{M}}|_{X^{(N_2)}})$.

Let f be the attaching map for an (N_2+1)-cell θ and e its corresponding basis element in the cellular chain group $C_{N_2+1}(X_{d,k}, X_{d,k}^{>1})$. Then

$$\mathfrak{o}(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}})(e) = [\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}]$$

is the homotopy class of the map represented by the composition

$$\partial\theta_j \xrightarrow{f|_{\partial\theta}} X^{(N_2)} \xrightarrow{\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}}} S(W_k \oplus U_k^{\oplus j}).$$

Since $\partial\theta$ and $S(W_k \oplus U_k^{\oplus j})$ are spheres of the same dimension N_2, the homotopy class $[\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}]$ is determined by the degree of the map $\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}$. Here we assume that the $\mathfrak{S}_k^{\pm}$-CW structure on $X_{d,k}$ is endowed with cell orientations, and in addition an orientation on the sphere $S(W_k \oplus U_k^{\oplus j})$ is fixed in advance. Therefore, the degree of the map $\nu \circ \psi_{\mathcal{M}} \circ f|_{\partial\theta}$ is well-defined.

Let $\alpha := \psi_{\mathcal{M}} \circ f|_{\partial\theta}$. In order to compute the degree of the map $\nu \circ \alpha$ and consequently the obstruction cocycle evaluated at e, fix the collection of measures as follows. Let $\mathcal{M}$ be the collection of masses $(I_1, \dots, I_j)$ where I_r is the mass concentrated on the segment $\gamma((t_r^1, t_r^2))$ of the moment curve in $\mathbb{R}^d$

$$\gamma(t) = (t, \tbinom{t}{2}, \tbinom{t}{3}, \dots, \tbinom{t}{d})^t,$$

such that

$$\ell < t_1^1 < t_1^2 < t_2^1 < t_2^2 < \cdots < t_j^1 < t_j^2,$$

for an integer ℓ, $0 \le \ell \le d-1$. The intervals $(I_1, \dots, I_j)$ determined by numbers $t_r^1 < t_r^2$ can be chosen in such a way that $0 \notin \mathrm{im}(\psi_{\mathcal{M}}|_{X^{(N_2)}})$. For every concrete situation in Section 3.5 this is verified directly.

Now consider the following commutative diagram:

$$\begin{tikzcd}
\partial\theta \arrow[r, "f|_{\partial\theta}"] \arrow[d] & X^{(N_2)} \arrow[r, "\psi_{\mathcal{M}}|_{X^{(N_2)}}"] \arrow[d] & W_k \oplus U_k^{\oplus j}\backslash\{0\} \arrow[r, "\nu"] \arrow[d] & S(W_k \oplus U_k^{\oplus j}) \\
\theta \arrow[r, "f"] & X^{(N_2+1)} \arrow[r, "\psi_{\mathcal{M}}|_{X^{(N_2+1)}}"] & W_k \oplus U_k^{\oplus j} &
\end{tikzcd}$$

where the vertical arrows are inclusions, and the composition of the lower horizontal maps is denoted by $\beta := \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ f$. Furthermore, let $B_\varepsilon(0)$ be a ball with center 0 in $W_k \oplus U_k^{\oplus j}$ of sufficiently small radius $\varepsilon > 0$. Set $\widetilde{\theta} := \theta\backslash\beta^{-1}(B_\varepsilon(0))$. Since $\dim\theta = \dim W_k \oplus U_k^{\oplus j}$ we can assume that the set of zeros $\beta^{-1}(0) \subset \operatorname{relint}\theta$ is finite, say of cardinality $r \geq 0$. Again, in every calculation presented in Section 3.5 this assumption is explicitly verified. The function β is a restriction of the test map and therefore *the points in $\beta^{-1}(0)$ correspond to arrangements of k hyperplanes $\mathcal{H}$ in* relint θ *that equipart $\mathcal{M}$*. Moreover, the facts that the measures are intervals on a moment curve and that each hyperplane of the arrangement from $\beta^{-1}(0)$ cuts the moment curve in d distinct points imply that each zero in $\beta^{-1}(0)$ is isolated and transversal. The boundary of $\widetilde{\theta}$ consists of the boundary $\partial\theta$ and r disjoint copies of N_2-spheres $S_1, \ldots, S_r$, one for each zero of β on θ. Consequently, the fundamental class of $\partial\theta$ is equal to the sum of fundamental classes $\sum[S_i]$ in $H_{N_2}(\widetilde{\theta};\mathbb{Z})$. Here the fundamental class of $\partial\theta$ is determined by the cell orientation inherited from the $\mathfrak{S}_k^{\pm}$-CW structure on $X_{d,k}$. The fundamental classes of $[S_i]$ are determined in such a way that the equality $[\partial\theta] = \sum[S_i]$ holds. Thus

$$\sum(\nu\circ\beta|_{\widetilde{\theta}})_*([S_i]) = (\nu\circ\beta|_{\widetilde{\theta}})_*([\partial\theta]) = (\nu\circ\alpha)_*([\partial\theta]) = \deg(\nu\circ\alpha)\cdot[S(W_k \oplus U_k^{\oplus j})].$$

Recall, we have fixed the orientation on the sphere $S(W_k \oplus U_k^{\oplus j})$ and so the fundamental class $[S(W_k \oplus U_k^{\oplus j})]$ is also completely determined. On the other hand,

$$\sum(\nu\circ\beta|_{S_i})_*([S_i]) = \Big(\sum\deg(\nu\circ\beta|_{S_i})\Big)\cdot[S(W_k \oplus U_k^{\oplus j})].$$

Hence $\deg(\nu\circ\alpha) = \sum\deg(\nu\circ\beta|_{S_i})$ where the sum ranges over all arrangements of k hyperplanes $\mathcal{H}$ in relint θ that equipart $\mathcal{M}$; consult [62, Prop. IV.4.5]. In other words,

$$\mathfrak{o}(\nu\circ\psi_{\mathcal{M}}|_{X^{(N_2)}})(e) = [\nu\circ\psi_{\mathcal{M}}\circ f|_{\partial\theta}] = \deg(\nu\circ\alpha)\cdot\zeta = \sum\deg(\nu\circ\beta|_{S_i})\cdot\zeta, \tag{3.5}$$

where $\zeta \in \pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \cong \mathbb{Z}$ is a generator, and the sum ranges over all arrangements of k hyperplanes $\mathcal{H}$ in relint θ that equipart $\mathcal{M}$.

If in addition we assume that all local degrees $\deg(\nu\circ\beta|_{S_i})$ are ±1 and that the number of arrangements of k hyperplanes $\mathcal{H}$ in relint θ that equipart $\mathcal{M}$ is odd, then we conclude that $\mathfrak{o}(\nu\circ\psi_{\mathcal{M}}|_{X^{(N_2)}})(e) \neq 0$. It will turn out that in many situations this information implies that the cohomology class $[\mathfrak{o}(\nu\circ\psi_{\mathcal{M}})]$ is not zero, and consequently the related $\mathfrak{S}_k^{\pm}$-equivariant map (3.4) does not exist, concluding the proof of corresponding Theorems 3.4, 3.5, and 3.6.

3.4 A regular cell complex model for the join configuration space

In this section, motivated by methods used in [19] and [31], we construct a regular $\mathfrak{S}_k^{\pm}$-CW model for the join configuration space $X_{d,k} = (S^d)^{*k} \cong S(\mathbb{R}^{(d+1)\times k})$ such that $X_{d,k}^{>1}$ is a $\mathfrak{S}_k^{\pm}$-CW subcomplex. Consequently, $(X_{d,k}, X_{d,k}^{>1})$ has the structure of a relative $\mathfrak{S}_k^{\pm}$-CW complex. For simplicity the cell complex we construct is denoted by $X := (X_{d,k}, X_{d,k}^{>1})$ as well. The cell model is obtained in two steps:

(1) the vector space $\mathbb{R}^{(d+1)\times k}$ is decomposed into a union of disjoint relatively open cones (each containing the origin in its closure) on which the $\mathfrak{S}_k^{\pm}$-action operates linearly permuting the cones, and then

(2) the open cells of a regular $\mathfrak{S}_k^{\pm}$-CW model are obtained as intersections of these relatively open cones with the unit sphere $S(\mathbb{R}^{(d+1)\times k})$.

The explicit relative $\mathfrak{S}_k^{\pm}$-CW complex we construct here is an essential object needed for the study of the existence of $\mathfrak{S}_k^{\pm}$-equivariant maps $X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j})$ via the relative equivariant obstruction theory of tom Dieck [34, Sec. II.3].

3.4.1 Stratifications by cones associated to an arrangement

The first step in the construction of the $\mathfrak{S}_k^{\pm}$-CW model is an appropriate stratification of the ambient space $\mathbb{R}^{(d+1)\times k}$. First we introduce the notion of the stratification of a Euclidean space and collect some relevant properties.

Definition 3.11. Let $\mathcal{A}$ be an arrangement of linear subspaces in a Euclidean space E. A *stratification of E (by cones) associated to $\mathcal{A}$* is a finite collection $\mathcal{C}$ of subsets of E that satisfies the following properties:

(i) $\mathcal{C}$ consists of finitely many non-empty relatively open polyhedral cones in E.

(ii) $\mathcal{C}$ is a partition of E, that is, $E = \biguplus_{C\in\mathcal{C}} C$.

(iii) The closure $\overline{C}$ of every cone $C \in \mathcal{C}$ is a union of cones in $\mathcal{C}$.

(iv) Every subspace $A \in \mathcal{A}$ is a union of cones in $\mathcal{C}$.

An element of the family $\mathcal{C}$ is called a *stratum*.

Example 3.12. Let E be a Euclidean space of dimension d, let L be a linear subspace of codimension r, where r is an integer with $1 \le r \le d$, and let $\mathcal{A}$ be the arrangement $\{L\}$. Choose a flag that terminates at L, that is, fix a sequence of linear subspaces in E

$$E = L^{(0)} \supset L^{(1)} \supset \cdots \supset L^{(r)} = L, \tag{3.6}$$

so that $\dim L^{(i)} = d - i$. The family $\mathcal{C}$ associated to the flag (3.6) consists of L and of the connected components of the successive complements

$$L^{(0)}\backslash L^{(1)},\ L^{(1)}\backslash L^{(2)},\ \dots\ , L^{(r-1)}\backslash L^{(r)}.$$

A $L^{(i)}$ is a hyperplane in $L^{(i-1)}$, each of the complements $L^{(i-1)}\backslash L^{(i)}$ has two connected components. This indeed yields a stratification by cones for the arrangement $\mathcal{A}$ in E.

Definition 3.13. Let $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$ be a collection of arrangements of linear subspaces in the Euclidean space E and let $(\mathcal{C}_1, \mathcal{C}_2 \ldots, \mathcal{C}_n)$ be the associated collection of stratifications of E by cones. The *common refinement* of the stratifications is the family

$$\mathcal{C} := \{C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset : C_i \in \mathcal{C}_i \text{ for all } i\}.$$

In order to verify that the common refinement of stratifications is again a stratification, we use the following elementary lemma.

Lemma 3.14. *Let $A_1, \ldots, A_n$ be relatively open convex sets in E that have non-empty intersection, $A_1 \cap \cdots \cap A_n \neq \emptyset$. Then the following relation holds for the closures:*

$$\overline{A_1 \cap \cdots \cap A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}.$$

Proof. The inclusion "$\subseteq$" follows directly. For the opposite inclusion take $x \in \overline{A_1} \cap \cdots \cap \overline{A_n}$. Choose a point $y \in A_1 \cap \cdots \cap A_n \neq \emptyset$ and consider the line segment $(x, y] := \{\lambda x + (1 - \lambda)y : 0 \leq \lambda < 1\}$. As each A_i is relatively open, the segment $(x, y]$ is contained in each of the A_i and consequently it is contained in $A_1 \cap \cdots \cap A_n$. Thus we obtain a sequence in this intersection converging to x, which implies that $x \in \overline{A_1 \cap \cdots \cap A_n}$. □

Proposition 3.15. *Given stratifications by cones $\mathcal{C}_1, \mathcal{C}_2 \ldots, \mathcal{C}_n$ associated to linear subspace arrangements $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, their common refinement is a stratification by cones associated to the subspace arrangement $\mathcal{A} := \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$.*

Proof. Properties (i) and (ii) of Definition 3.11 follow immediately from the definition of the common refinement. To verify property (iv), observe that a subspace $A_t \in \mathcal{A}_t$ is a union of strata from $\mathcal{C}_t$, say $A_t = \bigcup_s U_{t,s}$ where $U_{t,s} \in \mathcal{C}_t$. Hence taking the union of intersections $C_1 \cap \cdots \cap U_{t,s} \cap \cdots \cap C_n$ for all $C_i \in \mathcal{C}_i$ where $i \neq t$, and all $U_{t,s}$ gives A_t. Property (iii) follows from Lemma 3.14. □

Example 3.16. Let E be a Euclidean space of dimension d and let $\mathcal{A} = \{L_1, \ldots, L_s\}$ be an arrangement of linear subspaces of E. As in Example 3.12, for each of the subspaces L_i in the arrangement $\mathcal{A}$ fix a flag $L_i^{(s)}$ and form the corresponding stratifications $\mathcal{C}_1, \ldots, \mathcal{C}_s$. The common refinement of stratifications $\mathcal{C}_1, \ldots, \mathcal{C}_s$ is a stratification by cones associated to the subspace arrangement $\mathcal{A}$.

An arrangement of linear subspaces is *essential* if the intersection of the subspaces in the arrangement is $\{0\}$.

Proposition 3.17. *The intersection of a stratification $\mathcal{C}$ of E by cones associated to an essential linear subspace arrangement with the sphere $S(E)$ gives a regular CW-complex.*

Proof. The elements $C \in \mathcal{C}$ are relative open polyhedral cones. As $\{0\}$ is a stratum by itself, none of the strata contains a line through the origin. Thus $C \cap S(E)$ is an open cell, whose closure $\overline{C} \cap S(E)$ is a finite union of cells of the form $C' \cap S(E)$, so we get a regular CW complex. □

3.4.2 A stratification of $\mathbb{R}^{(d+1)\times k}$

Now we introduce the stratification of $\mathbb{R}^{(d+1)\times k}$ that will give us a $\mathfrak{S}_k^{\pm}$-CW model for $X_{d,k}$. One version of it, $\mathcal{C}$, arises from the construction in the previous section. However, we also give combinatorial descriptions of relatively-open convex cones in the stratification $\mathcal{C}'$ directly, for which the action of $\mathfrak{S}_k^{\pm}$ is evident. We then verify that $\mathcal{C}$ and $\mathcal{C}'$ coincide.

Stratification

Let elements $x \in \mathbb{R}^{(d+1)\times k}$ be written as $x = (x_1, \dots, x_k)$ where $x_i = (x_{t,i})_{t\in[d+1]}$ is the i-th column of the matrix x. Consider the arrangement $\mathcal{A}$ consisting of the following subspaces:

$$\begin{array}{rcll} L_r & := & \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_r = 0\}, & 1 \le r \le k \\ L^+_{r,s} & := & \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_r - x_s = 0\}, & 1 \le r < s \le k \\ L^-_{r,s} & := & \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_r + x_s = 0\}, & 1 \le r < s \le k. \end{array}$$

With each subspace we associate a flag:

(i) With $L_r = \{x_r = 0\}$ we associate

$$\mathbb{R}^{(d+1)\times k} \supset \{x_{1,r} = 0\} \supset \{x_{1,r} = x_{2,r} = 0\} \supset \cdots \supset \{x_{1,r} = x_{2,r} = \cdots = x_{d+1,r} = 0\},$$

(ii) With $L^+_{r,s} = \{x_r - x_s = 0\}$ we associate

$$\begin{aligned} \mathbb{R}^{(d+1)\times k} \supset \{x_{1,r} - x_{1,s} = 0\} \supset \{x_{1,r} - x_{1,s} = x_{2,r} - x_{2,s} = 0\} \supset \cdots \supset \\ \{x_{1,r} - x_{1,s} = x_{2,r} - x_{2,s} = \cdots = x_{d+1,r} - x_{d+1,s} = 0\}, \end{aligned}$$

(iii) $L^-_{r,s} = \{x_r + x_s = 0\}$ we associate

$$\begin{aligned} \mathbb{R}^{(d+1)\times k} \supset \{x_{1,r} + x_{1,s} = 0\} \supset \{x_{1,r} + x_{1,s} = x_{2,r} + x_{2,s} = 0\} \supset \cdots \supset \\ \{x_{1,r} + x_{1,s} = x_{2,r} + x_{2,s} = \cdots = x_{d+1,r} + x_{d+1,s} = 0\}. \end{aligned}$$

The construction from Example 3.12 shows how every subspace in $\mathcal{A}$ leads to a stratification by cones of $\mathbb{R}^{(d+1)\times k}$. The stratifications associated to the subspaces $L_r, L^+_{r,s}, L^-_{r,s}$ are denoted by $\mathcal{C}_r, \mathcal{C}^+_{r,s}, \mathcal{C}^-_{r,s}$, respectively. Now, if we apply Example 3.16 to this concrete situation we obtain the stratification by cones $\mathcal{C}$ of $\mathbb{R}^{(d+1)\times k}$ associated to the subspace arrangement $\mathcal{A}$. This means that each stratum of $\mathcal{C}$ is a non-empty intersection of strata from the stratifications $\mathcal{C}_r, \mathcal{C}^+_{r,s}, \mathcal{C}^-_{r,s}$ where $1 \le r < s \le k$.

Partition

Let us fix:

- a permutation $\sigma := (\sigma_1, \sigma_2, \dots, \sigma_k) \equiv (\sigma_1\sigma_2 \dots \sigma_k) \in \mathfrak{S}_k$, $\sigma\colon t \mapsto \sigma_t$,
- a collection of signs $S := (s_1, \dots, s_k) \in \{+1, -1\}^k$, and
- integers $I := (i_1, \dots, i_k) \in \{1, \dots, d+2\}^k$.

Furthermore, define x_0 to be the origin in $\mathbb{R}^{(d+1)\times k}$, $\sigma_0 = 0$ and $s_0 = 1$. Define

$$C^S_I(\sigma) = C^{s_1,\dots,s_k}_{i_1,\dots,i_k}(\sigma_1, \sigma_2, \dots, \sigma_k) \subseteq \mathbb{R}^{(d+1)\times k}$$

to be the set of all points $(x_1, \dots, x_k) \in \mathbb{R}^{(d+1)\times k}$, $x_i = (x_{1,i}, \dots, x_{d+1,i})$, such that for each integer t

with $1 \leq t \leq k$,

- if $1 \leq i_t \leq d+1$, then $s_{t-1}x_{i_t,\sigma_{t-1}} < s_t x_{i_t,\sigma_t}$ with $s_{t-1}x_{i',\sigma_{t-1}} = s_t x_{i',\sigma_t}$ for every $i' < i_t$,
- if $i_t = d+2$, then $s_{i_{t-1}}x_{\sigma_{t-1}} = s_{i_t}x_{\sigma_t}$.

Any triple $(\sigma|I|S) \in \mathfrak{S}_k \times \{1,\dots,d+2\}^k \times \{+1,-1\}^k$ is called a *symbol*. In the notation of symbols we abbreviate signs $\{+1,-1\}$ by $\{+,-\}$. The defining set of "inequalities" for the stratum $C_I^S(\sigma)$ is briefly denoted by:

$$\begin{aligned} C_I^S(\sigma) &= C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 <_{i_1} s_1 x_{\sigma_1} <_{i_2} s_2 x_{\sigma_2} <_{i_3} \cdots <_{i_k} s_k x_{\sigma_k}\}, \end{aligned}$$

where $y <_i y'$, for $1 \leq i \leq d+1$, means that y and y' agree in the first $i-1$ coordinates and at the i-th coordinate $y_i < y'_i$. The inequality $y <_{d+2} y'$ denotes that $y = y'$. Each set $C_I^S(\sigma)$ is the relative interior of a polyhedral cone in $(\mathbb{R}^{d+1})^k$ of codimension $(i_1 - 1) + \cdots + (i_k - 1)$, that is,

$$\dim C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) = (d+2)k - (i_1 + \cdots + i_k).$$

Let $\mathcal{C}'$ denote the family of strata $C_I^S(\sigma)$ defined by all symbols, that is,

$$\mathcal{C}' = \{C_I^S(\sigma) : (\sigma|I|S) \in \mathfrak{S}_k \times \{1,\dots,d+2\}^k \times \{+1,-1\}^k\}.$$

Different symbols can define the same set, and

$$C_I^S(\sigma) \cap C_{I'}^{S'}(\sigma) \neq \emptyset \iff C_I^S(\sigma) = C_{I'}^{S'}(\sigma).$$

In order to verify that the family $\mathcal{C}'$ is a partition of $\mathbb{R}^{(d+1)\times k}$ it remains to prove that it is a covering.

Lemma 3.18. $\bigcup \mathcal{C}' = \mathbb{R}^{(d+1)\times k}$.

Proof. Let $(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k}$. First choose signs $r_1,\dots,r_k \in \{+1,-1\}$ so that the vectors $r_1x_1,\dots,r_kx_k$ are greater or equal to $0 \in \mathbb{R}^{(d+1)\times k}$ with respect to the lexicographic order, that is, the first non-zero coordinate of each of the vectors r_ix_i is greater than zero. The choice of signs is not unique if one of the vectors x_i is zero. Next, record a permutation $\sigma \in \mathfrak{S}_k$ such that

$$0 <_{\text{lex}} r_{\sigma_1}x_{\sigma_1} <_{\text{lex}} r_{\sigma_2}x_{\sigma_2} <_{\text{lex}} \cdots <_{\text{lex}} r_{\sigma_k}x_{\sigma_k},$$

where $<_{\text{lex}}$ denotes the lexicographic order. The permutation σ is not unique if $r_ix_i = r_tx_t$ for some $i \neq t$. Define $s_i := r_{\sigma_i}$. Finally, collect coordinates i_t where vectors $s_{t-1}x_{\sigma_{t-1}}$ and $s_tx_{\sigma_t}$ first differ, or put $i_t = d+2$ if they coincide. Thus $(x_1,\dots,x_k) \in C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k)$. $\square$

Example 3.19. Let $d = 0$ and $k = 2$. Then the plane $\mathbb{R}^2$ is decomposed into the following cones. There are 8 open cones of dimension 2:

$$\begin{array}{rcl}
C_{1,1}^{+;+}(12) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < x_2\}, \\
C_{1,1}^{-;+}(12) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < -x_1 < x_2\}, \\
C_{1,1}^{+;-}(12) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < -x_2\}, \\
C_{1,1}^{-;-}(12) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < -x_1 < -x_2\}, \\
C_{1,1}^{+;+}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1\}, \\
C_{1,1}^{-;+}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < -x_2 < x_1\}, \\
C_{1,1}^{+;-}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < -x_1\}, \\
C_{1,1}^{-;-}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < -x_2 < -x_1\}.
\end{array}$$

Furthermore, there are 8 cones of dimension 1:

$$\begin{array}{rcccl}
C_{1,2}^{+;+}(12) &=& C_{1,2}^{+;+}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 = x_2\}, \\
C_{1,2}^{-;+}(12) &=& C_{1,2}^{+;-}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < -x_1 = x_2\}, \\
C_{1,2}^{+;-}(12) &=& C_{1,2}^{-;+}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 = -x_2\}, \\
C_{1,2}^{-;-}(12) &=& C_{1,2}^{-;-}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 < -x_1 = -x_2\}, \\
C_{2,1}^{+;+}(12) &=& C_{2,1}^{-;+}(12) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 = x_1 < x_2\}, \\
C_{2,1}^{+;-}(12) &=& C_{2,1}^{-;-}(12) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 = x_1 < -x_2\}, \\
C_{2,1}^{+;+}(21) &=& C_{2,1}^{-;+}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 = x_2 < x_1\}, \\
C_{2,1}^{+;-}(21) &=& C_{2,1}^{-;-}(21) &=& \{(x_1, x_2) \in \mathbb{R}^2 : 0 = x_2 < -x_1\}.
\end{array}$$

The origin in $\mathbb{R}^2$ is given by $C_{2,2}^{\pm;\pm}(12) = C_{2,2}^{\pm;\pm}(21)$. The example illustrates a property of our decomposition of $\mathbb{R}^{(d+1)\times k}$: There is a surjection from symbols to cones that is not a bijection, that is, different symbols can define the same cones.

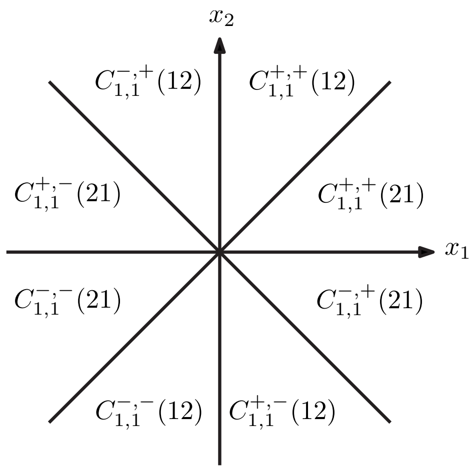


Figure 3.1: Illustration of the stratification in Example 3.19

Example 3.20. Let $d = 2$ and $k = 4$. The stratum associated to the symbol $(2143\,|\,2,3,1,4\,|+1,-1,+1,-1)$ can be described explicitly as follows.

$$\left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \end{pmatrix} \in (\mathbb{R}^3)^4 \;:\; \begin{array}{cccccccccc} 0 & = & x_{1,2} & = & -x_{1,1} & < & x_{1,4} & = & -x_{1,3} \\ 0 & < & x_{2,2} & = & -x_{2,1} & & x_{2,4} & = & -x_{2,3} \\ & & x_{3,2} & < & -x_{3,1} & & x_{3,4} & = & -x_{3,3} \end{array} \right\}.$$

In particular,

$$C^{+,-,+,-}_{2,3,1,4}(2143) = C^{+,-,-,+}_{2,3,1,4}(2134).$$

$\mathcal{C}$ and $\mathcal{C}'$ coincide

We proved that $\mathcal{C}$ is a stratification by cones of $\mathbb{R}^{(d+1)\times k}$, and that $\mathcal{C}'$ is a partition of $\mathbb{R}^{(d+1)\times k}$. Since both $\mathcal{C}$ and $\mathcal{C}'$ are partitions it suffices to prove that for every symbol $(\sigma|I|S)$ contained in $\mathfrak{S}_k \times \{1,\dots,d+2\}^k \times \{+1,-1\}^k$ the cone $C^S_I(\sigma) \in \mathcal{C}'$ also belongs to $\mathcal{C}$.

Consider the cone $C^S_I(\sigma)$ in $\mathcal{C}'$. It is determined by

$$\begin{aligned} C^S_I(\sigma) &= C^{s_1,\dots,s_k}_{i_1,\dots,i_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 <_{i_1} s_1 x_{\sigma_1} <_{i_2} s_2 x_{\sigma_2} <_{i_3} \cdots <_{i_k} s_k x_{\sigma_k}\}. \end{aligned}$$

The defining inequalities for $C^S_I(\sigma)$ imply that $(x_1,\dots,x_k) \in C^S_I(\sigma)$ if and only if

- $0 <_{\min\{i_1,\dots,i_a\}} s_a x_a$ for $1 \le a \le k$, and
- $s_a x_a <_{\min\{i_{a+1},\dots,i_b\}} s_b x_b$ for $1 \le a < b \le k$,

if and only if

- $(x_1,\dots,x_k)$ belongs to the appropriate one of two strata in the complement

$$L_a{}^{(\min\{i_1,\dots,i_a\}-1)} \setminus L_a{}^{(\min\{i_1,\dots,i_a\}-2)}$$

  of the stratification $\mathcal{C}_a$ depending on the sign s_a where $1 \le a \le k$, and
- $(x_1,\dots,x_k)$ belongs to the appropriate one of two strata in the complement

$$L^{s_a s_b\,(\min\{i_{a+1},\dots,i_b\}-1)}_{a,b} \setminus L^{s_a s_b}_{a,b}{}_{(\min\{i_{a+1},\dots,i_b\}-2)}$$

  of the stratification $\mathcal{C}^{s_a s_b}_{a,b}$ depending on the sign of the product $s_a s_b$ where $1 \le a < b \le k$. The product $s_a s_b$, appearing in the "exponent notation" of $L^{s_a s_b}_{a,b}$, is either "+" when the product $s_a s_b = 1$, or "−" when $s_a s_b = -1$.

Here we use the notation of Examples 3.12 and 3.16.

Thus we have proved that $C^S_I(\sigma) \in \mathcal{C}$ and consequently $\mathcal{C} = \mathcal{C}'$.

3.4.3 The $\mathfrak{S}_k^{\pm}$-CW model for $X_{d,k}$

The action of the group $\mathfrak{S}_k^{\pm}$ on the space $\mathbb{R}^{(d+1)\times k}$ induces an action on the family of strata $\mathcal{C}$ by as follows:

$$\begin{aligned} \pi \cdot C_I^S(\sigma) &= C_I^S(\pi\sigma), && (3.7)\\ \varepsilon_t \cdot C_I^S(\sigma) &= \varepsilon_t \cdot C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= C_{i_1,\dots,i_k}^{s_1,\dots,-s_t,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k), && (3.8)\end{aligned}$$

where $\pi \in \mathfrak{S}_k$ and $\varepsilon_1,\dots,\varepsilon_k$ are the canonical generators of the subgroup $(\mathbb{Z}/2)^k$ of $\mathfrak{S}_k^{\pm}$.

The $\mathfrak{S}_k^{\pm}$-CW complex that models $X_{d,k} = S(\mathbb{R}^{(d+1)\times k})$ is obtained by intersecting each stratum $C_I^S(\sigma)$ with the unit sphere $S(\mathbb{R}^{(d+1)\times k})$. Each stratum is a relatively open cone that does not contain a line. Therefore the intersection

$$D_I^S(\sigma) = D_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) := C_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \cap S(\mathbb{R}^{(d+1)\times k})$$

is an open cell of dimension $(d+2)k - (i_1+\dots+i_k) - 1$. The action of $\mathfrak{S}_k^{\pm}$ is induced by (3.7) and (3.8):

$$\begin{aligned} \pi \cdot D_I^S(\sigma) &= D_I^S(\pi\sigma), && (3.9)\\ \varepsilon_t \cdot D_I^S(\sigma) &= \varepsilon_t \cdot D_{i_1,\dots,i_k}^{s_1,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k) \\ &= D_{i_1,\dots,i_k}^{s_1,\dots,-s_t,\dots,s_k}(\sigma_1,\sigma_2,\dots,\sigma_k). && (3.10)\end{aligned}$$

Thus we have obtained a regular $\mathfrak{S}_k^{\pm}$-CW model for $X_{d,k}$. In particular, the action of the group $\mathfrak{S}_k^{\pm}$ on the space $\mathbb{R}^{(d+1)\times k}$ induces a cellular action on the model.

Theorem 3.21. *Let $d \geq 1$ and $k \geq 1$ be integers and let $N_1 = (d+1)k - 1$. The family of cells*

$$\{D_I^S(\sigma) : (\sigma|I|S) \neq (\sigma|d+2,\dots,d+2|S)\}$$

forms a finite regular N_1-dimensional $\mathfrak{S}_k^{\pm}$-CW complex $X := (X_{d,k}, X_{d,k}^{>1})$ that models the join configuration space $X_{d,k} = S(\mathbb{R}^{(d+1)\times k})$. It has
- *one full $\mathfrak{S}_k^{\pm}$-orbit in maximal dimension N_1, and*
- *k full $\mathfrak{S}_k^{\pm}$-orbits in dimension $N_1 - 1$.*

The (cellular) $\mathfrak{S}_k^{\pm}$-action on $X_{d,k}$ is given by (3.9) *and* (3.10). *Furthermore the collection of cells*

$$\{D_I^S(\sigma) : i_s = d+2 \text{ for some } 1 \leq s \leq k\}$$

is a $\mathfrak{S}_k^{\pm}$-CW subcomplex and models $X_{d,k}^{>1}$.

Example 3.22. Let $d, j \geq 1$ and $k \geq 2$ be integers with $dk = (2^k-1)j + \ell$ for an integer ℓ with $0 \leq \ell \leq d-1$. Consider the cell $\theta := D_{\ell+1,1,1,\dots,1}^{+,+,+,\dots,+}(1,2,3,\dots,k)$ of dimension $N_1 - \ell = N_2 + 1$ in $X_{d,k}$. It is determined by the following inequalities:

$$0 <_{\ell+1} x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

For the process of determining the boundary of θ, depending on value of ℓ, we distinguish the following cases.

(1) Let $\ell = 0$. Then $\theta := D^{+,+,+,\dots,+}_{1,1,1,\dots,1}(1,2,3,\dots,k)$. The cells of codimension 1 in the boundary of θ are obtained by introducing one of the following extra equalities:

$$x_{1,1} = 0\,, \quad x_{1,1} = x_{1,2}\,, \quad \dots \quad , x_{1,k-1} = x_{1,k}.$$

Each of these equalities will give two cells of dimension N_2, hence in total $2k$ cells of codimension 1, in the boundary of θ.

(a) The equality $x_{1,1} = 0$ induces cells:

$$\gamma_1 := D^{+,+,+,\dots,+}_{2,1,1,\dots,1}(1,2,3,\dots,k), \qquad \gamma_2 := D^{-,+,+,\dots,+}_{2,1,1,\dots,1}(1,2,3,\dots,k)$$

that are related, as sets, via $\gamma_2 = \varepsilon_1 \cdot \gamma_1$. Both cells γ_1 and γ_2 belong to the linear subspace

$$V_1 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = 0\}.$$

(b) The equality $x_{1,r-1} = x_{1,r}$ for $2 \le r \le k$ gives cells:

$$\gamma_{2r-1} := D^{+,+,+,\dots,+}_{1,\dots,1,2,1,\dots,1}(1,\dots,r-1,r,r+1,\dots,k),$$

$$\gamma_{2r} := D^{+,+,+,\dots,+}_{1,\dots,1,2,1,\dots,1}(1,\dots,r,r-1,r+1,\dots,k)$$

satisfying $\gamma_{2r} = \tau_{r-1,r} \cdot \gamma_{2r-1}$. In these cells the index 2 in the subscript $1,\dots,1,2,1,\dots,1$ appears at the position r. These cells belong to the linear subspace

$$V_r = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,r-1} = x_{1,r}\}.$$

Let e_θ denote a generator in $C_{N_2+1}(X_{d,k}, X^{>1}_{d,k})$ that corresponds to the cell θ. Furthermore let $e_{\gamma_1},\dots,e_{\gamma_{2k}}$ denote generators in $C_{N_2}(X_{d,k}, X^{>1}_{d,k})$ related to the cells $\gamma_1,\dots,\gamma_{2k}$.
The boundary of the cell θ is contained in the union of the linear subspaces $V_1,\dots,V_k$. Therefore we can orient the cells $\gamma_{2i-1}, \gamma_{2i}$ consistently with the orientation of V_i, $1 \le i \le k$, that is given in such a way that

$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_4}) + \dots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Consequently,

$$\partial e_\theta = (1 + (-1)^d \varepsilon_1) \cdot e_{\gamma_1} + \sum_{i=2}^{k} (1 + (-1)^d \tau_{i-1,i}) \cdot e_{\gamma_{2i-1}}. \tag{3.11}$$

(2) Let $\ell = 1$. Then $\theta := D^{+,+,+,\dots,+}_{2,1,1,\dots,1}(1,2,3,\dots,k)$. Now the cells in the boundary of θ are obtained by introducing extra equalities:

$$x_{2,1} = 0\,, \quad (0 =)\, x_{1,1} = x_{1,2}\,, \quad \dots \quad , x_{1,k-1} = x_{1,k}.$$

Each of these equalities, except for the second one, will give two cells of dimension N_2, which yields $2(k-1)$ cells in total, in the boundary of θ. The equality $x_{1,1} = x_{1,2}$ will give additional four cells in the boundary of θ.

(a) The equality $x_{2,1} = 0$ induces cells:

$$\gamma_1 := D^{+,+,+,\dots,+}_{3,1,1,\dots,1}(1,2,3,\dots,k), \qquad \gamma_2 := D^{-,+,+,\dots,+}_{3,1,1,\dots,1}(1,2,3,\dots,k)$$

that are related, as sets, via $\gamma_2 = \varepsilon_1 \cdot \gamma_1$. Notice that both cells γ_1 and γ_2 belong to the linear subspace

$$V_1 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = x_{2,1} = 0\}.$$

(b) The equality $x_{1,1} = x_{1,2}$ yields the cells

$$\gamma_3 := D^{+,+,+,\dots,+}_{2,2,1,\dots,1}(1,2,3,\dots,k), \quad \gamma_{31} := D^{+,-,+,\dots,+}_{2,2,1,\dots,1}(1,2,3,\dots,k),$$

$$\gamma_{32} := D^{+,+,+,\dots,+}_{2,2,1,\dots,1}(2,1,3,\dots,k), \quad \gamma_{33} := D^{-,+,+,\dots,+}_{2,2,1,\dots,1}(2,1,3,\dots,k).$$

They satisfy set identities $\gamma_{31} = \varepsilon_2 \cdot \gamma_3$, $\gamma_{32} = \tau_{1,2} \cdot \gamma_3$, and $\gamma_{33} = \varepsilon_1 \tau_{1,2} \cdot \gamma_3$. All four cells belong to the linear subspace

$$V_2 = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : 0 = x_{1,1} = x_{1,2}\}.$$

(c) The equality $x_{1,r-1} = x_{1,r}$ for $3 \le r \le k$ gives cells:

$$\gamma_{2r-1} := D^{+,+,+,\dots,+}_{2,\dots,1,2,1,\dots,1}(1,\dots,r-1,r,r+1,\dots,k),$$

$$\gamma_{2r} := D^{+,+,+,\dots,+}_{2,\dots,1,2,1,\dots,1}(1,\dots,r,r-1,r+1,\dots,k)$$

satisfying $\gamma_{2r} = \tau_{r-1,r} \cdot \gamma_{2r-1}$.
In these cells the second index 2 in the subscript $2,\dots,1,2,1,\dots,1$ appears at the position r. These cells belong to the linear subspace

$$V_r = \{(x_1,\dots,x_k) \in \mathbb{R}^{(d+1)\times k} : x_{1,1} = 0,\ x_{1,r-1} = x_{1,r}\}.$$

Again e_θ denotes a generator in $C_{N_2+1}(X_{d,k}, X^{>1}_{d,k})$ corresponding to θ. Let

$$e_{\gamma_1}, e_{\gamma_2}, e_{\gamma_3}, e_{\gamma_{31}}, e_{\gamma_{32}}, e_{\gamma_{33}}, e_{\gamma_4} \dots, e_{\gamma_{2k}}$$

denote generators in $C_{N_2}(X_{d,k}, X^{>1}_{d,k})$ related to the cells $\gamma_1, \gamma_2, \gamma_3, \gamma_{31}, \gamma_{32}, \gamma_{33}, \dots, \gamma_{2k}$.
The boundary of the cell θ, as before, is contained in the union of the linear subspaces $V_1, \dots, V_k$. Therefore we can orient cells consistently with the orientation of V_i, $1 \le i \le k$, that is given in such a way that

$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_{31}} + e_{\gamma_{32}} + e_{\gamma_{33}}) + \dots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Consequently,

$$\begin{aligned}\partial e_\theta \;=\;& (1+(-1)^{d-1}\varepsilon_1)\cdot e_{\gamma_1} + \\ & (1+(-1)^d\varepsilon_2+(-1)^d\tau_{1,2}+(-1)^{d+d}\varepsilon_1\tau_{1,2})\cdot e_{\gamma_3} + \\ & \sum_{i=3}^{k}(1+(-1)^d\tau_{i-1,i})\cdot e_{\gamma_{2i-1}}.\end{aligned} \tag{3.12}$$

(3) Let $2\le \ell\le d-1$. Then $\theta := D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k)$. The cells in the boundary of θ are now obtained by introducing following equalities:

$$x_{\ell+1,1}=0\,,\quad (0=)x_{1,1}=x_{1,2}\,,\quad \dots\quad x_{1,k-1}=x_{1,k}.$$

Each of them will give two cells of dimension N_2 in the boundary of θ, all together $2k$.

(a) The equality $x_{\ell+1,1}=0$ induces cells:

$$\gamma_1 := D^{+,+,+,\dots,+}_{\ell+2,1,1,\dots,1}(1,2,3,\dots,k),\qquad \gamma_2 := D^{-,+,+,\dots,+}_{\ell+2,1,1,\dots,1}(1,2,3,\dots,k)$$

that are related, as sets, via $\gamma_2=\varepsilon_1\cdot\gamma_1$. Both cells γ_1 and γ_2 belong to the linear subspace

$$V_1=\{(x_1,\dots,x_k)\in\mathbb{R}^{(d+1)\times k} : x_{1,1}=\cdots=x_{\ell+1,1}=0\}.$$

(b) The equality $(0=)x_{1,1}=x_{1,2}$ gives the cells

$$\gamma_3 := D^{+,+,+,\dots,+}_{\ell+1,2,1,\dots,1}(1,2,3,\dots,k),\qquad \gamma_4 := D^{+,-,+,\dots,+}_{\ell+1,2,1,\dots,1}(1,2,3,\dots,k)$$

that satisfy $\gamma_4=\varepsilon_2\cdot\gamma_3$. Both cells belong to the linear subspace

$$V_2=\{(x_1,\dots,x_k)\in\mathbb{R}^{(d+1)\times k} : x_{1,1}=\cdots=x_{\ell,1}=0,\ x_{1,1}=x_{1,2}\}.$$

(c) The equality $x_{1,r-1}=x_{1,r}$ for $3\le r\le k$ gives cells:

$$\gamma_{2r-1} := D^{+,+,+,\dots,+}_{\ell+1,\dots,1,2,1,\dots,1}(1,\dots,r-1,r,r+1,\dots,k),$$

$$\gamma_{2r} := D^{+,+,+,\dots,+}_{\ell+1,\dots,1,2,1,\dots,1}(1,\dots,r,r-1,r+1,\dots,k)$$

satisfying $\gamma_{2r}=\tau_{r-1,r}\cdot\gamma_{2r-1}$. In these cells the index 2 in the subscript $\ell+1,\dots,1,2,1,\dots,1$ appears at the position r. These cells belong to the linear subspace

$$V_r=\{(x_1,\dots,x_k)\in\mathbb{R}^{(d+1)\times k} : x_{1,1}=\cdots=x_{\ell,1}=0,\ x_{1,r-1}=x_{1,r}\}.$$

Again e_θ denotes a generator in $C_{N_2+1}(X_{d,k},X^{>1}_{d,k})$ that corresponds to the cell θ. Furthermore $e_{\gamma_1},\dots,e_{\gamma_{2k}}$ denote generators in $C_{N_2}(X_{d,k},X^{>1}_{d,k})$ related to the cells $\gamma_1,\dots,\gamma_{2k}$.
As before, the boundary of the cell θ is contained in the union of the linear subspaces $V_1,\dots,V_k$. Thus we can orient cells $\gamma_{2i-1},\gamma_{2i}$ consistently with the orientation of V_i, $1\le i\le k$, that is

given in such a way that

$$\partial e_\theta = (e_{\gamma_1} + e_{\gamma_2}) + (e_{\gamma_3} + e_{\gamma_4}) + \cdots + (e_{\gamma_{2k-1}} + e_{\gamma_{2k}}).$$

Hence

$$\partial e_\theta = (1 + (-1)^{d-\ell}\varepsilon_1) \cdot e_{\gamma_1} + (1 + (-1)^d \varepsilon_2) \cdot e_{\gamma_3} + \sum_{i=3}^{k} (1 + (-1)^d \tau_{i-1,i}) \cdot e_{\gamma_{2i-1}}. \tag{3.13}$$

The relations (3.11), (3.12) and (3.13) that will be essential in the proofs of Theorems 3.4 and 3.5.

3.4.4 The arrangements parametrized by a cell

In this section we describe all arrangements of k hyperplanes parametrized by the cell

$$\theta := D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1, 2, 3, \dots, k),$$

where $1 \le \ell \le d-1$. This description will be one of the key ingredients in Section 3.5 when the obstruction cocycle is evaluated on the cell θ.

Recall that the cell θ is defined as the intersection of the sphere $S(\mathbb{R}^{(d+1)\times k})$ and the cone given by the inequalities:

$$0 <_{\ell+1} x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

Consider the binomial coefficient moment curve $\hat{\gamma}\colon \mathbb{R} \longrightarrow \mathbb{R}^d$ defined by

$$\hat{\gamma}(t) = (t, \tbinom{t}{2}, \tbinom{t}{3}, \dots, \tbinom{t}{d})^t. \tag{3.14}$$

After embedding $\mathbb{R}^d \longrightarrow \mathbb{R}^{d+1}$, $(\xi_1, \dots, \xi_d)^t \longmapsto (1, \xi_1, \dots, \xi_d)^t$ it corresponds to the curve $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^{d+1}$ given by

$$\gamma(t) = (1, t, \tbinom{t}{2}, \tbinom{t}{3}, \dots, \tbinom{t}{d})^t.$$

Consider the following points on the moment curve γ:

$$q_1 := \gamma(0), \dots, q_{\ell+1} := \gamma(\ell). \tag{3.15}$$

Next, recall that each oriented affine hyperplane $\hat{H}$ in $\mathbb{R}^d$ (embedded in $\mathbb{R}^{d+1}$) determines the unique linear hyperplane H such that $\hat{H} = H \cap \mathbb{R}^d$, and almost vice versa. Now, the family of arrangements parametrized by the (open) cell θ is described as follows:

Lemma 3.23. *The cell $\theta = D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1, 2, 3, \dots, k)$ parametrizes all arrangements $(H_1, \dots, H_k)$ of k linear hyperplanes in $\mathbb{R}^{d+1}$, where the order and orientation are fixed appropriately such that*

- *$Q := \{q_1, \dots, q_\ell\} \subset H_1$,*
- *$q_{\ell+1} \notin H_1$,*
- *$q_1 \notin H_2, \dots, q_1 \notin H_k$, and*
- *$H_2, \dots, H_k$ have unit normal vectors with different (positive) first coordinates, that is, $|\{\langle x_2, q_1\rangle, \langle x_3, q_1\rangle, \dots, \langle x_k, q_1\rangle\}| = k-1$.*

Here $x_i \in S(\mathbb{R}^{(d+1)\times k})$ is a unit normal vector of the hyperplane H_i, for $1 \le i \le k$.

Proof. Observe that $\{q_1, \ldots, q_\ell\} \subset H_1$ holds if and only if $\langle x_1, q_1\rangle = \langle x_1, q_2\rangle = \cdots = \langle x_1, q_\ell\rangle = 0$ if and only if $x_{1,1} = x_{2,1} = \cdots = x_{\ell,1} = 0$: This is true since we have the binomial moment curve, so $q_i = \gamma(i-1)$ has only the first i coordinates non-zero.
Furthermore, $q_{\ell+1} \notin H_1$ holds if and only if $x_{\ell+1,1} \neq 0$; choosing an appropriate orientation for H_1 we can assume that $x_{\ell+1,1} > 0$.
The third condition is equivalent to $0 \notin \{\langle x_2, q_1\rangle, \langle x_3, q_1\rangle, \ldots, \langle x_k, q_1\rangle\}$, that is, $x_{1,2}, x_{1,3}, \ldots, x_{1,k} \neq 0$. Choosing orientations of $H_2, \ldots, H_k$ suitably this yields $x_{1,2}, x_{1,3}, \ldots, x_{1,k} > 0$.
Since the values $x_{1,2} = \langle x_2, q_1\rangle$, $x_{1,3} = \langle x_3, q_1\rangle$, ..., $x_{1,k} = \langle x_k, q_1\rangle$ are positive and distinct, we get $0 < x_{1,2} < x_{1,3} < \cdots < x_{1,k}$ by choosing the right order on $H_2, \ldots, H_k$. □

3.5 Proofs

3.5.1 Proof of Theorem 3.3

Let $d \geq 1$, $j \geq 1$, $\ell \geq 0$ and $k \geq 2$ be integers with the property that $dk = j(2^k - 1) + \ell$ for $0 \leq \ell \leq d-1$. Consider a collection of j ordered disjoint intervals $\mathcal{M} = (I_1, \ldots, I_j)$ along the moment curve γ. Let $Q = \{q_1, \ldots, q_\ell\} \subset \gamma$ be a set of ℓ predetermined points that lie to the left of the interval I_1. We prove Theorem 3.3 in two steps.

Lemma 3.24. *Let A be an ℓ-equiparting matrix, that is, a binary matrix of size $k \times j2^k$ with one row of transition count $d - \ell$ and all other rows of transition count d such that $A = (A_1, \ldots, A_j)$ for Gray codes $A_1, \ldots, A_j$ with the property that the last column of A_i is equal to the first column of A_{i+1} for $1 \leq i < j$. Then A determines an arrangement $\mathcal{H}$ of k affine hyperplanes that equipart $\mathcal{M} = (I_1, \ldots, I_j)$ and one of the hyperplanes passes through each point in Q.*

Proof. Without loss of generality we assume that the first row of the matrix A has transition count $d - \ell$ while rows 2 through k have transition count d. For a row a_s of the matrix A, denote by t_s its transition count, $1 \leq s \leq k$.

Place $j(2^k + 1)$ ordered points $q_{\ell+1}, \ldots, q_{\ell+j(2^k+1)}$ on γ, such that

$$I_i = [q_{\ell+(i-1)2^k+i},\; q_{\ell+i2^k+i}]$$

and each sequence of $2^k + 1$ points divides I_i into 2^k subintervals of equal length. *Ordered* refers to the property that $q_r = \gamma(t_r)$ if $t_1 < t_2 < \cdots < t_{j(2^k+1)}$.

We now define the hyperplanes in $\mathcal{H}$ by specifying which of the points they pass through and then choosing their orientations. Force the affine hyperplane H_1 to pass through all of the points in Q. For $s = 1, \ldots, i$, the affine hyperplane H_s passes through $x_{\ell+r+i}$ if there is a bit change in row a_s from entry r to entry $r+1$ for $(i-1)2^k < r \leq i2^k$. Orient H_s such that the subinterval $[q_r, q_{r+1}]$ is on the positive side of H_s if it corresponds to a 0-entry in a_s. Since each $A_1, \ldots, A_j$ is a Gray code, the arrangement $\mathcal{H}$ is indeed an equipartition. □

Lemma 3.25. *Every arrangement of k affine hyperplanes $\mathcal{H}$ that equiparts $\mathcal{M} = (I_1, \ldots, I_j)$ and where one of the hyperplanes passes through each point of Q induces a unique binary matrix A as in Lemma 3.24.*

Proof. Since $dk = j(2^k-1)+\ell$ and $0 \le \ell \le d-1$, the hyperplanes in $\mathcal{H}$ must pass through the points $q_{\ell+(i-1)2^k+i+1}, \dots, q_{\ell+i2^k+i-1}$ of the intervals I_i for $i \in \{1,\dots,j\}$. Recording the position of the subintervals $[q_{\ell+r}, q_{\ell+r+1}]$, for $r \neq i2^k+i$, with respect to each hyperplane leads to a matrix as in described in Lemma 3.24.

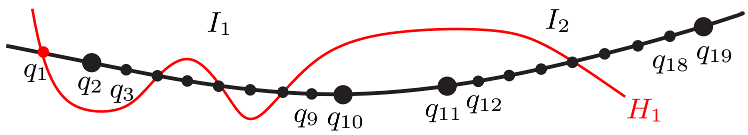


Figure 3.2: Illustration of one step in the proof of Lemma 3.24. Here H_1 is an affine hyperplane bisecting two intervals I_1 and I_2 on the 5-dimensional moment curve.

Thus the number of non-equivalent ℓ-equiparting matrices is the same as the number of arrangements of k affine hyperplanes $\mathcal{H}$ that equipart the collection of j disjoint intervals on the moment curve in $\mathbb{R}^d$, up to renumbering and orientation change of hyperplanes in $\mathcal{H}$, when one of the hyperplanes is forced to pass through ℓ prescribed points on the moment curve lying to the left of the intervals. This concludes the proof of Theorem 3.3. $\square$

3.5.2 Proof of Theorem 3.4

Let $j \ge 1$ and $k \ge 3$ with be integers and let $d = \lceil \frac{2^k-1}{k} j \rceil$ and $\ell = dk - (2^k-1)j$. In addition, assume that the number of non-equivalent ℓ-equiparting matrices of size $k \times j2^k$ is odd. In order to prove that $\Delta(j,k) \le d$ it suffices by Theorem 3.10 to prove that there is no $\mathfrak{S}_k^{\pm}$-equivariant map

$$X_{d,k} \longrightarrow S(W_k \oplus U_k^{\oplus j}),$$

whose restriction to $X_{d,k}^{>1}$ is $\mathfrak{S}_k^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,k}^{>1}}$ for $\mathcal{M} = (I_1, \dots, I_j)$. Following Section 3.3 we verify that the cohomology class

$$[\mathfrak{o}(g)] \in \mathcal{H}^{N_2+1}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big),$$

does not vanish, where $g = \nu \circ \psi_{\mathcal{M}}|_{X^{(N_2)}}$.

Consider the cell $\theta := D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k)$ of dimension $(d+1)k-1-\ell = N_2+1$ in $X_{d,k}$, as in Example 3.22. Let e_θ denote the corresponding basis element of the cell θ in the cellular chain group $C_{N_2+1}(X_{d,k}, X_{d,k}^{>1})$, and let h_θ be the attaching map of θ. This cell is cut out from the unit sphere $S(\mathbb{R}^{(d+1)\times k})$ by the following inequalities:

$$0 <_{\ell+1} x_1 <_1 x_2 <_1 \cdots <_1 x_k.$$

In particular, this means that the first ℓ coordinates of x_1 are zero, that is, $x_{1,1} = x_{2,1} = x_{3,1} = \cdots = x_{\ell,1} = 0$, and $x_{\ell+1,1} > 0$.

Let us fix ℓ points $Q = \{q_1, \dots, q_\ell\}$ on the moment curve (3.14) in $\mathbb{R}^{d+1}$ as it was done in (3.15): $q_1 := \gamma(0), \dots, q_\ell := \gamma(\ell-1)$. Then, by Lemma 3.23, the relative interior of $D^{+,+,+,\dots,+}_{\ell+1,1,1,\dots,1}(1,2,3,\dots,k)$ parametrizes the arrangements $\mathcal{H} = (H_1,\dots,H_k)$ for which orientations and order of the hyperplanes are fixed with H_1 containing all the points from Q. According to the formula (3.5) we have that

$$\mathfrak{o}(g)(e_\theta) = [\nu \circ \psi_{\mathcal{M}} \circ h_\theta|_{\partial\theta}] = \sum \deg(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta|_{S_i}) \cdot \zeta,$$

where as before $\zeta \in \pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \cong \mathbb{Z}$ is a generator, and the sum ranges over all arrangements of k hyperplanes in relint θ that equipart $\mathcal{M}$. Here, as before, S_i denotes a small N_2-sphere around a root of the function $\psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta$, that is, the point that parametrizes an arrangements of k hyperplanes in relint θ that equipart $\mathcal{M}$.
Now, the local degrees of the function $\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta$ are ± 1. Indeed, in a small neighborhood $U \subseteq$ relint θ around any root the test map $\psi_{\mathcal{M}}$ is a continuous bijection. Thus $\psi_{\mathcal{M}}|_{\partial U}$ is a continuous bijection into some N_2-sphere around the origin in $W_k \oplus U_k^{\oplus j}$ and by compactness of ∂U is a homeomorphism. Consequently,

$$\mathfrak{o}(g)(e_\theta) = \sum \deg(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta|_{S_i}) \cdot \zeta = \left(\sum \pm 1\right) \cdot \zeta = a \cdot \zeta, \tag{3.16}$$

where the sum ranges over all arrangements of k hyperplanes in relint θ that equipart $\mathcal{M}$. According to Theorem 3.3 the number of (± 1)'s in the sum (3.16) is equal to the number of non-equivalent ℓ-equiparting matrices of size $k \times j2^k$. By our assumption this number is odd and consequently $a \in \mathbb{Z}$ is an odd integer. We obtained that

$$\mathfrak{o}(g)(e_\theta) = a \cdot \zeta, \tag{3.17}$$

where $a \in \mathbb{Z}$ is an odd integer.

Remark 3.26. It is important to point out that the calculations and formulas up to this point also hold for $k = 2$. The assumption $k \geq 3$ affects the $\mathfrak{S}_k^{\pm} = (\mathbb{Z}/2)^k \rtimes \mathfrak{S}_k$ module structure on $\pi_{N_2}(S(W_k \oplus U_k^{\oplus j})) \cong \mathbb{Z}$. For $k \geq 2$ every generator ε_i of the subgroup $(\mathbb{Z}/2)^k$ acts trivially, while each transposition $\tau_{i,t}$, a generator of the subgroup $\mathfrak{S}_k$, acts as multiplication by -1 in the case $k \geq 3$, and as multiplication by $(-1)^{j+1}$ in the case $k = 2$.

Finally, we prove that $[\mathfrak{o}(g)]$ does not vanish and conclude the proof. This will be achieved by proving that the cocycle $\mathfrak{o}(g)$ is not a coboundary.
Let us assume to the contrary that $\mathfrak{o}(g)$ is a coboundary. Thus there exists a cochain

$$\mathfrak{h} \in \mathcal{C}^{N_2}_{\mathfrak{S}_k^{\pm}}\big(X_{d,k}, X_{d,k}^{>1}; \pi_{N_2}(S(W_k \oplus U_k^{\oplus j}))\big)$$

such that $\mathfrak{o}(g) = \delta\mathfrak{h}$, where δ denotes the coboundary operator.

(1) For $\ell = 0$ the relation (3.11) implies that

$$\begin{aligned} a \cdot \zeta &= \mathfrak{o}(g)(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta) \\ &= (1 + (-1)^d \varepsilon_1) \cdot \mathfrak{h}(e_{\gamma_1}) + \sum_{i=2}^{k} (1 + (-1)^d \tau_{i-1,i}) \cdot \mathfrak{h}(e_{\gamma_{2i-1}}) \\ &= (1 + (-1)^d) \cdot \mathfrak{h}(e_{\gamma_1}) + \sum_{i=2}^{k} (1 + (-1)^{d+1}) \cdot \mathfrak{h}(e_{\gamma_{2i-1}}) \\ &= 2b \cdot \zeta, \end{aligned}$$

for some integer b. Since a is an odd integer this is not possible, and therefore $\mathfrak{o}(g)$ is not a coboundary.

(2) For $\ell = 1$ the relation (3.12) implies that

$$\begin{aligned} a\cdot\zeta &= \mathfrak{o}(g)(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta)\\ &= (1+(-1)^{d-1}\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1}) + \\ &\quad (1+(-1)^d\varepsilon_2+(-1)^d\tau_{1,2}+(-1)^{d+d}\varepsilon_1\tau_{1,2})\cdot\mathfrak{h}(e_{\gamma_3}) + \\ &\quad \sum_{i=3}^{k}(1+(-1)^d\tau_{i-1,i})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &= (1+(-1)^{d-1})\cdot\mathfrak{h}(e_{\gamma_1}) + (1+(-1)^d+(-1)^{d+1}-1)\cdot\mathfrak{h}(e_{\gamma_3}) + \\ &\quad \sum_{i=3}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &= (1+(-1)^{d-1})\cdot\mathfrak{h}(e_{\gamma_1}) + \sum_{i=3}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &= 2b\cdot\zeta, \end{aligned}$$

for $b\in\mathbb{Z}$. Again we reached a contradiction, so $\mathfrak{o}(g)$ is not a coboundary.

(3) For $2\le\ell\le d-1$ the relation (3.13) implies that

$$\begin{aligned} a\cdot\zeta &= \mathfrak{o}(g)(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta)\\ &= (1+(-1)^{d-\ell}\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1}) + (1+(-1)^d\varepsilon_2)\cdot\mathfrak{h}(e_{\gamma_3}) + \\ &\quad \sum_{i=3}^{k}(1+(-1)^d\tau_{i-1,i})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &= (1+(-1)^{d-\ell})\cdot\mathfrak{h}(e_{\gamma_1}) + (1+(-1)^d)\cdot\mathfrak{h}(e_{\gamma_3}) + \\ &\quad \sum_{i=3}^{k}(1+(-1)^{d+1})\cdot\mathfrak{h}(e_{\gamma_{2i-1}})\\ &= 2b\cdot\zeta, \end{aligned}$$

for an integer b. Since a is an odd integer this is not possible. Again, $\mathfrak{o}(g)$ is not a coboundary. □

3.5.3 Proof of Theorem 3.5

Let $j\ge 1$ be an integer with $d=\lceil\frac{3}{2}j\rceil$ and $\ell = 2d-3j\le 1$.

The proof of this theorem is done in the footsteps of the proof of Theorem 3.4. In all three cases we rely on Theorem 3.10 and prove

- the nonexistence of $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2}\longrightarrow S(W_2\oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu\circ\psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ for $\mathcal{M}=(I_1,\dots,I_j)$; by
- evaluating the obstruction cocycle $\mathfrak{o}(g)$ for $g=\nu\circ\psi_{\mathcal{M}}|_{X^{(N_2)}}$ on cells $D_{1,1}^{+;+}(1,2)$ or $D_{2,1}^{+;+}(1,2)$, depending on ℓ being 0 or 1, using Theorem 3.3; and then
- prove that the cocycle $\mathfrak{o}(g)$ is not a coboundary, using boundary formulas from Example 3.22.

2-bit Gray codes

In order to evaluate the obstruction cocycle $\mathfrak{o}(g)$ on the relevant cells in the case $k = 2$ we need to understand (2×4)-Gray codes. These correspond to equipartitions of an interval I on the moment curve into four equal orthants by intersecting with two hyperplanes H_1 and H_2 in altogether three points of the interval. There are two such configurations: either H_1 cuts through the midpoint of I and H_2 separates both halves of I into equal pieces by two additional intersections, or the roles of H_1 and H_2 are reversed. In terms of Gray codes we can express this as follows.

Lemma 3.27. *There are two different 2-bit Gray codes that start with the zero column (or any other fixed binary vector of length 2):*

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Proof. The second column of the Gray code determines the rest of the code, and there are only two choices for a bit flip. □

This means that in the case $k = 2$ an ℓ-equiparting matrix A has a more compact representation: it is determined by the first column – a binary vector of length 2 – and j additional bits, one for each A_i, encoding whether the first bit flip in A_i is in the first or second row. These j bits *cannot be chosen independently* since there are restrictions imposed by the transition count.

Lemma 3.28. *Let $j \geq 1$ be an integer with $d = \lceil \frac{3}{2}j \rceil$ and $\ell = 2d - 3j \leq 1$.*
(1) *If $\ell = 0$, then the number of non-equivalent 0-equiparting matrices is equal to*

$$\tfrac{1}{2}\binom{j}{\frac{j}{2}}.$$

(2) *If $\ell = 1$, then the number of non-equivalent 1-equiparting matrices is equal to*

$$\binom{j}{\frac{j+1}{2}}.$$

Proof. We count the number of non-equivalent ℓ-equiparting matrices of the form $A = (A_1, \ldots, A_j)$ where A_i is a 2-bit Gray code. A (2×4)-Gray code with the first bit flip in the first row has in total two bit flips in the first row and one bit flip in the second row.
(1): Let $\ell = 0$. Then $2d = 3j$ and consequently j has to be even. The matrix A must have transition count d in each row. Thus half of the A_i's have the first bit flip in the first row. Consequently, 0-equiparting matrices A with a fixed first column are in bijection with $\frac{j}{2}$-element subsets of a set with j elements. By inverting the bits in each row we can fix the first column of A to be the zero vector. Additionally, we are allowed to interchange the rows. Up to this equivalence there are $\frac{1}{2}\binom{j}{j/2}$ such matrices.
(2): Let $\ell = 1$. Then $2d = 3j + 1$ and so j is odd. The matrix A must have transition count d in one row while transition count $d - 1$ in the remaining row. Without loss of generality we can assume that A have transition count d in the first row. Assume that r of the A_i's have the first bit flip in the first row. Consequently, $j - r$ of the A_i's have the first bit flip in the second row.

Now the transition count of the first row is $2r+j-r$ while the transition count of the second row is $r+2(j-r)$. The system of equations $2r+j-r=d$, $r+2(j-r)=d-1$ yields that $r=\frac{j+1}{2}$. Therefore, up to equivalence, there are $\binom{j}{r}$ such matrices. $\square$

The case $\ell = 0 \Leftrightarrow 2d = 3j$

Let $\theta := D^{+,+}_{1,1}(1,2)$, and let e_θ denote the related basis element of the cell θ in the top cellular chain group $C_{2d+1}(X_{d,2}, X^{>1}_{d,2})$ which, in this case, is equivariantly generated by θ. According to equation (3.16), which also holds for $k=2$ as explained in Remark 3.26,

$$\mathfrak{o}(g)(e_\theta) = \big(\sum \pm 1\big)\cdot\zeta = a\cdot\zeta, \tag{3.18}$$

where $\zeta\in\pi_{2d+1}(S(W_2\oplus U_2^{\oplus j}))\cong\mathbb{Z}$ is a generator, and the sum ranges over all arrangements of two hyperplanes in relint θ that equipart $\mathcal{M}$. Since θ parametrizes all arrangements $\mathcal{H}=(H_1,H_2)$ where orientations and order of hyperplanes are fixed, the sum in (3.18) ranges over all arrangements of two hyperplanes that equipart $\mathcal{M}$ where orientation and order of hyperplanes are fixed. Therefore, by Theorem 3.3, the number of (±1)'s in the sum of (3.18) is equal to the number of non-equivalent 0-equiparting matrices of size $2\times 4j$. Now, Lemma 3.28 implies that the number of (±1)'s in the sum of (3.18) is $\frac{1}{2}\binom{j}{j/2}$. Consequently, integer a is odd if and only if $\frac{1}{2}\binom{j}{j/2}$ is odd.

Assume that the cocycle $\mathfrak{o}(g)$ is a coboundary. Hence there exists a cochain

$$\mathfrak{h}\in\mathcal{C}^{2d}_{\mathfrak{S}_2^{\pm}}\big(X_{d,2}, X^{>1}_{d,2}\,;\,\pi_{2d}(S(W_2\oplus U_2^{\oplus j}))\big)$$

with the property that $\mathfrak{o}(g)=\delta\mathfrak{h}$. The relation (3.11) for $k=2$ transforms into

$$\partial e_\theta = (1+(-1)^d\varepsilon_1)\cdot e_{\gamma_1} + (1+(-1)^d\tau_{1,2})\cdot e_{\gamma_3}.$$

Thus we have that

$$\begin{aligned} a\cdot\zeta &= \mathfrak{o}(g)(e_\theta)=\delta\mathfrak{h}(e_\theta)=\mathfrak{h}(\partial e_\theta)\\ &= (1+(-1)^d\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^d\tau_{1,2})\cdot\mathfrak{h}(e_{\gamma_3})\\ &= (1+(-1)^d)\cdot\mathfrak{h}(e_{\gamma_1})+(1+(-1)^{d+j+1})\cdot\mathfrak{h}(e_{\gamma_3})\\ &= 2b\cdot\zeta. \end{aligned}$$

Consequently, $\mathfrak{o}(g)$ is not a coboundary if and only if a is odd if and only if $\frac{1}{2}\binom{j}{j/2}$ is odd. Having in mind the Kummer criterion Lemma 2.19 stated below we conclude that: A $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2}\longrightarrow S(W_2\oplus U_2^{\oplus j})$ whose restriction to $X^{>1}_{d,2}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu\circ\psi_{\mathcal{M}}|_{X^{>1}_{d,2}}$ does not exists *if and only is* $\mathfrak{o}(g)$ is not a coboundary *if and only if* a is an odd integer *if and only if* $\frac{1}{2}\binom{j}{j/2}$ is odd *if and only if* $j=2^t$ for $t\geq 1$.

Thus we have proved the case (ii) of Theorem 3.5. Moreover, since the primary obstruction $\mathfrak{o}(g)$ is the only obstruction, we have proved that a $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2}\longrightarrow S(W_2\oplus U_2^{\oplus j})$ whose restriction to $X^{>1}_{d,2}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu\circ\psi_{\mathcal{M}}|_{X^{>1}_{d,2}}$ exists if and only if j, an even integer, is not a power of 2.

The case $\ell = 1 \Leftrightarrow 2d = 3j+1$

Let $\theta := D^{+;+}_{2,1}(1,2)$, and again let e_θ denote the related basis element of the cell θ in the cellular chain group $C_{2d}(X_{d,2}, X^{>1}_{d,2})$ which, in this case, is equivariantly generated by two cells $D^{+;+}_{2,1}(1,2)$ and $D^{+;+}_{1,2}(1,2)$. Again, the equation (3.16) implies that

$$\mathfrak{o}(g)(e_\theta) = \big(\sum \pm 1\big)\cdot\zeta = a\cdot\zeta, \tag{3.19}$$

where $\zeta \in \pi_{2d+1}(S(W_2 \oplus U_2^{\oplus j})) \cong \mathbb{Z}$ is a generator, and the sum ranges over all arrangements of k hyperplanes in $\operatorname{relint}\theta$ that equipart $\mathcal{M}$. The cell θ parametrizes all arrangements $\mathcal{H} = (H_1, H_2)$ where H_1 passes through the given point on the moment curve and orientations and order of hyperplanes are fixed. Thus the sum in (3.19) ranges over all arrangements of two hyperplanes that equipart $\mathcal{M}$ where H_1 passes through the given point on the moment curve with order and orientation of hyperplanes being fixed. Therefore, by Theorem 3.3, the number of (±1)'s in the sum of (3.19) is the same as the number of non-equivalent 1-equiparting matrices of size $2\times 4j$. Again, Lemma 3.28 implies that the number of (±1)'s in the sum of (3.19) is $\binom{j}{(j+1)/2}$. The integer a is odd if and only if $\binom{j}{(j+1)/2}$ is odd if and only if $j = 2^t - 1$ for $t \geq 1$.

Assume that the cocycle $\mathfrak{o}(g)$ is a coboundary. Then there exists a cochain

$$\mathfrak{h} \in \mathcal{C}^{2d-1}_{\mathfrak{S}_2^\pm}\big(X_{d,2}, X^{>1}_{d,2}\,;\,\pi_{2d-1}(S(W_2 \oplus U_2^{\oplus j}))\big)$$

with the property that $\mathfrak{o}(g) = \delta\mathfrak{h}$. Now, the relation (3.12) for $k=2$ transforms into

$$\partial e_\theta = (1 + (-1)^{d-1}\varepsilon_1)\cdot e_{\gamma_1} + (1 + (-1)^d\varepsilon_2 + (-1)^d\tau_{1,2} + (-1)^{d+d}\varepsilon_1\tau_{1,2})\cdot e_{\gamma_3}.$$

Thus, having in mind that j has to be odd, we have

$$\begin{aligned}
a\cdot\zeta &= \mathfrak{o}(g)(e_\theta) = \delta\mathfrak{h}(e_\theta) = \mathfrak{h}(\partial e_\theta)\\
&= (1 + (-1)^{d-1}\varepsilon_1)\cdot\mathfrak{h}(e_{\gamma_1}) + \\
&\quad (1 + (-1)^d\varepsilon_2 + (-1)^d\tau_{1,2} + (-1)^{d+d}\varepsilon_1\tau_{1,2})\cdot\mathfrak{h}(e_{\gamma_3})\\
&= (1 + (-1)^{d-1})\cdot\mathfrak{h}(e_{\gamma_1}) + (1 + (-1)^d + (-1)^{d+j+1} + (-1)^{j+1})\cdot\mathfrak{h}(e_{\gamma_3})\\
&= (1 + (-1)^{d-1})\cdot\mathfrak{h}(e_{\gamma_1}) + (1 + (-1)^d + (-1)^d + 1)\cdot\mathfrak{h}(e_{\gamma_3})\\
&= \begin{cases} 2\mathfrak{h}(e_{\gamma_1}), & d \text{ odd}\\ 4\mathfrak{h}(e_{\gamma_3}), & d \text{ even.}\end{cases}
\end{aligned} \tag{3.20}$$

Now, we separately consider cases depending on parity of d and value of j.
(1) Let d be odd. Recall that a is odd if and only if $j = 2^t - 1$ for $t \geq 1$. Since $d = \frac{1}{2}(3j+1) = 3\cdot 2^{t-1} - 1$ and d is odd we have that for $j = 2^t - 1$, with $t \geq 2$, the integer a is odd and consequently $\mathfrak{o}(g)$ is not a coboundary. Thus a $\mathfrak{S}_2^\pm$-equivariant map $X_{d,2} \longrightarrow S(W_2 \oplus U_2^{\oplus j})$ whose restriction to $X^{>1}_{d,2}$ is $\mathfrak{S}_2^\pm$-homotopic to $\nu\circ\psi_{\mathcal{M}}|_{X^{>1}_{d,2}}$ does not exists. We have proved the case (ii) of Theorem 3.5 for $t \geq 2$.
(2) Let $d = 2$ and $j = 1 = 2^1 - 1$. Then the integer a is again odd and consequently cannot be divisible by 4 implying again that $\mathfrak{o}(g)$ is not a coboundary.Therefore a $\mathfrak{S}_2^\pm$-equivariant map

$X_{2,2} \longrightarrow S(W_2 \oplus U_2)$ whose restriction to $X_{2,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{2,2}^{>1}}$ does not exists. This concludes the proof of the case (ii) of Theorem 3.5.

(3) Let $d \geq 4$ be even. Now we determine the integer a by computing local degrees $\deg(\nu \circ \psi_{\mathcal{M}}|_{X^{(N_2+1)}} \circ h_\theta|_{S_i})$; see (3.16) and (3.19). We prove, almost identically as in the proof of Lemma 2.17, that all local degrees equal, either 1 or -1.

That local degrees of $\nu \circ \psi_{\mathcal{M}}|_\theta$ are ± 1 is simple to see since in a small neighborhood U in $\operatorname{relint}\theta$ around any root $\lambda u + (1-\lambda)v$ the test map $\psi_{\mathcal{M}}|_\theta$ is a continuous bijection. Indeed, for any vector $w \in W_2 \oplus U_2^{\oplus j}$, with sufficiently small norm, there is exactly one $\lambda u' + (1-\lambda)v' \in U$ with $\psi_{\mathcal{M}}(\lambda u' + (1-\lambda)v') = w$. Thus $\psi_{\mathcal{M}}|_{\partial U}$ is a continuous bijection into some $3j$-sphere around the origin of $W_2 \oplus U_2^{\oplus j}$ and by compactness of ∂U is a homeomorphism.

Next we compute the signs of the local degrees. First we describe a neighborhood of every root of the test map $\psi_{\mathcal{M}}$ in $\operatorname{relint}\theta$. Let $\lambda u + (1-\lambda)v \in \operatorname{relint}\theta$ with $\psi_{\mathcal{M}}(\lambda u + (1-\lambda)v) = 0$. Consequently $\lambda = \frac{1}{2}$. Denote the intersections of the hyperplane H_u with the moment curve by $x_1, \ldots, x_d$ in the correct order along the moment curve. Similarly, let $y_1, \ldots, y_d$ be the intersections of H_v with the moment curve. In particular, x_1 is the point q_1 that determines the cell θ; see Lemma 3.23. Choose an $\epsilon > 0$ such that ϵ-balls around $x_2, \ldots, x_d$ and around $y_1, \ldots, y_d$ are pairwise disjoint with the property that these balls intersect the moment curve only in precisely one of the intervals $I_1, \ldots, I_j$. Pairs of hyperplanes $(H_{u'}, H_{v'})$ with $\lambda u' + (1-\lambda)v' \in \operatorname{relint}\theta$ that still intersect the moment curve in the corresponding ϵ-balls parametrize a neighborhood of $\frac{1}{2}u + \frac{1}{2}v$. The local neighborhood consisting of pairs of hyperplanes with the same orientation still intersecting the moment curve in the corresponding ϵ-balls where the parameter λ is in some neighborhood of $\frac{1}{2}$. For sufficiently small $\epsilon > 0$ the neighborhood can be naturally parametrized by the product

$$(\tfrac{1}{2} - \epsilon, \tfrac{1}{2} + \epsilon) \times \prod_{i=2}^{2d} (-\epsilon, \epsilon),$$

where the first factor relates to λ, the next $d-1$ factors correspond to neighborhoods of the $x_2, \ldots, x_d$ and the last d factors to ϵ-balls around $y_1, \ldots, y_d$. A natural basis of the tangent space at $\frac{1}{2}u + \frac{1}{2}v$ is obtained via the push-forward of the canonical basis of $\mathbb{R}^{2d}$ as tangent space at $(\frac{1}{2}, 0, \ldots, 0)^t$.

Consider the subspace $Z \subseteq \operatorname{relint}\theta$ that consists all points $\lambda u + (1-\lambda)v$ associated to the pairs of hyperplanes (H_u, H_v) such that both hyperplanes intersect the moment curve in d points. In the space Z the local degrees only depend on the orientations of the hyperplanes H_u and H_v, but these are fixed since $Z \subseteq \operatorname{relint}\theta$. Indeed, any two neighborhoods of distinct roots of the test map $\psi_{\mathcal{M}}$ can be mapped onto each other by a composition of coordinate charts since their domains coincide. This is a smooth map of degree 1: the Jacobian at the root is the identity map. Let $\frac{1}{2}u + \frac{1}{2}v$ and $\frac{1}{2}u' + \frac{1}{2}v'$ be roots in Z of the test map $\psi_{\mathcal{M}}$ and let Ψ be the change of coordinate chart described above. Then $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{M}} \circ \Psi$ differ in a neighborhood of $\frac{1}{2}u + \frac{1}{2}v$ just by a permutation of coordinates. This permutation is always even by Claim 2.18.

The orientations of the hyperplanes H_u and H_v are fixed by the condition that $\frac{1}{2}u + \frac{1}{2}v \in \operatorname{relint}\theta$. Thus H_u and H_v are completely determined by the set of intervals that H_u cuts once. Let $A \subseteq \{1, \ldots, j\}$ be the set of indices of intervals $I_1, \ldots, I_h$ that H_u intersects once, and let $B \subseteq \{1, \ldots, j\}$ be the same set for H_v. Then Ψ is a composition of a multiple of $A \bigtriangleup B$ transpositions and, hence, an even permutation. This means that all the local degrees (± 1's) in the sum (3.19) are of the same sign, and consequently $a = \pm\binom{j}{(j+1)/2}$.

Now, since d is even the equality (3.20) implies that $a\cdot\zeta = 4b\cdot\zeta$. Thus, if $\mathfrak{o}(g)$ is a coboundary, a is divisible by 4. In the case $j = 2^t+1$ where $t \geq 2$, and $d = 3\cdot 2^{t-1}+2$ the Kummer criterion implies that the binomial coefficient $\binom{j}{(j+1)/2}$ is divisible by 2 but *not* by 4. Hence $\mathfrak{o}(g)$ is not a coboundary and a $\mathfrak{S}_2^{\pm}$-equivariant map $X_{d,2} \longrightarrow S(W_2 \oplus U_2^{\oplus j})$ whose restriction to $X_{d,2}^{>1}$ is $\mathfrak{S}_2^{\pm}$-homotopic to $\nu \circ \psi_{\mathcal{M}}|_{X_{d,2}^{>1}}$ does not exist. This concludes the final instance (iii) of Theorem 3.5. ☐

3.5.4 Proof of Theorem 3.6

We prove both instances of the Ramos conjecture $\Delta(2,3) = 5$ and $\Delta(4,3) = 10$ using Theorem 3.4. Thus in order to prove that

- $\Delta(2,3) = 5$ it suffices to show that the number of non-equivalent 1-equiparting matrices of size $3 \times 2 \cdot 2^3$ is odd, Proposition 3.30;
- $\Delta(4,3) = 10$ it suffices to show that the number of non-equivalent 2-equiparting matrices of size $3 \times 4 \cdot 2^3$ is also odd, Enumeration 3.31.

Consequently we turn our attention to 3-bit Gray codes. It is not hard to see that the following lemma holds.

Lemma 3.29. *Let $c_1 \in \{0,1\}^3$ be a choice of first column.*

(i) *There are* 18 *different* 3*-bit Gray codes $A = (c_1, c_2, \ldots, c_8) \in \{0,1\}^{3\times 8}$ that start with c_1. They have transition counts* $(3,2,2)$, $(3,3,1)$, *or* $(4,2,1)$.

(ii) *There are* 3 *equivalence classes of Gray codes that start with with c_1. The three classes can be distinguished by their transition counts.*

Proof. (i): Starting at a given vertex of the 3-cube, there are precisely 18 Hamiltonian paths. This can be seen directly or by computer enumeration.
(ii): Follows directly from (i), as all equivalence classes have size 6: If $c_1 = (0,0,0)^t$ then all elements in a class are obtained by permutation of rows. For other choices of c_1, they are obtained by arbitrary permutations of rows followed by the "correct" row bit-inversions to obtain c_1 in the first column. ☐

Proposition 3.30. *There are* 13 *non-equivalent* 1*-equiparting matrices that are of size* $3 \times (2 \cdot 2^3)$.

Proof. Let $A = (A_1, A_2)$ be a 1-equiparting matrix. This means that both A_1 and A_2 are 3-bit Gray codes and the last column of A_1 is equal to the first column of A_2. In addition, the transition counts cannot exceed 5 and must sum up to 14. Having in mind that A is a 1-equiparting matrix it follows that A must have transition counts $\{5,5,4\}$. Hence two of its rows must have transition count 5 and one row must have transition count 4. In the following a *realization* of transition counts is a Gray code with the prescribed transition counts.

Since we are counting 1-equiparting matrices up to equivalence we may fix the first column of A, and hence first column of A_1, to be $(0,0,0)^t$ and choose for A_1 one of the matrices from each of the 3 classes of 3-bit Gray codes described in Lemma 3.29(ii).

If A_1 has transition counts $(3,2,2)$, that is, the first row has transition count 3 while remaining rows have transition count 2, then its last column is $(1,0,0)^t$. The next Gray code A_2 in the matrix a can have transition counts $(2,3,2)$, $(2,2,3)$, or $(1,3,3)$, each having 2 realizations A_2, each with first column $(1,0,0)^t$.

If A_1 has transition $(3, 3, 1)$, then its last column is $(1, 1, 0)^t$. The Gray code A_2 can have transition counts $(2, 2, 3)$, having 2 realizations, or $(1, 2, 4)$, having 1 realization, or $(2, 1, 4)$, having one realization, each with first column $(1, 1, 0)^t$.

If A_1 has transition counts $(4, 2, 1)$, then its last column is $(0, 0, 1)^t$. The Gray code A_2 can have transition counts $(1, 2, 4)$, having 1 realization, or $(1, 3, 3)$, having 2 realizations, each with first column $(0, 0, 1)^t$.

In total we have $6 + 4 + 3 = 13$ non-equivalent 1-equiparting matrices $A = (A_1, A_2)$. □

Enumeration 3.31. *There are* 2015 *non-equivalent* 2*-equiparting matrices that are of size* $3\times 4\cdot 2^3$.

Proof. Using Lemma 3.29 we enumerate non-equivalent 2-equiparting matrices by computer. Let $A = (A_1, A_2, A_3, A_4)$ be a 2-equiparting matrix. It must have transition counts $\{10, 10, 8\}$. Similarly as above, A is constructed by fixing the first column to be $(0, 0, 0)^t$ and A_1 to be one representative from each of the 3 classes of Gray codes. Then all possible Gray codes for A_2, A_3, A_4 are checked, making sure that the last column of A_i is equal to the first column of A_{i+1} and that the transition counts of $A_1, \dots, A_4$ sum up to $\{10, 10, 8\}$. This leads to 2015 possibilities. □

This concludes the proof of Theorem 3.6.

Remark 3.32. By means of a computer we were able to calculate the number $N(j, k, d)$ of non-equivalent ℓ-equiparting matrices for several values of $j \geq 1$ and $k \geq 3$, where $d = \lceil \frac{2^k-1}{k} j \rceil$ and $\ell = dk - (2^k - 1)j$; see Table 3.1.

Number $N(j,k,d)$ of non-equiv ℓ-equiparting matrices given integers $j \geq 2$ and $k \geq 3$.				
j	k	ℓ	d	$N(j,k,d)$
2	3	1	5	13
3	3	0	7	60
4	3	2	10	2015
5	3	1	12	35040
6	3	0	14	185130
7	3	2	17	7572908
8	3	1	19	132909840
9	3	0	21	732952248
1	4	1	4	16
2	4	2	8	37964

Table 3.1: Here $d = \lceil \frac{2^k-1}{k} j \rceil$ and $\ell = dk - (2^k - 1)j$.

Part II

Tverberg-type theorems for matroids

Chapter 4

A counterexample and a proof

Abstract

Bárány, Kalai, and Meshulam recently obtained a topological Tverberg-type theorem for matroids, which guarantees multiple coincidences for continuous maps from a matroid complex to $\mathbb{R}^d$, if the matroid has sufficiently many disjoint bases. They make a conjecture on the connectivity of k-fold deleted joins of a matroid with many disjoint bases, which could yield a much tighter result – but we provide a counterexample already for the case of $k = 2$, where a tight Tverberg-type theorem would be a topological Radon theorem for matroids. Nevertheless, we prove a topological Radon theorem for the counterexample family of matroids by an index calculation, despite the failure of the connectivity-based approach.
Publication Remark. The results of this chapter are joint work with Pavle V. M. Blagojević and Günter M. Ziegler [26].

4.1 Introduction

Let $d \geq 1$ and $k \geq 1$ be integers and let $f\colon \Sigma \to \mathbb{R}^d$ be a continuous map from a non-trivial simplicial complex Σ to $\mathbb{R}^d$. A *Tverberg k-partition of f* is a collection $\{\sigma_1, \dots, \sigma_k\}$ of k pairwise disjoint faces of Σ such that $\bigcap_{i=1}^{k} f(\sigma_i) \neq \emptyset$. For fixed $d \geq 1$, the *topological Tverberg number* $\mathrm{TT}(\Sigma, d)$ is the maximal integer $k \geq 1$ such that every continuous map $f\colon \Sigma \to \mathbb{R}^d$ has a Tverberg k-partition. The topological Tverberg theorem due to Bárány, Shlosman, and Szűcs [10] implies that, if Σ is the d-skeleton $\Delta^{(d)}_{(k-1)(d+1)}$ of the simplex of dimension $(k-1)(d+1)$ and k is prime, then $\mathrm{TT}(\Delta^{(d)}_{(k-1)(d+1)}, d) = k$. For $k = 2$ this result is equivalent to the topological Radon theorem [6]. It follows from the work of Özaydin [64] that this result remains true when k is a prime power. Recently Frick [42], [25], using the "constraint method" [24] and building on work by Mabillard and Wagner [56], showed that if $k \geq 6$ is not a prime-power and $d \geq 3k+1$, then $\mathrm{TT}(\Delta_{(k-1)(d+1)}, d) < k$; see [8] for a recent survey.

Recently Bárány, Kalai, and Meshulam [9] gave lower bounds for the topological Tverberg number of a matroid, regarded as the simplicial complex of its independent sets. Let Σ be a matroid M of rank $d + 1$ with b disjoint bases, then [9, Thm. 1] asserts that $\mathrm{TT}(M, d) \geq \sqrt{b}/4$. If M is the uniform matroid $\Delta^{(d)}_{(k-1)(d+1)}$, then this result implies that $\mathrm{TT}(\Delta^{(d)}_{(k-1)(d+1)}, d) \geq \sqrt{k-1}/4$ for all integers $d, k \geq 1$.

The results of [10], [64], and [9, Thm. 1] mentioned above are all obtained by using a configuration space/test map scheme. In the *join scheme* used in [70] and [9] the configuration space X is the k-fold deleted join Σ^{*k}_{Δ} of the complex Σ and the test space Y is a sphere $S^{(k-1)(d+1)-1}$ of dimension $(k-1)(d+1)-1$. In the *product scheme* used in [10] the configuration space X is the k-fold deleted product $\Sigma^{\times k}_{\Delta}$ of the complex Σ and the test space Y is a sphere $S^{(k-1)d}$. If k is prime, then all spaces and in particular the two spheres admit free actions by the group $\mathbb{Z}/k$.

In order to obtain sharp results using a configuration space/test map scheme it is necessary to determine proof strategies for the nonexistence of an equivariant map from the configuration space X to the test space Y. One commonly used method is the *connectivity-based approach*, which can be applied if Y is a finite-dimensional CW complex on which the group acts freely: If one establishes that the connectivity of the space X is at least as high as the dimension of the space Y, then Dold's theorem [35] implies that an equivariant map $X \to Y$ cannot exist. For a more general version of Dold's theorem that is also applicable in this context see [77].

The connectivity-based approach (for k prime) yields tight bounds for the topological Tverberg number of $\Sigma = \Delta^{(d)}_{(k-1)(d+1)}$ with both the product scheme [10] and the join scheme [70]. The natural questions we are concerned with regard the more general case where Σ is a matroid M: What is the connectivity of the test spaces? Which results can/cannot be obtained via a connectivity-based approach? Having these questions in mind, Bárány, Kalai, and Meshulam formulated the following conjecture.

Conjecture 4.1 (Bárány, Kalai, and Meshulam 2016 [9, Conj. 4])**.** *For any integer $k \geq 1$ there exists an integer $n_k \geq 1$ depending only on k such that for any matroid M of rank $r \geq 1$ with at least n_k disjoint bases, the k-fold deleted join M^{*k}_{Δ} of the matroid M is $(kr-1)$-dimensional and $(kr-2)$-connected.*

For $k = 1$ the conjecture is true, since a matroid of rank r is pure shellable and hence in particular $(r-2)$-connected [17, Thm. 4.1]. Using the connectivity-based approach the conjecture would imply that for a matroid M of rank $d+1$ with $b \geq n_k$ disjoint bases the topological Tverberg number satisfies $\mathrm{TT}(M, d) \geq k$.

We prove the following theorem that gives a counterexample to the conjecture already in the case where $k = 2$.

Theorem 4.2 (Conjecture 4.1 fails for $k = 2$)**.** *There is a family of matroids M_r ($r \in \mathbb{Z}, r \geq 2$) such that each matroid M_r has rank r and r disjoint bases, while the 2-fold deleted join $(M_r)^{*2}_{\Delta}$ of M_r is is $(2r-1)$-dimensional and $(2r-3)$-connected, but not $(2r-2)$-connected.*

The family of matroids M_r $(r \geq 2)$ is a tight example for the failure of Conjecture 4.1 in the sense that if we increase the number of bases from r to $r+1$, then the 2-fold deleted join of the new complex is $(2r-2)$-connected; see Corollary 4.12. To prove Theorem 4.2 we first show that the complex $(M_r)^{*2}_{\Delta}$ is shellable for $r \geq 3$ using the notion of shellability for non-pure complexes due to Björner and Wachs [17], [18]; see Proposition 4.11. The crucial ingredient in the proof is Proposition 4.9, which shows that balanced subcomplexes of shellable balanced complexes are again shellable. The case $r = 2$ is treated separately; see Remark 4.10. We give a first proof of Theorem 4.2 by constructing a covering of $(M_r)^{*2}_{\Delta}$ by two subcomplexes; see Corollary 4.15. A second proof of Theorem 4.2 is a straightforward calculation involving only the combinatorics of $(M_r)^{*2}_{\Delta}$; see Section 4.3.5. This allows us to calculate the Betti numbers of $(M_r)^{*2}_{\Delta}$; see Corollary 4.16.

Using the connectivity-based approach one obtains that $\mathrm{TT}(M_r, d) \geq 2$ when $2r-3 \geq d$; see Corollary 4.23. However, despite the lower connectivity of the matroid M_r we still obtain a sharp topological Radon theorem for M_r by means of a Fadell–Husseini index argument that goes back to [20, Thm. 1] and [27, Thm. 4.2]; for the classical reference regarding the Fadell–Husseini index see [38]. Thus the following theorem is an example of a Tverberg-type result for a family of matroids that cannot be obtained via the connectivity-based approach.

Theorem 4.3 (Topological Radon theorem for M_r)**.** *Let $d \geq 1$ and $r \geq 3$ be integers such that $2r-2 \geq d$. Then the topological Tverberg number of the family of matroids M_r from Theorem 4.2 satisfies* $\mathrm{TT}(M_r, d) \geq 2$.

We summarize the remaining results of this chapter as follows.

- We show that [9, Cor. 3] in fact implies lower bounds for the topological Tverberg number $\mathrm{TT}(M, d)$ for matroids M of all ranks; see Corollary 4.17.
- We give upper bounds for the topological Tverberg number $\mathrm{TT}(M, d)$ in the case where the rank r of the matroid M is at most $d-2$; see Proposition 4.18.
- We show that the connectivity of the k-fold deleted product $M^{\times k}_{\Delta}$ of a matroid M of rank r with b disjoint bases is at least $r-2-\lfloor r(k-1)/b \rfloor$, when $k \geq 2$ and $b, r \geq k$. If $b \geq r(k-1)+1$, then $M^{\times k}_{\Delta}$ is not $(r-1)$-connected; see Theorem 4.20.
- Using Theorem 4.20 we establish the connectivity of the ordered configuration space of two particles in a matroid; see Corollary 4.21.

4.2 Preliminaries

4.2.1 Terminology

By a *simplicial complex* or simply *complex* we refer to a finite abstract simplicial complex or a geometric realization of a finite abstract simplicial complex. We require that any complex contains the empty set as a face of dimension -1. A *facet* of a complex is a face that is not contained in any other face. Let $\Sigma_1, \ldots, \Sigma_k$ be simplicial complexes with vertex sets $V_1, \ldots, V_k$. Then the *join* of the Σ_i is defined as the simplicial complex $\Sigma_1 * \cdots * \Sigma_k = \{\sigma_1 \sqcup \cdots \sqcup \sigma_k : \sigma_i \in \Sigma_i\}$ with vertex set equal to the disjoint union $\bigsqcup_{i=1}^k V_i$. Assume the vertex sets V_i are all contained in a common set V, then the *deleted join* of the Σ_i is defined as the simplicial complex $(\Sigma_1 * \cdots * \Sigma_k)_\Delta = \{\sigma_1 \sqcup \cdots \sqcup \sigma_k : \sigma_i \in \Sigma_i, \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j\}$ with vertex set $\bigsqcup_{i=1}^k V_i$. Let $\Sigma_i = \Sigma$ for $i = 1, \ldots, k$. Then $\Sigma^{*k} := \Sigma_1 * \cdots * \Sigma_k$ is the *k-fold join* of Σ and $\Sigma^{*k}_\Delta := (\Sigma_1 * \cdots * \Sigma_k)_\Delta$ is the *k-fold deleted join* of Σ. If $\sigma \subseteq V$, the *deletion* of σ from Σ is defined as $\Sigma \setminus \sigma = \{\tau \in \Sigma : \sigma \not\subseteq \tau\}$. We also denote $\Sigma \setminus \sigma$ by $\Sigma|(V \setminus \sigma)$ and refer to it as the *restriction* of Σ to the set $V \setminus \sigma$. The *link* of Σ with respect to a face $\sigma \in \Sigma$ is defined as $\Sigma/\sigma = \{\tau \in \Sigma : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma\}$. Given a geometric simplicial complex Σ, we define the *k-fold deleted product* $\Sigma^{\times k}_\Delta$ of Σ as the CW complex with cells given by products of relative interiors of (geometric) simplices $\sigma_i \in \Sigma$ of the form $\mathrm{relint}(\sigma_1) \times \cdots \times \mathrm{relint}(\sigma_k)$, where $\sigma_i \cap \sigma_j = \emptyset$ for all i, j with $1 \leq i < j \leq k$. The attaching maps for $\Sigma^{\times k}_\Delta$ are given by the products of the attaching maps of Σ. For additional terminology and results regarding simplicial complexes see Matoušek [59].

A *matroid* M with *ground set* E is a simplicial complex with vertices in E such that for every $A \subseteq E$ the restriction $M|A = \{\sigma \in M : \sigma \subseteq A\}$ is pure. We call a face of M an *independent set*.

We call a facet of M a *basis* and call the cardinality of a (any) basis the *rank* of M. Let m and n be integers with $0 \leq m \leq n$. Given a ground set E of cardinality n, the *uniform matroid* $U_{m,n}(E)$ is given by the collection of all subsets of E of cardinality at most m. Let $\Delta_{n-1}^{(m-1)}$ be $(m-1)$-skeleton of the simplex of dimension $n-1$. Then we have $\Delta_{n-1}^{(m-1)} = U_{m,n}(E)$. Given matroids $M_1, \dots, M_k$ with ground sets $E_1, \dots, E_k$, the *direct sum* $M_1 \oplus \cdots \oplus M_k$ of the family M_i is defined as the collection $\{I_1 \sqcup \cdots \sqcup I_k : \; I_i \in M_i\}$ and is a matroid with ground set $E_1 \sqcup \cdots \sqcup E_k$. The direct sum of a collection of matroids is equal to the join of the collection of matroids, viewed as simplicial complexes. For additional terminology and results regarding matroids see Oxley [63].

4.2.2 Non-pure shellability

Since some of the complexes we are interested in are non-pure, we use the notions of "non-pure shellability" introduced by Björner and Wachs [17], [18].

By [17, Def. 2.1] a *shelling* of a possibly non-pure finite simplicial complex Σ of dimension d is defined as a strict order "$\ll$" on the set $\mathcal{F}$ of facets of Σ such that for any facet $B \in \mathcal{F}$ of dimension $d' \leq d$ for which there exists a prior facet $A \in \mathcal{F}$ with $A \ll B$, the simplicial complex

$$B \cap \Big(\bigcup_{A \in \mathcal{F},\, A \ll B} A \Big)$$

defined by the intersection of B with the union of the previous facets (and their faces) is pure and $(d'-1)$-dimensional. This is equivalent to the following condition. For any two facets $A, B \in \mathcal{F}$ with $A \ll B$, there is a facet $C \in \mathcal{F}$ and a vertex $x \in B$ such that

$$C \ll B \quad \text{and} \quad A \cap B \subseteq B \cap C = B \setminus \{v\}. \tag{4.1}$$

For pure complexes Σ, the above definition coincides with the "usual" definition of shellability. A d-dimensional simplicial complex Σ is *shellable* if it has a shelling. It is *pure shellable* if it is pure and shellable.

4.3 Proof of the main result

4.3.1 The counterexample family M_r

Definition 4.4 (The counterexample family M_r)**.** Let $r \geq 2$ be an integer. Let E be a set of pairwise distinct elements v_i^j and w_j for $i = 1, \dots, r-1$ and $j = 1, \dots, r$. Define *blocks* E_i by

$$E_i = \{v_i^1, \dots, v_i^r\} \quad \text{for} \quad i = 1, \dots, r-1, \quad \text{and} \quad E_r = \{w_1, \dots, w_r\}.$$

Define a matroid $\widehat{M_r}$ by

$$\widehat{M_r} = U_{1,r}(E_1) \oplus \cdots \oplus U_{1,r}(E_{r-1}) \oplus U_{r,r}(E_r).$$

Then the matroid M_r with ground set E is defined as the $(r-1)$-skeleton of $\widehat{M_r}$, hence
$M_r = \{I \in \widehat{M_r} : \; |I| \leq r\}$.

The matroid M_r has rank r and has r pairwise disjoint bases of the form $\{v_1^j, \ldots, v_{r-1}^j, w_j\}$ for $j = 1, \ldots, r$. Faces of M_r are given by choosing at most r vertices in total and at most 1 vertex in each of the first $r-1$ blocks; see Figure 4.1.

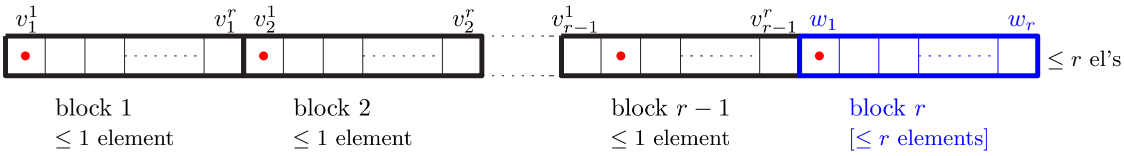


Figure 4.1: The matroid M_r.

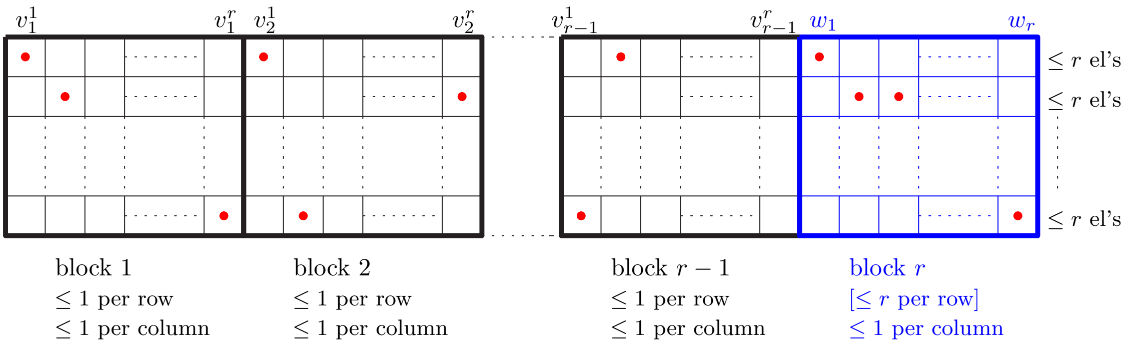


Figure 4.2: The k-fold deleted join $(M_r)^{*k}_{\Delta}$ with an example facet.

Consider the k-wise deleted join of the complex M_r, which we denote by $(M_r)^{*k}_{\Delta}$. We display the vertices of $(M_r)^{*k}_{\Delta}$ in k *rows*, based on the copy of M_r they belong to. We group the vertices of $(M_r)^{*k}_{\Delta}$ into r *blocks*; see Figure 4.2. A *column* of $(M_r)^{*k}_{\Delta}$ consists of the k copies of a fixed vertex $v \in E$. Faces of $(M_r)^{*k}_{\Delta}$ are given by choosing at most r vertices in each row, at most 1 vertex per column and at most 1 vertex in each row of each of the first $r-1$ blocks. Note that $(M_r)^{*k}_{\Delta}$ has dimension $d = 2r - 1$ and is not pure: Its facets have dimensions $d, d-1, \ldots, d-k+1$. See Figure 4.3 for an example facet of dimension $d - 1 = 8$ for $r = 5$ and $k = 2$.

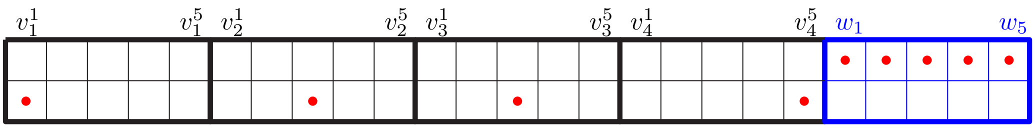


Figure 4.3: An 8-dimensional facet of the 9-dimensional complex $(M_5)^{*2}_{\Delta}$.

4.3.2 Shellability of subcomplexes of balanced complexes

To define a shelling of $(M_r)^{*2}_{\Delta}$ we use the existence of pure shellings of certain pure subcomplexes that we can describe as "balanced complexes." Let us recall the definition of a balanced complex.

Definition 4.5 (Stanley [74, Sec. 2])**.** Let $m \geq 1$ and $d \geq 0$ be integers and let $a = (a_1, \ldots, a_m)$ be an m-tuple of non-negative integers such that $a_1 + \cdots + a_m = d + 1$. Let Σ be a d-dimensional simplicial complex with vertex set V. Let $\mathcal{V} := (V_1, \ldots, V_m)$ be an ordered partition of V into pairwise disjoint sets V_i, called a *vertex coloring*. We call Σ a *balanced complex (of type a with respect to the partition* $\mathcal{V}$*)* if

(i) Σ is pure, and

(ii) for every facet $A \in \Sigma$ we have that $|A \cap V_i| = a_i$ for $i = 1, \ldots, m$.

We call Σ *completely balanced* if it is balanced of type $(1, \ldots, 1)$.

For example, the order complex of any graded poset is completely balanced. A simplicial complex is pure if and only if it is balanced of type $a = (a_1)$. Balanced complexes were introduced by Stanley in 1979 [74]. They have been studied in the context of posets [5], [16], simplicial polytopes [45], [50], and Cohen-Macaulay or shellable complexes [17], [61]. We point out that some authors use "balanced complex" to refer to a "completely balanced complex."

Balanced complexes are not necessarily pure shellable, as can be seen by taking any pure non-shellable d-dimensional complex. Given a balanced complex Σ consider its *type-selected subcomplex* Σ_T [14, p. 1858], which is the restriction of Σ to the set $\bigcup_{i \in T} V_i$ of vertices whose types (colors) are contained in the set $T \subseteq \{1, \ldots, m\}$. It was shown in [12] that any type-selected subcomplex of a pure shellable complex is pure shellable; see [14, Thm. 11.13] for a more general result. However, we are interested in the pure shellability of the following subcomplex.

Definition 4.6 (Balanced b-skeleton)**.** Let $m \geq 1$ be an integer and let Σ be a balanced d-complex of type $a = (a_1, \ldots, a_m)$ with vertex coloring $(V_1, \ldots, V_m)$ and let $b = (b_1, \ldots, b_m)$ be an m-tuple of integers with $0 \leq b_i \leq a_i$ for $i = 1, \ldots, m$. Then the complex Σ^b given by the faces F of Σ for which $|F \cap V_i| \leq b_i$ for $i = 1, \ldots, m$ is the *balanced b-skeleton of* Σ.

To show that balanced b-skeleta of pure shellable balanced complexes are pure shellable (Proposition 4.9) we use the existence of shellings of the skeleta of a shellable complex that are "compatible" with the original shelling of the complex. Let us formulate this as a definition.

Definition 4.7. Let Σ be a shellable simplicial complex of dimension $d \geq 0$ with shelling order "$\ll$". Let k be an integer with $0 \leq k \leq d$. Then a shelling $\ll'$ of the k-skeleton of Σ is *compatible with the shelling of* Σ if the following implication holds for any two k-faces $\overline{A}$ and $\overline{B}$ of Σ: If $\overline{A} \ll' \overline{B}$ and if A and B are the smallest (w.r.t. "$\ll$") d-faces of Σ containing $\overline{A}$ and $\overline{B}$, respectively, then $A \ll B$ or $A = B$.

The following lemma is true even for shellable complexes that are non-pure. We apply it only in the pure setting.

Lemma 4.8 (Björner and Wachs [17, Thm. 2.9])**.** *Let* Σ *be a shellable simplicial complex of dimension d. Let k be an integer with* $0 \leq k \leq d$*. Then there exists a shelling of the k-skeleton of* Σ *that is compatible with the shelling of* Σ*.*

We point out that the property of compatibility is transitive in the following sense. Let Σ be a shellable complex and let the shelling of its k-skeleton be compatible with the shelling of Σ. Then any shelling of its $(k-1)$-skeleton that is compatible with the shelling of its k-skeleton is also compatible with the shelling of Σ.

Proposition 4.9. *Let $m \geq 1$ be an integer and let Σ be a balanced d-complex of type $a = (a_1, \ldots, a_m)$ with vertex coloring $(V_1, \ldots, V_m)$ and let $b = (b_1, \ldots, b_m)$ be an m-tuple of non-negative integers with $0 \leq b_i \leq a_i$ for $i = 1, \ldots, m$. If Σ is pure shellable, then the balanced b-skeleton Σ^b of Σ is pure shellable.*

Proof. By induction and transitivity of compatibility, it suffices to prove the statement for $b_1 = a_1 - 1$ and $b_i = a_i$ for $i = 2\ldots, m$. Let d denote the dimension of Σ. Let "$\ll$" denote the shelling of Σ and, to simplify notation, let "$\ll$" also denote a compatible shelling of the $(d-1)$-skeleton of Σ, which exists by Lemma 4.8. We show that the restriction of this shelling to the subcomplex Σ^b is a shelling. Let $\overline{A}$ and $\overline{B}$ be two (balanced) facets of Σ^b such that $\overline{A} \ll \overline{B}$. We must show that there exists a $(d-1)$-face $\overline{C}$ of Σ and a vertex $v \in \overline{B}$ such that

$$\overline{C} \ll \overline{B} \text{ and } \overline{A} \cap \overline{B} \subseteq \overline{C} \cap \overline{B} = \overline{B} \setminus \{v\} \text{ with } |\overline{C} \cap V_i| = |b_i| \text{ for } i = 1, \ldots, m. \tag{4.2}$$

Let A and B denote the smallest (w.r.t. "$\ll$") facets of Σ containing $\overline{A}$ and $\overline{B}$, respectively. So, either $A = B$ or $A \ll B$. If $A = B$, then we are done, since $\overline{A}$ and $\overline{B}$ are codimension-one faces of the same facet, implying that they have all but one vertex in common. In this case $\overline{C} = \overline{A}$ satisfies Equation (4.2). Otherwise, if $A \ll B$, then by shellability of Σ there is a (balanced) facet $C \ll B$ of Σ that satisfies Equation (4.1). In particular C and B have all but one vertex in common. Both C and B have type a. This implies that the two vertices in $B \triangle C$ must have the same color. Hence if $\{v\} = B \setminus C$ and $\{w\} = C \setminus B$, then there is an $i_0 \in \{1, \ldots, m\}$ such that $v, w \in V_{i_0}$. Assume that v is not a vertex of $\overline{B}$. Then $\overline{B} \subset C$, leading to a contradiction to the minimality of B. Hence v is a vertex of $\overline{B}$. Now define $\overline{C} = B \cap C$. Then $\overline{C}$ is a face of Σ and $\overline{C} = \overline{B} \setminus \{v\} \cup \{w\}$. Since v and w are of the same type, $|\overline{C} \cap V_i| = |b_i|$ holds for all $i \in [m]$. The fact that $\overline{C}$ is contained in the facet $C \ll B$ of Σ and $\overline{B} \subset B$ is not contained in C implies that $\overline{C} \ll \overline{B}$. Hence $\overline{C}$ satisfies Equation (4.2). □

4.3.3 Shellability of the two-fold deleted join $(M_r)^{*2}_{\Delta}$

We make the following notational conventions. Let k and r be integers with $k \geq 2$ and $r \geq 2k-1$. We write a face A of $(M_r)^{*k}_{\Delta}$ as $A = (A_1, \ldots, A_k)$, where A_i lists the vertices used in the i-th row of $(M_r)^{*k}_{\Delta}$ for $i = 1, \ldots, k$. We write $A_i = (\overline{A_i}, A_i^r)$, where $\overline{A_i}$ lists the vertices of A_i contained in the first $r-1$ blocks of $(M_r)^{*k}_{\Delta}$ and A_i^r lists the vertices of A_i contained in the r-th block of $(M_r)^{*k}_{\Delta}$. If we need to clarify that a vertex v of Σ originates from row i, we write (v, i). We say that a vertex v of $(M_r)^{*k}_{\Delta}$ is *free for a face* A if there is no vertex of A in the column containing v, meaning that A contains neither v nor a copy of v.

Let $[r]$ refer to a zero-dimensional complex with r vertices. Then the deleted join $\Delta_{k,r} := [r]^{*k}_{\Delta}$ is a "chessboard complex" with k rows and r columns; see [82] for a detailed description. Each of the first $r-1$ blocks of $(M_r)^{*k}_{\Delta}$ is isomorphic to $\Delta_{k,r}$. The r-fold join $\Delta^{*r}_{k,r}$ is a subcomplex of $(M_r)^{*k}_{\Delta}$. The restriction of $\Delta^{*r}_{k,r}$ to the vertices of the first $r-1$ blocks is isomorphic to the $(r-1)$-fold join $\Delta^{*(r-1)}_{k,r}$. Denote $\Delta^{*(r-1)}_{k,r}$ by $\Sigma_{k,r-1}$. Color the vertices of $\Sigma_{k,r-1}$ based on the row i they are in. Let $a = (a_1, \ldots, a_k)$ with $a_i = r-1$ for $i = 1, \ldots, k$. Then $\Sigma_{k,r-1}$ is balanced of type a. For $b = (b_1, \ldots, b_k)$ with $0 \leq b_i \leq a_i$, the balanced b-skeleton $\Sigma^b_{k,r-1}$ of $\Sigma_{k,r-1}$ is the complex given by faces with at most b_i vertices in row i; see Definition 4.6.

In the following, let $k \geq 2$ and let $r \geq 2k-1$. Then the complex $\Delta_{k,r}^{*r}$ is shellable. This follows from the fact that the chessboard complex $\Delta_{k,r}$ is shellable for $r \geq 2k-1$ [82, Thm. 2.3] and that joins of shellable complexes are shellable (as one can shell the factors "lexicographically" [66, Prop. 2.4]). Since $\Sigma_{k,r-1} = \Delta_{k,r}^{*(r-1)}$ is a link of the complex $\Delta_{k,r}^{*r}$. Hence the shelling of $\Delta_{k,r}^{*r}$ induces a shelling of $\Sigma_{k,r-1}$.

Remark 4.10. For $r = 2$, the complex $(M_2)^{*2}_{\Delta}$ is not $(2r-2)$-connected, since its Euler characteristic is 2. Hence $(M_2)^{*2}_{\Delta}$ is not shellable. A calculation shows that its fundamental group is trivial.

Proposition 4.11 (Shellability of $(M_r)^{*2}_{\Delta}$)**.** *Let $r \geq 3$ be an integer. Then the 2-fold deleted join $(M_r)^{*2}_{\Delta}$ of the matroid M_r is shellable.*

Proof. For $x = (x_1, x_2) \in \mathbb{N}^2_{\geq 0}$ let $s(x) = (x_{i_1}, x_{i_2})$ be a reordering of the entries of x by decreasing value such that $x_{i_1} \geq x_{i_2}$. Let $<_l$ be the lexicographic order. Let $\prec$ be the strict order on $\mathbb{N}^2_{\geq 0}$ such that $x \prec y$ if and only if $s(x) <_l s(y)$ or both $s(x) = s(y)$ and $x <_l y$. Let "$\ll$" be a shelling of the subcomplex $\Delta_{2,r}^{*r}$. For facets A and B of $(M_r)^{*2}_{\Delta}$, let $b, x, y \in \mathbb{N}^2_{\geq 0}$ be defined by

$$x_i = |A_i^r|, \quad y_i = |B_i^r|, \quad \text{and} \quad b_i = \min\{r - x_i, r-1\} \quad \text{for } i = 1, 2.$$

If $B \notin \Delta_{2,r}^{*r}$, then let $A \ll B$, if any of the following three cases holds:
(a) $x \prec y$,
(b) $x = y$ and $A^r <_l B^r$,
(c) $A^r = B^r$ and $\overline{A} \ll \overline{B}$ for a fixed shelling of the balanced complex $\Sigma_{2,r-1}^b$.
In the following we show that "$\ll$" is indeed a shelling of $(M_r)^{*2}_{\Delta}$. Let $A \ll B$ be two facets of $(M_r)^{*2}_{\Delta}$. The goal is to find a facet C that satisfies Equation (4.1). We proceed case by case. For clarity, if $A \ll B$ due to (a), we write $A \ll_{(a)} B$, likewise for (b) and (c).
Case (a): $x \prec y$. Then there is a row $j \in \{1,2\}$ with $y_j > 1$, implying that $|B_j| > 1$. Hence there is a vertex $(b, j) \in B^r \setminus A^r$ and empty block $t < r$ of B in row j. We obtain C by switching (b, j) with a vertex in the empty block: Let (c, j) be any free vertex for B in row j and block t. Define $C = B \setminus \{(b,j)\} \cup \{(c,j)\}$. Observe that $(|C_1^r|, |C_2^r|) \prec y$. Thus $C \ll_{(a)} B$ and C satisfies Equation (4.1).
Case (b): $x = y$ and $A^r <_l B^r$. Assume $x_2 = 1$. Then $x_1 > 1$. Assume $A_1^r = B_1^r$. Then A^r and B^r each have one vertex in row 2 and these two vertices are distinct. Define $C = (\overline{B}, A^r)$. Since A^r and B^r only have two rows, C differs from B in only one vertex. (This is the point where this proof would fail if $k > 2$.) Hence $C \ll_{(b)} B$ and C satisfies 4.1. Assume $A_1^r \neq B_1^r$. Since $x_1 > 1$, there is an empty block $t < r$ of B in row 1. We switch vertices: Let $(b, 1)$ be any vertex in $B_1^r \setminus A_1^r$ and let $(c, 1)$ be any free vertex for B in block t and row 1. Define $C = B \setminus \{(b,1)\} \cup \{(c,1)\}$ and observe that $|C_1^r| = |B_1^r| - 1$ and $|C_2^r| = |B_2^r|$. Hence $C \ll_{(a)} B$ and C satisfies 4.1. Assume $x_2 > 1$, then there is an empty block $t < r$ of B in row 2. Again we switch vertices: Let $(b, 2)$ be any vertex in $B_2^r \setminus A_2^r$ and let $(c, 2)$ be any free vertex for B in block t and row 2. Define $C = B \setminus \{(b,2)\} \cup \{(c,2)\}$ and observe that $|C_2^r| = |B_2^r| - 1$ and $|C_1^r| = |B_1^r|$. Hence $C \ll_{(a)} B$ and C satisfies Equation (4.1).
Case (c): Since the balanced subcomplex $\Sigma_{2,r-1}^b$ is shellable by Proposition 4.9, there exists a facet $\overline{C}$ of $\Sigma_{2,r-1}^b$ with $\overline{C} \ll \overline{B}$ that satisfies Equation (4.2). Define $C = (\overline{C}, B^r)$. Then $C \ll_{(c)} B$ and C satisfies Equation (4.1). $\square$

For $r \geq 2$, define *blocks* E'_i by

$$E'_i = \{v_i^1, \dots, v_i^r, v_i^{r+1}\} \quad \text{for} \quad i = 1, \dots, r-1, \quad \text{and} \quad E'_r = \{w_1, \dots, w_r, w_{r+1}\},$$

for pairwise distinct v_i^j and w_i. Define a matroid M' with ground set $\bigcup_i E_i$ by

$$M' = U_{1,r+1}(E_1) \oplus \cdots \oplus U_{1,r+1}(E_{r-1}) \oplus U_{r+1,r+1}(E_r).$$

Now let M'_{r+1} be the $(r-1)$-skeleton of M'. Then M'_r is "built by the same principle" as M_r, but has $r+1$ instead of only r disjoint bases. Note that $(M'_r)^{*2}_\Delta$ is a pure complex for all $r \geq 2$ and contains the r-fold join $\Delta^{*r}_{2,r+1}$ as a subcomplex. From [82] we have that $\Delta^{*r}_{2,r+1}$ is shellable for all $r \geq 2$. By starting with a shelling of $\Delta^{*r}_{2,r+1}$ and repeating the proof of Proposition 4.11 for $(M'_r)^{*2}_\Delta$ instead of $(M_r)^{*2}_\Delta$ one obtains the following corollary.

Corollary 4.12. *For any integer $r \geq 2$, the 2-fold deleted join $(M'_r)^{*2}_\Delta$ of the matroid M'_r of rank r with $r+1$ disjoint bases is shellable and hence $(2r-2)$-connected.*

4.3.4 A covering of $(M_r)^{*2}_\Delta$

Next we give a topological description of $(M_r)^{*2}_\Delta$ via a covering by two subcomplexes. This yields a first proof of Theorem 4.2. In addition, the covering will allow us to determine the action of the group $\mathbb{Z}/2 := \langle t \rangle$ on cohomology needed for the proof of Theorem 4.3. Recall that the action of $\mathbb{Z}/2$ on $(M_r)^{*2}_\Delta$ is given by interchanging the factors of the join.

Definition 4.13. Let $r \geq 2$ be an integer. Define Σ_{2r-1} to be the subcomplex of $(M_r)^{*2}_\Delta$ induced by the facets of dimension $2r-1$ and their faces, and denote by Σ_{2r-2} the subcomplex of $(M_r)^{*2}_\Delta$ induced by the facets of dimension $2r-2$ and their faces.

Since each face of $(M_r)^{*2}_\Delta$ is contained in a facet of dimension $2r-1$ or $2r-2$, the two complexes Σ_{2r-1} and Σ_{2r-2} form a covering. The complex Σ_{2r-1} consists of faces that do not have more than $r-1$ vertices in either row of the last block. The complex Σ_{2r-2} consists of faces that in one row use only vertices in the last block and in the other row use no vertices in the last block. Hence Σ_{2r-2} has two connected components that we refer to as Σ^1_{2r-2} and Σ^2_{2r-2}; see Figure 4.4.

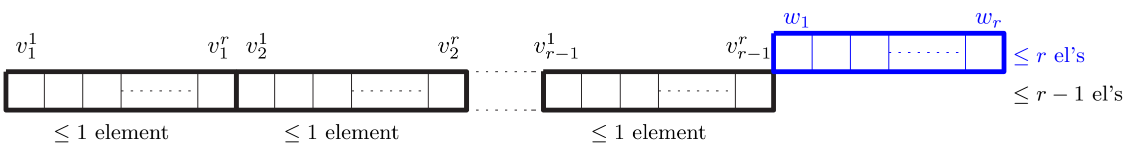


Figure 4.4: One connected component of Σ_{2r-2}.

Proposition 4.14. *Let $r \geq 3$ be an integer.*

(i) *The complex Σ_{2r-1} is a non-trivial wedge of $(2r-1)$-spheres, and a $\mathbb{Z}/2$-invariant subcomplex.*

(ii) *The complex Σ_{2r-2} is the disjoint union of two contractible spaces Σ^1_{2r-2} and Σ^2_{2r-2}. Moreover, $t \cdot \Sigma^1_{2r-2} = \Sigma^2_{2r-2}$, where t denotes the generator of the group $\mathbb{Z}/2$.*

(iii) *The intersection $\Sigma_{2r-1} \cap \Sigma_{2r-2} = (\Sigma_{2r-1} \cap \Sigma^1_{2r-2}) \cup (\Sigma_{2r-1} \cap \Sigma^2_{2r-2})$ is the disjoint union of two non-trivial wedges of $(2r-3)$-spheres, and $t \cdot (\Sigma_{2r-1} \cap \Sigma^1_{2r-2}) = (\Sigma_{2r-1} \cap \Sigma^2_{2r-2})$.*

Proof. (i) By [17, Lem. 2.6] we can reorder the facets of $(M_r)^{*2}_{\Delta}$ by decreasing dimension and obtain a shelling. This is done by first taking the facets of dimension $2r-1$ in the order given by "$\ll$", then taking the facets of dimension $2r-2$ again in the order given by "$\ll$", and so forth. This implies that Σ_{2r-1} is shellable. Since the chessboard complex $\Delta^{*r}_{2,r}$ consists of $(2r-1)$-facets that by definition of "$\ll$" are shelled first, homology facets of the chessboard complex are homology facets of Σ_{2r-1}. The chessboard complex $\Delta^{*r}_{2,r}$ is not contractible, and hence Σ_{2r-1} must be a non-trivial wedge of $(2r-1)$-spheres. Since the action of the group $\mathbb{Z}/2$ is simplicial and therefore preserves the dimension of the simplices we have that $t\cdot\Sigma_{2r-1}=\Sigma_{2r-1}$.
(ii) Each connected component of Σ_{2r-2} is isomorphic to the join $(M_r|S)*\Delta_{r-1}$ of the restriction $(M_r|S)$ and the simplex Δ_{r-1}, where $S=\bigcup_{i=1}^{r-1}E_i$. Hence each component is contractible. Furthermore, by direct inspection one sees that $t\cdot\Sigma^1_{2r-2}=\Sigma^2_{2r-2}$.
(iii) The intersection $\Sigma_{2r-1}\cap\Sigma_{2r-2}$ has two connected components. Its faces use in one row only $r-1$ vertices that are all contained in the last block. In the other row they use no vertices in the last block. Hence both components are isomorphic to the join $(M_r|S)*\Delta^{(r-2)}_{r-1}$, where $\Delta^{(r-2)}_{r-1}$ is the $(r-2)$-skeleton of the the simplex. The complex $(M_r|S)$ is a matroid of rank $r-1$. It is $(r-2)$-connected and has reduced Euler characteristic $(r-1)^{r-1}$. Hence each component of $\Sigma_{2r-1}\cap\Sigma_{2r-2}$ is a non-trivial wedge of $(2r-3)$-spheres. □

Corollary 4.15. *Let $r\geq 3$ be an integer. Then the deleted join $(M_r)^{*2}_{\Delta}$ is homotopy equivalent to a non-trivial wedge of spheres of dimensions $2r-1$ and $2r-2$.*

4.3.5 Proof of Theorem 4.2

The homotopy type of a shellable complex can be computed as follows. Let Σ be a shellable complex of dimension d. Define the *degree* of a face A of Σ by $\delta(A)=\max\{|F|\colon F\in\Sigma, A\subseteq F\}$. Thus $\delta(A)-1$ is the dimension of a largest facet containing A. Define the *f-triangle* $(f_{i,j}(\Sigma))_{0\leq i\leq j\leq d+1}$ of Σ by $f_{i,j}(\Sigma)=|\{A\in\Sigma\colon |A|=i,\delta(A)=j\}|$. Thus $f_{i,j}(\Sigma)$ is equal to the number of faces A of Σ of dimension $i-1$ that are contained in a largest facet of dimension $j-1$. For $j=0,1,\dots,d+1$ set

$$h_j(\Sigma)=(-1)^j\cdot\sum_{i=0}^{j}(-1)^i f_{i,j}(\Sigma)\ .$$

The vector $h(\Sigma)=(h_0(\Sigma),\dots,h_{d+1}(\Sigma))$ is the diagonal of the "h-triangle" of Σ [17, Def. 3.1]. By [17, Thm. 4.1], the homotopy type of Σ is a wedge of spheres, consisting of $h_j(\Sigma)$ copies of the $(j-1)$-sphere for $j=1,\dots,d+1$.

Proof of Theorem 4.2. The matroid M_r is of rank r and has r disjoint bases. For $r=2$ the complex $(M_r)^{*2}_{\Delta}$ is simply connected, but not 2-connected; see Remark 4.10. Let $r\geq 3$ in the following. Then the complex $(M_r)^{*2}_{\Delta}$ is shellable by Proposition 4.11. Hence by [17, Thm. 4.1] the homotopy type of $(M_r)^{*2}_{\Delta}$ is a wedge of spheres, consisting of h_j spheres of dimension $j-1$ for $j=1,\dots,2r$, where $h((M_r)^{*2}_{\Delta})=(h_0,\dots,h_{2r})$ is the diagonal of the h-triangle of $(M_r)^{*2}_{\Delta}$.

For $j=0,\dots,2r-2$, the entries h_j are zero, since $(M_r)^{*2}_{\Delta}$ has no facets of dimension j. Hence $(M_r)^{*2}_{\Delta}$ is $(2r-3)$-connected.

Let $j=2r-1$. We will show that $h_{2r-1}\neq 0$ and thus that $(M_r)^{*2}_{\Delta}$ is not $(2r-2)$-connected. The number h_{2r-1} is equal to the alternating sum $-f_{0,2r-1}+f_{1,2r-1}-\dots+f_{2r-1,2r-1}$ of the entries of the row $f_{2r-1}((M_r)^{*2}_{\Delta})$ of the f-triangle of $(M_r)^{*2}_{\Delta}$. Here, $f_{i,2r-1}$ denotes the cardinality of the set

$$\mathcal{F}_{i-1} := \{A \in (M_r)^{*2}_{\Delta} \colon |A| = i,\ \delta(A) = 2r-1\}$$

of faces of dimension $i-1$ of $(M_r)^{*2}_{\Delta}$ that are contained in a largest facet of dimension $2r-2$. Facets A of dimension $2r-2$ are given by choosing all vertices $\{w_1, \ldots, w_r\}$ in one row of $(M_r)^{*2}_{\Delta}$ and one vertex from each of the sets $\{v_i^1, \ldots, v_i^r\}$ for $i = 1, \ldots, r-1$ in the other row. This implies that $f_{i,2r-1} = 0$ for $i = 0, \ldots, r-1$ and that any face in $\mathcal{F}_{i-1}$ for $i \geq r$ must use all vertices $\{w_1, \ldots, w_r\}$. See Figure 4.3 for an example of a facet of dimension $2r-2 = 8$ for $r = 5$.

Let $S = \{v_i^1, \ldots, v_i^r : 1 \leq i \leq r-1\}$ and let $M_r|S$ be the restriction of M_r to the vertex set S. Then for $i = 0, \ldots, r-1$ there is a 2-to-1 surjection between the faces in $\mathcal{F}_{r+i-1}$ and the set of $(i-1)$-dimensional faces of $M_r|S$. This implies that

$$f_{r+i,2r-1}((M_r)^{*2}_{\Delta}) = 2\, f_i(M_r|S) \quad \text{for} \quad i = 0, \ldots, r-1,$$

where $f_i(M_r|S)$ is the number of $(i-1)$-faces of $M_r|S$. Hence

$$h_{2r-1} = (-1)^{r-1}\, 2\, \big(\chi(M_r|S) - 1\big),$$

where $\chi(M_r|S)$ denotes the Euler characteristic of $M_r|S$.

The complex $M_r|S$ is isomorphic to the $(r-1)$-fold join of the restriction $M_r|\{v_1^1, \ldots, v_1^r\}$, which in turn is isomorphic to a 0-dimensional complex with r vertices. Hence $M_r|S$ has Euler characteristic equal to $1 + (-1)^{r-1}(r-1)^{r-1}$. Thus $h_{2r-1} = 2\,(r-1)^{r-1}$, which is non-zero since $r \geq 2$. □

The missing value h_{2r} of the diagonal of the h-triangle can be calculated, similarly to the above, by calculating the complete f-vector of $(M_r)^{*2}_{\Delta}$. Both calculations are technical. Instead we give lower bounds.

Corollary 4.16. *Let $r \geq 3$ be an integer and let β_i denote the i-th reduced Betti number of $(M_r)^{*2}_{\Delta}$ for $i = 0, \ldots, 2r-1$. Then*

$$\beta_i = \begin{cases} 2\,(r-1)^{r-1} & \text{if } i = 2r-2 \\ 0 & \text{if } i \leq 2r-3 \end{cases} \qquad \text{and} \qquad \beta_{2r-1} \geq (r^2 - 3r + 1)^r\,.$$

Proof. Note that $\beta_i = h_{i+1}$ and that h_{2r} is equal to the number of $(2r-1)$-dimensional homology facets of $(M_r)^2_{\Delta}$. In the shelling of $(M_r)^2_{\Delta}$ the r-fold join $\Delta^{*r}_{2,r}$ of the chessboard complex is shelled first, implying that its homology facets are also homology facets of $(M_r)^2_{\Delta}$. The chessboard complex is pure and has Euler characteristic $1 - (r^2 - 3r + 1)^r$. Therefore $h_{2r} \geq (r^2 - 3r + 1)^r$. □

4.4 Further results

4.4.1 Bounds for the topological Tverberg number of matroids

Recall that the *topological Tverberg number* $\mathrm{TT}(M,d)$ of M is the largest integer $k \geq 1$ such that for every continuous map $f\colon M \to \mathbb{R}^d$, there is a collection $\{\sigma_1, \ldots, \sigma_k\}$ of k pairwise disjoint faces, called a Tverberg k-partition, such that $\bigcap_{i=1}^{k} f(\sigma_i) \neq \emptyset$.

Corollary 4.17 (Lower bounds for the topological Tverberg number)**.** *Let $b, d, r \geq 1$ be integers and let M be a matroid of rank r with b disjoint bases. Let $x = d+1$ for ease of notation. Let*

$$\ell(b,r,x) = \frac{2x + (r-x)b + \sqrt{(2x + b(r-x))^2 + 8bx^2}}{8x}.$$

If p is a prime power with

$$p \leq 2\ell(b,r,x),$$

then $\mathrm{TT}(M,d) \geq p$.

Proof. We use the join scheme and take a connectivity-based approach based on the lemma in [77] due to Volovikov, which can be seen as a generalization of Dold's theorem. If we show that the connectivity of the configuration space M_{Δ}^{*p} is at least as high as the dimension of the test space $S^{(p-1)(d+1)-1}$, then the result follows.

By [9, Cor. 3] the deleted join M_{Δ}^{*p} has connectivity at least $br/(\lceil b/p \rceil + 1) - 2$, implying that its connectivity is at least $\lceil br/(\lceil b/p \rceil + 1) \rceil - 2$. Hence it suffices to show that

$$\frac{br}{b/p + 2} - (p-1)(d+1) \geq 0.$$

This is equivalent to

$$-2xp^2 + (2x - xb + br)p + xb \geq 0, \tag{4.3}$$

which defines a negatively curved parabola in p with zeros

$$\frac{2x + (r-x)b + br \pm \sqrt{(2x + (r-x)b)^2 + 8bx^2}}{4x}.$$

Finally, we observe that

$$\frac{2x + (r-x)b - \sqrt{(2x + (r-x)b)^2 + 8bx^2}}{4x} \;\leq\; \ell(b,r,x) \;\leq\; p \;\leq 2\ell(b,r,x),$$

and hence p satisfies Equation (4.3). □

Upper bounds for the topological Tverberg number for matroids with codimension at least 3, meaning $r-1 \leq d-3$, can be obtained using the new sufficiency criterion [56, Thm. 7] due to Mabillard and Wagner for the nonexistence of Tverberg k-partitions for simplicial complexes with codimension 3. For a real number $x \geq 0$, we let $\lceil x \rceil_{\mathrm{npp}}$ denote the smallest integer $k \geq x$ that is not a prime-power.

Proposition 4.18 (Upper bounds for the topological Tverberg number)**.** *Let $d \geq 3$ and $r \geq 1$ be integers and let $r \leq d-2$. If M is a matroid of rank r, then*

$$\left\lceil \tfrac{d}{d-r+1} \right\rceil_{\mathrm{npp}} > \mathrm{TT}(M, d).$$

Proof. Let $k = \left\lceil \frac{d}{d-r+1} \right\rceil_{\mathrm{npp}}$. The dimension of the deleted product $M_\Delta^{\times k}$ is at most $k(r-1)$, which by choice of k is at most $(k-1)d$. Thus by [29, Cor. 5.2], which is a simple consequence of [64, Lem. 4.2], there exists an $\mathfrak{S}_k$-equivariant map from $M_\Delta^{\times k}$ to the sphere $S^{(k-1)d-1}$. (Here $\mathfrak{S}_k$ denotes the symmetric group on k letters.) Since M has dimension at most $d-3$, we can apply [56, Thm. 7] and get the existence of a continuous map $f\colon M \to \mathbb{R}^d$ that does not have a Tverberg k-partition. This implies that $k > TT(M, d)$. $\square$

Remark 4.19. Recently Paták [65] building on [44] proved several Tverberg-type results for matroids that are not directly related to [9, Thm. 1], including colored versions. We point out [65, Lem. 2]: Let M be a matroid of rank $r \geq 1$ with closure operator cl and let S be a subset of the ground set of M of cardinality at least $r(k-1)+1$. Then there exist pairwise disjoint subsets $S_1, \ldots, S_k$ of S such that $\mathrm{cl}\,\emptyset \subsetneq \mathrm{cl}S_1 \subseteq \cdots \subseteq \mathrm{cl}S_k$.

4.4.2 Connectivity of the deleted product of a matroid

In Theorem 4.20 we assume that the rank r of M is at least k, otherwise $M_\Delta^{\times k}$ can be empty. To simplify the statement of the theorem we assume that $k \leq b$, since then the dimension of $M_\Delta^{\times k}$ is equal to $(r-1)k$ and, in particular, is independent of the number of disjoint independent sets of lower cardinality. We point out, however, that the proof of Theorem 4.20 can be applied to the setting where $k < b$; see for example Corollary 4.22.

Theorem 4.20 (Connectivity bounds for the deleted product)**.** *Let $b, k, r \geq 2$ be integers with $r \geq k$ and $b \geq k$. Let M be a matroid of rank r with b disjoint bases and let $M_\Delta^{\times k}$ be the k-fold deleted product of M.*

(i) *Then the connectivity of $M_\Delta^{\times k}$ is at least*

$$r - 2 - \left\lfloor \frac{r(k-1)}{b} \right\rfloor .$$

(ii) *If $b \geq r(k-1)+1$, then $M_\Delta^{\times k}$ is $(r-2)$-connected, but not $(r-1)$-connected.*

The *ordered configuration space* $\mathrm{Conf}(X, n)$ of n particles in a topological space X is defined as the space $\{(x_1, \ldots, x_n) \in X^n \colon x_i \neq x_j \text{ for } i \neq j\}$. As Smale [72, Lem. 2.1] observed, in the case where $n = 2$ and $X = \Sigma$ is a finite simplicial complex, the 2-fold deleted product $\Sigma_\Delta^{\times 2}$ is a deformation retract of $\mathrm{Conf}(M, 2)$. This leads to the following corollary of Theorem 4.20.

Corollary 4.21. *Let $b, r \geq 2$ be integers and let M be a matroid of rank r with b disjoint bases. Then the configuration space $\mathrm{Conf}(M, 2) = \{(x, y) \in M^2 \colon x \neq y\}$ of two ordered particles in M is at least*

$$\left(r - 2 - \left\lfloor \frac{r}{b} \right\rfloor\right)\text{-}$$

connected and not $(r-1)$-connected, when $b \geq r+1$.

By [66, Thm. 3.2.1] any matroid M of rank r is pure shellable, implying that M is contractible or homotopy equivalent to a wedge of $(r-1)$-spheres. The reduced Euler characteristic $\widetilde{\chi}(M)$ of M can be computed using the Möbius function μ of the lattice of flats L of the dual matroid of M. By [13, Prop. 7.4.7] $\widetilde{\chi}(M)$ is zero if M has coloops (elements contained in every basis), and is otherwise equal to $(-1)^{r-1}|\mu_L(\hat{0},\hat{1})|$, which is non-zero. In fact $|\mu_L(\hat{0},\hat{1})|$ is non-zero for any geometric lattice [69, Thm. 4].

Proof of Theorem 4.20. (i) A cell of $M_\Delta^{\times k}$ is of the form

$$\operatorname{relint}(\sigma_1) \times \cdots \times \operatorname{relint}(\sigma_k),$$

where $\sigma_i \cap \sigma_j = \emptyset$ for all i, j with $1 \leq i < j \leq k$. Its dimension is given by the sum of the dimensions of the σ_i. Since the σ_i are vertex-disjoint and by assumption $k \leq b$, a product cell of maximal dimension uses rk vertices and has dimension $(r-1)k$.

We fix r and establish the connectivity of $M_\Delta^{\times k}$ by induction on k. Assume $k = 1$. If M has no coloops, it is homotopy equivalent to a wedge of $(r-1)$-spheres, else it is contractible.

Assume the statements of the theorem are true for $k-1$ for a fixed $k \geq 2$. Consider the projection p_{k-1} of the k-fold product M^k to the first $k-1$ coordinates. The map p_{k-1} restricts to a surjective continuous proper map

$$p_{k-1}\colon M_\Delta^{\times k} \longrightarrow M_\Delta^{\times k-1}.$$

Since $r \geq k$, both the domain and codomain of p_k are connected by induction. They are also locally compact, locally contractible separable metric spaces.

Let $x \in M_\Delta^{\times k-1}$ be a point and let faces $\sigma_1, \ldots, \sigma_{k-1} \in M$ be minimal under inclusion such that x is contained in the product $\operatorname{relint}(\sigma_1) \times \cdots \times \operatorname{relint}(\sigma_{k-1})$ of the relative interiors of the σ_i. Let $V_x = \operatorname{vert}(\sigma_1) \cup \cdots \cup \operatorname{vert}(\sigma_{k-1})$ be the union of the vertex sets of the σ_i. Assume $V_x = \{v_1, \ldots, v_n\}$. Then the preimage

$$\begin{aligned} p_{k-1}^{-1}(\{x\}) &\cong \{y \in M \colon \exists\, \sigma \in M \text{ s.t. } y \in \sigma \text{ and } \sigma \cap \sigma_i = \emptyset \text{ for } i = 1, \ldots, k-1\} \\ &= \{\sigma \in M \colon \{v_i\} \not\subseteq \operatorname{vert}(\sigma) \text{ for } i = 1, \ldots, n\}. \end{aligned}$$

Hence $p_{k-1}^{-1}(\{x\})$ is homeomorphic to (any geometric realization of) the successive deletion $M_x := M \setminus v_1 \setminus \cdots \setminus v_n$ of the vertices V_x from M. Any deletion of a matroid is again a matroid (see [63]), implying by induction that M_x is a matroid. Let r_x denote its rank. The total number n of vertices deleted is at most $r(k-1)$. Let

$$r_k = r - \left\lfloor \frac{r(k-1)}{b} \right\rfloor.$$

If V_x contains $\lfloor r(k-1)/b \rfloor$ vertices from b disjoint bases of M, then r_x can be equal to r_k, otherwise r_x is larger. Equality is given, if M has no other disjoint independent sets of cardinality greater than r_k. Thus M_x is a matroid of rank $r_x \geq r_k$. Hence $p_{k-1}^{-1}(\{x\})$ is locally contractible and either contractible (if M_x has coloops) or homotopy equivalent to a wedge of $(r_x - 1)$-spheres, which is at least $(r_k - 2)$-connected. By induction hypothesis $M_\Delta^{\times k}$ is $(r_{k-1} - 2)$-connected. Since $r_k \leq r_{k-1}$, the deleted product $M_\Delta^{\times k}$ is $(r_k - 2)$-connected by Smale's theorem, which is stated below.

Smale's Theorem [73]. *Let X and Y be connected, locally compact, separable metric spaces, and in addition let X be locally contractible. Let $f : X \longrightarrow Y$ be a surjective continuous proper map. If for every $y \in Y$ the preimage $f^{-1}(\{y\})$ is locally contractible and n-connected, then the induced homomorphism*

$$f_\# : \pi_i(X) \to \pi_i(Y)$$

is an isomorphism for all $0 \le i \le n$, and is an epimorphism for $i = n + 1$.

(ii) Let $b \ge (r-1)(k-1)+1$ for fixed r and fixed k. Then $r_k = r$. Hence $p_{\ell-1}^{-1}(\{x\})$ is a matroid of rank r for all $x \in M_\Delta^{\times \ell-1}$ and all ℓ with $2 \le \ell \le k$. For $\ell = 2$, the deleted product $M^{\times \ell-1}$ is equal to the matroid M, which is $(r-2)$-connected and not $(r-1)$-connected, since M has no coloops. By induction on ℓ the skeleton $M_\Delta^{\times \ell-1}$ is $(r-2)$-connected but not $(r-1)$-connected. Hence by Smale's theorem $M_\Delta^{\times \ell}$ is $(r-2)$-connected, but not $(r-1)$-connected. □

As a corollary we obtain a proof of the following result due to Bárány, Shlosman, and Szűcs.

Corollary 4.22 ([10, Lem. 1])**.** *Let r and k be integers with $r \ge k \ge 1$. Then the deleted product $(\Delta_{r-1})_\Delta^{\times k}$ of the simplex Δ_{r-1} of dimension $r-1$ is $(r-k-1)$-connected and not $(r-k)$-connected.*

Proof. The simplex Δ_{r-1} is a uniform matroid $U_{r,r}$ of rank r. It has one basis. Its dimension d_k is equal to $r-k$ for all $k \ge 1$. The fibers $p_{k-1}^{-1}(\{x\})$ are all contractible, since vertex-deletions of the simplex are contractible. Hence for all $k \ge 2$, Smale's theorem together with the Whitehead theorem implies that $(\Delta_{r-1})_\Delta^{\times k}$ and $\mathrm{sk}_{r-k}\big((\Delta_{r-1})_\Delta^{\times k}\big)$ are homotopy equivalent. In particular $(\Delta_{r-1})_\Delta^{\times k}$ is $(r-k-1)$-connected and not $(r-k)$-connected. □

4.4.3 A topological Radon-type theorem for M_r

The following topological Radon-type theorem for the family of matroids M_r $(r \ge 3)$ follows from Theorem 4.2 by using the join scheme and taking the connectivity-based approach.

Corollary 4.23. *Let $d \ge 1$ and $r \ge 3$ be integers such that $2r-3 \ge d$. Then $\mathrm{TT}(M_r, d) \ge 2$.*

Proof. By Theorem 4.2 the connectivity of the configuration space $(M_r)_\Delta^{*2}$ is $2r-3$, which is at least as high as the dimension of the test space S^d. □

We obtain a sharper result by using the join scheme and applying [20, Thm. 1], which is obtained by a Fadell–Husseini index calculation. We point out the following typo in [20, Thm. 1]: In the notation of the theorem, the roles of X and Y should be interchanged in the last sentence of the statement. See [27, Thm. 4.2] for a more general version that implies [20, Thm. 1].

Proof of Theorem 4.3. Without loss of generality let $d = 2r-2$. In order to prove the theorem using the join scheme, we need to show that there is no $\mathbb{Z}/2$-equivariant map $(M_r)_\Delta^{*2} \to S^d$, where the sphere is equipped with the antipodal action.

From Corollary 4.15 we have that $H^d((M_r)_\Delta^{*2}; \mathbb{F}_2) \ne 0$, and $H^i((M_r)_\Delta^{*2}; \mathbb{F}_2) = 0$ for all i with $1 \le i \le d-1$. Hence $(M_r)_\Delta^{*2}$ is not d-connected. Consequently the classical Dold theorem [35] cannot be applied. To prove nonexistence of a $\mathbb{Z}/2$-equivariant map we use [20, Thm. 1] For this it suffices to prove that the cohomology $H^d((M_r)_\Delta^{*2}; \mathbb{F}_2)$ is a free $\mathbb{F}_2[\mathbb{Z}/2]$-module where the action is induced by the $\mathbb{Z}/2$-action on $(M_r)_\Delta^{*2}$.

Indeed, consider the covering $\{\Sigma_{d+1}, \Sigma_d\} = \{\Sigma_{d+1}, \Sigma_d^1 \cup \Sigma_d^2\}$ of the complex $(M_r)^{*2}_\Delta$; see Section 4.3.4. The relevant part of the induced Mayer–Vietoris sequence in cohomology with $\mathbb{F}_2$-coefficients has the form:

$$H^{d-1}(\Sigma_{d+1}) \oplus H^{d-1}(\Sigma_d^1) \oplus H^{d-1}(\Sigma_d^2) \to H^{d-1}(\Sigma_{d+1} \cap \Sigma_d^1) \oplus H^{d-1}(\Sigma_{d+1} \cap \Sigma_d^2) \to$$

$$H^d((M_r)^{*2}_\Delta) \longrightarrow H^d(\Sigma_{d+1}) \oplus H^d(\Sigma_d^1) \oplus H^d(\Sigma_d^2).$$

From Proposition 4.14 we have that the subcomplexes Σ_d^1 and Σ_d^2 are contractible, and that Σ_{d+1} is d-connected. Thus the sequence simplifies to:

$$0 \to H^{d-1}(\Sigma_{d+1} \cap \Sigma_d^1) \oplus H^{d-1}(\Sigma_{d+1} \cap \Sigma_d^2) \to H^d((M_r)^{*2}_\Delta) \to 0.$$

Since, again by Proposition 4.14, the $\mathbb{Z}/2$-action interchanges the subcomplexes $\Sigma_{d+1} \cap \Sigma_d^1$ and $\Sigma_{d+1} \cap \Sigma_d^2$, we conclude that $H^d((M_r)^{*2}_\Delta; \mathbb{F}_2)$ is a free $\mathbb{F}_2[\mathbb{Z}/2]$-module, as claimed. □

4.4.4 Failure of shellability and vertex-decomposability for general k

For a definition of vertex-decomposability for possibly non-pure complexes see [18, Def. 11.1].

Proposition 4.24. *Let k and r be integers.*

(i) *For $k \geq 3$ and $r \geq 2k-1$ the complex $(M_r)^{*k}_\Delta$ is not shellable.*

(ii) *For $k \geq 2$ and $r \geq 2k-1$ the complex $(M_r)^{*k}_\Delta$ is not vertex-decomposable.*

Proof. (i) Let $r = 2k-1$. Consider the face

$$A = \{(v_1^i, i), \ldots, (v_{r-1}^i, i) : i = 1, \ldots, k-1\} \cup \{(v_1^r, k), \ldots, (v_{k-1}^r, k), (w_k, k), \ldots, (w_r, k)\}.$$

Hence A has in rows 1 to $k-1$ one vertex in each of the first $r-1$ blocks and in the k-th row $k-1$ vertices in the first $r-1$ blocks and $r-k+1$ vertices in the last block. The link $(M_r)^{*k}_\Delta/A$ of A is isomorphic to the square chessboard complex $\Delta_{k-1,k-1}$, which is not (pure) shellable by [43, Thm. 2]. However, links of shellable complexes must be shellable [18, Prop. 10.14].

(ii) To see that $(M_r)^{*k}_\Delta$ is not vertex-decomposable, we argue by contradiction. First we point out that it suffices to show this statement for $k = 2$, since by [18, Thm. 11.3], vertex-decomposability implies shellability. Assume there is a shedding sequence S for $(M_r)^{*2}_\Delta$. Consider only the deletions and let $s_0 \in S$ be the first vertex to be deleted from block r of $(M_r)^{*2}_\Delta$. Let M' be the complex given by successively deleting all vertices up to but not including s_0. By symmetry we may assume that $s_0 = (w_1, 2)$. Let d be the dimension of M'. Consider the link M'/s_0. A facet of the link has dimension $d-1$ and uses one vertex less in the second row than a d-dimensional facet of M'. It cannot use the vertices $(w_1, 1)$ or $(w_1, 2)$. Let A be a facet of M'/s_0 that uses the vertices $(w_i, 1)$ for $i = 2, \ldots, r$ in the first row of block r. Then the vertices $(w_i, 2)$ for $i = 1, \ldots, r$ in the second row of block r cannot be used by A, since they are either deleted ($i = 1$) or "blocked" ($i > 1$). Now consider the deletion $M' \setminus s_0$. It has dimension d. None of its facets use the vertex $(w_1, 2)$ and some of its facets have dimension $d-1$. In fact A is a facet of M'/s_0 of dimension $d-1$. Hence A is a facet of both the link and the deletion of M' with respect to s_0, thus violating the definition of vertex-decomposability [18, Def. 11.1]. □

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Summary

In this dissertation we present new applications of topological methods to problems in discrete geometry. The *topological proof strategy* we use goes back to Lovász' celebrated 1978 proof of the Kneser conjecture [55]: reduce the question of whether a geometric statement is true to the nonexistence of an equivariant map between a *test space* and a *configuration space*, where both spaces have the same non-trivial group acting on them.

The introduction (Chapter 1) is followed by two chapters in which the topological proof strategy is applied to the *Grünbaum–Hadwiger–Ramos hyperplane mass partition problem*, which is due to Grünbaum [46, Sec. 4.(v)], Hadwiger [48], and Ramos [68]: Given positive integers j, k the problem asks for the smallest dimension d such that any choice of j convex bodies in $\mathbb{R}^d$ or, more generally, any choice of j absolutely continuous finite Borel measures on $\mathbb{R}^d$ can be cut into 2^k equal pieces by k hyperplanes.

In Chapter 2 we give a critical review of the progress that has been made on the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem and point out mistakes and gaps in the articles [85], [68], [57], and [86]. This shows that the problem is still wide open. The main new result of Chapter 2 is a correct solution of the problem in the case of two hyperplanes and $2^t + 1$ measures. It is obtained by a degree calculation of a restriction of the test map.

In Chapter 3 we use a different approach based on relative equivariant obstruction theory to verify the solutions of the problem in the cases of two hyperplanes and $2^t - 1$ respectively $2^t + 1$ measures and obtain a correct solution of the problem in the case of two hyperplanes and 2^t measures. We also obtain solutions in the cases of three hyperlanes and two respectively four measures.

In Chapter 4 we study the problem, to what extent the well-known variant of the topological proof strategy based on the connectivity of the configuration space and a theorem by Dold [35] can be used to answer the question, when a matroid (viewed as a simplicial complex) can be mapped to $\mathbb{R}^d$ such that the images of no k pairwise disjoint faces intersect. An answer to this question would give rise to a Tverberg-type theorem for matroids. Our main result is a counterexample to a conjecture by Bárány, Kalai, and Meshulam [9, Conj. 4] concerning the connectivity of one of the two possible configuration spaces. Furthermore, we establish the connectivity of the other possible configuration space. Finally, we prove a tight Tverberg-type theorem for the family of matroids arising as counterexamples. Together, our results imply that the topological proof strategy based on the connectivity of the configuration space and Dold's thoerem does not lead to an optimal Tverberg-type result in the case of matroids.

Zusammenfassung

In dieser Dissertation werden neue Anwendungen von topologischen Methoden auf Probleme der diskreten Geometrie entwickelt. Die verwendete und untersuchte *topologische Lösungsstrategie* geht auf Lovász' bahnbrechende Arbeit [55] aus dem Jahr 1978 zurück und gestaltet sich wie folgt: Man reduziere die Frage der Richtigkeit der geometrischen Aussage auf die Frage der Nicht-Existenz einer equivarianten Abbildung zwischen einem *Konfigurationsraum* und einem *Testraum*, auf die jeweils diselbe nicht-triviale Gruppe wirkt.

Der Einleitung (Kapitel 1) folgen zwei Kapitel, die sich mit der Anwendung von Varianten der topologischen Lösungsstrategie auf das *Grünbaum–Hadwiger–Ramos hyperplane mass partition problem* beschäftigen, das auf Grünbaum [46, Sec. 4.(v)], Hadwiger [48] und Ramos [68] zurückgeht. In diesem Problem geht es um die Frage nach der kleinsten Dimension, in der sich bei beliebig vorgegebenen natürlichen Zahlen $j, k \geq 1$ jede Wahl von j konvexen Körpern, oder allgemeiner absolutstetigen endlichen Borelmaßen, durch k Hyperebenen in 2^k gleich große Teile zerlegen lassen. In Kapitel 2 werden der Fortschritt in Bezug auf dieses in weiten Teilen ungelöste Problem kritisch beleuchtet und Fehler sowie Beweislücken in den Aufsätzen [85], [68], [57] und [86] aufgedeckt. Das wesentliche neue Resultat dieses Kapitels ist ein korrekter Beweis für den Fall von zwei Hyperebenen und $2^t + 1$ Maßen. Der im Beweis verwendete Ansatz basiert auf einer Grad-Berechnung. In Kapitel 3 wird ein anderer Ansatz der topologischen Lösungsstrategie basierend auf relativer equivarianter Hindernistheorie verfolgt, mit dem die bereits bekannten Lösungen in den Fällen von zwei Hyperebenen und $2^t - 1$ Maßen und zwei Hyperebenen und $2^t + 1$ Maßen bestätigt werden, sowie erstmals für den Fall von zwei Hyperebenen und 2^t Maßen ein korrekter Beweis geliefert wird. Zwei weitere neue Lösungen für den Fall von drei Hyperebenen und zwei sowie vier Maßen werden erarbeitet.

Kapitel 4 beschäftigt sich mit der Anwendung einer bekannten Variante der topologischen Lösungsstrategie, die auf den topologischen Zusammenhangseigenschaften des Konfigurations- und Testraums und einem Satz von Dold [35] basiert, auf das Problem, ob sich ein Matroid (aufgefasst als Simplizialkomplex) stetig in den d-dimensionalen Raum abbilden lässt, ohne dass sich k paarweise disjunkte Seitenflächen im Bild schneiden. Dieses Problem kann als Frage aufgefasst werden, ob bzw. wie sich das *topological Tverberg theorem* [10] auf Matroide erweitern lässt. Hauptresultate dieses Kapitels sind die Widerlegung einer Vermutung von Bárány, Kalai und Meshulam [9, Conj. 4] bezüglich des topologischen Zusammenhangs eines der beiden möglichen Konfigurationsräume und die Berechnung des topologischen Zusammenhangs des anderen Konfigurationsraums. Ferner wird an einem Beispiel gezeigt, dass die Variante der Lösungsstrategie basierend auf dem Satz von Dold im Fall von Matroiden nicht zu optimalen Ergebnissen führt.

Eigenständigkeitserklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin vom 8. Januar 2007 versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die vorliegende Arbeit selbstständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht bereits zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den 10. Mai 2017 ....

(Albert Haase)