Boundary Value Problems
for Tri-harmonic Functions in the Unit Disc

DISSERTATION
des Fachbereichs Mathematik ind Informatik
der Freien Universität Berlin
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

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Tag der Disputation: 4 September, 2009
Vorgelegt von
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Juli 2009

To my parents

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Acknowledgements

I would like to express my deepest gratitude to Prof. Dr. H.Begehr for his support of my scientific work. His professionalism encouraged me to write the present thesis. I appreciate his diligence and patience while working on mathematical problems. He did influence a lot on enlarging my scientific knowledge.

My special thanks go to Prof. Dr. A.B.Tungatarov, who during many years was my scientific advisor in Kazakhstan. Due to his relations with Prof. H.Begehr I got the possibility to spend visits at FU Berlin. Permanent scientific consultations and discussions in the working group of Prof. A.B. Tungatarov formed me as a young scientist.

I record a note of special thanks to L.N.Gumilyev Eurasian National University for financial support of my Ph.D. studies.

Finally, I give my sincere thanks to my family for their permanent support and love.

Chapter 1

Introduction

The theory of boundary value problems for complex partial differential equations combines knowledged and methods from many fields of mathematics, i.e. complex analysis, partial differential equations, functional analysis, equation of mathematical physics etc. Initiated by B. Riemann and D. Hilbert the theory develops up to nowadays involving different research groups all over the world. In the classical theory of complex analysis, it is well known that harmonic functions are intimately connected with analytic functions. That is, for any real harmonic function, one can find an analytic function such that the harmonic function becomes its real part. In other words, any real harmonic function can be decomposed as a sum of an analytic function and its conjugate function which is an antianalytic function. The idea is simple but suitable and important because it constructs a bridge linking the two kinds of functions so that they can be mutually applied. In fact, the mutual applications are successfully realized in the classical theory of one complex variable. In this dissertation, one can find that the idea is valid for the generalized analogues of harmonic functions which are called tri-harmonic functions. Of course, analytic functions should also be generalized. It is fortunate that some generalized analogues for analytic functions have already been introduced by contribution from many mathematicians [7, 17, 21, 43]. Since complex analysis is closely related to mathematical physics, the theory of boundary value problems (simply, BVPs) in complex analysis were abundantly developed. Especially, the theory of boundary value problems for analytic functions is an important branch of function theory. Many mathematicians contributed to this field such as B. Riemann, D. Hilbert, N. I. Muskhelishvili, F. D. Gakhov, I. N. Vekua and their students. The initial investigations are due to B.Riemann and D. Hilbert. Deep developments were given by the BVPs school of the former Soviet Union. Except for analytic functions, the investigations were also devoted to particular partial differential equations, for example, the Bitsadze equation, elliptic partial differential equations with analytic coefficients and so on. There are many different types of BVPs which are called Riemann, Hilbert, Dirichlet, Schwarz, Neumann, Robin boundary value problems. Among them, the Riemann boundary value problem and the Hilbert boundary value problem are in the center of interest. The Dirichlet boundary value problem is connected to the Riemann boundary value problem. In this dissertation, we are mainly concerned with the Dirichlet and Neumann boundary value problems for the unit disc. The Schwarz problem is the simplest form of the Hilbert problem. The Neumann problem is related the Dirichlet problem. The Robin problem is contacted to the Dirichlet problem and the Neumann problem. In addition, for some special cases, e.g. the unit disc or the half plane, the Hilbert problem can be transformed to the Riemann problem. The present thesis contributes to the research subject initiated by Prof. Dr. H. Begehr and developed by his students and collaborators. A systematic investigation of boundary value problems for complex partial differential equations of arbitrary order on the base of integral representation formulas was initiated by H. Begehr. To start with, the basic boundary value problems for model equations are observed. The differential operator of a model equation consists of a product of powers of the complex Cauchy-Riemann operator $\partial_{\bar{z}}$ and its complex conjugate ∂_z. The main methods of the theory will be pointed on now.

The complex form of the Gauss theorem for the unit disc D on the complex plane C and an arbitrary function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ leads to the Cauchy-Pompeiu formula for analytic functions. The area integral appearing in the Cauchy-Pompeiu formula is called the Pompeiu operator. It plays an important role in treating boundary value problems for inhomogeneous complex partial differential equations. The properties of the Pompeiu operator were studied by I.N. Vekua [43]. If f belongs to $L_p(D;\mathbb{C}), p > 1$, then Tf possesses weak derivatives with respect to z and $\bar{z}$, moreover $\partial_{\bar{z}} Tf = f$, $\partial_z Tf =: \Pi f$, where Π is a singular integral being understood in the principle value sense. Integrals of such type are investigated in [32].

From the Cauchy-Pompeiu representation formula it follows that any function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ can be found by known values on the boundary and values of a first order derivative inside of the domain. On the other hand, for given $f \in L_p(D;\mathbb{C}), p > 1$, and $\gamma \in C(\partial D;\mathbb{C})$ a new function

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} \gamma(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} f(\zeta)\frac{d\xi}{d\eta}, \tag{1.0.1}$$

can be constructed according to this formula. The boundary integral is an analytic function, while the area integral represents the Pompeiu operator. Using the properties of the Pompeiu operator, the function w is seen to be a solution of the differential equation $w_{\bar{z}} = f$ in D, being understood in the weak sense. But the boundary values of the function defined in (1.0.1) in general differ from γ. Therefore the function w given by (1.0.1) is not the solution of the problem

$$w_{\bar{z}} = f \text{ in } D, \; w = \gamma \text{ on } \partial D,$$

which is called the Dirichlet boundary value problem for the inhomogeneous Cauchy-Riemann equation. This fact leads to the idea that the Cauchy-Pompeiu representation formula has to be modified in a proper way for being useful to treat boundary value problems. There are three basic boundary value problems for complex partial differential equations, namely, Schwarz, Dirichlet and Neumann problems. To find the solutions in explicit form they are investigated in particular domains, i.e. the unit disk, half planes, quarter planes, etc. For the unit disk the modified Cauchy-Pompeiu formula, which is known as Cauchy-Pompeiu-Schwarz-Poisson formula (or Schwarz-Poisson formula in the case of analytic functions), serves as the starting point in [13], where the solutions of the basic boundary value problems to first order equations are given. To solve boundary value problems for the inhomogeneous Cauchy-Riemann equation $w_{\bar{z}} = f$ the idea of I.N. Vekua is exploited, who suggested [43] to represent the solution of these problems in the form $w = \varphi + Tf$, with φ being an analytic function. By using the properties of the Pompeiu operator the boundary value problems for the inhomogeneous Cauchy-Riemann equation are reduced to the homogeneous case (see, e.g.[13]). Besides the three main boundary value problems listed above the Robin boundary value problem should be mentioned. This problem is the combination of Dirichlet and Neumann ones. The solutions of the particular Robin boundary value problem to the Cauchy-Riemann operator is given [19].

In Chapter 2 we present the main theorem of calculus, which is used to solve differential equations of first and second order under certain initial or boundary conditions in complex analysis. This chapter is written on the basis of some important papers of H. Begehr.

In Chapter 3, we obtain a tri-harmonic Green function for the unit disc and a tri-harmonic Neumann function for the unit disc. Moreover, we solve boundary value problems for the Poisson equation in the unit disc. In this chapter representation formulas for the tri-harmonic differential equation are given on the basis of some hybrid Green functions. We also discuss BVPs for the tri-harmonic differential equation, in particular the Dirichlet, Neumann and combined conditions.

In Chapter 4, we present some results on boundary value problems for higher order inhomogeneous complex partial differential equations in the unit disc. We begin with the higher order Pompeiu operators and then use these operators to study four classes of Dirichlet problems for the inhomogeneous equations.

The Appendix I devote to my results about the solutions of a class of complex partial differential equations of third order in the plane with Fuchs type differential operators. The solutions are constructed in explicit form and the Cauchy problem with prescribed growth at infinity is solved in unbounded angular domains within specified function classes.

Chapter 2

Boundary value problems for first and second order complex partial differential equations in the unit disc

A systematic investigation of basic boundary value problems for complex partial differential equations of arbitrary order is restricted to model equations. Four basic boundary value problems, namely, the Schwarz, the Dirichlet, the Neumann, the Robin problems for analytic functions and more generally for the inhomogeneous Cauchy-Riemann equation are investigated in the unit disc. The representation for the solutions and solvability conditions are given in explicit form. The fundamental tools are the Gauss theorem and the Cauchy-Pompeiu representation.

2.1 Notations and technical preliminaries

Let $\mathbb{C}$ be the complex plane of the variable $z = x+iy$, $x, y \in \mathbb{R}$. The extended complex plane is denoted by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The complex number $\bar{z} = x - iy$ is called the conjugate number to z. By $\operatorname{Re} z$, $\operatorname{Im} z$ the real and imaginary parts of z are denoted.

In complex analysis it is convenient to use the complex partial differential operators ∂_z and $\partial_{\bar{z}}$ defined by the real partial differential operators ∂_x and ∂_y as

$$2\partial_z = \partial_x - i\partial_y, \ 2\partial_{\bar{z}} = \partial_x + i\partial_y \tag{2.1.1}$$

Formally they are deducible by treating

$$z = x + iy, \ \bar{z} = x - iy, \ x, \, y \in \mathbb{R},$$

as independent variables using the chain rule of differentiation.

A complex-valued function $w = u + iv$ given by two real-valued functions u and v of the real variables x and y will be denoted by $w(z)$ although being rather a function of z and $\bar{z}$. In case when w is independent of $\bar{z}$ in an open set of the complex plane $\mathbb{C}$ it is an analytic function. It then is satisfying the Cauchy-Riemann system of first order partial differential equations

$$u_x = v_y, \ u_y = -v_x \tag{2.1.2}$$

This is equivalent to

$$w_{\bar{z}} = 0 \tag{2.1.3}$$

as follows from

$$2\partial_{\bar{z}} w = (\partial_x + i\partial_y)(u + iv) = \partial_x u - \partial_y v + i(\partial_x v + \partial_y u) \tag{2.1.4}$$

In that case also

$$\begin{aligned} 2\partial_z w = (\partial_x - i\partial_y)(u + iv) &= \partial_x u + \partial + yv + i(\partial_x v - \partial_y u) \\ &= 2\partial_x w = -2i\partial_y w = 2w' \end{aligned} \tag{2.1.5}$$

Using these complex derivatives the real Gauss divergence theorem for functions of two real variables being continuously differentiable in some regular domain, i.e. a bounded domain D with smooth boundary ∂D, and continuous in the closure $\overline{D} = D \cup \partial D$ of D, easily can be given in complex forms.

Main theorem of calculus. *Let w be analytic in z_0, $z_0 \in \mathbb{C}$, i.e. complex differentiable with respect to z in some neighborhood of z_0*

$$w(z) = w(z_0) + \int_{z_0}^{z} w'(\zeta) d\zeta. \tag{2.1.6}$$

Here the integration is taken along any rectifiable curve from z_0 to z in the neighborhood of z_0, i.e. the straight line. This result is evident because w' is continuous as it is a complex differentiable function itself.

Lemma 2.1.1 *Let w be analytic in z_0, $z_0 \in \mathbb{C}$, then for any $n \in \mathbb{N}$ and z in the neighborhood of z_0*

$$w(z) = \sum_{k=0}^{n} \frac{1}{k!} w^{(k)}(z_0)(z - z_0)^k + \frac{1}{n!} \int_{z_0}^{z} (z - \zeta)^n w^{(n+1)}(\zeta) d\zeta. \tag{2.1.7}$$

Proof. For $n = 0$ the representation (2.1.7) coincides with (2.1.6). Assume (2.1.7) holds for $n-1$ rather than for n i.e.

$$w(z) = \sum_{k=0}^{n-1} \frac{1}{k!} w^{(k)}(z_0)(z-z_0)^k + \frac{1}{(n-1)!} \int_{z_0}^{z} (z-\zeta)^{n-1} w^{(n)}(\zeta) d\zeta.$$

Applying (2.1.6) to $w^{(n)}(\zeta)$ shows for ζ in the neighborhood of z_0

$$w^{(n)}(\zeta) = w(z_0) + \int_{z_0}^{\zeta} w^{(n+1)}(t) dt.$$

Inserting this in the preceding formula and using integration by parts giving

$$\int_{z_0}^{z} (z-\zeta)^{n-1} \int_{z_0}^{\zeta} w^{(n+1)}(t) dt d\zeta = \frac{1}{n} \int_{z_0}^{z} (z-\zeta)^{n} w^{(n+1)}(\zeta) d\zeta.$$

leads to (2.1.7).

In real analysis from a representation formula like (2.1.7) the Taylor formula is deduced by applying the mean value theorem for integrals. This principle of iterating integral representation formulas can be applied also in case of partial differential operators being involved. However, the procedure is not restricted to this case or to hypercomplex analysis [10]. It can be applied in real analysis as well and this is done in Clifford analysis [8], [10], in particular in quaternionic analysis [10], [13]. For octonionic analysis the non-associativity causes some problems for higher order iterations [18].

The complex forms of the Gauss theorem.

The Gauss or Gauss - Ostrogradskii theorem is the main theorem of calculus in the case of several variables. the integral of the divergence of a vector field taken over a regular domain is expressed by the boundary and the (outward) normal direction. In case of the complex plane there are two kinds of divergence differential operator, the Cauchy-Riemann operator $2\partial_{\bar{z}} = \partial_x + i\partial_y$ and its complex conjugate, the anti-Cauchy-Riemann operator $2\partial_{\bar{z}} = \partial_x - i\partial_y$.

Gauss Theorem (complex forms)

Let $D \subset \mathbb{C}$ be a regular domain and $w \in \mathbb{C}^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$. Then

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(z) dz = \frac{1}{\pi} \int_{\mathbb{D}} w_{\bar{z}}(z) dxdy, \quad -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(z) d\bar{z} = \frac{1}{\pi} \int_{\mathbb{D}} w_z(z) dxdy. \tag{2.1.8}$$

A regular domain means a bounded domain with a piecewise smooth boundary.

From these Gauss theorems, see e.g. [7], [13], integral representation formulas are deduced by inserting a fundamental solution to the differential operators involved. This is for the Cauchy-Riemann operator (up to some constant factor) $1/\bar{z}$.

Choosing an arbitrary point $z \in D$ and applying the respective Gauss theorem to the function $w(\zeta)/(\zeta - z)$ in $D_\varepsilon = D \backslash \{\zeta : |\zeta - z| \leq \varepsilon\}$ for $0 < \varepsilon$ small enough and letting ε tend to zero leads to a representation formula. Similarly the second Gauss formula can be treated.

Cauchy-Pompeiu representations. Let D and w satisfy the assumptions from the Gauss theorem. Then

$$\frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{D} w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in D, \\ 0, & z \in \bar{\mathbb{C}} \backslash \bar{D}, \end{cases} \tag{2.1.9}$$

$$-\frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi} \int\limits_{D} w_{\zeta}(\zeta) \frac{d\xi d\eta}{\overline{\zeta - z}} = \begin{cases} w(z), & z \in D, \\ 0, & z \in \bar{\mathbb{C}} \backslash \bar{D}, \end{cases} \tag{2.1.10}$$

Here $\zeta = \xi + i\eta$ is used. The kernel function $1/(\zeta - z)$ is the Cauchy kernel. The integral operator

$$Tf(z) = -\frac{1}{\pi} \int\limits_{D} f(\zeta) \frac{d\xi d\eta}{\zeta - z}$$

is defined for $f \in L_p(D; \mathbb{C}), 1 \leq p$. It is called Pompeiu operator. Its properties are studied in detail in [43] in connection with the theory of generalized analytic functions. Tf is weakly differentiable (in distributional sense) with

$$\partial_{\bar{z}} Tf = f, \;\; \partial_z Tf = \Pi f$$

where

$$\Pi f(z) = -\frac{1}{\pi} \int\limits_{D} f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2}$$

is a singular integral operator of Calderon-Zygmund type to be taken as a Cauchy principle value integral. The properties of this Ahlfors-Beurling operator Π are also studied in [39].

In case w is analytic in D, i.e. $w_{\bar{z}} = 0$ in D, formula (2.1.9) is the Cauchy formula for analytic functions, one of the main tools in function theory and the source of many properties of analytic functions.

The representation formulas (2.1.9) and (2.1.10) are proper for iterations. This leads at the same time as well to higher order Cauchy-Pompeiu representation formulas as to fundamental solutions to the differential operators involved.

Theorem 2.1.2 *If $f \in L_1(D;\mathbb{C})$ then for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy + \int_D f(z)\varphi(z)dxdy = 0 \tag{2.1.11}$$

Here $C_0^1(D;\mathbb{C})$ denotes the set of complex-valued functions in D being continuously differentiable and having compact support in D, i.e. vanishing near the boundary.

Proof. From (2.1.9) and the fact that the boundary values of φ vanish at the boundary

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial D}\varphi(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D \varphi_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} = (T\varphi_{\bar{\zeta}})(z)$$

follows. Thus interchanging the order of integrations

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy = -\frac{1}{\pi}\int_D f(\zeta)\int_D \varphi_{\bar{z}}(z)\frac{dxdy}{\zeta - z}d\xi d\eta = -\int_D f(\zeta)\varphi(\zeta)d\xi d\eta$$

Formula (2.1.11) means that

$$\partial_{\bar{z}}Tf = f \tag{2.1.12}$$

in distributional sense.

Definition 2.1.3 *Let $f, g \in L_1(D;\mathbb{C})$. Then f is called generalized (distributional)derivative of g with respect to $\bar{z}$ if for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int_D g(z)\varphi_{\bar{z}}(z)dxdy + \int_D f(z)\varphi(z)dxdy = 0.$$

This derivative is denoted by $f = g_{\bar{z}} = \partial_{\bar{z}}g$.

In the same way generalized derivatives with respect to z are defined. In case a function is differentiable in the ordinary sense it is also differentiable in the distributional sense and both derivative coincide.

Sometimes solutions to differential equations in distributional sense can be shown to be differentiable in the classical sense. Then generalized solutions become classical solutions to the equation. An example is the Cauchy-Riemann system (2.1.3), see [43], [7].

More delecate is the differentiation of Tf with respect to z. For $z \in \mathbb{C}\backslash\overline{D}$ obviously Tf is analytic and its derivative

$$\partial_z Tf(z) = \Pi f(z) = -\frac{1}{\pi}\int\limits_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}. \tag{2.1.13}$$

That this holds in distributional sense also for $z \in D$ almost everywhere when $f \in L_p(D;\mathbb{C}),\ 1 < p$, and the integral on the right-hand side is understood as a Cauchy principal value integral

$$\int\limits_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2} = \lim_{\varepsilon\to 0}\int\limits_{D\backslash\overline{K_\varepsilon(z)}} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}$$

is a deep result of Calderon-Zygmund [32].

With respect to boundary value problems a modification of the Cauchy-Pompeiu formula is important in the case of the unit disc $\mathbb{D} = \{z : |z| < 1\}$.

Theorem 2.1.4 *Any $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ is representable as*

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \text{Re } w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int\limits_{|\zeta|=1} \text{Im } w(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta - z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1 - z\bar\zeta}\right)d\xi d\eta,\ |z| < 1. \tag{2.1.14}$$

Corollary 2.1.5 *Any $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \text{Re}w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{w_{\bar\zeta}(\zeta)}}{\bar\zeta}\frac{1 + z\bar\zeta}{1 - z\bar\zeta}\right)d\xi d\eta + i\text{Im } w(0),\ |z| < 1. \tag{2.1.15}$$

Proof. For fixed $z, |z| < 1$, formula (2.1.8) applied to $\mathbb{D}$ shows

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{\bar z d\zeta}{1-\bar z\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\bar\zeta}(\zeta)\frac{\bar z}{1-\bar z\zeta}d\xi d\eta = 0.$$

Taking the complex conjugate and adding this to (2.1.9) in the case $D = \mathbb{D}$ gives $|z| < 1$

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{\zeta w(\zeta)}{\zeta - z} + \overline{\frac{\overline{z w(\zeta)}}{\zeta - z}}\right)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta - z} + \frac{\overline{z w_{\bar\zeta}(\zeta)}}{1-\bar z\zeta}\right)d\xi d\eta,$$

where $\bar\zeta d\zeta = -\zeta\overline{d\zeta}$ for $|\zeta| = 1$ is used. This is (2.1.14). Subtracting $i\mathrm{Im}\ w(0)$ from (2.1.14) proves (2.1.15).

Remark 2.1.6 *For analytic functions (2.1.15) is the Schwarz-Poisson formula*

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\mathrm{Re}\ w(\zeta)\left(\frac{2\zeta}{\zeta - z} - 1\right)\frac{d\zeta}{\zeta} + i\mathrm{Im}\ w(0). \qquad (2.1.16)$$

The kernel

$$\frac{\zeta + z}{\zeta - z} = \frac{2\zeta}{\zeta - z} - 1$$

is called the Schwarz kernel. Its real part

$$\frac{\zeta}{\zeta - z} + \frac{\bar\zeta}{\overline{\overline{\zeta - z}}} - 1 = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

is the Poisson kernel. The Schwarz operator

$$S\varphi(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\varphi(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}$$

for $\varphi \in C(\partial\mathbb{D};\mathbb{R})$ is known to provide an analytic in $\mathbb{D}$ satisfying

$$\mathrm{Re}\ S\varphi = \varphi \ \ on\ \ \partial\mathbb{D}$$

see [42] in the sense

$$\lim_{z\to\zeta} S\varphi(z) = \varphi(\zeta),\ \ \zeta \in \partial\mathbb{D},$$

for z in $\mathbb{D}$ tending to ζ. Poisson has proved the respective representation for harmonic functions, i.e. to solutions for the Laplace equation

$$\triangle u = \partial_x^2 u + \partial_y^2 u = 0$$

in $\mathbb{D}$. $\mathrm{Re}\, w$ for analytic w is harmonic.

The Schwarz operator can be defined for other simply and even multi-connected domains, see e.g.[7].

Formula (2.1.15) is called the Cauchy-Schwarz-Poisson-Pompeiu formula. Rewriting it according to

$$w_{\bar z} = f \ \textit{ in } \ \mathbb{D}, \ \mathrm{Re}\, w = \varphi \ \textit{ on } \ \partial\mathbb{D}, \ \mathrm{Im}\, w(0) = c,$$

then

$$w(z) = \frac{1}{2\pi} \int\limits_{|\zeta|=1} \varphi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}$$

$$- \frac{1}{2\pi i} \int\limits_{|\zeta|<1} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar\zeta} \frac{1 + z\bar\zeta}{1 - z\bar\zeta} \right) d\xi d\eta + ic \tag{2.1.17}$$

is expressed by the given data. Applying the result of Schwarz it is easily seen that taking the real part on the right-hand side letting z tend to a boundary point ζ this tend to $\varphi(\zeta)$.

Differentiating with respect to $\bar z$ as every term on the right-hand side is analytic besides the T*-operator applied to f this gives $f(z)$. Also for $z = 0$ besides ic all other terms on the right-hand side are real.*

Hence, (2.1.17) is a solution to the so-called Dirichlet problem

$$w_{\bar z} = f \ \textit{ in } \ \mathbb{D}, \ \mathrm{Re}\, w = \varphi \ \textit{ on } \ \partial\mathbb{D}, \ \mathrm{Im}\, w(0) = c,$$

This shows how integral representation formulas serve to solve boundary value problems. The method is not restricted to the unit disc but in this case the solutions to the problems are given in an explicit way.

Second order Cauchy-Pompeiu representations. Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^2(D;\mathbb{C}) \cap C^1(\bar D;\mathbb{C})$. Then for $z \in D$

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} -$$

$$- \frac{1}{2\pi i} \int\limits_{\partial D} w_{\bar\zeta}(\zeta) \frac{\overline{\zeta - z}}{\zeta - z} d\zeta + \frac{1}{\pi} \int\limits_{D} w_{\zeta\bar\zeta}(\zeta) \frac{\overline{\zeta - z}}{\zeta - z} d\xi d\eta \tag{2.1.18}$$

and

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta-z}-$$

$$+\frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar\zeta}(\zeta)\log|\zeta-z|^2 d\bar\zeta+\frac{1}{\pi}\int\limits_{D} w_{\zeta\bar\zeta}(\zeta)\log|\zeta-z|^2 d\xi d\eta \tag{2.1.19}$$

Proof of (2.1.18). Inserting formula (2.1.9) applied to $w_{\bar z}$, i.e.

$$w_{\bar{\tilde\zeta}}(\tilde\zeta)=\frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar\zeta}(\zeta)\frac{d\zeta}{\zeta-\tilde\zeta}-\frac{1}{\pi}\int\limits_{D} w_{\bar\zeta\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-\tilde\zeta},$$

into (2.1.9)shows

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta-z}$$

$$-\frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar\zeta}(\zeta)\psi(z,\zeta)d\zeta+\frac{1}{\pi}\int\limits_{D} w_{\zeta\bar\zeta}(\zeta)\psi(z,\zeta)d\xi d\eta \tag{2.1.20}$$

with

$$\psi(z,\zeta)=\frac{1}{\pi}\int\limits_{D}\frac{1}{\zeta-\tilde\zeta}\frac{d\tilde\zeta d\tilde\eta}{\tilde\zeta-z}=-\frac{1}{\zeta-z}\frac{1}{\pi}\int\limits_{D}\left(\frac{1}{\tilde\zeta-\zeta}-\frac{1}{\tilde\zeta-z}\right)d\tilde\xi d\tilde\eta.$$

Formula (2.1.9) applied to the function $\bar z$ is

$$\bar z=\frac{1}{2\pi i}\int\limits_{\partial D}\bar{\tilde\zeta}\frac{d\tilde\zeta}{\tilde\zeta-z}-\frac{1}{\pi}\int\limits_{D}\frac{d\tilde\xi d\tilde\eta}{\tilde\zeta-z}.$$

Therefore

$$\frac{\overline{\zeta-z}}{\zeta-z}=\frac{1}{2\pi i}\int\limits_{\partial D}\frac{\bar{\tilde\zeta}d\tilde\zeta}{(\tilde\zeta-\zeta)(\tilde\zeta-z)}-\frac{1}{\pi}\int\limits_{D}\frac{d\tilde\xi d\tilde\eta}{(\tilde\zeta-\zeta)(\tilde\zeta-z)}=\tilde\psi(z,\zeta)+\psi(z,\zeta) \tag{2.1.21}$$

where

$$\tilde\psi(z,\zeta)=\frac{1}{2\pi i}\int\limits_{\partial D}\frac{\bar{\tilde\zeta}d\tilde\zeta}{(\tilde\zeta-\zeta)(\tilde\zeta-z)}$$

is an analytic function in both its variables outside ∂D.

From the Gauss theorem

$$\frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar\zeta}(\zeta)\tilde\psi(z,\zeta)d\zeta$$

$$= \frac{1}{\pi}\int\limits_D \partial_{\bar\zeta}[w_{\bar\zeta}(\zeta)\tilde\psi(z,\zeta)]d\xi d\eta = \frac{1}{\pi}\int\limits_D w_{\bar\zeta\bar\zeta}(\zeta)\tilde\psi(z,\zeta)d\xi d\eta$$

follows. Combining this with (2.1.20)and observing (2.1.21) proves (2.1.18)

In a similar way combining (2.1.10) applied to $w_{\bar\zeta}(\zeta)$ with (2.1.9) leads to (2.1.19). Here instead of the kernel (2.1.21)

$$\log|\zeta - z|^2 = \frac{1}{2\pi i}\int\limits_{\partial D}\log|\zeta-\tilde\zeta|^2\frac{d\tilde\zeta}{\tilde\zeta - z} - \frac{1}{\pi}\int\limits_D \frac{1}{\overline{\tilde\zeta-\zeta}}\frac{d\xi d\eta}{\zeta - z} \tag{2.1.22}$$

is used.

This follows from (2.1.21) applied to the weakly singular function $\log|\zeta - z|^2$ for fixed $\zeta \in D$, which can be justified by a limiting process as before considering $D_\varepsilon = D\backslash\{z : |z-\zeta| \leq \varepsilon\}$. The kernel functions in (2.1.21) and (2.1.22) are primitives of the Cauchy kernel with respect to the differential operators $\partial_{\bar z}$ and ∂_z respectively. For $z \neq 0$ there hold

$$\partial_{\bar z}\frac{\bar z}{z} = \frac{1}{z}, \quad \partial_z \log|z|^2 = \frac{1}{z}.$$

This is a principle for generating fundamental solutions for higher order differential operators from fundamental solutions of lower order ones. Just the respective primitives have to be determined. Thus the fundamental solution to the operator $\partial_{\bar z}^n$ is $\bar z^{n-1}/[z(n-1)!]$, to $\partial_{\bar z}^n$ it is $z^{n-1}/[\bar z(n-1)!]$, and to $\partial_z^m\partial_{\bar z}^n$ it is $z^{m-1}\bar z^{n-1}\log|z|^2/[(m-1)!(n-1)!]$. More natural and more convenient is for the last operator to use

$$\frac{z^{m-1}\bar z^{n-1}}{(m-1)!(n-1)!}\left(\log|z|^2 - \sum_{\mu=1}^{m-1}\frac{1}{\mu} - \sum_{\nu=1}^{n-1}\frac{1}{\nu}\right).$$

The difference of the last from the preceding one is just a function from the kernel of the operator $\partial_z^m\partial_{\bar z}^n$, i.e. a function annihilated by this operator. These fundamental solutions are part of a set of kernel functions of a hierarchy of higher order Pompeiu operators appearing when continuing the indicated iteration process, see [20]. Avoiding the general case for simplicity only two particular situations are listed the proofs of which follow again by induction.

Polyanalytic Cauchy-Pompeiu representation. Let $w \in C^n(D;\mathbb{C})\cap C^{n-1}(\bar D;\mathbb{C})$ for some $n \geq 1$. Then

$$w(z) = \sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int\limits_{\partial D}\partial_{\bar\zeta}^\nu w(\zeta)\frac{(\overline{z-\zeta})^\nu}{\nu!(\zeta - z)}d\zeta -$$

$$-\frac{1}{\pi}\int_D \partial_{\bar\zeta}^n w(\zeta)\frac{(\overline{z-\zeta})^{n-1}}{(n-1)!(\zeta-z)}d\xi d\eta \tag{2.1.23}$$

Polyharmonic Cauchy-Pompeiu representation. Let $w \in C^{2n}(D;\mathbb{C}) \cap C^{2n-1}(\bar{D};\mathbb{C})$ for some $n \geq 1$. Then

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{w(\zeta)}{\zeta - z}d\zeta + \sum_{\nu=1}^{n-1}\frac{1}{2\pi i}\int_{\partial D}\frac{(\zeta-z)^{\nu-1}\overline{(\zeta-z)}^{\nu}}{(\nu-1)!\nu!}\Big[\log|\zeta-z|^2 -$$

$$-\sum_{\rho=1}^{\nu-1}\frac{1}{\rho} - \sum_{\sigma=1}^{\nu}\frac{1}{\sigma}\Big](\partial_\zeta\partial_{\bar\zeta})^\nu w(\zeta)d\zeta \tag{2.1.24}$$

$$+\sum_{\nu=1}^{n}\frac{1}{2\pi i}\int_{\partial D}\frac{|\zeta-z|^{2(\nu-1)}}{(\nu-1)!^2}\left[\log|\zeta-z|^2 - 2\sum_{\rho=1}^{\nu-1}\frac{1}{\rho}\right]\partial_\zeta^{\nu-1}\partial_{\bar\zeta}^{\nu} w(\zeta)d\bar\zeta$$

$$+\frac{1}{\pi}\int_D \frac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\left[\log|\zeta-z|^2 - 2\sum_{\rho=1}^{k-1}\frac{1}{\rho}\right](\partial_\zeta\partial_{\bar\zeta})^n w(\zeta)d\xi d\eta.$$

Modifications of the Cauchy-Pompeiu representations.

Looking at the Cauchy-Pompeiu formulas, say at (2.1.9),(2.1.10), a function from $C^1(D;\mathbb{C}) \cup C(\bar{D};\mathbb{C})$ is obviously determined by its values on the boundary and one of its first order derivatives in the domain. If on the other hand a function $f \in L_1(D;\mathbb{C})$ and a $\gamma \in C(\partial D;\mathbb{C})$ are given this leads to a function

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{d\zeta}{\zeta-z} - \frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}. \tag{2.1.25}$$

As the boundary integral is analytic and the area integral is the Pompeiu operator w is a weak solution to the differential equation $w_{\bar z} = f$ in D. But in general the boundary values of w on ∂D differ from γ. (2.1.25) in general fails to be a weak solution to this so-called Dirichlet problem. This suggests to modify the Cauchy-Pompeiu formulas. Any solution to the equation $w_{\bar z} = f$ is of the form $w = \varphi + Tf$ with analytic φ. The reason is that $\varphi = w - Tf$ is a weak solution to $\varphi_{\bar z} = 0$. But this means φ is a classical solution i.e. analytic, see [43]. In order that w satisfies some boundary conditions just φ has to be chosen properly. This leads to boundary value problems for analytic functions.

There are four basic boundary value problems [13], [19]. In order to be explicit just some special domains will be considered, the unit disc $\mathbb{D} = \backslash\{z : |z| < 1\}$ and the upper half plane $\mathbb{H} = \backslash\{z : 0 < Imz\}$, the last one as unbounded being no regular domain. For the proofs with respect to the unit disc, see [13], [19], [35], for the upper half plane [35].

2.2 Boundary value problems for analytic functions

As was pointed out in connection with the Schwarz-Poisson formula in the case of the unit disc boundary value problems can be solved explicitly. For this reason this particular domain is considered. This will give necessary information about the nature of the problems considered. The simplest and therefore fundamental cases occur with respect to analytic functions.

Schwarz boundary value problem. Find an analytic function w in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying

$$\operatorname{Re} w = \gamma \ \text{ on } \ \partial\mathbb{D}, \ \operatorname{Im} w(0) = c$$

for $\gamma \in C(\partial\mathbb{D};\mathbb{R}), \ c \in \mathbb{R}$ given.

Theorem 2.2.1 *This Schwarz problem is uniquely solvable. The solution is given by the Schwarz formula*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic \tag{2.2.1}$$

The proof follows from the Schwarz-Poisson formula (2.1.16) together with a detailed study of the boundary behavior, see [42].

Dirichlet boundary value problem. Find an analytic function w in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying for given $\gamma \in C(\partial\mathbb{D};\mathbb{C})$

$$w = \gamma \ \text{ on } \ \partial\mathbb{D}.$$

Theorem 2.2.2 *This Dirichlet problem is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} = 0 \tag{2.2.2}$$

The solution is then uniquely given by the Cauchy integral

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta - z} \tag{2.2.3}$$

Remark 2.2.3 *This result is a consequence of the Plemelj-Sokhotzki formula, see e.g.[40], [36], [7]. The Cauchy integral (2.2.3) obviously provides an analytic function on $\mathbb{D}$ and on $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$, $\hat{\mathbb{C}}$ the Riemann sphere. The Plemelj-Sokhotzki formula states that for $|\zeta| = 1$*

$$\lim_{z\to\zeta,\ |z|<1} w(z) - \lim_{z\to\zeta,\ 1<|z|} wz = \gamma(\zeta).$$

In order that for any $|\zeta| = 1$

$$\lim_{z\to\zeta,\ |\zeta|<1} w(z) = \gamma(\zeta)$$

the condition $\lim\limits_{z\to\zeta,\ 1<|z|} w(z) = 0$ is necessary and sufficient. However, the Plemelj-Sokhotzki formula in its classical formulation holds if γ is Holder continuous. Nevertheless, for the unit disc Holder continuity is not needed, see [40].

Proof **1.** (2.2.2) is shown to be necessary. Let w be a solution to the Dirichlet problem. Then w is analytic in $\mathbb{D}$ having continuous boundary values

$$\lim_{z\to\zeta} w(z) = \gamma(\zeta) \tag{2.2.4}$$

for all $|\zeta| = 1$.

Consider for $1 > |z|$ the function

$$w(\frac{1}{\bar{z}}) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\bar{z}}{\overline{\zeta - z}}\frac{d\zeta}{\zeta}$$

From

$$w(z) - w(\frac{1}{\bar{z}}) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\right)\frac{d\zeta}{\zeta}$$

and the properties of the Poisson kernel for $|\zeta| = 1$ and (2.2.4) it is seen that $\lim\limits_{z\to\zeta} w(1/\bar{z})$ exists and

$$\lim_{z\to\zeta,\ |z|<1} w(z) - \lim_{z\to\zeta,\ 1<|z|} w(z) = \gamma(\zeta) \tag{2.2.5}$$

follows. Comparison with (2.2.4) shows $\lim\limits_{z\to\zeta} w(z) = 0$ for $1 < |z|$. As $w(\infty) = 0$ then the maximum principle for analytic functions tells that $w(z) \equiv 0$ in $1 < |z|$. This is condition (2.2.2).

2. The sufficiency of (2.2.2) follows at once from adding (2.2.2) to (2.2.3) leading to

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\overline{\zeta - z}} \right) \frac{d\zeta}{\zeta}$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1 \right) \frac{d\zeta}{\zeta}$$

Thus for $|\zeta| = 1$

$$\lim_{z \to \zeta,\ |z|<1} w(z) = \gamma(\zeta)$$

follows again from the properties of the Poisson kernel.

The third basic boundary value problem is based on the outward normal derivative at the boundary of a regular domain. This directional derivative on a circle $|z-a| = r$ is in the direction of the radius vector, i.e. the outward normal vector is $\nu = (z-a)/r$, and the normal derivative in this direction ν given by

$$\partial_\nu = \partial_r = \frac{z}{r}\partial_z + \frac{\bar{z}}{r}\partial_{\bar{z}}.$$

In particular for the unit disc $\mathbb{D}$

$$\partial_r = z\partial_z + \bar{z}\partial_{\bar{z}}.$$

Neumann boundary value problem. Find an analytic function w in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying for some $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$ and $c \in \mathbb{C}$

$$\partial_\nu w = 0 \text{ on } \partial\mathbb{D},\ w(0) = c.$$

Theorem 2.2.4 *This Neumann problem is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(1 - \bar{z}\zeta)\zeta} = 0 \qquad (2.2.6)$$

is satisfied. The solution then is

$$w(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta}. \qquad (2.2.7)$$

Proof. The boundary condition reduced to the Dirichlet condition is

$$zw'(z) = \gamma(z) \text{ for } \|z| = 1$$

because of the analyticity of w. Hence from the preceding result

$$zw'(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta - z}$$

if and only if for $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} = 0 \tag{2.2.8}$$

But as $zw'(z)$ vanished at the origin this imposes the additional condition

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta} = 0 \tag{2.2.9}$$

on γ. Then

$$w'(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(\zeta - z)\zeta}.$$

Integrating shows

$$w(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log \frac{\zeta - z}{\zeta} \frac{d\zeta}{\zeta}.$$

which is (2.2.7). Adding (2.2.8) and (2.2.9) leads to

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{1}{1 - \bar{z}\zeta} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\bar{\zeta}}{\overline{\zeta - z}} \frac{d\zeta}{\zeta} =$$

$$= -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\bar{\zeta}}{\overline{\zeta - z}} = 0$$

i.e. to (2.2.6). By integration this gives

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log(1 - \bar{z}\zeta) d\zeta = 0$$

A particular Robin boundary value problem. Find an analytic function in the unit disc $\mathbb{D}$, satisfying the boundary condition

$$w + \partial_\nu w = \gamma \quad \text{on} \quad \partial\mathbb{D} \tag{2.2.10}$$

for given $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ with ν being the outward normal vector to the boundary $\partial\mathbb{D}$.

Note that in our case the boundary condition (2.2.10) takes the form

$$w + zw_z = \gamma \quad \text{on} \quad \partial\mathbb{D}.$$

Let us set $g := w + zw_z$ and solve the Dirichlet problem

$$g_{\overline{z}} = 0 \quad \text{in} \quad \mathbb{D}, \tag{2.2.11}$$

$$g = \gamma \quad \text{on} \quad \partial\mathbb{D}. \tag{2.2.12}$$

According to Theorem (2.2.2) the problem (2.2.11), (2.2.12) is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = 0 \tag{2.2.13}$$

for all $|z| < 1$ and the solution is

$$g(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma(\zeta)}{\zeta - z}d\zeta. \tag{2.2.14}$$

Let $w(z) = \sum\limits_{k=0}^{\infty} w_k z^k$ be the Taylor expansion of w in $\mathbb{D}$. Then (2.2.14) gives us

$$\sum_{k=0}^{\infty}(k+1)w_k z^k = \sum_{k=0}^{\infty}\Big\{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma(\zeta)}{\zeta^{k+1}}d\zeta\Big\}z^k$$

from which we have

$$w_k = \frac{1}{k+1}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma(\zeta)}{\zeta^{k+1}}d\zeta$$

and

$$w(z) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\ln(1-z\overline{\zeta})}{z}d\zeta. \tag{2.2.15}$$

We have proved the following

Theorem 2.2.5 *The special Robin boundary value problem for analytic functions is solvable if and only if γ satisfies the condition (2.2.13) and the solution has the form (2.2.15).*

Let $\alpha \in C(\partial\mathbb{D}, \mathbb{C})$. Consider the Robin problem with the more general boundary condition

$$\alpha(z)w + zw_z = \gamma \quad \text{on} \quad \partial\mathbb{D}. \tag{2.2.16}$$

A modification of this general case is to reformulate (2.2.16) as a Neumann boundary condition. We write (2.2.16) in the form

$$zw_z = \gamma - \alpha(z)w.$$

According to Theorem (2.2.4) we get the unique solution

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \{\alpha(\zeta)w(\zeta) - \gamma(\zeta)\ln(1 - z\overline{\zeta})\}\frac{d\zeta}{\zeta} + w(0)$$

under the necessary and sufficient condition

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \alpha(\zeta)w(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta}$$

for $|z| < 1$.

Using the Taylor expansion of the function $\ln(1 - z\overline{\zeta})$ we have

$$w(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \{\gamma(\zeta) - \alpha(\zeta)w(\zeta)\}\overline{\zeta}^{k+1} d\zeta + w(0). \tag{2.2.17}$$

Example 2.2.6 *Let us take $\alpha(z) = z^n$. Then from (2.2.17) we get*

$$\begin{aligned} w(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k}\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Big\{\frac{\gamma(\zeta)}{\zeta^{k+1}} - \frac{w(\zeta)}{\zeta^{k+1-n}}\Big\} d\zeta + w(0) \\ &= \sum_{k=1}^{\infty} \frac{z^k}{k}\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma(\zeta)}{\zeta^{k+1}} d\zeta - \sum_{k=n}^{\infty} \frac{z^k}{k}\frac{w^{(k-n)}(0)}{(k-n)!} + w(0). \end{aligned}$$

Let $w(z) = \sum\limits_{k=0}^{\infty} w_k z^k$ in $\mathbb{D}$. Then we have

$$w_k = \frac{1}{2\pi i k} \int\limits_{|\zeta|=1} \frac{\gamma(\zeta)}{\zeta^{k+1}} d\zeta, \quad k = 1, \ldots, n-1,$$

$$w_k = \frac{1}{k}\Big(\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma(\zeta)}{\zeta^{k+1}} d\zeta - w_{k-n}\Big), \quad k = n, \ldots.$$

Hence, with $a_k = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta^{k+1}}$

$$w(z) = w_0 e^{-\frac{z^n}{n}} + \sum_{\nu=0}^{\infty} (-1)^\nu \sum_{k=1}^{\infty} \frac{a_k}{\prod\limits_{\sigma=0}^{\nu} (k+\sigma n)} z^{k+\nu n}$$

is the solution if and only if for $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(1-\overline{z}\zeta)\zeta} = 0.$$

Example 2.2.7 *For* $\alpha(z) = \overline{z}^n$ *we have*

$$w(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} \Big(\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma(\zeta)}{\zeta^{k+1}} d\zeta - w_{k+n} \Big) + w(0),$$

i.e.,

$$w_k = \frac{1}{k} \Big(\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma(\zeta)}{\zeta^{k+1}} d\zeta - w_{n+k} \Big), \quad k = 1, \ldots .$$

Hence, with a_k *as before*

$$\begin{aligned} w(z) &= \sum_{k=0}^{n} w_k z^k + \sum_{k=1}^{n} \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \prod_{\sigma=0}^{\nu-1} (k+\sigma n) w_k z^{k+\nu n} \\ &+ \sum_{\mu=0}^{\infty} \sum_{k=\mu n+1}^{(\mu+1)n} \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \prod_{\sigma=1}^{\nu-1} (k+\sigma n) a_k z^{k+\nu n} \end{aligned}$$

is the solution for arbitrary $w_0, w_1, \ldots, w_n$ *if and only if for* $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(1-\overline{z}\zeta)\zeta} = \sum_{k=0}^{n} w_k \overline{z}^{n-k}$$

Remark 2.2.8 *An other modification is to reformulate (2.2.16) as a Dirichlet boundary condition. To this end we solve the equation (2.2.16) with respect to* w *normalized by* $w(z_0) = 0$ *for a fixed* $z_0 \in \partial\mathbb{D}$. *It can be seen, see [19] that the solutions is*

$$w(z) = \exp\Big\{ -\int\limits_{z_0}^{z} \frac{\alpha(t)}{t} dt \Big\} \int\limits_{z_0}^{z} \exp\Big\{ \int\limits_{z_0}^{s} \frac{\alpha(t)}{t} dt \Big\} \frac{\gamma(s)}{s} ds, |z| = 1, \quad (2.2.18)$$

where the integrals are taken along arcs $\gamma_z \subset \partial\mathbb{D}$ with initial point z_0 and end point z; moreover, γ_z is positively oriented with respect to $\mathbb{D}$ for all $z \in \partial\mathbb{D}, z = z_0 e^{i\varphi}, 0 \leq \varphi \leq \pi$, and negatively for the remainder part of $\partial\mathbb{D}$.

Now we proceed in the same way as above: We find an analytic function in $\mathbb{D}$, satisfying the Dirichlet boundary condition (2.2.16). Using Theorem 2.2.2 we have

$$w(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{w(\zeta)}{\zeta - z} d\zeta, \quad |z| < 1,$$

under the 'if and only if' condition

$$\frac{1}{2\pi i} \int_{|\zeta|=1} w(\zeta) \frac{\overline{z}}{1 - \overline{z}\zeta} d\zeta = 0.$$

Next these boundary value problems will be studied for the inhomogeneous Cauchy-Riemann equation. Using the T-operator the problems will be reduced to the ones for analytic functions. Here in the case of the Neumann problem it will make a difference if the normal derivative on the boundary or only the effect of $z\partial_z$ on the function is prescribed.

2.3 Boundary value problems for the inhomogeneous Cauchy-Riemann equation

The Schwarz, the Dirichlet, the Neumann, the Robin boundary value problems for the inhomogeneous Cauchy-Riemann equation in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ are solved in this section. Using the definition and properties of the Pompeiu operator, the problems are reduced to the homogeneous case, or which is equivalent, to the boundary value problems for analytic functions.

Theorem 2.3.1 *The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f$$

in $\mathbb{D}$, $\operatorname{Re} w = \gamma$ *on* $\partial\mathbb{D}$, $\operatorname{Im} w(0) = c$ *for* $f \in L_p(\mathbb{D}; \mathbb{C}), 2 < p, \gamma \in C(\partial\mathbb{D}; \mathbb{R})$, $c \in \mathbb{R}$ *is uniquely solvable by the Cauchy-Schwarz-Pompeiu formula*

$$w(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic$$

$$- \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right] d\xi d\eta. \tag{2.3.1}$$

The Schwarz kernel function for the disc

$$\frac{\zeta + z}{\zeta - z} = \frac{2\zeta}{\zeta - z} - 1$$

is a simple modification of the Cauchy kernel function. Its real part

$$\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1$$

is the Poisson kernel function for discs.

By an inductive proof (2.1.24) is generalized in a nice way, [22].

Theorem 2.3.2 *The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \ \text{ in } \mathbb{D}, \ w = \gamma \ \text{ on } \partial\mathbb{D}$$

for $f \in L_p(\mathbb{D}; \mathbb{C}), 2 < p$ *and* $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$ *is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} = \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{\bar{z} d\xi d\eta}{1 - \bar{z}\zeta}. \tag{2.3.2}$$

The solution then is uniquely given by

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta - z}. \tag{2.3.3}$$

Representation (2.3.1) follows from (2.1.9) if the problem is solvable. The unique solvability is a consequence of Theorem (2.2.2). Applying condition (2.2.2) to the boundary value of the analytic function $w - Tf$ in $\mathbb{D}$, i.e. to $\gamma - Tf$ on $\partial\mathbb{D}$ gives (2.3.2) because of

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{1}{\pi} \int\limits_{|\tilde{\zeta}|} f(\tilde{\zeta}) \frac{d\tilde{\xi} d\tilde{\eta}}{\tilde{\zeta} - \zeta} \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} =$$

$$-\frac{1}{\pi} \int\limits_{|\tilde{\zeta}|} f(\tilde{\zeta}) \frac{1}{2\pi i} \int\limits_{|\zeta|} \frac{\bar{z}}{1 - \bar{z}\zeta} \frac{d\zeta}{\zeta - \tilde{\zeta}} d\tilde{\xi} d\tilde{\eta} = -\frac{1}{\pi} \int\limits_{|\tilde{\zeta}|=1} f(\tilde{\zeta}) \frac{\bar{z}}{1 - \bar{z}\tilde{\zeta}} d\tilde{\xi} d\tilde{\eta}$$

as is seen from the Cauchy formula.

Theorem 2.3.3 *The Neumann problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \;\; in \;\; \mathbb{D}, \;\; \partial_\nu w = \gamma \;\; on \;\; \partial\mathbb{D}, \;\; w(0) = c,$$

for $f \in C^\alpha(\bar{\mathbb{D}}; \mathbb{C})$, $0 < \alpha < 1$, $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$, $c \in C$ *is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(1-\bar{z}\zeta)\zeta} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} f(\zeta) \frac{d\bar{\zeta}}{1-\bar{z}\zeta} +$$

$$+ \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{\bar{z} f(\zeta)}{(1-\bar{z}\zeta)^2} d\xi d\eta = 0. \tag{2.3.4}$$

The unique solution then is

$$w(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma(\zeta) - \bar{\zeta} f(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{z f(\zeta)}{\zeta(\zeta - z)} d\xi d\eta. \tag{2.3.5}$$

Proof. The function $\varphi = w - Tf$ satisfies

$$\varphi_{\bar{z}} = 0 \;\text{ in }\; \mathbb{D}, \;\; \partial_\nu \varphi = \gamma - z\Pi f - \bar{z} f \;\text{ on }\; \partial\mathbb{D}, \;\; \varphi(0) = c - Tf(0).$$

As the property of the Π-operator, see [43], guarantee $\Pi f \in C^\alpha(\overline{\mathbb{D}}; \mathbb{C})$ for $f \in C^\alpha(\overline{\mathbb{D}}; \mathbb{C})$ Theorem (2.2.3) shows

$$\varphi(z) = c - Tf(0) - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma(\zeta) - \zeta \Pi f(\zeta) - \bar{\zeta} f(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta}$$

if and only if

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma(\zeta) - \zeta \Pi f(\zeta) - \bar{\zeta} f(\zeta)) \frac{d\zeta}{(1-\bar{z}\zeta)\zeta} = 0.$$

From

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \zeta \Pi f(\zeta) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} =$$

$$= -\frac{1}{\pi} \int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\log(1 - z\bar{\zeta})}{(\zeta - \tilde{\zeta})^2} d\zeta d\tilde{\xi} d\tilde{\eta} =$$

$$= \frac{1}{\pi} \int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\log(1 - z\bar{\zeta})}{(1 - \tilde{\zeta}\bar{\zeta})^2} d\bar{\zeta} d\tilde{\xi} d\tilde{\eta} = 0,$$

and

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Pi f(\zeta) \frac{d\zeta}{1 - \bar{z}\zeta} = -\frac{1}{\pi} \int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{1}{(\zeta - \tilde{\zeta})^2} \frac{d\zeta}{1 - \bar{z}\zeta} d\tilde{\xi} d\tilde{\eta} =$$

$$= -\frac{1}{\pi} \int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \partial_\zeta \frac{1}{1 - \bar{z}\zeta}|_{\zeta=\tilde{\zeta}} d\tilde{\xi} d\tilde{\zeta} = -\frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{z}{(1 - \bar{z}\zeta)^2} d\xi d\eta$$

the result follows.

Theorem 2.3.4 *The problem*

$$w_{\bar{z}} = f \ \text{ in } \mathbb{D}, \quad z w_z = \gamma \ \text{ on } \partial\mathbb{D}, \ w(0) = c,$$

is solvable for $f \in C^\alpha(\bar{\mathbb{D}}; \mathbb{C}), \ 0 < \alpha < 1, \ \gamma \in C(\partial\mathbb{D}; \mathbb{C}), \ c \in C$ *if and only*

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(1 - \bar{z}\zeta)\zeta} + \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{(1 - \bar{z}\zeta)^2} = 0.$$

The solution is then uniquely given as

$$w(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{z}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta(\zeta - z)}. \qquad (2.3.6)$$

A half-Neumann and the Neumann problem for the polyanalytic operator is investigated together with other boundary value problems in [21], [37]. As an example of a Robin boundary value problem, consisting of a linear combination of Dirichlet and Neumann boundary conditions a particular case is investigated in [19].

Theorem 2.3.5 *The Robin problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \ \text{ in } \mathbb{D}, \qquad (2.3.7)$$

$$w + \partial_\nu w = \gamma \ \text{ on } \partial\mathbb{D} \qquad (2.3.8)$$

is uniquely solvable for given $f \in L_1(\mathbb{D}; \mathbb{C}) \cap C(\partial\mathbb{D}; \mathbb{C})$, $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$ *if and only if for all* z, $|z| < 1$,

$$z\Big\{\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \{\gamma(\zeta) - \bar{\zeta} f(\zeta)\} \frac{d\zeta}{1 - \bar{z}\zeta} + \frac{1}{\pi} \int\limits_{\mathbb{D}} \frac{\bar{z}\zeta}{(1 - \bar{z}\zeta)^2} f(\zeta) d\xi d\eta\Big\} = 0, \quad (2.3.9)$$

and the solution is

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\overline{\zeta}f(\zeta) - \gamma(\zeta))\frac{\log(1-z\overline{\zeta})}{z}d\zeta - \frac{1}{\pi}\int\limits_{\mathbb{D}}\frac{1}{\zeta - z}f(\zeta)d\xi d\eta. \quad (2.3.10)$$

Proof. First observe that in the case of the unit disc the boundary condition (2.3.8) takes the form

$$w + zw_z + \overline{z}w_{\overline{z}} = \gamma \quad \text{on} \quad \partial\mathbb{D}. \quad (2.3.11)$$

The general solution of (2.3.7) is given by

$$w = \varphi - \frac{1}{\pi}\int\limits_{\mathbb{D}}\frac{1}{\zeta - z}f(\zeta)d\xi d\eta \quad (2.3.12)$$

for an analytic function φ in the unit disc. Putting the latter and

$$\begin{aligned} w_z &= \varphi_z - \frac{1}{\pi}\int\limits_{\mathbb{D}}\frac{1}{(\zeta - z)^2}f(\zeta)d\xi d\eta, \\ w_{\overline{z}} &= f \end{aligned}$$

in (2.3.11), we get

$$\varphi + z\varphi_z = \gamma + \frac{1}{\pi}\int\limits_{\mathbb{D}}\frac{\zeta}{(z-\zeta)^2}f(\zeta)d\xi d\eta - \overline{z}f(z) =: \tilde{\gamma}, \quad |z| = 1, \quad (2.3.13)$$

in the latter summand meaning the boundary values $f(z) = \lim\limits_{\mathbb{D}\ni\zeta\to z} f(\zeta)$.

Using the Theorem about the Robin boundary value problem for analytic functions, we find the unique φ satisfying (2.3.13) if and only if for $|z| < 1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\tilde{\gamma}(\zeta)\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} = 0, \quad (2.3.14)$$

and φ is given by the integral

$$\varphi(z) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\tilde{\gamma}(\zeta)\frac{\log(1-z\overline{\zeta})}{z}d\zeta,$$

or

$$\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\{\overline{\zeta}f(\zeta) - \gamma(\zeta)\}\frac{\log(1-z\overline{\zeta})}{z}d\zeta \\ &- \frac{1}{\pi}\int\limits_{\mathbb{D}}\left\{\frac{1}{2\pi i}\int\limits_{|s|=1}\frac{\log(1-z\overline{s})}{(s-\zeta)^2}ds\right\}\frac{\zeta}{z}f(\zeta)d\xi d\eta. \end{aligned}$$

The interior integral in the latter summand is equal to

$$-\frac{1}{2\pi i}\int\limits_{|s|=1}\frac{\log(1-z\overline{s})}{(1-\zeta\overline{s})^2}d\overline{s}=\overline{\frac{1}{2\pi i}\int |s|=1\frac{\log(1-\overline{z}s)}{(1-\overline{\zeta}s)^2}ds}=0.$$

Hence from (2.3.12) we get (2.3.10).

On the other hand from (2.3.14), using the relation

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(\zeta-s)^2(1-\overline{z}\zeta)}=\frac{\overline{z}}{(1-\overline{z}s)^2},\quad s\in\mathbb{D},$$

we get (2.3.9). $\square$.

2.4 Harmonic Green and Neumann functions and related boundary value problems for second order equations

Harmonic Green and Neumann functions are classical tools for treating the Dirichlet and the Neumann boundary value problems for the Poisson equation. They both arise in a very natural way in modifying the second order representation (2.1.23). This formula is unsymmetric. A dual formula is

$$w(z)=-\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta-z}}$$

$$-\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}w_\zeta(\zeta)\log|\zeta-z|^2d\zeta+\frac{1}{\pi}\int\limits_{\mathbb{D}}w_{\zeta\bar\zeta}(\zeta)\log|\zeta-z|^2d\xi d\eta \qquad (2.4.1)$$

Combining (2.1.23) and (4.1.1) would give a symmetric representation. But it would not meet the requirement to lead to solutions to proper boundary value problems. As a motivation the simplest case of a harmonic function, i.e. a solution to $w_{z\bar z}=0$, is considered in the unit disc.

Lemma 2.4.1 *The Dirichlet problem $w=0$ on $\partial\mathbb{D}$ for harmonic functions in the unit disc $\mathbb{D}$ is only trivially solvable.*

Proof. From $w_{z\bar z}=0$ the function w_z is seen to be analytic. Integrating w_z shows $w=\varphi+\overline{\psi}$ with analytic functions φ and ψ in $\mathbb{D}$. Without loss of generality $\psi(0)=0$ may be assumed. From the boundary condition $\varphi=-\overline{\psi}$

on $\partial\mathbb{D}$ follows. This is a Dirichlet problem for φ. By **Theorem 2.3.2** it is solvable if and only if

$$0 = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \overline{\psi(\zeta)}\frac{\bar{z}d\zeta}{1-z\zeta} =$$

$$\overline{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \psi(\zeta)\frac{d\zeta}{\zeta - z}} - \overline{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \psi(\zeta)\frac{d\zeta}{\zeta}} = \overline{\psi(z)} - \overline{\psi(0)} = \overline{\psi(z)}.$$

Hence ψ and thus φ vanishes in $\mathbb{D}$, i.e. so does w.

This proof does not require any maximum principle. The lemma shows that the Dirichlet problem for the Poisson equation $w_{z\bar{z}} = f$ is uniquely given. Hence, the terms in (2.1.23) and (4.1.1) with first order derivatives have to be avoided in order to get a representation formula related to the Dirichlet problem. This is done in the particular case of the unit disc. On the basis of the Gauss theorem

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_{\bar{\zeta}}(\zeta)\log|\zeta - z|^2 d\bar{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_{\bar{\zeta}}(\zeta)\log|1 - z\bar{\zeta}|^2 d\bar{\zeta} =$$

$$= -\frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta)\log|1 - z\bar{\zeta}|^2 d\xi d\eta + \frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\bar{\zeta}}(\zeta)\frac{\bar{z}}{1-\bar{z}\zeta}d\xi d\eta$$

and

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\bar{\zeta}}(\zeta)\frac{\bar{z}}{1-\bar{z}\zeta}d\xi d\eta = \frac{1}{\pi}\int\limits_{\mathbb{D}} \partial_{\bar{\zeta}}\left(\frac{\bar{z}w(\zeta)}{1-\bar{z}\zeta}\right)d\xi d\eta = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\bar{z}w(\zeta)}{1-\bar{z}\zeta}d\zeta$$

Inserting this into (2.1.23) gives

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\right)\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta)\log\left|\frac{1 - z\bar{\zeta}}{\zeta - z}\right|^2 d\xi d\eta. \tag{2.4.2}$$

The function

$$G_1(z,\zeta) = \log\left|\frac{1 - z\bar{\zeta}}{\zeta - z}\right|^2$$

is twice the Green function for the unit disc. It is harmonic in both variables with a characteristic logarithmic singularity for $z = \zeta$ and vanishes for one variable on the boundary. As the Poisson kernel appears in the boundary integral (4.1.2) leads to solutions of the Dirichlet problem.

Theorem 2.4.2 *The Dirichlet problem for the Poisson equation in the unit disc*

$$w_{z\bar{z}} = f \ \text{ in } \mathbb{D}, \ \varphi = \gamma \ \text{ on } \partial\mathbb{D}, \ f \in L_1(\mathbb{D};\mathbb{C}), \ \gamma \in C(\partial\mathbb{D};\mathbb{C}),$$

is uniquely solvable. The solution is

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\right)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta) G_1(z,\zeta) d\xi d\eta. \tag{2.4.3}$$

To modify (2.1.9) with respect to the Neumann boundary condition again the Gauss theorem is stressed.

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_\zeta(\zeta)\log|\zeta - z|^2 d\zeta = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_\zeta(\zeta)\log|1 - z\bar{\zeta}|^2 d\zeta =$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta)\log|1 - z\bar{\zeta}|^2 d\xi d\eta - \frac{1}{\pi}\int\limits_{\mathbb{D}} \partial_\zeta\left[w(\zeta)\frac{z}{1 - z\bar{\zeta}}\right] d\xi d\eta =$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta)\log|1 - z\bar{\zeta}|^2 d\xi d\eta + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{z w(\zeta)}{1 - z\bar{\zeta}} d\bar{\zeta} =$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta)\log|1 - z\bar{\zeta}|^2 d\xi d\eta - \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{z}{\zeta - z}\frac{d\zeta}{\zeta}.$$

Combining this with (2.1.9) shows

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\zeta w_\zeta(\zeta) + \bar{\zeta} w_{\bar{\zeta}}(\zeta))\log|\zeta - z|^2\frac{d\zeta}{\zeta} +$$

$$+ \frac{1}{\pi}\int\limits_{\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta)\log|(\zeta - z)(1 - z\bar{\zeta})|^2 d\xi d\eta. \tag{2.4.4}$$

Here the normal derivative of w appears in the second term and

$$N_1(z,\zeta) = -\log|(\zeta - z)(1 - z\bar{\zeta})|^2$$

is twice the negative Neumann function of the unit disc. As the Green function it is harmonic in both variables as long as $z \notin \zeta$ with the characteristic logarithmic singularity. Its normal derivative on the boundary is for, say, $z \in \mathbb{D}$

$$(\zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}})N_1(z,\zeta) = -\frac{\zeta}{\zeta - z} + \frac{\bar{z}\zeta}{1-\bar{z}\zeta} - \frac{\bar{\zeta}}{\overline{\zeta - z}} + \frac{z\bar{\zeta}}{1-z\bar{\zeta}} = -2.$$

The normalization condition is

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} N_1(z,\zeta)\frac{d\zeta}{\zeta} = -\frac{1}{\pi i}\int\limits_{\partial\mathbb{D}} \log|1-z\bar{\zeta}|^2\frac{d\zeta}{\zeta} = 0.$$

Theorem 2.4.3 *The boundary value problem for the Poisson equation in the unit disc*

$$w_{z\bar{z}} = f \;\; in \;\; \mathbb{D}, \;\; w = \gamma_0, \;\; w_z = \gamma_1 \;\; on \;\; \partial\mathbb{D},$$

for $f \in L_p(\mathbb{D};\mathbb{C}), 2 < p, \;\; \gamma)_0, \;\; \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$ *is uniquely solvable if and only if*

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{zd\bar{\zeta}}{1-z\bar{\zeta}} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\log(1-z\bar{\zeta})d\zeta$$

$$= \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\log(1-z\bar{\zeta})d\xi d\eta \qquad (2.4.5)$$

and

$$\frac{\bar{z}}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{1-\bar{z}\zeta} = \frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{1-\bar{z}\zeta}. \qquad (2.4.6)$$

The solution then is

$$w(z) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\log(1-z\bar{\zeta})d\zeta$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)(\log|\zeta - z|^2 - \log(1-\bar{z}\zeta))d\xi d\eta. \qquad (2.4.7)$$

Proof. The system

$$w_z = \omega, \;\; \omega_{\bar{z}} = f \;\; \text{in} \;\; \mathbb{D}, \;\; w = \gamma_0, \;\; \omega = \gamma_1 \;\; \text{on} \;\; \partial\mathbb{D}$$

is uniquely solvable if and only if

$$-\frac{z}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\bar{\zeta}}{1-z\bar{\zeta}} = \frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{1-z\bar{\zeta}},$$

$$\frac{z}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{1-\bar{z}\zeta}=\frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{1-\bar{z}\zeta}.$$

The solution then is

$$w(z)=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta-z}}-\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}},$$

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta-z}.$$

Inserting ω into the first condition gives

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{1-z\bar{\zeta}}=\frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\gamma_1(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|=1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(1-z\bar{\zeta})}d\tilde{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}f(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(1-z\bar{\zeta})}d\tilde{\xi}d\tilde{\eta}$$

with

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{z d\xi d\eta}{(\zeta-\tilde{\zeta})(1-z\bar{\zeta})}=-\log(1-z\bar{\tilde{\zeta}})-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta-\tilde{\zeta}}$$

$$=-\log(1-z\bar{\tilde{\zeta}})+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\bar{\zeta})}{1-\tilde{\zeta}\bar{\zeta}}\frac{d\bar{\zeta}}{\bar{\zeta}}$$

$$=-\log(1-z\bar{\tilde{\zeta}}).$$

Combining the two integral representations for w and ω leads to (4.1.6) as

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}}=-\frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\gamma_1(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(\overline{\zeta-z})}d\tilde{\zeta}$$

$$+\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}f(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(\overline{\zeta-z})}d\tilde{\xi}d\tilde{\eta},$$

where

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\overline{\zeta-z})(\zeta-\tilde{\zeta})}=\log|\tilde{\zeta}-z|^2-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log|\zeta-z|^2\frac{d\zeta}{\zeta-\tilde{\zeta}},$$

$$\log|\tilde{\zeta} - z|^2 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \log(1-\bar{z}\zeta)\frac{d\zeta}{\zeta - \tilde{\zeta}}$$

$$+\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \log(1-\bar{z}\zeta)\frac{d\bar{\zeta}}{\bar{\zeta}(1-\tilde{\zeta}\bar{\zeta})}$$

$$= \log|\zeta - z|^2 - \log(1-\bar{z}\tilde{\zeta}).$$

Remark 2.4.4 *In a similar way the problem*

$$w_{z\bar{z}} = f \ \ in\ \mathbb{D},\ w = \gamma_0,\ w_{\bar{z}} = \gamma_1 \ \ on\ \partial\mathbb{D}$$

with $f \in L_1(\mathbb{D};\mathbb{C})$, γ_0, $\gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$, *can be solved.*

Integral representations may not always be used to solve related boundary value problems as was done in the case of the Dirichlet problem with the formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\right) -$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\bar{\zeta}}(\zeta)\log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right|^2 d\xi d\eta$$

If w *is a solution to the Poisson equation* $w_{z\bar{z}} = f$ *in* $\mathbb{D}$ *satisfying* $\partial_\nu w = \gamma$ *on* $\partial\mathbb{D}$ *and being normalized by*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta} = c$$

for proper f *and* γ *then it may be presented as*

$$w(z) = c - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\log|\zeta - z|^2\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\log|(\zeta - z)(1-z\bar{\zeta})|^2 d\xi d\eta. \tag{2.4.8}$$

But this formula although providing always a solution to $w_{z\bar{z}} = f$ *does not for all* γ *satisfy the respective boundary behavior. Such a behavior is also known from the Cauchy integral.*

Theorem 2.4.5 *The Neumann problem for the Poisson equation in the unit disc*

$$w_{z\bar{z}} = f \ \text{ in } \mathbb{D}, \ \partial_\nu w = \gamma \ \text{ on } \partial\mathbb{D}, \ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta} = c, \tag{2.4.9}$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \ \gamma \in C(\partial\mathbb{D};\mathbb{C}), \ c \in \mathbb{C}$ *is solvable if only if*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{|\zeta|<1} f(\zeta)d\xi d\eta. \tag{2.4.10}$$

The unique solution is then given by (2.4.8).

Proof. As the Neumann function is a fundamental solution to the Laplace operator and the boundary integral is a harmonic function, (2.4.8) provides a solution to the Poisson equation. For checking the boundary behavior the first order derivatives have to be considered. They are

$$w_z(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{(\zeta - z)\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\left(\frac{1}{\zeta - z} + \frac{\bar{\zeta}}{1 - z\bar{\zeta}}\right)d\xi d\eta,$$

$$w_{\bar{z}}(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\overline{(\zeta - z\zeta)}} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\left(\frac{1}{\overline{\zeta - z}} + \frac{\zeta}{1 - \bar{z}\zeta}\right)d\xi d\eta,$$

so that

$$\partial_\nu w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\overline{\zeta - z}} - 1\right)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\left[\frac{z}{\zeta - z} + \frac{\bar{z}}{\overline{\zeta - z}} + \frac{z\bar{\zeta}}{1 - z\bar{\zeta}} + \frac{\bar{z}\zeta}{1 - \bar{z}\zeta}\right]d\xi d\eta$$

$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\overline{\zeta - z}} - 2\right)\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|=1} f(\zeta)\left[2 - \frac{z}{\zeta - z} - \frac{\bar{z}}{\overline{\zeta - z}} - \frac{1}{1 - z\bar{\zeta}} - \frac{1}{1 - \bar{z}\zeta}\right]d\xi d\eta$$

For $|z| = 1$ this is using the property of the Poisson kernel

$$\partial_\nu w(z) = \gamma(z) - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta} + \frac{2}{\pi}\int\limits_{|\zeta|=1} f(\zeta)d\xi d\eta.$$

Therefore $\partial_\nu w = \gamma$ on $|z| = 1$ if and only if condition (2.4.9) holds. At last the normalization condition has to be verified. It follows from $|\zeta - z| = |1 - z\bar{\zeta}|$ for $|z| = 1$ and

$$\frac{1}{2\pi i} \int\limits_{|z|=1} \log |1 - z\bar{\zeta}|^2 \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_{|z|=1} \log(1 - z\bar{\zeta}) \frac{dz}{z} -$$

$$-\frac{1}{2\pi i} \int\limits_{|z|=1} \log(1 - \bar{z}\zeta) \frac{d\bar{z}}{\bar{z}} = 0.$$

Chapter 3

Boundary value problems for the tri-harmonic complex partial differential operator in the unit disc

In order to treat boundary value problems for second order complex partial differential equations special kernel functions have to be constructed. The most important among them are Green and Neumann functions. All of them are certain fundamental solutions to the Laplace operator. The Green and Neumann functions are used to solve the Dirichlet, Neumann boundary value problems for the Poisson equation via corresponding integral representation formulas for solutions.

3.1 Cauchy - Pompeiu representation formulas

Definition 3.1.1 *The function* $G(z,\zeta) = (1\backslash 2)G_1(z,\zeta)$ *with*

$$G_1(z,\zeta) = \log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right|^2, \ z,\zeta \in \mathbb{D}, \ z \neq \zeta, \tag{3.1.1}$$

is called Green function of the Laplace operator for the unit disc.

Remark 3.1.2 *The Green function has the following properties. For any fixed* $\zeta \in \mathbb{D}$ *as a function of* z

(1) $G(z,\zeta)$ *is harmonic in* $\mathbb{D}\backslash\{\zeta\}$,

(2) $G(z,\zeta) + \log|\zeta - z|$ *is harmonic in* $\mathbb{D}$,

(3) $\lim\limits_{z\to t} G(z,\zeta) = 0$ *for all* $t \in \partial\mathbb{D}$,

(4) $G(z,\zeta) = G(\zeta, z)$ *for* $z, \zeta \in \mathbb{D},\ z \neq \zeta$.

Not any domain in the complex plane has a Green function. The existence of the Green function for a given domain $D \subset \mathbb{C}$ can be proved in the case when the Dirichlet problem for harmonic functions is solvable for D $(see, e.g. [7])$ For the unit disc the following result is shown.

Theorem 3.1.3 *Any* $w \in C^2(\mathbb{D};\mathbb{C}) \cap C^1(\bar{\mathbb{D}};\mathbb{C})$ *can be represented as*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} w(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1 \right) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int\limits_{|\zeta|<1} w_{\zeta\bar{\zeta}}(\zeta) G_1(z,\zeta) d\xi d\eta, \tag{3.1.2}$$

where $G_1(z,\zeta)$ *is defined in (3.1.1).*

Formulas (2.1.19) and (2.4.1) are both unsymmetric. Adding both gives some symmetric formula which is for the unit disc

$$\begin{aligned} w(z) &= \frac{1}{4\pi i} \int\limits_{|\zeta|=1} w(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} \right) \frac{d\zeta}{\zeta} \\ &- \frac{1}{4\pi i} \int\limits_{|\zeta|=1} (\zeta w_\zeta(\zeta) + \bar{\zeta} w_{\bar{\zeta}}(\zeta)) \log |\zeta - z|^2 \frac{d\zeta}{\zeta} \\ &+ \frac{1}{\pi} \int\limits_{|\zeta|<1} w_{\zeta\bar{\zeta}}(\zeta) \log |\zeta - z|^2 d\xi d\eta. \end{aligned} \tag{3.1.3}$$

Motivated by the procedure before, the Gauss Theorems are applied in a symmetric way to

$$\begin{aligned} &\frac{1}{\pi} \int\limits_{|\zeta|<1} w_{\zeta\bar{\zeta}}(\zeta) \log |\zeta - z|^2 d\xi d\eta \\ &= \frac{1}{2\pi} \int\limits_{|\zeta|<1} \Big\{ \partial_\zeta [w_{\bar{\zeta}}(\zeta) \log |1 - z\bar{\zeta}|^2] + \partial_{\bar{\zeta}} [w_\zeta(\zeta) \log |1 - z\bar{\zeta}|^2] \\ &+ \partial_{\bar{\zeta}} \left[w(\zeta) \frac{\bar{z}}{1 - \bar{z}\zeta} \right] + \partial_\zeta \left[w(\zeta) \frac{z}{1 - z\bar{\zeta}} \right] \Big\} d\xi d\eta \\ &= \frac{1}{4\pi i} \int\limits_{|\zeta|=1} (\zeta w_\zeta(\zeta) + \bar{\zeta} w_{\bar{\zeta}}(\zeta)) \log |\zeta - z|^2 \frac{d\zeta}{\zeta} \end{aligned}$$

$$+\frac{1}{4\pi i}\int\limits_{|\zeta|=1} w(\zeta)\left[\frac{\bar{z}\zeta}{1-\bar{z}\zeta}+\frac{z\bar{\zeta}}{1-z\bar{\zeta}}\right]\frac{d\zeta}{\zeta}$$

$$=\frac{1}{4\pi i}\int\limits_{|\zeta|=1}\log|\zeta-z|^2[\zeta w_\zeta(\zeta)+\bar{\zeta}w_{\bar{\zeta}}(\zeta)]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{4\pi i}\int\limits_{|\zeta|=1} w(\zeta)\left[\frac{z}{\zeta-z}+\frac{\bar{z}}{\overline{\zeta-z}}\right]\frac{d\zeta}{\zeta}.$$

Here are two possibilities. At first the second term in (3.1.3) can be eliminated giving

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\left(\frac{\zeta}{\zeta-z}+\frac{\bar{\zeta}}{\overline{\zeta-z}}-1\right)\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\bar{\zeta}}(\zeta)\log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2 d\xi d\eta,$$

i.e. (2.1.19).

Theorem 3.1.4 *Any function $w\in C^2(D;\mathbb{C})\cap C^1(\overline{D};\mathbb{C})$ can be represented by*

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(\zeta)\partial_{\bar{\zeta}}G_1(z,\zeta)d\bar{\zeta}-\frac{1}{\pi}\int\limits_{\mathbb{D}}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)G_1(z,\zeta)d\xi d\eta \qquad (3.1.4)$$

Proof. Let $z\in D$ and $\varepsilon>0$ be so small that $\overline{D_\varepsilon(z)}\subset D$,

$$D_\varepsilon(z)=\{\zeta\in\mathbb{C}:|\zeta-z|<\varepsilon\}$$

Let us denote $D_\varepsilon=D\backslash\overline{D_\varepsilon(z)}$ and consider

$$\frac{1}{\pi}\int\limits_{D_\varepsilon}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)G_1(z,\zeta)d\xi d\eta=$$

$$=\frac{1}{\pi}\int\limits_{D_\varepsilon}\left\{\partial_{\bar{\zeta}}\left[\partial_\zeta w(\zeta)G_1(z,\zeta)\right]-\partial_\zeta w(\zeta)\partial_{\bar{\zeta}}G_1(z,\zeta)\right\}d\xi d\eta$$

$$=\frac{1}{\pi}\int\limits_{D_\varepsilon}\left\{\partial_{\bar{\zeta}}\left[\partial_\zeta w(\zeta)G_1(z,\zeta)\right]-\right.$$

$$\left.\partial_\zeta\left[w(\zeta)\partial_{\bar{\zeta}}G_1(z,\zeta)\right]+w(\zeta)\partial_\zeta\partial_{\bar{\zeta}}G_1(z,\zeta)\right\}d\xi d\eta$$

$$= \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}_\varepsilon} \left[\partial_\zeta w(\zeta) G_1(z,\zeta) d\zeta + w(\zeta)\partial_{\bar\zeta} G_1(z,\zeta) d\bar\zeta\right]$$

$$+ \frac{1}{\pi} \int\limits_{D_\varepsilon} w(\zeta)\partial_\zeta \partial_{\bar\zeta} G_1(z,\zeta) d\xi d\eta$$

Introducing polar coordinates $\zeta = z + \varepsilon e^{i\theta}$ leads to

$$\int\limits_{|\zeta - z| = \varepsilon} w(\zeta)\partial_{\bar\zeta} G_1(z,\zeta) d\bar\zeta =$$

$$= i \int\limits_0^{2\pi} w(z + \varepsilon e^{it}) \left[\frac{\varepsilon z e^{-it}}{1 - z(\bar z + \varepsilon e^{-it})} + 1\right] dt$$

It tends to $2\pi i w(z)$ as $\varepsilon \to 0$.

Using this formula and polar coordinates gives

$$\int\limits_{|\zeta - z| = \varepsilon} \partial_\zeta w(\zeta) G_1(z,\zeta) d\zeta =$$

$$= i \int\limits_0^{2\pi} \partial_\zeta w(z + \varepsilon e^{it}) e^{it}\varepsilon \left[\log|1 - z(\bar z + \varepsilon e^{it})|^2 - \log|\varepsilon e^{it}|^2\right] dt$$

which tends to zero as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int\limits_{D_\varepsilon} \partial_\zeta \partial_{\bar\zeta} w(\zeta) G_1(z,\zeta) d\xi d\eta = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} w(\zeta)\partial_{\bar\zeta} G_1(z,\zeta) d\bar\zeta - w(z)$$

This proves (3.1.4).

Definition 3.1.5 *The function* $N(z,\zeta) = (1/2)N_1(z,\zeta)$ *with*

$$N_1(z,\zeta) = -\log|(\zeta - z)(1 - z\bar\zeta)|^2, \; z,\zeta \in \mathbb{D}, \; z \neq \zeta, \tag{3.1.5}$$

is called Neumann function of the Laplace operator for the unit disc.

Remark 3.1.6 *The Neumann function, sometimes [34] also called Green function of second kind or second Green function, has the properties*

(1) $N(z,\zeta)$ *is harmonic in* $z \in \mathbb{D}\backslash\{\zeta\}$,
(2) $N(z,\zeta) + \log|\zeta - z|$ *is harmonic in* $z \in \mathbb{D}$ *for any* $\zeta \in \overline{\mathbb{D}}$,
(3) $\partial_\nu N(z,\zeta) = -2$ *for* $z \in \partial\mathbb{D}, \; \zeta \in \mathbb{D}$,
(4) $N(z,\zeta) = N(\zeta, z)$ *for* $z,\zeta \in \mathbb{D}, \; z \neq \zeta$.
(5) $\frac{1}{2\pi i} \int_{|\zeta|=1} N(z,\zeta) \frac{d\zeta}{\zeta} = 0$.

For convenience sometimes some factors are introduced here. Both functions are related to the fundamental solution of the Laplacian. While the Green function vanishes on the boundary, i.e. for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$, the Neumann function satisfies

$$\partial_{\nu_z} N_1(z,\zeta) = (z\partial_z + \bar{z}\partial_{\bar{z}})N_1(z,\zeta) = -2.$$

Basic for a representation formula in terms of the Laplacian is

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \left\{ \frac{w(\zeta)}{\zeta - z} d\zeta + \log|\zeta - z|^2 w_{\bar{\zeta}}(\zeta) d\bar{\zeta} \right\} + \frac{1}{\pi}\int_{\mathbb{D}} \log|\zeta - z|^2 \partial_\zeta \partial_{\bar{\zeta}} w(\zeta) d\xi d\eta$$

see [7, 9, 20]. By symmetrization this becomes

$$w(z) = \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \left\{ \left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} \right) w(\zeta) - \log|\zeta - z|^2 \partial_{\nu_\zeta} w(\zeta) \right\} \frac{d\zeta}{\zeta}$$

$$+ \frac{1}{\pi}\int_{\mathbb{D}} \log|\zeta - z|^2 \partial_\zeta \partial_{\bar{\zeta}} w(\zeta) d\xi d\eta.$$

Adding

$$\frac{1}{\pi}\int_{\mathbb{D}} \log|1 - z\bar{\zeta}|^2 \partial_\zeta \partial_{\bar{\zeta}} w(\zeta) d\xi d\eta$$

$$= \frac{1}{2\pi}\int_{\mathbb{D}} \left\{ \partial_{\bar{\zeta}} \left[\log|1 - z\bar{\zeta}|^2 \partial_\zeta w(\zeta) \right] + \partial_\zeta \left[\log|1 - z\bar{\zeta}|^2 \partial_{\bar{\zeta}} w(\zeta) \right] \right.$$

$$- \partial_\zeta \left[\left(\partial_{\bar{\zeta}} \log|1 - z\bar{\zeta}|^2 \right) w(\zeta) \right] - \partial_{\bar{\zeta}} \left[\partial_\zeta \log|1 - z\bar{\zeta}|^2 w(\zeta) \right] +$$

$$\left. + 2\partial_\zeta \partial_{\bar{\zeta}} \left(\log|1 - z\bar{\zeta}|^2 \right) w(\zeta) \right\} d\xi d\eta$$

$$= \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \left\{ \log|1 - z\bar{\zeta}|^2 \partial_{\nu_\zeta} w(\zeta) + \left(\frac{\bar{z}\zeta}{1 - \bar{z}\zeta} + \frac{z\bar{\zeta}}{1 - z\bar{\zeta}} \right) w(\zeta) \right\} \frac{d\zeta}{\zeta}$$

leads to

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \left\{ w(\zeta) - \log|\zeta - z|^2 \partial_{\nu_\zeta} w(\zeta) \right\} \frac{d\zeta}{\zeta}$$

$$+ \frac{1}{\pi}\int_{\mathbb{D}} \log|(\zeta - z)(1 - z\bar{\zeta})|^2 \partial_\zeta \partial_{\bar{\zeta}} w(\zeta) d\xi d\eta.$$

This representation can be written as

$$w(z) = -\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \left\{\partial_{\nu_\zeta} N_1(z,\zeta)w(\zeta) - N_1(z,\zeta)\partial_{\nu_\zeta} w(\zeta)\right\}\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} N_1(z,\zeta)\partial_\zeta\partial_{\bar\zeta} w(\zeta)d\xi d\eta. \tag{3.1.6}$$

Theorem 3.1.7 *Any $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \partial_\nu w(\zeta)\log|\zeta - z|^2\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\bar\zeta}(\zeta)N_1(z,\zeta)d\xi d\eta. \tag{3.1.7}$$

This formula can also be written as

$$w(z) = -\frac{1}{4\pi i}\int\limits_{|\zeta|=1} [w(\zeta)\partial_{\nu_\zeta} N_1(z,\zeta) - \partial_\nu w(\zeta)N_1(z,\zeta)]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\bar\zeta}(\zeta)N_1(z,\zeta)d\xi d\eta. \tag{3.1.8}$$

Here the normal derivative appears in the second term while a new kernel function arises in the area integral. Formula (3.1.8) is (3.1.6). A second proof is given for (3.1.8).

Proof. Let $z \in D$ and $\varepsilon > 0$ be so small that $\overline{K_\varepsilon(z)} \subset D$,

$$K_\varepsilon(z) = \{\zeta \in \mathbb{C} : |\zeta - z| < \varepsilon\}.$$

We denote $D_\varepsilon = D\backslash\overline{K_\varepsilon(z)}$ and consider

$$\frac{1}{\pi}\int\limits_{D} N_1(z,\zeta)\partial_\zeta\partial_{\bar\zeta} w(\zeta)d\xi d\eta =$$

$$= \frac{1}{2\pi}\int\limits_{D_\varepsilon} \left[\partial_{\bar\zeta}\left[\partial_\zeta w(\zeta)N_1(z,\zeta)\right] + \partial_\zeta\left[\partial_{\bar\zeta} w(\zeta)N_1(z,\zeta)\right] -\right.$$

$$-\partial_\zeta w(\zeta)\partial_{\bar\zeta} N_1(z,\zeta) - \partial_{\bar\zeta} w(\zeta)\partial_\zeta N_1(z,\zeta)\big]\, d\xi d\eta =$$

$$= \frac{1}{4\pi i}\int\limits_{\partial D_\varepsilon} N_1(z,\zeta)\left[\partial_\zeta w(\zeta)d\zeta - \partial_{\bar\zeta} w(\zeta)d\bar\zeta\right] -$$

$$-\frac{1}{2\pi}\int\limits_{D_\varepsilon}\big[\partial_\zeta\left[w(\zeta)\partial_{\bar\zeta}N_1(z,\zeta)\right] + \partial_{\bar\zeta}\left[w(\zeta)\partial_\zeta N_1(z,\zeta)\right]$$

$$-2\partial_\zeta\partial_{\bar\zeta}N_1(z,\zeta)w(\zeta)\big]\, d\xi d\eta =$$

$$= \frac{1}{4\pi i}\int\limits_{\partial D} N_1(z,\zeta)\left[\partial_\zeta w(\zeta)d\zeta - \partial_{\bar\zeta} w(\zeta)d\bar\zeta\right] -$$

$$-\frac{1}{4\pi i}\int\limits_{|\zeta - z| = \varepsilon} N_1(z,\zeta)\left[\partial_\zeta w(\zeta)d\zeta - \partial_{\bar\zeta} w(\zeta)d\bar\zeta\right] -$$

$$-\frac{1}{4\pi i}\int\limits_{\partial D_\varepsilon} w(\zeta)\left[\partial_\zeta N_1(z,\zeta)d\zeta - \partial_{\bar\zeta} N_1(z,\zeta)d\bar\zeta\right] =$$

$$\frac{1}{4\pi i}\int\limits_{\partial D} N_1(z,\zeta)\left(\zeta\partial_\zeta + \bar\zeta\partial_{\bar\zeta}\right)w(\zeta)\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{4\pi i}\int\limits_{|\zeta - z| = \varepsilon} N_1(z,\zeta)\left[(\zeta - z)w_\zeta + \overline{(\zeta - z)}w_{\bar\zeta}(\zeta)\right]\frac{d\zeta}{\zeta - z} -$$

$$\frac{1}{4\pi i}\int\limits_{\partial D} w(\zeta)\left(\zeta\partial_\zeta + \bar\zeta\partial_{\bar\zeta}\right)N_1(z,\zeta)\frac{d\zeta}{\zeta} +$$

$$+\frac{1}{4\pi i}\int\limits_{|\zeta - z|}\left[(\zeta - z)\partial_\zeta N_1(z,\zeta) + \overline{(\zeta - z)}\partial_{\bar\zeta} N_1(z,\zeta)\right] w(\zeta)\frac{d\zeta}{\zeta - z}.$$

This gives formula (3.1.7), letting ε tend to zero, by the same arguments as have been used in the proof of **Theorem 3.1.4**.

Integral representation formula (3.1.7) is used to solve the Neumann problem for the Poisson equation, see **Theorem 2.4.5**

3.2 A tri-harmonic Green function for the unit disc

A tri-harmonic Green function is constructed in an explicit way for the unit disc of the complex plane by convoluting the harmonic with a bi-harmonic Green function. With this Green function an integral representation formula is developed for the try-harmonic operator. The Green function is the tool to solve the Dirichlet problem for the Poisson equation

$$\partial_z \partial_{\bar{z}} w = f \ \text{ in } \ D, w = \gamma \ \text{ on } \ \partial D,$$

where $f \in L_p(D;\mathbb{C})$, $2 < p$, $\gamma \in C(\partial D;\mathbb{C})$. The solution is unique and given by

$$w(z) = -\frac{1}{4\pi} \int_{\partial D} \partial_{v_\zeta} G_1(z,\zeta)\gamma(\zeta)ds_\zeta - \frac{1}{\pi}\int_D G_1(z,\zeta)f(\zeta)d\xi d\eta, \qquad (3.2.1)$$

where ∂_{v_ζ} denotes the outward normal derivative and s_ζ the arc length parameter on ∂D with respect to the variable ζ. The kernel $-\frac{1}{2}\partial_{v_\zeta} G_1(z,\zeta)$ is the Poisson kernel. In case of the unit disk $D = \mathbb{D} = \{|z| < 1\}$ it is

$$g_1(z,\zeta) = \frac{1}{1 - z\bar{\zeta}} + \frac{1}{1-\bar{z}\zeta} - 1$$

as in that case

$$G_1(z,\zeta) = \log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right|^2.$$

Despite of it's singularity $G_1(z,\zeta)$ can be inserted instead of $f(z)$ in the area integral in (3.2.1). Denoting

$$\widehat{G}_2(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta} \qquad (3.2.2)$$

and comparing this with formula (3.2.1) obviously $\widehat{G}_2(\cdot,\zeta)$ is the solution to the Dirichlet problem

$$\partial_z \partial_{\bar{z}} \widehat{G}_2(z,\zeta) = G_1(z,\zeta) \text{ in } D, \ \widehat{G}_2(z,\zeta) = 0 \text{ on } \partial D$$

for any $\zeta \in D$. That (3.2.2) in fact is the solution to this problem can be shown by considering

$$-\frac{1}{\pi}\int_{D_\varepsilon} G_1(z,\tilde{\zeta})\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}}\widehat{G}_2(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta},$$

where $D_\varepsilon = \{\tilde{\zeta} \in D : \varepsilon_1 < |\tilde{\zeta} - \zeta|, \varepsilon_2 < |\tilde{\zeta} - z|\}$ for small enough, positive $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Applying the Gauss theorem and letting the ε's tend to zero then (3.2.2) follows.

Evaluating (3.2.2) for $D = \mathbb{D}$ shows

$$\widehat{G}_2(z,\zeta) = |\zeta - z|^2 \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 +$$

$$+ \left(1 - |z|^2\right)\left(1 - |\zeta|^2\right) \left[\frac{\log(1 - z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta} \right] \qquad (3.2.3)$$

This biharmonic Green function differs from [3, 4, 6, 7, 8]

$$G_2(z,\zeta) = |\zeta - z|^2 \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 - \left(1 - |z|^2\right)\left(1 - |\zeta|^2\right), \qquad (3.2.4)$$

which is also a biharmonic Green function but not a primitive of $G_1(z,\zeta)$ with respect to the Laplasian $\partial_z \partial_{\bar{z}}$. Both these functions satisfy

• they are biharmonic in $z \in D \backslash \{\zeta\}$ for any $\zeta \in D$,

• adding $|\zeta - z|^2 \log |\zeta - z|^2$ produces a biharmonic function in $z \in D$ for any $\zeta \in D$,

• they are symmetric in z and ζ for $z \neq \zeta$.

However their boundary behaviors differ. While
$\widehat{G}_2(z,\zeta) = 0,\ \partial_z \partial \bar{z} \widehat{G}_2(z,\zeta) = 0$ for $z \in \partial D, \zeta \in D$
instead
$G_2(z,\zeta) = 0,\ \partial_{v_\zeta} G_2(z,\zeta) = 0$ for $z \in \partial D, \zeta \in D$
hold. From

$$\widehat{G}_2(z,\zeta) = -\frac{1}{\pi} \int_D \Big[\log|\tilde{\zeta} - z|^2 \log|\tilde{\zeta} - \zeta|^2 - \log|\tilde{\zeta} - z|^2 h_1(\tilde{\zeta}, \zeta) -$$

$$\log|\tilde{\zeta} - \zeta|^2 h_1(z, \tilde{\zeta}) + h_1(z, \tilde{\zeta}) h_1(\tilde{\zeta}, \zeta) \Big] \, d\tilde{\xi} d\tilde{\eta}$$

it is seen that

$$\widehat{G}_2(z,\zeta) = -|\zeta - z|^2 \log|\zeta - z|^2 + h_2(z,\zeta)$$

with a biharmonic $h_2(z,\zeta)$. This follows because $-\log|\zeta - z|^2$ is a fundamental solution to the Laplace operator $\partial_z \partial_{\bar{z}}$ and

$$\partial_z \partial_{\bar{z}} |\zeta - z|^2 \left[\log|\zeta - z|^2 - 2 \right] = \log|\zeta - z|^2.$$

Hence, $\widehat{G}_2(z,\zeta)$ is a smooth function, moreover it is obviously symmetric. Let

$$\hat{g}_2(z,\zeta) = -\frac{1}{2}\partial_{\nu_\zeta}\widehat{G}_2(z,\zeta) = -\frac{1}{\pi}\int\limits_D G_1(z,\tilde{\zeta})g_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}.$$

For $D = \mathbb{D}$ the $\hat{g}_2$ is explicitly given as

$$\hat{g}_2(z,\zeta) = (1-|z|^2)\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta} + 1\right],$$

see [17]. Proceeding with $\widehat{G}_2(z,\zeta)$ as before with $G_1(z,\zeta)$ leads to

$$\widehat{G}_3(z,\zeta) = -\frac{1}{\pi}\int\limits_D G_1(z,\zeta)\widehat{G}_2(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

For the unit disc it is

$$\widehat{G}_3(z,\zeta) = \frac{1}{4}|\zeta-z|^4\log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2 + h_3(z,\zeta).$$

Applying the Gauss theorems in complex form for a regular domain D and a continuously differentiable function w repeatedly leads to

$$\frac{1}{\pi}\int\limits_D (\partial_\zeta\partial_{\bar{\zeta}})^3 w(\zeta)\widehat{G}_3(z,\zeta)d\xi d\eta = \frac{1}{\pi}\int\limits_D \{\partial_{\bar{\zeta}}\left[\partial_\zeta^3\partial_{\bar{\zeta}}^2 w(\zeta)\widehat{G}_3(z,\zeta)\right] -$$

$$-\partial_\zeta^3\partial_{\bar{\zeta}}^2 w(\zeta)\partial_{\bar{\zeta}}\widehat{G}_3(z,\zeta)\}d\xi d\eta = \frac{1}{2\pi i}\int\limits_{\partial D}\partial_\zeta^3\partial_{\bar{\zeta}}^2 w(\zeta)\widehat{G}_3(z,\zeta)d\zeta -$$

$$-\frac{1}{\pi}\int\limits_D \{\partial_\zeta[\partial_\zeta^2\partial_{\bar{\zeta}}^2 w(\zeta)\partial_{\bar{\zeta}}\widehat{G}_3(z,\zeta)] - \partial_\zeta^2\partial_{\bar{\zeta}}^2 w(\zeta)\partial_\zeta\partial_{\bar{\zeta}}\widehat{G}_3(z,\zeta)\}d\xi d\eta =$$

$$= \frac{1}{2\pi i}\int\limits_{\partial D}\partial_\zeta^2\partial_{\bar{\zeta}}^2 w(\zeta)\partial_{\bar{\zeta}}\widehat{G}_3(z,\zeta)d\bar{\zeta} + \frac{1}{\pi}\int\limits_D \partial_\zeta^2\partial_{\bar{\zeta}}^2 w(\zeta)\widehat{G}_2(z,\zeta)d\xi d\eta =$$

$$= \frac{1}{2\pi i}\int\limits_{\partial D}(\partial_\zeta\partial_{\bar{\zeta}})^2 w(\zeta)\partial_{\bar{\zeta}}\widehat{G}_3(z,\zeta)d\bar{\zeta} + \frac{1}{2\pi i}\int\limits_{\partial D}[g_1(z,\zeta)w(\zeta) +$$

$$+\hat{g}_2(z,\zeta)w_{\zeta\bar{\zeta}}(\zeta)]\frac{d\zeta}{\zeta} - w(\zeta),$$

i.e.

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D}\left[g_1(z,\zeta)w(\zeta) + \hat{g}_2(z,\zeta)w_{\zeta\bar{\zeta}} - \bar{\zeta}\partial_{\bar{\zeta}}\widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^2 w(\zeta)\right]\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{D}\widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^3 w(\zeta)\,d\xi\,d\eta, \tag{3.2.5}$$

and similarly

$$\frac{1}{\pi}\int\limits_{D}(\partial_\zeta\partial_{\bar\zeta})^3 w(\zeta)\widehat{G}_3(z,\zeta)d\xi d\eta = \frac{1}{\pi}\int\limits_{D}\{\partial_\zeta[\partial_\zeta^2\partial_{\bar\zeta}^3 w(\zeta)\widehat{G}_3(z,\zeta)]-$$

$$-\partial_\zeta^2\partial_{\bar\zeta}^3 w(\zeta)\partial_\zeta\widehat{G}_3(z,\zeta)\}d\xi d\eta = -\frac{1}{2\pi i}\int\limits_{\partial D}\partial_\zeta^2\partial_{\bar\zeta}^3 w(\zeta)\widehat{G}_3(z,\zeta)d\bar\zeta-$$

$$-\frac{1}{\pi}\int\limits_{D}\{(\partial_\zeta\partial_{\bar\zeta})^2 w(\zeta)\widehat{G}_3(z,\zeta)\}d\xi d\eta =$$

$$= -\frac{1}{2\pi i}\int\limits_{\partial D}\partial_\zeta^2\partial_{\bar\zeta}^2 w(\zeta)\partial_\zeta\widehat{G}_3(z,\zeta)d\zeta + \frac{1}{\pi}\int\limits_{D}(\partial_\zeta\partial_{\bar\zeta})^2 w(\zeta)\widehat{G}_2(z,\zeta)d\xi d\eta,$$

so that

$$w = \frac{1}{2\pi i}\int\limits_{\partial D}[g_1(z,\zeta)w(\zeta) + \hat{g}_2(z,\zeta)w_{\zeta\bar\zeta}(\zeta) - \zeta\partial_\zeta\widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^2 w(\zeta)]\frac{d\zeta}{\zeta}-$$

$$-\frac{1}{\pi}\int\limits_{D}\widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^3 w(\zeta)d\xi d\eta \tag{3.2.6}$$

From (3.2.5) and (3.2.6) follows

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D}[g_1(z,\zeta)w(\zeta) + \hat{g}_2(z,\zeta)w_{\zeta\bar\zeta}(\zeta)-$$

$$-\frac{1}{2}(\zeta\partial_\zeta + \bar\zeta\partial_{\bar\zeta})\widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^2 w(\zeta)]\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{D}\widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^3 w(\zeta)d\xi d\eta$$

Then using the Cauchy-Pompeiu formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} w_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

results in a representation formula. So, we have

Lemma 3.2.1 *Any $w \in C^4(\mathbb{D}; C) \cap C^2(\bar{\mathbb{D}}; C)$ is representable by*

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} [g_1(z,\zeta)w(\zeta) + \hat{g}_2(z,\zeta)w_{\zeta\bar{\zeta}}(\zeta) + \hat{g}_3(z,\zeta)w_{\zeta\bar{\zeta}\zeta\bar{\zeta}}(\zeta)]\frac{d\zeta}{\zeta} -$$

$$(3.2.7)$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_3(z,\zeta)(\partial_\zeta \partial_{\bar{\zeta}})^3 w(\zeta) d\xi d\eta,$$

where, see [17]

$$g_1(z,\zeta) = \frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta} - 1,$$

$$\hat{g}_2(z,\zeta) = (1 - |z^2|)\left[\frac{1}{z\bar{\zeta}}\log(1 - z\bar{\zeta}) + \frac{1}{\bar{z}\zeta}\log(1 - \bar{z}\zeta) + 1\right],$$

$$\hat{g}_3(z,\zeta) = (1 - |z|^2)\left[\sum_{k=2}^{\infty}\frac{1}{k^2}\left((z\bar{\zeta})^{k-1} + (\bar{z}\zeta)^{k-1}\right) + 1\right] -$$

$$-\frac{1 - |z|^4}{2}\left[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\left((z\bar{\zeta})^{k-1} + (\bar{z}\zeta)^{k-1}\right) + \frac{1}{2}\right],$$

$$\widehat{G}_3(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\zeta)\widehat{G}_2(\tilde{\zeta},\zeta) d\tilde{\xi} d\tilde{\eta}.$$

This representation provides the solution to the Dirichlet problem.

Theorem 3.2.2 *The Dirichlet problem for the inhomogeneous tri-harmonic equation*

$$(\partial_z \partial_{\bar{z}})^3 w = f \quad in \quad \mathbb{D},$$

$$w = \gamma_0, \quad w_{z\bar{z}} = \gamma_1, \quad w_{z\bar{z}z\bar{z}} = \gamma_2 \quad on \quad \partial\mathbb{D}$$

is uniquely solvable for $f \in L_p(\mathbb{D}; C)$, $2 < p$, $\gamma_0, \ \gamma_1, \ \gamma_2 \in C(\partial\mathbb{D}; \mathbb{C})$ by

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} [g_1(z,\zeta)\gamma_0(\zeta) + \hat{g}_2(z,\zeta)\gamma_1(\zeta) + \hat{g}_3(z,\zeta)\gamma_2(\zeta)]\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_3(z,\zeta) f(\zeta) d\xi d\eta$$

The tri-harmonic function has the properties
$\bullet \widehat{G}_3(z,\zeta)$ is tri-harmonic for $z \in \mathbb{D}\backslash\{\zeta\}$

$$\bullet \widehat{G}_3(z,\zeta) = \frac{1}{4}\,|\zeta - z|^4 \log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right|^2 + h_3(z,\zeta) = \frac{1}{4}|\zeta - z|^4 G_1(z\zeta) + h_3(z,\zeta),$$

where $h_3(z,\zeta)$ is tri-harmonic for $z \in D,\ \zeta \in D$

$\bullet \widehat{G}_3(z,\zeta) = 0,\ \partial_z\partial_{\bar{z}}\widehat{G}_3(z,\zeta) = \widehat{G}_2(z,\zeta) = 0,$

$(\partial_z\partial_{\bar{z}})^2\widehat{G}_3(z,\zeta) = \widehat{G}_1(z,\zeta) = 0$ for $z \in \partial D,\ \zeta \in D$

$\bullet \widehat{G}_3(z,\zeta) = \widehat{G}_3(\zeta,z),\ \ z,\zeta \in \mathbb{D},\ z \neq \zeta.$

From these properties follow

$$h_3(z,\zeta) = \widehat{G_3}(z,\zeta) - \frac{1}{4}|\zeta - z|^4 \log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right| = 0 \text{ for } z \in \partial D,\ \zeta \in D$$

$$\partial_z\partial_{\bar{z}}h_3(z,\zeta) = \frac{1}{2}(1-|\zeta|^2)(|\zeta - z|^2 + 1 - |\zeta|^2),$$

$$(\partial_z\partial_{\bar{z}})^2 h_3(z,\zeta) = (1-|\zeta|^2)\left[\frac{1}{1-z\bar{\zeta}} + \frac{1}{1-\bar{z}\zeta}\right] + \frac{1}{2}(1-|\zeta|^2)^2\left[\frac{1}{(1-z\bar{\zeta})^2} + \frac{1}{(1-\bar{z}\zeta)^2}\right].$$

This function can be found by the next result, see [10].

Theorem 3.2.3 *The Dirichlet-2 problem*

$$(\partial_z\partial_{\bar{z}})^2 w = f \ \text{ in } \mathbb{D},\ w = \gamma_0,\ w_{z\bar{z}} = \gamma_2 \ \text{ on } \partial\mathbb{D}$$

is uniquely solvable for $f \in L_p(\mathbb{D};\mathbb{C}),\ 2 < p,\ \gamma_0, \gamma_2 \in C(\partial\mathbb{D};\mathbb{C})$ *by*

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} [g_1(z,\zeta)\gamma_0(\zeta) + \hat{g}_2(z,\zeta)\gamma_2(\zeta)]\,\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int_{\mathbb{D}} \widehat{G}_2(z,\zeta) f(\zeta) d\xi d\eta. \tag{3.2.8}$$

Here

$$\hat{g}_2(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}} G_1(z,\tilde{\zeta}) g_1(\tilde{\zeta},\zeta) d\tilde{\xi} d\tilde{\eta} =$$

$$= (1-|z|^2)\left[\frac{\log(1-z\tilde{\zeta})}{z\bar{\zeta}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta} + 1\right]$$

is satisfying

$$\partial_z\partial_{\bar{z}}\hat{g}_2(z,\zeta) = g_1(z,\zeta) \ \text{ for } z,\zeta \in \mathbb{D},\ \hat{g}_2(z,\zeta) = 0 \ \text{ for } z \in \partial\mathbb{D}, \zeta \in \mathbb{D}.$$

In order to find the tri-harmonic function $h_3(z,\zeta)$ use **Theorem3.2.3**. Then the solution will be found in the form

$$h_3(z,\zeta)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\{g_1(z,\tilde{\zeta})h_3(\tilde{\zeta},\zeta)+\hat{g}_2(z,\tilde{\zeta})\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}}h_3(\tilde{\zeta},\zeta)\}\frac{d\tilde{\zeta}}{\tilde{\zeta}}-$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}}\widehat{G}_2(z,\tilde{\zeta})(\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}})^2h_3(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}=$$

$$=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}g_1(z,\tilde{\zeta})h_3(\tilde{\zeta},\zeta)\frac{d\tilde{\zeta}}{\tilde{\zeta}}+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\hat{g}_2(z,\tilde{\zeta})\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}}h_3(\tilde{\zeta},\zeta)\frac{d\tilde{\zeta}}{\tilde{\zeta}}-$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}}\widehat{G}_2(z,\tilde{\zeta})(\partial_{\tilde{\zeta}}\partial_{\bar{\tilde{\zeta}}})^2h_3(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}=$$

$$=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}(1-|z|^2)\left[\frac{\log(1-z\bar{\tilde{\zeta}})}{z\bar{\tilde{\zeta}}}+\frac{\log(1-\bar{z}\tilde{\zeta})}{\bar{z}\tilde{\zeta}}+1\right]\frac{1}{2}(1-|\zeta|^2)\left[|1-\zeta\bar{\tilde{\zeta}}|^2+\right.$$

$$\left.+1-|\zeta|^2\right]\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\frac{1}{\pi}\int\limits_{\mathbb{D}}\widehat{G}_2(z,\tilde{\zeta})\left[(1-|\zeta|^2)\left(\frac{1}{1-\tilde{\zeta}\bar{\zeta}}+\frac{1}{1-\bar{\tilde{\zeta}}\zeta}\right)+\right.$$

$$\left.+\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\tilde{\zeta}\bar{\zeta})^2}+\frac{1}{(1-\bar{\tilde{\zeta}}\zeta)^2}\right)\right]d\tilde{\xi}d\tilde{\eta},$$

where

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}g_1(z,\tilde{\zeta})h_3(\tilde{\zeta},\zeta)\frac{d\tilde{\zeta}}{\tilde{\zeta}}=0,$$

because $h_3(z,\zeta)=0$ for $z\in\partial\mathbb{D},\zeta\in\mathbb{D}$.

So, we solve the boundary integral

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\left\{(1-|z|^2)\left[\frac{\log(1-z\bar{\tilde{\zeta}})}{z\bar{\tilde{\zeta}}}+\frac{\log(1-\bar{z}\tilde{\zeta})}{\bar{z}\tilde{\zeta}}+1\right]\times\right.$$

$$\left.\times\frac{1}{2}(1-|\bar{\zeta}|^2)\left[|1-\zeta\bar{\tilde{\zeta}}|^2+1-|\zeta|^2\right]\right\}\frac{d\tilde{\zeta}}{\tilde{\zeta}}=$$

$$=\frac{1}{4}\left(1-|z|^2\right)\left(1-|\zeta|^2\right)\left(z\bar{\zeta}+\bar{z}\zeta\right)-\left(1-|z|^2\right)\left(1-|\zeta|^2\right).$$

In the next step, we are solving the area integral

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})\left[(1-|\zeta|^2)\left(\frac{1}{1-\tilde{\zeta}\bar{\zeta}}+\frac{1}{1-\bar{\tilde{\zeta}}\zeta}\right)+\right.$$

$$\left.+\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\tilde{\zeta}\bar{\zeta})^2}+\frac{1}{(1-\bar{\tilde{\zeta}}\zeta)^2}\right)\right] d\tilde{\xi}d\tilde{\eta}$$

In order to evaluate this integral, it is divided into two integrals:

$1. -\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})\left[(1-|\zeta|^2)\left(\frac{1}{1-\tilde{\zeta}\bar{\zeta}}+\frac{1}{1-\bar{\tilde{\zeta}}\zeta}\right)\right] d\tilde{\xi}d\tilde{\eta} =$

$= -\frac{1}{2}(1-|z|^4)(1-|\zeta|^2)\left\{\left(\frac{1}{(z\bar{\zeta})^2}-\frac{1}{z\bar{\zeta}}\right)\log(1-z\bar{\zeta})+\right.$

$\left.+\left(\frac{1}{(\bar{z}\zeta)^2}-\frac{1}{\bar{z}\zeta}\right)\log(1-\bar{z}\zeta)+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right\}+$

$+(1-|z|^2)(1-|\zeta|^2)\sum\limits_{k=1}^{\infty}\frac{1}{k^2}\left[(z\bar{\zeta})^{k-1}+(\bar{z}\zeta)^{k-1}\right];$

$2. -\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})\left[\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\tilde{\zeta}\bar{\zeta})^2}+\frac{1}{(1-\bar{\tilde{\zeta}}\zeta)^2}\right)\right] d\tilde{\xi}d\tilde{\eta} =$

$= \frac{1}{4}(1-|z|^4)\,(1-|\zeta|^2)^2\left[\frac{\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}+\frac{\log(1-\bar{z}\zeta)}{(\bar{z}\zeta)^2}+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right]-$

$-\frac{1}{2}(1-|z|^2)\,(1-|\zeta|^2)^2\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}}+\frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right].$

Hence

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})\left[(1-|\zeta|^2)\left(\frac{1}{1-\tilde{\zeta}\bar{\zeta}}+\frac{1}{1-\bar{\tilde{\zeta}}\zeta}\right)+\right.$$

$$\left.+\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\tilde{\zeta}\bar{\zeta})^2}+\frac{1}{(1-\bar{\tilde{\zeta}}\zeta)^2}\right)\right] d\tilde{\xi}d\tilde{\eta} =$$

$$= -\frac{1}{2}(1-|z|^4)(1-|\zeta|^2)\left\{\left(\frac{1}{(z\bar{\zeta})^2}-\frac{1}{z\bar{\zeta}}\right)\log(1-z\bar{\zeta})+\right.$$

$$\left.+\left(\frac{1}{(\bar{z}\zeta)^2}-\frac{1}{\bar{z}\zeta}\right)\log(1-\bar{z}\zeta)+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right\}+$$

$$+(1-|z|^2)(1-|\zeta|^2)\sum_{k=1}^{\infty}\frac{1}{k^2}\left[(z\bar{\zeta})^{k-1}+(\bar{z}\zeta)^{k-1}\right]+$$

$$+\frac{1}{2}(1-|z|^4)\left(1-|\zeta|^2\right)^2\left[\frac{\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}+\frac{\log(1-\bar{z}\zeta)}{(\bar{z}\zeta)^2}+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right]-$$

$$-(1-|z|^2)\left(1-|\zeta|^2\right)^2\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}}+\frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right]$$

Hence, we get

$$h_3(z,\zeta)=\frac{1}{4}(1-|z|^2)(1-|\zeta|^2)(z\bar{\zeta}+\bar{z}\zeta)-(1-|z|^2)(1-|\zeta|^2)-$$

$$-\frac{1}{2}(1-|z|^4)(1-|\zeta|^2)\left\{\left(\frac{1}{(z\bar{\zeta})^2}-\frac{1}{z\bar{\zeta}}\right)\log(1-z\bar{\zeta})+\right.$$

$$\left.+\left(\frac{1}{(\bar{z}\zeta)^2}-\frac{1}{\bar{z}\zeta}\right)\log(1-\bar{z}\zeta)+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right\}+$$

$$+(1-|z|^2)(1-|\zeta|^2)\sum_{k=1}^{\infty}\frac{1}{k^2}\left[(z\bar{\zeta})^{k-1}+(\bar{z}\zeta)^{k-1}\right]+$$

$$+\frac{1}{4}(1-|z|^4)\left(1-|\zeta|^2\right)^2\left[\frac{\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}+\frac{\log(1-\bar{z}\zeta)}{(\bar{z}\zeta)^2}+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right]-$$

$$-\frac{1}{2}(1-|z|^2)\left(1-|\zeta|^2\right)^2\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}}+\frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right]$$

Simplifying this expression, we get

$$h_3(z,\zeta)=\frac{1}{4}(1-|z|^2)(1-|\zeta|^2)(z\bar{\zeta}+\bar{z}\zeta-4)-$$

$$-\frac{1}{4}(1-|z|^4)(1-|\zeta|^4)\left[\frac{\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}+\frac{\log(1-\bar{z}\zeta)}{(\bar{z}\zeta)^2}+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right]$$

$$+\frac{1}{2}(1-|z|^2)(1-|\zeta|^2)(|z|^2+|\zeta|^2)\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}}+\frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right]$$

$$+(1-|z|^2)(1-|\zeta|^2)\sum_{k=1}^{\infty}\frac{1}{k^2}\left[(z\bar{\zeta})^{k-1}+(\bar{z}\zeta)^{k-1}\right]$$

$$+\frac{1}{4}(1-|z|^4)\left(1-|\zeta|^2\right)^2\left[\frac{\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}+\frac{\log(1-\bar{z}\zeta)}{(\bar{z}\zeta)^2}+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right]-$$

$$-\frac{1}{2}(1-|z|^2)\left(1-|\zeta|^2\right)^2\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}}+\frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right]$$

Thus, we receive the following result

$$\widehat{G}_3(z,\zeta)=\frac{1}{4}|\zeta-z|^4\log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2+h_3(z,\zeta)=$$

$$=\frac{1}{4}|\zeta-z|^4G_1(z,\zeta)+\frac{1}{4}(1-|z|^2)(1-|\zeta|^2)(z\bar{\zeta}+\bar{z}\zeta-4)-$$

$$-\frac{1}{4}(1-|z|^4)(1-|\zeta|^4)\left[\frac{\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}+\frac{\log(1-\bar{z}\zeta)}{(\bar{z}\zeta)^2}+\frac{1}{z\bar{\zeta}}+\frac{1}{\bar{z}\zeta}\right]+$$

$$+\frac{1}{2}(1-|z|^2)(1-|\zeta|^2)(|z|^2+|\zeta|^2)\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}}+\frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right]+$$

$$+(1-|z|^2)(1-|\zeta|^2)\sum_{k=1}^{\infty}\frac{1}{k^2}\left[(z\bar{\zeta})^{k-1}+(\bar{z}\zeta)^{k-1}\right].$$

So, by iteration, using the solution to the Dirichlet problem for harmonic and bi-harmonic functions we received the solution to the Dirichlet problem for the tri-harmonic equation in form the (3.2.7).

3.3 A tri-harmonic Neumann function for the unit disc

While the Green function for the Laplacian of the unit disc is given as

$$G_1(z,\zeta)=\log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2,$$

the Neumann function is

$$N_1(z,\zeta)=-\log|(\zeta-z)(1-z\bar{\zeta})|^2.$$

Both functions are related to the fundamental solution of the Laplacian. While the Green function vanishes on the boundary, i.e. for $z\in\partial\mathbb{D},\zeta\in\mathbb{D}$, the Neumann function satisfies

$$\partial_{\nu_z}N_1(z,\zeta)=(z\partial_z+\bar{z}\partial_{\bar{z}})N_1(z,\zeta)=-2.$$

Neumann boundary conditions are given via outer normal derivatives ∂_ν. For the unit disc this is

$$\partial_\nu=z\partial_z+\bar{z}\partial_{\bar{z}}.$$

Typical for Neumann problems is that they are in general not well-posed. They are neither always solvable nor uniquely solvable. As well solvability conditions have to be determined as normalization conditions to be posed. The bi-harmonic Neumann function has the form, see [25, 26]

$$-N_2(z,\zeta) = |\zeta - z|^2 \left[\log|(\zeta - z)(1 - z\bar{\zeta})|^2 - 4\right] + 4\sum_{k=2}^{+\infty} \frac{(z\bar{\zeta})^k + (\bar{z}\zeta)^k}{k^2} +$$

$$+2\left[z\bar{\zeta} + \bar{z}\zeta\right]\log|1 - z\bar{\zeta}|^2 - (1+|z|^2)(1+|\zeta|^2)\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta}\right]$$

and satisfies the Neumann problem

$$\partial_z\partial_{\bar{z}} N_2(z,\zeta) = N_1(z,\zeta) \text{ in } \mathbb{D} \text{ for fixed } \zeta \in \overline{\mathbb{D}},$$

$$\partial_{\nu_z} N_2(z,\zeta) = 2(1 - |\zeta|^2) \text{ on } \partial\mathbb{D} \text{ for fixed } \zeta \in \overline{\mathbb{D}},$$

and the normalization condition

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} N_2(z,\zeta)\frac{dz}{z} = 0.$$

Moreover, N_2 is symmetric in z and ζ, $N_2(z,\zeta) = N_2(\zeta, z)$.

Theorem 3.3.1 *The Neumann problem*

$$(\partial_z\partial_{\bar{z}})^2 w = f \text{ in } \mathbb{D},\ \partial_\nu w = \gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}} w = \gamma_1 \text{ on } \partial\mathbb{D},$$

$$\frac{1}{2\pi i}\int_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta} = c_0,\ \frac{1}{2\pi i}\int_{|\zeta|=1} w_{\zeta\bar{\zeta}}(\zeta)\frac{d\zeta}{\zeta} = c_1$$

for $f \in L_p(\mathbb{D},\mathbb{C})$, $2 < p$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C}), c_0, c_1 \in \mathbb{C}$ *is uniquely solvable if and only if*

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{\zeta} = 2c_1 - \frac{2}{\pi}\int_{|\zeta|<1} (1 - |\zeta|^2) f(\zeta) d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_1(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{|\zeta|<1} f(\zeta) d\xi d\eta.$$

The solution is given as

$$w(z) = c_0 - (1 - |z|^2)c_1 + \frac{1}{4\pi i}\int_{|\zeta|=1} \{N_1(z,\zeta)\gamma_0(\zeta) + N_2(z,\zeta)\gamma_1(\zeta)\}\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} N_2(z,\zeta)f(\zeta)d\xi d\eta. \tag{3.3.1}$$

Definition 3.3.2 *The Neumann-3 function for the unit disc $\mathbb{D}$ is*

$$N_3(z,\zeta) = -\frac{1}{4}|\zeta - z|^4 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 + n_3(z,\zeta),$$

Where $n_3(z,\zeta)$ is tri-harmonic in both variables with proper boundary behavior. The properties of the third Neumann function are

$$\partial_z\partial_{\bar{z}}N_3(z,\zeta) = N_2(z,\zeta) \ \textit{in} \ \mathbb{D}\backslash\{\zeta\} \ \textit{for} \ \zeta \in \mathbb{D},$$

$$\partial_\nu N_3(z,\zeta) = -\frac{1}{2}(1 - |\zeta|^2)^2 - \frac{1}{2}\partial_\nu N_2(z,\zeta) \ \textit{on} \ \partial\mathbb{D} \ \textit{for} \ \zeta \in \mathbb{D},$$

where

$$\partial_{\nu_z} N_2(z,\zeta) = 2(1 - |\zeta|^2) \ \textit{on} \ \partial\mathbb{D} \ \textit{for} \ \zeta \in \mathbb{D},$$

so that

$$\partial_{\nu_z} N_3(z,\zeta) = -\left[\frac{1}{2}(1 - |\zeta|^2)^2 + (1 - |\zeta|^2)\right],$$

$$\frac{1}{2\pi i}\int\limits_{|z|=1} N_3(z,\zeta)\frac{dz}{z} = 0 \ \textit{for} \ \zeta \in \mathbb{D},$$

$$N_3(z,\zeta) = N_3(\zeta,z) \ \textit{for} \ z,\zeta \in \mathbb{D}.$$

It is important that the normal derivative of $N_3(z,\zeta)$ with respect to z does depend on ζ but not on z. In order to find $N_3(z,\zeta)$ in a proper way some particular Neumann problems are investigated.
Calculating

$$n_3(z,\zeta) = N_3(z,\zeta) + \frac{1}{4}|\zeta - z|^4 \log|(\zeta - z)(1 - z\bar{\zeta})|^2;$$

$$\partial_z n_3(z,\zeta) = \partial_z N_3(z,\zeta) - \frac{1}{2}(\zeta - z)(\overline{\zeta - z})^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 -$$

$$-\frac{1}{4}|\zeta - z|^4\left(\frac{1}{\zeta - z} + \frac{\bar{\zeta}}{1 - z\bar{\zeta}}\right),$$

$$\partial_z\partial_{\bar{z}} n_3(z,\zeta) = N_2(z,\zeta) + |\zeta - z|^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2 + 2|\zeta - z|^2 -$$

$$-\frac{1}{2}|\zeta - z|^2(1 - |\zeta|^2)\left(\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right),$$

$$(\partial_z\partial_{\bar z})^2 n_3(z,\zeta) = 6-(1-|\zeta|^2)\left(\frac{1}{1-z\bar\zeta}+\frac{1}{1-\bar z\zeta}\right)-$$
$$-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\bar z\zeta)^2}+\frac{1}{(1-\bar z\zeta)^2}\right),$$

for $|z|=1$ then

$$\partial_\nu n_3(z,\zeta) = -\frac{1}{2}(1-|\zeta|^2)^2-(1-|\zeta|^2)+$$
$$+\frac{1}{2}|\zeta-z|^2(2-z\bar\zeta-\bar z\zeta)\log|(\zeta-z)(1-z\bar\zeta)|^2+\frac{1}{2}|\zeta-z|^4,$$
$$\partial_\nu\partial_z\partial_{\bar z} n_3(z,\zeta) = 4(2-z\bar\zeta-\bar z\zeta)+2(2-z\bar\zeta-\bar z\zeta)\log|1-z\bar\zeta|^2-(1-|\zeta|^2)+$$
$$+\frac{1}{2}(1-|\zeta|^2)(2-z\bar\zeta-\bar z\zeta)-(1-|\zeta|^2)\left(\frac{1-\bar z\zeta}{1-z\bar\zeta}+\frac{1-z\bar\zeta}{1-\bar z\zeta}\right)$$

follows.

Next the first solvability conditions of **Theorem 3.3.2** is verified.

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\partial_\nu n_3(z,\zeta)\frac{dz}{z} = 2c_1-\frac{2}{\pi}\int\limits_{|z|<1}(1-|z|^2)\partial_z\partial_{\bar z} n_3(z,\zeta)dxdy$$

At the beginning consider the left-hand side

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\partial_\nu n_3(z,\zeta)\frac{dz}{z} = \frac{1}{2\pi i}\int\limits_{|z|=1}\left\{-\frac{1}{2}(1-|\zeta|^2)^2-(1-|\zeta|^2)+\right.$$
$$\left.+\frac{1}{2}|\zeta-z|^2(2-z\bar\zeta-\bar z\zeta)\log|(\zeta-z)(1-z\bar\zeta)|^2+\frac{1}{2}|\zeta-z|^4\right\}\frac{dz}{z} =$$
$$= -\frac{1}{2}(1-|\zeta|^2)^2-(1-|\zeta|^2)+3|\zeta|^2+\frac{1}{2}|\zeta|^4+3|\zeta|^2+\frac{1}{2}|\zeta|^4+1+4|\zeta|^2+|\zeta|^4 =$$
$$= 10|\zeta|^2+|\zeta|^4-1.$$

Also to solve the right-side of the condition, namely:

$$2c_1-\frac{2}{\pi}\int\limits_{|z|<1}(1-|z|^2)\partial_z\partial_{\bar z} n_3(z,\zeta)dxdy$$

where

$$c_1 = \frac{1}{2\pi i}\int\limits_{|z|=1}\partial_z\partial_{\bar z} n_3(z,\zeta)\frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Big\{ N_2(z,\zeta) + |\zeta - z|^2 \log|(\zeta - z)(1 - z\bar{\zeta})|^2$$

$$+2|\zeta - z|^2 - \frac{1}{2}|\zeta - z|^2(1 - |\zeta|^2)\left[\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right]\Big\}\frac{dz}{z} =$$

$$= 4|\zeta|^2 + 2 + 2|\zeta|^2 - \frac{1}{2}(1 - |\zeta|^2) - \frac{1}{2}(1 - |\zeta|^2) = 7|\zeta|^2 + 1$$

and solving separately:

$$\frac{2}{\pi} \int\limits_{|z|<1} (1 - |z|^2)(\partial_z \partial_{\bar{z}})^2 n_3(z,\zeta) dxdy =$$

$$= \frac{2}{\pi} \int\limits_{|z|<1} (1 - |z|^2) \Big\{ 6 - (1 - |\zeta|^2)\left(\frac{1}{1 - z\bar{\zeta}} + \frac{1}{1 - \bar{z}\zeta}\right)$$

$$-\frac{1}{2}(1 - |\zeta|^2)^2 \left(\frac{1}{(1 - \bar{z}\zeta)^2} + \frac{1}{(1 - z\bar{\zeta})^2}\right)\Big\} dxdy =$$

$$= 6 - 2(1 - |\zeta|^2) - (1 - |\zeta|^2)^2 = 3 + 4|\zeta|^2 - |\zeta|^4$$

it follows that

$$2c_1 - \frac{2}{\pi} \int\limits_{|\zeta|<1} (1 - |z|^2)(\partial_z \partial_{\bar{z}})^2 n_3(z,\zeta) d\xi d\eta = 10|\zeta|^2 + |\zeta|^4 - 1.$$

We have proved the validity of the first condition:

$$10|\zeta|^2 + |\zeta|^4 - 1 = 10|\zeta|^2 + |\zeta|^4 - 1.$$

In the next step we verify the second solvability condition of **Theorem 3.3.2**:

$$\frac{1}{2\pi i} \int\limits_{|z|=1} \partial_\nu \partial_z \partial_{\bar{z}} n_3(z,\zeta) \frac{dz}{z} = \frac{2}{\pi} \int\limits_{|z|<1} (\partial_z \partial_{\bar{z}})^2 n_3(z,\zeta) dxdy.$$

The left-hand side is

$$\frac{1}{2\pi i} \int\limits_{|z|=1} \partial_\nu \partial_z \partial_{\bar{z}} n_3(z,\zeta) \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \int\limits_{|z|=1} \Big\{ 4(2 - z\bar{\zeta} - \bar{z}\zeta) + 2(2 - z\bar{\zeta} - \bar{z}\zeta)\log|1 - z\bar{\zeta}|^2 - (1 - |\zeta|^2) +$$

$$+\frac{1}{2}(1-|\zeta|^2)(2-z\bar{\zeta}-\bar{z}\zeta)-(1-|\zeta|^2)\left(\frac{1-\bar{z}\zeta}{1-z\bar{\zeta}}+\frac{1-z\bar{\zeta}}{1-\bar{z}\zeta}\right)\Bigg\}\frac{dz}{z}=$$

$$=8+4|\zeta|^2-(1-|\zeta|^2)^2-(1-|\zeta|^2)(1-|\zeta|^2)=6+8|\zeta|^2-2|\zeta|^4.$$

Then evaluating the right-hand side of the condition shows

$$\frac{2}{\pi}\int\limits_{|\zeta|<1}(\partial_z\partial_{\bar{z}})^2 n_3(z,\zeta)dxdy=$$

$$=\frac{2}{\pi}\int\limits_{|\zeta|<1}\Bigg\{6-(1-|\zeta|^2)\left(\frac{1}{1-z\bar{\zeta}}+\frac{1}{1-\bar{z}\zeta}\right)-$$

$$-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-z\bar{\zeta})^2}+\frac{1}{(1-\bar{z}\zeta)^2}\right)\Bigg\}dxdy=$$

$$=12-4(1-|\zeta|^2)-2(1-|\zeta|^2)^2=8+4|\zeta|^2-2+4|\zeta|^2-2|\zeta|^4=6+8|\zeta|^2-2|\zeta|^4.$$

Hence the second conditions is satisfied, i.e.

$$6+8|\zeta|^2-2|\zeta|^4=6+8|\zeta|^2-2|\zeta|^4$$

In order to find the solution of **Theorem 3.3.2**, we also must calculate

$$c_0=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}n_3(z,\zeta)\frac{d\zeta}{\zeta}=$$

$$=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left\{N_3(z,\zeta)+\frac{1}{4}|\zeta-z|^4\log|(\zeta-z)(1-z\bar{\zeta})|^2\right\}\frac{d\zeta}{\zeta}$$

Evaluating this integral shows $c_0=2|\zeta|^2-\frac{3}{2}|\zeta|^4$.

Thus we have verified all the necessary and sufficient conditions of solvability of **Theorem 3.3.2**.
According to **Theorem 3.3.2** the function $n_3(z,\zeta)$ is given as

$$n_3(z,\zeta)=c_0+(1-|z|^2)c_1-\frac{1}{4\pi i}\int\limits_{|\tilde{\zeta}|=1}\Big\{N_1(z,\tilde{\zeta})\partial_\nu n_3(\tilde{\zeta},\zeta)+$$

$$
+N_2(z,\tilde\zeta)\partial_\nu\partial_{\tilde\zeta}\partial_{\bar{\tilde\zeta}} n_3(\tilde\zeta,\zeta)\Big\} \frac{d\tilde\zeta}{\tilde\zeta} - \frac{1}{\pi} \int\limits_{|\zeta|<1} N_2(z,\tilde\zeta) f(\tilde\zeta) d\tilde\xi d\tilde\eta =
$$

$$
= 2|\zeta|^2 - \frac{3}{2}|\zeta|^4 + (1-|z|^2)(7|\zeta|^2+1) + \frac{1}{4\pi i}\int\limits_{|\tilde\zeta|=1}\Big\{\Big(\log|(\tilde\zeta - z)(1-z\bar{\tilde\zeta})^2|\Big)
$$

$$
\times\Big(-\frac{1}{2}(1-|\zeta|^2)^2 - (1-|\zeta|^2) + \frac{1}{2}|\zeta-\tilde\zeta|^2(2-\zeta\bar{\tilde\zeta}-\bar\zeta\tilde\zeta)\log|(\zeta-\tilde\zeta)(1-\tilde\zeta\bar\zeta)|^2+
$$

$$
+\frac{1}{2}|\zeta-\tilde\zeta|^4\Big)\Big\}\frac{d\tilde\zeta}{\tilde\zeta} + \frac{1}{4\pi i}\int\limits_{|\zeta|=1}\Big\{\Big(|\tilde\zeta - z|^2[\log|(\tilde\zeta-z)(1-z\bar{\tilde\zeta})|^2 - 4]+
$$

$$
+4\sum_{k=2}^{+\infty}\frac{(z\bar{\tilde\zeta})^k+(\bar z\tilde\zeta)^k}{k^2} + 2(z\bar{\tilde\zeta}+\bar z\tilde\zeta)\log|1-z\bar{\tilde\zeta}|^2 - (1+|z|^2)(1+|\zeta|^2)\Big[\frac{\log(1-z\bar{\tilde\zeta})}{z\bar{\tilde\zeta}}+
$$

$$
+\frac{\log(1-\bar z\tilde\zeta)}{\bar z\tilde\zeta}\Big]\Big)\Big(4(2-\tilde\zeta\bar\zeta-\bar{\tilde\zeta}\zeta)+2(2-\tilde\zeta\bar\zeta-\bar{\tilde\zeta})\log|1-\tilde\zeta\bar\zeta|^2-(1-|\zeta|^2)+
$$

$$
+\frac{1}{2}(1-|\zeta|^2)(2-\tilde\zeta\bar\zeta-\bar{\tilde\zeta}\zeta)-(1-|\zeta|^2)\left(\frac{1-\bar{\tilde\zeta}\zeta}{1-\tilde\zeta\bar\zeta}+\frac{1-\tilde\zeta\bar\zeta}{1-\bar{\tilde\zeta}\zeta}\right)\Big)\Big\}\frac{d\tilde\zeta}{\tilde\zeta}-
$$

$$
+\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big\{\Big(\Big(|\tilde\zeta - z|^2[\log|(\tilde\zeta-z)(1-z\bar{\tilde\zeta})|^2 - 4]+4\sum_{k=2}^{+\infty}\frac{(z\bar{\tilde\zeta})^k+(\bar z\tilde\zeta)^k}{k^2}+
$$

$$
+2(z\bar{\tilde\zeta}+\bar z\tilde\zeta)\log|1-z\bar{\tilde\zeta}|^2 - (1+|z|^2)(1+|\zeta|^2)\Big[\frac{\log(1-z\bar{\tilde\zeta})}{z\bar{\tilde\zeta}}+
$$

$$
+\frac{\log(1-\bar z\tilde\zeta)}{\bar z\tilde\zeta}\Big]\Big)(6-(1-|\zeta|^2)\left(\frac{1}{1-\tilde\zeta\bar\zeta}+\frac{1}{1-\bar{\tilde\zeta}\zeta}\right)-
$$

$$
-\frac{1}{2}(1-|\zeta|^2)^2\left(\frac{1}{(1-\bar{\tilde\zeta}\zeta)^2}+\frac{1}{(1-\tilde\zeta\bar\zeta)^2}\right)\Big)\Big\}d\tilde\xi d\tilde\eta.
$$

3.4 Tri-harmonic hybrid Green-Neumann functions for the unit disc

Convolution of the biharmonic Green and the harmonic Neumann functions leads to a hybrid tri-harmonic Green function, also harmonic Green and the

biharmonic Neumann functions lead to a hybrid tri-harmonic Neumann function. Related boundary conditions are of Dirichlet-Neumann and Neumann-Dirichlet type. On the basis of the biharmonic Green function given by Almansi [6] two dual Dirichlet problems and a Dirichlet-Neumann problem arise.
Biharmonic Green and Neumann functions can be easily attained by convolution of the harmonic ones, see [14, 26, 27].

Definition 3.4.1 *Let for* $z, \zeta \in \mathbb{D},\ z \neq \zeta$*,*

$$H_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\tilde{\zeta})N_1(\tilde{\zeta},\zeta)d\tilde{\zeta}d\tilde{\eta}$$

This function is called a hybrid biharmonic Green function.

This hybrid biharmonic Green function is seen to be the solution to the Neumann problem

$$\partial_\zeta\partial_{\bar{\zeta}}H_2(z,\zeta) = G_1(z,\zeta) \text{ in } \mathbb{D},\ \partial_{\nu_\zeta}H_2(z,\zeta) = 2(1-|z|^2),$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} H_2(z,\zeta)\frac{d\zeta}{\zeta} = 0.$$

That the solvability condition (2.4.10) to this problem holds follows from (2.4.3) applied to the function $1-|z|^2$ in $\mathbb{D}$.

The characterizing properties of this function are

- $(\partial_z\partial_{\bar{z}})^2 H_2(z,\zeta) = 0$ for $z \in \mathbb{D}\backslash(\zeta), \zeta \in \mathbb{D}$
- $(\partial_z\partial_{\bar{z}})^2\left[H_2(z,\zeta) + |\zeta-z|^2\log|\zeta-z|^2\right] = 0$ for $z,\zeta \in \mathbb{D}$
- $H_2(z,\zeta) = 0,\ \partial_{\nu_z}\partial_z\partial_{\bar{z}}H_2(z,\zeta) = -2$ for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$
- $(\partial_z\partial_{\bar{z}})^2 H_2(z,\zeta) = 0$ for $\zeta \in \mathbb{D}\backslash(z), z \in \mathbb{D}$
- $(\partial_\zeta\partial_{\bar{\zeta}})^2\left[H_2(z,\zeta) + |\zeta-z|^2\log|\zeta-z|^2\right] = 0$ for $z,\zeta \in \mathbb{D}$
- $\partial_{\nu_\zeta}H_2(z,\zeta) = 2(1-|z|^2),\ \partial_\zeta\partial_{\bar{\zeta}}H_2(z,\zeta) = 0$ for $\zeta \in \partial\mathbb{D}, z \in \mathbb{D}$.

Evaluating the integrals shows

$$H_2(z,\zeta) = -|\zeta-z|^2\log|\zeta-z|^2 -$$

$$-(1-|z|^2)\left[4 + \frac{1-z\bar{\zeta}}{z\bar{\zeta}}\log(1-z\bar{\zeta}) + \frac{1-\bar{z}\zeta}{\bar{z}\zeta}\log(1-\bar{z}\zeta)\right] -$$

$$-\frac{(\zeta-z)(1-z\bar{\zeta})}{z}\log(1-z\bar{\zeta}) - \frac{\overline{(\zeta-z)}(1-\bar{z}\zeta)}{\bar{z}}\log(1-\bar{z}\zeta).$$

For the bi-Laplace operator there exist several different kinds of Green functions as there are more possibilities to prescribe the boundary behavior.

Hybrid Green-Neumann function

Similarly, for arbitrary domains

$$G_1N_1(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\tilde{\zeta})N_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

satisfies for $\zeta \in D$

$$\partial_z\partial_{\bar{z}}G_1N_1(z,\zeta) = N_1(z,\zeta) \text{ for } z \in D,\ G_1N_1(z,\zeta) = 0 \text{ for } z \in \partial D,$$

and hence for any $\zeta \in D$.
•$G_1N_1(z,\zeta)$ is biharmonic in $D\backslash\{\zeta\}$ and continuously differentiable on $\overline{D}\backslash\{\zeta\}$,
•$G_1N_1(z,\zeta) + |\zeta - z|^2 \log|\zeta - z|^2$ is biharmonic in z in the neighborhood of ζ,
•$G_1N_1(z,\zeta) = 0,\ \partial_{\nu_z}\partial_z\partial_{\bar{z}}G_1N_1(z,\zeta) = -\sigma(s)$ for $z = z(s) \in \partial D$.
Obviously, $G_1N_1(z,\zeta)$ is not symmetric in z and ζ. As a function of ζ it satisfies for any $z \in D$ besides the Poisson equation and Neumann boundary condition

$$\partial_\zeta\partial_{\bar{\zeta}}G_1N_1(z,\zeta) = G_1(z,\zeta) \text{ for } \zeta \in D,$$

$$\partial_{\nu_\zeta}G_1N_1(z,\zeta) = \frac{\sigma(s)}{\pi}\int_D G_1(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} \text{ for } \zeta = \zeta(s) \in \partial D,$$

the conditions
• • $G_1N_1(z,\zeta)$ is biharmonic in $D\backslash\{z\}$ and continuously differentiable on $\overline{D}\backslash\{z\}$,
• • $G_1N_1(z,\zeta) + |\zeta - z|^2 log|\zeta - z|^2$ is bi-harmonic in ζ in the neighborhood of z,
• • $\partial_{\nu_\zeta}G_1N_1(z,\zeta) = \frac{\sigma(s)}{\pi}\int_D G_1(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}$,
• • $\partial_\zeta\partial_{\bar{\zeta}}G_1N_1(z,\zeta) = 0$ for $\zeta = \zeta(s) \in \partial D$,
• • $\frac{1}{4\pi}\int_{\partial D} G_1N_1(z,\zeta)ds_\zeta = 0$.
If $N_1G_1(z,\zeta) = -\frac{1}{\pi}\int_D N_1(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$,
then the symmetry of $G_1(z,\zeta)$ and $N_1(z,\zeta)$ gives

$$N_1G_1(z,\zeta) = G_1N_1(z,\zeta) \text{ for } z,\zeta \in D, z \neq \zeta.$$

Some of the boundary value problems for the biharmonic operator can be decomposed into a system of two problems for the Laplace operator. They can be solved by iterating proper samples of formulas (2.5.4) and (2.5.3). Another method is to use partial integration for evaluating the area integral over the product of the bi-Laplasian applied to the unknown function w and the respective kernel function. The simplest way for proving the result below is by verification. However, for the uniqueness of the solutions and the necessity parts if any some extra argumentation is needed, see [15].

Theorem 3.4.2 *The Dirichlet-Neumann problem*

$$(\partial_z\partial_{\bar z})^2 w = f \ \ in \ \mathbb{D}, \ w = \gamma_0, \ \partial_\nu w_{z\bar z} = \gamma_3 \ \ on \ \partial\mathbb{D},$$

$$c_2 = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w_{z\bar z}(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_p(\mathbb{D};\mathbb{C})$, $2 < p$, $\gamma_3 \in C(\partial\mathbb{D};\mathbb{C})$, $c_2 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)d\xi d\eta.$$

The solution is

$$w(z) = (|z|^2-1)c_2 + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} g_1(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta}+$$

$$+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} H_2(z,\zeta)\gamma_3(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} H_2(z,\zeta)f(\zeta)d\xi d\eta.$$

For a proof, see [11].

Theorem 3.4.3 *The Neumann-Dirichlet problem*

$$(\partial_z\partial_{\bar z})^2 w = f \ \ in \ \mathbb{D}, \ \partial_\nu w = \gamma_1, \ w_{z\bar z} = \gamma_2 \ \ on \ \partial\mathbb{D},$$

$$c_0 = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} w(z)\frac{dz}{z}$$

is uniquely solvable for $f \in L_p(\mathbb{D};\mathbb{C})$, $2 < p$, $\gamma_1, \gamma_2 \in C(\partial\mathbb{D}), c_0 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\gamma_1(\zeta)+2\gamma_2(\zeta))\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)(1-|\zeta|^2)d\xi d\eta.$$

The solution is

$$w(z) = c_0 + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} N_1(z,\zeta)\gamma_1(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\zeta)\partial_{\nu_\zeta}H_2(\zeta,z) -$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)H_2(\zeta,z)d\xi d\eta.$$

For a proof, see [11].

Definition 3.4.4 *Let for* $z, \zeta \in \mathbb{D},\ z \neq \zeta,$

$$H_3(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})N_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

This function is called a hybrid tri-harmonic Green function.

This hybrid tri-harmonic Green function is seen to be the solution to the Dirichlet problem

$\partial_z\partial_{\bar{z}}H_3(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\zeta)N_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta} = H_2(z,\zeta),$

$H_2(z,\zeta) = 0$ for $z \in \partial\mathbb{D}$.

Moreover,

$(\partial_z\partial_{\bar{z}})^2H_3(z,\zeta) = \partial_z\partial_{\bar{z}}H_2(z,\zeta) = H_1(z,\zeta),$

$(\partial_z\partial_{\bar{z}})^3H_3(z,\zeta) = 0,$

$\partial_\zeta\partial_{\bar{\zeta}}H_3(z,\zeta) = \widehat{G}_2(z,\zeta),$

$(\partial_\zeta\partial_{\bar{\zeta}})^2H_3(z,\zeta) = G_1(z,\zeta),$

$(\partial_\zeta\partial_{\bar{\zeta}})^3H_3(z,\zeta) = 0.$

The characterizing properties of this function on the boundary are

•$H_3(z,\zeta) = 0$ for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$

•$\partial_z\partial_{\bar{z}}H_3(z,\zeta) = 0$ for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$

•$\partial_{\nu_z}(\partial_z\partial_{\bar{z}})^2H_3(z,\zeta) = \partial_{\nu_z}N_1(z,\zeta)$ for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$

•$\partial_{\nu_\zeta}H_3(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})\partial_{\nu_\zeta}N_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta} =$

$= \frac{2}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}$ for $\zeta \in \partial\mathbb{D},\ z \in \mathbb{D}$

•$\partial_\zeta\partial_{\bar{\zeta}}H_3(z,\zeta) = \widehat{G}_2(z,\zeta) = 0$ for $\zeta \in \partial\mathbb{D},\ z \in \mathbb{D}$

$(\partial_\zeta\partial_{\bar{\zeta}})^2H_3(z,\zeta) = G_1(z,\zeta) = 0$ for $\zeta \in \partial\mathbb{D},\ z \in \mathbb{D}$

Obviously, $H_3(z,\zeta) \neq H_3(\zeta,z)$ is not symmetric in z and ζ.

Theorem 3.4.5 *The Dirichlet-Neumann problem*

$$(\partial_z\partial_{\bar{z}})^3w = f \text{ in } \mathbb{D},\ w = \gamma_0,\ \partial_z\partial_{\bar{z}}w = \gamma_1,\ \partial_\nu\left(\partial_z\partial_{\bar{z}}\right)^2 w = \gamma_2 \text{ on } \partial\mathbb{D},$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \left(\partial_z\partial_{\bar{z}}\right)^2 w(\zeta)\frac{d\zeta}{\zeta} = c_1$$

for $f \in L_p(\mathbb{D};\mathbb{C})$, $2 < p$, $\gamma_0,\gamma_1,\gamma_2 \in C(\partial\mathbb{D};\mathbb{C})$, $c_1 \in \mathbb{C}$ *is uniquely solvable if and only if*

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)d\xi d\eta.$$

The solution is given as

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta) g_1(z,\zeta)\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\zeta)\hat{g}_2(z,\zeta)\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\zeta) H_3(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)H_3(z,\zeta)d\xi d\eta$$

Proof.
Decomposing the Dirichlet-2 problem and Neumann problem into the system

$$(\partial_z\partial_{\bar{z}})^2 w = \omega \text{ in } \mathbb{D},\ w = \gamma_0,\ \partial_z\partial_{\bar{z}} w = \gamma_1 \text{ on } \partial\mathbb{D},$$

having the solution

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \{\gamma_0(\zeta)g_1(z,\zeta) + \gamma_1(\zeta)\widehat{g}_2(z,\zeta)\}\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}} \omega(\zeta)\widehat{G}_2(z,\zeta)d\xi d\eta,$$

and the Neumann problem

$$\partial_z\partial_{\bar{z}}\omega = f \text{ in } \mathbb{D},\ \partial_\nu\omega = \gamma_2,\ \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \omega(\zeta)\frac{d\zeta}{\zeta} = c_1 \text{ on } \partial\mathbb{D},$$

with the solvability condition

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_2(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_{\mathbb{D}} f(\zeta)d\xi d\eta$$

and the solution

$$\omega(\zeta) = c_1 + \frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tilde{\zeta})N_1(\tilde{\zeta},\zeta)\frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi}\int_{\mathbb{D}} f(\tilde{\zeta})N_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

Inserting ω into the solution of the Dirichlet-2 problem,we get

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \{\gamma_0(\zeta)g_1(z,\zeta) + \gamma_1(\zeta)\widehat{g}_2(z,\zeta)\}\frac{d\zeta}{\zeta} - c_1\left(\frac{1}{\pi}\int_{\mathbb{D}} \widehat{G}_2(z,\zeta)d\xi d\eta\right) -$$

$$-\frac{1}{4\pi i}\int_{\partial\mathbb{D}} \gamma_2(\tilde{\zeta})\frac{1}{\pi}\int_{\mathbb{D}} \widehat{G}_2(z,\zeta)N_1(\tilde{\zeta},\zeta)d\xi d\eta\frac{d\tilde{\zeta}}{\tilde{\zeta}}$$

$$+\frac{1}{\pi}\int_{\mathbb{D}} f(\tilde{\zeta})\frac{1}{\pi}\int_{\mathbb{D}} \widehat{G}_2(z,\tilde{\zeta})N_1(\tilde{\zeta},\zeta)d\xi d\eta d\tilde{\xi}d\tilde{\eta} =$$

where

$$\frac{1}{\pi}\int_{\mathbb{D}} \widehat{G}_2(z,\zeta)d\xi d\eta = \frac{1}{4}(1-|z|^2)^2 + \frac{1}{2}(1-|z|^2)$$

Definition 3.4.6 *Let for $z,\zeta \in D,\ z \neq \zeta$,*

$$\tilde{H}_3(z,\zeta) = -\frac{1}{\pi}\int_D N_2(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

This function is also a hybrid tri-harmonic Green function.

The properties are

$\partial_z\partial_{\bar{z}}\tilde{H}_3(z,\zeta) = -\frac{1}{\pi}\int_D N_1(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta} = H_2(\zeta,z)$

$(\partial_z\partial_{\bar{z}})^2\tilde{H}_3(z,\zeta) = G_1(z,\zeta)$

$(\partial_z\partial_{\bar{z}})^3\tilde{H}_3(z,\zeta) = 0$

$\partial_\zeta\partial_{\bar{\zeta}}\tilde{H}_3(z,\zeta) = N_2(z,\zeta)$

$(\partial_\zeta\partial_{\bar{\zeta}})^2\tilde{H}_3(z,\zeta) = N_1(z,\zeta)$

$(\partial_\zeta\partial_{\bar{\zeta}})^3\tilde{H}_3(z,\zeta) = 0$

$\partial_{\nu_z}\tilde{H}_3(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}} \partial_{\nu_z}N_2(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$

$\partial_{\nu_z}\partial_z\partial_{\bar{z}}\tilde{H}_3(z,\zeta) = -\frac{1}{\pi}\int_D \partial_{\nu_z}N_1(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$ for $z \in \partial D, \zeta \in D$

$(\partial_z\partial_{\bar{z}})^2\tilde{H}_3(z,\zeta) = G_1(z,\zeta) = 0$ for $z \in \partial D, \zeta \in D$

$\tilde{H}_3(z,\zeta) = 0$ for $z \in \partial\mathbb{D}, \zeta \in D$

$\partial_{\nu_\zeta}\partial_\zeta\partial_{\bar{\zeta}}\tilde{H}_3(z,\zeta) = \partial_{\nu_\zeta}N_2(z,\zeta)$ for $z \in \partial D, \zeta \in D$

$\partial_{\nu_\zeta}(\partial_\zeta\partial_{\bar{\zeta}})^2\tilde{H}_3(z,\zeta) = \partial_{\nu_\zeta}N_1(z,\zeta)$ for $\zeta \in \partial D, z \in D$

$\tilde{H}_3(z,\zeta) \neq \tilde{H}_3(\zeta,z)$

Theorem 3.4.7 *The Dirichlet-Neumann problem*

$$(\partial_z\partial_{\bar{z}})^3 w = f \ \text{ in } \mathbb{D},\ w = \gamma_0,\ \partial_{\nu_z}\partial_z\partial_{\bar{z}}w = \gamma_1,\ \partial_{\nu_z}(\partial_z\partial_{\bar{z}})^2 w = \gamma_2 \ \text{ on } \partial\mathbb{D},$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \partial_z\partial_{\bar{z}}w(z)\frac{dz}{z} = c_1,\quad \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\partial_z\partial_{\bar{z}})^2 w(z)\frac{dz}{z} = c_2$$

is uniquely solvable for $f \in L_p(\mathbb{D};\mathbb{C}),\ 2 < p,\ \gamma_0,\gamma_1,\gamma_2 \in C(\partial\mathbb{D};\mathbb{C}),\ c_1,c_2 \in \mathbb{C}$ *by*

$$w(z) = -c_1(1-|z|^2) + c_2\left[\left(1-\frac{1}{2}|z|^2\right)^2\right] + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \gamma_0(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)\frac{d\zeta}{\zeta} +$$

$$+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \gamma_1(\zeta)H_2(z,\zeta)\frac{d\zeta}{\zeta} + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\zeta)\tilde{H}_3(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)\tilde{H}_3(z,\zeta)d\xi d\eta$$

Proof.
Rewriting the problem as the system

$$w_{z\bar{z}} = \omega \text{ in } \mathbb{D},\ w = \gamma_0 \text{ on } \partial\mathbb{D},$$
$$(\partial_z\partial_{\bar{z}})^2\omega = f \text{ in } \mathbb{D},\ \partial_\nu\omega = \gamma_1,\ \partial_\nu\partial_z\partial_{\bar{z}}\omega = \gamma_2 \text{ on } \partial\mathbb{D}$$

satisfying

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \omega(\zeta)\frac{d\zeta}{\zeta} = c_1,\ \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \omega_{\zeta\bar{\zeta}}(\zeta)\frac{d\zeta}{\zeta} = c_2,$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_1(\zeta)\frac{d\zeta}{\zeta} = 2c_2 - \frac{2}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)f(\zeta)d\xi d\eta,$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)d\xi d\eta$$

and combining its solutions

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \partial_{\nu_\zeta}G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\zeta)\omega(\zeta)d\xi d\eta$$

and

$$\omega(z) = c_1 - (1-|z|^2)c_2 + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_1(\zeta) + N_2(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} N_2(z,\zeta)f(\zeta)d\xi d\eta$$

$$\omega(\zeta) = c_1 - (1-|z|^2)c_2 - \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{N_1(\zeta,\tilde{\zeta})\gamma_1(\tilde{\zeta}) + N_2(\zeta,\tilde{\zeta})\gamma_2(\tilde{\zeta})\}\frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi}\int\limits_{\mathbb{D}} N_2(\zeta,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}$$

Will prove the result.
Inserting ω into the solution of the Dirichlet problem gives

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \partial_{\nu_\zeta}G_1(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} \big(c_1 - (1-|\zeta|^2)c_2 +$$

$$+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{N_1(\zeta,\tilde{\zeta})\gamma_1(\tilde{\zeta})+N_2(\zeta,\tilde{\zeta})\gamma_2(\tilde{\zeta})\}\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\frac{1}{\pi}\int\limits_{\mathbb{D}}N_2(\zeta,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}\Bigg)G_1(z,\zeta)d\xi d\eta,$$

where

$$\frac{1}{\pi}\int\limits_{\mathbb{D}}G_1(z,\zeta)d\xi d\eta=1-|z|^2,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)G_1(z,\zeta)d\xi d\eta=\left(1-\frac{1}{2}|z|^2\right)^2-\frac{1}{4}$$

then

$$w(z)=-c_1(1-|z|^2)+c_2\left((1-\frac{1}{2}|z|^2)^2-\frac{1}{4}\right)+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_0(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)\frac{d\zeta}{\zeta}+$$

$$+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\gamma_1(\tilde{\zeta})H_2(z,\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}}+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})\tilde{H}_3(z,\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\frac{1}{\pi}\int\limits_{\mathbb{D}}f(\zeta)\tilde{H}_3(z,\zeta)d\xi d\eta$$

Theorem 3.4.8 *The Neumann-Dirichlet problem*

$$(\partial_z\partial_{\bar{z}})^3=f\ \ in\ \mathbb{D},\ \partial_{\nu_z}w=\gamma_1,\ \partial_{\nu_z}\partial_z\partial_{\bar{z}}w=\gamma_2,\ (\partial_z\partial_{\bar{z}})^2w=\gamma_3\ \ on\ \partial\mathbb{D},$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}w(z)\frac{dz}{z}=c_1,\quad\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}w_{z\bar{z}}(z)\frac{dz}{z}=c_2$$

is uniquely solvable for $f\in L_p(\mathbb{D};\mathbb{C})$, $2<p$, $\gamma_1,\gamma_2,\gamma_3\in C(\partial\mathbb{D};\mathbb{C})$, $c_1,c_2\in\mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_3(\zeta)\frac{d\zeta}{\zeta}=2c_2+\frac{1}{\pi}\int\limits_{\mathbb{D}}f(\zeta)\left(2(1-\frac{1}{2}|\zeta|^2)-\frac{1}{2}\right)d\xi d\eta,$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\zeta)\frac{d\zeta}{\zeta}=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_3(\zeta)\frac{d\zeta}{\zeta}-\frac{2}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)f(\zeta)d\xi d\eta$$

by

$$w(z)=c_1-(1-|z|^2)c_2+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_1(\zeta)+N_2(z,\zeta)\gamma_2(\zeta)-$$

$$-\partial_{\nu_\zeta}\tilde{H}_3(z,\zeta)\gamma_3(\zeta)\Big\}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{\mathbb{D}}\tilde{H}_3(z,\zeta)f(\zeta)d\xi d\eta$$

Proof.
The problem is equivalent to the system

$$(\partial_z \partial_{\bar{z}})^2 w = \omega \ \text{ in } \ \mathbb{D}, \ \partial_\nu w = \gamma_1, \ \partial_\nu \partial_z \partial_{\bar{z}} w = \gamma_2 \ \text{ on } \ \partial\mathbb{D}$$

satisfying

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} w(\zeta) \frac{d\zeta}{\zeta} = c_1, \ \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w_{\zeta\bar{\zeta}}(\zeta) \frac{d\zeta}{\zeta} = c_2,$$

and

$$\omega_{z\bar{z}} = f \ \text{ in } \ \mathbb{D}, \ \omega = \gamma_3 \ \text{ on } \ \partial\mathbb{D}$$

Its solutions

$$\omega(z) = -\frac{1}{4\pi i} \int_{\partial\mathbb{D}} \partial_{\nu_\zeta} G_1(z,\zeta)\gamma_3(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int_{\mathbb{D}} G_1(z,\zeta) f(\zeta) d\xi d\eta$$

$$\omega(\zeta) = -\frac{1}{4\pi i} \int_{\partial\mathbb{D}} \partial_{\nu_{\tilde{\zeta}}} G_1(\zeta,\tilde{\zeta}) w(\tilde{\zeta}) \frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi} \int_{\mathbb{D}} G_1(\zeta,\tilde{\zeta}) \partial_{\tilde{\zeta}} \partial_{\bar{\tilde{\zeta}}} w(\tilde{\zeta}) d\tilde{\xi} d\tilde{\eta}$$

and

$$w(z) = c_1 - (1-|z|^2)c_2 - \frac{1}{4\pi i} \int_{\partial\mathbb{D}} \{N_1(z,\zeta)\gamma_1(\zeta) + N_2(z,\zeta)\gamma_2(\zeta)\} \frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi} \int_{\mathbb{D}} N_2(z,\zeta)\omega(\zeta) d\xi d\eta$$

Inserting ω into in the solution of the biharmonic Neumann problem

$$w(z) = c_1 - (1-|z|^2)c_2 + \frac{1}{4\pi i} \int_{\partial\mathbb{D}} \{N_1(z,\zeta)\gamma_1(\zeta) + N_2(z,\zeta)\gamma_2(\zeta)\} \frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi} \int_{\mathbb{D}} N_2(z,\zeta) \left(-\frac{1}{4\pi i} \int_{\partial\mathbb{D}} \partial_{\nu_{\tilde{\zeta}}} G_1(\zeta,\tilde{\zeta})\gamma_3(\tilde{\zeta}) \frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi} \int_{\mathbb{D}} G_1(\zeta,\tilde{\zeta}) f(\tilde{\zeta}) d\tilde{\xi} d\tilde{\eta} \right) d\xi d\eta$$

shows

$$w(z) = c_1 - (1-|z|^2)c_2 + \frac{1}{4\pi i} \int_{\partial\mathbb{D}} \{N_1(z,\zeta)\gamma_1(\zeta) + N_2(z,\zeta)\gamma_2(\zeta)\} \frac{d\zeta}{\zeta} +$$

$$+\frac{1}{4\pi i} \int_{\partial\mathbb{D}} \partial_{\nu_{\tilde{\zeta}}} \tilde{H}_3(z,\tilde{\zeta}) \frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{\pi} \int_{\mathbb{D}} f(\tilde{\zeta}) \tilde{H}_3(z,\tilde{\zeta}) d\xi d\eta$$

3.5 Boundary value problems for tri-harmonic differential equation

Theorem 3.5.1 *The tri-harmonic Dirichlet problem*

$$(\partial_z\partial_{\bar z})^3 w = f \ \text{ in } \mathbb{D},\ w=\gamma_0,\ \partial_z\partial_{\bar z}w=\gamma_1,\ (\partial_z\partial_{\bar z})^2 w=\gamma_2 \ \text{ on } \partial\mathbb{D}.$$

is uniquely solvable for $f\in L_p(\mathbb{D};\mathbb{C}),\ 2<p,\ \gamma_0,\gamma_1,\gamma_2\in C(\partial\mathbb{D};\mathbb{C})$ *by*

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}[g_1(z,\zeta)\gamma_0(\zeta)+\hat g_2(z,\zeta)\gamma_1(\zeta)+\hat g_3(z,\zeta)\gamma_2(\zeta)]\frac{d\zeta}{\zeta}-$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}}\hat G_3(z,\zeta)f(\zeta)d\xi d\eta. \tag{3.5.1}$$

Here

$$\hat g_3(z,\zeta)=-\frac{1}{\pi}\int\limits_{\mathbb{D}}G_1(z,\tilde\zeta)\hat g_2(\tilde\zeta,\zeta)d\tilde\xi d\tilde\eta$$

$$=(1-|z|^2)\left[\sum_{k=2}^{\infty}\frac{1}{k^2}\left((z\bar\zeta)^{k-1}+(\bar z\zeta)^{k-1}\right)+1\right]-$$

$$-\frac{1-|z|^4}{2}\left[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\left((z\bar\zeta)^{k-1}+(\bar z\zeta)^{k-1}\right)+\frac{1}{2}\right]$$

is satisfying, see [18]

$$\partial_z\partial_{\bar z}\hat g_3(z,\zeta)=-\frac{1}{2}\partial_{\nu_\zeta}\partial_z\partial_{\bar z}\hat G_3(z,\zeta)=-\frac{1}{2}\partial_{\nu_\zeta}\hat G_2(z,\zeta)=\hat g_2(z,\zeta)\ \text{ for } z,\zeta\in\mathbb{D},$$

$\hat g_3(z,\zeta)=0$ *for* $z\in\partial\mathbb{D},\zeta\in\mathbb{D}.$

Proof.
Decompose the problem into the system

$$\partial_z\partial_{\bar z}w=\omega \ \text{ in } \mathbb{D},\ w=\gamma_0 \ \text{ on } \partial\mathbb{D},$$

$$(\partial_z\partial_{\bar z})^2\omega=f \ \text{ in } \mathbb{D},\ \omega=\gamma_1,\ \omega_{z\bar z}=\gamma_2 \ \text{ on } \partial\mathbb{D},$$

This system has the unique solution

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}g_1(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{\mathbb{D}}G_1(z,\tilde\zeta)\omega(\tilde\zeta)d\tilde\xi d\tilde\eta$$

$$\omega(\tilde{\zeta}) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \Big(g_1(\tilde{\zeta},\zeta)\gamma_1(\zeta) + \hat{g}_2(\tilde{\zeta},\zeta)\gamma_2(\zeta)\Big)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} G_2(\tilde{\zeta},\zeta)f(\zeta)d\xi d\eta$$

Inserting ω into the formula for w gives (3.5.1).
As $\hat{g}_2$, $\hat{g}_3$ and $\widehat{G}_3$ vanish for $|z| = 1$ from the behavior of the Poisson formula

$$\lim_{z\to\zeta} w(z) = \gamma_0(\zeta)$$

follows for $|\zeta| = 1$. Applying the Laplace operator it is seen that

$$w_{z\bar{z}}(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \{g_1(z,\zeta)\gamma_1(\zeta) + \widehat{g}_2(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}} \widehat{G}_2(z,\zeta)f(\zeta)d\xi d\eta$$

Analogue as before

$$\lim_{z\to\zeta} w_{z\bar{z}}(z) = \gamma_1(\zeta)$$

follows and from

$$w_{z\bar{z}z\bar{z}}(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} g_1(z,\zeta)\gamma_2(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\zeta)f(\zeta)d\xi d\eta$$

$$\lim_{z\to\zeta} w_{z\bar{z}z\bar{z}}(z) = \gamma_2(\zeta)$$

Moreover, finally $(\partial_z\partial_{\bar{z}})^3 w = f$ is seen from the properties of G_1.

Theorem 3.5.2 *The tri-harmonic Neumann problem*

$$(\partial_z\partial_{\bar{z}})^3 w = f \ \textit{ in } \mathbb{D},\ f \in L_p(\mathbb{D};\mathbb{C}),\ 2 < p < +\infty,$$

$$\partial_\nu w = \gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}} w = \gamma_1\ \ \partial_\nu(\partial_z\partial_{\bar{z}})^2 w = \gamma_2 \ \textit{ on } \partial\mathbb{D},\ \gamma_0,\gamma_1,\gamma_2 \in C(\partial\mathbb{D};\mathbb{C}),$$

satisfying

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta} = c_0,\ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \partial_\zeta\partial_{\bar{\zeta}} w(\zeta)\frac{d\zeta}{\zeta} = c_1,\ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\partial_\zeta\partial_{\bar{\zeta}})^2 w(\zeta)\frac{d\zeta}{\zeta} = c_2$$

is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{d\zeta}{\zeta} = 2c_1 - c_2 - \frac{1}{16\pi i}\int\limits_{\partial\mathbb{D}} \gamma_2(\zeta)\frac{d\zeta}{\zeta} + \frac{1}{\pi}\int\limits_{\mathbb{D}} \left((1-|\zeta|^2)^2 - \frac{1}{2}\right) f(\zeta)d\xi d\eta,$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta} = c_1 - 2c_2 - \frac{2}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)f(\zeta)d\xi d\eta$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}}f(\zeta)d\xi d\eta$$

The solution is given as

$$w(z) = c_0 - c_1(1-|z|^2) - c_2\left(\frac{1}{4}(1-|z|^2)^2 + \frac{1}{2}(1-|z|^2)\right) + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_0(\zeta)+$$

$$+ N_2(z,\zeta)\gamma_1(\zeta) + N_3(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{\mathbb{D}}f(\zeta)N_3(z,\zeta)d\xi d\eta$$

Proof.
Rewriting the Neumann-3 problem as the system

$$(\partial_z\partial_{\bar{z}})^2 w = \omega \text{ in } \mathbb{D},\ \partial_\nu w = \gamma_0,\ \partial_\nu\partial_z\partial_{\bar{z}} w = \gamma_1 \text{ on } \partial\mathbb{D},$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}w(\zeta)\frac{d\zeta}{\zeta} = c_0,\ \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\partial_\zeta\partial_{\bar{\zeta}}w(\zeta)\frac{d\zeta}{\zeta} = c_1$$

and

$$\partial_z\partial_{\bar{z}}\omega = f \text{ in } \mathbb{D},\ \partial_\nu\omega = \gamma_2 \text{ on } \partial\mathbb{D},\ \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\omega(\zeta)\frac{d\zeta}{\zeta} = c_2$$

leads to the solvability conditions

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta} = 2c_1 - \frac{2}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)\omega(\zeta)d\xi d\eta \qquad (3.5.2)$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{\mathbb{D}}\omega(\zeta)d\xi d\eta \qquad (3.5.3)$$

The solution then is

$$w(z) = c_0 - (1-|z|^2)c_1 + \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{\gamma_0(\zeta)N_1(z,\zeta) + \gamma_1(\zeta)N_2(z,\zeta)\}\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{D}}\omega(\zeta)N_2(z,\zeta)d\xi d\eta \tag{3.5.4}$$

$$\omega(\zeta)=c_2+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})N_1(\zeta,\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\frac{1}{\pi}\int\limits_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}$$

Inserting ω into the (3.5.2) condition gives

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta}=2c_1-\frac{1}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)\left\{c_2+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})N_1(\zeta,\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}}-\right.$$

$$\left.-\frac{1}{\pi}\int\limits_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})f(\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}\right\}d\xi d\eta$$

with

$$\frac{1}{\pi}\int\limits_{\mathbb{D}}(1-|\zeta|^2)N_1(\zeta,\tilde{\zeta})d\xi d\eta=\frac{1}{2}\left(1-\frac{1}{2}|\zeta|^2\right)^2-\frac{1}{4}.$$

Then inserting ω into (3.5.4) shows

$$w(z)=c_0-(1-|z|^2)c_1-c_2\left(\frac{1}{\pi}\int\limits_{\mathbb{D}}N_2(z,\zeta)d\xi d\eta\right)+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{\gamma_0(\zeta)N_1(z,\zeta)+$$

$$+\gamma_1(\zeta)N_2(z,\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\gamma_2(\tilde{\zeta})\frac{1}{\pi}\int\limits_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})N_2(z,\tilde{\zeta})d\xi d\eta\frac{d\zeta}{\zeta}+$$

$$+\frac{1}{\pi}\int\limits_{\mathbb{D}}f(\tilde{\zeta})\frac{1}{\pi}\int\limits_{\mathbb{D}}N_1(\zeta,\tilde{\zeta})N_2(z,\zeta)d\xi d\eta d\tilde{\xi}d\tilde{\eta}$$

So,we get

$$w(z)=c_0-c_1(1-|z|^2)-c_2\left(\frac{1}{4}(1-|z|^2)^2+\frac{1}{2}(1-|z|^2)\right)+\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\{N_1(z,\zeta)\gamma_0+$$

$$+N_2(z,\zeta)\gamma_1(\zeta)+N_3(z,\zeta)\gamma_2(\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{\mathbb{D}}f(\zeta)N_3(z,\zeta)d\xi d\eta.$$

Chapter 4

Boundary value problems for higher order complex partial differential equations in the unit disc

4.1 Boundary value problems for the inhomogeneous polyanalytic equation

In this section we present how to proceed and what kind of boundary conditions can be posed. However, there is a variety of boundary conditions possible. All kinds of combinations of the three kinds, Schwarz, Dirichlet, Neumann conditions can be posed. As a simple example the Schwarz problem will be studied for the inhomogeneous polyanalytic equation, see [23]. Another possibility is the Neumann problem for the inhomogeneous polyharmonic equation, see [25], [26], and the Dirichlet problem, see [12].

Lemma 4.1.1 *For* $|z| < 1,\ |\tilde{\zeta}| < 1$ *and* $k \in \mathbb{N}_0$

$$\frac{1}{k+1}(\tilde{\zeta} - z + \overline{\tilde{\zeta} - z})^{k+1} = \frac{(-1)^{k+1}}{k+1}(z+\bar{z})^{k+1} -$$

$$-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1}{\zeta}\frac{\zeta+\tilde{\zeta}}{\zeta-\tilde{\zeta}} - \frac{1}{\bar{\zeta}}\frac{1+\tilde{\zeta}\bar{\zeta}}{1-\tilde{\zeta}\bar{\zeta}}\right)(\zeta - z + \overline{\zeta - z})^k d\xi d\eta. \tag{4.1.1}$$

Proof. The function $w(\tilde{\zeta}) = i(\tilde{\zeta} - z + \overline{\tilde{\zeta} - z})^{k+1}/(k+1)$ satisfies the Schwarz condition

$$w_{\bar{\tilde{\zeta}}}(\tilde{\zeta}) = i(\tilde{\zeta}-z+\overline{\tilde{\zeta} - z})^k \text{ in } \mathbb{D},\ \operatorname{Re} w(\tilde{\zeta}) = 0 \text{ on } \partial\mathbb{D},\ \operatorname{Im} w(0) = \frac{(-1)^{k+1}}{k+1}(z+\bar{z})^{k+1},$$

so that according to [13],

$$w(\tilde{\zeta}) = i\frac{(-1)^{k+1}}{k+1}(z+\bar{z})^{k+1} - \frac{i}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1}{\zeta}\frac{\zeta+\tilde{\zeta}}{\zeta-\tilde{\zeta}} - \frac{1}{\bar{\zeta}}\frac{1+\tilde{\zeta}\bar{\zeta}}{1-\tilde{\zeta}\bar{\zeta}}\right)(\zeta-z+\overline{\zeta-z})^k d\xi d\eta$$

This is (4.1.1).

Corollary 4.1.2 *For* $|z| < 1$ *and* $k \in \mathbb{N}_0$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1}{\zeta} - \frac{1}{\bar{\zeta}}\right)(\zeta - z + \overline{\zeta - z})^k d\xi d\eta = 0 \tag{4.1.2}$$

and

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z} - \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)(\zeta - z + \overline{\zeta - z})^k d\xi d\eta = \frac{(-1)^{k+1}}{k+1}(z+\bar{z})^{k+1} \tag{4.1.3}$$

Proof. (4.1.2) and (4.1.3) are particular cases of (4.1.1) for $\tilde{\zeta} = 0$ and $\tilde{\zeta} = z$, respectively.

Theorem 4.1.3 *The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc.*

$$\partial_{\bar{z}}^n = f \ \text{ in } \mathbb{D},\ \operatorname{Re}\, \partial_{\bar{z}}^\nu = \gamma_\nu \ \text{ on } \partial\mathbb{D},\ Im\partial_{\bar{z}}^\nu w(0) = 0,\ 0 \le \nu \le n-1,$$

is uniquely solvable for

$$f \in L_p(\mathbb{D};\mathbb{C}),\ 2 < p,\ \gamma_\nu \in C(\partial\mathbb{D};\mathbb{R}),\ c_\nu \in \mathbb{R},\ 0 \le \nu \le n-1.$$

The solution is

$$w(z) = i\sum_{\nu=0}^{n-1}\frac{c_\nu}{\nu!}(z+\bar{z})^\nu + \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{2\pi i\nu!}\int\limits_{|\zeta|=1}\gamma_\nu(\zeta)\frac{\zeta+z}{\zeta-z}(\zeta - z + \overline{\zeta - z})^\nu\frac{d\zeta}{\zeta}$$

$$+ \frac{(-1)^n}{2\pi(n-1)!}\int\limits_{|\zeta|<1}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)(\zeta - z + \overline{\zeta - z})^{n-1}d\xi d\eta. \tag{4.1.4}$$

Proof. For $n = 1$ formula (4.1.4) is just [13], (4.1.1). Assuming it holds for $n-1$ rather than for n the Schwarz problem is rewritten as the system

$$\partial_{\bar z}^{n-1} w = \omega \ \text{ in } \ \mathbb{D}, \ \text{Re}\ \partial_{\bar z}^{\nu} w = \gamma_\nu \ \text{ on } \ \partial\mathbb{D}, \ \text{Im}\ \partial_{\bar z}^{\nu} w(0) = c_\nu, \ 0 \le \nu \le n-2,$$

$$\omega_{\bar z} = f \ \text{ in } \ \mathbb{D}, \ \text{Re}\ \omega = \gamma_{n-1} \ \text{ on } \ \partial\mathbb{D}, \ \text{Im}\ \omega(0) = c_{n-1},$$

having the solution

$$w(z) = i\sum_{\nu=0}^{n-2} \frac{c_\nu}{\nu!}(z+\bar z)^\nu + \sum_{\nu=0}^{n-2} \frac{(-1)^\nu}{2\pi i \nu!} \int\limits_{|\zeta|=1} \gamma_\nu(\zeta)\frac{\zeta+z}{\zeta-z}(\zeta - z + \overline{\zeta - z})^\nu \frac{d\zeta}{\zeta}$$

$$+\frac{(-1)^n}{2\pi(n-2)!}\int\limits_{|\zeta|<1}\left(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right)(\zeta - z + \overline{\zeta - z})^{n-2} d\xi d\eta,$$

$$\omega(z) = ic_{n-2} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi d\eta.$$

Using

$$\frac{(-1)^{n-1}}{2\pi(n-2)!}\int\limits_{|\zeta|<1}\left(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right)(\zeta - z + \overline{\zeta - z})^{n-2} d\xi d\eta$$

$$= i\frac{c_{n-1}}{(n-1)!}(z+\bar z)_{n-1} + \frac{(-1)^{n-1}}{2\pi i(n-2)!}\int\limits_{|\tilde\zeta|=1}\gamma_{n-1}(\tilde\zeta)$$

$$\times\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{\tilde\zeta+\zeta}}{\overline{\tilde\zeta-\zeta}}\frac{1}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right)(\zeta - z + \overline{\zeta - z})^{n-2} d\xi d\eta \frac{d\tilde\zeta}{\tilde\zeta}$$

$$+\frac{(-1)^n}{2\pi(n-2)!}\int\limits_{|\tilde\zeta|<1}\frac{f(\tilde\zeta)}{\tilde\zeta}$$

$$\times\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{1+\bar\zeta\tilde\zeta}{1-\bar\zeta\tilde\zeta}\frac{1}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right)(\zeta - z + \overline{\zeta - z})^{n-2} d\xi d\eta d\tilde\xi d\tilde\eta$$

$$+\frac{(-1)^n}{2\pi(n-2)!}\int\limits_{|\tilde\zeta|<1}\frac{\overline{f(\tilde\zeta)}}{\bar{\tilde\zeta}}$$

$$\times\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1+\zeta\bar{\bar{\tilde\zeta}}}{1-\zeta\bar{\bar{\tilde\zeta}}}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{\tilde\zeta+\zeta}}{\overline{\tilde\zeta-\zeta}}\frac{1}{\bar{\tilde\zeta}}\frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}}\right)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta d\tilde\xi d\tilde\eta$$

follows. Because

$$\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\bar{\tilde\zeta}\tilde\zeta}{1-\bar{\tilde\zeta}\tilde\zeta}\frac{1}{\bar{\tilde\zeta}}\frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}}$$

$$=-\left(\frac{2\tilde\zeta}{\zeta-\tilde\zeta}+1\right)\left(\frac{2}{\zeta-z}-\frac{1}{\zeta}\right)+\left(\frac{2}{1-\bar{\tilde\zeta}\tilde\zeta}-1\right)\left(\frac{2z}{1-z\bar{\tilde\zeta}}+\frac{1}{\bar{\tilde\zeta}}\right)$$

$$-=\frac{4\tilde\zeta}{\tilde\zeta-z}\left(\frac{1}{\zeta-\tilde\zeta}-\frac{1}{\zeta-z}\right)+\frac{2}{\zeta-\tilde\zeta}-\frac{2}{\zeta}-\frac{2}{\zeta-z}+\frac{1}{\zeta}$$

$$+\frac{4z}{\tilde\zeta-z}\left(\frac{\tilde\zeta}{1-\tilde\zeta\bar{\tilde\zeta}}-\frac{z}{1-z\bar{\tilde\zeta}}\right)+\frac{2\tilde\zeta}{1-\tilde\zeta\bar{\tilde\zeta}}+\frac{2}{\bar{\tilde\zeta}}-\frac{2z}{1-z\bar{\tilde\zeta}}-\frac{1}{\bar{\tilde\zeta}}$$

$$=-2\frac{\tilde\zeta+z}{\tilde\zeta-z}\left(\frac{1}{\zeta-\tilde\zeta}-\frac{1}{\zeta-z}\right)+2\frac{\tilde\zeta+z}{\tilde\zeta-z}\left(\frac{\tilde\zeta}{1-\tilde\zeta\bar{\tilde\zeta}}-\frac{z}{1-z\bar{\tilde\zeta}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\tilde\zeta}}$$

$$=-2\frac{\tilde\zeta+z}{\tilde\zeta-z}\left(\frac{1}{\zeta-\tilde\zeta}-\frac{\tilde\zeta}{1-\tilde\zeta\bar{\tilde\zeta}}-\frac{1}{\zeta-z}+\frac{z}{1-z\bar{\tilde\zeta}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\tilde\zeta}}$$

$$=-\frac{\tilde\zeta+z}{\tilde\zeta-z}\left(\frac{2}{\zeta-\tilde\zeta}-\frac{1}{\zeta}-\frac{2\tilde\zeta}{1-\tilde\zeta\bar{\tilde\zeta}}-\frac{1}{\bar{\tilde\zeta}}-\frac{2}{\zeta-z}+\frac{1}{\zeta}+\frac{2z}{1-z\bar{\tilde\zeta}}+\frac{1}{\bar{\tilde\zeta}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\tilde\zeta}}$$

$$=-\frac{\tilde\zeta+z}{\tilde\zeta-z}\left(\frac{1}{\zeta}\frac{\zeta+\tilde\zeta}{\zeta-\tilde\zeta}-\frac{1}{\bar{\tilde\zeta}}\frac{1+\tilde\zeta\bar{\tilde\zeta}}{1-\tilde\zeta\bar{\tilde\zeta}}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\bar{\tilde\zeta}}\frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\tilde\zeta}}$$

and similarly

$$\frac{1+\zeta\bar{\bar{\tilde\zeta}}}{1-\zeta\bar{\bar{\tilde\zeta}}}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{\tilde\zeta+\zeta}}{\overline{\tilde\zeta-\zeta}}\frac{1}{\bar{\tilde\zeta}}\frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}}$$

$$=\left(\frac{2}{1-\zeta\bar{\bar{\tilde\zeta}}}-1\right)\left(\frac{2}{\zeta-z}-\frac{1}{\zeta}\right)-\left(\frac{2}{\overline{\zeta-\tilde\zeta}}-\frac{1}{\bar{\tilde\zeta}}\right)\left(\frac{2}{1-z\bar{\tilde\zeta}}-1\right)$$

$$=\frac{4}{1-z\bar{\bar{\tilde\zeta}}}\left(\frac{1}{\zeta-z}+\frac{\bar{\bar{\tilde\zeta}}}{1-\zeta\bar{\bar{\tilde\zeta}}}\right)-\left(\frac{2\bar{\bar{\tilde\zeta}}}{1-\zeta\bar{\bar{\tilde\zeta}}}+\frac{2}{\zeta}\right)$$

$$-\frac{2}{\zeta-z}+\frac{1}{\zeta}-\frac{4}{1-z\bar{\bar{\tilde{\zeta}}}}\left(\frac{1}{\overline{\overline{\zeta-\tilde{\zeta}}}}+\frac{z}{1-z\bar{\bar{\zeta}}}\right)+\frac{2z}{1-z\bar{\bar{\zeta}}}+\frac{2}{\bar{\bar{\zeta}}}+\frac{2}{\overline{\overline{\zeta-\tilde{\zeta}}}}-\frac{1}{\bar{\bar{\zeta}}}$$

$$=2\frac{1+z\bar{\bar{\tilde{\zeta}}}}{1-z\bar{\bar{\tilde{\zeta}}}}\left(\frac{1}{\zeta-z}+\frac{\bar{\bar{\tilde{\zeta}}}}{1-\zeta\bar{\bar{\tilde{\zeta}}}}-\frac{1}{\overline{\overline{\zeta-\tilde{\zeta}}}}-\frac{z}{1-z\bar{\bar{\zeta}}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\bar{\zeta}}}$$

$$=-\frac{1+z\bar{\bar{\tilde{\zeta}}}}{1-z\bar{\bar{\tilde{\zeta}}}}\left(\frac{2}{\overline{\overline{\zeta-\tilde{\zeta}}}}-\frac{1}{\bar{\bar{\zeta}}}-\frac{2\bar{\bar{\tilde{\zeta}}}}{1-\zeta\bar{\bar{\tilde{\zeta}}}}-\frac{1}{\zeta}-\frac{2}{\zeta-z}+\frac{1}{\zeta}+\frac{2z}{1-z\bar{\bar{\zeta}}}+\frac{1}{\bar{\bar{\zeta}}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\bar{\zeta}}}$$

$$=-\frac{1+z\bar{\bar{\tilde{\zeta}}}}{1-z\bar{\bar{\tilde{\zeta}}}}\left(\frac{1}{\bar{\bar{\zeta}}}\frac{\overline{\zeta+\tilde{\zeta}}}{\overline{\overline{\zeta-\tilde{\zeta}}}}-\frac{1}{\zeta}\frac{1+\zeta\bar{\bar{\tilde{\zeta}}}}{1-\zeta\bar{\bar{\tilde{\zeta}}}}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\bar{\bar{\zeta}}}\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right)-\frac{1}{\zeta}+\frac{1}{\bar{\bar{\zeta}}}$$

and applying (4.1.1), (4.1.2), and (4.1.3)

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\bar{\zeta}\tilde{\zeta}}{1-\bar{\zeta}\tilde{\zeta}}\frac{1}{\bar{\bar{\zeta}}}\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=-\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1}{\zeta}\frac{\zeta+\tilde{\zeta}}{\zeta-\tilde{\zeta}}-\frac{1}{\bar{\bar{\zeta}}}\frac{1+\tilde{\zeta}\bar{\zeta}}{1-\tilde{\zeta}\bar{\zeta}}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\bar{\bar{\zeta}}}\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\left[\frac{1}{n-1}(\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^{n-1}-\frac{(-1)^{n-1}}{n-1}(z+\bar{z})^{n-1}+\frac{(-1)^{n-1}}{n-1}(z+\bar{z})^{n-1}\right]$$

$$=\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\frac{1}{n-1}(\tilde{\zeta}-z+\tilde{\zeta}-z)^{n-1},$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\bar{\zeta}\tilde{\zeta}}{1-\bar{\zeta}\tilde{\zeta}}\frac{1}{\bar{\bar{\zeta}}}\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=-\frac{1+z\bar{\bar{\tilde{\zeta}}}}{1-z\bar{\bar{\tilde{\zeta}}}}\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left(\frac{1}{\bar{\bar{\zeta}}}\frac{\overline{\zeta+\tilde{\zeta}}}{\overline{\overline{\zeta-\tilde{\zeta}}}}-\frac{1}{\zeta}\frac{1-\bar{\bar{\tilde{\zeta}}}\zeta}{1-\bar{\bar{\tilde{\zeta}}}\zeta}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\bar{\bar{\zeta}}}\frac{1-z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=\frac{1+z\bar{\bar{\tilde{\zeta}}}}{1-z\bar{\bar{\tilde{\zeta}}}}\left[\frac{1}{n-1}(\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^{n-1}-\frac{(-1)^{n-1}}{n-1}(z+\bar{z})^{n-1}+\frac{(-1)^{n-1}}{n-1}(z+\bar{z})^{n-1}\right]$$

$$=\frac{1+z\bar{\bar{\tilde{\zeta}}}}{1-z\bar{\bar{\tilde{\zeta}}}}\frac{1}{n-1}(\tilde{\zeta}-z+\tilde{\zeta}-z)^{n-1},$$

then

$$\frac{(-1)^{n-1}}{2\pi(n-2)!}\int\limits_{|\zeta|<1}\left(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{\omega(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=i\frac{c_{n-1}}{(n-1)!}(z+\bar{z})_{n-1}+\frac{(-1)^{n-1}}{2\pi i(n-2)!}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{\zeta+z}{\zeta-z}(\zeta-z+\overline{\zeta-z})^{n-1}$$

$$+\frac{(-1)^{n}}{2\pi(n-1)!}\int\limits_{|\zeta|<1}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)(\zeta-z+\overline{\zeta-z})^{n-1}d\xi d\eta.$$

This proves formula (4.1.4).

Theorem 4.1.4 *The Dirichlet problem for the inhomogeneous poly-analytic equation in the unit disc*

$$\partial_{\bar{z}}^{n}=f \;\; in \;\; \mathbb{D}, \; \partial_{\bar{z}}^{\nu}=\gamma_\nu \;\; on \;\; \partial\mathbb{D}, \; 0\le\nu\le n-1,$$

is uniquely solvable for $f\in L_p(\mathbb{D};\mathbb{C})$, $2<p$, $\gamma_\nu\in C(\partial\mathbb{D};\mathbb{R})$, $0\le\nu\le n-1$, *if and only if for* $0\le\nu\le n-1$

$$\sum_{\lambda=\nu}^{n-1}\frac{\bar{z}}{2\pi i}\int\limits_{|\zeta|=1}(-1)^{\lambda-\nu}\frac{\gamma_\lambda(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta-z})^{\lambda-\nu}}{(\lambda-\nu)!}d\zeta$$

$$+\frac{(-1)^{n-\nu}\bar{z}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta-z})^{n-1-\nu}}{(n-1-\nu)!}d\xi d\eta=0. \tag{4.1.5}$$

The solution then is

$$w(z)=\sum_{\nu=0}^{n-1}\frac{(-1)^{\nu}}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_\nu(\zeta)}{\nu!}\frac{(\overline{\zeta-z})^{\nu}}{\zeta-z}d\zeta+$$

$$+\frac{(-1)^{n}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{(n-1)!}\frac{(\overline{\zeta-z})^{n-1}}{\zeta-z}d\xi d\eta. \tag{4.1.6}$$

Proof. For $n=1$ condition (4.1.5) coincides with (2.3.2) and (4.1.6) is (2.3.3). Assuming **Theorem 4.1.4** is provided for $n-1$ rather than for n the problem is decomposed into the system

$$\partial_{\bar{z}}^{n-1}w=\omega \text{ in } \mathbb{D}, \; \partial_{\bar{z}}^{\nu}w=\gamma_\nu \text{ on } \partial\mathbb{D}, \; 0\le\nu\le n-2,$$

$$\partial_{\bar z}\omega = f \text{ in } \mathbb{D},\ \partial_{\bar z}\omega = \gamma_{n-1} \text{ on } \partial\mathbb{D},$$

with the solvability conditions (4.1.5) for $0 \le \nu \le n-2$ and ω instead of f together with

$$\frac{\bar z}{2\pi i}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{d\zeta}{1-\bar z\zeta} - \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{1-\bar z\zeta} = 0$$

and the solutions (4.1.6) for $n-1$ instead of n and ω instead of f where

$$\omega(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{\zeta - z}.$$

Then for $0 \le \nu \le n-2$

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\omega(\zeta)}{1-\bar z\zeta}\frac{(\overline{\zeta - z})^{n-2-\nu}}{(n-2-\nu)!}d\xi d\eta$$

$$= \frac{1}{2\pi i}\int\limits_{|\tilde\zeta|=1}\gamma_{n-1}(\tilde\zeta)\psi_\nu(\tilde\zeta, z)d\tilde\zeta - \frac{1}{\pi}\int\limits_{|\tilde\zeta|<1} f(\tilde\zeta)\psi_\nu(\tilde\zeta, z)d\tilde\xi d\tilde\eta,$$

where

$$\psi_\nu(\tilde\zeta, z) = -\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{\zeta - z})^{n-2-\nu}}{(n-2-\nu)!(1-\bar z\zeta)}\frac{d\xi d\eta}{\zeta - \tilde\zeta}$$

$$= \frac{(\overline{\tilde\zeta - z})^{n-1-\nu}}{(n-1-\nu)!(1-\bar z\tilde\zeta)} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu)!(1-\bar z\zeta)}\frac{d\zeta}{\zeta - \tilde\zeta}$$

$$= \frac{(\overline{\tilde\zeta - z})^{n-1-\nu}}{(n-1-\nu)!(1-\bar z\tilde\zeta)}.$$

The last equality holds because

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta - z})^{n-1-\nu}}{(1-\bar z\zeta)(\zeta - \tilde\zeta)}d\zeta = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta - z})^{n-1-\nu}d\bar\zeta}{(\overline{\zeta - z})(1-\tilde\zeta\bar z)}$$

$$= -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{\zeta - z})^{n-2-\nu}}{(1-\tilde\zeta\bar z)}d\bar\zeta = 0.$$

Thus for $0 \le \nu \le n-2$

$$\sum_{\lambda=\nu}^{n-2} \frac{\bar z}{2\pi i} \int\limits_{|\zeta|=1} (-1)^{\lambda-\nu} \frac{\gamma_\lambda(\zeta)}{1-\bar z\zeta} \frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!} d\zeta$$

$$+\frac{(-1)^{n-1-\nu}\bar z}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{1-\bar z\zeta} \frac{(\overline{\zeta - z})^{n-2-\nu}}{(n-2-\nu)!} d\xi d\eta$$

$$= \sum_{\lambda=\nu}^{n-1} \frac{\bar z}{2\pi i} \int\limits_{|\zeta|=1} (-1)^{\lambda-\nu} \frac{\gamma_\lambda(\zeta)}{1-\bar z\zeta} \frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!} d\zeta$$

$$+\frac{(-1)^{n-\nu}\bar z}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{1-\bar z\zeta} \frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu)!} d\xi d\eta = 0.$$

This is (4.1.5). For showing (4.1.6) similarly

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{(\overline{\zeta - z})^{n-2}}{(n-2)!(\zeta - z)} d\xi d\eta$$

$$= \frac{1}{2\pi i} \int\limits_{|\tilde\zeta|=1} \gamma_{n-1}(\tilde\zeta)\psi_{n-1}(\tilde\zeta, z) d\tilde\zeta - \frac{1}{\pi} \int\limits_{|\tilde\zeta|<1} f(\tilde\zeta)\psi_{n-1}(\tilde\zeta, z) d\tilde\xi d\tilde\eta,$$

with

$$\psi_{n-1}(\tilde\zeta, z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{(\overline{\zeta - z})^{n-2}}{(n-2)!(\zeta - z)} \frac{d\xi d\eta}{\zeta - \tilde\zeta}$$

$$= -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{(\overline{\zeta - z})^{n-2}}{(n-2)!(\tilde\zeta - z)} \left(\frac{1}{\zeta - \tilde\zeta} - \frac{1}{\zeta - z} \right) d\xi d\eta = \frac{(\overline{\tilde\zeta - z})^{n-1}}{(n-1)!(\tilde\zeta - z)}$$

$$- \frac{1}{2\pi i (n-1)!(\tilde\zeta - z)} \int\limits_{|\zeta|=1} \left(\frac{(\overline{\zeta - z})^{n-1}}{(\zeta - \tilde\zeta)} - \frac{(\overline{\zeta - z})^{n-1}}{\zeta - z} \right) = \frac{(\overline{\tilde\zeta - z})^{n-1}}{(n-1)!(\tilde\zeta - z)}$$

$$+ \frac{1}{2\pi i (n-1)!(\tilde\zeta - z)} \int\limits_{|\zeta|=1} (\overline{\zeta - z})^{n-1} \left(\frac{1}{1 - \tilde\zeta\bar\zeta} - \frac{1}{1 - z\bar\zeta} \right) \frac{d\bar\zeta}{\bar\zeta} = \frac{(\overline{\tilde\zeta - z})^{n-1}}{(n-1)!(\tilde\zeta - z)}$$

Hence, $w(z)$ is equal to

$$\sum_{\nu=0}^{n-2} \frac{(-1)^\nu}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{(\overline{\zeta - z})^\nu}{\zeta - z} d\zeta + \frac{(-1)^{n-1}}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{(n-2)!} \frac{(\overline{\zeta - z})^{n-2}}{\zeta - z} d\xi d\eta$$

$$= \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{(\overline{\zeta - z})^\nu}{\zeta - z} d\zeta + \frac{(-1)^n}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{(n-1)!} \frac{(\overline{\zeta - z})^{n-1}}{\zeta - z} d\xi d\eta,$$

i.e. (4.1.6) is valid.

4.2 Polyharmonic Green and Neumann functions

As the Green function is used to modify the representation (2.1.9) and (2.1.10) a polyharmonic Green function serves for altering (2.1.24). Such a function is explicitly known [6] for the unit disc. For $\mathbb{D}$ it is [6], [12]

$$(n-1)!^2 G_n(z,\zeta) = |\zeta - z|^{2(n-1)} \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2 +$$

$$+ \sum_{\nu=1}^{n-1} \frac{(-1)^\nu}{\nu} |\zeta - z|^{2(n-1-\nu)} (1 - |z|^2)^\nu (1 - |\zeta|^2)^\nu, \tag{4.2.1}$$

while for the upper half plane $\mathbb{H}$ [35]

$$(n-1)!^2 G_n(z,\zeta) = |\zeta - z|^{2(n-1)} \log \left| \frac{\bar{\zeta} - z}{\zeta - z} \right|^2$$

$$+ \sum_{\nu=1}^{n-1} \frac{1}{\nu} |\zeta - z|^{2(n-1-\nu)} (z - \bar{z})^\nu (\zeta - \bar{\zeta})^\nu. \tag{4.2.2}$$

Introducing G_n into the representation (2.1.24) leads to

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} \Bigg\{ \left[g_1(z,\zeta) + \sum_{\nu=1}^{n-1} \frac{\bar{z}\zeta}{(1 - \bar{z}\zeta)^{\nu+1}} (1 - |z|^2)^\nu \right] w(\zeta) \frac{d\zeta}{\zeta}$$

$$+ \sum_{\nu=1}^{n-1} \frac{1}{\nu} g_\nu(z,\zeta)(1 - |z|^2)^\nu \partial_{\bar{\zeta}} w(\zeta) d\bar{\zeta}$$

$$+ \sum_{\mu=1}^{[\frac{n-1}{2}]} \sum_{\nu=2\mu}^{n-1} \sum_{\lambda=0}^{\nu-2\mu} \frac{1}{\mu!(\mu+1)!} \frac{\binom{\nu-\mu-1-\lambda}{\mu-1}}{\binom{\nu}{\mu+1}}$$

$$\times \left[g_{\lambda+1}(z,\zeta) + \lambda \frac{\bar{z}\zeta}{(1 - \bar{z}\zeta)^{\lambda+1}} \right] (1 - |z|^2)^\nu \partial_\zeta^\mu \partial_{\bar{\zeta}}^\mu w(\zeta) \frac{d\zeta}{\zeta}$$

$$+\sum_{\mu=1}^{[\frac{n}{2}]-1}\sum_{\nu=2\mu}^{n-1}\sum_{\lambda=0}^{\nu-2\mu}\frac{1}{\mu!(\mu+1)!}\frac{\binom{\nu-\mu-1-\lambda}{\mu-1}}{\binom{\nu}{\mu+1}}$$

$$\times g_\lambda(z,\zeta)(1-|z|^2)^\nu\partial_\zeta^\mu\partial_{\bar\zeta}^{\mu+1}w(\zeta)d\bar\zeta\Big\}-$$

$$-\frac{1}{\pi}\int_{\mathbb{D}}G_n(z,\zeta)\partial_\zeta^n\partial_{\bar\zeta}^n w(\zeta)d\xi d\eta, \tag{4.2.3}$$

where for $\nu\in\mathbb{N}$

$$g_\nu(z,\zeta)=\frac{1}{(1-z\bar\zeta)^\nu}+\frac{1}{(1-\bar z\zeta)^\nu}-1.$$

This representation formula corresponds to the Dirichlet problem

$$(\partial_z\partial_{\bar z})^n w=f \text{ in } \mathbb{D},\ (\partial_z\partial_{\bar z})^\mu w=\gamma_\mu \text{ on } \partial\mathbb{D},\ 0\le 2\mu\le n-1,$$

$$\partial_z^\mu\partial_{\bar z}^{\mu+1}w=\hat\gamma_\mu \text{ on } \partial\mathbb{D},\ 0\le 2\mu\le n-2. \tag{4.2.4}$$

A dual problem is

$$(\partial_z\partial_{\bar z})^n w=f \text{ in } \mathbb{D},\ (\partial_z\partial_{\bar z})^\mu w=\gamma_\mu \text{ on } \partial\mathbb{D},\ 0\le 2\mu\le n-1,$$

$$\partial_z^{\mu+1}\partial_{\bar z}^{\mu}w=\hat\gamma_\mu \text{ on } \partial\mathbb{D},\ 0\le 2\mu\le n-2. \tag{4.2.5}$$

and a combination of both

$$(\partial_z\partial_{\bar z})^n w=f \text{ in } \mathbb{D},\ (\partial_z\partial_{\bar z})^\mu w=\gamma_\mu \text{ on } \partial\mathbb{D},\ 0\le 2\mu\le n-1,$$

$$(\partial_z+\partial_{\bar z})(\partial_z\partial_{\bar z})^\mu=\hat\gamma_\mu \text{ on } \partial\mathbb{D},\ 0\le 2\mu\le n-2. \tag{4.2.6}$$

Another Dirichlet problem

$$(\partial_z\partial_{\bar z})^n w=f \text{ in } \mathbb{D},\ (z\partial_z+\bar z\partial_{\bar z})^\nu w=\gamma_\nu \text{ on } \partial\mathbb{D},\ 0\le\nu\le n-1. \tag{4.2.7}$$

The Dirichlet problem related to the representation (4.2.1) is treated in [27] and in the case of the upper half plane $\mathbb{H}$ in [35]. For the case of the unit disc the result it as follows.

Theorem 4.2.1 *The unique solution to the Dirichlet problem (4.2.6) for $f\in L_1(\mathbb{D};\mathbb{C})$, $\gamma_\mu\in C^{n-2\mu}(\mathbb{D};\mathbb{C})$, $\hat\gamma_\mu\in C^{n-1-2\mu}(\mathbb{D};\mathbb{C})$ in the distributional sense is given as*

$$w(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\Bigg\{\Bigg[g_1(z,\zeta)+\sum_{\nu=1}^{n-1}\frac{\bar z\zeta}{(1-\bar z\zeta)^{\nu+1}}(1-|z|^2)^\nu\Bigg]\gamma_0(\zeta)\frac{d\zeta}{\zeta}$$

$$+\sum_{\nu=1}^{n-1}\frac{1}{\nu}g_\nu(z,\zeta)(1-|z|^2)^\nu\hat{\gamma}_0(\zeta)d\bar{\zeta}$$

$$+\sum_{\mu=1}^{[\frac{n-1}{2}]}\sum_{\nu=2\mu}^{n-1}\sum_{\lambda=0}^{\nu-2\mu}\frac{1}{\mu!(\mu+1)!}\frac{\binom{\nu-\mu-1-\lambda}{\mu-1}}{\binom{\nu}{\mu+1}}$$

$$\times\left[g_{\lambda+1}(z,\zeta)+\lambda\frac{\bar{z}\zeta}{(1-\bar{z}\zeta)^{\lambda+1}}\right](1-|z|^2)^\nu\gamma_\mu(\zeta)\frac{d\zeta}{\zeta}$$

$$+\sum_{\mu=1}^{[\frac{n}{2}]-1}\sum_{\nu=2\mu}^{n-1}\sum_{\lambda=0}^{\nu-2\mu}\frac{1}{\mu!(\mu+1)!}\frac{\binom{\nu-\mu-1-\lambda}{\mu-1}}{\binom{\nu}{\mu+1}}$$

$$\times g_\lambda(z,\zeta)(1-|z|^2)^\nu\hat{\gamma}_\mu(\zeta)d\bar{\zeta}\}-\frac{1}{\pi}\int_{\mathbb{D}}G_n(z,\zeta)f(\zeta)d\xi d\eta. \qquad (4.2.8)$$

Besides these Dirichlet problems there is another Dirichlet problem for the polyharmonic operator arising from iterating the Dirichlet problem for the Laplacian. It is

$$(\partial_z\partial_{\bar{z}})^n w=f \ \text{ in } \mathbb{D},\ (\partial_z\partial_{\bar{z}})^\nu w=\gamma_\nu \ \text{ on } \partial\mathbb{D},\ 0\le\nu\le n-1. \qquad (4.2.9)$$

As in the preceding problem here are again n boundary conditions posed. This problem is equivalent to the system

$$\partial_z\partial_{\bar{z}}w=w_{\nu+1} \ \text{ in } \mathbb{D},\ w_\nu=\gamma_{\nu-1} \ \text{ on } \partial\mathbb{D},\ 0\le\nu\le n-1,$$

with $w_1=w,\ w_{n+1}=f$. Hence it is unconditionally and uniquely solvable. Composing the solutions to these Dirichlet problems to the Poisson equations gives the unique solution for the polyharmonic equation. It is

$$w(z)=\sum_{\nu-1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{\nu-1}(\zeta)\tilde{g}_\nu(z,\zeta)d\zeta-\frac{1}{\pi}\int_{\mathbb{D}}\tilde{G}_n(z,\zeta)f(\zeta)d\xi d\eta, \qquad (4.2.10)$$

where for $2\le\nu$

$$\tilde{g}_\nu(z,\zeta)=-\frac{1}{\pi}\int_{\mathbb{D}}\tilde{G}_{\nu-1}(z,\tilde{\zeta})g_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

with the Poisson kernel

$$g_1(z,\zeta)=\frac{\zeta}{\zeta-z}+\frac{\bar{\zeta}}{\overline{\zeta-z}}-1$$

and

$$\tilde{G}_\nu(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} \tilde{G}_{\nu-1}(z,\tilde{\zeta})G_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

with the harmonic Green function $\tilde{G}_1(z,\zeta) = G_1(z,\zeta)$.

The higher order Poisson kernels $\widehat{g}_\nu(z,\zeta)$ are investigated in the thesis [33]

Theorem 4.2.2 *The Dirichlet problem*

$$(\partial_z\partial_{\bar{z}})^\nu w = f \;\text{ in } \mathbb{D},\; (\partial_z\partial_{\bar{z}})^\nu w = \gamma_\nu \;\text{ on } \partial\mathbb{D},\; 0 \le \nu \le n-1$$

is uniquely solvable in the weak sense for

$$f \in L_p(\mathbb{D};\mathbb{C}),\, 2 < p,\; \gamma_\nu \in C(\partial\mathbb{D};\mathbb{C}),\; 0 \le \nu \le n-1.$$

The solution is given by (4.2.10).

Neither the higher order Green functions $\tilde{G}_n$ iterated convolutions of the Green function G_1, nor the convolutions of the Poisson kernel with these higher order Green functions are calculated in general. The first ones are [14], [17]

$$\tilde{G}_2(z,\zeta) = |\zeta - z|\log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right|^2$$
$$-(1-|z|^2)(1-|\zeta|^2)\left[\frac{1}{z\bar{\zeta}}\log(1-z\bar{\zeta}) + \frac{1}{\bar{z}\zeta}\log(1-\bar{z}\zeta)\right],$$
$$\tilde{g}_2(z,\zeta) = (1-|z|^2)\left[\frac{1}{z\bar{\zeta}}\log(1-z\bar{\zeta}) + \frac{1}{\bar{z}\zeta}\log(1-\bar{z}\zeta) + 1\right].$$

Other Dirichlet problems can be posed by combining the conditions in (4.2.4), (4.2.5), (4.2.7). The larger n is the more Dirichlet problems can be formulated and hence, the related polyharmonic Green functions exist.

Iterating the harmonic Neumann function $N_1(z,\zeta)$ in the same way leads to higher order (polyharmonic) Neumann functions

$$N_n(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} N_{n-1}(z,\tilde{\zeta})N_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}.$$

It has the properties

$$\partial_z\partial_{\bar{z}}N_n(z,\zeta) = N_{n-1}(z,\zeta) \;\text{ in } \mathbb{D},$$

$$\partial_{\nu_z}N_n(z,\zeta) = -\frac{2}{(n-1)!^2}(|\zeta|^2-1)^{n-1}$$

$$-\sum_{\mu=\left[\frac{n}{2}\right]} \frac{\mu!^2}{(n-1)!(n-1-\mu)!^2(2\mu-n+1)!}\partial_{\nu_z}N_{\mu+1}(z,\zeta) \ \ on\ \partial\mathbb{D}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} N_n(z,\zeta)\frac{dz}{z}=0$$

for $2 \le n,\ \gamma \in \mathbb{D}$, *and thus solves itself a Neumann problem. It also satisfies the higher order Neumann problem*

$$(\partial_z\partial_{\bar{z}})^{n-1}N_n(z,\zeta)=N_1(z,\zeta) \ \ in\ \mathbb{D},$$

$$\partial^{\sigma}_{\nu_z}N_n(z,\zeta)=\frac{2}{\pi}\int_{\mathbb{D}} N_{n-\sigma}(z,\zeta)dxdy \ \ on\ \partial\mathbb{D},\ 1\le\sigma\le n-1,$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} N_n(z,\zeta)\frac{dz}{z}=0,$$

for $2 \le n$ *and arbitrary* $\zeta \in \mathbb{D}$. *Using* **Theorem 2.4.5** *a generalization of this result follows inductively.*

Theorem 4.2.3 *Any* $w \in C^{2n}(\mathbb{D};\mathbb{C}) \cap C^{2n-1}(\bar{\mathbb{D}};\mathbb{C})$ *can be represented by*

$$w(z)=-\sum_{\mu=0}^{n-1}\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\{\partial_{\nu_\zeta}N_{\mu+1}(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^{\mu}w(\zeta)$$

$$-N_{\mu+1}(z,\zeta)\partial_\nu(\partial_\zeta\partial_{\bar{\zeta}})^{\mu}w(\zeta)\}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int_{\mathbb{D}} N_\mu(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^{\mu}w(\zeta)d\xi d\eta$$

This representation formula supplies a solution to the related higher order Neumann problem under some solvability conditions.

Theorem 4.2.4 *The Neumann-n problem*

$$(\partial_z\partial_{\bar{z}})^{n}=f \ \ in\ \mathbb{D},\ f\in L_p(\mathbb{D};\mathbb{C}) \ \ for\ \ 2<p<+\infty,$$

$$\partial_\nu(\partial_z\partial_{\bar{z}})^{\sigma}=\gamma_\sigma \ \ on\ \partial\mathbb{D},\ \gamma_\sigma\in C(\partial\mathbb{D};\mathbb{C}) \ \ for\ \ 0\le\sigma\le n-1,$$

satisfying

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(\partial_\zeta\partial_{\bar{\zeta}})^{\sigma}w(\zeta)\frac{d\zeta}{\zeta}=c_\sigma,\ c_\sigma\in\mathbb{C} \ \ for\ \ 0\le\sigma\le n-1,$$

is solvable if and only if $0 \le \sigma \le n-1$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_\sigma(\zeta)\frac{d\zeta}{\zeta} = \sum_{\mu=\sigma+1}^{n-1} \alpha_{\mu-\sigma} c_\mu - \frac{1}{\pi}\int\limits_{\mathbb{D}} \partial_{\nu_z} N_{n-\sigma}(z,\zeta) f(\zeta) d\xi d\eta. \tag{4.2.11}$$

Here $\alpha_1 = 2$ *and for* $3 \le k \le n$

$$\alpha_{k-1} = -\sum_{\mu=\left[\frac{k}{2}\right]} \frac{\mu!^2}{(k-1)!(k-1-\mu)!^2(2\mu-k+1)!}\alpha_\mu. \tag{4.2.12}$$

The solution is unique and given by

$$w(z) = \sum_{\mu=0}^{n-1}\left\{c_\mu \rho_{\mu+1} - \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} N_{\mu+1}(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta}\right\} - \frac{1}{\pi}\int\limits_{\mathbb{D}} N_n(z,\zeta) f(\zeta) d\xi d\eta, \tag{4.2.13}$$

where with $\partial_{\nu_z} = z\partial_z + \bar{z}\partial_{\bar{z}}$ *and* $\zeta \in \partial\mathbb{D}$

$$\rho_\mu(z) = -\frac{1}{2}\partial_{\nu_\zeta} N_\mu(z,\zeta) \ \textit{for}\ 1 \le \mu \le n \ \textit{and}\ z \in \mathbb{D}$$

Only $N_2(z,\zeta)$ *is explicitly calculated so far [25]. It is for* $\mathbb{D}$

$$-N_2(z,\zeta) = |\zeta - z|^2\left[\log|(\zeta - z)(1 - z\bar{\zeta})|^2 - 4\right] + 4\sum_{k=2}^{+\infty}\frac{(z\bar{\zeta})^k + (\bar{z}\zeta)^k}{k^2}$$

$$+2\left[z\bar{\zeta} + \bar{z}\zeta\right]\log|1 - z\bar{\zeta}|^2 - (1+|z|^2)(1+|\zeta|^2)\left[\frac{\log(1 - z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta}\right].$$

For $N_3(z,\zeta)$, see [38].

Appendix

Cauchy problem for a class of elliptic systems of third order in the plane with Fuchsian differential operator

The solutions of a class of complex partial differential equations of third order in the plane with a Fuchs type differential operator are constructed in explicit form and the Cauchy problem with prescribed growth at infinity is solved in unbounded angular domains within specified function classes, see [31].

Let $0 < \varphi \leq 2\pi$, $G = \{z = re^{i\varphi} : 0 \leq r < \infty,\ \ 0 \leq \varphi \leq \varphi_1\}$. Consider in G the equation

$$8f_1(\varphi)\bar{z}^3\frac{\partial^3 V}{\partial\bar{z}^3} + 4f_2(\varphi)\bar{z}^2\frac{\partial^2 V}{\partial\bar{z}^2} + 2f_3(\varphi)\bar{z}\frac{\partial V}{\partial\bar{z}} + f_4(\varphi)\overline{V} = f_5(\varphi)r^{\nu}, \quad \text{(A.1)}$$

$\nu > 0$ is a real parameter, $f_l(\varphi) \in C[0,\ \varphi_1]$, $(l = 1, 2, 3, 4, 5)$, $f_1(\varphi) \neq 0$;

$$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),\ \ \frac{\partial^k V}{\partial\bar{z}^k} = \frac{\partial}{\partial\bar{z}}\left(\frac{\partial^{k-1}V}{\partial\bar{z}^{k-1}}\right),\ \ (k = 2, 3).$$

Equation (A.1) is investigated for $f_1(\varphi) \equiv 0,\ f_2(\varphi) \equiv const \neq 0, f_3(\varphi) \equiv const \neq 0$ in [1, 2]and for $f_2(\varphi) \equiv f_2(\varphi) \equiv 0,\ f_3 \equiv const \neq 0$ in [5].
To find the solutions of equation these operators are used in polar coordinates

$$\frac{\partial}{\partial\bar{z}} = \frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi}\right),$$

$$\frac{\partial^2}{\partial\bar{z}^2} = \frac{e^{2i\varphi}}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{2i}{r}\frac{\partial^2}{\partial r\partial\varphi} - \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{2i}{r^2}\frac{\partial}{\partial\varphi}\right),$$

$$\frac{\partial^3}{\partial \bar{z}^3} = \frac{e^{3i\varphi}}{8}\left(\frac{\partial^3}{\partial r^3} + \frac{3i}{r}\frac{\partial^3}{\partial r^2 \partial \varphi} - \frac{3}{r^2}\frac{\partial^3}{\partial \varphi^2 \partial r} - \frac{i}{r^3}\frac{\partial^3}{\partial \varphi^3} - \right.$$
$$\left. -\frac{3}{r}\frac{\partial^2}{\partial r^2} - \frac{9i}{r^2}\frac{\partial^2}{\partial r \partial \varphi} + \frac{6}{r^3}\frac{\partial^2}{\partial \varphi^2} + \frac{3}{r^2}\frac{\partial}{\partial r} + \frac{8i}{r^3}\frac{\partial}{\partial \varphi}\right).$$

Equation (A.1) is rewritten in the form

$$f_1(\varphi)r^3\frac{\partial^3 V}{\partial r^3} + 3if_1(\varphi)r^2\frac{\partial^3 V}{\partial r^2 \partial \varphi} - 3f_1(\varphi)r\frac{\partial^3 V}{\partial r \partial \varphi^2} - if_1(\varphi)\frac{\partial^3 V}{\partial \varphi^3} +$$
$$+ (f_2(\varphi) - 3f_1(\varphi))r^2\frac{\partial^2 V}{\partial r^2} + (2f_2(\varphi) - 9f_1(\varphi))ir\frac{\partial^2 V}{\partial r \partial \varphi} + \tag{A.2}$$
$$+(6f_1(\varphi) - f_2(\varphi))\frac{\partial^2 V}{\partial \varphi^2} + (3f_1(\varphi) - f_2(\varphi) + f_3(\varphi))r\frac{\partial V}{\partial r} +$$
$$+(8f_1(\varphi) - 2f_2(\varphi) + f_3(\varphi))i\frac{\partial V}{\partial \varphi} + f_4(\varphi)\overline{V} = f_5(\varphi)r^\nu$$

The solutions of equation (A.2) are searched for in the Sobolev class [43]

$$W_p^3(G), \tag{A.3}$$

where $1 < p < \frac{2}{3-\nu}$, if $\nu < 3$ and $p > 1$, if $\nu \geq 3$.

One can, that the function

$$V(r,\ \varphi) = r^\nu \psi(\varphi) \tag{A.4}$$

represents a solution of equation (A.2) from class (A.3), if $\psi(\varphi) \in C^3[0,\ \varphi_1]$ is satisfying the equation

$$\psi''' + a_1(\varphi)\psi'' + a_2(\varphi)\psi' + a_3(\varphi)\psi = a_4(\varphi) - a_5(\varphi)\overline{\psi}, \tag{A.5}$$

where

$$a_1(\varphi) = \frac{6f_1(\varphi) - f_2(\varphi) - 3\nu f_1(\varphi)}{f_1(\varphi)}i,$$

$$a_2(\varphi) = \frac{\nu(9f_1(\varphi) - 2f_2(\varphi)) - 3\nu(\nu-1)f_1(\varphi) - 8f_1(\varphi) + 2f_2(\varphi) - f_3(\varphi)}{f_1(\varphi)},$$

$$a_3(\varphi) = \frac{\nu(\nu-1)(\nu-2)f_1(\varphi) + \nu(\nu-1)(f_2(\varphi) - 3f_1(\varphi)) + 3f_1(\varphi) - f_2(\varphi) + f_3(\varphi)}{f_1(\varphi)}i,$$

$$a_4(\varphi) = \frac{f_5(\varphi)}{f_1(\varphi)}i, \qquad a_5(\varphi) = \frac{f_4(\varphi)}{f_1(\varphi)}i$$

Let $\theta(\varphi) = \{\psi_1(\varphi),\, \psi_2(\varphi),\, \psi_3(\varphi)\}$ be a fundamental system of solutions of the homogeneous equation

$$\psi''' + a_1(\varphi)\psi'' + a_2(\varphi)\psi' + a_3(\varphi)\psi = 0 \tag{A.6}$$

Using the general solution of this equation in the form

$$\psi(\varphi) = c_1\psi_1(\varphi) + c_2\psi_2(\varphi) + c_3\psi_3(\varphi),$$

where c_l, $(l = 1, 2, 3)$ are arbitrary constants, by applying the method of variation of constants equation (A.5) becomes the integral equation

$$\psi(\varphi) = (B\psi)(\varphi) + cJ_0(\varphi) + G_0(\varphi), \tag{A.7}$$

where

$$J_k(\varphi) = \{J_{1,k}(\varphi),\, J_{2,k}(\varphi),\, J_{3,k}(\varphi)\},\; J_{1,0}(\varphi) = \psi_1(\varphi),$$

$$J_{2,0}(\varphi) = \psi_2(\varphi),\; J_{3,0}(\varphi) = \psi_3(\varphi),\; c = \{c_1,\, c_2,\, c_3\},$$

$$(B\psi)(\varphi) = \int\limits_0^{\varphi} b(\varphi,\tau)\overline{\psi(\tau)}d\tau, \quad G_0(\varphi) = \int\limits_0^{\varphi} g(\varphi,\tau)d\tau,$$

$$b(\varphi,\tau) = a_5(\tau)\cdot\gamma(\varphi,\tau), \quad g(\varphi,\tau) = a_4(\tau)\cdot\gamma(\varphi,\tau),$$

$$\gamma(\varphi,\tau) = \frac{1}{|\triangle(\varphi)|}((\psi_2(\tau)\psi_3'(\tau) - \psi_3(\tau)\psi_2'(\tau))J_{1,0}(\varphi) -$$

$$-(\psi_1(\tau)\psi_3'(\tau) - \psi_3(\tau)\psi_1'(\tau))J_{2,0}(\varphi) + (\psi_1(\tau)\psi_2'(\tau) - \psi_2(\tau)\psi_1'(\tau))J_{3,0}(\varphi)),$$

$$\triangle(\varphi) = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1'' & \psi_2'' & \psi_3'' \end{pmatrix},$$

$|\triangle(\varphi)|$ is the determinant of the matrix $\triangle(\varphi)$.
For solving equation (A.7) the functions and operators

$$J_{k,j}(\varphi) = \int\limits_0^{\varphi} b(\varphi,\tau)\overline{J_{k,j-1}(\tau)}d\tau, \quad G_k(\varphi) = \int\limits_0^{\varphi} b(\varphi,\tau)\overline{G_{k-1}(\tau)}d\tau, \quad (1 \le j < \infty),$$

$$(B(B^{k-1}f)(\varphi))(\varphi) = (B^k f)(\varphi), \quad (k = \overline{1,\infty}), \quad (B^0 f)(\varphi) = (Bf)(\varphi)$$

are used.

Applying the operator B to both sides of equation (A.7) gives an expression for the function $(Bf)(\varphi)$. Inserting it again into (A.7), we have

$$\psi(\varphi) = (B_3\psi)(\varphi) + c(J_0(\varphi) + J_2(\varphi)) + \overline{c}J_1(\varphi) + G_0(\varphi) + G_1(\varphi) + G_2(\varphi), \tag{A.8}$$

Continuing this process $2k+1$ times, we get

$$\psi(\varphi) = (B^{2k+1}\psi)(\varphi) + c\sum_{n=0}^{k} J_{2n}(\varphi) + \overline{c}\sum_{n=1}^{k} J_{2n-1}(\varphi) + \sum_{n=0}^{2k} G_n(\varphi). \quad \text{(A.9)}$$

As a consequence it is easy to check the inequalities

$$|(B^k\psi)(\varphi)| \le |b(\varphi,\, \tau)|_1^k \frac{\varphi^k}{k!}, \quad |J_{k,\,j}(\varphi)| \le |b(\varphi,\, \tau)|_1^j \frac{\varphi^j}{j!}, \quad \text{(A.10)}$$

where

$$|b(\varphi,\, \tau)|_1 = \max_{0\le\varphi,\,\tau\le\varphi_1} |b(\varphi,\, \tau)|.$$

If passing to the limit in the representations (A.9)as $k \longrightarrow \infty$, by virtue of (A.10) we receive

$$\psi(\varphi) = cQ(\varphi) + \overline{c}P(\varphi) + G(\varphi), \quad \text{(A.11)}$$

where

$$Q(\varphi) = (Q_1(\varphi),\, Q_2(\varphi),\, Q_3(\varphi)), \quad P(\varphi) = (P_1(\varphi),\, P_2(\varphi),\, P_3(\varphi)),$$

$$Q_j(\varphi) = \sum_{n=0}^{\infty} J_{j,\,2n}(\varphi), \quad P_j(\varphi) = \sum_{n=1}^{\infty} J_{j,\,2n-1}(\varphi), \quad G(\varphi) = \sum_{n=0}^{\infty} G_n(\varphi), \quad (j=1,2,3).$$

For these functions $Q_j(\varphi)$, $P_j(\varphi)$, $(j = \overline{1,2,3})$ and $G(\varphi)$ it is easy to check the relations

$$Q_j(\varphi) = J_{j,0}(\varphi) + \int_0^{\varphi} b(\varphi,\tau)\overline{P_j(\tau)}d\tau, \quad P_j(\varphi) = \int_0^{\varphi} b(\varphi,\tau)\overline{Q_j(\tau)}d\tau,$$

$$G(\varphi) = G_0(\varphi) + \int_0^{\varphi} b(\varphi,\tau)\overline{G(\tau)}d\tau,$$

$$Q_j^{(k)}(\varphi) = \psi_j^{(k)}(\varphi) + \int_0^{\varphi} b_{\varphi^k}^{(k)}(\varphi,\, \tau)\overline{P_j(\tau)}d\tau, \quad P_j^{(k)}(\varphi) = \int_0^{\varphi} b_{\varphi^k}^{(k)}(\varphi,\, \tau)\overline{Q_j(\tau)}d\tau,$$

$$G_j^{(k)}(\varphi) = \int_0^{\varphi} g_{\varphi^k}^{(k)}(\varphi,\, \tau)d\tau + \int_0^{\varphi} b_{\varphi^k}^{(k)}(\varphi,\, \tau)\overline{G(\tau)}d\tau, \quad (k=1,2), \quad \text{(A.12)}$$

$$Q_j'''(\varphi) = \psi_j'''(\varphi) - a_5(\varphi)\overline{P_j(\varphi)} + \int\limits_0^\varphi b_{\varphi^3}'''(\varphi,\, \tau)\overline{P_j(\tau)}d\tau,$$

$$P_j'''(\varphi) = -a_5(\varphi)\overline{Q_j(\varphi)} + \int\limits_0^\varphi b_{\varphi^3}'''(\varphi,\, \tau)\overline{Q_j(\tau)}d\tau, \quad (j = 1, 2, 3),$$

$$G_{\varphi^3}'''(\varphi) = a_4(\varphi) + \int\limits_0^\varphi g_{\varphi^3}'''(\varphi,\, \tau)d\tau - a_5(\varphi)\overline{G(\varphi)} + \int\limits_0^\varphi b_{\varphi^3}'''(\varphi,\, \tau)\overline{G(\tau)}d\tau.$$

It is also easy to check the equalities

$$b(\varphi, \varphi) = 0, \quad b_\varphi'(\varphi, \varphi) = 0, \quad b_{\varphi^2}''(\varphi, \varphi) = -b(\varphi),$$

$$\text{(A.13)}$$

$$g(\varphi, \varphi) = 0, \quad g_\varphi'(\varphi, \varphi) = 0, \quad g_\varphi''(\varphi, \varphi) = f_5(\varphi)$$

By using formula (A.12) and (A.13) we receive the equalities

$$P_j'''(\varphi) + a_1 P_j''(\varphi) + a_2 P_j'(\varphi) + a_3 P_j(\varphi) = -a_5(\varphi)\overline{Q_j(\varphi)},$$

$$Q_j'''(\varphi) + a_1 Q_j''(\varphi) + a_2 Q_j'(\varphi) + a_3 Q_j(\varphi) = -a_5(\varphi)\overline{P_j(\varphi)},$$

$$G'''(\varphi) + a_1 G''(\varphi) + a_2 G'(\varphi) + a_3 G(\varphi) = a_4(\varphi) - a_5(\varphi)\overline{G(\varphi)}.$$

Hence, we receive

$$\psi'(\varphi) = c\theta_\varphi' + c\int\limits_0^\varphi b_\varphi'(\varphi,\, \tau)\overline{P_j(\tau)}d\tau + \bar{c}\int\limits_0^\varphi b_\varphi'(\varphi,\, \tau)\overline{Q_j(\tau)}d\tau +$$

$$+ \int\limits_0^\varphi g_\varphi'(\varphi,\, \tau)d\tau + \int\limits_0^\varphi b_\varphi'(\varphi,\, \tau)\overline{G(\tau)}d\tau,$$

$$\psi''(\varphi) = c\theta_{\varphi^2}'' + c\int\limits_0^\varphi b_{\varphi^2}''(\varphi,\, \tau)\overline{P_j(\tau)}d\tau + \bar{c}\int\limits_0^\varphi b_{\varphi^2}''(\varphi,\, \tau)\overline{Q_j(\tau)}d\tau +$$

$$+ \int\limits_0^\varphi g_{\varphi^2}''(\varphi,\, \tau)d\tau + \int\limits_0^\varphi b_{\varphi^2}''(\varphi,\, \tau)\overline{G(\tau)}d\tau,$$

$$\psi'''(\varphi) = c\theta_{\varphi^3}''' - ca_5(\varphi)\overline{P_j(\varphi)} +$$

$$+c\int_0^{\varphi} b'''_{\varphi^3}(\varphi,\,\tau)\overline{P_j(\tau)}d\tau - \bar{c}a_5(\varphi)\overline{Q_j(\varphi)} + \bar{c}\int_0^{\varphi} b'''_{\varphi^3}(\varphi,\,\tau)\overline{Q_j(\tau)}d\tau$$

$$+\,a_4(\varphi) + \int_0^{\varphi} g'''_{\varphi^3}(\varphi,\,\tau)d\tau - a_5(\varphi)\overline{G(\varphi)} + \int_0^{\varphi} b'''_{\varphi^3}(\varphi,\,\tau)\overline{G(\tau)}d\tau. \qquad \text{(A.14)}$$

Since $J_{j,0}(\varphi)$, $b(\varphi,\tau)$, $g(\varphi,\,\tau)$ represent a solution of equation (A.6), then by virtue of formula (A.14) we see, that the function $\psi(\varphi)$, given by formula (A.11) is satisfying equation (A.5).

Using inequality (A.10), it is easy to receive the estimates

$$|Q_j(\varphi)| \le |\psi_j(\varphi_1)|ch(|b(\varphi,\,\tau)|_1), \quad |P_j(\varphi)| \le |\psi_j(\varphi_1)|sh(|b(\varphi,\,\tau)|_1),$$

$$\text{(A.15)}$$

$$|G(\varphi)| \le \varphi_1|g(\varphi,\,\tau)|_1 \exp(|b(\varphi,\,\tau)|_1), \quad (j = 1, 2, 3).$$

By estimate (A.15) we can assure that the function $V(r,\varphi)$, given by formulas (A.4), (A.11), is solving equation (A.1) in the class (A.3).

Thus, the following result holds.

Theorem 4.2.5 *Equation (A.1) has a solution in the class (A.3), which is given by formula (A.4), (A.11).*

Consider the Cauchy problem with prescribed growth at infinity for system (A.1).

Problem C. Find a solution of equation (A.1) from the class (A.3), satisfying the conditions

$$\alpha_{k1}V(r,\,0) + \alpha_{k2}\frac{\partial V}{\partial\varphi}(r,\,0) + \alpha_{k3}\frac{\partial^2 V}{\partial\varphi^2}(r,\,0) = \beta_k r^{\nu}, \quad (k = 1, 2, 3) \qquad \text{(A.16)}$$

$$|V(r,\,\varphi)| = O(r^{\nu}), \quad r \longrightarrow \infty, \qquad \text{(A.17)}$$

where a_{kj}, $(k = (1, 2, 3),\ j = (1, 2, 3))$ are given real numbers.

For solving problem **C** formulas (A.4), (A.11) are used. In that case (A.17) holds automatically. The constants c_1, c_2, c_3 in formula (A.11) are determined, in order that the solution of equation (A.1), represented in the form (A.4) and (A.11), satisfies condition (A.16). For that, insert function $V(r,\varphi)$ according to formulas (A.4), (A.11) into (A.16). Thus we get a system of linear algebraic equations in c_1, c_2, c_3:

$$(\alpha\triangle(0))c^T = \beta, \qquad \text{(A.18)}$$

where

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \quad c^T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Solving system (A.18) under $|\alpha\triangle(0)| \neq 0$, we receive

$$c^T = (\alpha\triangle(0))^{-1}\beta \tag{A.19}$$

Thus, the following result holds.

Theorem 4.2.6 *Let the roots of the characteristic equation (A.6) be mutually different and different from zero. Under $|\alpha\triangle(0)| \neq 0$ the Cauchy problem has a unique solution, which is given by formulas (A.4), (A.11) and (A.19).*

If $|\alpha\triangle(0)| = 0$ for the solvability of the algebraic systems (A.18) the conditions:

$$|A_1| = 0, \quad |A_2| = 0, \quad |A_3| = 0 \tag{A.20}$$

are necessary and sufficient. Here A_j is the matrix, which is received by replacing the i in matrix column of the matrix $\alpha\triangle(0)$ by the column β.

Theorem 4.2.7 *Let $|\alpha\triangle(0)| = 0$, then for the solvability of the Cauchy problem the condition (A.20) is necessary and sufficiently. In that case the Cauchy problem has an infinity number of solutions. They are given by formulas (A.4), (A.11), where c is determined from equation (A.18) under condition (A.20).*

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Zusammenfassung

Eine der grundlegenden partiellen Differentialgleichungen der mathematischen Physik ist die Poisson Gleichung. Sie ist der Prototyp für eine ganze Klasse von Modellgleichungen, den polyharmonischen Differentialgleichungen, auch n-Poisson Gleichungen genannt. Im zweidimensionalen Fall bietet sich die komplexe Analysis zur Behandlung an. Ihr großer Vorteil ist es, dass sich der Laplace Operator mittels des Cauchy-Riemann Operators faktorisieren lässt. Im Komplexen ist die n-Poisson Gleichungen in den letzten Jahren in der Gruppe von Professor Begehr wiederholt untersucht worden. In einer Reihe von Arbeiten sind verschiedene Randwertprobleme in unterschiedlichen Gebieten behandelt worden. Mittels eines natürlichen Iterationsprozesses lassen sich Cauchy-Pompeiusche Integraldarstellungsformeln beliebiger Ordnung gewinnen. Die sind allerdings zur Lösung von Randwertproblemen im Allgemeinen ungeeignet. Die Einführung polyharmonicher Green Funktionen transformiert diese Integraldarstellungen in solche, die gewisse Randwertprobleme lösen helfen. Je höher der Grand umso mehr polyharmonische Green Funktionen gibt es. Einige von ihnen lassen sich durch Faltung von Green Funktionen niedrigerer Ordnung gewinnen. Für spezielle Gebiete kann man diese Faltungsintegrale auswerten und so explizite poyharmonische Green Funktionen erhalten. In dieser Arbeit werden in der genannten Weise eine triharmonische Green und eine triharmonische Neumann Funktion für den Einheitskreis konstruiert. Die entsprechenden harmonischen und biharmonischen Funktionen sind explizit bekannt. Dies ist ein Beitrag zur induktiven Bestimmung von polyharmonischen Green und Neumann Funktionen beliebiger Ordnung für den Einheitskreis. Es gilt zunächst, die entsprechende Induktionsbehauptung aufzustellen. Mit Hilfe der gewonnen triharmonischen Green und Neumann Funktionen werden die zugehörigen Randwertprobleme gelöst. Darüber hinaus werden einige hybride triharmonische Green Funktionen eingeführt und die zugehörigen Randwertprobleme gelöst. In einem Anhang wird eine singuläre lineare komplexe partielle Differentialgleichung vom Fuchs Typ mit trianalytischem Hauptteiloperator behandelt.

Curriculum Vitae