Dissertation

Spectral approach to the axisymmetric evolution of Einstein's vacuum equations

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Diese Dissertation wird eingereicht beim
Fachbereich Mathematik und Informatik
der Freien Universität Berlin, sie wurde angefertigt am
Max-Planck-Institut für Gravitationsphysik
(Albert-Einstein-Institut) in Potsdam

2017

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Externer Gutachter: Prof. Dr. Jörg Frauendiener (University of Otago, Dunedin, New Zealand).
Datum der Disputation: 14. Dezember 2017.

Abstract

This thesis documents our study of Einstein's vacuum equations in spherical polar coordinates. Open questions in this setting concern applications like the understanding of gravitational collapse and conceptual matters such as the handling of the occurring coordinate singularity. We answer the conceptual aspects and demonstrate how they can be implemented numerically.

Our choice of coordinates allows a spectral approach. As basis functions we employ spin-weighted spherical harmonics. For most derivations and applications we assume hypersurface-orthogonal axisymmetry. This assumption leads to computational simplifications but is not a conceptual limitation.

We examine the eigenfunctions of the Laplace operator in spherical coordinates for quantities with different spin-weights and derive the consequences. A systematic investigation of the scalar wave equation in these coordinates leads to helpful insights for the regularization of the coordinate singularity at the origin and we confirm this numerically.

We show that a common gauge choice in axisymmetry is inappropriate for the expansion in spin-weighted harmonics and discuss alternatives. We derive Einstein's equations in axisymmetry in an appropriate gauge and solve the linearized equations exactly.

A recent formulation of Einstein's constraint equations regards them as an evolutionary system. We analyze the full set of equations and introduce modifications that allow us to derive two sets of locally well-posed problems.

Our numerical implementation uses a hybrid discretization consisting of finite difference techniques and the pseudo-spectral method. We simulate the derived equations and present a successful implementation of the parabolic-hyperbolic formulation of the nonlinear constraints. To do so we derive several possibilities to obtain initial data at the regular origin. We demonstrate further that our implementation is able to reproduce the exact linear solution in a fully constrained scheme.

The results obtained in this thesis offer a possible solution how to simulate Einstein's vacuum equations numerically in spherical polar coordinates with a regular origin. We present one of the first numerical studies of an evolutionary constraint solver.

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1. Introduction and overview

General relativity revolutionized our understanding of gravity and our paradigms of space, time and gravitation and, hence, represents a **scientific revolution**, according to the doctrine of Thomas Kuhn (1974). At present it seems to be the best-working model of gravity in reality. In its formulation many mathematical disciplines are involved including differential geometry, the theory of partial differential equations and geometric analysis. In the present thesis we want to gain a deeper understanding of its fundamental properties and to explore novel techniques to find solutions. Therefore these investigations are placed in the field of applied mathematics even though their major asset is in theoretical physics[1].

Similar as our characterization of mathematics and theoretical physics the following classification might not be unique but reflects our personal tendency. We want to characterize mathematical relativity as the mathematical field in which fundamental questions arising in general relativity are addressed with mathematical techniques, see Chruściel et al. (2010) for a sampler. On the other hand numerical mathematical relativity describes the application of numerical techniques in mathematical relativity. Under numerical relativity we understand

[1]There is no unique definition or characterization of mathematics but we want to advertise mathematics as an abstract science based on logical deductions and the application to fields like theoretical physics. In contrast to mathematics we want to interpret physics, in particular theoretical physics, as the natural science aiming to build models of reality and extracting predictions. Let us cite a few quotes in favor of our argumentation and supporting our point of view. We remark that there are also different standpoints. According to the Oxford dictionary (https://en.oxforddictionaries.com/definition/mathematics) mathematics is "[t]he abstract science of number, quantity, and space, either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering (applied mathematics)". Courant et al. (1996, preface to the 2$^{\text{nd}}$ edition): "[...] mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician." Richard Feynman (1994, page 49): "Mathematicians are only dealing with the structure of reasoning, and they do not really care what they are talking about. They do not even need to *know* what they are talking about [...]. But in physics you have to have an understanding of the connection of words with the real world." Karl Popper (1972, page 246) on natural science: "It is the task of the natural scientist to search for laws which will enable him to deduce predictions." Albert Einstein (1996, page 77): "Physics is an attempt conceptually to grasp reality as something that is considered to be independent of its being observed."

the numerical investigations of a relativistic theory including astrophysical applications. We discuss some aspects of numerical relativity and give references in sections 3.3 and the following. Accordingly, mathematical numerical relativity is the mathematical analysis of numerical relativity. It is not always possible to place a strict boundary between these fields, see Garfinkle (2017) for a recent review. We consider this thesis on that borderline between numerical mathematical and mathematical numerical relativity.

General relativity admits a very elegant and beautiful formulation as a geometrical theory, see chapter 3. Einstein's field equations are at the heart of the theory and we can write them in a very concise form, see section 3.2.2. Nevertheless in the standard formulation they are not accessible to the usual analytical and numerical techniques in a straightforward way. General relativity is reformulated in terms of the Cauchy formulation in section 3.3, such that standard theory of partial differential equations and its numerical techniques are applicable to the field equations.

Einstein's equations split into two sets of equations, evolution equations and constraints. All equations as a full set have the character of an overdetermined set of equations, i.e. not all equations need to be employed. There are different schemes depending on the amount of constraints that are incorporated in the set of equations applied. The system of constraint equations on the other hand is highly underdetermined. There is some freedom which variables are prescribed and which are to be solved for. Very common is to arrange the constraints as an elliptic system, see Cook (2000), Bartnik and Isenberg (2002). Quite recently a complementary approach was established where the constraints are formulated as evolutionary problem, see Rácz (2016a). We follow mainly the latter path and investigate the constraints as evolutionary system.

Einstein's equations relate gravity and matter[2]. There is a lot of interest in the case of pure gravitational interaction, both in astrophysics and in mathematical relativity. The field equations are nonlinear. The vacuum case without any kind of matter source is a demanding problem as such and is far from being understood completely. For example there are open questions concerning the gravitational collapse, especially its critical phenomena, see Gundlach and Martín-García (2007). We restrict our considerations in the thesis to the vacuum case, which allows us to focus on fundamental questions in the formulation.

[2]In the words of John Wheeler (1998, page 235): "Spacetime tells matter how to move; matter tells spacetime how to curve."

We use spherical polar coordinates in this thesis. For many applications in astrophysics or gravitational collapse the assumption of a spherical shape is intuitive. In addition the coordinates are beneficial for the use of adaptive methods, the implementation of horizon finders and they were employed in the first reported study Abrahams and Evans (1993) on critical vacuum collapse, see also the discussion in section 3.5. Another advantage of spherical coordinates and our main motivation is a different aspect though. Spherical polar coordinates allow for a spectral expansion in spherical harmonics as basis functions. For a tensor theory such as general relativity the spin-weight has to be taken into account for a correct formulation. We work out its explicit form and consequences in sections 2.2.4 and 2.4. A general disadvantage when using non-Cartesian coordinates is the occurrence of coordinate singularities even though we assume the spacetime to be fully regular. Especially from the numerical perspective such singularities become demanding.

We will show in section 2.5 how to tame and solve the problem. The key for the solution is an understanding of the mode structure of the eigenfunctions of the Laplace operator in spherical polar coordinates. We investigate the issue for the Laplacian applied to scalars, vectors and tensors. As a toy model we study the scalar wave equation in spherical coordinates in section 2.6 and show how the obtained insights are helpful for the numerical regularization.

In the main part of the thesis we assume axisymmetry. This assumption is not a fundamental restriction, but rather practically motivated. In our setting we are able to effectively reduce the problem by one dimension. Especially for the numerical implementation we save recourses, but also the analytical calculations become more manageable. The essential results are generalizable to the situation without symmetry but from the conceptual point of view the assumption is beneficial. We face the same problems as on the full 3+1-dimensional level and show how to solve them in the reduced example. Axisymmetry is the intermediate step between spherical symmetry and full general relativity. In vacuum the case of spherical symmetry is completely understood. Our framework allows to study the same phenomena as in the full theory including gravitational waves.

In the Cauchy formulation of general relativity the coordinate freedom is encoded in a so-called gauge choice for several free functions. The difference between a clever and a naive choice has a large influence on the mathematical structure and nature of the final set of equations. In section 4.3 we discuss the issue to some extent and show in particular that a very common and well-understood gauge for our situation is unfortunately incompatible with the desired spectral expansion in spin-weighted harmonics. We introduce different gauge choices that are appropriate for our needs. Here we benefit from our already published results in Schell and Rinne (2015).

The linearization of a nonlinear problem about known solutions allows already some insights into the conceptual difficulties. Especially from the computational perspective it is beneficial to start with simpler and easier to understand situations. In our case we face already the essential difficulties of the implementation on the linear level and show how to tackle them. In addition we are able to solve the linearized problem exactly. We address the derivation of a general solution to the linearized problem in section 4.6.

In the novel evolutionary approach to the constraints the corresponding set of equations forms an initial value problem. Hence a way to prescribe initial data is needed. We explore several methods to obtain them at the regular origin in chapter 5.

A very important issue for evolutionary sets of equations is their well-posedness as initial value problem. It requires to simulate systems that are not just weakly but strongly hyperbolic. There are positive results in the literature. For our entire system of evolution and evolutionary constraint equations it becomes a non-trivial task again. We show in section 4.7 how to arrange and modify the equations in our setting such that they result in a system consisting of a strongly hyperbolic and a parabolic-strongly hyperbolic system. So with an appropriate choice of initial data the problem looks promising from the mathematical perspective.

In section 5.2.1 we explain in more detail matters related to our code that we developed for the numerical implementation from scratch. In that situation it is important to check and to demonstrate that the numerical results are reliable. We perform and document these tests in chapter 5.

In the numerical studies it is one aim to show that our code is able to reproduce the exact solution. The spectral approach allows us to further reduce the linear equations to a 1+1-dimensional scheme where we model single modes. We demonstrate that we are indeed able to reproduce numerically the exact mode solution in section 5.3 and show the essentials of the simulations on the linear level in section 5.4. We describe our regularization procedure that is required to stabilize the formally singular evolution equations.

For the nonlinear level we build a solver for the constraints that uses the parabolic-hyperbolic formulation of the constraints. We demonstrate in section 5.5 that it is possible to use our techniques for obtaining initial data for the solver at the regular origin and to integrate the constraints. It seems to be for the first time that the parabolic-hyperbolic formulation is successfully implemented numerically.

We organize the thesis in the following way. In chapter 2 we review some necessary ingredients for differential equations for later use, including numerical issues and derive several components for the implementation. Of significant importance are our investigations concerning the eigenfunctions of the Laplacian and the numerical regularization of the wave equation, both in spherical polar coordinates. In chapter 3 we consider general relativity with a special focus on the aspects of interest for our thesis, in particular the evolutionary approach to the constraints. Chapter 4 presents our derivation of Einstein's vacuum equations in axisymmetry. We analyze certain aspects in detail, including implications of axisymmetry, gauges, the character as set of partial differential equations and derive an exact solution to the linearized problem. Our implementation of the resulting equations is discussed and the results are presented in chapter 5. We include our derivation of initial values for the constraint solver and demonstrate the successful numerical solution of the nonlinear constraints as parabolic-hyperbolic set of equations. We conclude and give an outlook in chapter 6. The appendix contains supplementary material, in particular lengthy expressions of our derivations that might disturb the flow of reading in the main text.

2. Differential equations and numerical methods

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2.1. Introduction

We begin with considerations of one of the essential mathematical ingredients for this thesis, which are differential equations, both ordinary and partial ones and some of their numerical techniques to obtain solutions. We briefly review some standard material and describe two numerical techniques applied by us, finite differences and the (pseudo-)spectral method. It is included in the thesis because it helps to fix the used notation and to clarify concepts that are used later. We derive spin-weighted harmonics and calculate expressions that are essential for further investigations. We also discuss issues related with code validation for later use. Our review of partial differential equations has a special focus on hyperbolic systems because it is essential for our further analysis. We also introduce our hybrid discretization. To apply it we derive explicit expressions for our angular derivatives and explain their implementation.

In the later part of the chapter we investigate certain issues and derive results related to partial differential equations which are essential for the remainder of the thesis. We examine in section 2.5 the eigenfunctions of the Laplace operator on $\mathbb{R}_{\geq 0} \times S^2$ for quantities with different spin-weights. These insights are essential for the implementation. In section 2.6 we deal with the scalar wave equation in spherical polar coordinates and derive techniques to tame the coordinate singularity at the regular origin. We also demonstrate our regularization scheme numerically.

2.2. Ordinary differential equations and their numerical methods

As a subclass of partial differential equations we start with the discussion of ordinary differential equations. Also we introduce techniques that are needed for the numerical implementation. Even though Einstein's equations form a coupled set of nonlinear partial differential equations there exist techniques which allow us to solve numerically essentially ordinary differential equations. The fact is very beneficial from the computational point of view.

There exists a huge amount of literature dealing with the general topic of this section, including textbooks on numerical analysis and computation. As good examples we list Press et al. (2007), Butcher (2003) and give more specialized references below.

2.2.1. Classification of ordinary differential equations

Definition 2.2.1. An **ordinary differential equation** F is a functional for a **variable** (a map $u : \mathbb{R} \to \mathbb{R},\ x \mapsto u(x)$[1]), that depends continuously on a coordinate, say x, where a derivative of u with respect to x may occur,

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R} &\to \mathbb{R}, \\ (x, u) &\mapsto F(x, \partial_x^n u, \ldots, \partial_x^1 u = \partial_x u, \partial_x^0 u = u) = 0. \end{aligned} \tag{2.1}$$

Definition 2.2.2. It is very common to classify the ordinary differential equation(2.1) with respect to several properties including the following.

- The highest derivative of u in equation (2.1) with non-vanishing contribution determines the **order** of the ordinary differential equation. Often in physics, including the field of general relativity, it is sufficient to limit oneself to order two.

- There are several levels of **linearity**

  - F is **linear** in u if it is so in u and all derivatives. That means that all derivatives $\partial_x^i u$, $i = 0, 1, \ldots$ form a linear basis and therefore F can be written as $F = g + c_0 u + c_1 \partial_x u + c_2 \partial_x^2 u + \ldots = 0$ where g (see below) and all c_i might be x-dependent but do not depend on the solution itself. The **superposition principle** can be applied for linear ordinary differential equations.

  - F is **quasilinear** if it is linear in the highest-order derivative. The coefficients may depend (even nonlinearly) on lower order derivatives though. For instance $\partial_x u\ \partial_x^2 u + (\partial_x u)^2 = 0$ is quasilinear.

  - F is **semilinear** if it is quasilinear and the coefficients of the highest derivative do not depend on the solution u and its derivatives. Lower order terms might contain derivatives though. For instance $x^2 \partial_x^2 u + (\partial_x u)^2 = 0$ is a semilinear ordinary differential equation.

  - F is **(fully) nonlinear** in u if it is so with respect to the highest derivative.

[1] It could be an interval in $\mathbb{R}$ as well, say $x \in [x_0, x_{\text{end}}]$. This is true in all considerations but will not be mentioned explicitly in the following.

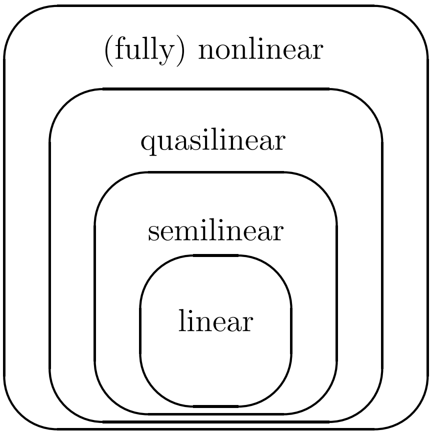


Figure 2.1.: A pictorial illustration of the concepts of linearity.

- We can separate the part of F in equation (2.1) that does not depend on u and its derivatives in the form

$$F = g(x) + f(x, \partial_x^n u, \dots, \partial_x^1 u = \partial_x u, \partial_x^0 u = u) = 0, \tag{2.2}$$

  where $f = 0$ for $u = 0$. The ordinary differential equation F is called **homogeneous** if $g \equiv 0$, otherwise **inhomogeneous**.

- Usually one is interested in the solution u for known F. The integration of equation (2.1) provides us with a "multi-parameter-family" of solutions, one parameter for each order. Therefore there remains some freedom which may be fixed. We call equation (2.1) an **initial value problem** if in addition to equation (2.1) some information at one "initial instance" x_0 of x is given[2], usually u and its derivatives are prescribed and called **initial data**. We call (2.1) a **boundary value problem** if in addition to (2.1) some information at the boundary of the domain $x \in [x_0, x_{\text{end}}] \subset \mathbb{R}$ is given. These might be a boundary conditions, for example of the form of
  - **Dirichlet**: some concrete value u_0 of the variable u is prescribed at the boundary in the form $u(x_0) = u_0$,
  - **Neumann**: some value u_0 for the derivative[3] for the variable u is prescribed at the boundary in the form $\partial_x u|_{x_0} = u_0$,
  - **Robin**: a combination of the above boundary conditions is given in the form $u(x_0) + f(x)\partial_x u|_{x_0} = u_0$,

[2]Often the coordinate labels the time and is denoted by t then. It need not necessarily be the case to have a physical time parameter for an initial value problem. It will be demonstrated later in the thesis.

[3]For general nontrivial domains the derivative along the normal of the boundary but we do not need to specify that issue here.

but can be replaced by other requirements like regularity for instance. We will later benefit from some concept like an “outgoing condition” that allows only information to leave the domain but prohibits any kind of reflection from the boundary.

Note that it is not an unambiguous notion. For example a first-order ordinary differential equation of the form $F = \partial_x u + f(x,u) = 0$ with some additional information $u(x_0)$ which can be interpreted both as initial data or as (Dirichlet) boundary value. We discuss that issue further section 2.2.2.

- Usually differential equations, ordinary differential equations as well as partial differential equations, are formulated as equations for some field on a given fixed background, the domain $\mathbb{R}$ or a subset of it for instance. In that way the equations are also open for standard computational methods and techniques. In the theory of general relativity the quantities or variables in the equations, so the physical fields, describe the geometry itself. There is no preferred background. Strictly speaking the solutions in the form of the basic fields are not unique solutions but a whole equivalence class of solutions. We will come back to that issue in definition 3.2.2. The resulting equations are called **geometric differential equations**. Even though quite often, as in general relativity, the field equations can be formulated in a very elegant way, they are usually not directly accessible to standard tools in numerical analysis but need to be “de-geometrized”. We will discuss these issues later in the thesis. The mathematical field that is concerned with topics on the interface between analytical techniques like differential equations and differential geometry is called **geometric analysis**.

Remarks 2.2.1.

- The definition 2.2.1 includes in particular algebraic equations where only the “0^{th} derivative” (no derivative) occurs.
- For the theory of ordinary differential equations one usually reduces the order of the ordinary differential equation to a first-order system of ordinary differential equations. It is always possible, due to the introduction of more variables though. We will make use of comparable reductions later on. Therefore one can consider the equivalent expression instead of equation (2.1)

$$
\begin{aligned}
f : \mathbb{R} \times \mathbb{R}^n \to R^n,\ u : \mathbb{R} \to \mathbb{R}^n \\
\partial_x u = f(x, u(x))
\end{aligned}
\tag{2.3}
$$

which should be understood as a system now and the variable u is a vector of possibly several variables.

A key result in the theory of ordinary differential equations is that for an initial value problem $\partial_x u = f(x, u)$ with given initial data a solution exists under relatively mild conditions (continuity assumptions, Peano's existence theorem). If in addition the **Lipschitz condition**

$$||f(x, u_1) - f(x, u_2)|| \leq L||u_1 - u_2|| \tag{2.4}$$

in the second argument with **Lipschitz constant** $L \in \mathbb{R}_{<\infty}$ is satisfied for all $u_1,\ u_2,\ x$, the solution is also unique (Picard–Lindelöf theorem). For a proof see the initial chapter of Hörmander (1997) (or Butcher (2003)) which can also be consulted for a very concise review of results for ordinary differential equations.

Definition 2.2.3. We call an initial value problem **stiff** if the Lipschitz constant L takes high values $\gg 1$, otherwise **non-stiff**.

It can happen that standard numerical methods (explicit, non-adaptive) result in instabilities even though the solution to a stiff problem is smooth. Stiffness can be associated with perturbations of a given solution, see Butcher (2003, section 112) for further discussions.

Example 2.2.1. Consider

$$f : [\epsilon, x_{\text{end}} > \epsilon] \times \mathbb{R} \to \mathbb{R},\ \epsilon > 0,\ g : \mathbb{R} \to \mathbb{R},\ u : \mathbb{R} \to \mathbb{R},$$
$$\partial_x u = f(x, u) = \frac{g(u)}{x} \tag{2.5}$$

where f denotes the right-hand side in general and g is of order $\mathcal{O}(x)$ for $x \to 0$. Because of the x^{-1} in the function f it is clear that the Lipschitz constant L has to grow arbitrarily large (only limited by ϵ which we shall assume to be arbitrarily small as usual) when x approaches 0. Hence equation (2.5) is a **stiff** equation (in the neighborhood of $x = 0$).

2.2.2. Cauchy problem

We have already seen that in addition to the actual differential equation (2.1) as such one should provide in general more information.

Definition 2.2.4. The additional initial data are called **Cauchy data** for a **Cauchy problem** (the problem of finding a solution for the equation with the provided data). They are prescribed on a **Cauchy (hyper-)surface**.

Interesting follow-up questions, in particular from the computational perspective, include

- if a solution u to F in equation (2.1) **exists** subject to possible Cauchy data,
- if so, if the solution u is **unique**[4] (that means that a particular member is chosen in the "multi-parameter-family" (compare definition 2.2.2) of solutions),
- and if so, if the solution u is "**stable**" with respect to the given data.

If we perturb the ordinary differential equation or the Cauchy data slightly the solution will be perturbed as well. It is of certain interest to know if the perturbations in the solution remain small or if they might lead to dramatic effects. This question is of particular interest if one considers a "real world problem", a problem that has some actual relevance in physics reality. All the data one obtains are not precise to an arbitrary degree. There are always inaccuracies of all kinds. To obtain a general statement about the situation one benefits from a formulation that is indeed stable. It is beneficial to know in advance if "small[5]" perturbations in prescribed data result in small perturbations of the solutions. Not all problems in nature are stable and form an interesting field of study as such.

Definition 2.2.5. A Cauchy problem is **well-posed** (otherwise **ill-posed**), iff

- there exists a solution (at least one),
- it is unique (at most one),
- it depends continuously on the Cauchy data.

The significance of well-posedness of the Cauchy problem is due to Hadamard (1902), see also Hadamard (1915, 1952).

2.2.3. Discretization and difference equation

The general finite difference technique is a standard procedure for the numerical integration. Huge parts of the current section are contained in many textbooks, see for example Press et al. (2007), Pang (2006). Some techniques based on the

[4] Here we state simply unique. Depending on the problem (for instance one might consider "geometric partial differential equations", see definition 2.2.2, like in general relativity) there might be a good reason that uniqueness is not desirable but one might be interested in "unique up to diffeomorphisms" in that context.

[5] The notion of "small" needs some further clarification of course.

standard one are also used in the numerical implementation and hence it makes sense to give a more or less self-contained and complete presentation here.

Consider a continuous map $\mathbb{R} \times \mathbb{R},\ x \mapsto u(x)$ depending on a continuous coordinate $x \in [x_0, x_I] \subset \mathbb{R}$. A way to model it, in particular on the computer, is to **discretize** it on a lattice[6] $x_i, i = 0, \ldots, I$ by assigning $u_i = u(x_i)$. In this chapter we restrict considerations to a **uniform grid** with step size $h \equiv \Delta x$, then

$$x_i = x_0 + ih. \tag{2.6}$$

Expressions for the difference operators

The basics for the centered finite difference scheme were already known in the 19th century. They were written down in a systematic way in Sheppard (1899) for instance, see also Fornberg (1988).

We need the derivative stencils only up to $\mathcal{O}(h^2)$. If interested in higher derivatives or higher orders the techniques explained below to obtain the stencils are directly generalizable. Basically one just need to take more grid points and higher Taylor expansions into account. One can get a flavor of it below when we consider one-sided stencils.

Proposition 2.2.1. For a smooth (or sufficiently often differentiable, so a member of the corresponding differentiability class) variable $u(x)$ (map $\mathbb{R} \mapsto \mathbb{R}$) one approximates the first two derivatives in the **centered difference stencil** as

$$\partial_x u\big|_{x_i} \equiv \partial_x u\big|_i = \frac{u(x_i + h) - u(x_i - h)}{2h} + \mathcal{O}(h^2), \tag{2.7}$$

$$\partial_x^2 u\big|_{x_i} \equiv \partial_x^2 u\big|_i = \frac{u(x_i + h) - 2u(x_i) + u(x_i - h)}{h^2} + \mathcal{O}(h^2) \tag{2.8}$$

and the approximation is accurate up to a **discretization error** $\mathcal{O}(h^2)$.

Proof. We switch frequently between the notation $u(x_i) \equiv u_i$, $u(x_i + h) \equiv u_{i+1}$ and therelike. Because of the smoothness we make use of the Taylor expansion in (both direction) at a given point x_i

$$u_{i+1} = u_i + \partial_x u\big|_i h + \frac{1}{2}\partial_x^2 u\big|_i h^2 + \frac{1}{6}\partial_x^3 u\big|_i h^3 + \mathcal{O}(h^4), \tag{2.9}$$

[6] This notation implies that the endpoints are explicitly included in the scheme. If one chooses a **staggered grid** one explicitly avoids to have a grid point at the boundary, in contrast to the **cell-centered grid**. Many options and combinations are possible for the boundaries.

$$u_{i-1} = u_i - \partial_x u\big|_i h + \frac{1}{2}\partial_x^2 u\big|_i h^2 - \frac{1}{6}\partial_x^3 u\big|_i h^3 + \mathcal{O}(h^4). \tag{2.10}$$

Now taking the difference of both equations gives

$$2\partial_x u\big|_i h = u_{i+1} - u_{i-1} + \mathcal{O}(h^3) \tag{2.11}$$

and adding both relations leads to

$$\partial_x^2 u\big|_i h^2 = u_{i+1} + u_{i-1} - 2u_i + \mathcal{O}(h^4) \tag{2.12}$$

and hence both claimed results are shown. □

Definition 2.2.6. The **one-sided forward/backward difference operator** (upper sign for forward, lower one for backward) of a variable u with respect to a coordinate x and of order $\mathcal{O}(h)$ (proven below in proposition 2.2.2) is defined as

$$D_x^{h\pm} u_i = \frac{\mp u_i \pm u_{i\pm 1}}{h}. \tag{2.13}$$

The same operator of order $\mathcal{O}(h^2)$ (proven below in proposition 2.2.2) is

$$D_x^{h^2\pm} u_i = \frac{\mp 3u_i \pm 4u_{i\pm 1} \mp u_{i\pm 2}}{2h}. \tag{2.14}$$

The h or h^2 as index in the operator D will be skipped in the labeling of the operator in the following and should be understood implicitly.

Proposition 2.2.2. Assume sufficiently smooth quantities. The operators in definition 2.2.6 are indeed of the claimed order. Further the second derivatives (denoted by the “2” in $D_x^{2\pm}$) read explicitly (here the order h or h^2 is already skipped)

$$D_x^{2\pm} u_i = \frac{u_i - 2u_{i\pm 1} + u_{i\pm 2}}{h^2} + \mathcal{O}(h), \tag{2.15}$$

$$D_x^{2\pm} u_i = \frac{2u_i - 5u_{i\pm 1} + 4u_{i\pm 2} - u_{i\pm 3}}{h^2} + \mathcal{O}(h^2). \tag{2.16}$$

Proof. We prove the statement only for the forward operator. The verification for the backward operator is obtained in a completely analogous way. Use the Taylor expansion

$$u_{i+1} = u_i + \partial_x u\big|_i h + \mathcal{O}(h^2) \tag{2.17}$$

and

$$D_x^{+} u_i = \partial_x u\big|_i = \frac{u_{i+1} - u_i}{h} + \mathcal{O}(h) \tag{2.18}$$

immediately follows. Consider the Taylor expansions at the two points

$$u_{i+1} = u_i + \partial_x u\big|_i h + \frac{1}{2}\partial_x^2 u\big|_i h^2 + \mathcal{O}(h^3), \tag{2.19}$$

$$\begin{aligned} u_{i+2} &= u_i + \partial_x u\big|_i 2h + \frac{1}{2}\partial_x^2 u\big|_i (2h)^2 + \mathcal{O}(h^3) \\ &= u_i + 2\partial_x u\big|_i h + 2\partial_x^2 u\big|_i h^2 + \mathcal{O}(h^3). \end{aligned} \tag{2.20}$$

Take four times equation (2.19) and subtract equation (2.20) to obtain

$$4u_{i+1} - u_{i+2} = 3u_i + 2\partial_x u\big|_i h + \mathcal{O}(h^3) \tag{2.21}$$

and hence the result for $D_x^{h^2+} u_i$ is shown. To obtain the result in equation (2.15) we can either take eight times equation (2.19) and subtract equation (2.20) or alternatively just take the composition

$$D_x^{2+} u_i = D_x^+ \left(D_x^+ u_i\right) \tag{2.22}$$

to get the result. Therefore we also know how to prove the rest and higher (in order of derivatives or accuracy in powers of h) results (either taking Taylor expansion with more and more points and terms or just compose the basic operators), in particular equation (2.16). □

Ghost-point techniques and boundary issues

It is clear that the techniques discussed above only work in the region of the grid where the boundary grid points are not involved. If the grid points on the boundaries are involved the discretization of the derivatives breaks down. Our strategy is to use **one-sided** finite difference approximations of the **same order** at the outer boundary (which corresponds to large radial distance at the "right side" of our computational domain) with the one-sided stencils defined in definition 2.2.6. The inner boundary is located at the origin $r = 0$, corresponding to $x_0 = 0$ in the current notation. There we make use of a different technique. We assume that our variables can be expanded in an appropriate basis. Then we are able to deal with a set of mode functions instead. The modes will have a definite parity and we can add artificial "**ghost points**". These are formal extensions of the domain with (in this case) negative grid numbers and the values at these points are determined by the parity. We have the following lemma.

Lemma 2.2.1. The value at the origin and at the first ghost points (labeled with u_{-1} and u_{-2} below) of a variable u with definite parity[7] and discretization u_i is given as

$$u_0 = \begin{cases} u_0, & \text{if } u \text{ even}, \\ 0, & \text{if } u \text{ odd}, \end{cases} \tag{2.23a}$$

$$u_{-1} = \begin{cases} u_1, & \text{if } u \text{ even}, \\ -u_1, & \text{if } u \text{ odd}, \end{cases} \tag{2.23b}$$

$$u_{-2} = \begin{cases} u_2, & \text{if } u \text{ even}, \\ -u_2, & \text{if } u \text{ odd}. \end{cases} \tag{2.23c}$$

Proposition 2.2.3. We summarize our finite difference stencils for the derivatives of a variable of definite parity $u(x)$ for a grid u_i and $i = 0, \ldots, N$ with step size h.

$$(\partial_x u)_{i,\ i=2,\ldots,N-1} = \frac{-u_{i-1} + u_{i+1}}{2h}, \tag{2.24a}$$

$$(\partial_x u)_0 = \begin{cases} 0, & \text{if } u \text{ even}, \\ \frac{u_1}{h}, & \text{if } u \text{ odd}, \end{cases} \tag{2.24b}$$

$$(\partial_x u)_1 = \begin{cases} \frac{-u_0+u_2}{2h}, & \text{if } u \text{ even}, \\ \frac{u_2}{2h}, & \text{if } u \text{ odd}, \end{cases} \tag{2.24c}$$

$$(\partial_x u)_N = \frac{u_{N-2} - 4u_{N-1} + 3u_{i+1}}{2h}, \tag{2.24d}$$

$$\left(\partial_x^2 u\right)_{i,\ i=2,\ldots,N-1} = \frac{u_{i-1} + 2u_i - u_{i+1}}{h^2}, \tag{2.24e}$$

$$\left(\partial_x^2 u\right)_0 = \begin{cases} \frac{-2u_0+2u_1}{h^2}, & \text{if } u \text{ even}, \\ 0, & \text{if } u \text{ odd}, \end{cases} \tag{2.24f}$$

$$\left(\partial_x^2 u\right)_1 = \begin{cases} \frac{u_0-2u_1+u_2}{h^2}, & \text{if } u \text{ even}, \\ \frac{-2u_1+u_2}{h^2}, & \text{if } u \text{ odd}, \end{cases} \tag{2.24g}$$

$$\left(\partial_x^2 u\right)_N = \frac{-u_{N-3} + 4u_{N-2} - 5u_{N-1} + 2u_N}{h^2}. \tag{2.24h}$$

Proof. Straightforward application of the one-sided stencil at the outer and lemma 2.2.1 at the inner boundary. □

[7] We call a variable $u(x)$ at x_0 **even** if $u(-x) = u(x)$ and **odd** if $u(-x) = -u(x)$ for x close to x_0.

Dissipation operator

For the implementation of several partial differential equations we will make use of a numerical technique of "**artificial dissipation**" which is added on the one side of the equation, see section 2.3.3, also for the reference. At this point we are in the position to define the corresponding dissipation operator and its boundary behavior. Its use and meaning will be discussed later. We are using a method of second order and therefore are after an operator of fourth order as will be discussed later. Recall the operators introduced in definition 2.2.6.

Definition 2.2.7. The **dissipation operator** Q_r of order r (the exponent denotes the power) is given as

$$Q_r = (-)^{r-1} 2^{-2r} h^{2r-1} D_x^{+r} D_x^{-r} \tag{2.25}$$

where h denotes the step size of the lattice in x-direction.

At the outer boundary we will set the dissipation to zero. If dissipation is applied at the innermost grid points we make use of ghost points there.

Proposition 2.2.4. For order $r = 2$ we have the following action of the dissipation operator applied to a variable u with lattice step size h (outermost point at N) for $i = 2, \dots, N-2$

$$(Q_2 u)_i = -2^{-4} h^3 D^{+2} D^{-2} = -\frac{u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}}{16h}. \tag{2.26}$$

and application of ghost points at the innermost points,

$$(Q_2 u)_0 = -\frac{u_{-2} - 4u_{-1} + 6u_0 - 4u_1 + u_2}{16h} = \begin{cases} -\frac{6u_0 - 8u_1 + 2u_2}{16h}, & \text{if } u \text{ even}, \\ 0, & \text{if } u \text{ odd}, \end{cases} \tag{2.27a}$$

$$(Q_2 u)_1 = -\frac{u_{-1} - 4u_0 + 6u_1 - 4u_2 + u_3}{16h} = \begin{cases} -\frac{-4u_0 + 7u_1 - 4u_2 + u_3}{16h}, & \text{if } u \text{ even}, \\ -\frac{5u_1 - 4u_2 + u_3}{16h}, & \text{if } u \text{ odd}, \end{cases} \tag{2.27b}$$

$$(Q_2 u)_2 = -\frac{u_0 - 4u_1 + 6u_2 - 4u_3 + u_4}{16h} = \begin{cases} -\frac{u_0 - 4u_1 + 6u_2 - 4u_3 + u_4}{16h}, & \text{if } u \text{ even}, \\ -\frac{-4u_1 + 6u_2 - 4u_3 + u_4}{16h}, & \text{if } u \text{ odd}. \end{cases} \tag{2.27c}$$

Proof. It is a direct application of definition 2.2.7 and the technique of ghost points as indicated. □

Euler discretization and simple integrators

Consider an initial value problem

$$\partial_x u = f(x, u), \tag{2.28}$$
$$u(x_0) = u_0. \tag{2.29}$$

The right-hand side is discretized as $f_i = f(x_i, u_i)$ with step-size h.

Definition 2.2.8. The **Euler method** for the integration of the initial value problem (2.28) is given as

$$u_{i+1} = u_i + hf_i. \tag{2.30}$$

Such a method is conventionally (see for example Press et al. (2007, section 17.1)) called to be of **order** p if its local truncation error is of order $\mathcal{O}(h^{p+1})$.

Remarks 2.2.2.
- The Euler method is a first-order method (as can be seen with the Taylor expansion in the proof of proposition 2.2.2).
- Even though it is relatively simple and intuitive it is not very often applied in practice. That is due to the low accuracy and problematic behavior for several types of ordinary differential equations (in particular stiff ones, see Press et al. (2007, section 17.1) and Butcher (2003, section 21 and following) for example).

Higher order integration and Runge-Kutta-Heun integration

In principle we can use for the numerical integration higher order stencils of higher order as derived at the beginning of this section, consult in particular definition 2.2.6 and generalizations of it. An often applied alternative (also by us in the numerics) are "predictor-corrector" methods where intermediate time steps are used. These are subsequently reached by Euler steps. Then one interpolates between the different primary integration steps to obtain an overall update. The Euler method is unsymmetric in the sense that the discretization is performed in one direction only (in equation (2.28) the forward discretization). The use of intermediate time steps symmetrizes the method and leads to a cancellation of the error terms.

The generalization of the previously discussed Euler method were worked out at the turn of the 19^{th} century, see Runge (1895), Heun (1900), Kutta (1901), see Butcher (2003) for a textbook devoted essentially to that topic.

Definition 2.2.9. A generalization of the Euler integration for an ordinary differential equation is to take intermediate time steps into account, evaluate it there and then interpolate to obtain results of higher accuracy. Those schemes are generally called **Runge-Kutta(-Heun) integration**. A prominent second-order scheme uses (in the notation introduced in connection to equation (2.28))

$$u_{i+1} = u_i + hf\left(x_i + \frac{h}{2}, u_i + \frac{h}{2}f_i\right). \tag{2.31}$$

The error term $\mathcal{O}(h^2)$ of the Euler steps cancels and the method has an truncation error of $\mathcal{O}(h^3)$ and is therefore of second-order, see again Press et al. (2007, section 17.1) and Butcher (2003, section 23 and following)

Remarks 2.2.3.

- There exist a huge amount of integrators of the type of Runge-Kutta-Heun, see the cited literature.
- Actually we implemented the generalization due to Shu (1998). For our interests the difference between the ordinary Runge-Kutta integrator and the one by proposed by Shu is negligible. Usually one performs a half step in time and updates with the knowledge of the quantities there by doing another half step of the variables. Shu suggested to do instead a full time step and evolve with the knowledge there to the next full time step. Then one averages between the partial results. It has advantages for the application for conservation laws. For our purposes we should consider the Runge-Kutta and the Shu integrator as equivalent.
- We concentrated our discussion on explicit schemes. There exist also implicit ones which are slightly more involved from the point of view of the implementation. They are also discussed in the given literature.
- In this line we should also mention a partially implicit scheme, see Cordero-Carrión and Cerda-Duran (2012) which we also used for numerical experiments. It seems to be better suited for stiff problems for example.

2.2.4. Spectral method for ordinary differential equations

We now turn to a different technique to tackle differential equations, for references consult for example Fornberg (1998), Boyd (2001), Grandclément and Novak (2009), Press et al. (2007)[8]. The very basic ideas of the expansion of quantities in certain basic functions and therefore a transformation

[8]For Press et al. (2007) make sure that it is a recent version.

to a different underlying space can be traced back at least to Fourier. The application to ordinary differential equations seems to go back to Lanczos even though more systematically only from the 1970s on, see the introduction of Fornberg (1998) for more historic discussions. In the field of numerical relativity it was presumably used for the first time in Bonazzola and Marck (1986, 1990). Nowadays there are many projects in numerical relativity using spectral methods including Bonazzola et al. (1999), Csizmadia et al. (2013), Szilágyi (2014).

We want to solve an ordinary differential equation of the form

$$F(u(x)) = g(x) \tag{2.32}$$

where F is a differential operator, usually subject to some boundary conditions. We require that the solution u is smooth.

Remark 2.2.1. Actually we should really assume smoothness (or better analyticity) here, so u should be infinitely often differentiable. "Discontinuities like shocks are bad – don't even try spectral methods" (Press et al. (2007, p. 1083)), even though other groups are more optimistic using multi-domain methods. Also discontinuities in higher derivatives are reported to cause problems.

In section 2.2.3 the variables and its derivatives were discretized locally to approximate the equation we wanted to solve to obtain a solution. Another approach is to approximate the solution **globally** by expanding it in a complete basis $\{\phi_\ell(x)\}$ as

$$u(x) \approx u_{(L)}(x) = \sum_{\ell=0}^{L-1} \hat{u}_\ell \phi_\ell(x). \tag{2.33}$$

Definition 2.2.10. The representation (2.33) is called **spectral expansion** and gives rise to the so-called **spectral method** with **spectral coefficients** $\hat{u}_\ell$. We call the real physical space where the $u(x)$ "live" **configuration space** and the one of the $\hat{u}_\ell$ **spectral space**.

The spectral coefficients $\hat{u}_\ell$ are independent of x, $\partial_x \hat{u}_\ell = 0$. Both descriptions are equivalent (bijective) and if one manages to transform easily (computationally speaking) between both spaces one may choose either of them depending on the purpose one has in mind; taking all kinds of nonlinear operations can easily be done for $u(x)$ while taking derivatives is straightforward in the coefficient space. Derivatives are written as

$$\partial_x u(x) \approx \partial_x u_{(L)}(x) = \sum_{\ell=0}^{L-1} \hat{u}_\ell \partial_x \phi_\ell(x). \tag{2.34}$$

Since $\{\phi_\ell(x)\}$ is a basis the derivative in equation (2.34) can also be re-expressed in that basis with different coefficients $\hat{v}_\ell$ as

$$\partial_x u_{(L)}(x) = \sum_{\ell=0}^{L-1} \hat{v}_\ell \phi_\ell(x). \tag{2.35}$$

The relation between both kinds of coefficients is

$$\hat{v}_m = \sum_{\ell=0}^{L-1} D_{m\ell} \hat{u}_\ell \tag{2.36}$$

with the **differentiation matrices** $D_{m\ell}$ (and similar for higher derivatives). The task is then to calculate those difference matrices.

The choice of appropriate basis functions is obviously one of the first tasks. It would be desirable if those approximated the solution well (rapid convergence of the sum (2.33) such that a moderate value of L is sufficient), the quantities like the difference matrix are easy to compute and that the transformation between the real physical **configuration space** where $u(x)$ lives and the **spectral space** of $\hat{u}_\ell$ is easy and fast. The choice is problem-depended, or nicely formulated in Boyd (2001, p. 10): "Geometry chooses the basis set". As a rough guideline we should take[9]:

- For periodic problems trigonometric functions are suitable, so expansion in a Fourier series. For the transformation so-called "fast Fourier transformations" exists.
- For non-periodic problems orthogonal polynomials of Jacobi type do a very good job. Prominent candidates are Chebyshev and Legendre polynomials discussed in section 2.2.5.

Another task is to find a technique to determine the expansion coefficients $\hat{u}_\ell$. Basically the aim of the game is to minimize the residual

$$R(x) = F(u(x)) - g(x) \tag{2.37}$$

of equation (2.32) while the boundary conditions are satisfied. The expansion coefficients of the residual are obtained with the difference matrices. The boundary conditions provide some requirements on the residual already. Therefore there remain less than L (let us assume more than zero boundary conditions) further relations for L coefficients $\hat{u}_\ell$ of the solution. There are three main techniques how to minimize the residual,

[9]Not following the moral principle Boyd (2001, p. 10) where Chebyshev polynomials are advertised.

- the **Tau method** satisfies the boundary conditions and makes the residual orthogonal[10] to as many of the basis functions as possible,
- the **Galerkin method** combines the chosen basis functions into a new set that automatically satisfies the boundary conditions and then makes the residual orthogonal to as many functions of the new set as possible,
- the **collocation method** requires that the boundary conditions are satisfied (as in the Tau method) and makes the residual zero at as many **collocation points** as possible.

The collocation method is also called **pseudo-spectral method** since it minimizes the residual at the collocation points in the configuration space and not in the spectral space. Therefore it is also very efficient in handling nonlinearities, so in particular a prominent option for nonlinear sets of equations like in general relativity. The value of the nonlinear expressions are just calculated on the collocation points in the configuration space. We will use a pseudo-spectral approach and therefore concentrate on that option, see the cited literature for the other choices. The task that remains is to choose a proper lattice of collocation point. There are again various options, see Press et al. (2007), Grandclément and Novak (2009) including

- **Gaussian quadrature collocation point lattice**, which are the roots of the Chebyshev polynomials (if Chebyshev polynomials are used as basis functions) and basically given by a cosine-distribution, the end-points are not included,
- **Gauss-Lobatto quadrature collocation point lattice**, which are extrema of the Chebyshev polynomials (if Chebyshev polynomials are used as basis functions) and also given by a cosine-distribution, here the end-points are included,
- an **equidistant** lattice.

In fact we will use an equidistant staggered grid in ϑ and Legendre-polynomials (see section 2.2.5) in $\cos\vartheta$. Therefore we use equidistant collocation points in ϑ.

Remarks 2.2.4.
- Usually spectral methods show exponential convergence properties for smooth settings.
- Pseudo-spectral methods can be interpreted as a kind of finite difference technique as well, see Fornberg (1998, chapter 3).

[10] Where the integral over the whole domain of the product of the residual with the basis polynomial multiplied with a certain weight is required to vanish.

2.2.5. Spin-weighted spherical harmonics

We will use in our setting spherical polar coordinates and in the angular directions a pseudo-spectral approach. Therefore we need an adapted set of basis functions living on the sphere S^2. General relativity is a tensor theory (see chapter 3.2) and so particular care should be taken. We will use quantities to which a **spin-weight**[11] can be assigned. The quantities take values in a complex bundle. From the computational perspective it is of advantage to use recursive derivations though and that is the approach we will follow here.

Jacobi polynomials

For the following derivation we mainly refer to Fornberg (1998). Jacobi polynomials play an important role in the Gaussian quadrature, see previously cited literature, in particular Press et al. (2007), Grandclément and Novak (2009). But also for the purpose of the spectral expansion and pseudo-spectral method they are an important ingredient. The most important examples of the general class of Jacobi polynomials are the Legendre and Chebyshev polynomials. While the latter are very popular as basis polynomials in general pseudo-spectral considerations. The former are key for spherical harmonics and therefore for this thesis. There are various ways to define those polynomials, for instance as a solution of a differential equation or using the orthogonality relation together with some normalization condition (see Fornberg (1998, Appendix A)). Here we take a very pragmatic point of view and define them using recursion relations. For the implementation this is a very straightforward approach.

Definition 2.2.11. Let $\alpha,\ \beta \in \mathbb{R}_{>-1}$ fixed. The **Jacobi polynomials** $P_n^{\alpha\beta}$ are maps (a different interval in $\mathbb{R}$ than $[-1,1]$ could be chosen as well)

$$P_n^{\alpha\beta} : [-1,1] \to \mathbb{R}, \tag{2.38a}$$

$$x \mapsto P_n^{\alpha\beta}(x) \tag{2.38b}$$

defined recursively. The first two polynomials are given as

$$P_0^{\alpha\beta}(x) = 1, \tag{2.39a}$$

$$P_1^{\alpha\beta}(x) = \frac{(2+\alpha+\beta)x+\alpha-\beta}{2}. \tag{2.39b}$$

[11] Very roughly speaking a scalar theory is connected with spin-weight 0, a vector theory (like electrodynamics) with spin-weight 1 and a theory like (linearized) gravity with spin-weight 2. There exist generalizations of that concept to "higher-spin theories" even though their physical significance is not that clear.

The remaining ones satisfy the formula

$$\begin{aligned}2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P^{\alpha\beta}_{n+1}+\big[(2n+\alpha+\beta+1)(\alpha^2-\beta^2)\\ +(2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)x\big]\,P^{\alpha\beta}_{n}\\ +2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P^{\alpha\beta}_{n-1}=0.\end{aligned} \tag{2.40}$$

Hence with known $P^{\alpha\beta}_{n-1}$ and $P^{\alpha\beta}_{n}$ we can calculate $P^{\alpha\beta}_{n+1}$. In the special case $\alpha=\beta=-\frac{1}{2}$ they are called **Chebyshev polynomials**, in the case $\alpha=\beta=0$ **Legendre polynomials**.

Lemma 2.2.2.

- The polynomials are orthogonal to each other, thus for $n\neq m$ they satisfy

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P^{\alpha\beta}_{n}(x)P^{\alpha\beta}_{m}(x)\mathrm{d}x=0 \tag{2.41}$$

  with **weights** α and β.

- They satisfy the ordinary differential equation in x

$$\begin{aligned}(1-x^2)\partial_x^2P^{\alpha\beta}_{n}(x)+[(\beta-\alpha)-(\alpha+\beta+2)x]\,\partial_xP^{\alpha\beta}_{n}(x)\\ +n(n+\alpha+\beta+1)P^{\alpha\beta}_{n}(x)=0.\end{aligned} \tag{2.42}$$

Proof. Both relations can be simply verified by plugging in the definition. □

Scalar spherical harmonics (spin-weight 0)

Scalar spherical harmonics play a particular role in quantum mechanics for the angular momentum and are discussed in atomic theory, see for example Weyl (1950), Louck (2006). The definition of spherical harmonics (especially of higher weight) is far from unique, there exist many conventions, see Sandberg (1978), Thorne (1980). We will follow the one in Sarbach and Tiglio (2001), Rinne (2009). The computations are essential for the implementation and hence included explicitly here.

Definition 2.2.12. The **(scalar) spherical harmonics** (**spin-weight** 0) are defined as a map

$$Y_{\ell m}(\vartheta,\varphi):[0,\pi]\times[0,2\pi],\;\;(\vartheta,\varphi)\mapsto Y_{\ell m}(\vartheta,\varphi) \tag{2.43}$$

with the use of the previously introduced Legendre polynomials

$$Y_{\ell m}(\vartheta, \varphi) = (-)^m \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sqrt{(1-x^2)^m} e^{im\varphi} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_\ell(x)|_{x=\cos\vartheta}. \quad (2.44)$$

The P_ℓ are the Legendre polynomials in definition 2.2.11 where the indices $\alpha = \beta = 0$ are skipped.

The following lemma is particularly beneficial if one wants to implement the functions and we will make use of these relations.

Lemma 2.2.3. The recursion relation for the Legendre polynomials in definition 2.2.12 are given as

$$P_0(x) = 1, \quad (2.45a)$$

$$P_1(x) = x, \quad (2.45b)$$

$$P_{\ell \geqq 2}(x) = \frac{(2\ell-1)xP_{\ell-1}(x) - (\ell-1)P_{\ell-2}(x)}{\ell}. \quad (2.45c)$$

For $m = 0$ the recursion relations for the scalar harmonics

$$Y_\ell(\vartheta) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\vartheta), \quad (2.46)$$

are

$$Y_0(\vartheta) = \sqrt{\frac{1}{4\pi}} = \text{const.}, \quad (2.47a)$$

$$Y_1(\vartheta) = \sqrt{\frac{3}{4\pi}} \cos\vartheta, \quad (2.47b)$$

$$Y_{\ell \geqq 2}(\vartheta) = \sqrt{\frac{2\ell+1}{4\pi}} \frac{(2\ell-1)\cos\vartheta \sqrt{\frac{4\pi}{2\ell-1}} Y_{\ell-1}(\vartheta) - (\ell-1)\sqrt{\frac{4\pi}{2\ell-3}} Y_{\ell-2}(\vartheta)}{\ell}. \quad (2.47c)$$

The recursion relation for the first derivatives in ϑ read

$$\partial_\vartheta Y_0(\vartheta) = 0, \quad (2.48a)$$

$$\partial_\vartheta Y_1(\vartheta) = -\sqrt{\frac{3}{4\pi}} \sin\vartheta, \quad (2.48b)$$

$$\partial_\vartheta Y_{\ell \geqq 2}(\vartheta) = \frac{1}{\ell} \Bigg[-(2\ell-1)\sin\vartheta \sqrt{\frac{4\pi}{2\ell-1}} Y_{\ell-1}(\vartheta) + (2\ell-1)\cos\vartheta \sqrt{\frac{4\pi}{2\ell-1}} \partial_\vartheta Y_{\ell-1}(\vartheta) - (\ell-1)\sqrt{\frac{4\pi}{2\ell-3}} \partial_\vartheta Y_{\ell-2}(\vartheta) \Bigg]. \quad (2.48c)$$

Proof. For the Legendre polynomials one uses definition 2.2.11, for the scalar harmonics definition 2.2.12 and for the derivatives $\partial_\vartheta P_\ell(\cos\vartheta) = -\sin\vartheta \partial_x P_\ell(x)\big|_{x=\cos\vartheta}$ (chain rule). □

Higher spin harmonics

Spherical harmonics with non-trivial spin-weight as required for a theory like general relativity and the authors below introduced handy ð-operators in the 1960s, see Newman and Penrose (1966), Goldberg et al. (1967). For a discussion of the ð-formalism with spin-weighted spherical harmonics and its impact for numerical relativity see Gomez et al. (1997) and because it is in relation with the later used formulation of the constraints we refer also to the notes by Rácz and Winicour (2016). Our conventions follow Rinne (2009). For the higher-order spin-weights it makes particular sense to be specific because of the mentioned ambiguities in the conventions in the literature.

As already stated in the definition 2.2.12 the ordinary spin-weighted harmonics are $Y_{\ell m}(\vartheta, \varphi) \equiv Y_{\ell m} = {}_0Y_{\ell m}$ where we dropped the angular dependency for simplicity but label the spin-weight 0 explicitly here.

Lemma 2.2.4. We just calculate the second partial derivative in ϑ of the spin-weight zero harmonic for later use

$$\partial_\vartheta^2 Y = -\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta Y - \ell(\ell+1). \tag{2.49}$$

Proof. The Legendre polynomials satisfy the differential relation

$$\partial_x^2 P_\ell(x) = \frac{2x}{(1-x^2)}\partial_x P_\ell(x) - \frac{\ell(\ell+1)}{(1-x^2)}P_\ell(x). \tag{2.50}$$

Hence we calculate (prefactor $\sqrt{(2\ell+1)/4\pi}$ ignored)

$$\partial_\vartheta Y = \partial_\vartheta P_\ell(\cos\vartheta) = -\sin\vartheta \partial_x P_\ell(x), \tag{2.51}$$

$$\Leftrightarrow \partial_x P_\ell(x) = -\frac{1}{\sin\vartheta}\partial_\vartheta Y, \tag{2.52}$$

$$\begin{aligned}\partial_\vartheta^2 Y = \partial_\vartheta^2 P_\ell(\cos\vartheta) &= -\cos\vartheta \partial_x P_\ell(x) + \sin^2\vartheta \partial_x^2 P_\ell(x),\\ &\overset{(2.50)}{=} -\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta Y - \ell(\ell+1)Y.\end{aligned} \tag{2.53}$$

□

Definition 2.2.13. The **"eth"-operators**[12] ð and $\bar{ð}$ are defined as

$$ð {}_sY_{\ell m} := \left[-\partial_\vartheta - \frac{i}{\sin\vartheta}\partial_\varphi + s\frac{\cos\vartheta}{\sin\vartheta}\right] {}_sY_{\ell m}, \tag{2.54a}$$

$$\bar{ð} {}_sY_{\ell m} := \left[-\partial_\vartheta + \frac{i}{\sin\vartheta}\partial_\varphi - s\frac{\cos\vartheta}{\sin\vartheta}\right] {}_sY_{\ell m} \tag{2.54b}$$

with imaginary unit i in front of the φ-derivative. They are used to lower or raise the **spin-weight** (can also be seen as definition of the spin-weight)

$${}_{s+1}Y_{\ell m} = [(\ell - s)(\ell + s + 1)]^{-1/2}\, ð\; {}_sY_{\ell m}, \tag{2.55a}$$

$${}_{s-1}Y_{\ell m} = -\,[(\ell + s)(\ell - s + 1)]^{-1/2}\, \bar{ð}\; {}_sY_{\ell m}. \tag{2.55b}$$

A **parity transformation** on the sphere is a map $(\vartheta, \varphi) \mapsto (\pi - \vartheta, \pi + \varphi)$. It flips the sign of one Cartesian coordinate as can be verified by a drawing. Therefore it corresponds to a space inversion. We call a quantity $f(\vartheta, \varphi)$ **even** (or **polar**) if it behaves as $f(\pi - \vartheta, \pi + \varphi) = (-)^\ell f(\vartheta, \varphi)$ under parity transformation and **odd** (or **axial**) if the factor is $(-)^{\ell+1}$ instead.

Definition 2.2.14. For the angular indices $A,\ B \in \{\vartheta, \varphi\}$ we define the gradient $Y_A = \hat{\nabla}_A Y = \partial_A Y$ (we denote here the derivative on the sphere with the hat $\hat{}$, the angular indices are raised and lowered with the metric on the sphere). With the two-dimensional Levi-Civita tensor on the sphere its dual $S_A = \hat{\epsilon}^B{}_A Y_B$ (indices $\ell,\ m$ suppressed but implicitly implied). We explicitly calculate some quantities on the sphere in appendix A.1.2. Further consider the trace-free part of the second covariant derivative $Y_{AB} = \left[\hat{\nabla}_A \hat{\nabla}_B Y\right]^{\text{tf}}$ ("tf" denoting trace-free) and second covariant derivative of the dual $S_{AB} = \frac{1}{2}(\hat{\nabla}_A S_B + \hat{\nabla}_B S_A)$ which has vanishing trace by construction. Note that the defined quantities carry indices ℓ, m like $Y_{\vartheta\vartheta,\ell m}$ for example (which are suppressed sometimes for readability).

We call the quantities

- $Y_\ell(\vartheta) =_0 Y_{\ell m}$ the **scalar harmonics**,
- $Y_{\vartheta,\ell}(\vartheta)$ the **vector harmonics** and
- $Y_{\vartheta\vartheta,\ell}(\vartheta)$ the **tensor harmonics**.

Proposition 2.2.5. For $m = 0$ ($\Leftrightarrow$ the quantities are independent of φ and we suppress the index $m = 0$) all spherical harmonics are given as

- the scalar harmonics $Y_\ell(\vartheta) =_0 Y_{\ell m}$,

[12] The lower case old-English respectively Icelandic letter ð was chosen to represent the derivative of the spin-weight and we follow that convention.

- the vector harmonics $Y_{\vartheta,\ell}(\vartheta) = \partial_\vartheta Y_\ell(\vartheta) = -\frac{1}{2}\sqrt{\ell(\ell+1)}({}_1Y - {}_{-1}Y)$ and
- the tensor harmonics $Y_{\vartheta\vartheta,\ell}(\vartheta) = -\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta Y_\ell(\vartheta) - \frac{\ell(\ell+1)}{2}Y_\ell(\vartheta) = \frac{1}{4}\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}({}_2Y + {}_{-2}Y)$.

Proof. The relation to the spin-weighted harmonics follows by direct calculation using definition 2.2.13 and the action $\partial_\varphi = 0$. We will calculate the explicit formulas for the spherical harmonics and show in particular that the odd (under parity transformation) harmonics vanish. Some calculations of quantities on S^2 (denoted again by a $\hat{}$) are given in appendix A.1.2.

Then the even vector harmonics, the gradient $Y_A = \hat{\nabla}_A Y = \partial_A Y$ are explicitly

$$Y_\varphi = \partial_\varphi Y = 0, \tag{2.56a}$$
$$Y_\vartheta = \partial_\vartheta Y. \tag{2.56b}$$

and the odd ones are the duals of the gradient, $S_A = \hat{\epsilon}^B{}_A Y_B$ and hence ($B \stackrel{!}{=} \vartheta$)

$$S_\varphi = \hat{\epsilon}^\vartheta{}_\varphi Y_\vartheta = \sin\vartheta\partial_\vartheta Y, \tag{2.57a}$$
$$S_\vartheta = \hat{\epsilon}^\vartheta{}_\vartheta Y_\vartheta = 0. \tag{2.57b}$$

For the tensor harmonics we start again with the even ones. The trace-free part of the second covariant derivative,

$$\begin{aligned} Y_{AB} = \left[Y_{AB} = \hat{\nabla}_A\hat{\nabla}_B Y\right]^{\mathrm{tf}} &= \hat{\nabla}_A\hat{\nabla}_B Y + \frac{1}{2}\ell(\ell+1)\hat{g}_{AB}Y \\ &= \partial_A\partial_B Y - \hat{\Gamma}^C{}_{AB}\partial_C Y + \frac{\ell(\ell+1)}{2}\hat{g}_{AB}Y. \end{aligned} \tag{2.58}$$

Hence we have

$$Y_{\vartheta\vartheta} = \partial_\vartheta^2 Y + \frac{\ell(\ell+1)}{2}Y \stackrel{(2.49)}{=} -\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta Y - \frac{\ell(\ell+1)}{2}Y, \tag{2.59a}$$
$$Y_{\vartheta\varphi} = Y_{\varphi\vartheta} = 0, \tag{2.59b}$$
$$Y_{\varphi\varphi} = \cos\vartheta\sin\vartheta\partial_\vartheta Y + \frac{\ell(\ell+1)}{2}\sin^2\vartheta Y = -\sin^2\vartheta Y_{\vartheta\vartheta}. \tag{2.59c}$$

The odd ones are the symmetrized covariant derivative of the odd vector harmonics,

$$S_{AB} = \frac{1}{2}(\hat{\nabla}_A S_B + \hat{\nabla}_B S_A) = \frac{1}{2}(\partial_A S_B + \partial_B S_A) - \hat{\Gamma}^C{}_{AB}S_C. \tag{2.60}$$

It immediately follows for the first term $B = \varphi$ and hence $A = \vartheta$ for non-vanishing contribution. The other way around for the second term. In the third term we need $C = \varphi$ and then $A \neq B$. Therefore

$$S_{\vartheta\vartheta} = S_{\varphi\varphi} = 0, \tag{2.61a}$$

$$\begin{aligned} S_{\vartheta\varphi} = S_{\varphi\vartheta} &= \frac{1}{2}\partial_\vartheta S_\varphi - \hat{\Gamma}^{\varphi}{}_{\varphi\vartheta} S_\varphi = \frac{1}{2}\cos\vartheta\partial_\vartheta Y + \frac{1}{2}\sin\vartheta\partial_\vartheta^2 Y - \cos\vartheta\partial_\vartheta Y \\ &= \frac{1}{2}\sin\vartheta\partial_\vartheta^2 Y - \frac{1}{2}\cos\vartheta\partial_\vartheta Y \overset{(2.49)}{=} -\cos\vartheta\partial_\vartheta Y - \frac{\ell(\ell+1)}{2}\sin\vartheta Y = \sin\vartheta Y_{\vartheta\vartheta}. \end{aligned} \tag{2.61b}$$

□

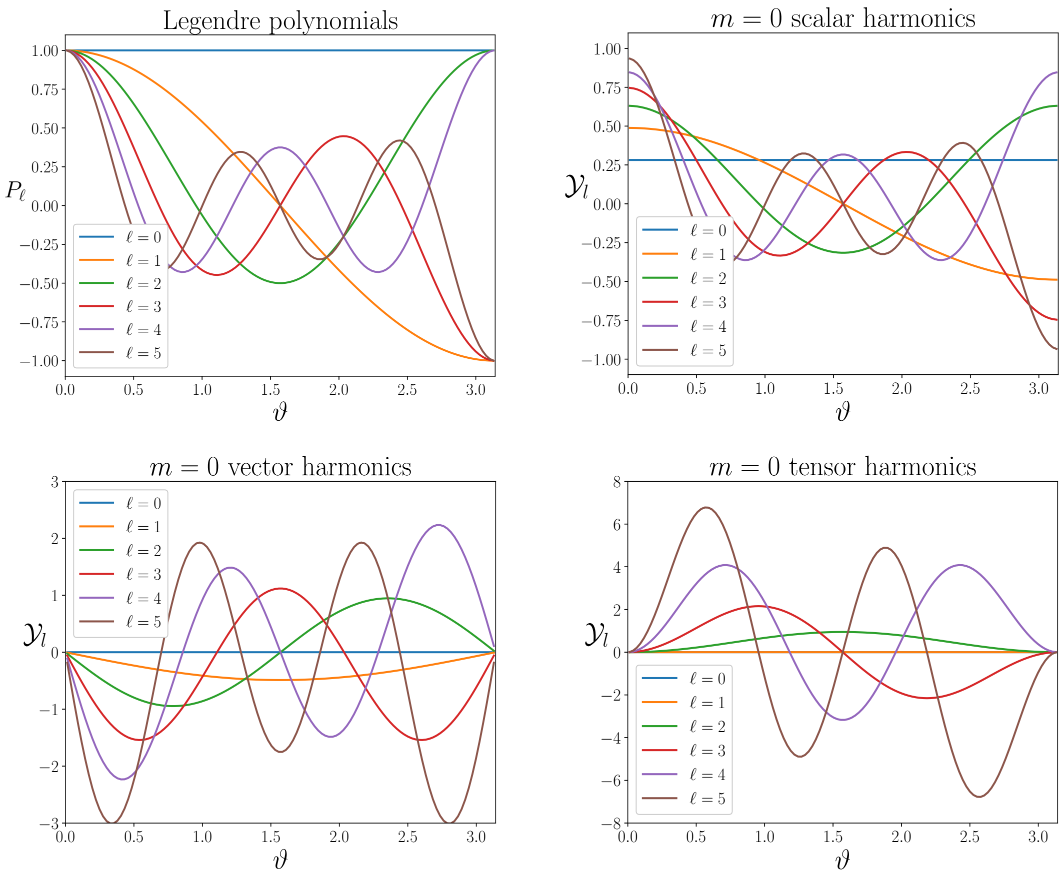


Figure 2.2.: The first few Legendre polynomials $P(x)$ for the argument $x = \cos\vartheta$ and spherical harmonics of the discussed types $Y(\vartheta)$ (spin-weight 0), $Y_\vartheta(\vartheta)$ (spin-weight 1) and $Y_{\vartheta\vartheta}(\vartheta)$ (spin-weight 2).

We will speak in the remainder often just of scalar, vector and tensor quantities if their angular part expands in the corresponding way. The reader should keep in mind that they are in fact (combinations of positive and negative) spin-weighted quantities, see definition 2.2.13.

2.2.6. Validation of the numerical method

For the implementation of differential equations on the computer we have to approximate the data. We discussed already two ways of doing so, the method of finite differences in section 2.2.3 where variables and differential operators are evaluated on a discrete grid with finite step size and the spectral method in section 2.2.4 where the solution is approximated by a discrete set of coefficients for given basis functions. There exist more techniques, which are in some sense related to the finite difference method such as the finite volume method (see LeVeque (1992) for example) or the finite element method (see for example Johnson (1982)). We will not discuss these approaches here.

As already mentioned spectral methods usually have a very good convergence behavior. In our simulations the error will be dominated (for reasonable choices of parameters) by the one caused by using finite differences. The analysis of numerical errors is a standard topic in numerical analysis, for a good source consult for example Richtmyer and Morton (1967).

In general there are several kinds of numerical errors. One aim for a numerical analyst is to understand those different sources of inaccuracies. Here we will list some of them following Rezzolla and Zanotti (2013, chapter 8) which can be consulted for more details.

- Error of **machine precision**: the computer uses a different representation for floating-point numbers than we use in the thesis. Label a floating-point representation of a rational number α on the computer as $\mathrm{fp}(\alpha)$. Then the inaccuracy can be represented as $\mathrm{fp}(\alpha) \mapsto \mathrm{fp}(\alpha) + \epsilon_{\mathrm{mp}}$. In the scripting language Python the error ϵ_{mp} is usually of the order of 10^{-16}.

- **Round-off** error: Due to an accumulation of machine-precision error when performing several floating-point operations one gets this kind of error. Usually it is related or can be estimated with the machine precision error.

- **Truncation error**: This is an error of different nature than the ones discussed above. Due to the discretization one truncates quantities to a prescribed order, see section 2.2.3. It exists in a local and a global version and is of high importance in the validation of the numerical computation and discussed in more detail in the following.

Definition 2.2.15. Consider a differential equation which is supposed to be satisfied on the continuum level and its discretized version obtained by taking finite differences. The discretized version has a **local truncation error** due to the Taylor expansion, see section 2.2.3. The **global truncation error** is defined as the norm[13] of the local error over the whole domain. It is resolution-dependent ($\Leftrightarrow$ step size-dependent) and denoted by ϵ_h for step size h. The solution to the equation is said to be **consistent** if

$$\lim_{h\to 0} \epsilon_h = 0. \tag{2.65}$$

The **global error** $\mathrm{err}(h)$ is the norm of the difference between discrete and continuum[14] solution. The discretization is **convergent** if the global error tends to zero with h^p for a p^{th}-order scheme.

It was shown already in Richardson (1911) that for uniform step size h and a centered stencil of second order the global error for an equation which is supposed to vanish on the continuum level (a vanishing constraint or the difference between numerical and exact solution) behaves as

$$\mathrm{err}(h) = ch^2 + \text{ higher even powers of } h \tag{2.66}$$

[13] Let us briefly discuss several issues related to norms. We restrict to the one-dimensional case but generalizations are straightforward. The p-norm of a continuum quantity $u(x)$ in the interval $x \in [a, b]$ is

$$||u||_p := \left(\frac{1}{b-a} \int_a^b |u(x)|^p \mathrm{d}x \right)^{1/p}. \tag{2.62}$$

Its discrete analog reads (for N grid points)

$$||u||_p := \left(\frac{1}{N} \sum_{n=0}^{N} |u_n|^p \right)^{1/p}. \tag{2.63}$$

Of particular importance is the discrete 2-norm

$$||u||_2 := \sqrt{\frac{1}{N} \sum_{n=0}^{N} |u_n|^2}. \tag{2.64}$$

and often in the remainder of the thesis it is implicitly understood that the discrete case $p = 2$ is implied.

[14] Of course in practice the solution on the continuum level is not always known. If it is the numerical solution is more of theoretical interest.

with some h-independent constant c. In general the exponent 2 is replaced by p for a p^{th}-order system. The expansion is known as **Richardson expansion**.

Proposition 2.2.6. Consider for a system with Richardson expansion two resolutions for the same spatial domain, namely the original one and one with twice the number of grid points. They correspond to step sizes h and $h/2$. For a 2^{nd}-order scheme the numerical implementation is convergent if four times the global error of the doubled resolution is not larger than the error of the ordinary resolution,

$$4 \times \operatorname{err}\left(\frac{h}{2}\right) \geq \operatorname{err}(h). \tag{2.67}$$

Otherwise the convergence regime is not (yet) reached and the code is not (yet) convergent.

Proof. In general the global error behaves like $\operatorname{err}(h) = ch^\alpha$ for resolution-independent constants c and $\alpha \in \mathbb{R}$ (analytically $\equiv 2$ for a 2^{nd}-order scheme). Then

$$4 \times \operatorname{err}\left(\frac{h}{2}\right) = 4c\left(\frac{h}{2}\right)^\alpha = 2^{2-\alpha}\operatorname{err}(h). \tag{2.68}$$

Hence it converges if $\alpha \geq 2$ and it fails to do so for $\alpha < 2$. Therefore a 2^{nd}-order scheme should have a global error that must decrease by at least a factor of four when doubling the resolution. □

We will make use of these convergence tests in form of residual tests later on to show that the code works as it should.

If there is no vanishing relations because, for instance, the exact solution is not known, one can use three resolutions to perform a "self-convergence test" instead, see for example Baumgarte and Shapiro (2010), Rezzolla and Zanotti (2013).

2.3. Partial differential equations and their numerical methods

Quite some of the techniques to solve ordinary differential equations can be generalized to partial differential equations as we will see. Also conceptual issues like the notion of well-posedness can be applied.

There exist a huge amount of general literature about partial differential equations that were helpful for the current chapter including John (1982), Copson (1975), Courant and Hilbert (1962), Evans (2010), Renardy and Rogers (1993) and from the numerical perspective Tveito and Winther (2005), Thomas (1995, 1999). With a focus on the application in general relativity see Rendall (2008), Geroch (1996). We also mention Alinhac (2009) as a good introduction of hyperbolic partial differential equations with an emphasis on recent techniques applied for the study of general relativity and Hörmander (1997) where nonlinear hyperbolic equations are discussed. We refer to more specialized literature in the corresponding sections.

2.3.1. Classifications of partial differential equations

Recall definition 2.2.2 for the classification of ordinary differential equations.

Definition 2.3.1. A **partial differential equation** F is a functional for a variable, say u (or more variables), that depends continuously on coordinates, say x and y (or more coordinates), where derivatives of the variables with respect to the coordinates may occur in various combinations (with respect to order, linearity and similar properties). As example to generalize equation (2.1) (here only up to second order which is an usual restriction in physics and enough for our purposes)

$$F(x, y, u, \partial_x u, \partial_y u, \partial_x^2 u, \partial_x \partial_y u, \partial_y^2 u) = 0 \tag{2.69}$$

In general one deals with systems of partial differential equations again.

All the statements given in definition 2.2.2 for ordinary differential equations can be transfered to partial differential equations as well.

Definition 2.3.2. A general form of a **semilinear** (note that is in particular quasilinear) second-order partial differential equation in two coordinates x and y for only one variable $u(x, y)$ is

$$A\partial_x^2 u + B\partial_x \partial_y u + C\partial_y^2 u + \text{ “lower-order terms” } = 0 \tag{2.70}$$

where A, B, C do not depend on the solution u (put presumably on the coordinates x and y) and “lower-order terms” refers to all kinds of terms of order less or equal one including the inhomogeneity. The **symbol** of equation (2.70) is then defined as

$$F(x, y, u, \xi, \eta) = A\xi^2 + B\xi\eta + C\eta^2 + \text{ “lower order terms” } = 0. \tag{2.71}$$

The **principal part** of the symbol is defined as the terms in equation (2.71) without the lower-order terms,

$$F_{\text{pp}}(x,y,u,\xi,\eta) = A\xi^2 + B\xi\eta + C\eta^2$$
$$\overset{\text{repr.}}{=} (\xi\ \eta)\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \equiv v^\dagger M v. \tag{2.72}$$

We call the equation (2.70)

- **elliptic** if M is strictly definite (both eigenvalues of M come with the same sign) $\Leftrightarrow B^2 - 4AC < 0$,
- **parabolic** if M is degenerate (eigenvalue 0 of M occurs) $\Leftrightarrow B^2 - 4AC = 0$,
- **hyperbolic** if M is indefinite (both eigenvalues of M have different signs) $\Leftrightarrow B^2 - 4AC > 0$

in the corresponding domain in x, y.

Remarks 2.3.1.
- Ordinary differential equations are also partial differential equations. They are of hyperbolic type. This does not become obvious in the definition 2.3.2 but can be deduced with the fact that an ordinary differential equation has the maximal number of characteristics and the application of the equivalent of proposition 2.3.1 below.

- Often in the literature the symbol is complexified. We do not follow that convention but use the real version instead.

- Equations can be of several types for different values of the coefficients.

- The motivation for the particular names are connected to the conic sections which becomes quite obvious in the definition above. In alternative concepts which are used for the definition that aspect is sometimes lost.

- Equations of either parabolic or hyperbolic type are also called **evolutionary** equations.

We also give an alternative but essentially equivalent definition which makes use of the concept of characteristics. These are basically special curves determining the solution. The speed of propagation of information of the equation is related to the eigenvalues of the principal part. So it is possible to map the determination of the type of equation to an eigenvalue problem. Here we are just interested in the existence of the eigenvalues and its number. We will consider coupled systems of equations soon. There we will use that approach to refine the theory.

Definition 2.3.3. A solution $u(x, y$ to a partial differential equation of the form (2.69) can be given as a parametrized curve in the x-y-plane as a **solution curve** $y(x)$. With **characteristics** of the partial differential equation we identify the family of curves $y(x)$ along which the solution u is constant.

In the cited standard literature the equivalent to the following proposition can be found.

Proposition 2.3.1. At a point (x, y) a second-order quasilinear partial differential equation in two variables has either zero, one or two characteristics. The equation is

- **hyperbolic** if it has two characteristics,
- **parabolic** if it has one characteristic and
- **elliptic** if there are no characteristics.

The solution curves are determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B}{2A} \pm \frac{B}{2A}\sqrt{B^2 - 4AC}. \tag{2.73}$$

Depending on the discriminant $D := B^2 - 4AC$ there exists zero, one or two characteristics. Consider the discriminant D and observe that the cases listed in the proposition and in definition 2.3.2 coincide.

Examples 2.3.1. Because it is helpful later on let us include a few linear standard examples. Here we give them in two dimensions, generalizations are straightforward. The Laplace operator Δ (see definition 2.5.1 for the general case, here it is an abbreviation for the second derivative $\Delta = \partial_y^2$ or in two dimensions $\Delta = \partial_x^2 + \partial_y^2$) will be examined in section 2.5 but is used already here for convenience.

- **Poisson equation**:

$$\Delta u = \partial_x^2 u + \partial_y^2 u = f(u). \tag{2.74}$$

  If the inhomogeneity $f(u) \equiv 0$ the equation is called **Laplace equation**. The matrix representation in definition 2.3.2 of the principal part is $M = \mathrm{diag}(1, 1)$ and hence the equation is elliptic. It is an example for a "timeless" equation (infinite speed of propagation) and is typical for equilibrium phenomena. It does not make sense to discuss it as an evolutionary Cauchy problem.

- **Heat equation**:

$$\Delta u - \partial_x = \partial_y^2 u - \partial_x u = 0. \tag{2.75}$$

  The matrix representation of the principal part is $M = \mathrm{diag}(1, 0)$ and hence the equation is parabolic. The equation is typical for transport phenomena. The described process is irreversible (and hence marks a "direction of time").

- **Wave equation**

$$\Box u := \Delta u - \partial_x^2 u = \partial_y^2 u - \partial_x^2 u = 0. \tag{2.76}$$

  The matrix representation of the principal part is $M = \mathrm{diag}(1, -1)$ and hence the equation is hyperbolic. It is typical for reversible evolutionary phenomena and has a finite speed of propagation (here 1).

So far we have considered for the characterization of the type in definition 2.3.2 equations in one variable u only. The classification can be extended to coupled systems of equations. Let us generalize the definition of the type of a partial differential equation to systems of two variables. This is sufficient for the thesis but can be further generalized of course. Following again the standard references we can state the following proposition.

Proposition 2.3.2. The solution curves $y(x)$ of a coupled system of quasilinear partial differential equations of first order for two variables u_1 and u_2 in two coordinates x and y have either zero, one or two characteristics.

The part of the system leading to the principal part can be written in a matrix representation for a vector[15] $u = (u_1, u_2)^\dagger$ as

$$F_{\mathrm{pp}} = A_x \partial_x u + A_y \partial_y u \tag{2.77}$$

with x, y-dependent matrices $A_x, A_y \in \mathcal{M}(2 \times 2, \mathbb{R})$. The solution curve are determined by a quadratic equation. The number of characteristics is given by the sign of the discriminant

$$\begin{aligned} D := \; & \left(A_{x,11}A_{y,22} - A_{x,21}A_{y,12} + A_{x,22}A_{y,11} - A_{x,12}A_{y,21}\right)^2 \\ & - 4\left(A_{x,11}A_{x,22} - A_{x,12}A_{x,21}\right)\left(A_{y,11}A_{y,22} - A_{y,12}A_{y,21}\right). \end{aligned} \tag{2.78}$$

[15]We denote throughout the thesis the transposed vector or matrix with a superscript $\dagger$. Frequently the symbol refers in addition to the complex conjugation and the transposed object is denoted with superscript T. Since we are usually dealing with real quantities these operations coincide.

Definition 2.3.4. A coupled system of quasilinear partial differential equations of first order for two variables u_1 and u_2 in two coordinates x and y is called

- **hyperbolic** if it has two characteristics $\Leftrightarrow$ $D > 0$ in equation (2.78),
- **parabolic** if it has one characteristic $\Leftrightarrow$ $D = 0$ in equation (2.78) and
- **elliptic** if there are no characteristics $\Leftrightarrow$ $D < 0$ in equation (2.78).

2.3.2. Some statements and techniques about systems of partial differential equations

While it is a rather feasible task to understand a single partial differential equation the development of a solid theory for coupled systems of partial differential equations is a more involved issue. In general it is hard to find a complete theory concerning systems. The classification of the type of the system was already given in section 2.3.1. In the current section we just give an selection of statements which are used later and point to some literature for further results.

For the discussion of evolutionary problems there exist some valuable sources including Gustafsson et al. (1995), Kreiss and Busenhart (2001), Kreiss and Lorenz (2004), Ascher (2008). Systems in a first-order formulation with constant coefficients and periodic boundary conditions are quite well understood, also their numerical properties. The more general the system is the harder it is to prove statements about it.

General relativity, the topic of interest in this thesis, is a geometric system of coupled partial differential equations of second order. Therefore there is quite some freedom in the formulation of equations. Its full analysis and the obtaining of results, in particular statements about well-posedness, is a highly non-trivial task. Having well-posedness statements for a formulation of Einstein's equations as an initial boundary value problem is a desirable aim but usually hard to achieve. Some methods and results for general relativity are reviewed in Reula (1998), Stewart (1998) and more recently in Reula and Sarbach (2011), Choquet-Bruhat (2009).

For a few specific formulations in general relativity there are results (positive as well as negative). Especially if boundary conditions are taken into account as well it is even harder to obtain results. It is an active field of research in mathematical relativity, reviews include Sarbach and Tiglio (2005, 2012). In the field of numerical relativity there is some experience with that issue. There exist attempts with formulations that turned out to be highly unfortunate for the implementation. A lot of effort, time and computational power was spent. Essential was a lack of mathematical understanding of the underlying equations and their consequences.

The fact that standard formulation of general relativity, the so-called ADM formulation (see Arnowitt et al. (1962, 2008)), is not well-posed was only shown around the turn of the millenium. Kreiss and Ortiz (2002) show analytically that the equations are not well-posed when linearized about a flat background. Numerically it became clear that their might be a serious problem already in the 1990s. See Kidder et al. (2001), Calabrese et al. (2002), Sarbach et al. (2002), Bona et al. (2003) where these numerical issues are discussed and analytically understood and also working alternatives are introduced. One difficulty in the analysis is that the equations are first-order in the time parameter but usually second-order in the spatial coordinates. Often an additional first-order reduction is introduced, see Gundlach and Martín-García (2006), Hilditch and Richter (2015). Consider in this respect also the latest edition of Gustafsson et al. (1995).

Our strategy later on will be to analyze and understand as much as possible with the discussed methods.

For the discussion of Einstein's equations (but not only there) the notion of constraints is important.

Definition 2.3.5. Given a set of partial differential equations that contain evolutionary equations (of parabolic or hyperbolic type in the time variable t) of order m. Those partial differential equations in the set which do not contain time derivatives of m^{th} order are called **constraints** for obvious reasons. The remaining equations (with m^{th} time derivative) are connected with the evolution of the system. If the constraints are preserved under the time evolution ($\Leftrightarrow$ if satisfied initially they remain satisfied) we say "the **constraints propagate**".

As we will show later on the Cauchy formulation of general relativity is dealing with a constrained system of partial differential equations. In fact there are more equations than variables (the set of equations will be called overdetermined, see definition 2.3.8). So there is some liberty to replace some evolution equations by solving constraints instead (or vice versa). Clearly the initial data set has to satisfy the constraints. Since at least analytically the constraints indeed propagate there are several options for the evolution.

Definition 2.3.6. For a constrained system of partial differential equations and initial data that satisfy those constraints we call the time evolution a

- **free scheme** if no constraints are solved during the evolution,

- **fully constrained scheme** if all constraints are explicitly solved during the evolution,
- **partially constrained scheme** if some but not all constraints are explicitly solved during the evolution.

We should mention that even though one can use the Bianchi identity to show that the constraints are propagated on the analytical level the argument is not necessarily true on the numerical level. There are so-called constraint-violating modes which can lead to a blow-up and ultimately to a crash of the code. It is a serious issue in numerical relativity. A prominent and often applied method to circumvent that problem is to use some way of constraint damping. Already Detweiler (1987) pointed out that violations in the constraints might be controlled by additional terms in the evolution equations, Frittelli (1997) investigates the addition of constraints further. See in addition Bona et al. (2003) for the introduction of the so-called "$Z4$-formulation" where constraints (or its covariant derivative) are added to Einstein's equations. In a fully constrained scheme constraint violations are obviously no problem at all. The evolution equations which are not explicitly employed can be used to monitor the accuracy of the scheme.

There exist all version of schemes in definition 2.3.6 in the numerical relativity community. See Bardeen and Piran (1983), Stark and Piran (1985) for early attempts of partially constrained formulations and Evans (1989) for a fully constrained evolution. Cordero-Carrión et al. (2008), Bonazzola et al. (2004) describe a fully constrained scheme for full general relativity without symmetries.

In the following paragraph we will explicitly label the time coordinate with t. Afterwards for the discussion of the parabolic system (actually just for a parabolic equation) we switch back to the more general x-y-notation. The reason will become clear later on.

Hyperbolic systems

We consider a system of partial differential equations of hyperbolic form (according to definition 2.3.4). Again we just concentrate on the part leading to the principal part, so the equation looks like the one in equation (2.77). If the matrices which are now denoted as A_t and A_x are of full rank it can equivalently be written with a single matrix $A = -A_t^{-1} A_x$ as

$$\partial_t u = A \partial_x u. \tag{2.79}$$

The generalization to more than a single spatial direction is straightforward, see equation (2.80). Hyperbolicity as such as in definition 2.3.4 (called "weak" in the following) just depends on the principal part, so is completely determined by the highest derivatives. To guarantee that the lower-order terms can indeed be ignored a stronger version of hyperbolicity is required. That the ignored terms do not spoil the behavior is needed for a stable evolution.

Definition 2.3.7. Let $u \in \mathbb{R}^m$ be a vector of m variables. The d spatial coordinates in addition to the time coordinate t (so all in all a $d+1$-dimensional situation) are given by $x \in \mathbb{R}^d$ (with derivatives ∂_i in the direction x^i). Consider the system

$$\partial_t u = \sum_{i=1}^{d} A_i \partial_i u \tag{2.80}$$

with constant matrices $A_i \in \mathcal{M}(m \times m, \mathbb{R})$. Take a d-dimensional real unit vector[16] denoted by ω. Define the linear combination of coefficient matrices $\mathcal{P} = \omega^i A_i$.

- The system (2.80) is **(weakly) hyperbolic** if for all linear combinations $(\forall\omega)$ the matrix $\mathcal{P}$ has real eigenvalues[17].
- The system (2.80) is **strongly hyperbolic** if $\forall\omega$ the matrix $\mathcal{P}$ is diagonalizable (with corresponding matrix S) with real eigenvalues and both S and the inverse S^{-1} depend smoothly on ω. An equivalent statement is that $\forall\omega$ there is a complete set of eigenvectors.
- The system (2.80) is **strictly hyperbolic** if it is strongly hyperbolic and all eigenvalues are distinct.
- The system (2.80) is **symmetric hyperbolic** if $\forall\omega\exists H$ which is Hermitian[18] and positive definite (called "symmetrizer") such that the matrix $H\mathcal{P}$ is Hermitian in any direction, so for all possible ω we have $H\mathcal{P} = (H\mathcal{P})^\dagger$.

For a higher-order system we assign the same characterization if it **admits** a first-order reduction that has the corresponding property. We do not require that all arbitrary reductions come with the same behavior.

[16] That means $\omega \in \mathbb{R}^d$ such that $|\omega| = \sqrt{\sum_{i=1}^d \omega_i^2} = 1$.

[17] It is quite easy to see that this notion is equivalent to the notion of hyperbolicity in definition 2.3.4. The discriminant resulting from the characteristic polynomial for equation (2.79) is identical to the discriminant D in (2.78)

[18] That means $\bar{H}^T = H$ (equivalently "self-adjoint") which reduces to $H^T = H$ for real coefficient matrices, so just being symmetric and hence in our convention $H^\dagger = H$. The requirement of symmetry in the definition is in fact just the demand that the product will be Hermitian, $H\mathcal{P} = (H\mathcal{P})^\dagger$. The right-hand side can be written as $(H\mathcal{P})^\dagger = \mathcal{P}^\dagger H^\dagger = \mathcal{P}^\dagger H$ which justifies the name symmetrizer for H.

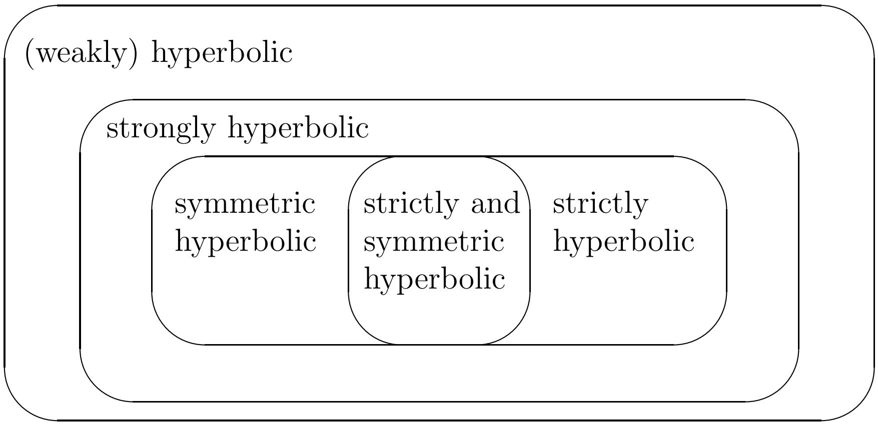


Figure 2.3.: For the illustration of the concepts of hyperbolicity, highly influenced by figure 4.3.1 in Gustafsson et al. (1995).

Clearly if we only have one spatial direction everything becomes simpler because there is just one direction for the unit normal.

Remark 2.3.1. Here we define symmetric hyperbolicity by the existence of a symmetrizer which makes the system symmetric. Often this is just called symmetrizable hyperbolic which is, strictly speaking, correct. There is no unique way in the literature to define the concept. We mainly choose the definition above for aesthetic reasons.

The definition implies that for a hyperbolic system where the coefficient matrix $\mathcal{P}$ is diagonalizable the system is in fact strongly hyperbolic. Strong hyperbolicity guarantees well-posedness for an initial value problem as explained in the cited literature about evolutionary problems. As already mentioned that means in particular for strongly hyperbolic systems that the lower-order terms which are not contained in the principal part do not cause problems. Therefore it is possible to ignore those terms in the analysis. The essential problem for weakly hyperbolic systems is that even though they are hyperbolic in nature one can find lower-order terms that make the system totally ill-posed. Therefore it is always desirable to have a strongly hyperbolic system, especially for the numerical implementation. The nice feature for symmetric hyperbolic systems is that they allow for the definition of an energy (not necessarily of physical meaning) in a natural way. The notion of strict hyperbolicity has a more formal character.

The eigenvalues are also called “characteristic speeds” and their properties are closely related to the classification of partial differential equations in proposition 2.3.1. If there are non-real eigenvalues the equation cannot be hyperbolic at all. If all of them are real, so we have the maximum number of characteristics, the equation is hyperbolic, at least in the weak sense. On the other hand that does not imply that the initial value problem is well-posed. If it is strongly hyperbolic it admits a well-posed initial value problem. The eigenvalues and their reality are a manifestation of the finite speed of propagation that is inherent in a hyperbolic equation.

As we will see Einstein's equations are of second order. The standard Cauchy formulation splits the equations into first-order in time evolution equations and constraints (consequently zeroth-order time derivatives). In general the set of evolution equations for a set of variables u has therefore the form

$$\partial_t u = Q^{ij}(u)\partial_i\partial_j u + R^i(u)\partial_i u + S(u). \tag{2.81}$$

For a constant-coefficient system the matrices Q and R are independent of the solution. In the theory of such a second-order in space system there are quite some results but it is by far not as developed as first-order systems. Therefore one possibility (and taken for instance in Gundlach and Martín-García (2006), Hilditch and Richter (2015)) is to reduce the second-order system to one of first order only. That increases the number of variables and introduces further "auxiliary" constraints but brings the system closer to solid theory of systems of partial differential equations. Therefore we will consider in the analysis a system (or at least its principal part) again in the first-order reduction, see the end of definition 2.3.7.

We already discussed at the beginning of the current section the impact of several notions of hyperbolicity on the Cauchy formulation in general relativity. We will analyze the hyperbolicity in chapter 4 for our set of equations and discuss here the 2+1-dimensional wave equation in spherical coordinates because it is helpful for the later understanding.

Example 2.3.1. We show that the (2+1)-dimensional scalar wave equation in spherical coordinates is strongly (even symmetric and strictly) hyperbolic for $r \geq 0$.

The equation has the principal part

$$\partial_t^2\phi = \partial_r^2\phi + r^{-2}\partial_\vartheta^2\phi. \tag{2.82}$$

The equation admits a first-order reduction with the auxiliary variables $\Pi := \partial_t\phi$, $W := \partial_r\phi$ and $V := r^{-1}\partial_\vartheta\phi$ and becomes therefore a system of coupled equations. There is some advantage to include the r^{-1} in the definition (remember $r \neq 0$)[19].

[19] At the end it can be boiled down again to the difference between the already symmetric system and symmetrizable ones. Since we decided to choose a definition of symmetric hyperbolicity that basically ignores these differences it is of rather cosmetic nature though.

Therefore we have then the system

$$
\begin{aligned}
\partial_t \Pi &= \partial_r W + r^{-1}\partial_\vartheta V, && (2.83a)\\
\partial_t W &= \partial_r \Pi, && (2.83b)\\
\partial_t V &= r^{-1}\partial_\vartheta \Pi. && (2.83c)
\end{aligned}
$$

So we write it with the vector $u^\dagger = (\Pi, W, V)$ as

$$
\partial_t u = A_r \partial_r u + A_\vartheta \partial_\vartheta u \tag{2.84}
$$

with coefficient matrices

$$
A_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A_\vartheta = \begin{pmatrix} 0 & 0 & r^{-1} \\ 0 & 0 & 0 \\ r^{-1} & 0 & 0 \end{pmatrix}. \tag{2.85}
$$

With the unit vector $\omega = (\omega_r, \omega_\vartheta)^\dagger$ we define the matrix

$$
\mathcal{P} = \omega_r A_r + \omega_\vartheta A_\vartheta = \begin{pmatrix} 0 & \omega_r & r^{-1}\omega_\vartheta \\ \omega_r & 0 & 0 \\ r^{-1}\omega_\vartheta & 0 & 0 \end{pmatrix} \tag{2.86}
$$

The eigenvalues and corresponding eigenvectors of $\mathcal{P}$ are

$$
\begin{aligned}
\lambda_1 &= \sqrt{\omega_r^2 + r^{-2}\omega_\vartheta^2} : \left(1, \frac{\omega_r}{\lambda_1}, \frac{r^{-1}\omega_\vartheta}{\lambda_1}\right)^\dagger, && (2.87a)\\
\lambda_2 &= 0 : (0, -\omega_\vartheta, r\omega_r)^\dagger, && (2.87b)\\
\lambda_3 &= -\sqrt{\omega_r^2 + r^{-2}\omega_\vartheta^2} = -\lambda_1 : \left(1, \frac{\omega_r}{\lambda_3}, \frac{r^{-1}\omega_\vartheta}{\lambda_3}\right)^\dagger. && (2.87c)
\end{aligned}
$$

The eigenmatrix reads

$$
S = \begin{pmatrix} 1 & 0 & 1 \\ \frac{\omega_r}{\lambda_1} & -\omega_\vartheta & \frac{\omega_r}{\lambda_3} \\ \frac{r^{-1}\omega_\vartheta}{\lambda_1} & r^2\omega_r & \frac{r^{-1}\omega_\vartheta}{\lambda_3} \end{pmatrix} \tag{2.88}
$$

and $S^{-1}\mathcal{P}S$ diagonalizes the matrix in a continuous manner. Because of the normalization of ω all eigenvalues are distinct and real. Both coefficient matrices are real and symmetric and therefore Hermitian. Hence all kinds of requirements for (strict and symmetric) hyperbolicity are satisfied. Therefore the wave equation can be formulated in such a way that it lies in the central oval of figure 2.3, so in the overlap of strict and symmetric hyperbolicity.

Parabolic equations

We briefly consider also parabolic systems, in fact only systems consisting of a single equation. This restriction is sufficient for our purposes.

Appropriate sources are basically the same as for hyperbolic systems, see for example Gustafsson et al. (1995), Kreiss and Lorenz (2004). In chapter 4 we will analyze Einstein's equations and formulate the Hamiltonian constraint as a parabolic equation, the momentum constraint as hyperbolic system. We will see in section 4.7 that the principal parts of momentum and Hamiltonian constraint decouple. Therefore the separate results can be combined.

Often parabolic equations play no role in the discussion of Einstein's equations, even though there are also numerous exceptions. As will be discussed in section 3.4.1 the constraints are often formulated as elliptic equations which makes the full set of equations (the constraints and the evolution equations for the remaining variables) an elliptic-hyperbolic system. We employ a hyperbolic-parabolic formulation of the constraints. The "time"-coordinate will be the radial direction r then. So even though it sounds unfamiliar to discuss an evolutionary problem (a parabolic equation) in purely spatial coordinates we will do so in the following.

Consider a parabolic equation of second order in two coordinates for the variable $u = u(x, y)$ of the form

$$\partial_x u = k\partial_y^2 u + \text{“lower-order terms”}, \tag{2.89}$$

so its principal part is just the heat equation, see example 2.3.1. We will show the irreversible character of the heat equation which means that parabolic equations usually suggest intuitively a direction of the "arrow of time".

Proposition 2.3.3. Given equation (2.89) and appropriate initial values. For $k > 0$ the initial value problem is well-posed, for $k < 0$ the solution to equation (2.3.3) blows up.

Proof. We prove the statement for the heat equation which follows more or less directly John (1982, chapter 7). Consider the wave solution $\exp i(\lambda x + \xi y)$ for $\lambda, \xi \in \mathbb{R}_{\neq 0}$ which is a solution for equation (2.89) if $i\lambda + k\xi^2 = 0$. So the solution reads $u = \exp(-k\xi^2 x + i\xi y)$. Therefore the amplitude is $|u| = \sqrt{u\bar{u}} = \exp(-k\xi^2 x)$. Consider the positive x-direction, so initial data given at x_0 and then the evolution continues in $x > x_0$-direction. Therefore for $k < 0$ the amplitude decays and for $k > 0$ the amplitude grows without bounds and blows up ($k = 0$ trivial). The general case is again obtained by superposition in Fourier series. $\square$

Technique of frozen coefficients

In general systems do not have constant coefficients in front of the derivatives of the variables. Again the theory of partial differential equations is less developed for systems with variable coefficients. Since we are (fortunately) only dealing with quasilinear systems the principal part is at least linear in the highest derivatives of the solution. Nevertheless the equations might contain dependencies on the coordinates for example, even on the linear level.

If the coefficients of A in front of the part leading to the principal part of an equation like equation (2.79) are not constant but variable one can consider the coefficient matrix A as a single point $(x_0, y_0, \ldots)$, hence "**freeze**" the coefficients. Therefore we expect local statements only. Those are based on the theory for constant coefficient problems.

In the nonlinear case the variable coefficients might in addition depend on the solution (in a quasilinear way though). It is possible to freeze again with a known solution and to obtain local statements in a neighborhood of the known solution. Since our formulation of Einstein's equations later on suggests a foliation of the flat spacetime in spherical polar coordinates freezing about that solution seems appropriate. Hence the analysis should be the same as considering the linearization about flat spacetime (and then freezing the remaining variable coefficients r and ϑ). Of course we expect only local results.

The general idea is that if all frozen coefficient problems are well-posed, the variable coefficient problem is so as well, see again Gustafsson et al. (1995), Kreiss and Lorenz (2004) for discussions and more solid formulations, also Kreiss and Winicour (2006) for an application to Einstein's equations. On the other hand, as for example discussed in Ascher (2008, chapter 5.1), stability of the frozen coefficient problem does not necessarily imply stability of the variable coefficient problem. While from the rigorous point of view the issue is more involved the more pragmatic one gives some hope that freezing helps indeed.

Elliptic systems

In principle elliptic equations play a prominent role in most formulations of Einstein's equations as can be seen in many discussions in numerous publications on general relativity. For instance the Newtonian limit for weak-field gravity results in Newton's equation for the gravitational potential[20]. As we will discuss at several instances in the remainder of the thesis many gauge choices result in elliptic equations for the constraints. Early results towards the well-posedness statements of general relativity use elliptic equations and its regularity theory. Also in numerical relativity gauge conditions resulting in elliptic equations are very prominent and probably the most frequently used ones.

Since we use in this thesis a different approach without elliptic equations at all we basically refer to the literature for the theory of elliptic equations and systems. Except most textbooks on partial differential equations which contain material about elliptic equations we explicitly mention Gilbarg and Trudinger (2001) for a whole textbook devoted to elliptic problems, and with a focus on ellipticity in general relativity the textbook Rendall (2008) and the review Dain (2006).

Under- and overdetermined systems

Another issue concerning systems of partial differential equations deserves attention because it also is important for Einstein's equations.

Definition 2.3.8. We call a system of partial differential equations with more (less) variables than equations **underdetermined (overdetermined)**.

Example 2.3.2. Consider the linear scalar equation in two coordinates $(x, y) \in \mathbb{R}^2$

$$(\partial_x^2 + \partial_y^2)u_1 + (\partial_x^2 - \partial_y^2)u_2 + (\partial_x - \partial_y^2)u_3 + u_4 = 0. \tag{2.90}$$

To obtain a solution for one of the variables we have to prescribe the remaining three variables and possibly initial or boundary conditions.

According to the classification in definition 2.3.1 the equation (2.90) is elliptic if we solve it for u_1 and prescribe u_2, u_3 and u_4 (it is the Poisson equation, see example 2.3.1). To obtain a solution, boundary conditions for u_1 need to be given in addition.

The equation (2.90) is hyperbolic if we solve for u_2 and prescribe u_1, u_3 and u_4 (it is the wave equation, see example 2.3.1). To be able to integrate it, boundary conditions in one variable, say y need to be given and initial conditions in the other variable, then x, are required. x plays the role of a "time" variable.

[20] It is a (if not **the**) Poisson equation, so an inhomogeneous scalar Laplace equation, see example 2.3.1. The proportionality factor in Einstein's equation is chosen such that the Newtonian limit turns out correctly.

Solving equation (2.90) for u_3 the equation is parabolic with "time" variable x and we need to prescribe u_1, u_2 and u_4 (it is the heat equation, see example 2.3.1). Boundary conditions for u_3 in y need to be given and in addition initial condition at some x_0 are required. For a stable integration the "evolution" should be outward in x then (see proposition 2.3.3).

Finally if we interpret equation (2.90) as equation for u_4 it is algebraic (of zeroth order) and we need to prescribe u_1, u_2 and u_3. u_4 is then completely determined by these variables.

We will see that in the Cauchy formulation of general relativity there are in fact more variables than equations due to the coordinate freedom. Therefore we are dealing in principle with an underdetermined system. Because of the coordinate freedom one has the liberty to make additional choices, later on called "gauge choices" which might help to bring the system in a desired form from the point of view of the theory of partial differential equations. On the other hand the constraint system as such extracted from Einstein's equations is a highly overdetermined as we will also discuss later.

2.3.3. Numerics for partial differential equations

There are several issues and techniques we should discuss concerning the numerical implementation of partial differential equations. Note that we make use of techniques allowing the reduction to ordinary differential equations. Therefore the statements in section 2.2.6 should also be taken into account.

Stability analysis In numerical analysis the stability of a difference scheme is an important topic. Here we are just able to scratch the surface, more material is given for example in the references cited at the beginning of the section. We are already aware of the fact that for the continuous problem convergence of the solution to the true solution is of huge importance for well-posedness. For the discretized problem stability takes a comparable role. That those are essentially equivalent concepts is shown by the **Lax theorem** (or **Lax-Richtmyer theorem** Lax and Richtmyer (1956)). It says that for a **consistent**[21] discretization scheme and a well-posed initial (boundary) value problem both issues (convergence and stability) are equivalent, see Richtmyer and Morton (1967) for instance.

[21]This means that the truncation error goes to zero for decreasing step size(s).

Explicit stability analyses are included in most textbooks on numerical investigations for partial differential equations and on numerical relativity (see beginning of section 2.3 and section 3.3 for references). For example in Baumgarte and Shapiro (2010, chapter 6.2.3) the von Neumann stability analysis for the linear **advection equation** on a uniform grid with step sizes Δt and Δx is explicitly given. Here we just state the result.

The von Neumann stability analysis reveals the necessary stability criterion

$$\frac{c\Delta t}{\Delta x} \leq 1. \tag{2.91}$$

Definition 2.3.9. The necessary condition (2.91) is called **CFL condition**[22] and the factor $\Delta t/\Delta x$ is usually called **Courant factor**.

In our numerical simulation we choose usually a Courant factor of the order 1/10, the spatial resolution is determined by the number of grid points and the outer boundary (the inner one is $r = 0$). Hence the time step is determined as

$$\Delta t = \frac{1}{10}\Delta x. \tag{2.92}$$

Numerical dissipation Our aim is to solve continuous problems. We already discussed in section 2.2.3 the issue of discretization of the continuous variables and their derivatives on a grid. The discretization of the derivatives in two spatial dimension, our "hybrid approach" and consequences for the derivative will be discussed afterwards.

For the numerical evolution of hyperbolic equations it is a very common feature that for a large class of discretization schemes the high-frequency components are not accurately propagated. Because of the finite difference approximation there are solutions with low wavelengths (low with respect to the spatial grid size that is again denoted by h here) which have no physical origin. The suppressing of those high-frequency modes from the numerical solution is the task of the artificial dissipation which acts as a low-pass filter.

We use the so-called Kreiss-Oliger dissipation, see Kreiss and Oliger (1973, in particular chapter 9).

Definition 2.3.10. The making use of the **Kreiss-Oliger dissipation** is to map the evolution equation to a modified equation,

$$\partial_t u = f(u) \mapsto \partial_t u = f(u) + Q_r(u). \tag{2.93}$$

[22] After Richard Courant, Kurt Friedrichs and Hans Levy who introduced the concept in Courant et al. (1928).

The **dissipation operator** Q_r of order $2r$ was introduced already in definition 2.2.7. Formally it might look like the type of equation has changed.This is not the case because the dissipation operator Q_r contains the step size h, see definition 2.2.7 and vanishes in the continuum limit. Therefore it is just a "numerical trick" to block high frequencies and the type of equation is not affected.

Since we are using a second-order accurate scheme we need an operator of 4^{th} order, hence $r = 2$. The consequences and the finite difference properties are worked out in proposition 2.2.4 in section 2.2.3.

Method of lines

In the numerical implementation we will make use of the so-called **method of lines**. It is a very powerful method and of great value in that field. We have seen already in example 2.3.1 that it also simplifies the wave equation.

Consider a general partial differential equation of first order in (at least) one coordinate (the "temporal one", here x) for the variable $u(x, y)$ of the form (generalizations to higher orders in y and other coordinates is straightforward)

$$\partial_x u(x, y) = f(x, y, u, \partial_y u, \partial_y u). \tag{2.94}$$

If we have knowledge of all quantities at one instance in x (at one "line" in the x-y-plane), say at x^n, we can completely determine the values of the right-hand side of equation (2.94), so f^n, with all derivatives in y. Then we can consider equation (2.94) as an ordinary differential equation for x and use an integrator like those discussed in 2.2.3. For instance we can use an Euler integrator (or better more sophisticated ones like Runge-Kutta)

$$u^{n+1} = u^n + \Delta x f^n \tag{2.95}$$

to obtain the values for $u(x^{n+1}, y)$ at the next instance of the x-coordinate. Therefore we decomposed the problem of finding a solution to $u(x, y)$ of partial differential equations to solving ordinary differential equations "line by line", thus the name **method of lines**.

Hybrid disrectization

Our aim is to solve Einstein's equations in axisymmetry with vanishing twist. Even though we start of course with the full 3+1-dimensional theory we have, due to the assumed symmetry, effectively a dimensional reduction to 2+1 dimensions. Hence in adapted coordinates we will not have the φ-dependence any more, see our derivation in chapter 4. We will choose spherical polar coordinates (see A.1.2) for the formulation, so our variables are of the form $u(t, r, \vartheta)$. For the ϑ-direction we will use a spectral expansion and the implementation of the pseudo-spectral method. On the two-sphere S^2 an appropriate expansion basis are the spherical harmonics as discussed in section 2.2.5.

Since we will not deal with a scalar theory but general relativity, which is a tensor theory, the harmonics look slightly more complicated but are numerically tractable. In principle it is also possible to choose a spectral expansion in t and r as well, for attempts see for example Hennig (2013), Panosso Macedo and Ansorg (2014). An obvious disadvantage is that the number of grid points for those coordinates are significantly higher than in the ϑ-direction. It implies that the approximations do not converge fast enough to the real variables in the whole interval. These disadvantages can be overcome by using “domain decompositions”, see Grandclément and Novak (2009).

We decided to follow a different approach, namely use an **hybrid discretization**. That means that we use a (pseudo-)spectral method in ϑ and finite differences as discussed in section 2.2.3 in t and r. In section 2.4 we will derive the consequences for the derivatives.

2.4. The spatial derivatives

As previously discussed in section 2.3.3 we will use a hybrid discretization for the implementation. Therefore we assume that our variables admit a spectral expansion in eigenfunctions of the Laplace operator on the sphere S^2 (that issue will be addressed in more detail in section 2.5). In this section we derive the consequences for the derivatives and the matrix representations for all derivative operations as needed for the implementation. Here we focus on the spatial part in r and ϑ.

2.4.1. The derivatives in ϑ

The expansion in spherical harmonics allows us to express the angular derivatives explicitly. We should discuss the different quantities (scalar harmonics $Y_\ell(\vartheta)$, vector harmonics $Y_{\vartheta,\ell}(\vartheta)$ and tensor harmonics $Y_{\vartheta\vartheta,\ell}(\vartheta)$, see section 2.2.5) separately. We saw in proposition 2.2.5 that for the mode number $m = 0$ (as we will consider) there are only these three kinds of harmonics. In section 4.4 we will arrange all variables such that they have a definite expansion in one of these types.

We introduce an additional notation. Let us denote by $\mathcal{Y}_\ell(\vartheta)$ a general set of basic functions which corresponds to one of the mentioned cases. Sometimes we omit the ϑ-dependence and the index ℓ but it is always implicitly understood.

The tensor harmonics $Y_{\vartheta\vartheta}$ are a specific combination of scalar Y and vector harmonics Y_ϑ, namely

$$Y_{\vartheta\vartheta} = -\frac{\ell(\ell+1)}{2}Y - \frac{\cos\vartheta}{\sin\vartheta}Y_\vartheta. \tag{2.96}$$

Hence any variable can be written as a specific combination of Y and Y_ϑ. An arbitrary variable u (restricting to the spatial part only) expands as

$$u(r,\vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_\ell(r)\mathcal{Y}_\ell(\vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_\ell(r)Y_\ell(\vartheta) = \sum_{\ell=0}^{L-1} \left[\hat{v}_\ell(r)Y_\ell(\vartheta) + \hat{w}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\right], \tag{2.97}$$

where $\hat{v}_\ell$ and $\hat{w}_\ell$ are to be determined in the following.

In the following we just state the propositions without proofs. They can be varified by direct calculation.

Scalar quantity

Proposition 2.4.1. For a scalar quantity (implies $\hat{w}_\ell = 0$ in equation (2.97))

$$u(r,\vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_\ell(r)Y_l(\vartheta) = \sum_{\ell=0}^{L-1} \left[\hat{u}_\ell(r)Y_\ell(\vartheta) + 0Y_{\vartheta,\ell}(\vartheta)\right] \tag{2.98}$$

the derivatives are

$$\partial_r u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[\partial_r\hat{u}_\ell(r)Y_\ell(\vartheta) + 0Y_{\vartheta,\ell}(\vartheta)\right], \tag{2.99a}$$

$$\partial_r^2 u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[\partial_r^2\hat{u}_\ell(r)Y_\ell(\vartheta) + 0Y_{\vartheta,\ell}(\vartheta)\right], \tag{2.99b}$$

$$\partial_\vartheta u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[0Y_\ell(\vartheta) + \hat{u}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\right], \tag{2.99c}$$

$$\partial_\vartheta^2 u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[-\ell(\ell+1)\hat{u}_\ell(r) Y_\ell(\vartheta) - \frac{\cos\vartheta}{\sin\vartheta} \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) \right]$$
$$\equiv \sum_{\ell=0}^{L-1} \hat{u}_\ell(r) \left[\mathfrak{S}_1 + \mathfrak{V}_1 \right], \tag{2.99d}$$
$$\partial_r \partial_\vartheta u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[0 Y_\ell(\vartheta) + \partial_r \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) \right]. \tag{2.99e}$$

Here and in the following $\mathfrak{S}_1$, $\mathfrak{V}_1$ and similar expressions stand for the specific combination of the prefactor and the (scalar or vector) spherical harmonic that was calculated. So for example we have

$$\mathfrak{S}_1 = -\ell(\ell+1) Y_\ell(\vartheta) \tag{2.100a}$$
$$\text{and } \mathfrak{V}_1 = -\frac{\cos\vartheta}{\sin\vartheta} Y_{\vartheta,\ell}(\vartheta). \tag{2.100b}$$

Vector quantity

Proposition 2.4.2. For a vector quantity (implies $\hat{v}_\ell = 0$ in equation (2.97))

$$u(r,\vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) = \sum_{\ell=0}^{L-1} \left[0 Y_\ell(\vartheta) + \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) \right] \tag{2.101}$$

the derivatives are

$$\partial_r u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[0 Y_\ell(\vartheta) + \partial_r \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) \right], \tag{2.102a}$$
$$\partial_r^2 u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[0 Y_\ell(\vartheta) + \partial_r^2 \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) \right], \tag{2.102b}$$
$$\partial_\vartheta u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[-\ell(\ell+1)\hat{u}_\ell(r) Y_\ell(\vartheta) - \frac{\cos\vartheta}{\sin\vartheta} \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta) \right]$$
$$\equiv \sum_{\ell=0}^{L-1} \hat{u}_\ell(r) \left[\mathfrak{S}_1 + \mathfrak{V}_1 \right], \tag{2.102c}$$
$$\partial_\vartheta^2 u(r,\vartheta) = \sum_{\ell=0}^{L-1} \left[+\ell(\ell+1) \frac{\cos\vartheta}{\sin\vartheta} \hat{u}_\ell(r) Y_\ell(\vartheta) \right.$$

$$+ \left(1 - \ell(\ell+1) + 2\frac{\cos^2\vartheta}{\sin^2\vartheta}\right)\hat{u}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\Bigg]$$

$$\equiv \sum_{\ell=0}^{L-1} \hat{u}_\ell(r)\left[\mathfrak{S}_2 + \mathfrak{V}_2\right], \tag{2.102d}$$

$$\partial_r\partial_\vartheta u(r,\vartheta) = \sum_{\ell=0}^{L-1}\left[-\ell(\ell+1)\partial_r\hat{u}_\ell(r)Y_\ell(\vartheta) - \frac{\cos\vartheta}{\sin\vartheta}\partial_r\hat{u}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\right]$$

$$\equiv \sum_{\ell=0}^{L-1}\partial_r\hat{u}_\ell(r)\left[\mathfrak{S}_1 + \mathfrak{V}_1\right]. \tag{2.102e}$$

Tensor quantity

Especially from the computational point of view a tensor quantity can be seen as a specific combination of the previously discussed scalar and vector ones.

Proposition 2.4.3. A tensor quantity is (here we have in general both contributions $\hat{v}_\ell$ and $\hat{w}_\ell$ non-vanishing in equation (2.97))

$$u(r,\vartheta) = \sum_{\ell=0}^{L-1} u_\ell(r)\left(Y_{\vartheta\vartheta}\right)_l(\vartheta) = \sum_{\ell=0}^{L-1}\left[-\frac{\ell(\ell+1)}{2}\hat{u}_\ell(r)Y_\ell(\vartheta) - \frac{\cos\vartheta}{\sin\vartheta}\hat{u}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\right]. \tag{2.103}$$

In the notation of equation (2.97) we can identify

$$\hat{v}_\ell = -\frac{\ell(\ell+1)}{2}\hat{u}_\ell, \tag{2.104a}$$

$$\hat{w}_\ell = -\frac{\cos\vartheta}{\sin\vartheta}\hat{u}_\ell. \tag{2.104b}$$

The derivatives are

$$\partial_r u(r,\vartheta) = \sum_{\ell=0}^{L-1}\left[-\frac{\ell(\ell+1)}{2}\partial_r\hat{u}_\ell(r)Y_\ell(\vartheta) - \frac{\cos\vartheta}{\sin\vartheta}\partial_r\hat{u}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\right], \tag{2.105a}$$

$$\partial_r^2 u(r,\vartheta) = \sum_{\ell=0}^{L-1}\left[-\frac{\ell(\ell+1)}{2}\partial_r^2\hat{u}_\ell(r)Y_\ell(\vartheta) - \frac{\cos\vartheta}{\sin\vartheta}\partial_r^2\hat{u}_\ell(r)Y_{\vartheta,\ell}(\vartheta)\right], \tag{2.105b}$$

$$\partial_\vartheta u(r,\vartheta) = \sum_{\ell=0}^{L-1}\Bigg[\ell(\ell+1)\frac{\cos\vartheta}{\sin\vartheta}\hat{u}_\ell(r)Y_\ell(\vartheta)$$

$$
\begin{aligned}
&+ \left(1 - \frac{\ell(\ell+1)}{2} + 2\frac{\cos^2\vartheta}{\sin^2\vartheta}\right) \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta)\Bigg] \\
&\equiv \sum_{\ell=0}^{L-1} \hat{u}_\ell(r) \left[\mathfrak{S}_2 + \mathfrak{V}_3\right], \qquad (2.105c)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\vartheta^2 u(r,\vartheta) = \sum_{\ell=0}^{L-1} \Bigg[&\left(\frac{\ell^2(\ell+1)^2}{2} - 3\ell(\ell+1)\frac{\cos^2\vartheta}{\sin^2\vartheta} - 2\ell(\ell+1)\right) \hat{u}_\ell(r) Y_\ell(\vartheta) \\
&+ \frac{\cos\vartheta}{\sin\vartheta}\left(\frac{3\ell(\ell+1)}{2} - 5 - 6\frac{\cos^2\vartheta}{\sin^2\vartheta}\right) \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta)\Bigg] \qquad (2.105d) \\
\equiv \sum_{\ell=0}^{L-1} &\hat{u}_\ell(r) \left[\mathfrak{S}_3 + \mathfrak{V}_4\right], \qquad (2.105e)
\end{aligned}
$$

$$
\begin{aligned}
\partial_r\partial_\vartheta u(r,\vartheta) = \sum_{\ell=0}^{L-1} \Bigg[&\ell(\ell+1)\frac{\cos\vartheta}{\sin\vartheta} \partial_r \hat{u}_\ell(r) Y_\ell(\vartheta) \\
&+ \left(1 - \frac{\ell(\ell+1)}{2} + 2\frac{\cos^2\vartheta}{\sin^2\vartheta}\right) \partial_r \hat{u}_\ell(r) Y_{\vartheta,\ell}(\vartheta)\Bigg] \\
\equiv \sum_{\ell=0}^{L-1} &\partial_r \hat{u}_\ell(r) \left[\mathfrak{S}_2 + \mathfrak{V}_3\right]. \qquad (2.105f)
\end{aligned}
$$

Sometimes the following proposition is of use.

Proposition 2.4.4. The vector harmonics vanish for $\ell = 0$ as do the tensor harmonics for $\ell = 0$ and 1.

Proof. We calculated in section 2.2.5 that for $\ell = 0$ the scalar harmonics $Y_{\ell=0}$ are a constant, hence the vector harmonics are $Y_{\vartheta,\ell=0} = 0$. We use equation (2.103) as the formula for tensor harmonics. For $\ell = 1$ we have also seen that the ϑ-dependence is $Y_{\ell=1} \sim \cos\vartheta$ and therefore the vector harmonics are $Y_{\vartheta,\ell=1} \sim -\sin\vartheta$ with the same prefactor. Those facts lead easily to the conclusion that the tensor harmonics for both $\ell = 0$ and $\ell = 1$ have to vanish. □

2.4.2. The transformation between point and spectral space

A general variable u (we restrict here to the spatial part in r and ϑ again) is in general given in the physical point space as $u(r, \vartheta)$. On a finite lattice in r ($N+1$ points) and ϑ (L points, called collocation points, see section 2.2.4) we represent it as matrix

$$\mathbf{u} = u_{ij} \in \mathcal{M}(N+1 \times L, \mathbb{R}). \tag{2.106}$$

In the previous section 2.4 we calculated the spatial derivatives explicitly. For that purpose the representation in modes in the spectral space is of advantage. It is given as (remember the notation $\mathcal{Y}_\ell(\vartheta)$ for a general harmonic, see section 2.4)

$$u(r, \vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_\ell(r) \mathcal{Y}_\ell(\vartheta) = \sum_{\ell=0}^{L-1} (\hat{u}_{\ell i})^\dagger M_{\ell j}. \tag{2.107}$$

The dagger $\dagger$ needs some explanation. We prefer to write the modes as $\hat{u}_\ell(r)$. That suggests in a matrix representation the index ℓ before i (the latter corresponding to the $(N+1)$-lattice in the coordinate r). For the multiplication in equation (2.107) we need the transposed of that quantity though (and since we are working in the real field the operation can be denoted by $\dagger$).

We write the mode decomposition as a matrix equation

$$\mathbf{u} = \hat{\mathbf{u}}^\dagger \cdot \mathbf{M}, \tag{2.108}$$

where $\hat{\mathbf{u}} = \hat{u}_{\ell i} \in \mathcal{M}(L \times N+1, \mathbb{R})$ the matrix where the modes (each dependent on r) are stored in (and with the transposed matrix $\hat{\mathbf{u}}^\dagger \in \mathcal{M}(N+1 \times L, \mathbb{R})$ of course) and $\mathbf{M} = M_{\ell j} = \mathcal{Y}_\ell(\vartheta_j) \in \mathcal{M}(L \times L, \mathbb{R})$ the matrix representation for the spherical harmonics.

If we are interested in the modes $\hat{u}_\ell(r)$ we just calculate

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}^{\dagger^\dagger} = \left(\hat{\mathbf{u}}^\dagger \mathbf{M}\mathbf{M}^{-1}\right)^\dagger = \left(\mathbf{u} \cdot \mathbf{M}^{-1}\right)^\dagger. \tag{2.109}$$

For given spectral cutoff L the matrix $\mathbf{M}$ and its inverse can be calculated, see the following. In the implementation those matrices need only be calculated once, afterwards they are just recalled.

We are interested in obtaining the modes from a given variable u we calculate the transpose of

$$\hat{\mathbf{u}}^\dagger = \mathbf{u}\mathbf{M}^{-1}. \tag{2.110}$$

$\mathbf{u}$ is a fully ranked matrix, in general all entries non-vanishing, $\hat{\mathbf{u}}^\dagger$ the transposed entries of the mode decomposition, $\mathbf{M}^{-1}$ (with entries $(M^{-1})_{j\ell}$) has to take care of some properties coming from the spherical harmonics (compare proposition 2.4.4 and the following discussion);

- if the $\ell = 0$ mode vanishes, $(\mathbf{M}^{-1})_{j0} \equiv 0$ (0^{th} column),
- if the $\ell = 0$ and $\ell = 1$ modes vanish, $(\mathbf{M}^{-1})_{j0} = \mathbf{M}^{-1})_{j1} \equiv 0$ (0^{th} and 1^{th} column),
- if ϑ_0 should be ignored, $(\mathbf{M}^{-1})_{0\ell} \equiv 0$ (0^{th} row),
- if ϑ_{L-1} should be ignored, $(\mathbf{M}^{-1})_{L-1,\ell} \equiv 0$ ($(L-1)^{\text{th}}$ row).

There might be situations (see below) where we have more grid points in ϑ-direction than freedom to reduce the residuum, compare the discussions for the (pseudo-)spectral approach in section 2.2.4. Hence it does make sense in those situations to "ignore" a grid point for one ϑ-value for the minimization of the residual.

Transformation matrices

In the following we calculate the transformation matrices $\mathbf{M}$ and the inverse explicitly. As discussed in section 2.2.3 we take a staggered grid in ϑ with L collocation points ($j = 0, \ldots, L-1$);

$$\vartheta_j = \left(j + \frac{1}{2}\right)\frac{\pi}{L} = \frac{1}{2}\frac{\pi}{L}, \frac{3}{2}\frac{\pi}{L}, \ldots, \left(L - \frac{1}{2}\right)\frac{\pi}{L} = \pi\left(1 - \frac{1}{2L}\right). \tag{2.111}$$

Our transformation matrices are of the form $(L \times L)$,

$$M_{\ell j} = \mathcal{Y}_\ell(\vartheta_j) = \begin{pmatrix} \mathcal{Y}_0(\vartheta_0) & \mathcal{Y}_0(\vartheta_1) & \cdots & \mathcal{Y}_0(\vartheta_{L-1}) \\ \mathcal{Y}_1(\vartheta_0) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathcal{Y}_{L-1}(\vartheta_0) & \cdots & \cdots & \mathcal{Y}_{L-1}(\vartheta_{L-1}) \end{pmatrix}. \tag{2.112}$$

Remember the notation introduced in section 2.4 where $\mathcal{Y}_\ell(\vartheta)$ represents either a scalar $Y_\ell(\vartheta)$, a vector $Y_{\vartheta,\ell}(\vartheta)$ or a tensor $Y_{\vartheta\vartheta,\ell}(\vartheta)$ spherical harmonic.

The inverse is $(M^{-1})_{j\ell}$. From proposition 2.4.4 we know that $(\partial_\vartheta Y)_{\ell=0} = 0$ and $(Y_{\vartheta\vartheta})_{\ell=0} = (Y_{\vartheta\vartheta})_{\ell=1} = 0$. Hence our three different transformation matrices for the transformation from point space to configuration space are given as follows.

Scalar quantity

$$M_{\ell j} = Y_\ell(\vartheta_j) = \begin{pmatrix} Y_0(\vartheta_0) & Y_0(\vartheta_1) & \cdots & Y_0(\vartheta_{L-1}) \\ Y_1(\vartheta_0) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ Y_{L-1}(\vartheta_0) & \cdots & \cdots & Y_{L-1}(\vartheta_{L-1}) \end{pmatrix}. \tag{2.113}$$

The matrix has rank L, thus full rank and inversion is possible, leading again to a $L \times L$ matrix $(M^{-1})_{j\ell}$.

Vector quantity

$$M_{\ell j} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (\partial_\vartheta Y)_1(\vartheta_0) & (\partial_\vartheta Y)_1(\vartheta_1) & \cdots & (\partial_\vartheta Y)_1(\vartheta_{L-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (\partial_\vartheta Y)_{L-1}(\vartheta_0) & (\partial_\vartheta Y)_{L-1}(\vartheta_1) & \cdots & (\partial_\vartheta Y)_{L-1}(\vartheta_{L-1}) \end{pmatrix}. \tag{2.114}$$

The matrix has rank $L-1$, it is in principle not invertible. For the application of $\mathbf{M}$ in equation (2.108) the individual modes $\hat{u}_\ell \leftrightarrow x_{li}$ are given. By the structure of the matrix multiplication the zeroth row in $\mathbf{M}$ takes automatically care of ignoring every entry of the zeroth mode. So technically, the entries can be arbitrary. We mention that the $\ell = 0$-mode of the vector quantity has zero-mean.

The inversion (needed in equation (2.110) when a two-dimensional variable is given and we are interested in the decomposition in modes) is more involved. The fact that the zeroth mode vanishes for the vector harmonics means that we have “more grid points than contributing modes”. This implies we can just ignore one grid point and obtain a $(L-1) \times (L-1)$ submatrix which should have full rank and be invertible. There is an ambiguity in the choice of that grid point for the pseudo-spectral approach. Let us simply choose the last one, ϑ_{L-1}.

We obtain the submatrix

$$\tilde{M}_{\ell j} = \begin{pmatrix} (\partial_\vartheta Y)_1(\vartheta_0) & \cdots & (\partial_\vartheta Y)_1(\vartheta_{L-2}) \\ \vdots & \ddots & \vdots \\ (\partial_\vartheta Y)_{L-1}(\vartheta_0) & \cdots & (\partial_\vartheta Y)_{L-1}(\vartheta_{L-2}) \end{pmatrix}. \tag{2.115}$$

The submatrix has itself full rank and is therefore invertible. Inverting it (numerically) gives a $(L-1 \times L-1)$ matrix $\tilde{\mathbf{M}}^{-1}$ with entries $(\tilde{M}^{-1})_{j\ell}$. Building the full “pseudo-inverted” matrix requires to embed $\tilde{\mathbf{M}}^{-1}$ into a $L \times L$ matrix

$\hat{M}^{-1}$. There we set the L^{th} row equals zero. It means the values $u(r, \vartheta_{L-1})$ are ignored in the mode decomposition. $\tilde{\mathbf{M}}^{-1}$ has the form ($*$ stands for some entry that is in general non-vanishing)

$$(\hat{M}^{-1})_{j\ell} = \begin{pmatrix} 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.116}$$

Hence the "pseudo-unity matrix" reads $\mathbb{1} = \mathbf{M} \cdot \mathbf{M}^{-1} = \operatorname{diag}(0, 1, \ldots, 1)$.

Tensor quantity

$$M_{\ell j} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ (Y_{\vartheta\vartheta})_2(\vartheta_0) & (Y_{\vartheta\vartheta})_2(\vartheta_1) & \cdots & (Y_{\vartheta\vartheta})_2(\vartheta_{L-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (Y_{\vartheta\vartheta})_{L-1}(\vartheta_0) & (Y_{\vartheta\vartheta})_{L-1}(\vartheta_1) & \cdots & (Y_{\vartheta\vartheta})_{L-1}(\vartheta_{L-1}) \end{pmatrix}. \tag{2.117}$$

The matrix has rank $L-2$ and is not invertible. Exactly in parallel to the vector transformation the $(L-2) \times (L-2)$ submatrix which is obtained by ignoring (for instance) the first and the last column (equivalently grid points ϑ_0 and ϑ_{L-1}), namely

$$\tilde{M}_{j\ell} = \begin{pmatrix} (Y_{\vartheta\vartheta})_2(\vartheta_1) & \cdots & (Y_{\vartheta\vartheta})_2(\vartheta_{L-2}) \\ \vdots & \ddots & \vdots \\ (Y_{\vartheta\vartheta})_{L-1}(\vartheta_1) & \cdots & (Y_{\vartheta\vartheta})_{L-1}(\vartheta_{L-2}) \end{pmatrix}. \tag{2.118}$$

has full rank, can be inverted and gives $\tilde{\mathbf{M}}^{-1}$.

Again, the full "pseudo-inverted" matrix is obtained by setting the 0^{th} and $(L-1)^{\text{th}}$ row equals zero (hence the values at ϑ_0 and ϑ_{L-1} of u are ignored in the mode decomposition).

$\tilde{\mathbf{M}}^{-1}$ has the form

$$(\hat{M}^{-1})_{j\ell} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.119}$$

Hence the "pseudo-unity matrix" reads $\mathbb{1} = \mathbf{M} \cdot \mathbf{M}^{-1} = \mathrm{diag}(0, 0, 1, \ldots, 1)$.

2.4.3. The derivatives in the spatial coordinates

Here we give some explicit formulas for spatial derivatives for our 2-dimensional (in r and ϑ) variable $u(r, \vartheta)$ represented as matrix with entries u_{ij}. We state again just the propositions, the proofs can be easily obtained by direct calculation. Compare with sections 2.2.3 and 2.2.5.

Derivatives in r

We have for $r \in [0, R]$ the grid points $r_i = i\Delta r,\ i = 0 \ldots N$, so $N + 1$ points. We know that each individual spherical harmonic $\mathcal{Y}_\ell$ implies a definite parity in r at the origin for the modes $\hat{u}_\ell(r)$. Therefore we can use on the mode level the parity property at the inner boundary $r = 0$. In general we consider superpositions of modes. Hence we have no parity information and cannot make use of it. Therefore we use at the inner boundary a one-sided stencil of the same accuracy as the centered stencil for the rest of the grid.

Proposition 2.4.5. The derivative of u in r is given by the centered stencil which is accurate up to $\mathcal{O}(\Delta r^2)$ as

$$\partial_r u_{ij} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta r},\ i = 1, \ldots, N, \tag{2.120a}$$

$$\partial_r u_{0j} = \frac{-3u_{0j} + 4u_{1j} - u_{2j}}{2\Delta r}, \tag{2.120b}$$

$$\partial_r u_{Nj} = \frac{u_{N-2,j} - 4u_{N-1,j} + 3u_{Nj}}{2\Delta r}. \tag{2.120c}$$

The second derivatives are

$$\partial_r^2 u_{ij} = \frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{\Delta r^2},\ i = 1, \ldots, N, \tag{2.121a}$$

$$\partial_r^2 u_{0j} = \frac{2u_{0,j} - 5u_{1j} + 4u_{2,j} - u_{3j}}{\Delta r^2}, \tag{2.121b}$$

$$\partial_r^2 u_{Nj} = \frac{-u_{N-3,j} + 4u_{N-2,j} - 5u_{N-1,j} + 2u_{Nj}}{\Delta r^2}. \tag{2.121c}$$

Derivatives in ϑ

We use the relation $\mathbf{u} = \hat{\mathbf{u}}^\dagger \mathbf{M}$ and $\hat{\mathbf{u}} = (\mathbf{u}\mathbf{M}^{-1})^\dagger \Rightarrow \hat{\mathbf{u}}^\dagger = \mathbf{u}\mathbf{M}^{-1}$ (recall equations (2.108) and (2.109) in section 2.4.2 above).

Proposition 2.4.6. We have the following scheme for the derivatives of u in ϑ

$$\partial_\vartheta \mathbf{u} = \partial_\vartheta(\hat{\mathbf{u}}^\dagger \mathbf{M}) = \hat{\mathbf{u}}^\dagger \partial_\vartheta \mathbf{M} = \mathbf{u}\mathbf{M}^{-1}\partial_\vartheta \mathbf{M}, \tag{2.122a}$$

$$\partial_\vartheta^2 \mathbf{u} = \partial_\vartheta^2(\hat{\mathbf{u}}^\dagger \mathbf{M}) = \hat{\mathbf{u}}^\dagger \partial_\vartheta^2 \mathbf{M} = \mathbf{u}\mathbf{M}^{-1}\partial_\vartheta^2 \mathbf{M}, \tag{2.122b}$$

$$\partial_\vartheta \partial_r \mathbf{u} = \partial_\vartheta \partial_r(\hat{\mathbf{u}}^\dagger \mathbf{M}) = (\partial_r \hat{\mathbf{u}}^\dagger)(\partial_\vartheta \mathbf{M}) = \partial_r(\mathbf{u}\mathbf{M}^{-1})\partial_\vartheta \mathbf{M} = (\partial_r \mathbf{u})\mathbf{M}^{-1}\partial_\vartheta \mathbf{M}. \tag{2.122c}$$

In the following we determine for each quantity the matrices which determine the derivatives. For a given variable in the point space the matrices calculate and return the point values of the derivatives. They are supposed to be implemented for the numerical studies. It is a simple calculation and the corresponding proof is in some sense included already in the expression.

1st derivative in ϑ: $\partial_\vartheta \mathbf{u} = \mathbf{u}\mathbf{M}^{-1}\partial_\vartheta \mathbf{M}$

Proposition 2.4.7. For a scalar quantity $\mathbf{u}$ the corresponding matrix is

$$(Y_\ell)^{-1}\partial_\vartheta Y_\ell = (Y_\ell)^{-1}(Y_\vartheta)_\ell \tag{2.123}$$

where the last object are just the vector harmonics. For a vector quantity $\mathbf{u}$ the matrix reads

$$((Y_\vartheta)_\ell)^{-1}\partial_\vartheta Y_\vartheta = ((Y_\vartheta)_\ell)^{-1}(\mathfrak{S}_1 + \mathfrak{V}_1). \tag{2.124}$$

Recall the notation on pages 53 and following for the quantities as $\mathfrak{S}_1$. The tensor quantity $\mathbf{u}$ has the matrix

$$((Y_{\vartheta\vartheta})_\ell)^{-1}\partial_\vartheta Y_{\vartheta\vartheta} = ((Y_{\vartheta\vartheta})_\ell)^{-1}(\mathfrak{S}_2 + \mathfrak{V}_3). \tag{2.125}$$

2^{nd} derivative in ϑ: $\partial_\vartheta^2 \mathbf{u} = \mathbf{u}\mathbf{M}^{-1}\partial_\vartheta^2\mathbf{M}$

Proposition 2.4.8. For a scalar quantity $\mathbf{u}$ the corresponding matrix is

$$(Y_\ell)^{-1}\partial_\vartheta Y_\ell = (Y_\ell)^{-1}(\mathfrak{S}_1 + \mathfrak{V}_1). \tag{2.126}$$

For a vector quantity $\mathbf{u}$ the matrix reads

$$((Y_\vartheta)_\ell)^{-1}\partial_\vartheta Y_\vartheta = ((Y_\vartheta)_\ell)^{-1}(\mathfrak{S}_2 + \mathfrak{V}_2). \tag{2.127}$$

The tensor quantity $\mathbf{u}$ has the matrix

$$((Y_{\vartheta\vartheta})_\ell)^{-1}\partial_\vartheta Y_{\vartheta\vartheta} = ((Y_{\vartheta\vartheta})_\ell)^{-1}(\mathfrak{S}_3 + \mathfrak{V}_4). \tag{2.128}$$

Derivatives in r and ϑ: $\partial_\vartheta\partial_r\mathbf{u} = (\partial_r\mathbf{u})\mathbf{M}^{-1}\partial_\vartheta^2\mathbf{M}$ For the implementation this derivative is quite easy. Because of the product structure of the variable both derivatives can be calculated successively.

2.5. Laplace operator for $\mathbb{R}_{\geq 0} \times S^2$

A central assumption of our formulation is that we require that our variables admit a spectral decomposition in eigenfunctions of the Laplace operator on the sphere S^2. In this section we show that for a non-scalar theory the spin-weight is important, compare section 2.2.5. In the following we do not consider quantities on the sphere but in the full ball $\mathbb{R}_{\geq 0} \times S^2$. That allows us to derive the leading r-behavior at the origin which will be beneficial for the implementation. Some results of this section seem to be known in principle as they are listed in Thorne (1980, Part 1, section II F) even though it is hard to find any explicit calculations[23]. In the respect of the the current section also Nakamura (1984) is of interest, see also Nagar and Rezzolla (2005).

We consider the Laplace equation on the sphere in spherical polar coordinates, see appendix A.1.2. Given the orthogonal basis $(\partial_r, \partial_\vartheta, \partial_\varphi)$ the components of the metric tensor are $f_{ij} = \operatorname{diag}(1, r^2, r^2\sin^2\vartheta)$. Its non-vanishing Christoffel symbols are calculated in appendix A.1.2 as well. In the following we restrict to hypersurface-orthogonal axisymmetry, see section 3.5, so we suppress basically the φ-dependence.

[23]The group in Meudon published similar calculations as the ones we present here, see Bonazzola et al. (2004), Grandclément and Novak (2009), Novak et al. (2010). They use a different basis, $(\partial_r, r^{-1}\partial_\vartheta, (r\sin\vartheta)^{-1}\partial_\varphi)$. The advantage is that the metric tensor then reads $f_{ij} = \operatorname{diag}(1, 1, 1)$, so the basis is even orthonormal. The disadvantage is that it might be a less common choice. Therefore their results differ from ours and are not directly comparable.

Definition 2.5.1. For an n-dimensional manifold the **Laplace operator** or **Laplacian** Δ is defined as top-dimensional second covariant derivative,

$$\Delta := \nabla^2 = f^{ij}\nabla_i\nabla_j, \tag{2.129}$$

summation over $i, j = 1, \ldots, n$ is implied. The **Laplace equation** for some quantity ψ is $\Delta\psi = 0$. A solution of the Laplace equation is called **harmonic**. The Laplace-Beltrami operator is the Laplace operator for the $(n-1)$-dimensional hypersurface, also denoted by Δ.

Proposition 2.5.1. In Cartesian coordinates (here in general n dimensions, usually we restrict to $n = 3$) the Laplace operator reads

$$\Delta := \sum_{i=1}^{n} \partial_{x_i}^2 \stackrel{n=3}{=} \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{2.130}$$

In n spherical polar coordinates (so in $\mathbb{R} \times S^{n-1}$) the Laplace operator can be written by using the Laplace-Beltrami operator $\Delta_{S^{n-1}}$ and a radial contribution as

$$\Delta_{\mathbb{R}_{\geq 0} \times S^{n-1}} = \partial_r^2 + (n-1)r^{-1}\partial_r + r^{-2}\Delta_{S^{n-1}}. \tag{2.131}$$

The Laplace-Beltrami operator can be written in a recursive way

$$\Delta_{S^{n-1}} = \partial_\vartheta^2 + (n-2)\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta + \frac{1}{\sin^2\vartheta}\Delta_{S^{n-2}}. \tag{2.132}$$

In the case of $n = 3$ the Laplace operator is

$$\Delta_{\mathbb{R}_{\geq 0} \times S^2} = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2} \tag{2.133}$$

with Laplace-Beltrami operator

$$\Delta_{S^2} = \partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta + \frac{1}{\sin^2\vartheta}\partial_\varphi^2. \tag{2.134}$$

The Laplace equation is elliptic.

Proof. In Cartesian coordinates it is trivial and we refer to example 2.3.1. For other coordinates we can make use of the transformations as given in appendix A.1.2. □

In the following we investigate the Laplace operator for a scalar, vector and a symmetric two tensor. We restrict ourself to the three-dimensional case, so to $\mathbb{R} \times S^2$ and assume hypersurface-othogonal axisymmetry, see section 3.5. This is not a conceptual limitation but just easier in the calculation and sufficient for our needs in the remainder. Its generalization to the case without axisymmetry or to higher dimensions is straightforward.

2.5.1. Scalar Laplace equation

Proposition 2.5.2. The **regular** solutions to the **scalar** Laplace equation (in axisymmetry) are given by the **solid spherical harmonics**

$$\sum_{\ell=0}^{L-1} r^\ell \bar{\phi}_\ell Y_\ell(\vartheta) \tag{2.135}$$

with (scalar) Y_ℓ (see section 2.2.5) and L constants $\bar{\phi}_\ell$.

Proof. In spherical polar coordinates the Laplace equation for a scalar function ϕ can be calculated as

$$\begin{aligned}\Delta_{\mathbb{R}_{\geq 0}\times S^2}\phi &= f^{ij}\nabla_i\partial_j\phi = f^{ij}\partial_i\partial_j\phi - f^{ij}\Gamma^k{}_{ij}\partial_k\phi \\ &= \partial_r^2\phi + 2r^{-1}\partial_r\phi + r^{-2}\left[\partial_\vartheta^2\phi + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta\phi\right].\end{aligned} \tag{2.136}$$

Making the product ansatz

$$\phi(r,\vartheta) = \sum_{\ell=0}^{L-1}\hat{\phi}_\ell(r)Y_\ell(\vartheta) \equiv \hat{\phi}_\ell Y_\ell \tag{2.137}$$

gives

$$\begin{aligned}&\left[\partial_r^2 + 2r^{-1}\partial_r + r^{-2}\Delta_{S^2}\right]\hat{\phi}_\ell Y_\ell \\ &= \left[\partial_r^2 + 2r^{-1}\partial_r - r^{-2}\ell(\ell+1)\right]\hat{\phi}_\ell Y_\ell = 0.\end{aligned} \tag{2.138}$$

The ordinary differential equation in r has the general solution for $\hat{\phi}_\ell$ in the form

$$\hat{\phi}_\ell = \bar{\phi}_\ell^+ r^\ell + \bar{\phi}_\ell^- r^{-(\ell+1)}. \tag{2.139}$$

Regularity at the origin requires $\bar{\phi}_\ell^-$ to vanish and therefore the regular solution looks like the one claimed above (where we dropped the superscript $+$). □

2.5.2. Vector Laplace equation

Since we are more interested in a one-form with components β_i than in a vector β^i we consider directly the covariant components.

Proposition 2.5.3. The **regular** solutions of the **vector** Laplace equation in hypersurface-orthogonal axisymmetry ($\beta_\varphi = 0$) are given by the spin-one-weighted components

$$\beta_r = \left(\bar{\beta}_1 r^{\ell+1} + \bar{\beta}_2 r^{\ell-1}\right) Y_\ell(\vartheta), \tag{2.140a}$$
$$\beta_\vartheta = \left(\bar{\beta}_1 r^{\ell+2} + \bar{\beta}_2 r^{\ell}\right) Y_{\vartheta,\ell}(\vartheta) \tag{2.140b}$$

with constants $\bar{\beta}_1$ and $\bar{\beta}_2$ (and $\bar{\beta}_2$ vanishes for $\ell = 0$).

Proof. Straightforward calculation of the covariant derivatives of the components of the one-form leads[24] to

$$\Delta_{\mathbb{R}_{\geq 0}\times S^2}\beta_r = \partial_r^2\beta_r + 2r^{-1}\partial_r\beta_r + r^{-2}\left[\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\right]\beta_r$$
$$- 2r^{-2}\beta_r - 2r^{-3}\left[\partial_\vartheta + \frac{\cos\vartheta}{\sin\vartheta}\right]\beta_\vartheta, \tag{2.141a}$$
$$\Delta_{\mathbb{R}_{\geq 0}\times S^2}\beta_\vartheta = \partial_r^2\beta_r + r^{-2}\left[\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\right]\beta_r - r^{-2}\left(1 + \frac{\cos\vartheta}{\sin\vartheta}\right)\beta_\vartheta + 2r^{-1}\partial_\vartheta\beta_r, \tag{2.141b}$$
$$\Delta_{\mathbb{R}_{\geq 0}\times S^2}\beta_\varphi = 0. \tag{2.141c}$$

Again with the product ansatz

$$\beta_r(r,\vartheta) = \sum_{\ell=0}^{L-1}\hat{\beta}_{r,\ell}(r)Y_\ell(\vartheta) \equiv \hat{\beta}_{r,\ell}Y_\ell, \tag{2.142a}$$
$$\beta_\vartheta(r,\vartheta) = \sum_{\ell=0}^{L-1}\hat{\beta}_{\vartheta,\ell}(r)Y_{\vartheta,\ell}(\vartheta) \equiv \hat{\beta}_{\vartheta,\ell}Y_{\vartheta,\ell}, \tag{2.142b}$$
$$\beta_\varphi(r,\vartheta) = 0 \tag{2.142c}$$

we arrive at a coupled set of linear ordinary differential equations for the components

$$\partial_r^2\hat{\beta}_r + 2r^{-1}\partial_r\hat{\beta}_r - \ell(\ell+1)r^{-2}\hat{\beta}_r - 2r^{-2}\hat{\beta}_r + 2\ell(\ell+1)r^{-3}\hat{\beta}_\vartheta = 0, \tag{2.143a}$$

[24] Here we see the problems if we compare with the results obtained by the group in Meudon cited at the beginning of the section. Their choice results in different factors which makes the comparison unpleasant.

$$\partial_r^2 \hat{\beta}_\vartheta - \ell(\ell+1) r^{-2} \hat{\beta}_\vartheta + 2r^{-1} \hat{\beta}_r = 0. \tag{2.143b}$$

One way to solve the coupled system is to decouple it which gives one ordinary differential equation of fourth order,

$$-\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{2} r^{-3} \hat{\beta}_\vartheta + \frac{\ell(\ell+1)}{r} \partial_r^2 \hat{\beta}_\vartheta - 2\partial_r^3 \hat{\beta}_\vartheta - \frac{r}{2} \partial_r^4 \hat{\beta}_\vartheta = 0 \tag{2.144}$$

with solution

$$\hat{\beta}_{r,\ell} = -c_1(\ell+1) r^{-(\ell+2)} + \ell c_2 r^{-\ell} - (\ell+1) c_3 r^{\ell+1} + \ell c_4 r^{\ell-1}, \tag{2.145a}$$

$$\hat{\beta}_{\vartheta,\ell} = c_1 r^{-(\ell+1)} + \ell c_2 r^{-(\ell-1)} + c_3 r^{\ell+2} + c_4 r^{\ell-1}. \tag{2.145b}$$

Regularity requires $c_1 = 0$ and $c_2 = 0$ (for $\ell \geq 1$) and therefore leads to the claimed result. In addition we can read off that $\hat{\beta}_{r,\ell} = \mathcal{O}(r^{\ell-1})$ for $\ell \neq 0$ and $\mathcal{O}(r^1)$ for $\ell = 0$ and $\hat{\beta}_{\vartheta,\ell} = \mathcal{O}(r^\ell)$ close to the origin. □

2.5.3. Tensor Laplace equation

Theorem 2.5.1. Consider a symmetric two-tensor M_{ij} in hypersurface-orthogonal axisymmetry (in particular $M_{\varphi r} = 0 = M_{\varphi\vartheta}$) which is harmonic (solution of the Laplace equation). The components of the solution of the Laplace equation can be combined such that its angular part expands (see section 2.2.5 for the definition of the spherical harmonics) in

- two scalar components with $\mathcal{Y}_\ell = Y_\ell$,
- a vector component with $\mathcal{Y}_\ell = Y_{\vartheta,\ell}$,
- and a tensor component with $\mathcal{Y}_\ell = Y_{\vartheta\vartheta,\ell}$

and each of these quantities, say u, expands in contributions of spin-weight 0 and spin-weight ± 2 as

$$u = \sum \left(\bar{u}_\ell r^{\ell-2} + \bar{v}_\ell r^\ell + \bar{w}_\ell r^{\ell+2} \right) \mathcal{Y}_\ell \tag{2.146}$$

with constants $\bar{u}_\ell$, $\bar{v}_\ell$ and $\bar{w}_\ell$.

Proof. Calculating the second covariant derivatives for the two-tensor gives (here we skip the subscript $\mathbb{R}_{\geq 0} \times S^2$ of the Laplacian)

$$\Delta M_{rr} = \partial_r^2 M_{rr} + 2r^{-1} \partial_r M_{rr} + r^{-2} \left[\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta} \partial_\vartheta \right] M_{rr} - 4r^{-2} M_{rr}$$

$$- 4r^{-3}\left[\partial_\vartheta + \frac{\cos\vartheta}{\sin\vartheta}\right]M_{\vartheta r} + 2r^{-4}\left[M_{\vartheta\vartheta} + \frac{M_{\varphi\varphi}}{\sin^2\vartheta}\right], \tag{2.147a}$$

$$\Delta M_{\vartheta r} = \partial_r^2 M_{\vartheta r} + r^{-2}\left[\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta + \frac{\cos^2\vartheta}{\sin^2\vartheta}\right]M_{\vartheta r} - 4r^{-2}M_{\vartheta r}$$
$$+ 2r^{-1}\partial_\vartheta M_{rr} - 2r^{-3}\left[\partial_\vartheta M_{\vartheta\vartheta} + \frac{\cos\vartheta}{\sin\vartheta}\left(M_{\vartheta\vartheta} - \frac{M_{\varphi\varphi}}{\sin^2\vartheta}\right)\right], \tag{2.147b}$$

$$\Delta M_{\vartheta\vartheta} = \partial_r^2 M_{\vartheta\vartheta} - 2r^{-1}\partial_r M_{\vartheta\vartheta} + r^{-2}\left[\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta\right]M_{\vartheta\vartheta}$$
$$- 2r^{-2}\frac{\cos^2\vartheta}{\sin^2\vartheta}M_{\vartheta\vartheta} + 4r^{-1}\partial_\vartheta M_{\vartheta r} + 2M_{rr} + 2r^{-2}\frac{\cos^2\vartheta}{\sin^2\vartheta}\frac{M_{\varphi\varphi}}{\sin^2\vartheta}, \tag{2.147c}$$

$$\Delta M_{\varphi\varphi} = \partial_r^2 M_{\varphi\varphi} - 2r^{-1}\partial_r M_{\varphi\varphi} + r^{-2}\left[\partial_\vartheta^2 - 3\frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta\right]M_{\varphi\varphi}$$
$$+ 2r^{-2}\frac{M_{\varphi\varphi}}{\sin^2\vartheta} + 4r^{-1}\sin\vartheta\cos\vartheta M_{\vartheta r} + 2\sin^2\vartheta M_{rr} + 2r^{-2}\cos^2\vartheta M_{\vartheta\vartheta} \tag{2.147d}$$

and $\Delta M_{\varphi r} = 0 = \Delta M_{\varphi\vartheta}$. The remaining components follow from symmetry. We expand the components in equation (2.147) in eigenfunctions on the two-sphere S^2, referring to section 2.2.5 for the notation of our harmonics,

$$\Delta M_{rr}(r,\vartheta) = A(r)Y(\vartheta), \tag{2.148a}$$
$$\Delta M_{\vartheta r} = rB(r)Y_\vartheta(\vartheta), \tag{2.148b}$$
$$\Delta M_{\vartheta\vartheta} = r^2\left(-\frac{A(r)}{2}Y(\vartheta) + C(r)Y_{\vartheta\vartheta}(\vartheta)\right), \tag{2.148c}$$
$$\Delta M_{\varphi\varphi} = r^2\sin^2\vartheta\left(-\frac{A(r)}{2}Y(\vartheta) - C(r)Y_{\vartheta\vartheta}(\vartheta)\right) \tag{2.148d}$$

where we make use of the notation and abbreviations (for example summation is automatically understood) as above. Considering further $\Delta M_{\vartheta r}$ and the combinations $\Delta M_{\vartheta\vartheta} \pm \Delta M_{\varphi\varphi}/\sin^2\vartheta$ (alternatively ΔM_{rr} instead of the combination with "+") the Laplace equation leads to

$$\left\{r^2\partial_r^2 A(r) + 2r\partial_r A(r) - [6+\ell(\ell+1)]A(r) + 4l(\ell+1)B(r)\right\}Y(\vartheta) = 0, \tag{2.149a}$$
$$\left\{r^2\partial_r^2 B(r) + 2r\partial_r B(r) - [4+\ell(\ell+1)]B(r) + 3A(r)\right.$$
$$\left.+[(\ell+1)-2]C(r)\right\}Y_\vartheta(\vartheta) = 0, \tag{2.149b}$$
$$\left\{r^2\partial_r^2 C(r) + 2r\partial_r C(r) - [-2+\ell(\ell+1)]C(r) + 4B(r)\right\}Y_{\vartheta\vartheta}(\vartheta) = 0. \tag{2.149c}$$

To find a solution of equation (2.149) we use the ansatz $A(r) = r^\alpha\bar{A}$, $B(r) = r^\beta\bar{B}$ and $C(r) = r^\gamma\bar{C}$ with constants in r (for each ℓ-mode) denoted with a bar. Then equation (2.149) becomes (without the angular part, but still for each ℓ-mode a separate set of equations)

$$4\ell(\ell+1)\bar{B}r^{\beta} + [-6 - \ell(\ell+1) + \alpha(\alpha+1)]\bar{A}r^{\alpha} = 0, \tag{2.150a}$$

$$3\bar{A}r^{\alpha} + [-4 - \ell(\ell+1) + \beta(\beta+1)]\bar{B}r^{\beta} + [-2 + \ell(\ell+1)]\bar{C}r^{\gamma} = 0, \tag{2.150b}$$

$$4\bar{B}r^{\beta} + [2 - \ell(\ell+1) + \gamma(\gamma+1)]\bar{C}r^{\gamma} = 0. \tag{2.150c}$$

We are just interested in non-trivial solutions which implies $\alpha = \beta = \gamma$. Only a few values for the exponents lead to consistent solutions, let us list the regular ones[25]:

- $\alpha = \beta = \gamma = \ell$ implies $\bar{A} = 2\ell(\ell+1)\bar{B}/3$ and $\bar{C} = -2\bar{B}$,
- $\alpha = \beta = \gamma = \ell - 2$ implies $\bar{A} = \ell\bar{B}$ and $\bar{C} = \bar{B}/(\ell-1)$,
- $\alpha = \beta = \gamma = \ell + 2$ implies $\bar{A} = -(\ell+1)\bar{B}$ and $\bar{C} = -\bar{B}/(\ell+2)$.

Therefore the relation (2.146) is shown which completes the proof. □

Corollary 2.5.1. Close to the origin the regular solutions to the tensor Laplace equation consist of only very few modes. In particular up to $\mathcal{O}(\Delta r^2)$ we have for the first two grid points the relations (with notation as in theorem 2.5.1),

$$\text{at } r = 0: \; u = \bar{v}_0\mathcal{Y}_0 + \bar{u}_2\mathcal{Y}_2, \tag{2.151}$$

$$\text{at } r = \Delta r: u = \bar{v}_0\mathcal{Y}_0 + \bar{u}_2\mathcal{Y}_2 + (\bar{v}_1\mathcal{Y}_1 + \bar{u}_3\mathcal{Y}_3)\,\Delta r + \mathcal{O}(\Delta r^2). \tag{2.152}$$

For vector and tensor quantities the expressions even simplify further because of the vanishing of the lowest spherical harmonics, see proposition 2.4.4.

Proof. Using theorem 2.5.1 we write the variable in the form of a series

$$\begin{aligned} u &= \sum_{\ell=0}^{L-1} \hat{u}_\ell \mathcal{Y}_\ell = \sum_\ell \left(\bar{u}_\ell r^{\ell-2} + \bar{v}_\ell r^{\ell} + \bar{w}_\ell r^{\ell+2}\right)\mathcal{Y}_\ell \\ &= \left(\bar{u}_0 r^{-2} + \bar{v}_0 r^0 + \bar{w}_0 r^2\right)\mathcal{Y}_0 + \left(\bar{u}_1 r^{-1} + \bar{v}_1 r^1 + \bar{w}_1 r^3\right)\mathcal{Y}_1 \\ &\quad + \left(\bar{u}_2 + \bar{v}_2 r^2 + \bar{w}_2 r^4\right)\mathcal{Y}_2 + \left(\bar{u}_3 r^1 + \bar{v}_3 r^3 + \bar{w}_3 r^5\right)\mathcal{Y}_3 \\ &\quad + \left(\bar{u}_4 r^2 + \bar{v}_4 r^4 + \bar{w}_4 r^6\right)\mathcal{Y}_4 + \dots . \end{aligned} \tag{2.153}$$

Regularity at the origin requires that $\bar{u}_0 = 0 = \bar{u}_1$. Therefore close to the origin we can write

$$u = \bar{v}_0\mathcal{Y}_0 + \bar{w}_2\mathcal{Y}_2 + (\bar{v}_1\mathcal{Y}_1 + \bar{w}_3\mathcal{Y}_3)\,r + \mathcal{O}(r^2) \tag{2.154}$$

and the claimed relations follow. □

[25] It is interesting to note that there are formal solutions with exponents $\ell \pm 1$ as well. They come with a negative sign in the exponent and are thus irregular.

Therefore in the numerics it is always possible to apply a filter that makes use of corollary 2.5.1 which means that all the other modes can be set to zero at the innermost grid points in the (pseudo-)spectral formulation.

Note in particular that the variables may be ϑ-dependent at the origin which is just a single point at $r = 0$. This is due to the tensorial character of the equation. For a scalar equation for example (see section 2.5.1 there is just a single contribution (for each mode) proportional to r^ℓ. Therefore at $r = 0$ the only possible ϑ-dependence is in $\mathcal{Y}_0$ which turns out (see section 2.2.5) to be ϑ-independent though.

2.6. Axisymmetric scalar wave equation in spherical polar coordinates

The aim for this thesis is the study of Einstein's equations in spherical polar coordinates. A subset of these equations have some wave-like character. They can be seen as a nonlinear generalization of the wave equation. In this section we study the scalar wave equation in these coordinates where one is faced with some similar problems. The results derived here and its generalizations are very helpful for later studies of Einstein's equations. See also Frauendiener (2002) where the discretization in axisymmetry and the stability of the wave equation is examined.

2.6.1. Some analytical considerations and consequences for the numerical implementation

Following Gundlach et al. (2013) where the first steps of this section are already included we consider now the more general situation of n spatial dimensions (usually $n = 3$ and $S^{n-1} = S^2$). Let us consider the $n+1$-dimensional wave equation[26] in spherical coordinates $(t, r, \vartheta, \varphi, \ldots)$ for a scalar[27] quantity ϕ (see for the Laplacian also definition 2.5.1),

$$\Box_{\mathcal{M}^{(1,1)}\times S^{n-1}}\phi = \left(\partial_r^2 + (n-1)r^{-1}\partial_r + r^{-2}\triangle_{S^{n-1}} - \partial_t^2\right)\phi = 0. \tag{2.155}$$

[26] In equation (2.155) we introduced already the **d'Alembert operator** $\Box_{\mathcal{M}^{(1,1)}\times S^{n-1}} := \left[\triangle_{\mathbb{R}_{\geq 0}\times S^{n-1}} - \partial_t^2\right]$.

[27] It would be interesting to extend the analysis to more sophisticated quantities than scalars like those we considered in section 2.5.

Let ϕ be regular at the origin and analytic[28]. We make use of the eigenfunctions and -values of the Laplace operator on $S^{n-1}, n \geq 2$,

$$\triangle_{S^{n-1}} Y_{\ell m \dots} = -\ell(\ell + n - 2) Y_{\ell m \dots}. \tag{2.156}$$

That suggests an expansion in (spin-weight 0) spherical harmonics, see section 2.2.5,

$$\phi(t, r, \vartheta, \varphi, \dots) = \sum_{\ell m \dots} \hat{\phi}(t, r)_{\ell m \dots} Y_{\ell m \dots} \tag{2.157}$$

and the wave equation reads on the mode level

$$\Box_{\mathcal{M}^{(1,1)} \times S^{n-1}} \phi = \sum_{\ell m \dots} \hat{\Box}_{\mathcal{M}^{(1,1)}} \hat{\phi}_{\ell m \dots} Y_{\ell m \dots} = 0 \tag{2.158}$$

where

$$\begin{aligned} \hat{\Box}_{\mathcal{M}^{(1,1)}} \hat{\phi}_{\ell m \dots} &= \left[-\partial_t^2 + \hat{\triangle}_{S^{n-1}} \right] \hat{\phi}_{\ell m \dots} \\ = \left[-\partial_t^2 + r^{-(n-1)} \partial_r \left(r^{n-1} \partial_r \right) - \frac{\ell(\ell + n - 2)}{r^2} \right] \hat{\phi}_{\ell m \dots} &= 0 \end{aligned} \tag{2.159}$$

must be satisfied for each mode. In the standard case ($n = 3,\ S^2$) we are dealing with the equation

$$\left[-\partial_t^2 + r^{-2} \partial_r \left(r^2 \partial_r \right) - \frac{\ell(\ell + 1)}{r^2} \right] \hat{\phi}_{\ell m} = 0. \tag{2.160}$$

Proposition 2.6.1. $\hat{\phi}_{\ell m \dots}$ is regular at the origin if and only if $\hat{\phi}_{\ell m \dots} = r^\ell \bar{\phi}_{\ell m \dots}$ and $\bar{\phi}_{\ell m \dots} = \sum_{k \geq 0, k \text{ even}} c_k r^k$.

Proof. We use the wave equation (2.158) and insert a general Taylor expansion of $\hat{\phi}$ (analyticity and regularity at the origin required), $\hat{\phi} = \sum_k \frac{a_k}{k!} r^k$ where the a_k dependent on t only. We demand the coefficients of the resulting power series in r to vanish. It gives the result in the proposition. The other direction is trivial. $\square$

Making explicit use of a stencil for the derivative in r allows us to determine the first few grid-points **algebraically**. So instead of applying the field equation at the first few innermost grid points we can determine their value equivalently by using some algebraic relations instead. This seems to be beneficial when the field equation turns out to be formally singular. In some sense we push the inner boundary further out. This needs to be done mode by mode since the relations are not the same for all modes.

[28] **Analyticity** means that there is a neighborhood where its Taylor series converges.

Proposition 2.6.2. For the zeroth grid point we have the algebraic relation

$$u_0 = \begin{cases} \frac{4u_1 - u_2}{3}, \ \ell = 0, \\ 0, \ \ell \neq 0. \end{cases} \tag{2.161}$$

Proof. Since the modes are $u = \mathcal{O}(r^\ell)$ at the origin the statement $u_0 = 0$ is obviously true $\forall \ell \geq 1$. The mode for $\ell = 0$ has an even expansion at the origin, $u = ar^0 + br^2 + \ldots$ for some constant numbers $a, b \in \mathbb{R}$. If we denote the step size by Δr we conclude (up to higher powers in Δr)

$$u_0 = a \ \ \rightarrow a = u_0, \tag{2.162a}$$

$$u_1 = a + b\Delta r^2 \ \ \rightarrow b = \frac{u_1 - u_0}{\Delta r^2}, \tag{2.162b}$$

$$u_2 = a + 4b\Delta r^2. \tag{2.162c}$$

The claimed result follows. □

Alternatively one could use a one-sided stencil of the same order for the derivative at the origin and since, as we will show shortly, the first derivative vanishes for $\ell = 0$ the same result follows.

Remark 2.6.1. Going further one can also use a higher-order approximation, namely at the origin $u = ar^0 + br^2 + cr^4 + \ldots$, then taking into account the first three grid points to determine the zeroth one. Then we obtain

$$u_0 = \frac{15u_1 - 6u_2 + u_3}{10}. \tag{2.163}$$

If the numerical division by r is required in the equations it is absolutely necessary to algebraically determine the zeroth grid point (if the grid is vertex-centered, for a staggered grid it might be possible to use the field equation though).

As an example for the algebraic relations for the grid points in the neighborhood of the origin let us take a second-order accurate stencil. Therefore the first derivatives at the zeroth grid point read

$$\partial_r u|_0 = \frac{u_1 - u_{-1}}{2h}, \tag{2.164a}$$

$$\partial_r^2 u|_0 = \frac{u_1 - 2u_0 + u_{-1}}{h^2}, \tag{2.164b}$$

$$\partial_r^3 u|_0 = \frac{-u_{-2} + 2u_{-1} - 2u_1 + u_2}{2h^3}, \tag{2.164c}$$

$$\partial_r^4 u|_0 = \frac{u_{-2} - 4u_{-1} + 6u_0 - 4u_1 + u_2}{h^4}. \tag{2.164d}$$

Proposition 2.6.3. As long as the Taylor expansion is valid and we use a second-order accurate scheme we have the following hierachy

$$\forall \ell \geq 1: \; u_0 = 0, \tag{2.165a}$$

$$\forall \ell \geq 3: \; u_1 = 0, \tag{2.165b}$$

$$\forall \ell \geq 5: \; u_2 = 0, \tag{2.165c}$$

$$\ldots .$$

Proof. In proposition 2.6.1 we saw that the modes behave as $u = \mathcal{O}(r^\ell)$ close to the origin. Therefore the derivatives behaves as $\partial_r u = \mathcal{O}(r^{\ell-1})$ there and vanishes at the origin for $\ell \geq 1$. For higher derivatives we have similar statements for higher ℓ.

Further we use the fact that we know the parity of the mode function and together with the ghost-point-technique (see section 2.2.3) we can express the first few grid points with negative index by their positive pendant, $u_{-i} = \pm u_i$.

Combining the vanishing of the derivatives at the origin and the stencil for the derivative gives the desired result. □

Remark 2.6.2. We should compare our results with the ones obtained in Csizmadia et al. (2013). The basic statement ("replace innermost grid points by algebraic relations") is equivalent. They use the regularity of the Laplacian and parity of the modes to derive $u|_0 = 0 \; \forall \ell \geq 1$ and $\partial_r u|_0 = 0 \; \forall \ell \neq 1$. Together with their fourth-order stencil it results in conditions for the zeroth (origin), first (next to origin) and second grid point.

2.6.2. Numerical confirmation of a stable evolution

Concerning the coding and the implementation consider also the remarks in section 5.2.1.

For the implementation we use (similarly as in example 2.3.1) a first-order in time formulation for the wave equation and $n = 3, \; m = 0$ (scalar wave equation in axisymmetry), with $\hat{\phi} \equiv \gamma$ (observe the similarity with the Cauchy formulation of general relativity in section 3.3),

$$\partial_t \gamma = -2K, \tag{2.166a}$$

$$\partial_t K = -\frac{1}{2}\left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2}\right]\gamma. \tag{2.166b}$$

Proposition 2.6.4. For the posed problem we can even give the exact solution with the ansatz (superscript + for ingoing solution and − for outgoing)

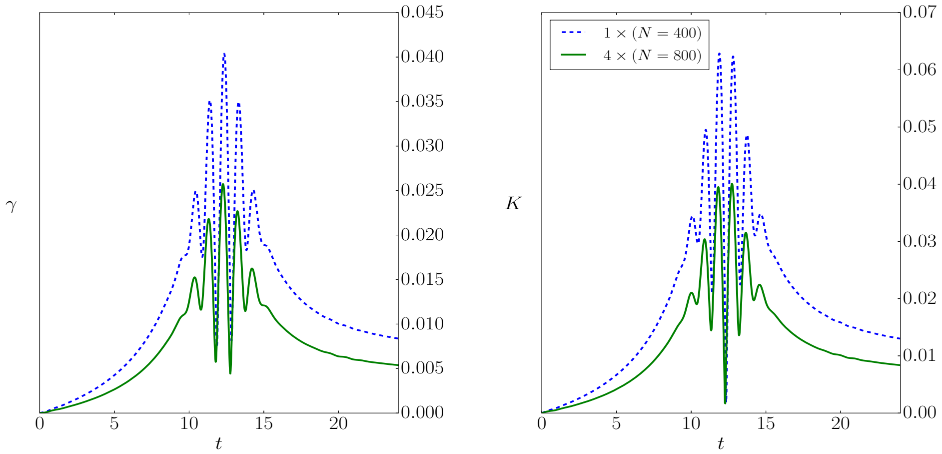
$$\phi^{\pm}(t,r) = \sum_{j=0}^{\ell} r^{j-\ell-1} c_j G^{(j)}(x) \text{ and } c_j = \frac{(-2)^{j-\ell}(2\ell-j)!}{(\ell-j)!j!} \tag{2.167}$$

and generating function $G(x), x = r \pm t$ and jth derivative $G^{(j)}$. Compare with the exact solution we obtain for linearized Einstein's equations in section 4.6.

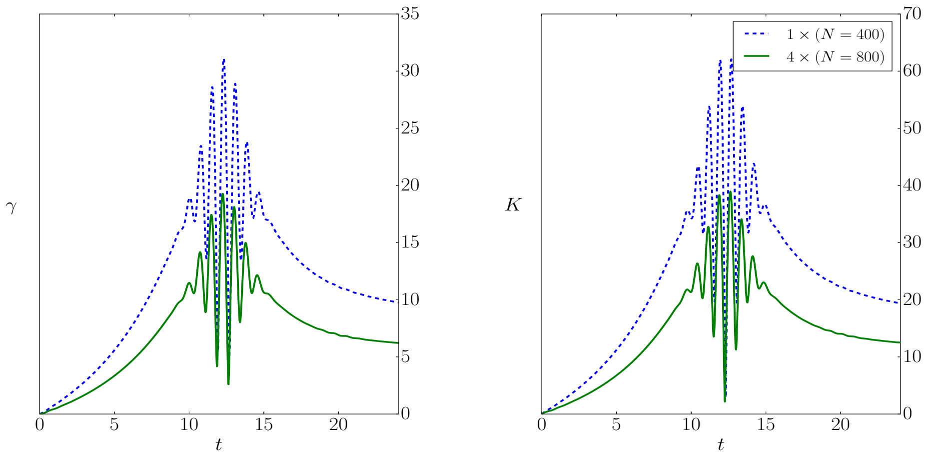
Proof. We just have to plug the ansatz in the wave equation and see that it is indeed a solution. The regular solution γ is given as $\gamma = \phi^+ - \phi^-$. It simplifies the choice of initial data and the convergence tests. □

The term $\ell(\ell+1)/r^2$ in the time evolution of K becomes problematic for high ℓ (and high spatial resolution). If we just use the necessary (for a vertex-centered grid) condition (2.161) the simulation quickly (usually after just a few time steps) leads to a blow up in the origin (the points very close to the origin).

Controlling some innermost grid points with algebraic relations in equation (2.165) stabilizes the origin. As an example (we choose to display $\ell = 5$ and $\ell = 11$) consider figure 2.4. Since we are using a second-order stencil we expect the error to reduce with a factor of four if we double the resolution. This seems to be more than achieved. Observe that the scale is significantly different for the higher ℓ-mode.



(a) For $\ell = 5$.



(b) For $\ell = 11$.

Figure 2.4.: We show the convergence plots for the wave equation. We display the L2-norm of the difference between numerical and exact solution for two resolutions. The higher resolution is multiplied with four (second-order scheme).

We can also calculate the quotient between the norms of different resolutions. For a second-order scheme we expect the quotient to be at least four for a convergent implementation. In figure 2.5 we see that the factor of four is indeed more than achieved. We just plot the quotient for $\ell = 5$, the plot for $\ell = 11$ looks very similar. The code is in fact overconverging.

In these simulation it is in fact enough to determine the zeroth, first and second grid point algebraically (setting it to zero). Going higher, say $\ell = 37$ for instance, already requires the third grid point to be included in the considerations. This again fails for some higher ℓ which, for example, requires the controlling of the fourth grid point for $\ell = 47$. One should remark that the last statements with some particular ℓ are not at all sharp. For instance the stencil and the particular formulation of the problem should have some influence.

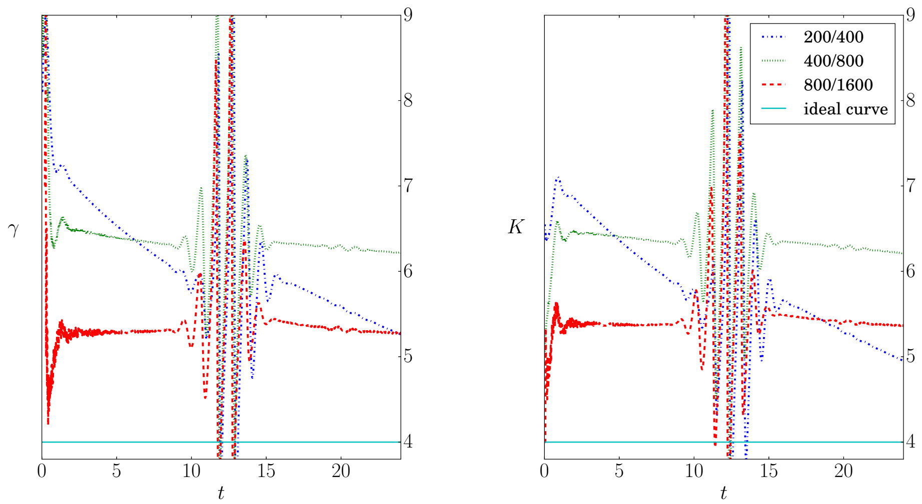


Figure 2.5.: The quotient between the L2-norms plotted in figure 2.4, here for $\ell = 5$.

2.6.3. Generalizations, in particular to situations without parity

Let us generalize the results obtained in section 2.6.1 to the situation that we will consider in chapter 5. The major difference is that we have no parity information any more.

Nevertheless we can derive a statement how to express the zeroth grid point at the origin **algebraically** with the values of the variable at neighboring grid points close to the origin instead of applying the formally singular evolution equations.

Proposition 2.6.5. The value u_0 at the zeroth grid point for a variable u can be expressed algebraically. With step size Δr up to an error $\mathcal{O}(\Delta r^2)$ we have

$$u_0 = u_2 - 2u_1, \tag{2.168}$$

up to an error $\mathcal{O}(\Delta r^3)$ it reads

$$u_0 = 3u_1 - 3u_2 + u_3 \tag{2.169}$$

with u_i denoting the value at the i^{th} grid point.

Proof. Consider the Taylor expansion at the origin of the variable u up to $\mathcal{O}(\Delta r)$,

$$u(r = 0) = u_0 = a, \tag{2.170a}$$
$$u(r = \Delta r) = u_1 = a + b\Delta r + \ldots, \tag{2.170b}$$
$$u(r = 2\Delta r) = u_2 = a + 2b\Delta rh + \ldots. \tag{2.170c}$$

Up to $\mathcal{O}(\Delta r)$ we have

$$u(r = 0) = u_0 = a, \tag{2.171a}$$
$$u(r = \Delta r) = u_1 = a + b\Delta r + c\Delta r^2 + \ldots, \tag{2.171b}$$
$$u(r = 2\Delta r) = u_2 = a + 2b\Delta rh + 4c\Delta r^2 + \ldots, \tag{2.171c}$$
$$u(r = 3\Delta r) = u_3 = a + 3b\Delta r + 9c\Delta r^2 + \ldots. \tag{2.171d}$$

We cut the expressions and neglect the indicated further terms. With these relations we express the grid point u_0 as in the equations in the proposition. □

So we are always able to use that relation to determine the innermost point u_0 without using the (formally singular) field equations. Generalizations of the proposition 2.6.5 are possible.

Actually from the numerical point of view we have better experience with a filter correcting the variables close to the origin. The filter decomposes the variable into modes and determines the innermost point on the mode level before transforming back. According to section 2.5 we know the behavior in r close to the origin of our variables. Most points can be controlled with proposition 2.6.3. Nevertheless for the simulation of Einstein's equations we can also apply an additional trick.

Proposition 2.6.6. For mode functions $\hat{u}$ that behave close to the origin like $\mathcal{O}(r^1)$ (i.e. for a tensor quantity the modes with mode number $\ell = 1$ and $\ell = 3$, see section 2.5) we set the zeroth grid point to zero, $u_0 = 0$, and the first one to (satisfied up to an error $\mathcal{O}(\Delta r^2)$ in step size Δr)

$$u_1 = \frac{1}{2} u_2 \tag{2.172}$$

or (satisfied in fact up to $\mathcal{O}(\Delta r^2)$)

$$u_1 = \frac{4u_2 - u_3}{5}. \tag{2.173}$$

Proof. The vanishing of u_0 follows from parity. For u_1 we proceed as for the proof of proposition 2.6.2. $\hat{u}$ is an odd function and expands close to the origin as $\hat{u} \approx ar^1 + br^3$ for some constant numbers $a, b \in \mathbb{R}$. Therefore with step size Δr we have (up to $\mathcal{O}(\Delta r^3)$)

$$u_1 = a\Delta r + \dots, \tag{2.174a}$$
$$u_2 = 2a\Delta r + \dots. \tag{2.174b}$$

Up to $\mathcal{O}(\Delta r^3)$ these relations imply the result

$$u_1 = \frac{u_2}{2}. \tag{2.175}$$

Generalizing up to $\mathcal{O}(\Delta r^5)$ we write

$$u_1 = a\Delta r + b\Delta r^3 + \dots, \tag{2.176a}$$
$$u_2 = 2a\Delta r + 8b\Delta r^3 + \dots, \tag{2.176b}$$
$$u_3 = 3a\Delta r + 27b\Delta r^3 + \dots. \tag{2.176c}$$

Hence up to $\mathcal{O}(\Delta r^5)$ we have the remaining result

$$u_1 = \frac{4u_2 - u_3}{5}. \tag{2.177}$$

□

Again generalizations are possible in a straightforward way, for instance to other modes to determine the first non-vanishing grid point there.

2.6.4. Boundary conditions for the wave equation

The situation of interest is the one of isolated systems where the solution is supposed to be asymptotically flat (will be addressed in section 3.3.2). It corresponds to an unbounded domain. Hence no boundary conditions are required. Nevertheless since for a numerical study an infinite domain is troublesome one approach is to assume an artificial boundary at a finite distance[29] which requires to choose some boundary conditions there.

[29] An alternative is the use of conformally compactified techniques, see Frauendiener (2004) for a review. A recent approach was developed by Moncrief and Rinne (2009), numerically confirmed in Rinne (2010) and applied for example in Baake and Rinne (2016). See Reula and Sarbach (2011), Sarbach and Tiglio (2005, 2012) for reviews on boundary conditions in numerical relativity.

Remark 2.6.3. Already in section 2.2.1 we discussed several possible boundary conditions. Since we are mainly interested in the transition of a wave package through the origin we could for the numerical confirmation above use a homogeneous Dirichlet condition at the outer boundary, i.e. just set it to zero. This method would obviously fail if some information (the outgoing wave package) reached the boundary. Since we have the exact solution to the simulated problem we use an inhomogeneous Dirichlet condition at the outer boundary, namely we just take the value from the exact solution.

Since it is of advantage for Einstein's equations we discuss another method as well. The **Bjørhus projection method** was introduced in Bjørhus (1995), see in addition Sarbach and Tiglio (2012). Absorbing boundary conditions are examined in Sarbach (2007), Rinne et al. (2009).

The aim is to allow the information to travel out but there are no artificial or spurious reflections. Since the information travels, as discussed in section 2.3.2, on characteristics with the characteristic speed we need to transform to those directions and impose the boundary conditions there (and then transform back to obtain the boundary conditions for our fields).

In the current section we will demonstrate the technique for the wave equation for a scalar field ϕ. Both for the wave equation in spherical coordinates as above and for Einstein's equations we have to derive boundary conditions for large distances in the r-direction. The fall-off behavior in r is determined independent of the angle ϑ. Consider the contribution $\ell = 0$ in equation (2.160). We do not have to consider the angular part of the wave equation (2.160) and restrict to the equation

$$\partial_t^2 \phi = \partial_r^2 \phi + \frac{2}{r} \partial_r \phi. \tag{2.178}$$

We perform a first-order reduction with $\Pi := \partial_t \phi$ and $\xi := \partial_r \phi$ such that

$$\partial_t \Pi = \partial_r \xi + \frac{2}{r} \xi. \tag{2.179}$$

The time derivatives of the auxiliary variables read $\partial_t \xi = \partial_r \Pi$ and $\partial_t \Pi = \partial_r \xi + 2\xi/r$. Consider the eigenfunctions $v_\pm := \Pi \pm \xi$ with derivatives $\partial_t v_\pm = \pm \partial_r v_\pm + 2\xi/r$. We require that the scalar field is purely outgoing at the outer boundary, say at R (equations on the boundary are here denoted by $\hat{=}$). The general solution for $\ell = 0$ to the wave equation is given by $\phi(t,r) = r^{-1} f(t \pm r)$ (see the literature cited at the beginning of section 2.3). Therefore we allow at the boundary only $r\phi \mathrel{\hat{=}} f(t-r)$. That implies

$\partial_t(r\phi) + \partial_r(r\phi) \mathrel{\hat{=}} 0$. Taking the time derivative of that relation and reexpressing variables to v_+ gives the relation

$$\partial_t v_+ \mathrel{\hat{=}} -\frac{\pi}{r} \mathrel{\hat{=}} -\frac{\pi}{R}. \tag{2.180}$$

We keep v_- as it is but replace the derivative of v_+ at the boundary accordingly. Therefore we have the following relations at the outer boundary

$$\partial_t \varphi \mathrel{\hat{=}} \Pi, \tag{2.181a}$$

$$\partial_t \Pi \mathrel{\hat{=}} -\frac{1}{2}\frac{\Pi}{R} + \frac{\partial_r \xi}{2} + \frac{\xi}{R} - \frac{\partial_r \Pi}{2}, \tag{2.181b}$$

$$\partial_t \xi \mathrel{\hat{=}} -\frac{1}{2}\frac{\Pi}{R} - \frac{\partial_r \xi}{2} - \frac{\xi}{R} + \frac{\partial_r \Pi}{2}. \tag{2.181c}$$

We will use a first-order in time but second-order in space reduction later on (similar to the one discussed in the numerical verification of the wave equation), so the boundary conditions need to be chosen accordingly.

3. General relativity and its Cauchy formulation

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3.1. Introduction

The central aim of the thesis is a better understanding of certain aspects in general relativity. Therefore it is of some value to discuss its basics and fundamentals at least to a small extent. We give a very concise introduction into the main ingredients, in particular the Cauchy formulation. To obtain sets of equations that are accessible for the theory of partial differential equations as discussed in chapter 2 we need to perform so-called gauge choices. Several gauge conditions are also discussed. Even though we rely on well-known facts from the literature the material and sketched derivations are fundamental for the understanding of the thesis. There exists a large amount of literature dealing with several parts of the content of the chapter in much more detail. The goal is that the presentation helps the trained reader to get familiar with the used notation and to guide the unfamiliar reader through the literature to understand the scientific contribution in the remainder of the thesis.

A substantial part of the thesis deals with the analysis and implementation of the constraint solver. It is needed to obtain initial data for Einstein's equations. Therefore we discuss known results in some more detail and present some of our own calculations and attempts. Most of our calculations in that direction are postponed to the next chapter though. In the last part we provide a review on symmetry-reduced situations in general relativity with a focus on numerical simulations and critical phenomena. It is intended to guide the unfamiliar reader through the literature.

As a convention we label spacetime indices (running from 0 till 3) with Greek letters $\mu, \nu, \lambda, \ldots$ and spatial indices (running from 1 till 3) with Latin indices $i, j, k, \ldots$. Often the dimension as superscript for the metric tensor such as ${}^4g$, ${}^3\gamma$ is omitted. We do not distinguish carefully between the line element and the metric tensor but follow the usual convention to use these phrases interchangeably. We adopt "Einstein's summation convention"[1], saying that we imply summation over repeated covariant (subscript) and contravariant (superscript) index in one term, for example for a spatial two-tensor M

$$M_i{}^i \equiv \sum_{i=1}^{3} M_i{}^i. \tag{3.1}$$

3.2. General relativity

We already mentioned a large amount of literature[2] that is devoted to general as well as particular aspects of general relativity. We refer to these sources for more details and restrict ourselves to absolutely fundamental issues.

[1] It was introduced by Einstein himself as simplification in Einstein (1916, equation (7)), see also Gutfreund and Renn (2015, page 59 and 97).

[2] Let us in the footnote give some standard and helpful references concerning the theory of general relativity. They include Weinberg (1972), Hawking and Ellis (1973), Misner et al. (1973), Schutz (1980), Wald (1984), Stewart (1991), Choquet-Bruhat (2009, 2015). From the mathematical perspective we recommend in particular Nakahara (2003), O'Neill (1983). We mention also the more historic accounts Weyl (1952), Pauli (1981), Bergmann (1976), Schrödinger (1950) which are still good introductions and interesting from the historic perspective and in this respect Schouten (1954) for the corresponding calculus.

3.2.1. Spacetime

Definition 3.2.1. A **spacetime** is a 4-dimensional time-orientable (usually even globally hyperbolic) Lorentzian manifold (index $(3, 1)$, see appendix A.1.1) with torsion-free connection that is compatible with the metric.

Remarks 3.2.1.

- The definite choice of dimension four is not necessary mathematically but physically motivated. Considerations with more than four dimensions are not unpopular in several fields of research. Relativistic toy theories with less dimensions are far easier to understand but seem to be ruled out empirically.

- Historically the "causal requirement" of time-orientability was not included in the definition of a spacetime. We do it here because time-orientability is necessary for the Cauchy formulation of general relativity in section 3.3, see O'Neill (1983) or literature cited later. Often global hyperbolicity (introduced generally by Leray (1952) and discussed in general relativity in Geroch (1971)), so the existence of a Cauchy hypersurface, is assumed additionally. For a discussion of different causal concepts and the "causal ladder" see Minguzzi and Sánchez (2006).

- A **Lorentzian manifold** is a **pseudo-Riemannian manifold**. Therefore we have some necessary structure (topology, differentiability, metric, etc.) in our space. We try to illustrate the path from a topological space to the concept of a spacetime by adding several levels of structure in figure 3.1.

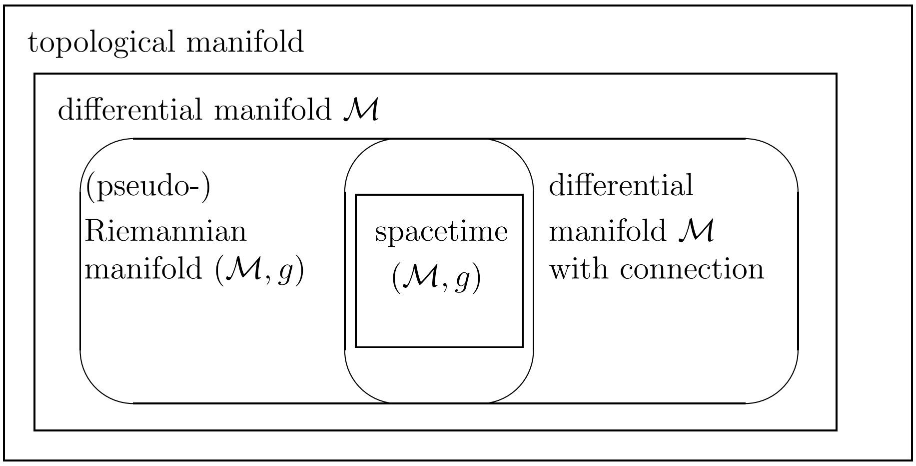

Figure 3.1.: An illustration how structure is added to a very general manifold and it becomes a spacetime eventually.

- The requirements on the connection restrict it uniquely to the **Levi-Civita connection**, so in a coordinate representation to the Christoffel symbols (see appendices A.1.1 and A.1.2). For extended theories in that respect, in particular theories with torsion see the review Hehl et al. (1995).
- Non-Eulcidean geometry has a long and interesting history, nicely recounted for example in Bonola (2007). Its systematic development in the nineteenth century culminated in the seminal work by Bernhard Riemann (1868). The foundation for the tensor calculus (necessary structure for a theory like general relativity) was only developed at the turn of the twentieth century. When Einstein worked on the theory of general relativity the calculus was not yet on a solid basis. The fact that the underlying mathematical theory was in its infancy in those days should be seen as one of the major difficulties of finding the correct field equations. It was a lucky coincidence that Einstein had several competent colleagues and supporters assisting him in that respect, see also the remarks in section 3.2.2. Reich (1994) recounts the history of the development of tensor calculus.

We cite an interesting result that is of some importance, in particular in respect to our later gauge choice, see section 4.3.

Proposition 3.2.1. A globally hyperbolic spacetime admits a "Cauchy orthonormal splitting", that implies that the spacetime $(\mathcal{M},{}^4 g)$ is representable in a product structure

$$\mathcal{M} = \mathbb{R} \times \Sigma, \ \ g = -\alpha^2 \mathrm{d}t^2 + \gamma \tag{3.2}$$

with some function α and spatial Cauchy hypersurface $(\Sigma,{}^3 \gamma)$.

Proof. For a more precise statement and the proof see Minguzzi and Sánchez (2006), Müller and Sánchez (2011). □

3.2.2. Einstein's field equations

So far we have introduced the notion of a spacetime. Essential for the theory of general relativity are in addition the field equations.

Definition 3.2.2. Einstein's field equations are

$$\mathrm{Ric} - \frac{1}{2} g \mathrm{R} = 8\pi \mathrm{T} \tag{3.3}$$

With **stress-energy tensor** T and components $T_{\mu\nu}$ (symmetric two-tensor describing all non-gravitational fields). In coordinates the components of the field equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{3.4}$$

Some of the quantities in the expressions above are defined in appendix A.1.1.

The **theory of general relativity** is Einstein's equations on a spacetime. The physical field is the metric tensor, which determines the spacetime structure. The motion of test particles follows **geodesics** (extremal curves in spacetime).

The field equations are **geometric** partial differential equations. A **solution** of Einstein's equations is given by an equivalence class $[g]$ (equivalent up to diffeomorphism). Often a representative of the class, i.e. the metric tensor $g_{\mu\nu}$, is considered to be the solution (this not completely correct usage will be adopted in the remainder).

Remarks 3.2.2.
- Instead of postulating the field equations there are several ways to motivate or derive Einstein's equations:
  - They can be derived with an **action principle** (see textbooks at the beginning of the chapter), which is very appealing for physicists.
  - They can be motivated slightly heuristically with the use of the **equivalence principle**. That follows more the historic line and is also applied in many of the cited textbooks.
  - David Lovelock (1969, 1971) showed (generalizing ideas in Weyl (1952)) that Einstein's equations are the unique field equations under mild conditions (second order equations in four spacetime dimensions that are covariantly constant[3], assumptions that can all be easily motivated) when a cosmological constant[4] is added.
- Since the field equations in definition 3.2.2 are so central for the topic of the thesis we give a few comments on its history:

[3] In that ansatz the covariant constancy is postulated (and motivated by the fact that the energy-momentum tensor is covariantly constant and therefore it should also hold for the left-hand side). If the field equations are derived or postulated in a different way one can easily show that $\nabla_\mu G^{\nu\mu} = 0$ holds. This identity is referred to as **Bianchi identity**.

[4] We assume the cosmological constant to vanish, which is also a plausible assumption in our setting.

- Albert Einstein published the field equations[5] (3.4) in a small note Einstein (1915) in November 1915. From the historic perspective very important is the highly extended version Einstein (1916) though (see also the translations Lorentz et al. (1952), Gutfreund and Renn (2015)[6]).

- It took Einstein roughly a decade (it depends where one puts the start of the search) to find the correct field equations. The development is sketched in Norton (1989), Gutfreund and Renn (2015). After the establishment of the special theory of relativity in 1905 the next mentionable progress came in 1907 while Einstein prepared a review article about his 1905 theory. He realized a central idea of the general theory referred to as equivalence principle. In the following years till 1912, in particular during his appointment in Prague (his time in Prague is reviewed in Bičák (2014)), Einstein worked out the essential ingredients for the foundation of general relativity. In particular the necessity of the covariant formulation and the dual role that is played by the ten components of the metric tensor (its components describe both the spacetime geometry and are the gravitational potential). In the following years Einstein transformed these insights into the final theory. Important steps include the "Zurich notebook" (where he learns the basics of tensor calculus, see Straumann (2011)) and the "Entwurf theory" together with Grossmann (for a translation see Klein et al. (1995, page 151–188)) which was continued by Einstein to the "formal foundations of general relativity" (see Einstein (1914)) where the correct vacuum equations are already included.

- Einstein finalized the search for the field equations in November 1915. Exactly at the same time David Hilbert found the field equations from an action principle as well, see Hilbert (1916). It is still under debate how they influenced each other (the fact that they did is out of question) and who should be given credit for finding the equations first. While Mehra (1974), Corry et al. (1997) seem to favor Einstein,

[5] In Einstein (1915) the field equations have a slightly different but equivalent form as in equation (3.4) (essentially the "trace-term" is on the right-hand side with the matter contribution). Only in 1918 Einstein adopted the more common form (3.4), see Gutfreund and Renn (2015, page 103).

[6] Observe that in the translation Lorentz et al. (1952), which has some importance because a previous edition was already published in 1923, the abstract is missing. There Einstein refers to other scientific work for the mathematical foundation. In the annotated translation Gutfreund and Renn (2015) the authors include the abstract and give background information.

Wuensch (2005) exonerate Hilbert from copying essential ideas from Einstein. It is not clear if the questions will be resolved by historians of science but the debate forms an interesting story.

  – The research in general relativity in the following years has an interesting history. The theory was almost immediately confirmed but it was not a mainstream field of science for a long time. Only late after the second world war a community formed that more or less systematically examined the theory further. Eisenstaedt (2006, 1989), Blum et al. (2015) tell some of the history.

- The theory seems to be an excellent model of reality describing gravitational phenomena and passed numerous verifications experimentally and observationally (see Ashtekar et al. (2015) for relatively recent surveys). Also the long desired direct detection of gravitational waves and the existence of black holes is claimed to be given and in excellent agreement with predictions of general relativity, see Abbott et al. (2016) which belongs to the recent highlights in the whole field of science.
- Even though there are hardly any violations of general relativity known we should mention that as such general relativity does not seem to be in agreement with the theory of quantum mechanics (which also seems to be necessary for a description of reality). This can be seen in the different roles of time in both theories for instance. The theory of **quantum gravity** is concerned with the marriage of both fields but is not the topic of this thesis, see Kiefer (2012) for a review of ideas in that field.

3.2.3. Comments on perturbation theory and linearized gravity

Quite often one is interested in perturbations about a given solution. If everything in the theory behaves sufficiently smoothly, perturbatively derived results deliver already some insights about the behavior of the solution. Consult Breuer (1975), Chandrasekhar (1983) for elaborated introductions into the topic and results for general relativity. Here we will focus on some technical aspects that we will apply in chapter 4.

Definition 3.2.3. A **perturbation** of a solution of Einstein's equations is given as

$$g_{\mu\nu} \mapsto g_{\mu\nu} + \epsilon \tilde{g}_{\mu\nu} \tag{3.5}$$

where $\tilde{g}_{\mu\nu}$ is the perturbation and we explicitly introduced the **strength** or **amplitude** ϵ. In this way we constructed a **one-parameter family** of solutions.

Given the perturbed solution one can insert it and derive the behavior of all further geometric quantities, see standard textbooks at the beginning of the section. In general we get expressions with various powers of the amplitude ϵ. Restricting to terms of the order ϵ^0 gives back the original unperturbed results. Just keeping terms up to the linear order ϵ^1 gives results of the linearized theory. Quite a lot of the difficulties, especially for the implementation, are already present on the linear level. Therefore it is very beneficial to consider and solve these conceptual problems already on the linear level. We will follow that strategy in the remaining chapters of the thesis.

The major part of the thesis deals with perturbations about flat spacetime. That means that all quantities are given in the form

$$u = u_{\text{flat}} + \epsilon \tilde{u} \tag{3.6}$$

with (possibly nonlinear) perturbation $\tilde{u}$.

When deriving the equations one deals also with functions of these quantities, say $f(u)$. As a general rule for the perturbation of f one uses for a small parameter ϵ

$$f(u) = f(u_{\text{flat}}) + \epsilon \frac{\partial f}{\partial u}\bigg|_{u_{\text{flat}}} \tilde{u}. \tag{3.7}$$

For example for $u_{\text{flat}} \in \{0, 1\}$ the inverse quantity u^{-1} is perturbed as

$$u^{-1} = u_{\text{flat}} - \epsilon \tilde{u}. \tag{3.8}$$

Remark 3.2.1. In the implementation it makes sense to store just the perturbations of flat spacetime, the part $\epsilon\tilde{u}$. The flat contribution (which will be either 1 or 0, see section 4.4) will be added when evaluating the equations.

3.3. Cauchy formulation of general relativity

General relativity as introduced in section 3.2 is a geometric theory of spacetime with field equations determining the geometry itself. The equations are an example of a set of geometric partial differential equations (see definition 2.2.2). There is no preferred concept of “space” or “time” any more, only some notion of “spacelike” and “timelike” remains. On the other hand our intuition in physics teaches that we have some instance where we prescribe some initial data. The data are integrated in some direction of time to obtain the whole physical spacetime picture. Seen from the point of view of partial differential equations, in particular from the computational perspective, we seek for some definite, non-geometric formulation in equations that can be (numerically) integrated in "time" starting from an "initial surface" ("space").

The present section sketches the derivation of the formulation. Spacetime is decomposed into "space" and "time" and Einstein's equations are written as a first-order in time set of differential equations. Not for all geometric settings of spacetimes is the foliation possible and hence meaningful. One basically has to require that it is indeed possible. It means that there should exist a continuous timelike vector field along which spacetime is decomposable into a level-set of non-intersecting spacelike hypersurfaces. This is the case if the spacetime is time-orientable. Then one chooses one orientation which assures that the timelike vector field exists. For every globally hyperbolic spacetime the Cauchy formulation can be obtained, see Minguzzi and Sánchez (2006), Müller and Sánchez (2011). Usually the "zeroth" coordinate x^0 acts as timelike coordinate t.

In a series of publications reviewed in Arnowitt et al. (1962) the decomposition of the field variables was established for the canonical formulation of general relativity. We follow here more the review York (1979). Further consult Gourgoulhon (2012), Alcubierre (2008), Baumgarte and Shapiro (2010), Poisson (2004), Rezzolla and Zanotti (2013), Shibata (2016), Bona et al. (2009) for more material on this section. From the mathematical perspective see Friedrich and Rendall (2000), Ringström (2009, 2015) for the Cauchy formulation of general relativity, its origin and achievements. The (early) history of the Cauchy problem is reviewed in Stachel (1992), Choquet-Bruhat (2014).

While the Cauchy formulation is the standard framework for numerical relativity we should mention that there are alternatives. For example consider the characteristic formulation in Stewart and Friedrich (1982), Winicour (1998).

Proposition 3.3.1. Four of the ten Einstein's equations are **constraints**, that means that those four equations do not contain highest (here second) time derivatives.

Proof. Let the time coordinate be $x^0 \equiv t$. With the Bianchi identity we have

$$0 = \nabla_\mu G^{\nu\mu} = \partial_\mu G^{\nu\mu} + \Gamma^{\nu}{}_{\lambda\mu} G^{\lambda\mu} + \Gamma^{\mu}{}_{\lambda\mu} G^{\nu\lambda}. \tag{3.9}$$

The first term on the right-hand side of equation (3.9) can be separated into

$$\partial_\mu G^{\nu\mu} = \partial_0 G^{\nu 0} + \partial_i G^{\nu i} \tag{3.10}$$

where we recall that the Latin indices are spatial only. Therefore

$$\partial_t G^{\nu 0} = -\partial_i G^{\nu i} - \Gamma^{\nu}{}_{\lambda\mu} G^{\lambda\mu} - \Gamma^{\mu}{}_{\lambda\mu} G^{\nu\lambda}. \tag{3.11}$$

Since ∂_i denotes a spatial derivative and $G^{\mu\nu}$ only contains second derivatives we see that the right-hand side of equation (3.11) contains at most second derivatives in t, hence also the left-hand side. Therefore $G^{\nu 0}$ cannot have second time derivatives. Hence the relations given in the zeroth row/column in the matrix representation are constraints. □

3.3.1. Foliation of spacetime and decomposition of Einstein's equations

In this section we are very brief and only sketch the derivations, the unfamiliar reader is referred to the given literature at the beginning of the section.

- Let us label the zeroth coordinate x^0 as time t. Its normal n^μ is orthogonal to $t =$ constant hypersurfaces. Essentially it describes our time evolution. The **lapse function** $\alpha := -||\nabla_\mu t||$ encodes the freedom in the choice of the time parameter.
- n^μ allows to define the completely **spatial 3-metric**[7] (or **first fundamental form**)

  $$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu \tag{3.12}$$

  where g is the spacetime metric. It is used for the projection of spacetime tensor field into the spatial hypersurface.
- The purely spatial **extrinsic curvature** (or **second fundamental form**) is a symmetric tensor

  $$K_{\mu\nu} := -\gamma_\mu{}^\lambda \gamma_\nu{}^\kappa \nabla_\lambda n_\kappa = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}. \tag{3.13}$$

  It is essentially the "time derivative" of the spatial metric. The **mean curvature** K is the trace of the extrinsic curvature,

  $$\mathrm{tr}K := \mathrm{tr}(K_{\mu\nu}) = \mathrm{tr}(K_{ij}) = \gamma_{ij} K_i{}^j = -\mathcal{L}_n \ln\sqrt{\det\gamma}. \tag{3.14}$$

[7] It is by definition an object in the spacetime. Its contraction with the normal vector field n^μ vanishes. Therefore it is a completely spatial tensor field in the three-dimensional Riemannian manifold which justifies to switch from spacetime indices $\mu, \nu \ldots$ to spatial ones $i, j, \ldots$ and explains the name. This transition from Lorentzian (spacetime) indices to Riemannian (spatial) ones will also be applied in the following.

Therefore (and since $\sqrt{\gamma}\mathrm{d}^3x$ is the volume element on the Riemannian hypersurface) tr$K$ is a measure of the change of proper volume along the unit normal

- With the equations by Gauss, Codazzi, Mainardi and Ricci we project the spacetime quantities into spatial ones and along n^μ. We denote the connection compatible with γ_{ij} with D_i. The constraints (see proposition 3.3.1), given by the $G_{0\mu}$-components of Einstein's equations are the so-called

$$\textbf{Hamiltonian constraint } R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 16\pi\rho \text{ and the} \tag{3.15}$$

$$\textbf{momentum constraint } D_j\left(K^{ij} - \gamma^{ij}\mathrm{tr}K\right) = 8\pi S_i \tag{3.16}$$

with matter sources defined as the corresponding projections of the stress-energy tensor as measured by a normal observer, $\rho := n_\mu n_\nu T^{\mu\nu}$ and $S_\mu := -\gamma_\mu{}^\nu n^\lambda T_{\nu\lambda}$ (also $S_{\mu\nu} := \gamma_\mu{}^\lambda \gamma_\nu{}^\rho T_{\lambda\rho}$). Note that we used purely spatial indices because the projection of the equations along the timelike vector field vanishes. The evolution equations along the general timelike vector field n^μ are given as (here R_{ij} is the spatial Ricci tensor)

$$\mathcal{L}_n\gamma_{ij} = -2K_{ij}, \tag{3.17a}$$

$$\begin{aligned}\mathcal{L}_n K_{ij} = & -\frac{1}{\alpha}D_iD_j\alpha + R_{ij} - 2K_{ik}K_j{}^k + \mathrm{tr}K K_{ij} \\ & - 8\pi\left[S_{ij} - \frac{1}{2}\gamma_{ij}(\mathrm{tr}S - \rho)\right]\end{aligned} \tag{3.17b}$$

with matter sources given above and $\mathrm{tr}S := \mathrm{tr}S_{\mu\nu} = S_\mu{}^\mu$.

- Choosing a coordinate basis there is some further freedom in the choice of the timelike vector field (as can be seen as $t^\mu\nabla_\mu t = 1$),

$$t^\mu = \alpha n^\mu + \beta^\mu \tag{3.18}$$

where β^μ is an arbitrary purely spatial vector field (hence in the following with spatial index β^i) called the **shift vector**. The line element in the chosen coordinates reads

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\mathrm{d}t\right) \tag{3.19}$$

and the evolution equations

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \frac{1}{2}D_{(i}\beta_{j)}, \tag{3.20a}$$

$$\partial_t K_{ij} = \alpha\left(R_{ij} - 2K_{ik}K_j{}^k + \mathrm{tr}K K_{ij}\right) - D_iD_j\alpha$$

$$\begin{aligned}
&- 8\pi\alpha \left[S_{ij} - \frac{1}{2}\gamma_{ij}(\mathrm{tr}S - \rho) \right] \\
&+ \beta^k \partial_k K_{ij} + K_{ik}\partial_j \beta_k + K_{kj}\partial_i \beta_k, \qquad (3.20b)\\
\partial_t K &= \gamma^{ij} D_i D_j \alpha + \alpha \left[K_{ij}K^{ij} + 4\pi(\mathrm{tr}S + \rho) \right] + \beta^i D_i \mathrm{tr}K, \qquad (3.20c)\\
\partial_t \ln \sqrt{\det \gamma} &= - \alpha \mathrm{tr}K + D_i \beta^i. \qquad (3.20d)
\end{aligned}$$

Note that the last two evolution equations are not independent of the former ones.

Therefore we can summarize Einstein's equations in the Cauchy formulation as the Hamiltonian (3.15) and momentum constraint (3.16) together with the evolution equations (3.20).

3.3.2. Comments on asymptotic flatness

For a large class of objects in general relativity (not for cosmological spacetimes though) the concept of **isolated objects** is important. Mathematically formulated the concept implies some statement about the asymptotic structure, namely that sufficiently far away (there exists the concept in either the null or the spacelike direction) from the system of interest the space approaches the flat one. "Flat" corresponds to Minkowskian spacetime for a Lorentzian manifold and the Euclidean space for a Riemannian manifold. For the notion of "approaches" one defines some fall-off rate of the metric towards the flat one, usually it is required to behave in Cartesian coordinates for large distance r like

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) + \mathcal{O}(r^{-1}). \qquad (3.21)$$

There exist different definitions of asymptotically flat, usually not equivalent to each other.

For the 3+1-foliated spacetime picture we follow the coordinate approach and prescribe for our variables some specific fall-off condition, see Jaramillo and Gourgoulhon (2011), Gourgoulhon (2012).

Definition 3.3.1. Given a globally hyperbolic spacetime $(\mathcal{M}, g)$ foliated by a family of spacelike hypersurfaces $(\Sigma_t, \gamma_t, K_t)$, $t \in \mathbb{R}$. It is **asymptotically flat** if $\forall$ spatial slices Σ_t $\exists$ a Riemannian background metric f such that

- f is Riemann-flat ($\mathrm{Riem}_f = 0$) on $\Sigma_t \setminus \mathcal{B}$ where $\mathcal{B}$ is a compact ball, the **strong field region**,

- on $\Sigma_t \setminus \mathcal{B}$ $\exists$ a Cartesian-like coordinate system $(x^i) = (x, y, z)$ such that $f_{ij} = \mathrm{diag}(1, 1, 1)$ and the radial distance $r := \sqrt{x^2 + y^2 + z^2}$ can grow arbitrary large, the region $r \to \infty$ is called **spatial infinity**[8] i^0,
- at spatial infinity the variables show the following fall-off behavior,

$$\gamma_{ij} = f_{ij} + \mathcal{O}(r^{-1}), \qquad \gamma_{ij,k} = \mathcal{O}(r^{-2}), \tag{3.22}$$
$$K_{ij} = \mathcal{O}(r^{-2}), \qquad K_{ij,k} = \mathcal{O}(r^{-3}), \tag{3.23}$$
$$\alpha = 1 + \mathcal{O}(r^{-1}), \qquad \partial_i \alpha = \mathcal{O}(r^{-2}), \tag{3.24}$$
$$\beta_i = \mathcal{O}(r^{-1}), \qquad \partial_i \beta_j = \mathcal{O}(r^{-2}). \tag{3.25}$$

Remarks 3.3.1.

- The asymptotic behavior of the gauge quantities α and β_i is not always included in the definition but included here to accomodate the requirement (3.21), see also Beig and Ó Murchadha (1987).
- The fall-off behavior of the derivatives makes sure that there are no gravitational waves at spatial infinity i^0.
- The discussed structures are related to the (problem of the) definition of energy, mass and angular momentum in general relativity, see the review Szabados (2004).

3.3.3. Gauge conditions

In the derivation of the decomposed Einstein's equations, the constraints in equations (3.15) and (3.16) and the evolution equations in equation (3.20), we encoded the coordinate freedom in the liberty to choose the "gauge functions", the lapse α and the components of the shift vector β^i.

The constraints are independent of the gauge parameters but to evolve from one spacelike slice to the next we have to impose coordinate conditions. One choice for the time coordinate is free, the "slicing" condition, and three choices of the spatial coordinates, the spatial gauges. Numerous options are available.

Algebraic choices for some components are possible as well as more geometric alternatives leading to partial differential equations. In the following only a small selection is picked emphasizing different characters of the gauge choices. For more choices consult the literature given at the beginning of the section.

[8] There are other concepts of infinity in general relativity as well, the infinite future (or past) in the light-like direction is denoted as **future (or past) null infinity** $\mathscr{I}^{\pm}$ and the infinite future (or past) in the time-like direction is denoted as **future (or past) timelike infinity** $i^{\pm}$.

We should mention that the term "gauge freedom" might be misleading. It encodes the coordinate independence of general relativity. General relativity can be formulated as a gauge theory though, see for example Göckeler and Schücker (1987).

Slicing conditions

Definition 3.3.2. Setting the proper time of a normal observer equal to the coordinate time,

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \alpha = 1 \tag{3.26}$$

is called **geodesic slicing**.

Often geodesic slicing is applied together with the geodesic gauge condition, see definition 3.3.4. The observer is freely falling. This means that she follows timelike geodesics. In the presence of gravity the geodesics tend to focus, which lets them run into a singularity. The coordinates are tied to the geodesics and form a coordinate singularity. At the singularity the equations break down. That makes this gauge unpopular at least for strong gravitational fields.

Definition 3.3.3. Requiring that the trace of the extrinsic curvature, the mean curvature $\mathrm{tr}K$, remains constant,

$$\mathrm{tr}K = K_i{}^j = \mathrm{const}, \;\; \partial_t \mathrm{tr}K = 0, \tag{3.27}$$

is called **constant mean curvature** slicing. The particular choice of vanishing constant,

$$\mathrm{tr}K = 0 = \partial_t \mathrm{tr}K, \tag{3.28}$$

is called **maximal slicing**.

Proposition 3.3.2. Constant mean curvature slicing leads to an elliptic equation for the lapse α.

Proof. We use the vacuum evolution equation in (3.20) for the mean curvature

$$\begin{aligned} \partial_t \mathrm{tr}K = -D^2\alpha + \alpha K_{ij}K^{ij} + \beta^i D_i \mathrm{tr}K \overset{!}{=} 0 \\ \Leftrightarrow D^2\alpha = \alpha K_{ij}K^{ij} + \beta^i D_i \mathrm{tr}K. \end{aligned} \tag{3.29}$$

This is a scalar Poisson equation and hence elliptic, see example 2.3.1. Inclusion of matter results in an additional source term. □

Remarks 3.3.2.
- In the case of maximal slicing equation (3.29) reduces to $D^2\alpha = \alpha K_{ij}K^{ij} = \alpha R$ where the vacuum Hamiltonian constraint (3.15) with maximal slicing is used in the last step.
- The elliptic equation for the lapse is of second order and needs boundary conditions.
- The name is due to the fact that the slicing maximizing the volume (it was called "minimal" in Lichnerowicz (1944) though).
- There is also the possibility to introduce an approximate maximal slicing in the sense that $\mathrm{tr}K$ might be non-vanishing but is "driven back" to zero by requiring

$$\partial_t \mathrm{tr}K = -c\,\mathrm{tr}K, \;\; c \in \mathbb{R}_0. \tag{3.30}$$

  Together with the evolution equation for $\mathrm{tr}K$ it leads to an "evolutionary" equation for α, actually an parabolic equation. This gauge is called **K driver condition**, see Balakrishna et al. (1996).

Spatial gauge

Definition 3.3.4. The requirement of vanishing shift is called **geodesic gauge** (or canonical gauge[9]). Together with geodesic slicing the resulting coordinates are called **Gaussian normal coordinates**.

Remarks 3.3.3.
- This condition implies that the coordinates stay at rest with respect to the normal observer.
- The results concerning the Cauchy orthogonal splitting in proposition 3.2.1 should be taken into account for geodesic slicing.
- It is possible to require elliptic gauge conditions for the shift vector (see also section 3.4.1). The **minimal strain** (basically the time-derivative of the spatial metric) or **minimal distortion** (trace-free part of the strain) are prominent examples. Again they can be cast into an evolutionary scheme as well (**Gamma driver condition**, with a similar approximate requirement and an "artificial" evolutionary equation for the shift β^i, see Alcubierre and Brügmann (2001).

Some gauges which are more relevant in symmetry-reduced situation, in particular in axisymmetry, are discussed in section 4.3.

[9] It was called "canonical form of the space-time metric" in Christodoulou and Klainerman (1993, page 5).

3.4. Initial data for Einstein's constraint equations

We saw in section 3.3 that Einstein's equations can be split into two fundamentally different sets of equations, the (spacetime) evolution equations and the constraint equations (no time derivatives). In the present section we concentrate on the system of constraint equations itself on the Riemannian Cauchy hypersurface. We restrict our considerations to vacuum spacetimes, so we set the matter terms in equations (3.15) and (3.16) to zero. In principle the inclusion of matter is a straightforward exercise and just leads to additional source terms in the equations. See Isenberg (2014) for a review.

Proposition 3.4.1 (Vacuum constraints)**.** Every spacelike hypersurface in the 3+1-vacuum decomposition has to satisfy the Hamiltonian and momentum constraint,

$$\mathcal{H} := R + K^2 - K_{ij}K^{ij} = 0, \tag{3.31a}$$
$$\mathcal{C}^i := D_j\left(K^{ij} - \gamma^{ij}K\right) = 0. \tag{3.31b}$$

Remarks 3.4.1.

- The gauge parameters α and β do not appear in the constraints. This is clear by construction (they describe the lapse between proper and coordinate time and the spatial shift between different hyperspaces, but are not responsible to determine the geometry of the spatial slices) and can also be seen in the decomposed Lagrangian formulation of the theory where the quantities appear as Lagrange multipliers, see Arnowitt et al. (1962).

  The disappearance of the gauge parameters is not true for all formulations. For example they appear in the constraints in the conformal thin-sandwich formulation (see York (1999)).

- The Hamiltonian constraint (3.31a) contains the components of the extrinsic curvature only algebraically (without derivatives), but non-linearly $\Rightarrow$ in some sense naturally the Hamiltonian is an equation for the components of the spatial metric (those components come with second spatial derivatives due to the spatial Ricci scalar R).

- The momentum constraint (3.31b) is essentially the divergence of the extrinsic curvature, hence a first-order equation in the components of K_{ij}.

- Both constraints in theorem 3.4.1 as such (without the remaining evolution equations) form a (coupled) system of four equations involving 12 variables (the components of γ_{ij} and K_{ij}). In that respect it is a highly **underdetermined** system. The discussions in section 2.3.2 indicate that the character of the system of partial differential equations is not automatically fixed but subject to several possible interpretations. An essential question is which data should be prescribed (for the constraint solver) and which data should be determined.

Even though the constraints are given in a geometric fashion and the type of the equations is not fixed, they are quite often arranged and considered as an elliptic system which we will discuss next. We used an elliptic formulation of the constraints for some time and gained some experience in the numerical implementation of the solver. Therefore we developed the techniques for our situation which are discussed in the following.

3.4.1. Constraints as elliptic system

There are many ways to obtain elliptic equations from the constraints, see in particular section 4.7.2 where the possibility of elliptic constraints is discussed for the formulation used in this thesis. For convenience we discuss in the current section one particular way which deals as a prototype for the very famous "elliptic method". This method has a long history in the discussion of the Cauchy problem in general relativity. It was initiated by the French school with the Hamiltonian constraint as an elliptic equation and was later extended to the momentum constraint which culminates in the "Bowen-York" method, see Bowen and York (1980). Good sources for the following section include Bartnik and Isenberg (2002) for a mathematical treatment and Cook (2000), Alcubierre (2008), Baumgarte and Shapiro (2010) for a more numerical perspective. We also discuss the method formulated in spherical coordinates, see as possible sources papers by the group in Meudon, for example Bonazzola et al. (1999, 2004).

A disadvantage of the elliptic method is that there are not many results for spacetimes without constant mean curvature slicing. This is problematic for instance for the Kerr spacetime, see Garat and Price (2000). In Dain and Friedrich (2001) the existence of solutions to the constraint equations is shown for a large class of spacetimes assuming maximal slicing. See for example Andersson and Moncrief (2003) for a mathematical analysis of a well-posed elliptic-hyperbolic formulation in general relativity.

Hamiltonian constraint

The systematic study of the Hamiltonian constraint was initiated by the French school in Paris. Charles Racine considered in his thesis Racine (1934) the constraints in maximal slicing and with a conformally related metric. His choice was an Euclidean reference metric. That result was later generalized by André Lichnerowicz in Lichnerowicz (1944) to a Riemannian reference metric and the resulting equation is therefore often called "Lichnerowicz equation".

The Hamiltonian constraint is essentially a differential equation for **one** component of the spatial metric, the remaining components (as well as the extrinsic curvature if the Hamiltonian constraint is considered as a single equation) have to be prescribed. It is not immediately obvious which component is the desired one to be solved for. We will discuss several choices in section 4.7 and see that the type of the equation is highly dependent on that choice.

Let us consider in general the possibility to prescribe a reference metric $\bar{\gamma}_{ij}$ which is **conformally related** to the physical metric γ_{ij} as

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \Leftrightarrow \bar{\gamma}_{ij} = \psi^{-4} \gamma_{ij} \tag{3.32}$$

with **conformal factor** ψ which is an arbitrary non-vanishing function of the coordinates. Usually the conformal reference metric is a well-known metric adapted to the physical situation. The conformal method itself is a famous technique in general relativity for dealing with the whole set of Einstein's equations, see for example Frauendiener (2004) or the recent textbook Valiente Kroon (2016) for reviews.

Proposition 3.4.2. The vacuum Hamiltonian constraint becomes a non-linear (but semilinear) **elliptic** equation for the conformal factor ψ,

$$8\bar{D}^2\psi - \psi\bar{R} + \psi^5 \left(K_{ij}K^{ij} - \mathrm{tr}K^2\right) = 0. \tag{3.33}$$

Proof. The relation (3.32) allows us to calculate the metric connection $\bar{\Gamma}^i{}_{jk}$ of $\bar{\gamma}_{ij}$ (hence the covariant derivative $\bar{D}$ is known), the Riemann tensor $\bar{R}^i{}_{jkl}$ and the Ricci quantities $\bar{R}_{ij}$ and $\bar{R}$ with respect to the conformal metric $\bar{\gamma}$ and to express the physical unbarred quantities in terms of the barred conformal ones. The principal part of the equation is given by the conformal Laplacian $\bar{D}^2$ and thus the ellipticity follows with example 2.3.1. □

Momentum constraint

Early studies of the system of constraints where usually restricted to maximal slicing. It was in the early 1970s that James York (1974) realized that **constant mean curvature** slicing results in a linear momentum constraint $D_j K^{ij} = J^i$ with source $J^i = 0$ in vacuum leading to a semilinear elliptic system for the constraints. Together with Jeffrey Bowen he formulated the so-called Bowen-York method Bowen and York (1980) which will be discussed in the following. It is based on the tensor decomposition as derived by York (1974) which is based itself on studies concerning the spacetime decomposition by Deser (1967). Besides being successful for numerical implementations and well-understood mathematically one disadvantage of the method is that only very limited results exist for a slicing beyond constant mean curvature.

We consider the momentum constraint (3.31b) as an equation for three components of the extrinsic curvature. Usually one considers again a conformally related tensor. The discussion is basically analogous to the one presented here and can be followed in the cited literature.

Proposition 3.4.3. The vacuum momentum constraint can be cast into an elliptic system of equations of second order.

Proof. We present the proof in a constructive way. Introduce the trace-free part of the extrinsic curvature

$$A^{ij} := K^{ij} - \frac{1}{3}\gamma^{ij}\mathrm{tr}K. \tag{3.34}$$

Quite often the "elliptic method" comes with maximal slicing as a gauge condition. In that case A^{ij} is identical to K^{ij}. Following York (1974) we split the symmetric, trace-free tensor into its transverse-traceless and a longitudinal part[10] as

$$A^{ij} = A^{ij}_{\mathrm{TT}} + A^{ij}_{\mathrm{L}}. \tag{3.36}$$

[10] A tensor $A^{\mu\nu}$ is called **transverse** if it is purely spatial ($A^{\mu 0} = 0$) and the spatial components are divergence-free in space ($\partial_i A^{ij}$). The conditions imply that A^{ij} is also transverse to its direction of propagation (see for example Misner et al. (1973)). The part of the tensor after subtracting its transverse contribution and the spatial trace is called **longitudinal** tensor $A^{\mu\nu}_{\mathrm{L}}$. Therefore any tensor can be decomposed into transverse-traceless and longitudinal part and its spatial trace $\mathrm{tr}A = A_i{}^i$,

$$A^{\mu\nu} = A^{ij}_{\mathrm{TT}} + A^{\mu\nu}_{\mathrm{L}} + \mathrm{tr}A. \tag{3.35}$$

Often the TT-part of the tensor is prescribed and we have to solve for A_{L}^{ij}. To do so we introduce a **vector potential** W^i such that A_{L}^{ij} can be written as its **vector gradient** (or **longitudinal operator**)

$$A_{\mathrm{L}}^{ij} = (\mathbb{L}W)^{ij} := \frac{1}{2} D^{(i} W^{j)} - \frac{2}{3} \gamma^{ij} D_k W^k . \tag{3.37}$$

Inserting into the vacuum momentum constraint (3.31b) leads to

$$D_j K^{ij} = D^2 W^i + \frac{1}{3} D^i D_j W^j + \gamma^{ik} R_{kj} W^j = 0 \tag{3.38}$$

where D^2 is the Laplace operator $D^2 = D_i D^i$ and we used the relation for the Ricci tensor. This leads to a vector Poisson equation (therefore the ellipticity is shown) of the form

$$D^2 W^i + \frac{1}{3} D^i D_j W^j = S^i := D_j K^{ij} - \gamma^{ik} R_{kj} W^j \tag{3.39}$$

The source term S^i in the Poisson equation explicitly depends on the solution W^i. Therefore computationally an iterative solver should be applied. The final equations (3.39) form a **coupled** set of partial differential equations for the components of W^i, even in Cartesian coordinates. □

With the introduction of the vector potential we managed to write the momentum constraint in a manifestly elliptic form. To solve it numerically one would benefit from writing it in a decoupled form. The price one has to pay is the introduction of additional scalar potentials. In Cartesian coordinates it is sufficient to introduce only one more scalar potential which implies that four scalar elliptic equations have to be solved.

There are at least two schemes to decouple the constraint, see Grandclément et al. (2001). Historically Bowen and York (1980) were the first to propose a suitable scheme. Another scheme, slightly different but conceptually comparable, was used in Oohara and Nakamura (1997) and is also discussed and analyzed in Grandclément et al. (2001). As long as the Laplacian operator and the (dual) gradient operator commute, $[D^i, D^2] = 0$, for a scalar field one can decouple relation (3.39) and iteratively solve the momentum constraint. The commuting $[D^2, D^i]\chi = 0$ in Cartesian and spherical polar coordinates can be verified by direct calculation. For the method in Oohara and Nakamura (1997) one needs in addition the commuting of these operators for a vector field, $[D^2, D^i]X^i = 0$ which is also satisfied for both Cartesian and spherical coordinates as can be verified by computation.

Bowen York We decompose the vector field W^i into two parts $W^i = X^i + D^i\chi$ with

$$D^2\chi = -\frac{1}{4}D_i X^i \text{ and} \tag{3.40a}$$
$$D^2 X^i = S^i. \tag{3.40b}$$

This is equivalent to equation (3.39) since

$$\begin{aligned} D^2W^i + \frac{1}{3}D^iD_jW^j &= D^2X^i + \frac{1}{3}D^iD_jX^j + D^2D_i\chi + \frac{1}{3}D^iD_jD^j\chi \\ &\overset{[D^2,D^i]\chi=0}{=} S^i + \frac{1}{3}D^iD_jX^j + \frac{4}{3}D^iD^2\chi \overset{(3.40a)}{=} S^i \end{aligned} \tag{3.41}$$

and therefore one has the iterative scheme

0. calculate the initial source $S^i = D_jK^{ij} - \gamma^{ik}R_{kj}W^j$, for example on a flat background,
1. solve equation (3.40b) $D^2X^i = S^i$ for X^i,
2. solve equation (3.40a) $D^2\chi = -\frac{1}{4}D_iX^i$ for χ,
3. obtain $W^i = X^i + D^i\chi$,
4. obtain $A^{ij}_{\mathrm{L}} = \frac{1}{2}D^{(i}W^{j)} - \frac{2}{3}\gamma^{ij}D_kW^k$, $A^{ij} = A^{ij}_{\mathrm{TT}} + A^{ij}_{\mathrm{L}}$ and $K^{ij} := A^{ij} - \frac{1}{3}\gamma^{ij}\mathrm{tr}K$ (the trace $\mathrm{tr}K$ and A^{ij}_{TT} were prescribed in this scheme),
5. calculate the new source term and start with point 1 again until some abortion condition is reached.

There remains one obvious problem though, namely in point 1 of the iterative scheme where we have to solve the Poisson equation $D^2X^i = S^i$ for the vector field X^i. It obviously decouples in Cartesian coordinates since the Laplace operator is diagonal there.

In a different coordinate system, in particular in spherical polar coordinates, cross-terms appear in equations like equation (3.40b) where one solves the vector Poisson equation $D^2X^i = S^i$ in those coordinates, which cause a problem. The vector Laplace operator might contain cross-terms in non-Cartesian coordinates. Therefore one can sometimes read, e.g. in Grandclément et al. (2001), that the process is only applicable in Cartesian coordinates. This statement is not completely true though. There are situations where the introduction of additional scalar potentials allows to decouple the corresponding equation in a way that just scalar elliptic equations need to be solved. See Bonazzola et al. (2004) for a general discussion on a spherically symmetric background.

Also for our situation (hypersurface-orthogonal axisymmetry in spherical coordinates) we have positive experience in the process of decoupling. Nevertheless the introduction of the vector potentials is problematic concerning the correct number of degrees of freedom. Therefore we decided to abandon that approach.

3.4.2. The constraints as evolutionary system

As already said the constraints as such form an underdeterminded system of partial differential equations. In section 2.3.2 we discussed a general example with the help of a single scalar equation. It is clear that the previously discussed "elliptic method" is not the only possibility, even though it might be the most popular one and it is an often used scheme in the numerical relativity community. The constraints were studied systematically as an evolutionary system of equations only in recent years in a series of papers by István Rácz, see Rácz (2014a,b, 2015, 2016a), even though the idea of the constraint system as non-elliptic equations might be older.

There are various motivations to study alternatives to the elliptic method. For example the implementation is slightly more straightforward for evolutionary partial differential equations and the issue of boundary conditions is also different in those formulations as will be discussed below. Large collaborations in numerical relativity report on "spurious gravitational junk radiation" that is believed to be caused by the initial data, see Chu (2014) for example. Besides its mathematical and numerical attractiveness and the curiosity to explore some alternatives there are also some researchers who hope that a (in parts drastically) different formulation might also bring some clarification concerning such issues. Nevertheless even though there are some nice mathematical properties concerning these formulations there is no guarantee that the resulting formulation is well-behaving with respect to the full set of Einstein's equations (with including spacetime evolution). The lack of understanding might be related to the fact that this particular subfield is rather new and far less developed than the traditional elliptic-hyperbolic formulation of Einstein's equations. In the standard approach some solid understanding took several decades and significant human resources.

The aim of this section is to review ideas and some central results, see Rácz (2015) for a source of some parts of it. In the literature quite often the slightly more general situation of the general $n+1$-dimensional ($n \geq 3$) either

Riemannian or Lorentzian case[11] is discussed. For our purposes it is sufficient to concentrate on the 3+1-dimensional Lorentzian case since that is the one of interest in general relativity. The Riemannian hypersurface (initial slice) Σ is further decomposed into a one-parameter family of homologous surfaces $\Sigma = \mathbb{R} \times \mathcal{S}_\rho$, which gives an additional $(n-1)+1$-split of the n-dimensional spatial hypersurface. The parameter called ρ in publications like Rácz (2015) (which corresponds to the radial coordinate r in the current thesis) plays the role of the "time" parameter in the evolutionary scheme. The remaining variables (which are not dealt with in the constraint solver) are called "freely specifiable" and are prescribed for the solver (for instance they can be obtained by the (spacetime) evolution equations of Einstein's equations).

Two schemes for solving the constraints are slightly distinct and particularly attractive and therefore discussed in more details. They allow for (at least local) well-posedness statements as "initial value problems" which sounds alluring for the implementation.

The parabolic-hyperbolic scheme

In this scheme the Hamiltonian constraint is cast into a parabolic differential equation for the quantity that is connected with the lapse of the additional $(n-1)+1$-decomposition of the Riemannian surface (the rr-component of the metric in the spacetime formulation). The momentum constraint can be formulated as a set of equations of the form

$$A_\rho \partial_\rho u + \sum_{k=x_2,x_3} A_k \partial_k u + B = 0 \tag{3.42}$$

where u is a vector of components of the extrinsic curvature tensor of the $(n-1)+1$-decomposition (and ρ the radial coordinate). The momentum constraint is solved for the components of u. The matrices A_ρ, A_{x_2}, A_{x_3} are the coefficient matrices for the partial derivatives in ρ and the remaining spacelike directions x_2 and x_3, B the inhomogeneity of the set of equations. Both A and B are determined by the freely specifiable data. Remarkable is that the coefficient matrices are symmetric and A_ρ is in addition positive definite for $\rho \geq \rho_0 > 0$. Therefore the system (3.42) is in the form of a **first order symmetric hyperbolic system**. "Initial values" need to be chosen at ρ_0.

Theorem 3.4.1 (Theorem 4.1 in Rácz (2015))**.** Under suitable assumptions (basically some smoothness requirements) the parabolic-hyperbolic system

[11] Concerning the spacetime, the n-dimensional hypersurface is Riemannian, of course.

provides locally in an interval $[\rho_0, \rho_0 + \epsilon]$ for some $\epsilon > 0$ a unique smooth solution and the solution together with the freely specifiable data satisfy the vacuum constraint equations in proposition 3.4.1 in the corresponding interval.

For the purpose of this thesis it is interesting to remark that we can extract the information that nonlinear perturbations of Minkowski spacetime form a (locally) well-posed system when suitable initial data are given. A proposal to use the formulation to obtain initial data for a spacetime with a binary system of Kerr black holes is given in Rácz (2016b) and continued in Rácz (2016c). As far as we are aware there is no successful implementation of that proposal yet. For an example of a parabolic-hyperbolic formulation of a constraint system in practice see sections 4.5 and 4.7 and for the implementation section 5.5.

The algebraic-hyperbolic scheme

The Hamiltonian constraint can also be considered as an algebraic equation for a component, say κ, of the extrinsic curvature (basically κ is, except for some prefactor, the rr-component of the extrinsic curvature). Note that, since in the Hamiltonian constraint (3.31a) the extrinsic curvature components appear quadratically, it should be expected that in the linearization about the flat spacetime (where the extrinsic curvature vanishes) the procedure fails.

More precisely it becomes clear that equation (2.8) of Rácz and Winicour (2015) (which is a stability condition for components of the extrinsic curvature that needs to be satisfied by it) is not satisfied for the flat Minkowski space. If it was we would have a first-order strongly hyperbolic system for components of the extrinsic curvature and similar well-posedness results would be applicable.

Jeffrey Winicour (2017) addresses the problem in more detail. He shows that for a large class of applications in numerical relativity the algebraic-hyperbolic approach carries a lot of risk or is even impossible. Therefore one should either change the setting (for example use non-spherical coordinates, assume the existence of spacetime singularities) or, as we will do in the following, avoid that method in those situation.

As we will see explicitly in section 4.7 the nonlinear Hamiltonian constraint can be interpreted as an algebraic equation for the rr-component of the extrinsic curvature which will correspond to a variable that will be called $K_{\mathrm{s}1}$ in our formulation. The linearized equations do not even contain quantities of the extrinsic curvature (as it should be due to the quadratic appearence as discussed above). Therefore in the remainder of the thesis we will not consider the algebraic-hyperbolic formulation but remark that it seems to be an interesting possibility in general and for instance for perturbations about a Schwarzschild background it seems to be an attractive alternative, see Rácz and Winicour (2015). Remarkably the "initial values" for the solver can also be given at the outer boundary (so they can be obtained for an isolated system by making use of asymptotic flatness and therefore some fall-off condition) and the integration goes inwards. Hence the more subtle issue of finding appropriate initial values at the inner boundary (typically an event/apparent horizon or the origin itself) is avoided.

In a recent paper Beyer et al. (2017) nonlinear axisymmetric perturbations of the Schwarzschild black hole are considered. Since the reference spacetime is given by the Schwarzschild metric the results in Winicour (2017) do not apply and the origin is avoided. The authors prescribe initial data at the horizon and study the evolutionary solver numerically. They report problems to reproduce asymptotically flat data in the generic case. If these problems are avoided when starting at the origin is left open.

3.5. Symmetry-reduced situations in general relativity

Quite often one is, especially for more conceptual studies like the one reported in this thesis, first interested in "symmetry-reduced situations". It means that one assumes some symmetry that is admitted by the spacetime (and/or the corresponding tensor fields on it, see discussion below). It leads to computational simplifications, both for analytical considerations and the numerical implementation. Nevertheless the spacetime is, as always in our considerations, a 4-dimensional Lorentzian manifold. Symmetry-reduction has nothing to do with lower-dimensional gravity, even though the resulting problem is sometimes also reduced by some dimension.

Basically the material of this section is covered quite well by Hawking and Ellis (1973), Chandrasekhar (1983), Wald (1984), Stewart (1991) for example, in parts also by more mathematically orientated literature like Nakahara (2003), O'Neill (1983) for instance.

Definition 3.5.1. A spacetime has an **isometry** if its metric tensor is invariant ("symmetric") under displacement along integral curves of a smooth vector field ξ, hence if the **Lie derivative**[12] satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \tag{3.43}$$

Proposition 3.5.1. For a spacetime with Levi-Civita connection (see section 3.2) the condition $\mathcal{L}_\xi g_{\mu\nu} = 0$ is equivalent to the so-called **Killing's equation**

$$\nabla_{(\mu}\xi_{\nu)} = 0. \tag{3.44}$$

A solution ξ_ν (or rather $\xi^\mu = g^{\mu\nu}\xi_\nu$) is called **Killing vector field**.

Proof.

$$0 = \mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \nabla_\lambda g_{\mu\nu} + g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\lambda\nu}\nabla_\mu \xi^\lambda \overset{\text{metricity}}{=} 2\nabla_{(\mu}\xi_{\nu)}. \tag{3.45}$$

□

Definition 3.5.2. A spacetime that admits a time coordinate t with corresponding everywhere timelike vector field ∂_t such that ∂_t generates an isometry, $\mathcal{L}_{\partial_t} g_{\mu\nu} = 0$, is called **stationary**. A stationary spacetime which is orthogonal to each hypersurface $t = \text{constant}$ ($g_{ta} = 0$ for a any spatial index) is called **static**.

Definition 3.5.3. Consider the x-y-plane (in Cartesian coordinates and the azimuthal angle $\varphi = \arctan y/x$ (see appendix A.1.2 for comments on the division when x becomes zero) and the vector field $\xi = \partial_\varphi = x\partial_y - y\partial_x$. The **orbit** of ξ is the collection of events resulting from the action of the transformation. If ξ is a Killing vector field with closed orbits[13],

$$\mathcal{L}_{\partial_\varphi} g_{\mu\nu} = 0, \tag{3.46}$$

then we call the spacetime **axisymmetric**. An axisymmetric spacetime whose Killing vector field is orthogonal to each hypersurface $\varphi = \text{constant}$ (i.e. $g_{\varphi A} = 0$ for A any non-φ-component, so $g_{\varphi A} = \delta_{\varphi A} g_{\varphi\varphi}$) is called **hypersurface-orthogonal axisymmetric**.

Remark 3.5.1. The components of the Killing vector field ξ in definition 3.5.3 vanishes at the origin $\xi = x\partial_y - y\partial_x = 0$ in Cartesian coordinates. This is not the case in spherical coordinates, $\xi = \partial_\varphi \neq 0$.

[12] We do not discuss the the Lie derivative here. It is done in most textbooks on general relativity and we recommend in particular Schutz (1980).

[13] Sometimes this requirement is dropped. It prohibits for example the existence of discontinuities in the orbits as it might occur for objects like cosmic strings. We will not go into further details.

The following definitions are due to Geroch (1971) (there in more generality) and the decomposition where one "divides out" the spacelike orbits of an isometry is usually called **Geroch-decomposition**. In section 3.3 we discussed the 3+1-decomposition and there is some similarity with a "Riemannian version" of the Geroch reduction, here in one dimension less (in Rinne (2005)[chapter 3] both decomposition are considered together which was introduced in Maedaetal80).

Definition 3.5.4. Consider a spacetime $(\mathcal{M}, g)$ with isometry. The isometry transformations (symmetries of the metric tensor) form a group. If this isometry group contains a subgroup that is (at least isomorphic to) $SO(3)$ and the orbit of the subgroup are two-spheres[14], then the spacetime is called **spherically symmetric**.

Proposition 3.5.2 (Birkhoff Jebsen theorem)**.** For a smooth spherically symmetric spacetime the vacuum solution to Einstein's equations (with vanishing cosmological constant) is diffeomorphic to the exterior Schwarzschild solution[15].

Proposition 3.5.2 is often called **Birkhoff's theorem** even though Jørg Tofte Jebsen (2005) should be given credit for the discovery, see the historic note Johansen and Ravndal (2006) for a discussion. The proof is very elementary and included in many courses and textbooks on general relativity, see Choquet-Bruhat (2009, pages 74, 75).

Proposition 3.5.2 implies that for spherical symmetric general relativity in vacuum every solution of the system is diffeomorphic to a representative of the Schwarzschild family, so basically completely determined by one number, the mass parameter. So in spherical symmetry the whole dynamics is given by the matter field, for instance gravitational collapse is driven by the matter source in that situation. In particular there is no gravitational radiation in spherical symmetry. Thus if we are interested in vacuum general relativity the assumption of spherical symmetry is "too strong", the situation is automatically static and hence understood. Therefore in vacuum one should study situations with less symmetry, for instance axisymmetry.

In fact it was shown in Bičák and Pravdová (1998) that it is not possible to assume any further reasonable symmetry in vacuum when demanding

[14]The requirement concerning the orbits being two-spheres might be relaxed. Then one gets quotients of two-spheres (including the projective plane) as solutions as well. One might use some asymptotic condition or an orientability requirement to exclude these "exotic" cases.

[15]The (exterior) **Schwarzschild metric** is named after Karl Schwarzschild Schwarzschild (1916). It should better be called "member of the Schwarzschild one-parameter family of metrics", depending on the value of the constant which is a mass parameter of the spacetime.

gravitational radiation in axisymmetric vacuum[16] spacetime. Therefore axisymmetry is the "least" symmetric situation when studying gravitational radiation in vacuum general relativity.

The definitions and properties we discussed above for the whole spacetime, therefore for symmetries of the metric, can be translated to arbitrary tensor fields (in particular including scalar and vector fields). So for example we define an axisymmetric tensor field in the following way.

Definition 3.5.5. Given a general tensor field $M_{\mu\nu}$. $M_{\mu\nu}$ is **axisymmetric** if the Lie-derivative along $\xi = \partial_\varphi$ vanishes,

$$\mathcal{L}_\xi M_{\mu\nu} = \xi^\lambda \nabla_\lambda M_{\mu\nu} + M_{\mu\lambda} \nabla_\nu \xi^\lambda + M_{\lambda\nu} \nabla_\mu \xi^\lambda = 0. \tag{3.47}$$

We will make use of these properties of tensor fields in section 4.2.

Numerical implementations in spherical symmetry and critical phenomena in general relativity The assumption of spherical symmetry is very attractive from the computational point of view, essentially due to the fact that in adapted coordinates (spherical polar coordinates are appropriate, see appendix A.1.2) the originally 3+1-dimensional equations are reduced to a 1+1-dimensional system. Calculations become tractable and the numerical implementation is easier and cheaper (measured in computational time, electricity, occupation of computer resources or similar quantities).

As we have seen the dynamical situation in vacuum is trivial though. Therefore an introduction of some matter field is inevitable if one is interested in the dynamical situation. There are numerous possibilities including (scalar) fields, fluids, collisionless particles discussed in the corresponding literature.

Here we should put special emphasis on the toy model of a scalar field because it is of special importance, even though physically of limited relevance in reality. Its simplicity makes it very prominent in numerical relativity.

An important breakthrough in the field of numerical relativity was achieved by Matthew Choptuik (1993) where he considers a collapsing massless scalar field in spherically symmetric general relativity. In the following we will say a few words about his discovery and about critical phenomena in gravitation in general. See in addition the reviews Choptuik (1992, 1994) with far more details than in Choptuik (1993), especially concerning the implementation, and the review Gundlach and Martín-García (2007) for critical collapse in general relativity.

[16] Actually the authors assume electrovacuum.

It is a well-known fact that there are at least two endstates for perturbations (gravitational or matter perturbations) of Minkowski spacetime. It was proven in Christodoulou and Klainerman (1993) that weak perturbations (see the cited book for a mathematically precise definition) of perturbations decay. They disperse to null infinity[17] and decay which results in the statement that the Minkowski space is nonlinearly stable. On the other hand it is well known that sufficiently strong perturbations will form a black hole due to the universal attractive nature of general relativity. Therefore there should be an intermediate point between those cases at the onset of black hole formation. This is the point of interest for critical studies in general relativity[18].

Essential was the high accuracy that was obtained with the help of adaptive mesh refinement to zoom into regions of interest, which was applied in numerical relativity for the first time for this project. Nowadays it is a widely used tool in numerical relativity. Also the introduction of numerical dissipation (discussed in section 2.3.3) was important.

Choptuik considered a one-parameter family of massless scalar fields coupled to a spherically symmetric gravitational field. He used polar slicing (implying for the components ${K_\vartheta}^\vartheta = {K_\varphi}^\varphi = 0$, see definition 4.3.2). Because of the high accuracy he was able to determine the critical value very precisely and studied sub- as well as supercritical solutions.

The analysis of the subcritical simulations revealed the existence of a **universal critical solution** at the onset of black hole formation. In the vicinity of that critical solution there is a finite number of echos. The echos are scale-periodic with again an universal "echoing exponent". The critical solution is **discretely self-similar**[19] which means there exists an invariance of the solution under rescaling of time and the radial coordinate.

[17] This dispersion is due to the asymptotic structure of Minkowski space and very essential for the proof. This is in contrast to the so-called anti-de Sitter space (which has a boundary) which is believed to be basically unstable, see Bizoń and Rostworowski (2011), and Maliborski and Rostworowski (2013) for an example of an "island of stability".

[18] As the (unconfirmed) story goes the mathematician Demetrios Christodoulou asked the numerical relativist Matt Choptuik in the late 1980s: "Matt, what happens at the threshold?" which initiated this important study. Some statements in this direction are included in the introduction of Choptuik (1994) which confirms that a question in pure mathematical relativity led with the help of numerical mathematical relativity to a very important contribution in numerical relativity. This might be the first result where numerical tools were absolutely indispensable in mathematical relativity and a milestone in numerical relativity. In the following years some (semi-)analytical understanding (basically using perturbative techniques and the theory of dynamical systems) was achieved as well, in large parts due to Carsten Gundlach, see Gundlach and Martín-García (2007).

[19] There exist also continuously self-similar solutions.

If a black hole is formed in the supercritical regime an infinitesimally small mass of the black hole is possible. Its mass is determined by

$$m_{\mathrm{bh}} \sim (p - p_*)^\gamma \tag{3.48}$$

with parameter p of the one-parameter family of scalar fields (and critical value p_*) and **universal** exponent γ (whose numerical value is $\approx .37$) which is independent of the particular family of initial data.

This type of critical collapse was later called "type II" critical collapse, see Gundlach and Martín-García (2007). Consequently there is also a "type I" critical collapse, now with stationary critical solutions with time-periodicity. It occurs if a mass scale in the field equations is relevant. In that case there exist a finite black hole mass and a mass gap.

For completeness let us mention that there exists also an additional "type III" critical collapse discovered in Choptuik et al. (1999) with a critical solution being an unstable (due to a coupling to a Yang-Mills matter field) static black hole, see also Rinne (2014) for more recent investigations.

Due to the Jebsen-Birkhoff theorem 3.5.2 it is obvious that in spherical symmetry the collapse is driven by the coupling to the matter contribution, the "right-hand side" of Einstein's equation (3.3). It is not so clear which role is played by the "left-hand side" as such, so by gravity itself with vanishing matter field[20]. Also the investigations of the role of angular momentum in the critical collapse is of huge interest and cannot be studied in spherical symmetry[21]. There is some recent progress in that direction, see Baumgarte and Gundlach (2016), Gundlach and Baumgarte (2016).

There are a number of immediate follow-up questions, including

1. are the phenomena in the critical collapse a curiosity of the high symmetry assumption?
2. are the phenomena in the critical collapse a curiosity of the scalar field?
3. is the matter relevant for the critical collapse or can it occur as a pure gravitational effect as well?

[20] On https://alanrendall.wordpress.com/2009/04/25/respecting-the-matter-in-general-relativity/ one can read: "Einstein himself is often quoted as having said that the left hand side of his equations is made of marble while the right hand side is made of wood". We also failed to find sources for Alan Rendall's quote.

[21] We only mention the possibility of a charged scalar field that might act as a toy model for angular momentum.

The answer to points 1 and 2 was basically found in the last two and a half decades and is basically in the negative. There are a few studies showing critical phenomena in less symmetry and driven by a matter field. Also a large number of simulations with different matter models. Concerning these points consult Gundlach and Martín-García (2007) for a fairly complete review until 2007, in particular about the matter sources, and Choptuik et al. (2015) for a more recent review where references concerning point 1 are included. That the answer to point 3 is in the affirmative was shown in the same year as Choptuik (1993) by Andrew Abrahams and Charles Evans in Abrahams and Evans (1993) who considered vacuum collapse and found critical phenomena as well. It is remarkable (and will be discussed below) that since then further attempts to this problem remained unsuccessful[22]. Therefore there is some justification to consider point 3 in the list above to be open. Further attempts in that direction are considered in the paragraph below.

Numerical implementations in axial symmetry Between the highly idealized situation of spherical symmetry and the full 3+1-dimensional framework the study of axisymmetry is suitable. We discussed above that, under certain assumptions, axisymmetry[23] is the least symmetry one should assume when departing from spherical symmetry. Actually axisymmetry is of immense importance in mathematical relativity in the study of a single isolated system. The two-parameter family of Kerr solutions[24] is believed to be the final state of all (uncharged) black holes, even though a manifest proof of that conjecture is one of the major open problems in mathematical relativity and subject of recent research.

Especially from the numerical point of view axisymmetry is attractive because it is possible to reduce the dimension effectively by one. One results in a 2+1-dimensional situation.

Therefore it is not surprising that the celebrated "birth of numerical relativity"

[22] The code was never published and also details about the implementation are difficult to find in the literature. Nevertheless consider Evans et al. (1986) for some details on earlier implementations that led to the one used for Abrahams and Evans (1993).

[23] Along that line it is obvious that hypersurface-orthogonal axisymmetry can be considered as the "simplest" situation (in some respect) one has to consider when studying gravitational perturbations in vacuum spacetimes.

[24] Discovered in 1963 by Roy Kerr (1963) and discussed in basically all textbooks on general relativity, for a solid mathematical treatment (even though slightly outdated) see Chandrasekhar (1983).

Hahn and Lindquist (1964)[25] was an axisymmetric study. In the following two to three decades the focus of the field of numerical relativity was on axisymmetrical simulations. A central reason is that the computational resources were so limited that it seemed beneficial to concentrate on the reduced version first. For reviews on early successes in axisymmetric numerical relativity like head-on collisions and the evolution of so-called Brill waves[26] see for example the contributions of Eardley, Smarr, Eppley in Smarr (1979) and the early review on axisymmetric numerical relativity Bardeen and Piran (1983). Also the results by the Japanese group, reviewed in Nakamura et al. (1987), should be mentioned here.

The focus switched to "full general relativity" (in the sense of no symmetry assumption), in particular simulations with the aim of the extraction of gravitational waves of binary systems (black holes as well as neutron stars) in the 1990s. Numerical relativity groups combined forces in projects like the "Binary Black Hole Alliance", a High-Performance Computing and Communications Grand Challenge project founded by the National Science Foundation, see Matzner et al. (1995)[27]. Finally we have large successful numerical relativity collaborations[28].

Only a subfield remained interested in axisymmetric simulations until today and some particular results are discussed below. Nowadays a large motivation for studying axisymmetric general relativity is to test and demonstrate conceptual issues and to develop new tools with potential applications in full general relativity. It is usually easier to handle and to implement. Also the simulations are faster and therefore cheaper. As will be demonstrated in this thesis axisymmetry is a well-suited testbed for new techniques and fundamental studies. It deals as appropriate "intermediate step" between the sometimes oversimplifying spherical symmetry and general relativity without symmetries. Basically all results in this thesis are restricted to vacuum

[25]One should remark here that the pioneering paper marks indeed the beginning of the era of numerical relativity and it is justified to be celebrated but actually their simulations are unstable and, as the history of numerical relativity taught us, their conclusion that "the numerical solution of the Einstein field equations presents no insurmountable difficulties" is incorrect to a very large degree.

[26]Brill waves are pure gravitational waves as introduced in Brill (1959) (there for time-symmetric initial data) and were the system studied in Abrahams and Evans (1993).

[27]While the aim of the project was to tackle the full 3+1-dimensional problem of the coalescence and merger of a binary black hole they report here on a successful head-on collision of two black holes which assumes axisymmetry. Also a mentionable result is the numerical confirmation of the "pair of pants", the non-time-symmetric horizon structure of two merging black holes.

[28]We list SpEC (https://www.black-holes.org/code/SpEC.html), Einstein toolkit (https://einsteintoolkit.org/), Cactus (http://cactuscode.org/) for examples.

hypersurface-orthogonal axisymmetry but are more or less directly generalizable to the 3+1-dimensional situation. On the other hand it allows for the implementation and demonstration of new techniques, even offside large collaborations and with moderate effort. It will be interesting to see these techniques in action in situations with less symmetry.

Gravitational collapse in vacuum Since critical collapse simulations seem to be out of range of full vacuum general relativity it is appropriate to discuss attempts in that direction in axisymmetry. Spherical symmetry is not suited for the discussion because in vacuum the situation is already understood and there is no dynamics, see proposition 3.5.2. As already mentioned in discussions above on page 111 pure gravitational collapse is not completely understood yet (there are, as we will discuss below, exceptional papers but as we argued basically the situation of critical vacuum collapse can be considered to be open). The results of this thesis might be seen as a step towards that direction.

Pure gravitational waves in vacuum, the already mentioned Brill waves, were numerically studied in the late 70s, see Eppley (1979) for a review from those days. There he discusses their studies using cylindrical coordinates with quasi-isotropic gauge and maximal slicing. The simulations are restricted to the subcritical case, i.e. with a low amplitude of the initial wave. The aim is rather to address conceptual issues. For example they aim to understand the numerical calculation of an energy flux or a mass of such spacetimes.

Shoken Miyama (1981) seems to be the first who describes both the sub- and supercritical evolution of numerical vacuum collapse correctly. He uses cylindrical coordinates as well and geodesic slicing but a non-vanishing shift. The resolution and accuracy does not allow to investigate the critical regime though.

Very interesting are the studies of Richard Stark and Tsvi Piran (James Bardeen seems to be involved as well) who consider axisymmetric spacetimes in spherical coordinates for vacuum as well as matter. In Appendix A of Bardeen and Piran (1983) they derive regularity conditions for the origin which seems to be used in the numerical simulations. In Stark and Piran (1985) both sub- and supercritical configurations for vacuum waves are suggested as test cases for the code[29]. Nevertheless in a more detailed paper Stark and Piran (1987) only weak

[29]The possibility to investigate the critical regime is not suggested though and the main part of the letter deals with spacetimes with matter. Interesting to remark is that they even observe some kind of universality of the wave form (see footnote 22 in Stark and Piran (1985)).

gravitational waves are successfully evolved. It would be interesting to know if their code was also able to handle the formation of a collapse to a black hole and if the critical regime could be investigated.

The already mentioned letter by Abrahams and Evans (1993) reports on critical phenomena in vacuum spacetimes. Hints to the existence of critical phenomena in vacuum collapse were already included in Abrahams and Evans (1992), so the first critical simulations for a scalar field and vacuum were really performed in parallel. The authors developed their axisymmetric code over many years and had apparently a very strong implementation at that time. They use spherical polar coordinates and a quasi-isotropic spatial gauge condition together with maximal slicing. A key point in the implementation is their moving mesh algorithm which allows for higher accuracy in the important areas of the simulation. Therefore they are not just able to simulate sub- and supercritical configurations but also to investigate the critical regime. They find a similar mass scaling (3.48) with same exponent .37 as for the collapse of the scalar field. Also the echoing of solutions is observed and further, in the follow-up paper Abrahams and Evans (1994), evidence for universality is given[30]. It is remarkable that up to now there is no known independent confirmation of their studies and reported results. In view of the increased computational power between the machines in the early 90s and today and the number of people working in numerical relativity this is somewhat surprising.

At the turn of the millennium full 3+1-dimensional codes became available. In Alcubierre et al. (2000) the authors use the Cactus code to investigate gravitational collapse. They use cylindrical coordinates and a combination of maximal slicing and vanishing shift. They find that strong waves form a black hole and weak ones disperse. They give a rough estimate of the critical amplitude between sub- and supercritical configurations but are not able to investigate that regime further and to find critical phenomena.

Garfinkle and Duncan (2001) study axisymmetric gravitational waves in cylindrical coordinates with maximal slicing and in the quasi-isotropic spatial gauge. For the evolution they introduce regularized variables in the form of special combinations of several quantities to deal with the coordinate singularity at the axis and they compactify the spacetime which allows to include spatial infinity in the computational domain. Their formulation is partially constrained. The resolution only allows a rough estimate of the critical value. They do not see indications for the formation of naked singularities.

[30] And the critical exponent is reported to be .36 now.

Also in cylindrical coordinates Choptuik et al. (2003) investigate axisymmetric spacetimes. Again they apply maximal slicing and quasi-isotropic coordinates. A fully constrained formulation is used. They use a similar regularization procedure as in Garfinkle and Duncan (2001). The authors do not really investigate critical collapse but postpone it to possible future studies.

Both Garfinkle and Duncan (2001) and Choptuik et al. (2003) solve the Hamiltonian constraint. In Rinne (2008) it was pointed out that their formulations have some problems concerning uniqueness and that issue was corrected. Similarly as the other two projects Rinne (2008) uses cylindrical coordinates and the combination of maximal slicing and quasi-isotropic coordinates. While in a previous study Rinne (2005) (which was not necessarily restricted to vacuum) uses a free evolution it is now performed in a fully constrained scheme. The possibility of a collapse to a black hole is confirmed but similarly as in previous studies the simulations are not able to explore the critical regime.

A very different approach to vacuum collapse was presented by Evgeny Sorkin in Sorkin (2011). There Brill waves are evolved in axisymmetry using cylindrical coordinates and the generalized harmonic formulation of general relativity. He focuses on the subcritical regime and reports on the possibility to approach the critical value to a high degree. He reports on a similar scaling behavior for curvature invariants as for the mass in (3.48) in supercritical simulations. Also echoing of solutions seem to be present as well as universality with respect to the choice of initial data and specific coordinate conditions. Nevertheless the critical solution seems to be different than the one in Abrahams and Evans (1993). It is not completely ruled out that the observations are due to an unphysical coordinate effect.

In a more recent investigation Hilditch et al. (2013) study vacuum spacetimes in axisymmetry in "moving puncture" coordinates. They obtain their results with two basically independent codes, one with spherical polar coordinates and one with cylindrical coordinates with "1+log slicing" and "Gamma driver shift" (see remark 3.3.3). In Hilditch et al. (2013) they focus on the issue of the moving puncture coordinates and do not address numerically the issue of critical collapse. In Hilditch et al. (2016) some of the previous authors present a new implementation of a pseudo-spectral code. Inspired by the SpEC code they use quite similar techniques, for example generalized harmonic coordinates. Even though the implementation is able to handle situations without symmetry they focus on axisymmetry for applications. The authors demonstrate that they can evolve the supercritical regime and that their excision technique works but the investigations of critical collapse are postponed to forthcoming work. Recently the same group published a preprint Hilditch et al. (2017) where they determine the critical parameter but cannot investigate the critical regime to the accuracy that is needed. They use again cylindrical coordinates.

To summarize there are quite a lot of attempts to the collapse of gravitational waves in vacuum. Most work is devoted to cylindrical coordinates with little success concerning the exploration of the critical regime. There is basically just one remarkable exception. On the other hand it seems to make sense that for the collapse of vacuum perturbations the use of spherical coordinates should be favored since it sounds more promising from the physical perspective. At least the earliest critical vacuum simulations were performed in spherical coordinates.

It seems intuitive to use spherical polar coordinates for collapse scenarios. If using adaptive methods they are also more natural. One can benefit from results from large collaborations that do not use cylindrical coordinates. This thesis should also be seen as a contribution to a deeper understanding of the background that is needed to implement spacetime simulations in spherical coordinates with a regular origin. We promote the use of spherical polar coordinates.

4. Vacuum axisymmetry in spherical coordinates

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4.1. Introduction

The present chapter contains some of the main results of the thesis. Its focus is the formulation and analysis of Einstein's vacuum equations in axisymmetry. Some aspects of the general axisymmetric framework (including the definition) and the discussion of some published results in the literature were already addressed in section 3.5. Here we explicitly derive Einstein's nonlinear equations in spherical polar coordinates. The aim is to adapt the situation such that an expansion in spin-weighted spherical harmonics is suitable as discussed in section 2.2.5. The mathematical analysis, which we prepared in chapter 2, is a major issue of this chapter.

We use spherical polar coordinates. They imply a coordinate singularity for $\vartheta = 0$ and $\vartheta = \pi$, so on the coordinate axis in adapted coordinates. Even though not necessary conceptually, we assume for computational reasons axisymmetry, so the coordinate axis is the axis of symmetry. We work out and derive its consequences. While known in cylindrical coordinates the relations seem to be unpublished for spherical polar coordinates. Their derivation has consequences on the definition of our variables in section 4.4.

Our approach to use a spectral expansion in spin-weighted harmonics has an influence on possible gauge choices. We work out the consequences and present our results in section 4.3. Our derivation of Einstein's field equations in the considered situation is presented in section 4.5, including its linearization about the flat solution and the reduction to the 1+1-dimensional mode equations. We present our derivation of the exact solution of the linearized problem in section 4.6.

We present our analysis of the derived equations in section 4.7, an eminent part of the current chapter. We discuss several optional possibilities and present our modifications that allow a formulation of the equations consisting of a strongly hyperbolic and a parabolic-strongly hyperbolic subset.

4.2. Axisymmetry and implications

We assume hypersurface-orthogonal axisymmetry, hence the existence of a Killing vector field ∂_φ, see section 3.5. While a lot of the results of the thesis are general or straightforwardly generalizable this is obviously not the case for the current section. We will see that the assumption of axisymmetry already reveals some information about the behavior of the variables. The aim is to keep the framework as general as possible (concerning the generalization to a situation without symmetry). We will not have to use the results of the current section in the numerical studies in section 5. On the other hand we benefited during the development of the code from having the relations. Further there might be situations where one is just interested in axisymmetry and the relations are of use. Also in the general case the knowledge of the leading term r (radial coordinate) in the corresponding power series close to the origin is of considerable value.

The essential step was performed for cylindrical coordinates and published in 2005 in Rinne (2005), Rinne and Stewart (2005). For spherical polar coordinates the calculations are similar. The basic ingredient is “elementary flatness” which is used on the axis to derive conditions for the variables. These allow then to deduce a power series for the variables close to the axis.

Some similar considerations can be already found in Garfinkle and Duncan (2001) and Choptuik et al. (2003) and ideas in that direction are already in Miyama (1981). See also Ruiz et al. (2008). We are not aware of these results being transformed to spherical polar coordinates which might be caused by the fact that most formulations use cylindrical coordinates, see the discussions in section 3.5.

Recall the definition 3.5.3 of axisymmetry. The following definition of regularity on the axis incorporates the essential idea of elementary flatness, see Wilson and Clarke (1996) and Synge (1964, chapter VII) for more details.

Definition 4.2.1. Given an axisymmetric tensor field T. It is **regular on the axis** if it is so in Cartesian coordinates, that means if it has a convergent Taylor expansion with respect to x and y close to the axis of symmetry. In particular this defines **regularity at the origin**.

Proposition 4.2.1. A symmetric tensor field M on a spacetime $(\mathcal{M},\ {}^4g)$ which is axisymmetric and regular on the axis has, close to the axis of symmetry, the following behavior in cylindrical coordinates (t, ρ, φ, z)

$$M_{\mu\nu} = \begin{pmatrix} A & \rho D & \rho^2 F & B \\ \rho D & H + \rho^2 J & \rho^3 K & \rho E \\ \rho^2 F & \rho^3 K & \rho^2(H - \rho^2 J) & \rho^2 G \\ B & \rho E & \rho^2 G & C \end{pmatrix} \tag{4.1}$$

and close to the origin the following behavior in spherical polar coordinates $(t, r, \vartheta, \varphi)$. The components of the corresponding matrix $M_{\mu\nu}$ are

$$M_{tt} = A, \tag{4.2a}$$
$$M_{tr} = \cos\vartheta B + \sin^2\vartheta r D, \tag{4.2b}$$
$$M_{t\vartheta} = \cos\vartheta\left(-rB + \sin\vartheta r^2 D\right), \tag{4.2c}$$
$$M_{t\varphi} = \sin^2\vartheta r^2 F, \tag{4.2d}$$
$$M_{rr} = \cos^2\vartheta C + \sin^2\vartheta H + 2\sin^2\vartheta\cos\vartheta r E + \sin\vartheta\cos\vartheta r^2 J, \tag{4.2e}$$
$$M_{r\vartheta} = \sin\vartheta r\left[\cos\vartheta(H - C) + (\cos^2\vartheta - \sin^2\vartheta) r E - 2\sin^2\vartheta\cos\vartheta r^2 J\right], \tag{4.2f}$$
$$M_{r\varphi} = \sin^2\vartheta\left(\cos\vartheta + \sin^2\vartheta r\right) G \tag{4.2g}$$
$$M_{\vartheta\vartheta} = -2\sin^2\vartheta\cos\vartheta r E + \sin\vartheta\cos\vartheta r^2 J + \sin^2\vartheta r^2 C + \cos^2\vartheta r^2 H. \tag{4.2h}$$

$$M_{\vartheta\varphi} = \sin^3\vartheta r^3(-1+\cos\vartheta r)G \tag{4.2i}$$
$$M_{\varphi\varphi} = \sin^2\vartheta r^2\left(H - 2\sin^2\vartheta r^2 J\right) \tag{4.2j}$$

All functions denoted by a capital letter are functions dependent on higher powers in r or ρ respectively. Note that there exist a special relation for some of the coordinates close to the axis/origin. That relation reads

- in Cartesian coordinates

$$(x^2-y^2)(M_{xx}-M_{yy}) + 4xyM_{xy} = \mathcal{O}\left((x^2+y^2)^2\right), \tag{4.3}$$

- in cylindrical coordinates

$$\rho^2 M_{\rho\rho} - M_{\varphi\varphi} = \mathcal{O}\left(\rho^4\right) \tag{4.4}$$

- and in spherical polar coordinates

$$\sin^4\vartheta r^2 M_{rr} + 2\sin^3\vartheta\cos\vartheta M_{r\vartheta} + \sin^2\vartheta\cos^2\vartheta M_{\vartheta\vartheta} - M_{\varphi\varphi} = \mathcal{O}\left(r^4\right). \tag{4.5}$$

Proof. The essential calculations where performed in Rinne (2005, Chapter 2) for the cylindrical coordinates which we essentially copy here. We assume elementary flatness so locally close to the axis we use Cartesian coordinates. The transformations to spherical coordinates are explicitly given in appendix A.1.2. The Killing vector field reads

$$\partial_\varphi = -y\partial_x + x\partial_y \tag{4.6}$$

which is defined in the whole $\mathbb{R}^2$-plane. Then that expression is inserted into the Killing equation $\mathcal{L}_{\partial_\varphi} M = 0$ in Cartesian coordinates. Because M is axisymmetric we can determine the Taylor expansion. For further details see Rinne (2005). In Cartesian coordinates (t,x,y,z) the components of M need to behave close to the axis as

$$M_{tt} = A,\ M_{tx} = xD - yF,\ M_{ty} = yD + xF,\ M_{tz} = B \tag{4.7a}$$
$$M_{xx} = H + (x^2-y^2)J - 2xyG,\ M_{xy} = 2xyJ + (x^2-y^2)G,\ M_{xz} = xE - yG \tag{4.7b}$$
$$M_{yy} = H - (x^2-y^2)J + 2xyG,\ M_{yz} = xG + yE,\ M_{zz} = C \tag{4.7c}$$

where the capital letters $A, B, \ldots J$ denote functions of (t, x^2+y^2, z). The remaining entries follow by symmetry. We notice immediately that the "special relation" (4.3) holds.

For the transformation from one coordinates system $(x^A, x^B, \dots)$ to another one $(x^i, x^j, \dots)$ we use the formula (summation implied)

$$M_{ij} = \frac{\partial x^A}{\partial x^i}\frac{\partial x^B}{\partial x^j} M_{AB}. \tag{4.8}$$

Therefore in cylindrical coordinates (t, ρ, φ, z) we get exactly equation (4.1) and the "special relation" (4.4).

The transformation from Cartesian to spherical polar coordinates results in equation (4.2) and the "special relation" (4.5) holds. □

Remark 4.2.1. The leading order behavior in ρ (or r) of the metric may be obtained in a slightly easier way as derived in Ruiz et al. (2008). The authors use some invariance properties at the axis in the x-y-plane to find the leading order of the components there. Then the results can be transformed into the desired coordinate system. It does not seem to be possible to obtain the specific behavior as with the method in Rinne (2005) and the "special relation" with that method though.

Remark 4.2.2. The results in proposition 4.2.1 cannot be applied for the case without symmetry. As already said there is some motivation to keep the formulation as general as possible. The motivation for axisymmetry has basically computational reasons. In section 4.6 we will derive the exact solution to Einstein's linearized equations in axisymmetry. We can confirm that the analytical solution satisfies the behavior in proposition 4.2.1.

4.3. Gauges in axisymmetry

We discuss a few possible gauge choices suitable for axisymmetry in the following, including the one we will use in the remainder and motivate our choice. We considered the gauges in spherical polar coordinates (see appendix A.1.2) even though they can be considered in the form of cylindrical coordinates as well. Symmetry reduction was discussed in section 3.5 and the current section can be seen as kind of continuation of that one. Further gauge conditions which are not particular for symmetry-reduced situations are discussed in section 3.3.3. Some of the presented material is covered in Baumgarte and Shapiro (2010).

4.3.1. The quasi-isotropic gauge

Axisymmetry is a symmetry with respect to the Killing vector field ∂_φ, see section 3.5. In spherical symmetry isotropic coordinates can be used. Then the spatial metric is conformal to the flat one. Since this is not possible in axisymmetry the idea is to choose a gauge that models the isotropic gauge "as good as possible".

Definition 4.3.1. In spherical symmetry we call coordinates where the spatial part is conformal to the flat Euclidean space,

$$\mathrm{d}l^2 = \psi^4(r)\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) \text{ with } \mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2 \tag{4.9}$$

and conformal factor $\psi(t,r)$ (where the exponent is just chosen for convenience) **isotropic coordinates**. The quasi-isotropic gauge is defined by conditions for the components of the spatial metric

$$\gamma_{r\vartheta} = 0 = \gamma_{r\varphi},\ \partial_t\gamma_{r\vartheta} = 0 = \partial_t\gamma_{r\varphi}, \tag{4.10a}$$
$$(r^2\gamma_{rr} - \gamma_{\vartheta\vartheta})\gamma_{\varphi\varphi} + \gamma_{\vartheta\varphi}^2 = 0,\ \partial_t\left((r^2\gamma_{rr} - \gamma_{\vartheta\vartheta})\gamma_{\varphi\varphi} + \gamma_{\vartheta\varphi}^2\right) = 0. \tag{4.10b}$$

One can easily see that in spherical symmetry the quasi-isotropic gauge reduces to the use of isotropic coordinates. In hypersurface-orthogonal axisymmetry, so without twist and rotation, the component $\gamma_{\vartheta\varphi}$ vanishes. Then the aim of the gauge condition, namely to separate $\gamma_{\varphi\varphi}$, is achieved completely.

This is a widely used gauge in axisymmetric simulations, see Abrahams and Evans (1993), Garfinkle and Duncan (2001), Choptuik et al. (2003), Rinne (2008) and also analytically well studied, see Dain (2011) for a review.

Proposition 4.3.1 (Schell and Rinne (2015))**.** When using the expansion of the metric variables in spin-weighted spherical harmonics in hypersurface-orthogonal axisymmetry the quasi-isotropic gauge is unfortunate in the sense that it is only compatible with spherical symmetry and should therefore be abandoned.

Proof. The expansion of the relevant components of the spatial metric γ_{ij} is (compare section 2.2.5)

$$\gamma_{rr} = HY, \tag{4.11a}$$
$$\gamma_{\vartheta\vartheta} = r^2\left(K - \frac{\ell(\ell+1)}{2}G\right)Y - r^2\frac{\cos\vartheta}{\sin\vartheta}GY_\vartheta = r^2\left(KY + GY_{\vartheta\vartheta}\right), \tag{4.11b}$$

$$\gamma_{\varphi\varphi} = r^2 \sin^2\vartheta \left(K + \frac{\ell(\ell+1)}{2} G\right) Y + r^2 \cos\vartheta \sin\vartheta G Y_\vartheta$$
$$= r^2 \sin^2\vartheta \left(KY - GY_{\vartheta\vartheta}\right), \tag{4.11c}$$

where H, K and G are functions of t and r only and a sum over the mode number ℓ is implicitly implied. Applying the expansion of the metric coefficients (4.11) to the quasi-isotropic condition (4.10), one finds

$$r^2 \left(H - K + \frac{\ell(\ell+1)}{2} G\right) Y + r^2 \frac{\cos\vartheta}{\sin\vartheta} G Y_\vartheta = 0, \tag{4.12}$$

which implies $G = 0$ and hence $H = K$. Thus only one degree of freedom for the spatial metric remains. Therefore the only situation that is compatible with this choice is the one of spherical symmetry. □

The result can also be understood on a conceptual level. The gauge tells you to separate $\gamma_{\vartheta\vartheta}$ and $\gamma_{\varphi\varphi}$ but the spin-weighted harmonics combine those components. That sounds like a contradiction and is confirmed by the calculation above. In order to repair this issue we came up with a new gauge condition in Schell and Rinne (2015).

Proposition 4.3.2. If we keep the diagonal gauge as before but use as remaining condition

$$\gamma_{\vartheta\vartheta} = r^4 \sin^2\vartheta \; \gamma^{\varphi\varphi} (\gamma_{rr})^2 = r^4 \sin^2\vartheta \; \frac{(\gamma_{rr})^2}{\gamma_{\varphi\varphi}} \tag{4.13}$$

(and its preservation under time evolution) the situation is well-suited together with the expansion in spherical harmonics.

Proof. The proposed gauge contains a nonlinear condition. In order to apply the expansion in spherical harmonics we linearize condition (4.13) about a flat background,

$$\gamma_{\vartheta\vartheta} = 2r^2 \gamma_{rr} - \frac{\gamma_{\varphi\varphi}}{\sin^2\vartheta} \tag{4.14}$$

and hence, again by using the expansion (4.11),

$$2r^2 (K - H) Y = 0. \tag{4.15}$$

Therefore $H = K$ and G arbitrary are two remaining degrees of freedom, which shows that these conditions are indeed well suited. □

Therefore this new gauge condition sounds quite attractive for the numerical implementation. In the development of the code and also as guiding element for analytical considerations one highly benefits from the knowledge of an exact solution of the problem. We will derive it in section 4.6. In the derivation, more precisely in equations (4.71) below and also discussed there, one has to solve integrals. The solution of the integrals can be written down in closed form if there are some vanishing components of the shift. Since the exact solution was so essential for the project we decided to abandon this gauge choice as well even though we are convinced that it is suitable for a pseudo-spectral implementation of the situation.

For the remainder of the thesis we use the geodesic (or canonical) gauge, discussed in section 3.3.3. That implies that we take vanishing shift $\beta^i = 0$ and set the lapse $\alpha = 1$. Therefore there is no further freedom for putting conditions on the spatial metric.

4.3.2. Radial gauge and polar slicing

We mention a related gauge condition and polar slicing here, see for example standard textbooks on numerical relativity cited at the beginning of section 3.3. It was used in Choptuik (1993) in spherical symmetry where it implies also vanishing shift.

Definition 4.3.2. The **radial gauge** requires

$$\gamma_{r\vartheta} = 0 = \gamma_{r\varphi}, \;\; \partial_t \gamma_{r\vartheta} = 0 = \partial_t \gamma_{r\varphi}, \tag{4.16a}$$

$$\gamma_{\vartheta\vartheta}\gamma_{\varphi\varphi} - \gamma_{\vartheta\varphi}^2 = r^4 \sin^2\vartheta, \;\; \partial_t \left(\gamma_{\vartheta\vartheta}\gamma_{\varphi\varphi} - \gamma_{\vartheta\varphi}^2\right) = 0. \tag{4.16b}$$

Polar slicing is given as

$$K_\vartheta{}^\vartheta + K_\varphi{}^\varphi = 0. \tag{4.17}$$

Another reason to refer to it is that it would be interesting to explore its consequences for our general setting (expansion in spin-weighted spherical harmonics). The disadvantage is that the shift vector does not vanish for the radial gauge which presumably implies that there are problems to write down the exact solution in a closed form, see the discussions in section 4.6. The exact solution is of great value for the implementation and development of the code but in general not necessary of course. A very interesting observation concerning the polar slicing is that the given combination of extrinsic curvature components transforms as a scalar quantity concerning the spherical harmonics (spin-weight 0), see section 4.4. Its vanishing by gauge allows to remove one of the evolution equations that will turn out to be formally singular at the origin, see section 4.5. Of course we should also mention that the radial gauge seems to be unfortunate for the study of black hole spacetimes.

4.4. Choice of variables

The variables in our gauge consist of the non-vanishing components of the spatial metric γ_{ij} and the extrinsic curvature K_{ij}, the gauge quantities α and β^i are trivial ($\alpha = 1$ and $\beta^i = 0$). It is beneficial to arrange the variables in a clever way for the derivations of the equations which should be implemented later. There are two major demands the choice should match;

- the perturbation part of the variables should expand in an appropriate way in spherical harmonics as derived in sections 2.2.5 and 2.5, the eigenfunctions of the Laplacian operator for the quantity,
- close to the axis the variables should behave in a certain way, namely the leading order should be $\mathcal{O}(r^0)$ close to the axis, as derived in section 4.2.

The first item in the list of demands implies that the components should be combined in a specific way (namely such that their linearization expands in spherical harmonics), the second item determines the power of r in the definition.

Definition 4.4.1. For hypersurface-orthogonal axisymmetry (in particular the contributions in $\vartheta\varphi$ and $r\varphi$ vanish) we define our variables in the following way,

$$\gamma_{\mathrm{s1}} = \gamma_{rr}, \tag{4.18a}$$

$$\gamma_{\mathrm{s2}} = \frac{\sqrt{\gamma_{\vartheta\vartheta}\gamma_{\varphi\varphi}}}{r^2 \sin\vartheta}, \tag{4.18b}$$

$$\gamma_{\mathrm{v}} = \frac{\gamma_{r\vartheta}}{r}, \tag{4.18c}$$

$$\gamma_{\mathrm{t}} = \sin\vartheta \sqrt{\frac{\gamma_{\vartheta\vartheta}}{\gamma_{\varphi\varphi}}} = \sin\vartheta\sqrt{\gamma_{\vartheta\vartheta}\gamma^{\varphi\varphi}}, \tag{4.18d}$$

$$K_{\mathrm{s1}} = -\frac{1}{2}K_r{}^r, \tag{4.18e}$$

$$K_{\mathrm{s2}} = \frac{1}{2}\left(K_\vartheta{}^\vartheta + K_\varphi{}^\varphi\right), \tag{4.18f}$$

$$K_{\mathrm{v}} = r^{-1}K_\vartheta{}^r, \tag{4.18g}$$

$$K_{\mathrm{t}} = \frac{1}{2}\left(K_\vartheta{}^\vartheta - K_\varphi{}^\varphi\right). \tag{4.18h}$$

Proposition 4.4.1. All quantities in equation (4.18) are $\mathcal{O}(r^0)$ close to the origin $r = 0$. Their flat contribution is either 0 or 1. Further, as the labeling intuitively suggests, the variables are arranged such that the ones called "scalar" have a linearization that expands in scalar harmonics, and the corresponding statements for the "vector" and "tensor" labels. Maximal slicing corresponds to $K_{\mathrm{s}1} = K_{\mathrm{s}2}$. The transformations back is obvious or reads for the angular components of the spatial metric

$$\gamma_{\vartheta\vartheta} = r^2 \gamma_{\mathrm{s}2}\gamma_{\mathrm{t}}, \tag{4.19a}$$

$$\gamma_{\varphi\varphi} = r^2 \sin^2\vartheta \frac{\gamma_{\mathrm{s}2}}{\gamma_{\mathrm{t}}}. \tag{4.19b}$$

Proof. The $\mathcal{O}(r^0)$ behavior follows from the results in section 4.2. The flat metric (see appendix A.1.2) in spherical coordinates with basis $(\partial_r, \partial_\vartheta, \partial_\varphi)$ is given as $\eta_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\vartheta)$. Inserting that metric in equation (4.18) shows that the flat contribution of $\gamma_{\mathrm{s}1}$, $\gamma_{\mathrm{s}2}$ and γ_{t} is indeed 1, the rest vanishes. Linearization as discussed in section 3.2.3 (write all variables as $u = u_{\text{flat}} + \epsilon\tilde{u}$ with the linearization $\tilde{u}$ and keep just terms up to $\mathcal{O}(\epsilon^1)$) about flat background shows that the variables behave as

$$\gamma_{\mathrm{s}1} = 1 + \epsilon\tilde{\gamma}_{\mathrm{s}1} = 1 + \epsilon\tilde{\gamma}_{rr}, \tag{4.20a}$$

$$\gamma_{\mathrm{s}2} = 1 + \epsilon\tilde{\gamma}_{\mathrm{s}2} = 1 + \frac{\epsilon}{2}\left(\frac{\tilde{\gamma}_{\vartheta\vartheta}}{r^2} + \frac{\tilde{\gamma}_{\varphi\varphi}}{r^2\sin^2\vartheta}\right), \tag{4.20b}$$

$$\gamma_{\mathrm{v}} = 0 + \epsilon\tilde{\gamma}_{\mathrm{v}} = \frac{\epsilon}{r^2}\tilde{\gamma}_{r\vartheta}, \tag{4.20c}$$

$$\gamma_{\mathrm{t}} = 1 + \epsilon\tilde{\gamma}_{\mathrm{t}} = 1 + \frac{\epsilon}{2}\left(\frac{\tilde{\gamma}_{\vartheta\vartheta}}{r^2} - \frac{\tilde{\gamma}_{\varphi\varphi}}{r^2\sin^2\vartheta}\right), \tag{4.20d}$$

$$K_{\mathrm{s}1} = 0 + \epsilon\tilde{K}_{\mathrm{s}1} = \epsilon\tilde{K}_{rr}, \tag{4.20e}$$

$$K_{\mathrm{s}2} = 0 + \epsilon\tilde{K}_{\mathrm{s}2} = \frac{\epsilon}{2}\left(\frac{\tilde{K}_{\vartheta\vartheta}}{r^2} + \frac{\tilde{K}_{\varphi\varphi}}{r^2\sin^2\vartheta}\right), \tag{4.20f}$$

$$K_{\mathrm{v}} = 0 + \epsilon\tilde{K}_{\mathrm{v}} = \frac{\epsilon}{r}\tilde{K}_{\vartheta r}, \tag{4.20g}$$

$$K_{\mathrm{t}} = 0 + \epsilon\tilde{K}_{\mathrm{t}} = \frac{\epsilon}{2}\left(\frac{\tilde{K}_{\vartheta\vartheta}}{r^2} - \frac{\tilde{K}_{\varphi\varphi}}{r^2\sin^2\vartheta}\right). \tag{4.20h}$$

With the expansion as in section 2.2.5 the correct expansion in the claimed spherical harmonics follows. The trace of the extrinsic curvature reads

$$\mathrm{tr}K = K_r{}^r + K_\vartheta{}^\vartheta + K_\varphi{}^\varphi = 2\left(K_{\mathrm{s}2} - K_{\mathrm{s}1}\right) \tag{4.21}$$

and therefore the statement about maximal slicing is shown. The relations (4.19) follow by inserting them into the expressions in equation (4.18). □

For later convenience we calculate the determinant of the spatial metric in the defined variables.

Proposition 4.4.2. The determinant for the spatial metric is in our variables

$$\det\gamma = r^4\sin^2\vartheta\left(\gamma_{\mathrm{s1}}\gamma_{\mathrm{s2}}^2 - \gamma_{\mathrm{s2}}\gamma_{\mathrm{v}}^2\gamma_{\mathrm{t}}^{-1}\right) \tag{4.22}$$

and its linearization about flat spacetime

$$\det\tilde{\gamma} = r^4\sin^2\vartheta\left(\tilde{\gamma}_{\mathrm{s1}} + 2\tilde{\gamma}_{\mathrm{s2}}\right). \tag{4.23}$$

Proof. Straightforward calculation of the determinant of a 3×3-matrix with some vanishing entries leads to $\det\gamma = \gamma_{rr}\gamma_{\vartheta\vartheta}\gamma_{\varphi\varphi} - \gamma_{r\vartheta}^2\gamma_{\varphi\varphi}$ and therefore, by plugging in, to the nonlinear result. Linearization leads to the second formula. □

4.5. Formulation of Einstein's equations

4.5.1. Nonlinear equations

With the definition 4.4.1 of the nonlinear variables we can derive Einstein's equations in the Cauchy formulation (section 3.3) in these variables. A few remarks are in order concerning the momentum constraint. In the Cauchy formulation it reads

$$\mathcal{C}^i = D_j\left(K^{ij} - \gamma^{ij}\mathrm{tr}K\right) = 0. \qquad \text{(3.31b rev.)}$$

We will see later in section 4.7.2 that it is indeed possible for the original momentum constraint to be cast in a hyperbolic form. Unfortunately it is not hyperbolic for the variables we desire it to be solved for. Therefore we will derive a modification in theorem 4.7.3 which behaves exactly as we like it to do. In the following derivation we refer with the modified version of the momentum constraint to the relation (note the index μ)

$$\mathcal{C}^{i,\mu} := D_j\left(K^{ij} + (\mu-1)\gamma^{ij}\mathrm{tr}K\right) = \mu\gamma^{ij}D_j\mathrm{tr}K. \tag{4.24}$$

We will basically only consider the original version with $\mu = 0$ and the case $\mu = 2$,

$$\mathcal{C}^{i,\mu=2} = D_j\left(K^{ij} + \gamma^{ij}\mathrm{tr}K\right) = 2\gamma^{ij}D_j\mathrm{tr}K. \tag{4.25}$$

The meaning will become clear in section 4.7.2.

Theorem 4.5.1. The Cauchy formulation of hypersurface-orthogonal axisymmetric Einstein's equations for the variables in definition 4.4.1 with vanishing shift $\beta^i = 0$ and lapse $\alpha = 1$ (geodesic gauge, see section 3.3.3) results in four evolution equations for the components of the spatial metric

$$\begin{aligned}
\partial_t \gamma_{\mathrm{s1}} &= 4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ -2\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^{-1}\ \gamma_{\mathrm{t}}^{-1}\ \gamma_{\mathrm{v}}\ -4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^{-1}\ \gamma_{\mathrm{t}}^{-1}\ \gamma_{\mathrm{v}}^2 \\
&\quad -2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^{-1}\ \gamma_{\mathrm{t}}^{-1}\ \gamma_{\mathrm{v}}^2\ -2\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^{-1}\ \gamma_{\mathrm{t}}^{-1}\ \gamma_{\mathrm{v}}^2, &\quad (4.26a)\\
\partial_t \gamma_{\mathrm{s2}} &= -2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ -K_{\mathrm{v}}\ \gamma_{\mathrm{t}}^{-1}\ \gamma_{\mathrm{v}}, &\quad (4.26b)\\
\partial_t \gamma_{\mathrm{v}} &= -2\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ -2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{v}}\ -2\ K_{\mathrm{t}}\ \gamma_{\mathrm{v}}, &\quad (4.26c)\\
\partial_t \gamma_{\mathrm{t}} &= -2\ K_{\mathrm{t}}\ \gamma_{\mathrm{t}}\ -K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^{-1}\ \gamma_{\mathrm{v}}, &\quad (4.26d)
\end{aligned}$$

and four evolution equations for the extrinsic curvature components. We write the latter in the form (all components have structurally the same behavior, therefore a specific coefficient is skipped in the following equation)

$$\partial_t K = \frac{\mathcal{K}}{\kappa} \tag{4.27}$$

and list, because they are rather lengthy, the corresponding $\mathcal{K}$ in appendix A.2.1.

The prefactor is the same in all cases and reads

$$\kappa = r^{-4}\gamma_{\vartheta\vartheta}^2\left(\gamma_{rr}\gamma_{\vartheta\vartheta} - \gamma_{r\vartheta}^2\right)^2 = r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \left(\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ -\gamma_{\mathrm{v}}^2\right)^2. \tag{4.28}$$

We get two further (not independent) evolution equations for the trace of the extrinsic curvature,

$$\begin{aligned}
&\partial_t \mathrm{tr}K = 4\ K_{\mathrm{s1}}^2 + 2\ K_{\mathrm{s2}}^2 + 2\ K_{\mathrm{t}}^2 \\
&+ \frac{2\ K_{\mathrm{v}}\ \left[K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ + (2K_{\mathrm{s1}}\ + K_{\mathrm{s2}}\ + K_{\mathrm{t}})\ \gamma_{\mathrm{v}}\right]}{\gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}}
\end{aligned} \tag{4.29}$$

and the determinant of the spatial metric,

$$\partial_t \ln\sqrt{\det\gamma} = 2(K_{\mathrm{s1}} - K_{\mathrm{s2}}) \tag{4.30}$$

Further we get three nontrivial constraints[1]. In fact we are not interested in the original momentum constraint but in the modified versions according to theorem 4.7.3 with factor $\mu = 2$. The constraints and their modifications are listed in appendix A.3.

[1] The fourth one, the φ-component of the momentum constraint is identically satisfied due to the axisymmetry assumption.

Proof. Starting point are the equations which are listed in section 3.3 that need to be transformed to the variables in definition 4.4.1. The evolution equations for the spatial metric with vanishing shift are

$$\partial_t \gamma_{ij} = -2K_{ij}. \tag{4.31}$$

Using the definition 4.4.1 and the usual rules for the derivatives. For example we have

$$\partial_t \gamma_{s2} = \frac{1}{r^2 \sin^2 \vartheta} \left(\sqrt{\frac{\gamma_{\varphi\varphi}}{\gamma_{\vartheta\vartheta}}} \partial_t \gamma_{\vartheta\vartheta} + \sqrt{\frac{\gamma_{\vartheta\vartheta}}{\gamma_{\varphi\varphi}}} \partial_t \gamma_{\varphi\varphi} \right) \tag{4.32}$$

results in the listed evolution equations. For the source-free (vacuum) evolution equations for the extrinsic curvature with vanishing shift we calculate

$$\partial_t K_{ij} = \left(R_{ij} + 2K_{ik} K^k{}_j + \mathrm{tr}K \;\; K_{ij} \right). \tag{4.33}$$

All tensor components need to be written in the form of the variables used in definition 4.4.1. Therefore we have to express the purely covariant and contravariant components of the extrinsic curvature tensor in terms of the mixed components. For example

$$K^{rr} = \gamma^{rr} K_r{}^r + \gamma^{r\vartheta} K_\vartheta{}^r. \tag{4.34}$$

The components of the spatial metric have to be expressed in covariant form. For example we get

$$\gamma^{rr} = \frac{\gamma_{\vartheta\vartheta}}{\gamma_{rr}\gamma_{\vartheta\vartheta} - \gamma_{r\vartheta}^2}. \tag{4.35}$$

Taking these rules into account for all components and writing the resulting equations in the defined variables leads to the claimed equations. The evolution equation for $\mathrm{tr}K$ can be calculated as

$$\begin{aligned} \partial_t \mathrm{tr}K = K_{ij}K^{ij} &= (K_r{}^r)^2 + (K_\vartheta{}^\vartheta)^2 + (K_\varphi{}^\varphi)^2 \\ +2\frac{\gamma_{rr}}{\gamma_{\vartheta\vartheta}}(K_\vartheta{}^r)^2 &- 2\frac{\gamma_{r\vartheta}}{\gamma_{\vartheta\vartheta}} K_r{}^r K_\vartheta{}^r + 2\frac{\gamma_{r\vartheta}}{\gamma_{\vartheta\vartheta}} K_\vartheta{}^r K_\vartheta{}^\vartheta \end{aligned} \tag{4.36}$$

The Hamiltonian constraint (3.15)

$$\mathcal{H} = R + \mathrm{tr}K^2 - K_{ij}K^{ij} = 0 \tag{4.37}$$

needs to expressed in the defined variables as well. The **original** ($\mu = 0$) and the **modified** momentum constraint in equation (4.119) with $\mu = 2$ read (the upper sign for the original, the lower one for the modified constraint)

$$\mathcal{C}^{i,\mu} = D_j \left(K^{ij} \mp \gamma^{ij} \mathrm{tr}K \right) = \partial_i K^{ij} + \Gamma^i{}_{jk} K^{kj} + \Gamma^j{}_{jk} K^{ik} \mp \gamma^{ij} \partial_j \mathrm{tr}K = 0. \tag{4.38}$$

We have the spectral expansion in spherical harmonics in mind. It is beneficial to express the momentum constraint in the covariant version $\mathcal{C}_i^{\mu} = \gamma_{ij}\mathcal{C}^{j,\mu}$. We are free, since the constraint equations are supposed to vanish, to multiply them with some factor. For the equations claimed in theorem 4.5.1 we multiply

- the Hamiltonian constraint with
$$\frac{1}{2}r^{-4}\gamma_{\vartheta\vartheta}^2\left(\gamma_{rr}\gamma_{\vartheta\vartheta}-\gamma_{r\vartheta}^2\right)^2 = \frac{r^2}{2}\gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\left(\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ -\ \gamma_{\mathrm{v}}^2\right)^2, \tag{4.39}$$
- the r-component of the momentum constraint with
$$r^{-5}\gamma_{\vartheta\vartheta}^2\left(\gamma_{rr}\gamma_{\vartheta\vartheta}-\gamma_{r\vartheta}^2\right) = r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\left(\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ -\ \gamma_{\mathrm{v}}^2\right), \tag{4.40}$$
- and the ϑ-component of the momentum constraint with
$$-r^{-6}\gamma_{\vartheta\vartheta}^2\left(\gamma_{rr}\gamma_{\vartheta\vartheta}-\gamma_{r\vartheta}^2\right) = -\gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\left(\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ -\ \gamma_{\mathrm{v}}^2\right). \tag{4.41}$$

Note that the factor is basically the same as the prefactor in front of the time evolution of the extrinsic curvature in equation (4.28). The explicit expressions for the Hamiltonian constraint and both versions ($\mu = 0$ and $\mu = 2$) of the momentum constraint are explicitly listed in appendix A.3. □

4.5.2. Linear equations

The nonlinear equations are supposed to be solved as perturbations about the Minkowski spacetime. For many purposes it makes sense to consider the linear problem first. The equations are far shorter and hence comprehensible and easier to handle, both from the analytical and the numerical point of view. When developing the code it is reasonable to start with the easier model before going to the more complicated situation. Therefore we will derive the linearized version of the equations in theorem 4.5.1 in this section.

Theorem 4.5.2. The linearization about the flat Minkowski background of the equations in theorem 4.5.1 are given as follows. The evolution equations are

$$\partial_t\tilde{\gamma}_{\mathrm{s1}} = 4\ \tilde{K}_{\mathrm{s1}} \tag{4.42a}$$
$$\partial_t\tilde{\gamma}_{\mathrm{s2}} = -\ 2\ \tilde{K}_{\mathrm{s2}} \tag{4.42b}$$
$$\partial_t\tilde{\gamma}_{\mathrm{v}} = -\ 2\ \tilde{K}_{\mathrm{v}} \tag{4.42c}$$
$$\partial_t\tilde{\gamma}_{\mathrm{t}} = -\ 2\ \tilde{K}_{\mathrm{t}} \tag{4.42d}$$
$$\partial_t\tilde{K}_{\mathrm{s1}} = -\ \frac{\frac{\cos\vartheta}{\sin\vartheta}\ \tilde{\gamma}_{\mathrm{v}}}{2\ r^2} + \frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde{\gamma}_{\mathrm{s1}}}{4\ r^2} - \frac{\partial_\vartheta\tilde{\gamma}_{\mathrm{v}}}{2\ r^2}$$

$$
\begin{aligned}
&+\frac{\partial_\vartheta^2\tilde\gamma_{\rm s1}}{4\ r^2}-\frac{\partial_r\tilde\gamma_{\rm s1}}{2\ r}+\frac{\partial_r\tilde\gamma_{\rm s2}}{r}-\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_r\tilde\gamma_{\rm v}}{2\ r}\\
&-\frac{\partial_r\partial_\vartheta\tilde\gamma_{\rm v}}{2\ r}+\frac{1}{2}\ \partial_r^2\tilde\gamma_{\rm s2} \qquad (4.42e)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\tilde K_{\rm s2}=&\ \frac{\tilde\gamma_{\rm s1}}{r^2}-\frac{\tilde\gamma_{\rm s2}}{r^2}-\frac{\tilde\gamma_{\rm t}}{r^2}+\frac{3\ \frac{\cos\vartheta}{\sin\vartheta}\ \tilde\gamma_{\rm v}}{2\ r^2}\\
&-\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm s1}}{4\ r^2}-\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm s2}}{2\ r^2}+\frac{3\ \frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm t}}{2\ r^2}\\
&+\frac{3\ \partial_\vartheta\tilde\gamma_{\rm v}}{2\ r^2}-\frac{\partial_\vartheta^2\tilde\gamma_{\rm s1}}{4\ r^2}-\frac{\partial_\vartheta^2\tilde\gamma_{\rm s2}}{2\ r^2}+\frac{\partial_\vartheta^2\tilde\gamma_{\rm t}}{2\ r^2}\\
&+\frac{\partial_r\tilde\gamma_{\rm s1}}{2\ r}-\frac{2\ \partial_r\tilde\gamma_{\rm s2}}{r}+\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_r\tilde\gamma_{\rm v}}{2\ r}\\
&+\frac{\partial_r\partial_\vartheta\tilde\gamma_{\rm v}}{2\ r}-\frac{1}{2}\ \partial_r^2\tilde\gamma_{\rm s2} \qquad (4.42f)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\tilde K_{\rm v}=&\ -\frac{\tilde\gamma_{\rm v}}{r^2}+\frac{\partial_\vartheta\tilde\gamma_{\rm s1}}{2\ r^2}+\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_r\tilde\gamma_{\rm t}}{r}\\
&-\frac{\partial_r\partial_\vartheta\tilde\gamma_{\rm s2}}{2\ r}+\frac{\partial_r\partial_\vartheta\tilde\gamma_{\rm t}}{2\ r} \qquad (4.42g)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\tilde K_{\rm t}=&\ -\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \tilde\gamma_{\rm v}}{2\ r^2}+\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm s1}}{4\ r^2}+\frac{\partial_\vartheta\tilde\gamma_{\rm v}}{2\ r^2}\\
&-\frac{\partial_\vartheta^2\tilde\gamma_{\rm s1}}{4\ r^2}-\frac{\partial_r\tilde\gamma_{\rm t}}{r}-\frac{\frac{\cos\vartheta}{\sin\vartheta}\ \partial_r\tilde\gamma_{\rm v}}{2\ r}\\
&+\frac{\partial_r\partial_\vartheta\tilde\gamma_{\rm v}}{2\ r}-\frac{1}{2}\ \partial_r^2\tilde\gamma_{\rm t}, \qquad (4.42h)
\end{aligned}
$$

$$\partial_t{\rm tr}\tilde K=0, \qquad (4.42i)$$

$$\partial_t\det\tilde\gamma=0. \qquad (4.42j)$$

The linear Hamiltonian constraint reads

$$
\begin{aligned}
\tilde{\mathcal H}=&\ \tilde\gamma_{\rm s1}\ -\tilde\gamma_{\rm s2}\ -\tilde\gamma_{\rm t}\ +2\ \frac{\cos\vartheta}{\sin\vartheta}\ \tilde\gamma_{\rm v}\ -\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm s1}\\
&-\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm s2}\ +\frac{3}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \partial_\vartheta\tilde\gamma_{\rm t}\ +2\ \partial_\vartheta\tilde\gamma_{\rm v}\ -\frac{1}{2}\ \partial_\vartheta^2\tilde\gamma_{\rm s1}\\
&-\frac{1}{2}\ \partial_\vartheta^2\tilde\gamma_{\rm s2}\ +\frac{1}{2}\ \partial_\vartheta^2\tilde\gamma_{\rm t}\ +r\ \partial_r\tilde\gamma_{\rm s1}\ -3\ r\ \partial_r\tilde\gamma_{\rm s2}\\
&+r\ \frac{\cos\vartheta}{\sin\vartheta}\ \partial_r\tilde\gamma_{\rm v}\ +r\ \partial_r\partial_\vartheta\tilde\gamma_{\rm v}\ -r^2\ \partial_r^2\tilde\gamma_{\rm s2}=0, \qquad (4.43)
\end{aligned}
$$

the r-component of the original linear momentum constraint is

$$\tilde{\mathcal C}_r^{\mu=0}=-4\ \tilde K_{\rm s1}\ -2\ \tilde K_{\rm s2}\ +\frac{\cos\vartheta}{\sin\vartheta}\ \tilde K_{\rm v}$$

$$+\partial_\vartheta \tilde{K}_{\mathrm{v}} \; - 2\; r\; \partial_r \tilde{K}_{\mathrm{s2}} = 0 \tag{4.44}$$

and the ϑ-component gives

$$\begin{aligned}\tilde{\mathcal{C}}_\vartheta^{\mu=0} = 2\; &\frac{\cos\vartheta}{\sin\vartheta}\; \tilde{K}_{\mathrm{t}} \; + 3\; \tilde{K}_{\mathrm{v}} \; + 2\; \partial_\vartheta \tilde{K}_{\mathrm{s1}} \\ &- \partial_\vartheta \tilde{K}_{\mathrm{s2}} \; + \partial_\vartheta \tilde{K}_{\mathrm{t}} \; + r\; \partial_r \tilde{K}_{\mathrm{v}} = 0.\end{aligned} \tag{4.45}$$

The modified version (with $\mu = 2$) of the momentum constraint reads in the r-component

$$\begin{aligned}\tilde{\mathcal{C}}_r^{\mu=2} &= -4\; \tilde{K}_{\mathrm{s1}} \; - 2\; \tilde{K}_{\mathrm{s2}} \; + \frac{\cos\vartheta}{\sin\vartheta}\; \tilde{K}_{\mathrm{v}} \\ &+\partial_\vartheta \tilde{K}_{\mathrm{v}} \; - 4\; r\; \partial_r \tilde{K}_{\mathrm{s1}} \; + 2\; r\; \partial_r \tilde{K}_{\mathrm{s2}} = 0\end{aligned} \tag{4.46}$$

and for the ϑ-component

$$\begin{aligned}\tilde{\mathcal{C}}_\vartheta^{\mu=2} = 2\; &\frac{\cos\vartheta}{\sin\vartheta}\; \tilde{K}_{\mathrm{t}} \; + 3\; \tilde{K}_{\mathrm{v}} \; - 2\; \partial_\vartheta \tilde{K}_{\mathrm{s1}} \\ &+3\; \partial_\vartheta \tilde{K}_{\mathrm{s2}} \; + \partial_\vartheta \tilde{K}_{\mathrm{t}} \; + r\; \partial_r \tilde{K}_{\mathrm{v}} = 0.\end{aligned} \tag{4.47}$$

Proof. We simply have to linearize all variables about the flat background in the form $u = u_{\mathrm{flat}} + \epsilon \tilde{u}$ (with flat contributions as in proposition 4.4.1), insert those in the nonlinear equations of theorem 4.5.1 and ignore all nonlinear contributions $\mathcal{O}(\epsilon^{\geq 2})$. □

4.5.3. Equations on the mode level

One of the purposes of the definition of our variables in definition 4.4.1 was that we can expand them in eigenfunctions of the Laplace operator on the sphere (see section 2.5), thus in the corresponding spherical harmonics. We have seen that we can expand all linear variables in the following way,

$$\tilde{u}(t, r, \vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_\ell(t, r) \mathcal{Y}_\ell(\vartheta) \equiv \hat{u}_\ell \mathcal{Y}_\ell \tag{4.48}$$

where we cut the series after L contributions and $\mathcal{Y}_\ell$ are the spherical harmonics as given in section 2.2.5, so either the scalar, vector or tensor harmonics. We take the liberty to skip the dependencies on coordinates and the sum in most of the following expressions on the mode level. We hope that the equations become more readable but the meaning should be unambiguous.

Theorem 4.5.3. The expansion of Einstein's linear equations in the corresponding spherical harmonics lead to a set of decoupled (in the ℓ-modes) 1+1-dimensional equations for each ℓ-mode. Explicitly the evolution equations read

$$\partial_t\hat{\gamma}_{\mathrm{s1}} = 4\hat{K}_{\mathrm{s1}}, \tag{4.49a}$$

$$\partial_t\hat{\gamma}_{\mathrm{s2}} = -2\hat{K}_{\mathrm{s2}}, \tag{4.49b}$$

$$\partial_t\hat{\gamma}_{\mathrm{v}} = -2\hat{K}_{\mathrm{v}}, \tag{4.49c}$$

$$\partial_t\hat{\gamma}_{\mathrm{t}} = -2\hat{K}_{\mathrm{t}}, \tag{4.49d}$$

$$\begin{aligned}\partial_t\hat{K}_{\mathrm{s1}} = {}& \frac{1}{r^2}\left[-\left(\frac{\ell(\ell+1)}{4}\right)\hat{\gamma}_{\mathrm{s1}} + \frac{\ell(\ell+1)}{2}\hat{\gamma}_{\mathrm{v}}\right] \\ &+ \frac{1}{r}\left[-\frac{1}{2}\partial_r\hat{\gamma}_{\mathrm{s1}} + \partial_r\hat{\gamma}_{\mathrm{s2}} + \frac{\ell(\ell+1)}{2}\partial_r\hat{\gamma}_{\mathrm{v}}\right] + \frac{1}{2}\partial_r^2\hat{\gamma}_{\mathrm{s2}},\end{aligned} \tag{4.49e}$$

$$\begin{aligned}\partial_t\hat{K}_{\mathrm{s2}} = {}& \frac{1}{r^2}\left[\left(1+\frac{\ell(\ell+1)}{4}\right)\hat{\gamma}_{\mathrm{s1}} + \left(-1+\frac{\ell(\ell+1)}{2}\right)\hat{\gamma}_{\mathrm{s2}}\right. \\ &\left. -\frac{1}{2}\left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{\gamma}_{\mathrm{t}} - \frac{3\ell(\ell+1)}{2}\hat{\gamma}_{\mathrm{v}}\right] \\ &+ \frac{1}{r}\left[\frac{1}{2}\partial_r\hat{\gamma}_{\mathrm{s1}} - 2\partial_r\hat{\gamma}_{\mathrm{s2}} - \frac{\ell(\ell+1)}{2}\partial_r\hat{\gamma}_{\mathrm{v}}\right] - \frac{1}{2}\partial_r^2\hat{\gamma}_{\mathrm{s2}},\end{aligned} \tag{4.49f}$$

$$\begin{aligned}\partial_t\hat{K}_{\mathrm{v}} = {}& \frac{1}{r^2}\left[\frac{1}{2}\hat{\gamma}_{\mathrm{s1}} - \hat{\gamma}_{\mathrm{v}}\right] \\ &+ \frac{1}{r}\left[-\frac{1}{2}\partial_r\hat{\gamma}_{\mathrm{s2}} + \frac{1}{2}\left(1-\frac{\ell(\ell+1)}{2}\right)\partial_r\hat{\gamma}_{\mathrm{t}}\right]\end{aligned} \tag{4.49g}$$

$$\partial_t\hat{K}_{\mathrm{t}} = \frac{1}{r^2}\left[-\frac{1}{2}\hat{\gamma}_{\mathrm{s1}} + \hat{\gamma}_{\mathrm{v}}\right] + \frac{1}{r}\left[-\partial_r\hat{\gamma}_{\mathrm{t}} + \partial_r\hat{\gamma}_{\mathrm{v}}\right] - \frac{1}{2}\partial_r^2\hat{\gamma}_{\mathrm{t}}, \tag{4.49h}$$

$$\partial_t \mathrm{tr}\hat{K} = 0, \tag{4.49i}$$

$$\partial_t \det\hat{\gamma} = 0. \tag{4.49j}$$

The Hamiltonian constraint is

$$\begin{aligned}\hat{\mathcal{H}} = {}& \left(1+\frac{\ell(\ell+1)}{2}\right)\hat{\gamma}_{\mathrm{s1}} + \left(-1+\frac{\ell(\ell+1)}{2}\right)\hat{\gamma}_{\mathrm{s2}} \\ &-\frac{1}{2}\left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{\gamma}_{\mathrm{t}} - 2\ell(\ell+1)\hat{\gamma}_{\mathrm{v}} + r\partial_r\hat{\gamma}_{\mathrm{s1}} \\ &-3r\partial_r\hat{\gamma}_{\mathrm{s2}} - \ell(\ell+1)r\partial_r\hat{\gamma}_{\mathrm{v}} - r^2\partial_r^2\hat{\gamma}_{\mathrm{s2}} = 0\end{aligned} \tag{4.50}$$

and the original momentum constraint has the components in r

$$\hat{\mathcal{C}}_r^{\mu=0} = -4\,\hat{K}_{\mathrm{s1}} - 2\,\hat{K}_{\mathrm{s2}} - \ell(\ell+1)\,\hat{K}_{\mathrm{v}} - 2\,r\partial_r\hat{K}_{\mathrm{s2}} = 0 \tag{4.51}$$

and ϑ

$$\hat{\mathcal{C}}_{\vartheta}^{\mu=0} = 2\ \hat{K}_{\mathrm{s}1} + \left(1 - \frac{\ell(\ell+1)}{2}\right) \hat{K}_{\mathrm{t}} - \hat{K}_{\mathrm{s}2} + 3\ \hat{K}_{\mathrm{v}} + r\partial_r \hat{K}_{\mathrm{v}} = 0. \tag{4.52}$$

The modified momentum constraint (as always we choose $\mu = 2$) has the components in r

$$\hat{\mathcal{C}}_{r}^{\mu=2} = -4\ \hat{K}_{\mathrm{s}1} - 2\ \hat{K}_{\mathrm{s}2} - \ell(\ell+1)\ \hat{K}_{\mathrm{v}} - 4\ r\partial_r \hat{K}_{\mathrm{s}1} + 2\ r\partial_r \hat{K}_{\mathrm{s}2} = 0 \tag{4.53}$$

and ϑ

$$\hat{\mathcal{C}}_{\vartheta}^{\mu=2} = -2\ \hat{K}_{\mathrm{s}1} + \left(1 - \frac{\ell(\ell+1)}{2}\right) \hat{K}_{\mathrm{t}} + 3\ \hat{K}_{\mathrm{s}2} + 3\ \hat{K}_{\mathrm{v}} + r\partial_r \hat{K}_{\mathrm{v}} = 0. \tag{4.54}$$

Proof. One just has to perform the expansion in equation (4.48) for each variable, insert it in the linear equations in theorem 4.5.2 and one gets the claimed equations. The decoupling of each mode is obvious. □

4.6. Exact solution to the linear problem

For the actual numerical implementation it is very beneficial to have some analytical knowledge of an exact solution to the problem. It simplifies comparisons and demonstrations of convergence, allows to localize problems when making use of that knowledge and deals as a guide for the implementation. We will show that it is possible to solve the problem for the linearized perturbations analytically and to give the solution in closed form.

There are basically two major approaches to perturbation theory in general relativity. One, the **Regge Wheeler Zerilli formalism** is based on the analysis of perturbations of non-rotating spacetimes like the Minkowski and Schwarzschild solution. Perturbations in the metric coefficients are studied. The perturbations fall in two different classes or sectors, see Chandrasekhar (1983). Some induce a dragging of the inertial frame and are therefore connected with rotations. They are called **odd** (or **axial**). The other class impart no such rotations and the perturbations are called **even** (or **polar**). It was initiated for the odd sector in Regge and Wheeler (1957) and later continued for the even sector in Zerilli (1970).

A second and complementary approach is the **Teukolsky formalism** in Teukolsky (1973). Here perturbations in the Weyl and Maxwell scalars are studied in the Newman Penrose formalism. It is naturally applicable to rotating spacetimes but it is harder to extract the metric perturbations. It is suited for the study of so-called algebraically special spacetimes (including rotating black holes) but will not be considered further. Consult Chandrasekhar (1983) for a detailed monograph concerning these topics, see also Nagar and Rezzolla (2005).

In this section we concentrate on non-rotating perturbations of the Minkowski spacetime in the Regge Wheeler Zerilli formalism as were considered in Sarbach and Tiglio (2001). The purpose of the current section is to follow closely Rinne (2009) where the formalism was used to generalize the $\ell = 2$ solution by Teukolsky (1982)[2] to arbitrary higher ℓ and to apply the techniques developed there to the situation we consider in this thesis. See in that respect also the less-known paper Nakamura (1984) for an alternative derivation with general ℓ and m.

Since we assume axisymmetry and hypersurface-orthogonality we can concentrate on the even (or polar) sector. This is a "closed sector" in the sense that there is no coupling or mixing with the odd sector. Extensions to the odd sector are straightforwardly possible following Sarbach and Tiglio (2001), Rinne (2009) though.

This section is organized in a rather pragmatic way and retraces to some extent the actual computation as we implemented it with the help of computer algebra. The example $\ell = 2$ (essentially Teukolsky's solution in our choice of variables) is explicitly included.

We assume a background structure $\mathcal{M}^4 = \mathcal{M}^2 \times S^2$ of the Minkowski space with Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin^2 \vartheta)$. The general metric in "**TT gauge**"[3] with even perturbations (see also section 3.2.3) about the flat background in spherical polar coordinates reads

$$g = \eta + \tilde{g} \tag{4.55}$$

[2] Note the difference between the Teukolsky solution in Teukolsky (1982), which is a solution for the $\ell = 2$-mode of perturbations of Minkowski spacetime and the Teukolsky equation in Teukolsky (1973) which is an essential ingredient in the second approach to the perturbation theory in general relativity that was mentioned above.

[3] The **transverse-traceless** gauge is characterized by $\tilde{g}_{t\mu} = 0$ (transverse) and vanishing spatial trace $\mathrm{tr}\tilde{g} = \eta^{ij}\tilde{g}_{ij} = 0$ and has numerous applications in general relativity, see also footnote 10 on page 99.

with (compare Rinne (2009, equation (4)))

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & AY & rBY_\vartheta & r\sin\vartheta BY_\varphi \\ 0 & rBY_\vartheta & r^2\left(-\frac{1}{2}AY + CY_{\vartheta\vartheta}\right) & r^2\sin\vartheta CY_{\vartheta\varphi} \\ 0 & r\sin\vartheta BY_\varphi & r^2\sin\vartheta CY_{\vartheta\varphi} & r^2\sin^2\vartheta\left(-\frac{1}{2}AY - CY_{\vartheta\vartheta}\right) \end{pmatrix}. \tag{4.56}$$

Here we use a rather abstract notation for the spherical harmonics as introduced in section 2.2.5 and label with capital letters $A,\ B,\ C,\dots$ functions of t and r.

Regge Wheeler Zerilli scalar Φ

Definition 4.6.1. The **generalized Regge Wheeler Zerilli scalar** Φ is a solution of the “**master equation**” (equation (28) in Sarbach and Tiglio (2001) and equation (14) in Rinne (2009)), which is for the flat spacetime obtained by a wave equation for a scalar function $\psi := r^{-1}\Phi \Leftrightarrow \Phi = r\psi$.

The wave equation in spherical polar coordinates on the mode level for a scalar function ψ in t and r reads (see section 2.6)

$$\partial_t^2\psi - \partial_r^2\psi - 2r^{-1}\partial_r\psi + r^{-2}\ell(\ell+1)\psi = 0 \tag{4.57}$$

and with $\psi := r^{-1}\Phi$ we have the master equation

$$r^{-1}\left[\partial_t^2\Phi - \partial_r^2\Phi + r^{-2}\ell(\ell+1)\Phi\right] = 0. \tag{4.58}$$

It is easy to verify that for arbitrary $\ell \geq 2$ the Regge Wheeler Zerilli scalar can be built from a “generating function” $G_\pm \equiv G(r \pm t)$ which depends just on either $x = r - t$ (outgoing) or $x = r + t$ (ingoing) and takes the form

$$\Phi(t,r) = \sum_{j=0}^{\ell} c_j r^{j-\ell} G_\pm^{(j+2)}(x) \tag{4.59}$$

$$\text{for } c_j := \frac{(-2)^{j-1}(2\ell-j)!}{(\ell-j)!j!} \tag{4.60}$$

and $G_\pm^{(j)} = \partial_x^j G_\pm(x)$ and $x = r \pm t$. Of course the Regge-Wheeler-Zerilli scalar is not unique. Every superposition is again a solution which is clear because of the linearity of the equation.

Example 4.6.1. For $\ell = 2$ and general generating function $G(x)$ (with $x = r + t$ for the ingoing contribution and $x = r - t$ for the outgoing one) the Regge Wheeler Zerilli scalar (4.59) for even perturbations is

$$\Phi(t, r) = \frac{3}{r^2} G^{(2)}(x) - \frac{3}{r} G^{(3)}(x) + G^{(4)}(x). \tag{4.61}$$

In the remainder of the section we follow closely Rinne (2009, section 3.3). Since we use capital letters from the beginning of the alphabet for the angular coordinates on the sphere S^2 we use here $M,\ N, \ldots$ to denote t and r in $\mathcal{M}^2$.

Zerilli one-form From the Regge Wheeler Zerilli scalar we build the **Zerilli one-form** which is basically the corresponding component of the gradient multiplied with $\lambda := (\ell - 1)/(\ell + 2)$,

$$Z_M := \lambda \Delta_M \Phi \tag{4.62}$$

Example 4.6.2. For $\ell = 2$ we continue with example 4.6.1 and the components of the Zerilli one-form read

$$Z_t = -\frac{12}{r^2} G^{(3)}(x) + \frac{12}{r} G^{(4)}(x) - 4G^{(5)}(x), \tag{4.63a}$$

$$Z_r = -\frac{24}{r^3} G^{(2)}(x) + \frac{24}{r^2} G^{(3)}(x) - \frac{12}{r} G^{(4)}(x) + 4G^{(5)}(x). \tag{4.63b}$$

Gauge-invariant potential $K^{(\mathrm{inv})}$ The gauge-invariant potential (compare Sarbach and Tiglio (2001, section II B 1) for the introduction of the invariant amplitudes) is defined to be

$$K^{(\mathrm{inv})} = -\frac{\ell(\ell + 1)}{r} \Phi - \frac{2}{\lambda} Z_r. \tag{4.64}$$

Example 4.6.3. With the previous calculations the potential in our example is

$$K^{(\mathrm{inv})} = -\frac{6}{r^3} G^{(2)}(x) + \frac{6}{r^2} G^{(3)}(x) - 2G^{(5)}(x). \tag{4.65}$$

One-form built from Zerilli one-form and $K^{(\mathrm{inv})}$ In an intermediate step we define the one-form C_M built with the Zerilli one-form and the gradient of $K^{(\mathrm{inv})}$,

$$C_M := Z_M + r \nabla_M K^{(\mathrm{inv})}. \tag{4.66}$$

Example 4.6.4. With the calculations above the two components are

$$C_t = -\frac{6}{r^2}G^{(3)}(x) + \frac{6}{r}G^{(4)}(x) - 4G^{(5)}(x) + 2rG^{(6)}(x), \tag{4.67a}$$

$$C_r = -\frac{6}{r^3}G^{(2)}(x) + \frac{6}{r^2}G^{(3)}(x) - \frac{6}{r}G^{(4)}(x) + 4G^{(5)}(x) - 2rG^{(6)}(x). \tag{4.67b}$$

Gauge-invariant potentials $H^{(\mathrm{inv})}_{MN}$ The relation between the just defined one-forms C_M and the perturbations $H^{(\mathrm{inv})}_{MN}$ is

$$C_M = H^{(\mathrm{inv})}_{MN}\nabla^N r. \tag{4.68}$$

We require by gauge the trace of the perturbation to vanish, hence the tensor $H^{(\mathrm{inv})}_{MN}$ should be trace-free. Thus ${H_t}^{t(\mathrm{inv})} + {H_r}^{r(\mathrm{inv})} = -H^{(\mathrm{inv})}_{tt} + H^{(\mathrm{inv})}_{rr} = 0$ implies the relations

$$H^{(\mathrm{inv})}_{tt} = H^{(\mathrm{inv})}_{rr} = C_r, \tag{4.69a}$$

$$H^{(\mathrm{inv})}_{tr} = H^{(\mathrm{inv})}_{rt} = C_t. \tag{4.69b}$$

Gauge parameters p_M The perturbations H_{MN} are related with the gauge-invariant perturbations $H^{(\mathrm{inv})}_{MN}$ by the gauge parameters p_M as

$$H_{MN} = H^{(\mathrm{inv})}_{MN} + 2\nabla_{(M}p_{N)}. \tag{4.70}$$

The transverse gauge-condition $\tilde{g}_{0\mu} = 0$ implies $H_{tt} = H_{rr} = 0$.

Remark 4.6.1. We intend to write down the solution in closed form. To do so it is important that we choose the transverse gauge and hence (at least) $\beta_r = 0$ (which corresponds to the tr-component of the metric). Otherwise the following integrals cannot be expressed in closed form and the final expressions would contain integrals. In fact this is a major motivation for the application of the geodesic (or canonical) gauge, besides its frequent use in mathematical relativity and the fact that it is rather well understood mathematically.

The remaining relations can be expressed as

$$p_t = -\frac{1}{2}\int H^{(\mathrm{inv})}_{tt}\mathrm{d}t = -\frac{\ell(\ell+1)}{2r}\int \Phi\mathrm{d}t + \int \partial_r\Phi\mathrm{d}t + r\int \partial_r^2\Phi\mathrm{d}t, \tag{4.71a}$$

$$\begin{aligned} p_r = -\int (H^{(\mathrm{inv})}_{tt} + \partial_r p_t)\mathrm{d}t = \\ 2\Phi + 2r\partial_r\Phi - \frac{\ell(\ell+1)}{2r^2}\iint \Phi\mathrm{d}t\mathrm{d}t + \frac{\ell(\ell+1)}{2r}\iint \partial_r\Phi\mathrm{d}t\mathrm{d}t \\ -2\iint \partial_r^2\Phi\mathrm{d}t\mathrm{d}t - r\iint \partial_r^2\Phi\mathrm{d}t\mathrm{d}t. \end{aligned} \tag{4.71b}$$

Example 4.6.5. With the Regge Wheeler Zerilli scalar calculated in example 4.6.1 for $\ell = 2$ the gauge parameters are

$$p_t = \frac{3}{r^3}G^{(1)}(x) - \frac{3}{r^2}G^{(2)}(x) + \frac{3}{r}G^{(3)}(x) - 2G^{(4)}(x) + rG^{(5)}(x), \tag{4.72a}$$
$$p_r = \frac{9}{r^4}G^{(0)}(x) - \frac{9}{r^3}G^{(1)}(x) + \frac{3}{r}G^{(3)}(x) - 2G^{(4)}(x) + rG^{(5)}(x). \tag{4.72b}$$

Gauge quantities G, Q_M, $\mathfrak{K}$ and H_{MN} The TT-gauge implies $Q_t = 0$ and we already derived $H_{tt} = H_{rr} = 0$ above. The remaining gauge quantities are[4]

$$G = -\frac{2}{r^2}\int p_r \mathrm{d}t = \frac{\ell(\ell+1)}{r^3}\iint \Phi \mathrm{d}t\mathrm{d}t - \frac{2}{r^2}\iint \partial_r\Phi \mathrm{d}t\mathrm{d}t - 2\iint \partial_r^2\Phi \mathrm{d}t\mathrm{d}t, \tag{4.73a}$$
$$Q_r = p_r + \frac{r^2}{2}\partial_r G, \tag{4.73b}$$
$$\mathfrak{K} = K^{(\mathrm{inv})} + \frac{2}{r}p_r - \frac{\ell(\ell+1)}{2}G, \tag{4.73c}$$
$$H_{rr} = H_{rr}^{(\mathrm{inv})} + 2\nabla_r p_r. \tag{4.73d}$$

Example 4.6.6. For $\ell = 2$ the gauge quantities are

$$G = -\frac{6}{r^5}G^{(0)}(x) + \frac{6}{r^4}G^{(1)}(x) - \frac{6}{r^3}G^{(2)}(x) + \frac{4}{r^2}G^{(3)} - \frac{2}{r}G^{(4)}(x) + rG^{(5)}(x), \tag{4.74a}$$
$$Q_r = \frac{24}{r^4}G^{(0)}(x) - \frac{24}{r^3}G^{(1)}(x) + \frac{12}{r^2}G^{(2)}(x) - \frac{4}{r}G^{(3)}, \tag{4.74b}$$
$$\mathfrak{K} = \frac{36}{r^5}G^{(0)}(x) - \frac{36}{r^4}G^{(1)}(x) + \frac{12}{r^2}G^{(2)}(x), \tag{4.74c}$$
$$H_{rr} = -\frac{72}{r^5}G^{(0)}(x) + \frac{72}{r^4}G^{(1)}(x) - \frac{24}{r^2}G^{(2)}(x) \tag{4.74d}$$

Translation of metric perturbation to the used variables Combing the obtained results and the general even perturbation of the metric (4.56) for the TT-gauge results in the ℓ-modes of our variables as defined in definition 4.4.1,

$$\hat{\gamma}_{\mathrm{s}1} = H_{rr}, \;\; \hat{\gamma}_{\mathrm{s}2} = \mathfrak{K}, \tag{4.75a}$$
$$\hat{\gamma}_{\mathrm{t}} = \frac{Q_r}{r}, \;\; \hat{\gamma}_{\mathrm{t}} = G, \tag{4.75b}$$

[4] In difference to Sarbach and Tiglio (2001), Rinne (2009) we use $\mathfrak{K}$ instead of K for the gauge amplitude. We hope to avoid confusion because we use K to denote the extrinsic curvature. There should be no clash of notation for the invariant amplitude $K^{(\mathrm{inv})}$.

$$\hat{K}_{\rm s1} = \frac{\partial_t H_{rr}}{4}, \;\; \hat{K}_{\rm s2} = -\frac{\partial_t K}{2}, \tag{4.75c}$$
$$\hat{K}_{\rm v} = -\frac{\partial_t Q_r}{2r}, \;\; \hat{K}_{\rm t} = -\frac{\partial_t G}{2}. \tag{4.75d}$$

Note that we use explicitly the notation with hat for the modes.

Proposition 4.6.1. The examples can lead to a regular solution to Einstein's linearized equations in section 4.5.2.

Proof. The calculations in the examples result in a mode solution for $\ell = 2$ as

$$\hat{\gamma}_{\rm s1} = -\frac{72}{r^5}G^{(0)}(x) + \frac{72}{r^4}G^{(1)}(x) - \frac{24}{r^3}G^{(2)}(x), \tag{4.76a}$$
$$\hat{\gamma}_{\rm s2} = \frac{36}{r^5}G^{(0)}(x) - \frac{36}{r^4}G^{(1)}(x) + \frac{12}{r^3}G^{(2)}(x), \tag{4.76b}$$
$$\hat{\gamma}_{\rm t} = \frac{24}{r^5}G^{(0)}(x) - \frac{24}{r^4}G^{(1)}(x) + \frac{12}{r^3}G^{(2)}(x) - \frac{4}{r^2}G^{(3)}(x), \tag{4.76c}$$
$$\hat{\gamma}_{\rm t} = -\frac{6}{r^5}G^{(0)}(x) + \frac{6}{r^4}G^{(1)}(x) - \frac{6}{r^3}G^{(2)}(x) + \frac{4}{r^2}G^{(3)}(x) - \frac{2}{r}G^{(4)}(x), \tag{4.76d}$$
$$\hat{K}_{\rm s1} = \frac{18}{r^5}G^{(1)}(x) - \frac{18}{r^4}G^{(2)}(x) + \frac{6}{r^3}G^{(3)}(x), \tag{4.76e}$$
$$\hat{K}_{\rm s2} = \frac{18}{r^5}G^{(1)}(x) - \frac{18}{r^4}G^{(2)}(x) + \frac{6}{r^3}G^{(3)}(x), \tag{4.76f}$$
$$\hat{K}_{\rm v} = \frac{12}{r^5}G^{(1)}(x) - \frac{12}{r^4}G^{(2)}(x) + \frac{6}{r^3}G^{(3)}(x) - \frac{2}{r^2}G^{(4)}(x), \tag{4.76g}$$
$$\hat{K}_{\rm t} = -\frac{3}{r^5}G^{(1)}(x) + \frac{3}{r^4}G^{(2)}(x) - \frac{3}{r^3}G^{(3)}(x) + \frac{2}{r^2}G^{(4)}(x) - \frac{1}{r}G^{(5)}(x). \tag{4.76h}$$

Note that $\hat{K}_{\rm s1} = \hat{K}_{\rm s2}$ which reflects the vanishing of the trace of K in the linear situation. Clearly the variables are singular at the origin $r = 0$ (as well as the Regge-Wheeler-Zerilli scalar in example 4.6.1). For the linear problem at hand we apply the superposition principle. We consider a combination of in- and outgoing solution[5]. One easily confirms that the combination with opposite relative sign turns out to be regular (as can be shown using Taylor expansion), both for the scalar Φ and a mode of a variable, $\hat{u}_\ell$,

$$\Phi = \Phi^- - \Phi^+, \tag{4.77a}$$
$$\hat{u}_\ell = u_\ell^- - u_\ell^+. \tag{4.77b}$$

Inserting the solution into the equations in theorem 4.5.2 shows that it is indeed a solution.

[5] This also makes sense from the physical point of view: everything that is "going into the center" (which is a totally regular point of the spacetime) has to "go out" again.

The full linear 2+1-dimensional solution therefore is

$$\tilde{u} = \sum_{\ell} \hat{u}_\ell(t,r)\mathcal{Y}_\ell = \hat{u}_2(t,r)\mathcal{Y}_2 \tag{4.78}$$

with the multiplication with the corresponding spin-weighted spherical harmonics (which represents here either scalar, vector or tensor contribution, compare section 2.2.5). Extensions to all higher modes are possible in a straightforward manner. We recommend to use computer algebra (see footnote 4 on page 169 for our choice).

Having all modes the linear variables are obtained as

$$u(t,r,\vartheta) = \hat{u}_\ell\mathcal{Y}_\ell = \sum_{\ell=0}^{L-1} \hat{u}_\ell(t,r)\mathcal{Y}_\ell(\vartheta) \tag{4.79}$$

with spin-weighted spherical harmonics $\mathcal{Y}_\ell$ and those form a regular solution of the linear system. □

The technique to find the solution is only valid for $\ell \geq 2$. See again Sarbach and Tiglio (2001) for comments on the static and stationary modes $\ell = 0, 1$.

Even though the solution can be shown to be regular it contains formally singular terms. In particular for the implementation one benefits from a different representation close to the origin. Using Taylor expansion there shows that all variables behave like $\mathcal{O}(r^\ell)$ close to the origin. We arranged the variables in definition 4.4.1 accordingly. Further the knowledge of a manifestly regular representation close to the origin helps for the comparison between exact and numerical solution.

We will explicitly show the formula for the Teukolsky example ($\ell = 2$) including the Taylor expansion close to the origin. Because it is rather lengthy we postpone it to the appendix A.4. For growing ℓ the scheme outlined above is applicable in exactly the same way. The expressions become longer though and therefore they are not explicitly included in the thesis.

We see in appendix A.4 that the modes for $\ell = 2$ show a behavior of $\mathcal{O}(r^0) = \mathcal{O}(r^{\ell-2})$ close to the origin. With the insights from section 2.5 we conclude that the exact solution as derived here corresponds to a spin-2 (gravitational) contribution of the perturbation (there we discussed also spin-0 solutions going with $\mathcal{O}(r^\ell)$ that is encoded in the eigenfunctions of the Laplacian applied to a symmetric two-tensor, which is not represented by the derived solution).

4.7. Some analysis of Einstein's equations

For the analysis of the resulting equations we start with the least complicated situation, the 1+1-dimensional mode equations. This chapter should be seen highly connected with the numerical studies in chapter 5. The analytical basics are settled here, which are supposed to be confirmed later. When doing numerics, especially when developing your own code, it makes sense to start with the easiest model problem and then to explore more advanced options. We follow that route also in the current chapter for the analytical developments.

4.7.1. Analysis on the mode level

In principle we do not need the 1+1-dimensional analysis when the results for the 2+1-dimensional situation are known. Nevertheless there are some different statements since on the mode level there is no ϑ-dependence and some problems appear only in that direction. On the mode level the constraints are just ordinary differential equations. Hence the analysis is simpler for that situation. The analysis for the 1+1-dimensional evolution equations (derivatives in t and r) gives already some intuition for the 2+1-dimensional analysis in section 4.7.2.

The evolution equations

We want to implement the evolution equations as given in equation (4.49) in theorem 4.5.3 by using the method of lines as described in section 2.3.3. The essential part in that method is to integrate an inhomogeneous ordinary differential equation. The inhomogeneity (the right-hand side of equation (2.94)) is calculated with the quantities of the previous time step as discussed there. The evolution equations for the components of the spatial metric are rather simple in the sense that they are all of the form $\partial_t \hat{\gamma} \sim \hat{K}$ for some specific components of the spatial metric and the extrinsic curvature. In particular those equations are manifestly regular. Therefore the numerical implementation, which consists of simple assignments in the end, is expected to cause no numerical problems at all. The same is true for the evolution equation for the trace of the extrinsic curvature which is trivial on the linear level and hence also for the mode equations.

The situation is different for the equations for the components of the extrinsic curvature. Those equations contain a coordinate singularity. The equations are all of the form

$$\partial_t \hat{K} = \frac{f}{r^2} + \frac{g}{r} + h \tag{4.80}$$

where the quantities f, g and h can be read off in theorem 4.5.3. Even though Einstein's equations are regular (except if an event or apparent horizon forms which is a priori not the case in our studies) the equations become singular. The choice of coordinates is responsible for that feature, similarly as the wave equation becomes singular in non-Cartesian coordinates as seen in section 2.6. Also the exact solution of the linear system shows that all the variables are in fact regular, see section 4.6. It is obvious that one has to deal with that issue when implementing the equations on the computer. We discuss it further in section 5.3. Here we examine the mathematical structure of the continuum equations.

Analysis of the evolution equations Our aim is to use a fully (or at least partially) constrained scheme, see definition 2.3.6. Therefore some of the variables are obtained by the constraints and do not have to be updated explicitly by the evolution equations[6]. We will see that we obtain either the variables $\{\hat{\gamma}_{\mathrm{s1}}, \hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{v}}\}$ or $\{\hat{\gamma}_{\mathrm{s1}}, \hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{v}}\}$ from solving the constraints, the other updates should be taken from the evolution equations. We examine first that the evolution equation for $\hat{K}_{\mathrm{s1}}$ should better not be taken into account.

Proposition 4.7.1. The evolution equations for the set $\{\hat{\gamma}_{\mathrm{s2}}, \hat{\gamma}_{\mathrm{v}}, \hat{\gamma}_{\mathrm{t}}, \hat{K}_{\mathrm{s1}}, \hat{K}_{\mathrm{t}}\}$ form a weakly hyperbolic system which fails to be strongly hyperbolic in the standard reduction.

Proof. As discussed in section 2.3.2 we perform a first-order reduction for the analysis. Therefore we introduce two additional variables (and two constraints as subsidiary system), $\hat{W}_{\mathrm{s2}} := \partial_r \hat{\gamma}_{\mathrm{s2}}$ and $\hat{W}_{\mathrm{t}} := \partial_r \hat{\gamma}_{\mathrm{t}}$. We restrict to the principal part and analyze the system

$$\partial_t \hat{\gamma}_{\mathrm{s2}} = 0, \tag{4.81a}$$
$$\partial_t \hat{\gamma}_{\mathrm{v}} = 0, \tag{4.81b}$$
$$\partial_t \hat{\gamma}_{\mathrm{t}} = 0, \tag{4.81c}$$

[6]Nevertheless one can apply a free evolution and then, on the next time level, overwrite those variables explicitly with the solutions of the constraint solver. That might have advantages if one uses an iterative constraint solver for instance and the initial guess for that solver is then presumably closer to the solution as the possible guess from the previous time step. Usually the constraint solver is much more involved from the computational perspective (solving elliptic partial differential equations or, as in our case, stiff evolutionary equations). The solution to the evolution equations is obtained by an assignment which is numerically easier.

$$\partial_t \hat{K}_{\mathrm{s1}} = \frac{1}{2}\partial_r \hat{W}_{\mathrm{s2}}, \tag{4.81d}$$

$$\partial_t \hat{K}_{\mathrm{t}} = -\frac{1}{2}\partial_r \hat{W}_{\mathrm{t}}, \tag{4.81e}$$

$$\partial_t \hat{W}_{\mathrm{s2}} = 0, \tag{4.81f}$$

$$\partial_t \hat{W}_{\mathrm{t}} = -2\partial_r \hat{K}_{\mathrm{t}}. \tag{4.81g}$$

The system represented in a form such that definition 2.3.7 can be used with a coefficient matrix A and a vector $u = (\hat{\gamma}_{\mathrm{s2}}, \hat{\gamma}_{\mathrm{v}}, \hat{\gamma}_{\mathrm{t}}, \hat{K}_{\mathrm{s1}}, \hat{K}_{\mathrm{t}}, \hat{W}_{\mathrm{s2}}, \hat{W}_{\mathrm{t}})^\dagger$ is

$$\partial_t u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{pmatrix} \partial_r u. \tag{4.82}$$

The matrix A has the five times degenerate eigenvalue 0 and both 1 and -1 and is **not** diagonalizable. According to definition 2.3.7 this system is therefore only weakly hyperbolic but fails to be strongly hyperbolic in the given reduction. □

Thus one should not directly implement these equations. We want to obtain some positive statements though and try to understand where the problematic contribution comes from.

It is easy to show that the evolution equations for the subsystem $\{\hat{\gamma}_{\mathrm{t}}, \hat{K}_{\mathrm{t}}\}$ only or the subsystem $\{\hat{\gamma}_{\mathrm{s2}}, \hat{K}_{\mathrm{s2}}\}$ only are strongly (even strictly and symmetric) hyperbolic. Also adding $\hat{\gamma}_{\mathrm{v}}$ does not change the property of strong hyperbolicity. For the implementation we aim to formulate the set of evolution equations in a way that is in fact strongly hyperbolic. The following proposition is true on the 1+1-dimensional level but will not be generalizable to the 2+1-dimensional level.

Proposition 4.7.2. The evolution equations for the set $\{\hat{\gamma}_{\mathrm{s2}}, \hat{\gamma}_{\mathrm{v}}, \hat{\gamma}_{\mathrm{t}}, \hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{t}}, \hat{W}_{\mathrm{s2}}, \hat{W}_{\mathrm{t}}\}$ form a strongly (actually also symmetric but not strictly) hyperbolic system.

Proof. Again we perform a first-order reduction for the analysis. Therefore we introduce two additional variables (and two constraints as subsidiary system), $\hat{W}_{\mathrm{s2}} := \partial_r \hat{\gamma}_{\mathrm{s2}}$ and $\hat{W}_{\mathrm{t}} := \partial_r \hat{\gamma}_{\mathrm{t}}$. Restricting to the principal part the system reads

$$\partial_t \hat{\gamma}_{\mathrm{s2}} = 0, \tag{4.83a}$$

$$\partial_t \hat{\gamma}_{\mathrm{v}} = 0, \tag{4.83b}$$
$$\partial_t \hat{\gamma}_{\mathrm{t}} = 0, \tag{4.83c}$$
$$\partial_t \hat{K}_{\mathrm{s2}} = -\frac{1}{2}\partial_r \hat{W}_{\mathrm{s2}}, \tag{4.83d}$$
$$\partial_t \hat{K}_{\mathrm{t}} = -\frac{1}{2}\partial_r \hat{W}_{\mathrm{t}}, \tag{4.83e}$$
$$\partial_t \hat{W}_{\mathrm{s2}} = -2\partial_r \hat{K}_{\mathrm{s2}}, \tag{4.83f}$$
$$\partial_t \hat{W}_{\mathrm{t}} = -2\partial_r \hat{K}_{\mathrm{t}}. \tag{4.83g}$$

Again we rewrite it in a form such that definition 2.3.7 can be used for a coefficient matrix A and a vector $u = (\hat{\gamma}_{\mathrm{s2}}, \hat{\gamma}_{\mathrm{v}}, \hat{\gamma}_{\mathrm{t}}, \hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{t}}, \hat{W}_{\mathrm{s2}}, \hat{W}_{\mathrm{t}})^\dagger$. The equation reads

$$\partial_t u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{pmatrix} \partial_r u. \tag{4.84}$$

The coefficient matrix has the following eigenvalues and corresponding eigenvectors:

$$0 : (1,0,0,0,0,0,0)^\dagger, \tag{4.85a}$$
$$0 : (0,1,0,0,0,0,0)^\dagger, \tag{4.85b}$$
$$0 : (0,0,1,0,0,0,0)^\dagger, \tag{4.85c}$$
$$1 : (0,0,0,1,0,-2,0)^\dagger, \tag{4.85d}$$
$$1 : (0,0,0,0,1,0,-2)^\dagger, \tag{4.85e}$$
$$-1 : (0,0,0,1,0,2,0)^\dagger, \tag{4.85f}$$
$$-1 : (0,0,0,0,1,0,2)^\dagger, \tag{4.85g}$$

which form a complete set of vectors. As we recognize the matrix A has degenerate eigenvalue and therefore the system cannot be strictly hyperbolic. Consider the Hermitean matrix $H = \mathrm{diag}(1,1,1,4,4,1,1)$. We easily calculate

$$HA^\dagger = A^\dagger H \tag{4.86}$$

and therefore, according to definition 2.3.7, the system is symmetric hyperbolic. □

Remark 4.7.1. It is important to note that this scheme is unfortunately **not** generalizable to the 2+1-dimensional (linear) situation. There the ϑ-derivatives spoil the strong hyperbolicity of the evolutions as we will see in section 4.7.2 (just in r-direction the analysis remains valid). Therefore the well-posedness is an artifact of the use of the spectral expansion for the 1+1-dimensional level.

With regard to the remark 4.7.1 we do the analysis as well for a different set of variables even though there does not seem to be any particular need for that choice on the 1+1-dimensional level. We will motivate the choice in connection with theorem 4.7.2.

Theorem 4.7.1. Define the variables

$$\hat{\gamma}_\varphi := \hat{\gamma}_{\mathrm{s2}} - \hat{\gamma}_{\mathrm{t}}, \tag{4.87a}$$
$$\hat{K}_\varphi := \hat{K}_{\mathrm{s2}} - \hat{K}_{\mathrm{t}}. \tag{4.87b}$$

The evolution equations for the set $\{\hat{\gamma}_\varphi, \hat{\gamma}_{\mathrm{v}}, \hat{\gamma}_{\mathrm{t}}, \hat{K}_\varphi, \hat{K}_{\mathrm{t}}\}$ form a strongly (actually also symmetric but not strictly) hyperbolic system.

Proof. Again we perform a first-order reduction for the analysis. Therefore we introduce two additional variables (and two constraints as subsidiary system), $\hat{W}_\varphi := \partial_r \hat{\gamma}_\varphi$ and $\hat{W}_{\mathrm{t}} := \partial_r \hat{\gamma}_{\mathrm{t}}$. Concentrating on the principal part the system reads

$$\partial_t \hat{\gamma}_\varphi = 0, \tag{4.88a}$$
$$\partial_t \hat{\gamma}_{\mathrm{v}} = 0, \tag{4.88b}$$
$$\partial_t \hat{\gamma}_{\mathrm{t}} = 0, \tag{4.88c}$$
$$\partial_t \hat{K}_\varphi = -\frac{1}{2}\partial_r^2 \hat{\gamma}_\varphi, \tag{4.88d}$$
$$\partial_t \hat{K}_{\mathrm{t}} = -\frac{1}{2}\partial_r \hat{W}_{\mathrm{t}}, \tag{4.88e}$$
$$\partial_t \hat{W}_\varphi = -2\partial_r \hat{K}_\varphi, \tag{4.88f}$$
$$\partial_t \hat{W}_{\mathrm{t}} = -2\partial_r \hat{K}_{\mathrm{t}}. \tag{4.88g}$$

Here we recognize exactly the same structure as for the choice of variables in proposition 4.7.2, namely two wave equations. Therefore the remaining proof follows exactly the lines of the one for proposition 4.7.2. □

Outer boundary conditions for the evolution equations There are several choices for the outer boundary conditions of the evolution equation. We want to model an isolated system, which implies that all variables are asymptotically flat, see the discussion in section 3.3.2. In the ideal situation we would set the outer boundary R to $r \to \infty$ and do not need any condition there, compare the discussion at the end of section 2.6.4. We will cut the spatial domain instead and introduce an artificial boundary and therefore need to choose boundary conditions.

If boundary conditions are needed the probably easiest possibilities are Dirichlet conditions – either the homogeneous one by just setting the variable to zero there or, since we have the exact solution for the linear system, the inhomogeneous Dirichlet condition by setting the variable equal to the value of the exact solution. Since one is primarily interested in ingoing wave packages and the transition through the origin those choices are not that bad for some test runs. Nevertheless for a long term evolution one should do better, for instance apply the Bjørhus (1995) projection method (as discussed in section 2.6.4 for the wave equation) to our situation.

We will need boundary conditions for $\hat{K}_{\mathrm{t}}$ and the auxiliary variable $\hat{K}_\varphi$ (see theorem 4.7.1). Both sets of equations are wave-like and have the same structure in the r-direction. Therefore we will discuss them in the same manner, namely consider the first-order in time and second-order in space system

$$\partial_t \hat{\gamma} = -2\hat{K}, \tag{4.89a}$$

$$\partial_t \hat{K} = \frac{f}{r^2} + \frac{g}{r} - \frac{1}{2}\partial_r^2 \hat{\gamma} \tag{4.89b}$$

where we denote our variables collectively with $\hat{\gamma}$ and $\hat{K}$.

In an analogous way as in section 2.6.4 we define an auxiliary variable[7] $\hat{\xi} := \partial_r \hat{\gamma}$ and do the same steps in the derivation as before. As in section 2.6.4 we project on the eigenfunctions $v_\pm := \hat{K} \pm \hat{\xi}$. We leave the evolution equations for v_- as it is and replace at the outer boundary the one for v_+ by the condition of an outgoing wave. The result of the projection method is that we should keep the evolution equation for $\hat{\gamma}$ as it is (no special boundary condition) and at the outer boundary $r = R$ set (again denoted by $\hat{=}$)

$$\partial_t \hat{K} \mathrel{\hat{=}} -\frac{\hat{K}}{2R} + \frac{f}{2R^2} + \frac{g}{2R} - \frac{\partial_r^2 \hat{\gamma}}{4} - \frac{\partial_r \hat{K}}{2}. \tag{4.90}$$

[7] In section 2.6.4 we used another auxiliary variable for the time derivative of $\hat{\gamma}$ because we also reduced the second-order equation in time. We do not have to do the reduction in time because our formulation is already first-order in time.

The constraints

The Hamiltonian constraint is an equation only for components of the spatial metric and the momentum constraint forms a set of equations only for components of the extrinsic curvature. They decouple and hence we can discuss them separately. This remains true on the 2+1-dimensional level.

The momentum constraint On the 1+1-dimensional level the constraints (the whole system of equations is first-order in time and consequently the constraints have no time derivatives) are just ordinary differential equations in r. We will postpone the analysis essentially to the linear level in section 4.7.2. Here we will just give a few comments on numerical experiments with the equations on the mode level, in particular because we spent a considerable amount of time and effort on these experiments.

The momentum constraint (in the original and the modified version) are derived in section 4.5.3. We use the parameter $\mu = 2$ (see section 4.5.3) throughout. Since on the linear level the trace vanishes, the added inhomogeneity on the right-hand side of the momentum constraint vanishes though.

maximal slicing Maximal slicing (see definition 3.3.3) implies an equality of the two scalar components of the extrinsic curvature, $\hat{K}_{s1} = \hat{K}_{s2} \equiv \hat{K}_s$. The components of the momentum constraint on the mode level are then

$$\hat{\mathcal{C}}_r = -6\ \hat{K}_s - \ell(\ell+1)\ \hat{K}_v - 2\ r\partial_r \hat{K}_s = 0 \tag{4.91a}$$

$$\text{and } \hat{\mathcal{C}}_\vartheta = \hat{K}_s + \left(1 - \frac{\ell(\ell+1)}{2}\right) \hat{K}_t + 3\ \hat{K}_v + r\partial_r \hat{K}_v = 0. \tag{4.91b}$$

We will see in section 4.7.2 that solving the momentum constraint for $u^\dagger = (\hat{K}_s, \hat{K}_v)$ corresponds to elliptic partial differential equations on the 2+1-dimensional linear level. The expansion in spherical harmonics is basically just a numerical technique for solving the equations.

It might be interesting to remark that we were able to obtain positive numerical results on the mode level (and unfortunately only there) for the elliptic choice and maximal slicing. For the solution of the elliptic equation we used a Newton-Raphson solver, see for example Press et al. (2007). Since the equations (4.91a) and (4.91b) are coupled we used two different schemes. Both seemed to be working:

- We can decouple explicitly the two components and solve a second-order equation with the solver and an algebraic assignment. This procedure is valid on the linear level but the decoupling does not work in general on the nonlinear level.
- We can use an iterative scheme, so first the single equation (4.91a) is solved for $\hat{K}_{\mathrm{s}}$ with fixed $\hat{K}_{\mathrm{v}}$, then equation (4.91b) for $\hat{K}_{\mathrm{v}}$ with fixed $\hat{K}_{\mathrm{s}}$ and the process is iterated. Since one takes as "initial guess" the solution of the previous time step there is some hope that the guess is not that far away from the correct solution and the iterative solver converges. Remarkably that is exactly what we were able to observe on the 1+1-dimensional mode level. Since the convergence on the 2+1-dimensional level was unfortunately problematic we had to give up that attempt finally.

The second option for the momentum constraint that we will discuss for maximal slicing in section 4.7.2 leads on the 2+1-dimensional level to a system for the variables $\{\hat{K}_{\mathrm{v}}, \hat{K}_{\mathrm{t}}\}$.

We will see in section 4.7.2 that for this interpretation the momentum constraint is parabolic. It is evolutionary in the "time" coordinate ϑ on the linear level. After expansion in spherical harmonics the equations translate into assignments for the mode coefficients of the corresponding variables. The scalar component of the equation is solved for the vector component of the extrinsic curvature, the vector component of the equation for the tensor component of the extrinsic curvature. This sounds a little uncommon.In addition special care is needed for the lowest modes.

Having just assignments instead of differential equations to solve sounds attractive of course. Even though we do not really feel comfortable with these issues we have, just on the **linear** level, positive results (not just in 1+1 dimensions but also in 2+1 dimensions[8]). Nevertheless it is far from clear if the scheme is generalizable to the nonlinear level. In fact there does not seem to be a straightforward procedure to achieve it and therefore this approach was dropped by us as well.

We should add that it might be more than just a curiosity that this choice leads to some drastically simplified procedures, not just as a conceptual example of the freedom of underdetermined systems, but also with some potential applications.

[8] We should remark that on the linear level it is still possible to benefit from the mode equations on the 1+1-dimensional level and their decoupling. In some sense one just has to solve a huge set of equations instead of single equations, one for each mode, and then to transform between configuration and spectral space. In fact, if one is just interested in the linear level one could decide to work entirely in the spectral space. The situation is drastically simplified if one restricts to the linear level but many interesting effects and features are lost then.

Non-maximal slicing For this possibility the modification of the momentum constraint becomes important. The hyperbolic choice for the momentum constraint is the most interesting one for the remainder of the thesis. We explicitly deal with two scalar components of the extrinsic curvature tensor and we saw in the previous discussion that it is indeed important that we did not combine them by demanding maximal slicing. So we consider the momentum constraint as a coupled set for the variable $u^\dagger = (\hat{K}_{\mathrm{s1}}, \hat{K}_{\mathrm{v}})$ or $u^\dagger = (\hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{v}})$. On the mode level those equations reduce to a coupled set of stiff ordinary differential equations in the coordinate r. In principle they can be directly integrated. For the determination of initial values for this process and numerical results we refer to section 5.3.

Hamiltonian constraint The linear Hamiltonian constraint is an equation involving all components of the spatial metric and only them (no mixing with the extrinsic curvature). In section 4.5.3 we will analyze the equation for each component of the spatial metric and realize that several different choices exist. On the 1+1-dimensional level equation (4.50) is just a single ordinary differential equation. Depending on the choice of variable it is on the mode level either

1. a second-order ordinary differential equation for $\hat{\gamma}_{\mathrm{s2}}$ (elliptic in general) or
2. a first-order ordinary differential equation for $\hat{\gamma}_{\mathrm{s1}}$ (parabolic in general) or
3. a first-order ordinary differential equation for $\hat{\gamma}_{\mathrm{v}}$ (hyperbolic in general) or
4. an algebraic assignment for $\hat{\gamma}_{\mathrm{t}}$ (an ordinary differential equation in general).

The last point is remarkable. On the full linear level it will be an ordinary differential equation in the coordinate ϑ (of second order). Our numerical technique – expanding in a spectral basis (spherical harmonics) in the coordinate ϑ – reduces the actual equation that is to solve to a (set of decoupled, one for each mode) equation which is just an assignment at the end, hence numerically cheap. Nevertheless, similarly as the discussion for the momentum constraint as a parabolic system, there are some features which seem to be odd. For the modes $\ell = 0$ and 1 the component $\hat{\gamma}_{\mathrm{t}}$ drops out of the equation (4.50) so it cannot be solved for these components.

In the successful implementation of the system on the linear level (1+1 as well as 2+1 dimensions) that was mentioned above the algebraic choice for $\hat{\gamma}_{\mathrm{t}}$ was actually taken. In addition the first two modes were solved for $\hat{\gamma}_{\mathrm{s2}}$ as elliptic equations (using the equations on the mode level). Remarkably it produced convergent results even though we lack a fundamental understanding of it. Since we could not see an immediate way to generalize the attractive numerical technique to the nonlinear situation we decided to drop the choice as well.

The elliptic choice, i.e. consider the Hamiltonian constraint as an equation for $\hat{\gamma}_{s2}$, sounds quite intuitive (or at least common) from the point of view of a numerical relativist. It led to positive results with the Newton-Raphson solver on the linear 1+1-dimensional mode level and seemed also to be promising on the 2+1-dimensional linear level. Nevertheless we were not able to build, on the 2+1-dimensional level, a convergent solver for all the constraints. Most probably the non-convergence of the coupled system in the momentum constraint caused these problems. Also there is no proof of the constraint system to be well posed in our formulation. Therefore we decided to continue with the parabolic choice of the Hamiltonian constraint for $\hat{\gamma}_{s1}$ which corresponds to solve a first-order ordinary differential equation in the "time" coordinate r. It sounds more reasonable to solve a scalar equation (that expands in scalar spherical harmonics) for a component that expands as a scalar quantity as well. In particular the choice is in agreement with the suggestions in Rácz (2016a) as the parabolic-hyperbolic solver.

4.7.2. Analysis on the linear level

If we consider the constraint equations in 4.5.2 we recognize easily that the Hamiltonian constraint is an equation for all components of the spatial metric and the momentum constraint forms a coupled system involving all components of the extrinsic curvature. There is, in the constraints, no coupling between those sets of variables on the linear level. On the linear level the coupling only appears through the evolution equations. We will see in section 4.7.3 that this essentially does not change on the nonlinear level. Recall that the constraints as such form an underdetermined system, see the discussions in section 3.4. We start again with the examination of the evolution equations.

Evolution equations

We are aiming for a fully or partially constrained formulation of Einstein's equations which implies that the constraints or parts of them are to be solved on each time level and therefore give an update of, in our axisymmetric situation, up to three variables. In the discussion of the section we obtain promising statements for two sets of variables, both in the parabolic-hyperbolic formulation. Therefore, as we will see, we want to solve the constraints for either

$\{\tilde{\gamma}_{s1}, \tilde{K}_{s1}, \tilde{K}_v\}$ or $\{\tilde{\gamma}_{s1}, \tilde{K}_{s2}, \tilde{K}_v\}$. The remaining variables are supposed to be updated by the evolution equations.

In some sense it seems to be natural to expect the spacetime dynamics to be governed by the gravitational, spin-two-weighted perturbations, so the angular part of the variables, compare with section 2.5. These are encoded in our formulation by a scalar and the tensor contribution, namely the s2 and tensor components of the spatial metric and extrinsic curvature. Another argument for this choice is that, since we want to solve the Hamiltonian constraint for $\tilde{\gamma}_{s1}$, we would include both evolution equations for the canonical pair of variables. We will see another motivation for this set in the analysis below.

In the rest of the discussion of the evolution equations we will assume $r > 0$ and hence exclude the origin explicitly. From the numerical perspective it seems to be justified because we deal with the grid point at $r = 0$ in an algebraic manner. It implies that all statements below are valid everywhere except at the origin.

We can show that the evolution equations for the canonical pair $\{\tilde{\gamma}_{s2}, \tilde{K}_{s2}\}$ form a strongly hyperbolic (even symmetric and strictly) system. The principal part of the evolution equations for the pair $\{\tilde{\gamma}_{s1}, \tilde{K}_{s1}\}$ does not contain r-derivatives. Even though it has the character of a wave equation in the ϑ-direction the missing r-derivatives prohibit the continuous diagonalizability and the system is not strongly hyperbolic. The same is true for the pair $\{\tilde{\gamma}_t, \tilde{K}_t\}$ with interchanged roles of the coordinates. We show next that the coupling of more evolution equations for components of the extrinsic curvature tensor can be problematic.

Proposition 4.7.3. The coupled system of evolution equations for s1 and tensor components is weakly hyperbolic, but in the standard reduction not strongly hyperbolic. The same is true for the evolution equations for s2 and tensor components.

Proof. Consider the usual first-order reduction $\tilde{W}_t := \partial_r \tilde{\gamma}_t$ and $\tilde{V}_{s1} := r^{-1}\partial_\vartheta \tilde{\gamma}_{s1}$. The principal part for the vector $u^\dagger = (\tilde{K}_{s1}, \tilde{K}_t, \tilde{W}_t, \tilde{V}_{s1})$ can be written as

$$\partial_t u = A_r \partial_r u + A_\vartheta \partial_\vartheta u \tag{4.92}$$

with coefficient matrices

$$A_r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_\vartheta = -r^{-1} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}. \tag{4.93}$$

To show that the system is strongly hyperbolic we have to consider **all** linear combinations

$$\mathcal{P} = \omega_r A_r + \omega_\vartheta A_\vartheta \tag{4.94}$$

and show that all those matrices are diagonalizable. In other words if we can find **one** linear combination such that the matrix is not diagonalizable the system cannot be strongly hyperbolic. Choose $\omega_r = \omega_\vartheta = 1/\sqrt{2}$ (which guarantees that the normalization condition is satisfied). Then the matrix reads

$$\mathcal{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4}r^{-1} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{4}r^{-1} \\ 0 & -2 & 0 & 0 \\ 4r^{-1} & 0 & 0 & 0 \end{pmatrix}. \tag{4.95}$$

Therefore $\mathcal{P}$ is not diagonalizable even though the eigenvalues are real. Hence in the given standard reduction the system is only weakly but not strongly hyperbolic.

For the second system we perform again the usual first-order reduction $\tilde{V}_{\mathrm{s2}} := r^{-1}\partial_\vartheta \tilde{\gamma}_{\mathrm{s2}},\ \tilde{V}_{\mathrm{t}} := r^{-1}\partial_\vartheta \tilde{\gamma}_{\mathrm{t}}$ and $\tilde{W}_{\mathrm{s2}} := \partial_r \tilde{\gamma}_{\mathrm{s2}},\ \tilde{W}_{\mathrm{t}} := \partial_r \tilde{\gamma}_{\mathrm{t}}$. The principal part for the vector $u^\dagger = (\tilde{K}_{\mathrm{s2}}, \tilde{K}_{\mathrm{t}}, \tilde{V}_{\mathrm{s2}}, \tilde{V}_{\mathrm{t}}, \tilde{W}_{\mathrm{s2}}, \tilde{W}_{\mathrm{t}})$ can be written as

$$\partial_t u = A_r \partial_r u + A_\vartheta \partial_\vartheta u \tag{4.96}$$

with coefficient matrices

$$A_r = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_\vartheta = r^{-1} \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.97}$$

Similar as above we need to find **one** linear combination such that

$$\mathcal{P} = \omega_r A_r + \omega_\vartheta A_\vartheta \tag{4.98}$$

is not diagonalizable. Here $\omega_\vartheta = 1$ and $\omega_r = 0$ does the job. Therefore in the standard reduction the system is not strongly hyperbolic. □

Now we are after a complete set of remaining variables that form a strongly hyperbolic set of evolution equations. Even though some single pair of canonically conjugated variables might form as such a promising system, it need not be the case when coupled to another set. The art is to formulate it such that the coupled system has the desired properties.

Remark 4.7.2. Observe that the term that spoils strong hyperbolicity seems to be connected to the second ϑ-derivatives in the evolution equation of $\tilde{K}_{\mathrm{s2}}$. Similar terms are contained in the principal part of the Hamiltonian constraint (4.43). Repairing the issue by subtracting the Hamiltonian constraint from the evolution equation actually results in an elliptic system for the s2 terms which is not what we want for the evolution equations.

Hyperbolization of evolution system As we have seen there is no direct and straightforward choice of variables that guarantees a strongly hyperbolic evolution on the 2+1-dimensional level. A strongly hyperbolic evolution scheme is necessary in order to have a promising set of equations for the numerical evolution. We will demonstrate in the following a procedure that results in a strongly hyperbolic set of equations.

The key step is to transform the variables for the evolution to a different set of variables which is indeed strongly hyperbolic. The disadvantage is that the introduced variables do not have a definite expansion in spherical harmonics. That should not be any problem since we basically need the expansion in spherical harmonics for the calculation of the derivatives. For the evolution we have to add the corresponding terms together, regardless of their expansion behavior in spherical harmonics. We can do this at the collocation points in the configuration space. We have seen above that the offending term for the strongly hyperbolic evolution seems to be the ϑ-derivative of the scalar component which "comes with the wrong sign". When considering instead of the s2 components new variables which correspond to the $\varphi\varphi$-components we can repair that issue.

Theorem 4.7.2. We define the following variables (a combination of definite spin-weights)

$$\tilde{\gamma}_{\varphi} := \tilde{\gamma}_{\mathrm{s2}} - \tilde{\gamma}_{\mathrm{t}}, \tag{4.99a}$$

$$\tilde{K}_{\varphi} := \tilde{K}_{\mathrm{s2}} - \tilde{K}_{\mathrm{t}}. \tag{4.99b}$$

In these variables the evolution equations for $\{\tilde{\gamma}_{\mathrm{v}}, \tilde{\gamma}_{\varphi}, \tilde{\gamma}_{\mathrm{t}}, \tilde{K}_{\varphi}, \tilde{K}_{\mathrm{t}}\}$ form a strongly (even symmetric) hyperbolic system.

Proof. In the new variables the principal part of the evolution equations of interest are

$$\partial_t \tilde{\gamma}_{\mathrm{v}} = -2\tilde{K}_{\mathrm{v}}, \tag{4.100a}$$

$$\partial_t \tilde{\gamma}_{\varphi} = -2\tilde{K}_{\varphi}, \tag{4.100b}$$

$$\partial_t \tilde{\gamma}_{\mathrm{t}} = -2\tilde{K}_{\mathrm{t}}, \tag{4.100c}$$

$$\partial_t \tilde{K}_\varphi = -\frac{1}{2}\partial_r^2 \tilde{\gamma}_\varphi - \frac{1}{2} r^{-2} \partial_\vartheta^2 \tilde{\gamma}_\varphi, \tag{4.100d}$$
$$\partial_t \tilde{K}_{\mathrm{t}} = -\frac{1}{2}\partial_r^2 \tilde{\gamma}_{\mathrm{t}} + \frac{1}{2} r^{-1} \partial_r \partial_\vartheta \tilde{\gamma}_{\mathrm{v}}. \tag{4.100e}$$

As usual we perform a first-order reduction by introduction of auxiliary variables as

$$\tilde{W}_\varphi := \partial_r \tilde{\gamma}_\varphi, \tag{4.101a}$$
$$\tilde{W}_{\mathrm{t}} := \partial_r \tilde{\gamma}_{\mathrm{t}}, \tag{4.101b}$$
$$\tilde{V}_\varphi := r^{-1} \partial_\vartheta \tilde{\gamma}_\varphi, \tag{4.101c}$$
$$\tilde{V}_{\mathrm{v}} := r^{-1} \partial_\vartheta \tilde{\gamma}_{\mathrm{v}}. \tag{4.101d}$$

Then we can write the system for the vector $u = (\tilde{K}_\varphi, \tilde{K}_{\mathrm{t}}, \tilde{W}_\varphi, \tilde{W}_{\mathrm{t}}, \tilde{V}_\varphi, \tilde{V}_{\mathrm{v}})^\dagger$ in the form

$$\partial_t u = A_r \partial_r u + A_\vartheta \partial_\vartheta u \tag{4.102}$$

with coefficient matrices

$$A_r = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_\vartheta = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2}r^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2r^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.103}$$

We consider all linear combinations in the form

$$\mathcal{P} := \omega_r A_r + \omega_\vartheta A_\vartheta. \tag{4.104}$$

The eigenvalues and corresponding eigenvectors of $\mathcal{P}$ are

$$\lambda_0 = 0 : (0, 0, \omega_\vartheta, 0, -\omega_r)^\dagger, \tag{4.105a}$$
$$\lambda_1 = -\sqrt{\omega_r^2 + r^{-2}\omega_\vartheta^2} : \left(1, 0, \frac{2\omega_r}{\lambda_1}, 0, \frac{2\omega_\vartheta}{\lambda_1}\right)^\dagger, \tag{4.105b}$$
$$\lambda_2 = \sqrt{\omega_r^2 + r^{-2}\omega_\vartheta^2} : \left(1, 0, \frac{2\omega_r}{\lambda_2}, 0, \frac{2\omega_\vartheta}{\lambda_2}\right)^\dagger, \tag{4.105c}$$
$$\lambda_3 = \omega_r : (0, 1, 0, -2, 0)^\dagger, \tag{4.105d}$$
$$\lambda_4 = -\omega_r : (0, 1, 0, 2, 0)^\dagger, \tag{4.105e}$$
$$\lambda_5 = 0 : (0, 0, 0, 0, 0, 1)^\dagger. \tag{4.105f}$$

The eigenvectors form the columns of the eigenmatrix S and we can diagonalize the system as $S^{-1}\mathcal{P}S = \mathrm{diag}(0, \lambda_1, \lambda_2, \omega_r, -\omega_r, 0)$. Since the diagonalization is continously dependent on the parameters of the unit normal $\omega = (\omega_r, \omega_\vartheta)$ the system is strongly hyperbolic. On the other hand the eigenvalues are not necessarily distinct, consider for example $\omega_r = 1$ and $\omega_\vartheta = 0$. Consider further the Hermitian matrix

$$H = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{4.106}$$

wich is obviously Hermitian. By direct computation one can show that it is a symmetrizer by calculating

$$H\mathcal{P} = \mathcal{P}^\dagger H. \tag{4.107}$$

Therefore the system is symmetric hyperbolic. □

Outer boundary conditions for the evolution equations The discussion of the outer boundary conditions for the 2+1-dimensional simulations is essentially exactly the same as for the 1+1-dimensional equations in section 4.7.1. Except of homogeneous and inhomogeneous Dirichlet conditions we also implemented the Bjørhus projection method in Bjørhus (1995).

Since the derivation is exactly the same we just state the result. Again we denote by $\tilde{K}$ either $\tilde{K}_\mathrm{t}$ or $\tilde{K}_\varphi$ and correspondingly for $\tilde{\gamma}$. The outer boundary condition reads as in section 4.7.1

$$\partial_t \tilde{K} = -\frac{\tilde{K}}{2R} + \frac{f}{2R^2} + \frac{g}{2R} - \frac{\partial_r^2 \tilde{\gamma}}{4} - \frac{\partial_r \tilde{K}}{2}. \tag{4.108}$$

There are again no special boundary conditions for $\tilde{\gamma}$.

The constraints

We obtained in theorem 4.7.2 a set of strongly hyperbolic evolution equations. It is not obvious that one can extend the set to a larger one by including further variables (or new combinations of them) and to find a first-order reduction such that the system is still strongly hyperbolic. To update the remaining variables we can use the constraints instead in a partially or fully constrained scheme. In line with the discussions in section 2.3.2 the type of the constraints depends on the choice of variables we want to solve them for. It would be particularly good if we were able to obtain the variables K_{v} and K_{s1} since those evolution equations seem to be problematic.

The Hamiltonian constraint The linear Hamiltonian constraint reads

$$\begin{aligned}\tilde{\mathcal{H}} = \ &\tilde{\gamma}_{\mathrm{s1}} - \tilde{\gamma}_{\mathrm{s2}} - \tilde{\gamma}_{\mathrm{t}} + 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\tilde{\gamma}_{\mathrm{v}} - \frac{1}{2}\frac{\cos\vartheta}{\sin\vartheta}\,\partial_\vartheta\tilde{\gamma}_{\mathrm{s1}} \\ &-\frac{1}{2}\frac{\cos\vartheta}{\sin\vartheta}\,\partial_\vartheta\tilde{\gamma}_{\mathrm{s2}} + \frac{3}{2}\,\frac{\cos\vartheta}{\sin\vartheta}\,\partial_\vartheta\tilde{\gamma}_{\mathrm{t}} + 2\,\partial_\vartheta\tilde{\gamma}_{\mathrm{v}} - \frac{1}{2}\partial_\vartheta^2\tilde{\gamma}_{\mathrm{s1}} \\ &-\frac{1}{2}\partial_\vartheta^2\tilde{\gamma}_{\mathrm{s2}} + \frac{1}{2}\partial_\vartheta^2\tilde{\gamma}_{\mathrm{t}} + r\,\partial_r\tilde{\gamma}_{\mathrm{s1}} - 3\,r\,\partial_r\tilde{\gamma}_{\mathrm{s2}} \\ &+ r\,\frac{\cos\vartheta}{\sin\vartheta}\,\partial_r\tilde{\gamma}_{\mathrm{v}} + r\,\partial_r\partial_\vartheta\tilde{\gamma}_{\mathrm{v}} - r^2\,\partial_r^2\tilde{\gamma}_{\mathrm{s2}} = 0. \end{aligned} \qquad (4.43\ \text{rev.})$$

Proposition 4.7.4. Depending for which variable the Hamiltonian constraint is solved for, it has a different character, namely

- for $\tilde{\gamma}_{\mathrm{s2}}$ it is an elliptic equation,
- for $\tilde{\gamma}_{\mathrm{s1}}$ it is a parabolic equation,
- for $\tilde{\gamma}_{\mathrm{v}}$ it is a hyperbolic equation and
- for $\tilde{\gamma}_{\mathrm{t}}$ an ordinary differential equation.

Proof. The linear Hamiltonian constraint is a second-order equation in r and ϑ. We write the principal part as $A\partial_r^2 u + 2B\partial_r\partial_\vartheta u + C\partial_\vartheta^2 u$.

Considered as an equation for $\tilde{\gamma}_{\mathrm{s2}}$ the highest derivatives in ϑ and r come with the same sign (-), hence $A = -r$, $C = -1/2$ and $B = 0$ and the equation is a scalar elliptic equation.

On the other hand considered as an equation for $\tilde{\gamma}_{\mathrm{s1}}$, the highest derivative (second order) appears with a prefactor of 0 in ∂_r^2 ($\Leftrightarrow$ there is no $\partial_r^2\tilde{\gamma}_{\mathrm{s1}}$-term). Therefore $A = B = 0$ and $C = -1/2$ and it is a scalar parabolic equation for $\tilde{\gamma}_{\mathrm{s1}}$.

Taking it as an equation for $\tilde{\gamma}_{\mathrm{v}}$ the only second-order derivatives come with $\partial_r\partial_\vartheta$ and therefore $A = C = 0$ and $B = r$ and it is a scalar hyperbolic equation for $\tilde{\gamma}_{\mathrm{v}}$.

The last option is to consider it as an equation for $\tilde{\gamma}_{\mathrm{t}}$. We recognize that there are no r-derivatives of $\tilde{\gamma}_{\mathrm{t}}$ in equation (4.43). It is a second-order ordinary differential equation in the coordinate ϑ. □

Momentum constraint In our axisymmetric setting we have two non-vanishing components of the momentum constraint, in r and in ϑ.

Proposition 4.7.5. The original momentum constraint forms a symmetric hyperbolic system for the components K_{s2} and K_{v}. If we assume maximal slicing it appears naturally as an elliptic system for the scalar and vector component of the extrinsic curvature. At the origin the set degenerates to a parabolic one.

Proof. Consider the principal part of equations (4.44) and (4.45),

$$\partial_\vartheta \tilde{K}_{\mathrm{v}} - 2r\partial_r \tilde{K}_{\mathrm{s2}} = 0, \tag{4.109a}$$

$$-\partial_\vartheta \tilde{K}_{\mathrm{s2}} + r\partial_r \tilde{K}_{\mathrm{v}} = 0. \tag{4.109b}$$

For $u^\dagger = (K_{\mathrm{s2}}, K_{\mathrm{v}})$ we write the set in the form

$$A_r \partial_r u + A_\vartheta \partial_\vartheta u = 0 \tag{4.110}$$

with

$$A_r = \begin{pmatrix} -2r & 0 \\ 0 & r \end{pmatrix}, \; A_\vartheta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.111}$$

Therefore the quantity $D = B^2 - 4AC$ in definition 2.3.2 reads $D8r^2 > 0$ and the system is hyperbolic for all values of $r \neq 0$ (we freeze the variable coefficient r and consider the cases $r = 0$ and $r > 0$ separately) and parabolic at the origin. Written in the form

$$\partial_r u = B\partial_\vartheta u = \begin{pmatrix} 0 & \frac{1}{2}r^{-1} \\ r^{-1} & 0 \end{pmatrix} \partial_\vartheta u \tag{4.112}$$

the matrix B has the following eigenvalues and corresponding eigenvectors,

$$-r^{-1}\frac{1}{\sqrt{2}} : \; (1, -\sqrt{2})^\dagger, \tag{4.113a}$$

$$r^{-1}\frac{1}{\sqrt{2}} : \; (1, \sqrt{2})^\dagger. \tag{4.113b}$$

We can consider the case $r = 0$ separately and in the following $r \neq 0$. The eigenvalues are real and distinct and therefore the system is strictly hyperbolic. Consider $H = \mathrm{diag}(2, 1)$, then one can calculate

$$HB = B^\dagger H \tag{4.114}$$

and hence H is a symmetrizer and the system is also shown to be symmetric hyperbolic.

In the case of maximal slicing by definition $\mathrm{tr}\tilde{K} = 0$ and it implies (as seen in proposition 4.4.1) $\tilde{K}_{\mathrm{s}1} = \tilde{K}_{\mathrm{s}2} \equiv \tilde{K}_{\mathrm{s}}$. The principal part of the momentum constraint is

$$\partial_\vartheta \tilde{K}_{\mathrm{v}} - 2r\tilde{K}_{\mathrm{s}} = 0, \tag{4.115a}$$

$$\partial_\vartheta \tilde{K}_{\mathrm{s}} + r\partial_r \tilde{K}_{\mathrm{v}}. \tag{4.115b}$$

The matrices in the representation (4.111) are now

$$A_r = \begin{pmatrix} -2r & 0 \\ 0 & r \end{pmatrix}, \; A_\vartheta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.116}$$

Hence $D = -8r^2 < 0$ and according to definition 2.3.2 (consider the cases $r = 0$ and $r > 0$ separately) the system is elliptic for $r > 0$ and parabolic at the origin. □

We show a lemma before starting with the next part of the analysis.

Lemma 4.7.1. For the component ${K_r}^\vartheta$ (observe that we want to use the "other order" of indices ${K_\vartheta}^r$ in definition 4.4.1 for K_{v}) we have the relation

$${K_r}^\vartheta = \frac{\gamma^{r\vartheta}\gamma_{rr}{K_r}^r + \gamma^{\vartheta\vartheta}\gamma_{rr}{K_\vartheta}^r + \gamma^{\vartheta\vartheta}\gamma_{r\vartheta}{K_\vartheta}^\vartheta}{1 - \gamma_{r\vartheta}\gamma^{\vartheta r}}. \tag{4.117}$$

If we are just interested in the principal part (for K-components) and linearize about the flat solution it simplifies to ${\tilde{K}_r}{}^\vartheta = r^{-2}{\tilde{K}_\vartheta}{}^r$.

Proof. It is essentially a straightforward calculation

$${K_r}^\vartheta = \gamma^{\vartheta i}\gamma_{rj}{K_i}^j = \gamma^{r\vartheta}\gamma_{rr}{K_r}^r + \gamma^{\vartheta\vartheta}\gamma_{rr}{K_\vartheta}^r + \gamma^{\vartheta\vartheta}\gamma_{r\vartheta}{K_\vartheta}^\vartheta + \gamma^{\vartheta r}\gamma_{r\vartheta}{K_r}^\vartheta \tag{4.118}$$

which leads to the result. Inserting the flat contribution for the γ-quantities gives the further result. □

According to the analysis of the evolution equations above we would like to solve the momentum constraint for the components $K_{\mathrm{s}1}$ and K_{v} though because $K_{\mathrm{s}2}$ might be obtained by the evolution equations while that looks troublesome for $K_{\mathrm{s}1}$.

To do so we add multiples of the gradient of the trace of the extrinsic curvature to both sides of the momentum constraint. On the one side it will change the character of the equation for some variables, on the other side it will be considered as an inhomogeneity. We included the modification for the derivation of the equations in theorem 4.5.2 and provide the motivation for it now. In addition we derived an evolution equation for $\mathrm{tr}\tilde{K}$ which allows us to update its value by the evolution. Since, as we will see, that equation does not contribute to the principal part, it does not spoil the analysis for the evolution equations in theorem 4.7.2, but we will benefit from it in the momentum constraint. The dealing with the already determined right-hand sight of the equation as source term reminds of the driver gauge conditions, see Balakrishna et al. (1996) and section 3.3.3.

For the next important theorem we start in fact with the more general nonlinear constraint and use for the analysis a linearization about the known flat solution. Therefore the result is only locally valid for the nonlinear setting but globally for the linear level.

Theorem 4.7.3. Consider for a real number $\mu \in \mathbb{R}$ the **modified momentum constraint**

$$\mathcal{C}_i^{\mu} := \gamma_{ij} D_k (K^{jk} - \gamma^{jk}\mathrm{tr}K) + \mu\gamma_{ij} D_k \left(\gamma^{jk}\mathrm{tr}K\right) = \mu\gamma_{ij} D_k (\gamma^{jk}\mathrm{tr}K). \tag{4.119}$$

On the left-hand side the additional trace-term will be used to modify the momentum constraint and its character. On the right-hand side it will be calculated as a source term where the trace is updated by some evolution equation. Therefore the source will not contribute to the principal part.

Depending on the value of μ the character of the system changes. At the origin the system degenerates to a parabolic one. Considered as a system for $\{K_{\mathrm{s1}}, K_{\mathrm{v}}\}$ it is parabolic for $\mu \in \{0, 1\}$, elliptic for $\mu \in (0, 1)$ and hyperbolic elsewhere. Considered as a system for $\{K_{\mathrm{s2}}, K_{\mathrm{v}}\}$ it is parabolic for $\mu = 1$ and hyperbolic for all other choices. The term on the right-hand side is just μ times the gradient of the trace.

Proof. We again calculate the principal part of the modified momentum constraint (with the right-hand side of equation (4.119) as inhomogeneity $\Rightarrow$ no contribution to the principal part) and linearize about the flat solution,

$$\begin{aligned} &\gamma_{ij} D_k (K^{jk} - \gamma^{jk}\mathrm{tr}K) + \mu\gamma_{ij} D_k \left(\gamma^{jk}\mathrm{tr}K\right) \\ &\overset{\mathrm{pp}}{=} \partial_k {K_i}^k + (\mu - 1)\gamma_{ij}\gamma^{jk}\partial_k \mathrm{tr}K \overset{\mathrm{lin}}{=} \partial_k {K_i}^k + (\mu - 1)\partial_i \mathrm{tr}K. \end{aligned} \tag{4.120}$$

In components we have (making use of lemma 4.7.1 and multiplication of the first component with r)

$$\text{in } r : r\partial_r {K_r}^r + r^{-1}\partial_\vartheta {K_r}^\vartheta + (\mu - 1) r\partial_r {K_r}^r + (\mu - 1) r\partial_r ({K_\vartheta}^\vartheta + {K_\varphi}^\varphi)$$

$$\begin{aligned}
&\stackrel{\text{pp}}{=} -2\mu r\partial_r K_{\text{s1}} + 2(\mu-1)r\partial_r K_{\text{s2}} + \partial_\vartheta K_{\text{v}} && (4.121\text{a})\\
\text{in } \vartheta : {}& \partial_r K_\vartheta{}^r + \partial_\vartheta K_\vartheta{}^\vartheta + (\mu-1)\partial_\vartheta K_r{}^r + (\mu-1)\partial_\vartheta(K_\vartheta{}^\vartheta + K_\varphi{}^\varphi)\\
&\stackrel{\text{pp}}{=} r\partial_r K_{\text{v}} + (2\mu-1)\partial_\vartheta K_{\text{s2}} - 2(\mu-1)\partial_\vartheta K_{\text{s1}}. && (4.121\text{b})
\end{aligned}$$

We write the system again in the form

$$A_r\partial_r u + A_\vartheta\partial_\vartheta u = 0. \tag{4.122}$$

For $u = (K_{\text{s1}}, K_{\text{v}})^\dagger$ the matrices read

$$A_r = \begin{pmatrix} -2\mu r & 0 \\ 0 & r \end{pmatrix}, \; A_\vartheta = \begin{pmatrix} 0 & 1 \\ -2(\mu-1) & 0 \end{pmatrix}. \tag{4.123}$$

Therefore the quantity D in the classification 2.3.2 reads $D = 16\mu(\mu-1)r^2$. We can again discuss the cases $r = 0$ and $r > 0$ separately. Then all the single cases considered in the theorem immediately follow.

Now we consider the modified momentum constraint as system for $u = (K_{\text{s2}}, K_{\text{v}})^\dagger$. The corresponding matrices now read

$$A_r = \begin{pmatrix} 2(\mu-1)r & 0 \\ 0 & r \end{pmatrix}, \; A_\vartheta = \begin{pmatrix} 0 & 1 \\ 2(\mu-1) & 0 \end{pmatrix}. \tag{4.124}$$

Therefore we have $D = 16(\mu^2 - 2\mu + 1)r^2$ which is positive except for the mentioned cases.

Since the connection should be compatible with the metric and the trace $\text{tr}K$ is a scalar the covariant derivative is equal to the ordinary one we have for the right-hand side of the modified momentum constraint

$$\mu\gamma_{ij}D_k(\gamma^{jk}\text{tr}K) = \mu\gamma_{ij}\gamma^{jk}D_k\text{tr}K = \mu\gamma_i^k\partial_k\text{tr}K = \mu\partial_i\text{tr}K. \tag{4.125}$$

□

Therefore we are able to manipulate the momentum constraint such that it is a hyperbolic system for a proper choice of the parameter μ. Throughout we will consider the case $\mu = 2$. The modified momentum constraint is hence hyperbolic for the desired variables K_{s1} and K_{v} except at the origin.

Summary

According to the theorem 4.7.2 we should aim for the following scheme for the update from time level $n \mapsto n+1$:

- Start with the given set of variables $\left\{\tilde{\gamma}_{\mathrm{s1}}, \tilde{\gamma}_{\mathrm{s2}}, \tilde{\gamma}_{\mathrm{v}}, \tilde{\gamma}_{\mathrm{t}}, \tilde{K}_{\mathrm{s1}}, \tilde{K}_{\mathrm{s2}}, \tilde{K}_{\mathrm{v}}, \tilde{K}_{\mathrm{t}}\right\}$ at time n.
- Transform to the auxiliary variables $\tilde{\gamma}_{\varphi}$ and $\tilde{K}_{\varphi}$ as defined in equation (4.100).
- Apply the evolution equations for $\left\{\tilde{\gamma}_{\varphi}, \tilde{\gamma}_{\mathrm{v}}, \tilde{\gamma}_{\mathrm{t}}, \tilde{K}_{\varphi}, \tilde{K}_{\mathrm{t}}\right\}$ to update those variables from $n \mapsto n+1$.
- Transform back to $\tilde{\gamma}_{\mathrm{s2}}$ and $\tilde{K}_{\mathrm{s2}}$.
- Apply the constraint solver with the already updated variables at $n+1$ to obtain the quantities $\left\{\tilde{\gamma}_{\mathrm{s1}}, \tilde{K}_{\mathrm{s1}}, \tilde{K}_{\mathrm{v}}\right\}$ at $n+1$.
- Start again to evolve from $n+1 \mapsto n+2$.

Final system We showed above that the evolution equations admit a formulation that is strongly hyperbolic. In addition the constraints form a well-posed hyperbolic-parabolic system. Therefore the suggested constrained scheme looks very promising for the numerical implementation.

Nevertheless we will see in section 5.3 that the numerical integration of the parabolic Hamiltonian constraint is troublesome. If the freely specifiable data for the constraints are taken from the exact linear solution the constraint solver converges. Since the real data from the evolution which form the source terms for the constraints are obtained numerically they have some inaccuracies. Even though those deviations converge away in the continuum limit they are responsible for some spurious oscillations in the solution of the stiff parabolic equation using a direct solver as discussed in section 5.3.

Supplementary observations In the development of the code we benefited from several further observations. For completeness we list them here as well.

Lemma 4.7.2. On the linear level the spatial slices are in fact maximal for all times on the analytical level. This is equivalent to $\tilde{K}_{\mathrm{s1}} = \tilde{K}_{\mathrm{s2}}$.

Proof. Because of the definition of the variables and its consequences, see proposition 4.4.1, $\mathrm{tr}\tilde{K} = 2(\tilde{K}_{s2} - \tilde{K}_{s1})$. The evolution equation for the trace in theorem 4.5.2 implies $\mathrm{tr}\tilde{K} = \text{const.}$ under time evolution. Since initially the exact solution sets that constant to zero the trace vanishes for all times. □

Lemma 4.7.3. On the linear level the relation $\tilde{\gamma}_{s1} + 2\tilde{\gamma}_{s2} = 0$ is always satisfied.

Proof. The evolution equation involving the determinant of the metric in theorem 4.5.2 reads $\partial_t \ln \sqrt{\det \tilde{\gamma}} = -\mathrm{tr}\tilde{K} = 0$ according to lemma 4.7.2. By the chain rule we have $\det \tilde{\gamma} = \text{const.}$ in time. Since the constant is initially zero from the exact solution it is conserved in the evolution and therefore vanishes for all times. The linearization of the determinant was calculated in proposition 4.4.2 to be $\det \tilde{\gamma} = r^4 \sin^2 \vartheta \, (\tilde{\gamma}_{s1} + 2\tilde{\gamma}_{s2})$. □

Remark 4.7.3. Both results in the lemmata 4.7.2 and 4.7.2 can be confirmed by the exact solution derived in section 4.6 (in that case for the mode functions but it directly translates when multiplying the functions with the spherical harmonics), see also appendix A.4.

4.7.3. Some remarks on the nonlinear level

We performed the essential steps for the nonlinear equations already in section 4.7.2. Here we just summarize our findings from the analysis there for completeness.

For the analysis of the linear equations in section 4.7.2 we were able to consider the Hamiltonian constraint as equation involving the spatial metric only and the momentum constraint as equations for the extrinsic curvature only. Also the right-hand sides of the evolution equations for the spatial metric were entirely given by the extrinsic curvature and vice versa. The next lemma shows that these considerations basically remain true.

Lemma 4.7.4. The evolution equations of the spatial metric only contribute trivially to the principal part and the only relevant principal part contribution to the evolution equations for the extrinsic curvature comes from the Ricci tensor. For the Hamiltonian constraint only the Ricci scalar contains derivatives and is the only contribution to the principal part. For the momentum constraint $\gamma_{ij} D_k (K^{jk} - \gamma^{jk} \mathrm{tr} K) = 0$ both terms have to be taken into account.

Proof. These statements follow directly from the final equations in the Cauchy formulation in section 3.3. Since the shift vanishes there are no derivatives on the right-hand side of $\partial_t \gamma_{ij}$. Vanishing shift also reduces the terms for the evolution equations of the extrinsic curvature. The Ricci tensor contains second derivatives of the metric and hence gives the only contribution to the principal part of the evolution equations, the same is true for the Ricci scalar in the Hamiltonian constraint. □

The Ricci tensor is not multiplied with any component of the extrinsic curvature in the evolution equations. Thus lemma 4.7.4 implies that we should not expect any quantity of the extrinsic curvature tensor K to be multiplied with derivatives of γ.

For the nonlinear equations we consider nonlinear perturbations of the flat solution. Therefore in a neighborhood of the known solution we linearize about flat spacetime. The global results for the linear equations are then directly generalizable to the nonlinear setting but are only valid locally. Actually we did the analysis for the modified momentum constraint in theorem 4.7.3 already along these lines.

Hence the analysis reveals that the constraints solved for the set $\{\gamma_{\mathrm{s1}}, K_{\mathrm{s1}}, K_{\mathrm{v}}\}$ and the evolution equations solved for the set $\{\gamma_{\mathrm{v}}, \gamma_{\mathrm{t}}, \gamma_{\varphi\varphi}, K_{\mathrm{t}}, K_{\varphi}{}^{\varphi}\}$ form two locally well-posed sets of evolutionary equations and are therefore promising for the implementation. The missing s2 variables can be obtained by algebraic relations involving the angular contributions of the corresponding evolved variables (the $\varphi\varphi$ and tensor quantities).

At the end of section 4.7.2 we derived some supplementary relations. For completeness let us generalize these findings as well. On the nonlinear level the trace of the extrinsic curvature is not constant in time anymore. The corresponding evolution equation is given in equation (4.5.1).

Proposition 4.7.6. There exists an additional evolution equation for the determinant of the spatial metric.

Proof. In the evolution equations (3.20) we also have an evolution equation essentially for the determinant of the spatial metric. With the chain rule it can be written as

$$\partial_t \det \gamma = -2 \det \gamma \ \mathrm{tr} K. \tag{4.126}$$

The trace $\mathrm{tr}K$ is algebraically given as $K_r{}^r + K_\vartheta{}^\vartheta + K_\varphi{}^\varphi = 2\left(K_{\mathrm{s2}} - K_{\mathrm{s1}}\right)$. In proposition 4.4.2 we calculated the determinant to be

$$\det \gamma = \gamma_{rr}\gamma_{\vartheta\vartheta}\gamma_{\varphi\varphi} - \gamma_{r\vartheta}^2\gamma_{\varphi\varphi} = r^4 \sin^2\vartheta \left(\gamma_{\mathrm{s1}}\gamma_{\mathrm{s2}}^2 - \gamma_{\mathrm{s2}}\gamma_{\mathrm{v}}^2\gamma_{\mathrm{t}}^{-1}\right). \tag{4.127}$$

Its time derivative contains derivatives of the components of the metric. Those can be expressed with the evolution equations (3.20) or (4.26) respectively. Note that for different gauge conditions spatial derivatives of the shift vector would appear (which vanishes for the geodesic gauge). □

Remarks on the boundary condition of the evolution equations The treatment of the outer boundary conditions for the nonlinear evolution equations is basically exactly the same as for both (2+1- and 1+1-dimensional) linear evolution equations in sections 4.7.1 and 4.7.2. Hence the Bjørhus projection method is again applicable.

Since the structure of the derivation is exactly the same we only state the result here. With K we denote either K_{t} or K_φ and correspondingly for γ. The outer boundary condition reads (there are again no special boundary conditions for γ itself)

$$\partial_t K = -\frac{K}{2R} + \frac{f}{2R^2} + \frac{g}{2R} - \frac{\partial_r^2 \gamma}{4} - \frac{\partial_r K}{2}. \tag{4.128}$$

Here the terms f and g have to be adapted to the nonlinear situation. Therefore they are more complicated from the computational perspective but similar from the conceptual one. Conceptually they can be considered in the same manner as in section 4.7.2 and are compatible with the outgoing characteristics.

5. Numerical studies in vacuum axisymmetry

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5.1. Introduction

In chapter 4 we analyzed Einstein's vacuum equations in axisymmetry and proposed a promising scheme. It consists of a strongly hyperbolic subset of evolution equations and the constraints as a strongly hyperbolic-parabolic set for the remaining variables. It remains to verify that it is indeed possible to actually implement the equations and to obtain numerically solutions to Einstein's equations in this setting. We address this point in the current chapter.

We start with some remarks on the code and implementation itself and the verification of the essentials of the implementation. The tests are of significant importance for an implementation that was developed from scratch. Then we discuss separately the linear 1+1- and the essentials of the 2+1-dimensional situation and principal aspects of the nonlinear equations.

We split the set of equations in evolution equations and the constraints. The constraint solver delivers the initial values for the spacetime evolution. In the evolutionary formulation of the constraints we need to provide also initial values at the origin for the integration of the constraints. We discuss several possibilities to derive these initial data and demonstrate that the full parabolic-hyperbolic initial value solver is working "everywhere" as it is supposed to do on all levels. For these simulations we take the freely specifiable variables from the exact analytical solution.

Finally the spacetime evolution equations are taken into account. We demonstrate that our code is able to reproduce essentially the known exact linear solution numerically. The handling of the coordinate singularity in the evolution equations is a serious issue and we show how we address the problem.

5.2. Some remarks on the implementation and basic code verifications

5.2.1. General remarks on the implementation

The code that was used for our simulations was developed and written by us from scratch. As language for the implementation we chose the scripting language Python[1]. We will indicate when special (public) libraries and packages[2] are used.

The code was developed as single core application. Smaller runs were performed on an ordinary workstations but quite often we made use of a cluster[3].

[1] There is a huge number of valuable sources and introductions. We particularly benefited from the ones designed for its scientific use, see Langtangen (2008), Stewart (2014), and online documentations. While it might be true that Python is not the ultimately optimal language concerning the final simulation time it is extremely elegant. Time needed for the development is shorter, at least that is our experience. This makes Python a good choice for more conceptual and fundamental studies like the ones presented in the thesis.

[2] In general we made use of the packages NumPy (http://www.numpy.org/), SciPy (https://www.scipy.org/) and for the visualization Matplotlib (https://matplotlib.org/).

[3] Mainly we used the cluster "Datura" run by the Albert Einstein Institute for Gravitational Physics (http://www.aei.mpg.de/).

Even though there are good and strong arguments to do so it is not our intention to make the code public. In general we agree that even not perfectly documented or well-developed codes or snippets can help others for their further development. There are also scientific reasons that prevent publication. It is never certain that there are no bugs, typos or flaws included. In general the probability is higher the smaller the number of people developing and maintaining the code. There is the tendency to "black-box" tools (using them without the detailed understanding of their function). This makes perfectly sense if those tools are well tested but might become problematic if the number of developers is (very) small. We see it in these cases as the scientific duty to explain as detailed as meaningful how things are implemented and which crucial tools and parameters are used but to encourage other researchers to imitate (or to do better based on the given achievements or further developments) the implementation independently.

Even though all derivations in the thesis can be done without, we highly benefited from the use of computer algebra packages[4].

5.2.2. Some essential verifications and numerical tests

In the following we confirm that our numerical implementation of the derivative operators works as expected. For the r-derivatives we work on the linear mode level and construct an equation that is supposed to vanish in the continuum limit, compare with the discussion in section 2.2.6. For example for some variable u we consider

$$\partial_r^k u\big|_{\text{num}} - \partial_r^k u\big|_{\text{exact}}, \;\; k \in \{1, 2\}. \tag{5.1}$$

To have a non-trivial test case we choose for u an arbitrary variable from the exact solution, say $u = \hat{\gamma}_{\text{s1}}$ for $\ell = 2$ as given explicitly in appendix A.4. We can also simply calculate the exact derivatives in r, $\partial_r^1 u|_{\text{exact}}$ and $\partial_r^2 u|_{\text{exact}}$, and check that the residuals of equation (5.1) converge to zero in the continuum limit. Since we are using a second-order formulation we expect quadratic convergence. The results for several resolutions are listed in table 5.1 where the L2-norm of equation (5.1) is given. The expected factor of four is achieved when doubling the resolution.

[4]We use mainly Mathematica (Wolfram (1999)) including xAct (http://www.xact.es/) and Sage (http://www.sagemath.org/).

N	Δr	1^{st} derivative	2^{nd} derivative
100	.2	257.2×10^{-6}	297.3×10^{-6}
200	.1	65.0×10^{-6}	75.0×10^{-6}
400	.05	16.3×10^{-6}	18.8×10^{-6}
800	.025	4.1×10^{-6}	4.7×10^{-6}

Table 5.1.: The norms of the errors of the derivative operators in equation (5.1) for different step sizes. One can see that in the continuum limit $\Delta r \to 0$ the values converge to zero with the expected quadratic convergence.

We should also verify that the analytic solution is indeed the exact solution for the derived linear equations. Since the essential part of the exact solution is given in t and r we perform the convergence test on the 1+1-dimensional level. If we show that the exact solution inserted in the field equations results in residuals that converge away in the continuum limit we give a strong indication that both the exact solution is correct and the implementation of the equations (in particular the derivatives) are behaving as they should. The spatial discretization in r is of second order and therefore should converge quadratically with decreasing step size. To test the evolution equations as well we use for the time discretization also a second-order discretization, here a forward stencil (see section 2.2.3). Therefore we consider the relation

$$(\partial_t u)^n - \mathrm{rhs}_u^n = \frac{-3u^n + 4u^{n+1} - u^{n+2}}{2\Delta t} - \mathrm{rhs}_u^n \to 0 \tag{5.2}$$

at time step n where rhs_u^n denotes the right-hand side of the evolution equation for the variable u at time step n. The evolution equations for the spatial metric should converge in a trivial way, for the extrinsic curvature also the spatial derivatives are involved. We test everything with the exact solution for $\ell = 2$ as derived in section 4.6 with peak localized at $r = 10$. Throughout our Courant factor (see definition 2.3.9) is 1/10 which determines the time step when the spatial grid spacing is prescribed.

N	Δr	Δt	$\partial_t\hat{\gamma}_{\mathrm{s1}}$	$\partial_t\hat{\gamma}_{\mathrm{s2}}$	$\partial_t\hat{\gamma}_{\mathrm{v}}$	$\partial_t\hat{\gamma}_{\mathrm{t}}$	$\partial_t\hat{K}_{\mathrm{s1}}$	$\partial_t\hat{K}_{\mathrm{s2}}$	$\partial_t\hat{K}_{\mathrm{v}}$	$\partial_t\hat{K}_{\mathrm{t}}$
100	.2	.02	5.70	2.85	20.96	259.5	232.7	223.1	1322.5	9127.4
200	.1	.01	1.43	.71	5.25	65.0	59.0	56.5	335.0	2306.
400	.05	.005	.36	.18	1.31	16.3	14.8	14.2	84.1	578.
800	.025	.0025	.09	.04	.33	4.1	3.7	3.6	21.	145.
1600	.0125	.00125	.02	.01	.08	1.	.9	.9	5.	36.

Table 5.2.: The norm of the residuals (multiplied by 10^{-6}) of the field equations with respect to the exact solution for different resolutions for the test of the evolution equations. One sees that in the continuum limit the residuals converge to zero with the expected quadratic convergence.

N	Δr	Δt	$\hat{\mathcal{H}}$	$\hat{\mathcal{C}}_r^{\mu=0}$	$\hat{\mathcal{C}}_\vartheta^{\mu=0}$
100	.2	.02	43.0	2.95	12.37
200	.1	.01	10.9	.74	3.13
400	.05	.005	2.7	.18	.79
800	.025	.0025	.7	.05	.20
1600	.0125	.00125	.2	.01	.05

Table 5.3.: The norm of the residuals (multiplied by 10^{-3}) of the field equations with respect to the exact solution for different resolutions for the constraints. We see that in the continuum limit the residuals converge to zero with the expected quadratic convergence.

We see in tables 5.2 and 5.3 that all relations converge quadratically with increasing resolution (and hence decreasing step size). We notice that the residuals of the evolution equations for $\hat{\gamma}_{\mathrm{s1}}$ and $\hat{\gamma}_{\mathrm{s2}}$ in table 5.2 differ only by a factor of two. This is no surprise because of the relation derived in lemma 4.7.2. That the residuals for the evolution equations for $\hat{K}_{\mathrm{s1}}$ and $\hat{K}_{\mathrm{s2}}$ differ (even though we derived in lemma 4.7.2 that they are equal analytically) is caused by different expressions on the right-hand sides of the evolution equations. The constraint equations converge in the continuum limit but it is remarkable that the residuals for the constraints converge with a factor that is larger. For reasonable resolutions the error is still quite large (the values in table 5.3 are only multiplied with 10^{-3}, not with 10^{-6} as for the evolution equations).

5.2.3. Test with the exact solution in the nonlinear constraints

In the process of the numerical implementation it is of huge advantage if one has knowledge of the exact solution of the problem and one can aim to reproduce known results. It shows that the implementation is indeed correct. We derived in section 4.6 the exact solution to the linearized problem, which allowed us to do some consistency checks.

Also the knowledge of the exact solution allows us to verify that the coding of the nonlinear equations is correct and one solves the correct equations. We write the nonlinear variables in the form

$$u = u_{\text{flat}} + \epsilon u_{\text{pert}} \tag{5.3}$$

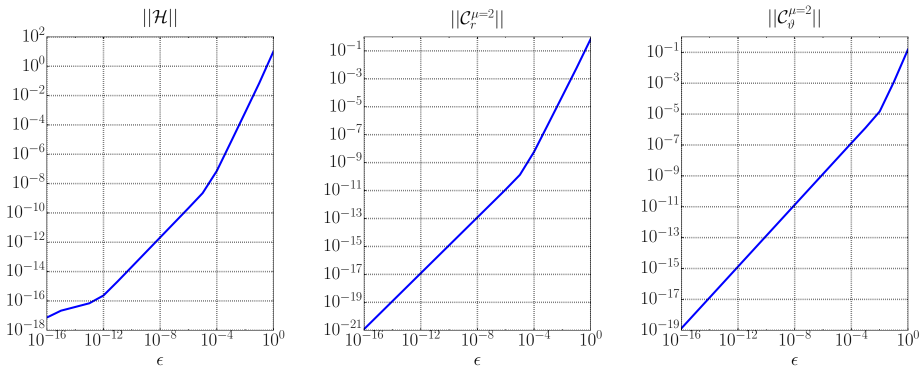
with an amplitude ϵ determining the strength of the perturbation. It is usually considered to be small, $\epsilon \ll 1$, see also section 3.2.3. Inserting the exact linear solution $\tilde{u}$ in the expansion (5.3), i.e. $u = u_{\text{flat}} + \epsilon\tilde{u}$, and evaluating the nonlinear equations (abbreviated as nonlin) with these variables results in relations of the form

$$\text{nonlin}(u) = A\epsilon^0 + B\epsilon^1 + C\epsilon^2 + \dots \tag{5.4}$$

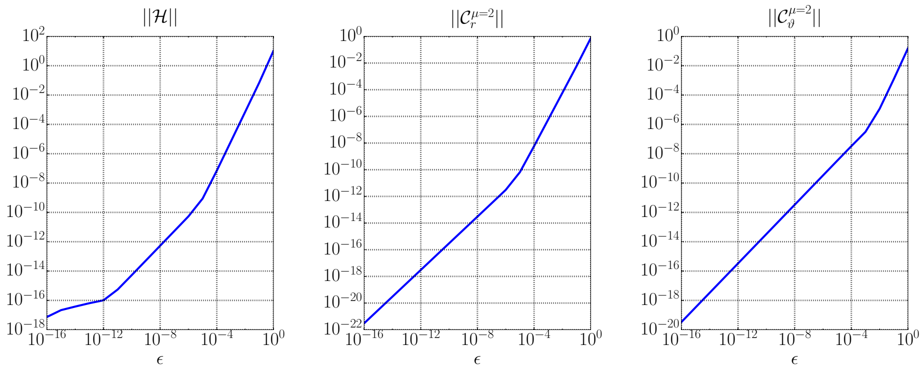
with contributions $A,\ B,\ C\dots$. If the coding of the equations is correct we expect the following behavior:

- The flat contribution A is based on the flat values and should vanish up to roughly machine precision for equations that are supposed to vanish.
- The linear contribution B is solved by the exact solution and therefore should vanish analytically. Numerically it will cause some contribution due to the truncation error but it should converge to zero with decreasing step size as discussed above. The nonlinear errors are suppressed with ϵ^2 provided ϵ is small enough. Hence in this regime we expect a linear convergence for $\epsilon \to 0$.
- In the intermediate regime for ϵ the nonlinear contribution C (and possible higher terms) are expected to give the dominant contribution. Even analytically they do not vanish since we inserted the solution to the linear problem only. In that regime we expect a quadratic (with respect to ϵ) convergence to zero. Therefore a (double-)logarithmic plot is well suited to test the described behavior.

We implemented the modified momentum constraint with parameter $\mu = 2$ in theorem 4.7.3. The right-hand side of those equations is proportional to the derivative of the trace of the extrinsic curvature. Since we insert the linear solution where the trace is exactly zero the inhomogeneous part of the modified momentum constraint vanishes exactly.



(a) The norm of the residuals of the constraints versus ϵ for a spatial resolution $N = 800$.



(b) The norm of the residuals of the constraints versus ϵ for a spatial resolution $N = 1600$.

Figure 5.1.: The exact linear solution $\tilde{u}$ inserted in the form $u = u_{\text{flat}} + \epsilon\tilde{u}$ in the nonlinear constraints (the Hamiltonian constraint $\mathcal{H}$ and the two components of the modified momentum constraint, $\mathcal{C}_r^{\mu=2}$ and $\mathcal{C}_\vartheta^{\mu=2}$) for decreasing ϵ for two resolutions.

Our expectations and the results in figure 5.1 match and we observe that the numerics satisfy the predictions. The transition between the two regimes is visible in figure 5.1. The results are improving with growing resolution. At some instance for the amplitude ϵ (between $\epsilon = 10^{-12}$ and 10^{-16}) we observe that the machine error seems to prohibit further improvements. This is due to the (nonlinear) combination of several terms.

To summarize we provide evidence that the numerical implementation of the derivatives and the equations behave as expected on the 1+1-dimensional level and we are ready to discuss the numerics of the solution of the field equations.

5.3. The linear mode level

Our aim is to use a fully constrained evolution. In any case we need to solve all the constraints on the initial slice. Therefore we will first demonstrate that we are able to solve the constraints as an evolutionary system "everywhere" (that means for an arbitrary initial time) when we take the freely specifiable variables from the exact solution. Then we will turn to the full spacetime evolution.

Even though a standard solver seems to be able to solve the equations we experienced some difficulties in solving the Hamiltonian constraint when the source terms are not given by the exact analytical solution. The numerical integration of a stiff parabolic equation is apparently non-trivial. We will address these problems and also a possible solution in the following.

5.3.1. The constraint solver

We want to solve the constraints as an evolutionary system as discussed in section 3.4.2. On the linear level the Hamiltonian and momentum constraint completely decouple. The former is an equation for the components of the spatial metric, the latter for the components of the extrinsic curvature. We solve the Hamiltonian constraint (4.50) as a parabolic equation for $\hat{\gamma}_{\mathrm{s1}}$ and the modified momentum constraint as a coupled hyperbolic system for either $\{\hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{v}}\}$ or $\{\hat{K}_{\mathrm{s1}}, \hat{K}_{\mathrm{v}}\}$.

We take the freely specifiable data from the exact solution as derived in section 4.6 with a generating function[5] $G(t,r) = G(x) = A\exp(-x^2/2)$, $x = r \pm t$ and amplitude $A = 1$. We choose for the demonstration the mode $\ell = 2$ and hence we can use the quantities given explicitly in appendix A.4.

[5]In the simulations we are interested in a wave package evolving through the origin. Usually the radial domain is $r \in [0, R = 20]$. To determine an ingoing package we have to choose a negative time in the regular exact solution. If initially $t = -10$ it implies that the outgoing terms are almost entirely suppressed and only the ingoing contributions remain. For $t > 0$ the ingoing contributions are suppressed and the outgoing ones are of significant value. These observations are important for the determination of the exact solution. In the plots we show the simulation time. Therefore the plots have always a positive time axis.

The constraints form in a neighborhood of the origin a stiff (see definition 2.2.3) set of ordinary differential equations. For its integration we make use of the SciPy solver `ODEint`[6]. It is basically a wrapper for `lsoda` from the FORTRAN library `odepack`, see Hindmarsh (1983). That is a collection of solvers for systems of ordinary differential equations that can handle stiff and non-stiff systems. It uses in parts adaptive methods for the step size and (a combination of) explicit and implicit integrators. For the inclusion of the inhomogeneous part we use the SciPy package `interp1d`[7] for the interpolation to arbitrary grid points (required by `ODEint`). On the 1+1-dimensional mode level we are successful already with the default settings of the solver, which have to be modified later.

In addition we need to find a way to prescribe "initial data" close to the origin. As seen in section 4.7.1 the solver is supposed to start at $r = \epsilon > 0$ and we need also the value at the origin since we are using a vertex-centered grid. Hence we need to determine initial data at the grid points $r = 0$ and $r = \Delta r$.

Remark 5.3.1. The parabolic equation only allows an integration outwards. In a partially constrained scheme where we only solve the hyperbolic momentum constraint or for a separate solver for the Hamiltonian constraint it would be an alternative to start the integration for the momentum constraint at the outer boundary. There one can use asymptotic flatness (see definition 3.3.1) to determine initial data. Then the integration can be performed inwards. It would be interesting to see if the solution reproduces regular data at the origin. Since we have a proper way to determine initial values at the regular origin we will apply the integration outwards.

In general we distinguish two cases. We show that the solver converges for both sets of variables, for $\{\hat{\gamma}_{\mathrm{s1}}, \hat{K}_{\mathrm{s2}}, \hat{K}_{\mathrm{v}}\}$ and for $\{\hat{\gamma}_{\mathrm{s1}}, \hat{K}_{\mathrm{s1}}, \hat{K}_{\mathrm{v}}\}$

5.3.1.1. Initial data for the constraint solver from Taylor expansion

For the initial values we make use of the Taylor expansion of the variables close to the origin. We discussed two promising formulations of the momentum constraint in section 4.7.1. Depending on which scalar component of the extrinsic curvature the equations are solved for, we have to compute the Taylor expansion using the remaining ones. The aim is to express the first two grid points of the desired variables in terms of the freely specifiable data to obtain a value at the origin where the equation is degenerate. These are the initial values for the evolutionary solver.

[6] See https://docs.scipy.org/doc/scipy-0.18.1/reference/generated/scipy.integrate.odeint.html.
[7] See https://docs.scipy.org/doc/scipy-0.19.0/reference/generated/scipy.interpolate.interp1d.html.

Proposition 5.3.1. For **even** modes ($\leftrightarrow$ mode parameter ℓ is even) the first two values for $\hat{\gamma}_{\mathrm{s1}}$ are given as

$$\hat{\gamma}_{\mathrm{s1}}\big|_0 = \frac{1}{2+\ell(\ell+1)}\Big\{4\ell(\ell+1)\hat{\gamma}_{\mathrm{v}}\big|_0 + [2-\ell(\ell+1)]\hat{\gamma}_{\mathrm{s2}}\big|_0 + \left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\,\hat{\gamma}_{\mathrm{t}}\big|_0\Big\}, \tag{5.5a}$$

$$\hat{\gamma}_{\mathrm{s1}}\big|_{\Delta r} = \hat{\gamma}_{\mathrm{s1}}\big|_0 + \frac{\Delta r^2}{6+\ell(\ell+1)}\Big\{[8-\ell(\ell+1)]\hat{\gamma}_{\mathrm{s2}}|_{\Delta r} \left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{\gamma}_{\mathrm{t}}|_{\Delta r} + 8\ell(\ell+1)\;\hat{\gamma}_{\mathrm{v}}|_{\Delta r}\Big\}, \tag{5.5b}$$

(5.5c)

The momentum constraint as equation for $\hat{K}_{\mathrm{s2}}$ and $\hat{K}_{\mathrm{v}}$ has initial values

$$\hat{K}_{\mathrm{s2}}\big|_0 = \frac{1}{3\ell(\ell+1)-6}\Big\{[12+2\ell(\ell+1)]\hat{K}_{\mathrm{s1}}\big|_0 - \left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{K}_{\mathrm{t}}\big|_0\Big\}, \tag{5.6a}$$

$$\hat{K}_{\mathrm{v}}\big|_0 = \frac{1}{6}\Big\{-2\hat{K}_{\mathrm{s2}}\big|_0 + [\ell(\ell+1)-2]\,\hat{K}_{\mathrm{t}}\big|_0\Big\}, \tag{5.6b}$$

$$\hat{K}_{\mathrm{s2}}\big|_{\Delta r} = \hat{K}_{\mathrm{s2}}\big|_0 + \hat{K}_{\mathrm{s1}}|_{\Delta r}\Delta r^2, \tag{5.6c}$$

$$\hat{K}_{\mathrm{v}}\big|_{\Delta r} = \hat{K}_{\mathrm{v}}\big|_0 + \frac{\Delta r^2}{5}\Big\{-\hat{K}_{\mathrm{s1}}|_{\Delta r} + \left[\frac{\ell(\ell+1)}{2}-1\right]\hat{K}_{\mathrm{t}}|_{\Delta r}\Big\}. \tag{5.6d}$$

Considered as an equation for $\hat{K}_{\mathrm{s1}}$ and $\hat{K}_{\mathrm{v}}$ the corresponding quantities read

$$\hat{K}_{\mathrm{s1}}\big|_0 = \frac{1}{2\,[\ell(\ell+1)+6]}\Big\{3[\ell(\ell+1)-2]\hat{K}_{\mathrm{s2}}\big|_0 + \left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{K}_{\mathrm{t}}\big|_0\Big\}, \tag{5.7a}$$

$$\hat{K}_{\mathrm{v}}\big|_0 = \frac{1}{\ell(\ell+1)+6}\Big\{-8\hat{K}_{\mathrm{s2}}\big|_0 + [\ell(\ell+1)-2]\,\hat{K}_{\mathrm{t}}\big|_0\Big\}, \tag{5.7b}$$

$$\hat{K}_{\mathrm{s1}}\big|_{\Delta r} = \hat{K}_{\mathrm{s1}}\big|_0 + \frac{\Delta r^2}{2\,[\ell(\ell+1)+30]}\left[(10+3\ell(\ell+1))\,\hat{K}_{\mathrm{s2}}\big|_{\Delta r} + \left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{K}_{\mathrm{t}}|_{\Delta r}\right], \tag{5.7c}$$

$$\hat{K}_{\mathrm{v}}\big|_{\Delta r} = \hat{K}_{\mathrm{v}}\big|_0 + \frac{\Delta r^2}{\ell(\ell+1)+30}\left[-16\hat{K}_{\mathrm{s2}}\big|_{\Delta r} + 3\,[\ell(\ell+1)-2]\,\hat{K}_{\mathrm{t}}|_{\Delta r}\right]. \tag{5.7d}$$

For **odd** modes ($\leftrightarrow$ mode parameter ℓ is odd) the first two values of $\hat{\gamma}_{\mathrm{s1}}$ and the value at the origin of the extrinsic curvature are

$$\hat{\gamma}_{\mathrm{s1}}\big|_0 = 0, \tag{5.8a}$$
$$\hat{K}_{\mathrm{s2}}\big|_0 = 0, \tag{5.8b}$$
$$\hat{K}_{\mathrm{v}}\big|_0 = 0, \tag{5.8c}$$
$$\begin{aligned}\hat{\gamma}_{\mathrm{s1}}\big|_{\Delta r} = \frac{\Delta r}{4+\ell(\ell+1)}&\{[8-\ell(\ell+1)]\hat{\gamma}_{\mathrm{s2}}|_{\Delta r} \\ &\left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{\gamma}_{\mathrm{t}}|_{\Delta r}+6\ell(\ell+1)\ \hat{\gamma}_{\mathrm{v}}|_{\Delta r}\Big\}.\end{aligned} \tag{5.8d}$$

If the constraint is solved for $\hat{K}_{\mathrm{s2}}$ and $\hat{K}_{\mathrm{v}}$ we have

$$\hat{K}_{\mathrm{s2}}\big|_{\Delta r} = -\frac{8\Delta r}{\ell(\ell+1)}\hat{K}_{\mathrm{s1}}|_{\Delta r}, \tag{5.9a}$$
$$\hat{K}_{\mathrm{v}}\big|_{\Delta r} = \frac{\Delta r}{6}\left\{4\hat{K}_{\mathrm{s1}}|_{\Delta r}\ +[\ell(\ell+1)-2]\,\hat{K}_{\mathrm{t}}|_{\Delta r}\right\}. \tag{5.9b}$$

The momentum constraint considered for $\hat{K}_{\mathrm{s1}}$ and $\hat{K}_{\mathrm{v}}$ gets initial values

$$\hat{K}_{\mathrm{s1}}\big|_{\Delta r} = \frac{\Delta r}{2\,[16+\ell(\ell+1)]}\left\{3\ell(\ell+1)\hat{K}_{\mathrm{s2}}|_{\Delta r}+\left(\ell+\frac{\ell^2}{2}-\ell^3-\frac{\ell^4}{2}\right)\hat{K}_{\mathrm{t}}|_{\Delta r}\right\}, \tag{5.10a}$$
$$\hat{K}_{\mathrm{v}}\big|_{\Delta r} = \frac{\Delta r}{16+\ell(\ell+1)}\left\{-12\hat{K}_{\mathrm{s2}}|_{\Delta r}\ +[2\ell(\ell+1)-4]\,\hat{K}_{\mathrm{t}}|_{\Delta r}\right\}. \tag{5.10b}$$

Proof. It follows from proposition 4.4.1 that even modes expand as $\hat{u} = u_0 + u_2 r^2 + \ldots$ and odd ones as $\hat{u} = u_1 r + \ldots$ close to the origin. Inserting these expansions in the constraints in theorem 4.5.3 on the mode level leads to the relations above. On the mode level it is clear that we can explicitly express the variables by decoupling the equations. □

As discussed we prescribe data (the freely specifiable data for the constraint solver) with support away from the origin $r = 0$ and also from the outer boundary (here $r = R = 20$), we start with the exact solution for $\ell = 2$, see section 4.6 and appendix A.4. As generating function we choose $G = A\exp(-x^2/2)$ for $x = r \pm t$ and amplitude $A = 1$ an localize the Gaussian initially at $r = 10$ (or $r = 12$), see also footnote 5 on page 174. At that time instance we give the exact linear solution for those variables which are not supposed to be solved by the constraints (but are instead intended to be updated by the evolution equations later). Then we solve the constraints with initial data near the origin as derived before. Afterwards we proceed to the next time level and start again with the scheme, exactly in the same manner as before. So there is no time evolution yet, even though the plots might suggest the opposite. The aim is to show that the residuals of the constraints vanish in the continuum limit, see the discussion in section 2.2.6.

5.3.1.1.1. Constraint solver for $\{\hat{\gamma}_{\mathbf{s1}}, \hat{K}_{\mathbf{s2}}, \hat{K}_{\mathbf{v}}\}$

We plot the results in figure 5.2. There we solve the constraints for $\ell = 2$ at each time instance for $\hat{\gamma}_{\mathrm{s1}}$, $\hat{K}_{\mathrm{s2}}$ and $\hat{K}_{\mathrm{v}}$, the remaining variables are taken from the exact solution. We obtain the initial values for the evolutionary solver with the Taylor expansion in proposition 5.3.1 for the momentum constraint and the Hamiltonian constraint. We see that already for moderate resolutions (the dash-dotted blue line) the quotient is not too far away of the desired curve (the solid turquois line), even though probably not yet in the convergent regime. For higher resolution (dotted green and dashed red lines) the curves approach the desired value.

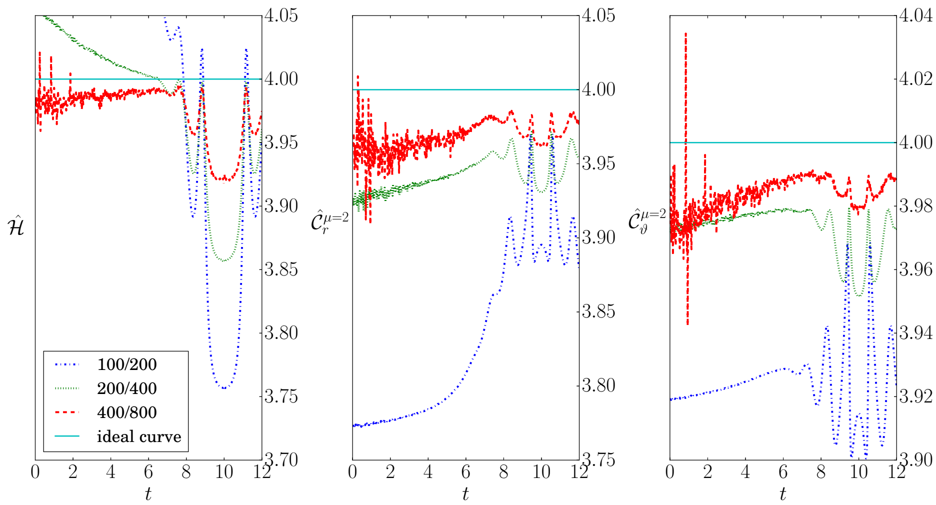


Figure 5.2.: We show the quotients of the L2-norms of the residuals of the constraints (Hamiltonian constraint $\hat{\mathcal{H}}$ and the components of the modified momentum constraint, $\hat{\mathcal{C}}_r^{\mu=2}$ and $\hat{\mathcal{C}}_\vartheta^{\mu=2}$). We calculate the quotient between two resolutions, one having twice as many grid points (hence half of the step size). Therefore we expect that the quotient approaches four (solid turquois line in the plots).

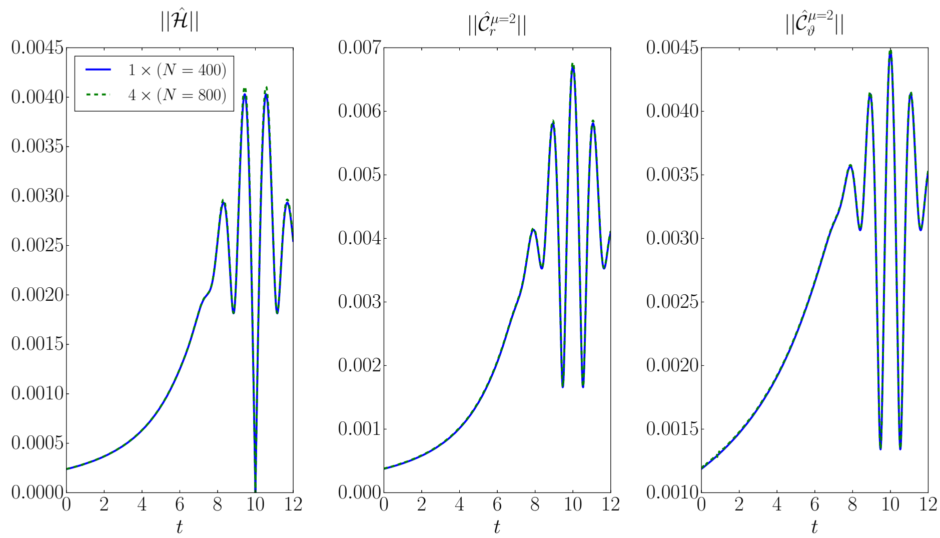


Figure 5.3.: We plot the norm of the residuals of the constraints for $N = 400$ (blue solid line) and four times the residuals for $N = 800$ (dashed green line). The superscript $\mu = 2$ is omitted for the presentation. The curves are almost indistinguishable and therefore show convergence.

For high resolutions there are several outliers. These are due to the representation showing the convergence factor where division by small numbers is involved. Observe that we chose the scale of the axis such that we plot only a rather small region significantly away from zero. As a further demonstration we display in figure 5.3 the norm of the residuals of the constraints for two resolutions. Since we multiply the value of the higher resolution (corresponding to a finer grid) with a factor four we expect that the two plotted curve coincide. This is indeed the case in figure 5.3.

5.3.1.1.2. Constraint solver for $\{\hat{\gamma}_{\mathbf{s1}}, \hat{K}_{\mathbf{s1}}, \hat{K}_{\mathbf{v}}\}$

We show the convergence plots in figure 5.4. Both the discussion and the results are absolutely analogous as in the case where we solved the constraints for $\hat{\gamma}_{\text{s1}}$, $\hat{K}_{\text{s2}}$ and $\hat{K}_{\text{v}}$ presented in figures 5.2 and 5.3 and discussed there. Hence we are brief for the current case. In the Hamiltonian constraint the value at the transition through the origin (corresponding to the simulation time $t = 10$ in the plot) is indistinguishable from zero for $N = 800$ and therefore the division produces a useless result. Hence we should ignore the value at $t = 10$ and therefore the spike in the red dashed curve. The constraint solver convergences in the continuum limit.

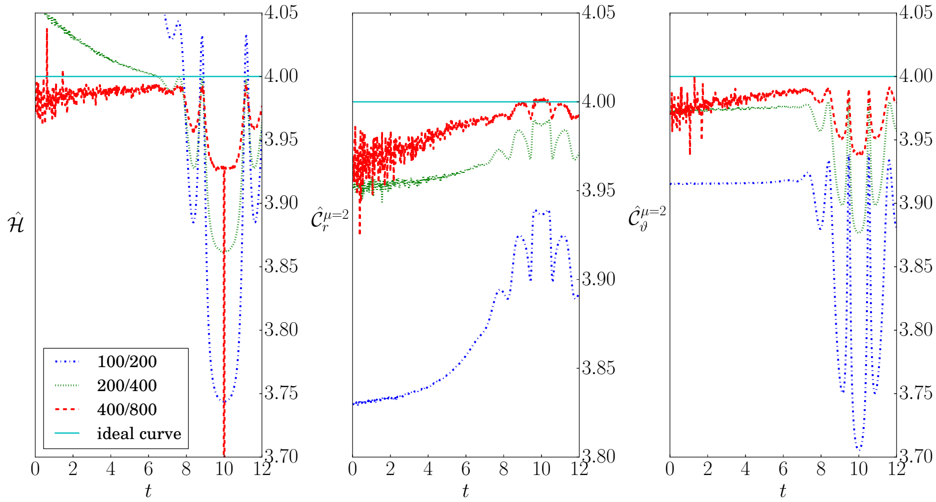


Figure 5.4.: We show the quotients of the L2-norms of the residuals of the constraints. The quotient is calculated for two resolutions, one having twice as many grid points. We expect that the quotient approaches four (solid line in the plots).

5.3.1.1.3. Taylor expansion and evolution equation for initial data

If we want to use the fully constrained scheme in the nonlinear case as well we have, at least for the Hamiltonian constraint, to use a different approach to obtain initial data. The nonlinearity of the Hamiltonian constraint does not allow us to obtain the value of the corresponding variable on the first grid points from the Taylor expansion. The momentum constraint is in fact semilinear (see definition 2.2.2) in the components of the extrinsic curvature. Therefore a similar procedure using the Taylor expansion can be applied for the momentum constraint even in the nonlinear case. For the initial values for $\hat{\gamma}_{s1}$ we can use the free time evolution instead. Since the corresponding equation is free of coordinate singularities (this is also true on the nonlinear level) we expect no serious problems for the free evolution at the 0^{th} and 1^{st} grid point in that equation. The corresponding evolution equation (4.49) reads $\partial_t\hat{\gamma}_{s1} = 4\hat{K}_{s1}$.

We just mention here (and use it later on) that it is also possible to use only for the first two grid points the evolution equations for $\hat{\gamma}_{s1}$. In contrast to the equations for the extrinsic curvature components the evolution equation is fully regular for the spatial metric (this remains true on the nonlinear level).

5.3.1.2. Using the decoupling of the constraints

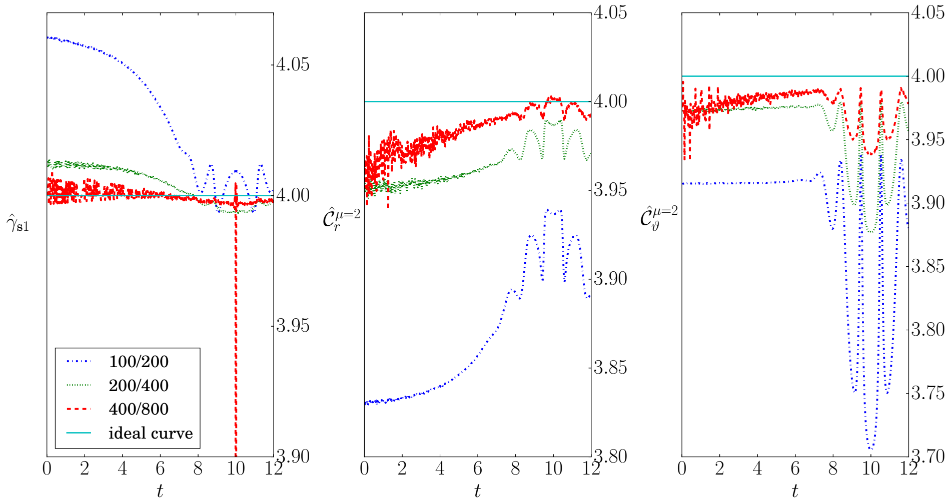


Figure 5.5.: We show the quotients of the L2-norms of the residuals of the constraints. Since the Hamiltonian constraint is satisfied identically (see the discussion in the main text) we display instead the convergence of the corresponding variable $\hat{\gamma}_{s1}$ to the exact solution. The quotient is shown for two resolutions, one having twice as many grid points, and is expected to approach four (solid turquois line in the plots). The spike at the transition through the origin (at $t = 10$) for the convergence plot for $\hat{\gamma}_{s1}$ can be ignored for the same reason as discussed above.

On the linear level the constraints decouple into two independent sets. The Hamiltonian constraint is an equation for the components of the spatial metric only, the momentum constraint forms a coupled system for the components of the extrinsic curvature. One can make use of it and solve the momentum constraint as hyperbolic equations with the solver `ODEint` and the Hamiltonian constraint separately. We experimented for the Hamiltonian constraint with an iterative Newton-Raphson solver (see for example Press et al. (2007)). It is only a single equation and implicit solvers for parabolic partial differential equations are not uncommon. For a linear equation we expect that the solver solves the equation already in the first iteration.

As a demonstration we show in figure 5.5 the convergence of the independent solvers. The results are basically the same as above, the only difference is that the Hamiltonian constraint is satisfied up to the desired accuracy (it is a linear equation, hence it is satisfied after one iteration already and its violation is indistinguishable from zero). Nevertheless we keep in mind that the decoupling of the constraints is not satisfied any more on the nonlinear level.

Summary

We can conclude that our constraint solver works as desired on the 1+1-dimensional mode level. This is true for reasonable spatial resolution of at least $N = 200$. For a spatial domain of $r \in [0, R = 20]$ it corresponds to a step size of maximally $\Delta r = 1/10$. On the 1+1-dimensional mode level we presented several different choices for the obtaining of initial data close to the origin and numerical solvers.

5.3.2. The fully constrained evolution

All evolution equations are implemented using the method of lines, see section 2.3.3. It implies that an integration scheme for ordinary differential equations is suitable for the time integration. Our choice is a version of the Runge-Kutta algorithm of second order, namely the integrator by Shu (1998) (see the discussion in section 2.2.3). We use a fully constrained formulation where the constraints are explicitly solved on each time slice. Thus it makes sense to implement the solver explicitly from scratch instead of using an already implemented integrator like the `ODEint` integrator discussed above.

In principle quite a number of other integrators should be working as well, including higher-order methods. We experimented with some different possibilities without noticing any substantial difference that is worth to be mentioned. Therefore for all the numerical results presented in the following we use the Shu integrator. We remark that it is for our applications essentially the Runge-Kutta integrator of second order.

The evolution equations split into two sets of equations. For the components of the spatial metric they are fully regular. The ones for the extrinsic curvature contain a coordinate singularity at the origin.

Applying the formally singular evolution equations for the components of the extrinsic curvature requires to divide by r in the limit $r \to 0$. This is problematic for the vertex-centered lattice with grid point at $r = 0$. Instead of applying the field equations at the origin we make use of proposition 2.6.5 and determine the zeroth grid point algebraically. In proposition 2.6.5 we derived for the modes $\ell = 0$ and $\ell = 2$ the relations

$$u_0 = u_2 - 2u_1 \qquad \text{(2.168 rev.)}$$

$$\text{and } u_0 = 3u_1 - 3u_2 + u_3 \qquad \text{(2.169 rev.)}$$

up to an error of $\mathcal{O}(\Delta r^2)$ and $\mathcal{O}(\Delta r^3)$ respectively (for step size Δr). For all the other modes we can just set the value at u_0 to zero (up to the same accuracy).

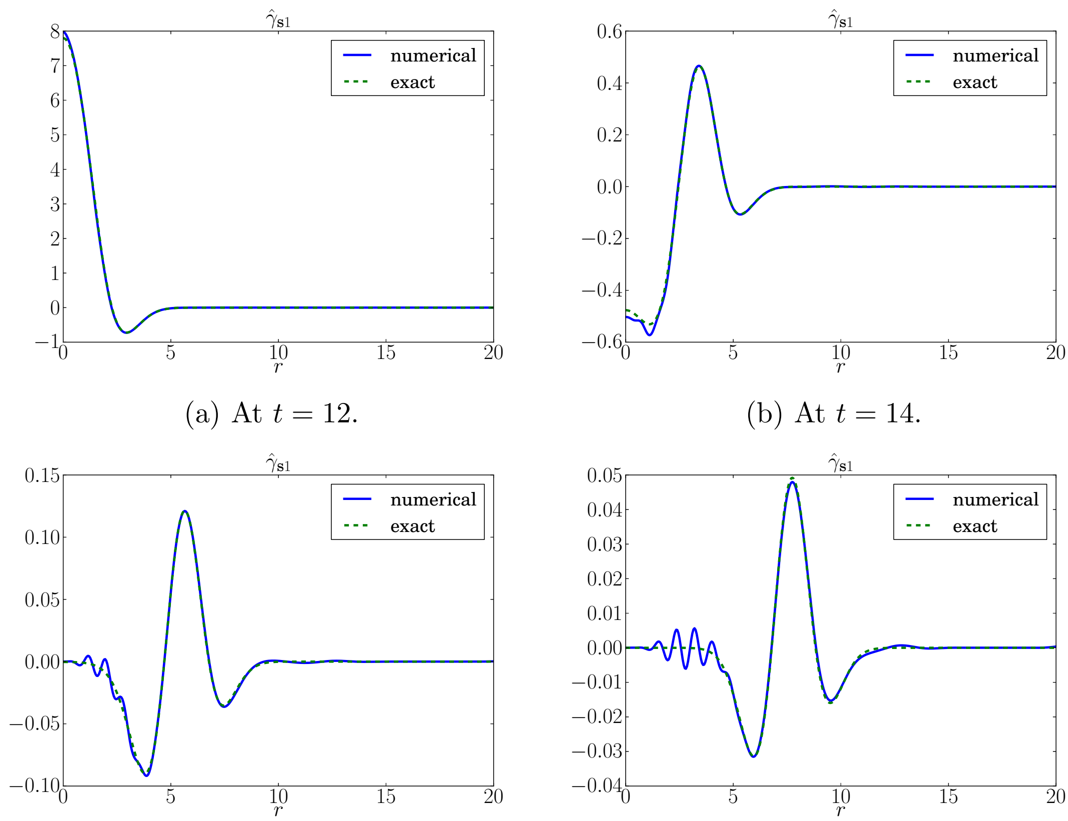
5.3.2.1. Failure of the `ODEint` solver for the parabolic equation

We derived in section 4.7.1 a promising formulation consisting of strongly hyperbolic evolution equations together with a well-posed (for suitable initial data as discussed above) evolutionary constraint system. The latter one is a parabolic-hyperbolic system on the linear level.

We know that the constraint solver as such converges everywhere. In the fully constrained scheme we consider a similar situation as above. Instead of taking the exact solution as source terms in the constraint solver, we employ the set of evolution equations to evolve the remaining variables. When the evolution equations are used the source terms for the solver contain naturally some discretization errors.

For the constraint solver (hyperbolic-parabolic system) we use `ODEint` with initial data determined by the Taylor expansion for all equations (see discussion above). As we can see in the snapshots of the evolution in figure 5.6 there is some problem after the wave package transited through the origin. The visible oscillations are responsible for non-convergent results. The reason for the failure is caused by the Hamiltonian constraint. Apparently solving a stiff parabolic equation is more involved and results in the oscillations when the source terms contain some numerical error.

We remark that if, instead of solving the Hamiltonian constraint, the quantity $\hat{\gamma}_{s1}$ is taken from the exact solution the convergence results are looking good.



(a) At $t = 12$.

(b) At $t = 14$.

(c) At $t = 16$.

(d) At $t = 18$.

Figure 5.6.: We show some snapshots for the fully constrained scheme as desired. We choose an outer boundary at $R = 20$ and a resolution of $N = 400$ points. The initial ($t = 0$) peak is located at $r = 10$, directed inwards and determined by the exact solution for $\ell = 2$. Therefore the transition through the origin corresponds to simulation time $t = 10$ and the displayed snapshots are taken after the transition.

5.3.2.2. Convergent fully constrained evolution

To circumvent the problems in the evolution we switch to the alternative solvers for the constraints. We keep using the `ODEint` solver for the momentum constraint (we obtain the initial data with the Taylor expansion) and apply the Newton-Raphson solver for the Hamiltonian constraint (compare with the discussion for the results in figure 5.5). On the 2+1-dimensional level the Hamiltonian constraint is a parabolic equation. It is not uncommon to use an implicit solver for a parabolic equation.

For the simulations the outer boundary is again at $R = 20$ (we use the Bjørhus projection method to implement the boundary condition for the evolution equations, see section 4.7.1). We initialize the wave package at $r = 10$ and direct it inwards. As initial data (concerning the full spacetime evolution) we use the exact solution. Otherwise we take the exact solution only for comparison in the convergence tests and it plays no role for the numerics any longer.

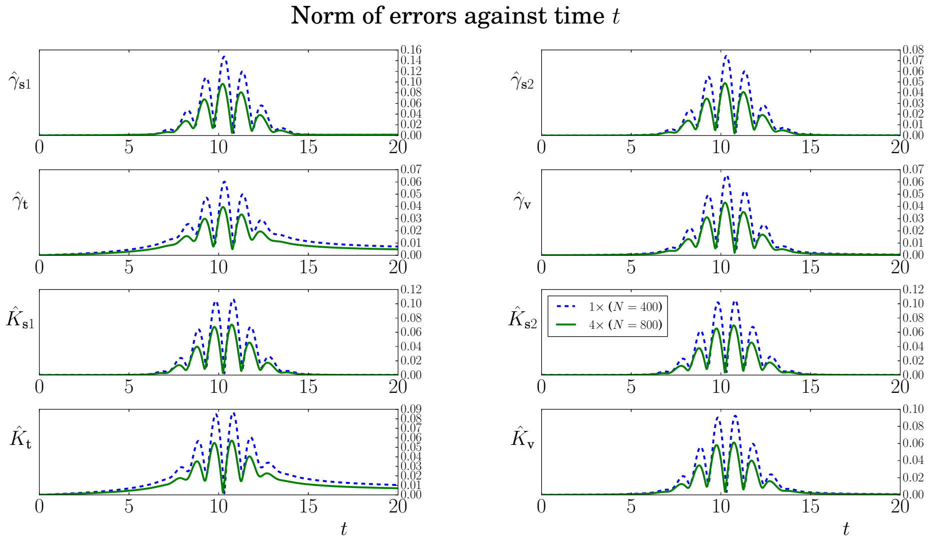


Figure 5.7.: At each time instance (horizontal axis labels time t) we show the L2-norm of the difference between the numerically calculated variable and the exact solution. The values for the higher resolution (solid green line) are multiplied with four. Hence we expect for convergence the solid green line to lie below or on top of the dashed blue line.

In figure 5.7 we see the results and observe that the numerical values of the variables converge to the exact solution. As long as four times the values for the higher resolution are less or equal than the values for the lower resolution (factor 1/2) it is a verification that the numerical solution converges to the exact solution in the continuum limit. In fact the convergence seems to be even slightly faster than the desired factor.

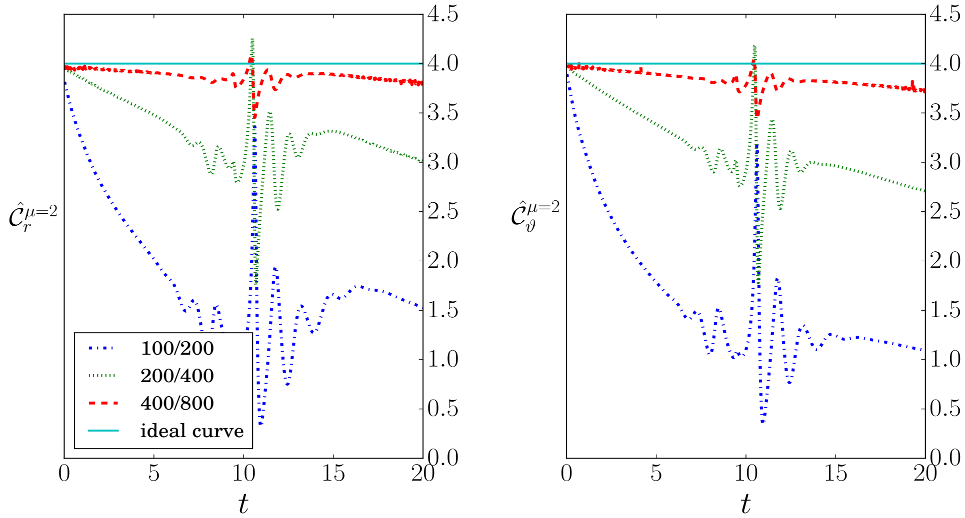


Figure 5.8.: We show the quotients of the L2-norms of the residuals of the momentum constraint. The Hamiltonian constraint is satisfied due to the application of the iterative solver. The quotient is calculated for two resolutions, one having twice as many grid points. Ideally the quotient would be equal to four (solid turquois line in the plots).

The violations in the Hamiltonian constraint are indistinguishable from zero, hence the Newton-Raphson iteration solves that equation appropriately. We display the momentum constraint in figure 5.8. Observe that the displayed violations are significantly larger than for the constraint solvers with exact inhomogeneities above. Nevertheless the figure shows that in the continuum limit the momentum constraint converges to zero and hence it is satisfied in the fully constrained scheme as well.

5.4. The linear 2+1-dimensional level

5.4.1. The constraint solver

The constraints form close to the origin a stiff (see definition 2.2.3) system of partial differential equations in the "time" direction r. We use a spectral expansion in known basis functions in the ϑ-direction and have knowledge of the action of the derivatives with respect to ϑ. Hence we can implement the constraints with a solver for ordinary differential equations with respect to the coordinate r. Similar as for the 1+1-dimensional case in section 5.3 we make use of the `ODEint` solver explained already there. We remark that we had to experiment with the tolerance of the solver and found that for the 2+1-dimensional situation decreasing the default value (roughly 10^{-8}) to 10^{-10} improved the solution substantially.

Again we mention that we are working with the modified version of the momentum constraint with parameter $\mu = 2$ in theorem 4.7.3. Since the trace of the extrinsic curvature vanishes on the linear level and the exact solution is used for the freely specifiable data, the inhomogeneity of the modified momentum constraint vanishes.

To test the constraint solver, we take all the remaining variables (either the set $\{\tilde{\gamma}_{\mathrm{s}2}, \tilde{\gamma}_{\mathrm{v}}, \tilde{\gamma}_{\mathrm{t}}, \tilde{K}_{\mathrm{s}1}, \tilde{K}_{\mathrm{t}}\}$ or $\{\tilde{\gamma}_{\mathrm{s}2}, \tilde{\gamma}_{\mathrm{v}}, \tilde{\gamma}_{\mathrm{t}}, \tilde{K}_{\mathrm{s}2}, \tilde{K}_{\mathrm{t}}\}$), the "freely specifiable data", from the exact solution. We excite one mode, say the $\ell = 2$-mode (and use the explicit form in appendix A.4). We generate $\hat{u}_{\ell=2}(t, r)$ with a generating function $G(t, r) = G(x) = A \exp(-x^2/2)$, $x = r \pm t$ and amplitude $A = 1$ and build the exact linear solution as $\tilde{u}(t, r, \vartheta) = \sum_{\ell=0}^{L-1} \hat{u}_{\ell=1}(t, r)\mathcal{Y}_\ell(\vartheta)$, see again footnote 5 on page 174.

For the demonstration that the constraint solver does the task as expected we give at each point in the r-direction the freely specifiable data and apply the constraint solver for the remaining variables, similarly as in the 1+1-dimensional situation.

No matter if one is interested in a free or (partially or fully) constrained scheme, on the initial slice all constraints need to be satisfied. We encountered several problems with the Hamiltonian constraint, which is a stiff parabolic equation. Nevertheless it is important to demonstrate that wherever the spacetime evolution is supposed to start we are able to apply the full constraint solver. We show that we can solve the initial constraints as evolutionary equations everywhere with the `ODEint` solver.

In the evolutionary approach one needs to find a way to obtain initial values for the solver. There are several ways to obtain initial values at the origin (strictly speaking the evolutionary scheme starts at $r = \epsilon > 0$ but the methods also give the value at $r = 0$).

The first possibility on the linear level is to obtain the values by explicitly using the mode structure. There we used the Taylor expansion in the mode equations as described in section 5.3. This approach will not be generalizable (since we do not have the explicit mode equations on the nonlinear level) and we will not consider it in full detail. To obtain positive results with this approach is no surprise since the modes decouple on the linear level.

Secondly, we can use the special mode structure of the gravitational perturbations derived in section 2.5 which is due to the spin-weighted structure. We will discuss and demonstrate the method for the solution of one set of variables, here for $\{\tilde{\gamma}_{\mathrm{s1}}, \tilde{K}_{\mathrm{s2}}, \tilde{K}_{\mathrm{v}}\}$.

The third and most promising option is the Taylor expansion at the origin of the full linear constraints. That option will be generalized to the nonlinear level and we will show it for both sets of variables for which the parabolic-hyperbolic formulation forms a well-posed formulation.

The initial value for $\tilde{\gamma}_{\mathrm{s1}}$ can be obtained by applying the corresponding evolution equation, see theorem 4.5.2. Since it is fully regular we do not expect any problems. This remains true on the nonlinear level as well. Therefore we apply the technique in the following numerical experiments when we use an evolutionary solver.

5.4.1.1. Initial data – using mode equations

On the linear level we can obtain, as on the mode level in section 5.3, the initial data with the help of the constraints on the mode level. With $\hat{u}_\ell(r=0)$ and $\hat{u}_\ell(r=\Delta r)$ obtained there we build as usual with the multiplication of the corresponding spherical harmonics the variable $\tilde{u}$. Unfortunately the method is not generalizable to the nonlinear situation since the equations of the constraints cannot be given in terms of the individual modes there. Therefore we will not consider the method further but remark that we obtained positive results on the linear level.

5.4.1.2. Initial data – using structure of eigenfunctions of the Laplacian

In section 2.5 we examined the eigenfunctions of the Laplace operator in spherical polar coordinates for spin-weighted quantities. We use the fact that the variables have the following structure

$$\tilde{u} = \sum_\ell \hat{u}_\ell \mathcal{Y}_\ell = \sum_\ell \left(\bar{u}_\ell r^{\ell-2} + \bar{v}_\ell r^\ell + \bar{w}_\ell r^{\ell+2}\right) \mathcal{Y}_\ell \qquad \text{(2.146 rev.)}$$

with constant (in r and ϑ, not in t) coefficients $\bar{u}_\ell, \bar{v}_\ell, \bar{w}_\ell$. Because of regularity at the origin $\bar{u}_0$ and $\bar{u}_1$ should vanish. Hence (see corollary 2.5.1)

$$\tilde{u}(r=0) = \bar{v}_2\mathcal{Y}_2 + \bar{v}_0\mathcal{Y}_0, \qquad (2.151 \text{ rev.})$$

$$\tilde{u}(r=\Delta r) = \bar{v}_2\mathcal{Y}_2 + \bar{v}_0\mathcal{Y}_0 + (\bar{v}_1\mathcal{Y}_1 + \bar{v}_3\mathcal{Y}_3)\,\Delta r + \mathcal{O}(\Delta r^2). \qquad (2.152 \text{ rev.})$$

On the linear level the aim is to reproduce the exact solution derived in section 4.6. Hence we restrict to the gravitational perturbation with spin-weight 2,

$$\tilde{u}(r=0) = \bar{v}_2\mathcal{Y}_2, \tag{5.11a}$$

$$\tilde{u}(r=\Delta r) = \bar{v}_2\mathcal{Y}_2 + \bar{v}_1\mathcal{Y}_1\Delta r + \mathcal{O}(\Delta r^2). \tag{5.11b}$$

Thus we have to solve at $r=0$ the linear momentum constraint (interpreted as equations for $\tilde{K}_{\mathrm{s2}}$ and $\tilde{K}_{\mathrm{v}}$),

$$\text{for } \tilde{\mathcal{C}}_r^{\mu=2}:\ \bar{K}_{\mathrm{s2},\ell=2}\big|_0 + 3\bar{K}_{\mathrm{v},\ell=2}\big|_0 = -2\bar{K}_{\mathrm{s1},\ell=2}\big|_0, \tag{5.12a}$$

$$\text{and for } \tilde{\mathcal{C}}_\vartheta^{\mu=2}:\ 3\bar{K}_{\mathrm{s2},\ell=2}\big|_0 + 3\bar{K}_{\mathrm{v},\ell=2}\big|_0 = 2\bar{K}_{\mathrm{s1},\ell=2}\big|_0 + 2\bar{K}_{\mathrm{t},\ell=2}\big|_0. \tag{5.12b}$$

We write it as

$$\mathbf{a}_1\bar{K}_{\mathrm{s1},\ell=2}\big|_0 + \mathbf{a}_2\bar{K}_{\mathrm{v},\ell=2}\big|_0 = \mathbf{y} \tag{5.13}$$

with $\mathbf{a}_1 = (1,3)^\dagger$, $\mathbf{a}_2 = (3,3)^\dagger$ and $\mathbf{y} = (-2\bar{K}_{\mathrm{s1},\ell=2}\big|_0, 2\bar{K}_{\mathrm{s1},\ell=2}\big|_0 + 2\bar{K}_{\mathrm{t},\ell=2}\big|_0)^\dagger$. The coefficient matrix has full rank in the linear case and should therefore give a unique solution.

We write equation (5.13) in the form

$$A\mathbf{x} = \mathbf{y} \tag{5.14}$$

where $\mathbf{y}$ as in equation (5.13) and $\mathbf{x} = (\bar{K}_{\mathrm{s1},\ell=2}\big|_0, \bar{K}_{\mathrm{v},\ell=2}\big|_0)^\dagger$. We know explicitly the coefficient matrix A which consists of $\mathbf{a}_1 = (1,3)^\dagger$ and $\mathbf{a}_2 = (3,3)^\dagger$ and hence reads

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}. \tag{5.15}$$

To solve the linear system (5.14) we have to invert A,

$$A^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 3 \\ 3 & 1 \end{pmatrix}. \tag{5.16}$$

On the nonlinear level the matrix inversion cannot be written out that easily because the coefficients are not constant as in the linear case. There we make use of a numerical routine. If we want to use a numerical inversion on the linear level as well we can use the SciPy module `lstsq`[8] for the matrix inversion. It is a routine that solves equation (5.14) iteratively by minimizing the L2-norm of $||\mathbf{y} - A\mathbf{x}||$.

For $r = \Delta r$ we take the r-derivative of the momentum constraint and then consider the resulting relations at $r = 0$, so we have

$$\begin{aligned} \text{for } \tilde{\mathcal{C}}_r^{\mu=2} &: \frac{\cos\vartheta}{\sin\vartheta}\partial_r\tilde{K}_\mathrm{v} + \partial_\vartheta\tilde{K}_\mathrm{v} \ - 8\partial_r\tilde{K}_\mathrm{s1} = 0, && (5.17\text{a})\\ \text{and for } \tilde{\mathcal{C}}_\vartheta^{\mu=2} &: 3\partial_r\partial_\vartheta\tilde{K}_\mathrm{s2} + 4\partial_r\tilde{K}_\mathrm{v} \ - 2\partial_r\partial_\vartheta\tilde{K}_\mathrm{s1} \\ &\quad + 2\frac{\cos\vartheta}{\sin\vartheta}\partial_r\tilde{K}_\mathrm{t} - 2\partial_r\partial_\vartheta\tilde{K}_\mathrm{t} = 0. && (5.17\text{b}) \end{aligned}$$

Again the equations are evaluated at $r = 0$ where all the variables only have contributions in several particular modes, see section 2.5. Therefore we can consider the equations on the mode level, namely just in $\ell = 0$ and $\ell = 2$ (in particular ℓ is even). The finite-difference expression up to the order of interest is (see section 2.2.3)

$$\partial_r\hat{u}_\ell\big|_i = \frac{\hat{u}_\ell\big|_{i-1} - 2\hat{u}_\ell\big|_i + \hat{u}_\ell\big|_{i+1}}{2\Delta r} + \mathcal{O}(\Delta r^2) \tag{5.18}$$

and will be evaluated at $r = 0$. Hence we can use the parity for the modes (see proposition 4.4.1), so $\hat{u}_\ell\big|_{-1} = \hat{u}_\ell\big|_1$ since ℓ is even. Therefore we have

$$\partial_r\hat{u}_\ell\big|_0 = \frac{\hat{u}_\ell\big|_1 - \hat{u}_\ell\big|_0}{\Delta r} + \mathcal{O}(\Delta r^2). \tag{5.19}$$

The equations for the momentum constraint at $r = \Delta r$ then read

$$\begin{aligned} \text{for } \tilde{\mathcal{C}}_r^{\mu=2} &: 3\bar{K}_{\mathrm{v},\ell=2}\big|_1 = \ 3\bar{K}_{\mathrm{v},\ell=2}\big|_0 \ - 4\left(\bar{K}_{\mathrm{s1},\ell=2}\big|_1 - \bar{K}_{\mathrm{s1},\ell=2}\big|_0\right), && (5.20\text{a})\\ \text{and for } \tilde{\mathcal{C}}_\vartheta^{\mu=2} &: 3\bar{K}_{\mathrm{s2},\ell=2}\big|_1 + 4\bar{K}_{\mathrm{v},\ell=2}\big|_1 = \ 3\bar{K}_{\mathrm{s2},\ell=2}\big|_0 + 4\bar{K}_{\mathrm{v},\ell=2}\big|_0 \\ &\quad + 2\left(\bar{K}_{\mathrm{t},\ell=2}\big|_1 - \bar{K}_{\mathrm{t},\ell=2}\big|_0\right) \ + 2\left(\bar{K}_{\mathrm{s1},\ell=2}\big|_1 - \bar{K}_{\mathrm{s1},\ell=2}\big|_0\right). && (5.20\text{b}) \end{aligned}$$

It can again be written as a matrix equation like (5.14) for $\mathbf{x} = (\bar{K}_{\mathrm{s1},\ell=2}\big|_1, \bar{K}_{\mathrm{v},\ell=2}\big|_1)^\dagger$, here the matrix is

$$A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}. \tag{5.21}$$

[8]See https://docs.scipy.org/doc/scipy-0.19.0/reference/generated/scipy.integrate.odeint.html.

The explicit inverse reads

$$A^{-1} = \frac{1}{9}\begin{pmatrix} -4 & 3 \\ 3 & 0 \end{pmatrix} \tag{5.22}$$

or can be obtained numerically again with the help of `lstsq` for example. The system can be solved in the same way as for $r = 0$.

As already discussed we take initial data (concerning the spacetime evolution) at some instance such that the wave package is located away from the origin $r = 0$ and also from the outer boundary (which we place at $r = R = 20$), we start at $r = 10$. At that time instance we solve the constraints in the full spatial domain. Afterwards we proceed in a similar fashion as in section 5.3 for the mode equations. We repeat the process at the next time step where the peak of the wave package is located slightly closer to the origin. The aim is to show that the residuals of the constraints vanish in the continuum limit, see section 2.2.6.

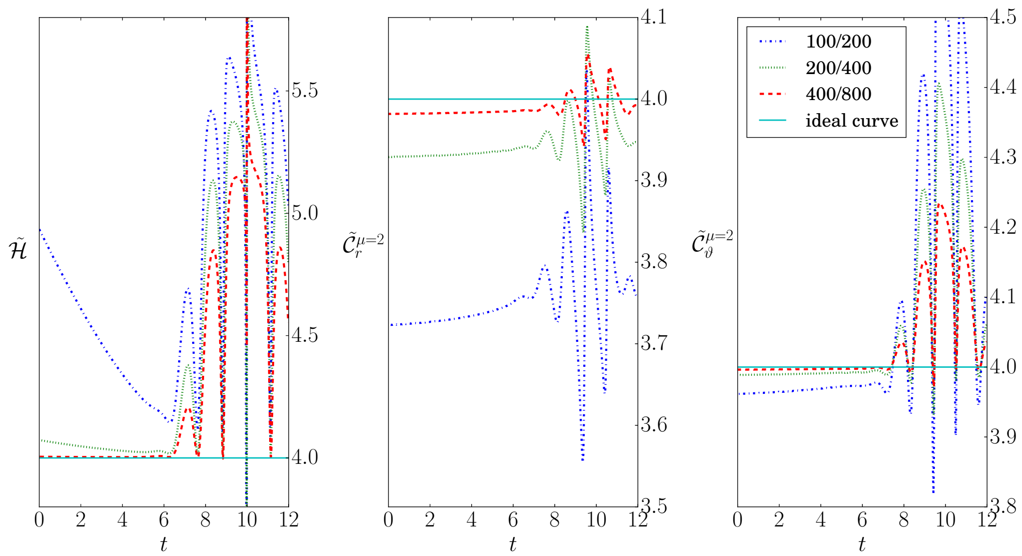


Figure 5.9.: We show the result after we solve the constraints at each time instance for $\{\tilde{\gamma}_{s1}, \tilde{K}_{s2}, \tilde{K}_{v}\}$, taking the remaining variables from the exact solution. We obtain the initial data for the evolutionary solver `ODEint` with the special mode structure as discussed in the main text. For similar considerations as in section 5.3 we show the quotients between several resolutions. The spectral cutoff is at $L = 12$, the tolerance of the solver is set to 10^{-10}.

We see in figure 5.9 that the described constraint solver produces convergent results. Actually, especially for the Hamiltonian constraint, some results seems to be over-converging.

5.4.1.3. Initial data – using Taylor expansion

We obtain the initial data close to the origin for the momentum constraint as hyperbolic set of equations from the Taylor expansion of the linear equations. This option looks like the most promising approach and will be discussed and numerically demonstrated for both sets of variables, i.e. when solving the constraints for either $\{\tilde{\gamma}_{\mathrm{s}1}, \tilde{K}_{\mathrm{s}2}, \tilde{K}_{\mathrm{v}}\}$ or $\{\tilde{\gamma}_{\mathrm{s}1}, \tilde{K}_{\mathrm{s}1}, \tilde{K}_{\mathrm{v}}\}$. We use the evolution equation for $\tilde{\gamma}_{\mathrm{s}1}$ for the first innermost grid points. It provides us with the initial data for the Hamiltonian constraint.

We will demonstrate that the parabolic-hyperbolic solver works and produces convergent results "everywhere" (i.e. wherever the initial solver is supposed to produce the initial values) in the cases when the momentum constraint is interpreted as a set of equations for $\tilde{K}_{\mathrm{s}2}$ and $\tilde{K}_{\mathrm{v}}$ and for $\tilde{K}_{\mathrm{s}1}$ and $\tilde{K}_{\mathrm{v}}$.

We make use of the Taylor expansion of the variables close to the origin. All variables are arranged such that (see proposition 4.4.1) close to the origin they and their derivatives have a behavior in r like

$$u = u_{\mathrm{flat}} + u_{\mathrm{pert}} = u_{\mathrm{flat}} + u_0 + u_1 r + \mathcal{O}(r^2), \tag{5.23a}$$
$$\partial_r u = u_1 + 2u_2 r + \mathcal{O}(r^2), \tag{5.23b}$$
$$\partial_r^2 u = 2u_2 + 6u_3 r + \mathcal{O}(r^2), \tag{5.23c}$$

all coefficients u_i are r-independent. Inserting these expansions in the linear momentum constraint in theorem 4.5.2 leads to equations close to the origin of the form (again the coefficients are r-independent)

$$\tilde{\mathcal{C}}_r^{\mu=2} = \tilde{\mathcal{C}}_{r0} + \tilde{\mathcal{C}}_{r1} r + \mathcal{O}(r^2) \tag{5.24a}$$
$$\text{and } \tilde{\mathcal{C}}_\vartheta^{\mu=2} = \tilde{\mathcal{C}}_{\vartheta 0} + \tilde{\mathcal{C}}_{\vartheta 1} r + \mathcal{O}(r^2). \tag{5.24b}$$

These are enough relations to determine the missing values of $\tilde{K}_{\mathrm{s}2}$ and $\tilde{K}_{\mathrm{v}}$ (or of $\tilde{K}_{\mathrm{s}1}$ and $\tilde{K}_{\mathrm{v}}$ respectively) at the grid points $r = r_0 = 0$ and $r = r_1 = \Delta r$. Evaluating equation (5.24a) at the origin $r = 0$ leads to the relations

$$\tilde{\mathcal{C}}_{r0} = -4\ \tilde{K}_{\mathrm{s}1}|_0\ - 2\ \tilde{K}_{\mathrm{s}2}|_0\ + \frac{\cos\vartheta}{\sin\vartheta}\ \tilde{K}_{\mathrm{v}}|_0\ + \partial_\vartheta \tilde{K}_{\mathrm{v}}|_0 = 0 \tag{5.25a}$$

$$\text{and } \tilde{\mathcal{C}}_{\vartheta 0} = 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\tilde{K}_{\rm t}|_0 \;+3\,\tilde{K}_{\rm v}|_0 \;-2\,\partial_\vartheta \tilde{K}_{\rm s1}|_0 \;+3\,\partial_\vartheta \tilde{K}_{\rm s2}|_0 \;+\partial_\vartheta \tilde{K}_{\rm t}|_0 = 0. \tag{5.25b}$$

The second set of relations for the terms proportional to r is

$$\tilde{\mathcal{C}}_{r1} = -8\,\tilde{K}_{\rm s1}|_1 \;+\frac{\cos\vartheta}{\sin\vartheta}\,\tilde{K}_{\rm v}|_1 \;+\partial_\vartheta \tilde{K}_{\rm v}|_1 = 0 \tag{5.26a}$$

$$\text{and } \tilde{\mathcal{C}}_{\vartheta 1} = 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\tilde{K}_{\rm t}|_1 \;+4\,\tilde{K}_{\rm v}|_1 \;-2\,\partial_\vartheta \tilde{K}_{\rm s1}|_1 \;+3\,\partial_\vartheta \tilde{K}_{\rm s2}|_1 \;+\partial_\vartheta \tilde{K}_{\rm t}|_1 = 0. \tag{5.26b}$$

5.4.1.3.1. Momentum constraint as system for $\tilde{K}_{\rm s2}$ and $\tilde{K}_{\rm v}$

We write the system of equations that need to be solved as

$$a_1 + b_1 \tilde{K}_{\rm s2}|_0 + c_1 \tilde{K}_{\rm v}|_0 + d_1 \partial_\vartheta \tilde{K}_{\rm s2}|_0 + e_1 \partial_\vartheta \tilde{K}_{\rm v}|_0 = 0, \tag{5.27a}$$
$$a_2 + b_2 \tilde{K}_{\rm s2}|_0 + c_2 \tilde{K}_{\rm v}|_0 + d_2 \partial_\vartheta \tilde{K}_{\rm s2}|_0 + e_2 \partial_\vartheta \tilde{K}_{\rm v}|_0 = 0, \tag{5.27b}$$
$$a_3 + b_3 \tilde{K}_{\rm s2}|_1 + c_3 \tilde{K}_{\rm v}|_1 + d_3 \partial_\vartheta \tilde{K}_{\rm s2}|_1 + e_3 \partial_\vartheta \tilde{K}_{\rm v}|_1 = 0, \tag{5.27c}$$
$$a_4 + b_4 \tilde{K}_{\rm s2}|_1 + c_4 \tilde{K}_{\rm v}|_1 + d_4 \partial_\vartheta \tilde{K}_{\rm s2}|_1 + e_4 \partial_\vartheta \tilde{K}_{\rm v}|_1 = 0 \tag{5.27d}$$

with $a_1 = -4\tilde{K}_{\rm s1}|_0$, $a_2 = 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\tilde{K}_{\rm t}|_0 \;+\partial_\vartheta \tilde{K}_{\rm t}|_0 \;-2\,\tilde{K}_{\rm s2}|_0$, $a_3 = -8\,\tilde{K}_{\rm s1}|_1$, $a_4 = 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\tilde{K}_{\rm t}|_1 \;+\partial_\vartheta \tilde{K}_{\rm t}|_1 \;-2\,\tilde{K}_{\rm s2}|_1$, $b_1 = -2$, $c_1 = c_3 = \frac{\cos\vartheta}{\sin\vartheta}$, $c_2 = d_2 = d_4 = 3$, $c_4 = 4$, $e_1 = e_3 = 1$ and the remaining coefficients vanish. Therefore the equations have the form

$$Au = b \tag{5.28}$$

with linear operator A build with coefficients $b, \ldots, e$.

For the solution we use the Python solver `bicgstab`[9]. Conjugate gradient methods are iterative methods to solve linear systems $Au = b$ for unknown u and require the matrix A to be symmetric and positive definite. Biconjugate gradient methods are a generalization to non-symmetric matrices A. Both are discussed in Press et al. (2007). The improvement to a stabilized biconjugate gradient method (abbreviated "bicgstab") was suggested in van der Vorst (1992). It has faster and smoother convergence properties.

An advantage of the Python solver is that the matrix A does not need to be hardcoded as a matrix. It is implemented as an object called "linear operator"[10],

[9] See https://docs.scipy.org/doc/scipy-0.14.0/reference/generated/scipy.sparse.linalg.bicgstab.html.
[10] See https://docs.scipy.org/doc/scipy-0.14.0/reference/generated/scipy.sparse.linalg.LinearOperator.html

i.e. a map $x \mapsto Ax$, which is easier to code than the nonconstant matrix itself. A has no constant coefficients but is computed during the runtime with the variable coefficients of that instance in the simulation.

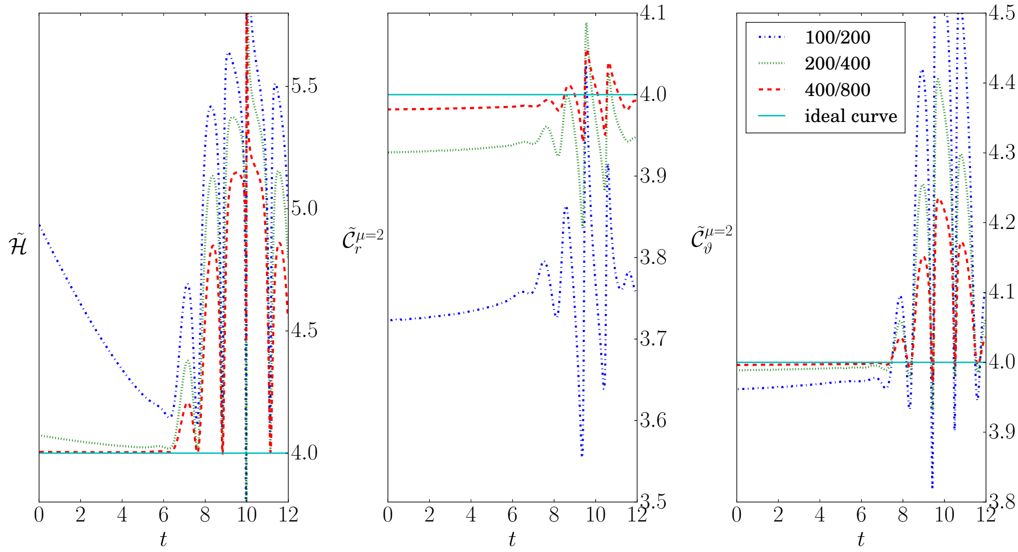


Figure 5.10.: We show the result after we solve the constraints at each time instance for $\{\tilde{\gamma}_{\rm s1}, \tilde{K}_{\rm s2}, \tilde{K}_{\rm v}\}$, the remaining variables are taken from the exact solution. We use the discussed Taylor expansion for the initial data for the momentum constraint and the free evolution of $\tilde{\gamma}_{\rm s1}$ for the Hamiltonian constraint. We solve the whole system with the `ODEint` solver. Again we plot the quotient between different resolutions. The spectral cutoff is again at $L = 12$, the tolerance of the solver is set to 10^{-10}.

Using the discussed scheme to obtain initial values and solving all the constraints with `ODEint` results in the residuals in figure 5.10. It is remarkable that the results are indistinguishable from the ones obtained before with the use of the special structure of the modes and plotted in figure 5.9. On the other hand the solver as such is identical, only the techniques to determine the initial values at the origin differ. We interpret the similarities in the results as numerical evidence that both procedures work fine.

5.4.1.3.2. Momentum constraint as system for $\tilde{K}_{s1}$ and $\tilde{K}_v$

The momentum constraint interpreted as a system for $\tilde{K}_{s1}$ and $\tilde{K}_v$ results essentially in a very similar system as the one in equation (5.27), only some variables are switched and the coefficients are slightly changed. Namely we have the system

$$a_1 + b_1\tilde{K}_{s1}|_0 + c_1\tilde{K}_v|_0 + d_1\partial_\vartheta\tilde{K}_{s1}|_0 + e_1\partial_\vartheta\tilde{K}_v|_0 = 0, \tag{5.29a}$$
$$a_2 + b_2\tilde{K}_{s1}|_0 + c_2\tilde{K}_v|_0 + d_2\partial_\vartheta\tilde{K}_{s1}|_0 + e_2\partial_\vartheta\tilde{K}_v|_0 = 0, \tag{5.29b}$$
$$a_3 + b_3\tilde{K}_{s1}|_1 + c_3\tilde{K}_v|_1 + d_3\partial_\vartheta\tilde{K}_{s1}|_1 + e_3\partial_\vartheta\tilde{K}_v|_1 = 0, \tag{5.29c}$$
$$a_4 + b_4\tilde{K}_{s1}|_1 + c_4\tilde{K}_v|_1 + d_4\partial_\vartheta\tilde{K}_{s1}|_1 + e_4\partial_\vartheta\tilde{K}_v|_1 = 0 \tag{5.29d}$$

with $a_1 = -2\tilde{K}_{s2}|_0$, $a_2 = 2\ \frac{\cos\vartheta}{\sin\vartheta}\ \tilde{K}_t|_0\ + \partial_\vartheta\tilde{K}_t|_0\ + 3\ \tilde{K}_{s2}|_0$, $a_3 = 0$, $a_4 = 2\ \frac{\cos\vartheta}{\sin\vartheta}\ \tilde{K}_t|_1\ + \partial_\vartheta\tilde{K}_t|_1\ + 3\ \tilde{K}_{s2}|_1$, $b_1 = -4$, $b_3 = -8$, $c_1 = c_3 = \frac{\cos\vartheta}{\sin\vartheta}$, $c_2 = 3$, $c_4 = 4$, $d_2 = d_4 = -2$, $e_1 = e_3 = 1$ and the remaining coefficients vanish. Again the equations can be written in the same form as above,

$$Au = b \tag{5.28 rev.}$$

We can apply the same routine to solve the system.

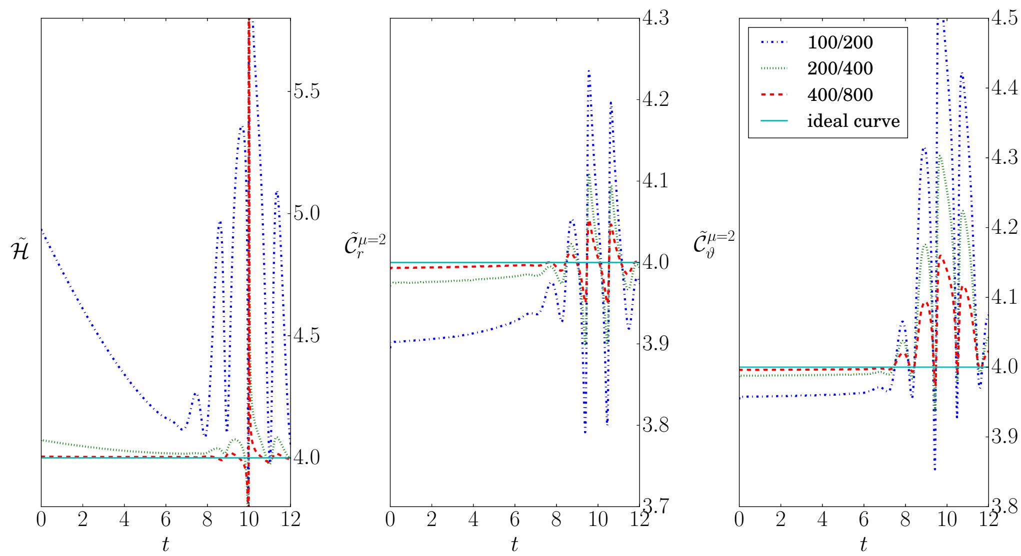


Figure 5.11.: We show the result as before after we solve the constraints at each time instance for $\{\tilde{\gamma}_{s1}, \tilde{K}_{s1}, \tilde{K}_v\}$. Except for the different variable we solve for everything else is the same as for the situation in figure 5.10.

We show the results in figure 5.11. Clearly the situation is essentially the same as in the discussion for figure 5.10 above. There is absolutely no difference concerning the (on the linear level) decoupled Hamiltonian constraint. The curves for the momentum constraint are slightly different than in figure 5.10 due to the fact that we consider a different set of variables. The overall picture, the convergence of the constraint solver in the continuum limit, remains.

5.4.1.3.3. Using the decoupling of the constraints

On the 1+1-dimensional level we experienced some problems with the `ODEint` solver for the parabolic equation. On the linear level the Hamiltonian and momentum constraints decouple. We can use this fact and apply the `ODEint` solver for the momentum constraint and the Newton-Raphson method for the Hamiltonian one. On the nonlinear level the equations do not decouple. As generalization for this case we propose to use an iterative routine that solves first the Hamiltonian constraint and then the momentum constraint (or the other way around). The process is then iterated until a prescribed abortion is reached. Since the quantities should not differ drastically from one time level to the next we expect that the procedure should converge.

In figure 5.12 we display the results for the solver that uses the decoupling of the Hamiltonian and momentum constraints. We observe that for the momentum constraint the result is exactly as above and hence it shows convergence in the continuum limit. This is clear because we applied the same solver for the completely decoupled equations.

The Hamiltonian constraint is solved by the Newton-Raphson method at the first iteration. It is a linear equation and therefore the constraint is satisfied up to the numerical threshold after one application of the routine. Therefore we plot the convergence of the variable $\tilde{\gamma}_{s1}$ again. We see again a spike at the transition through the origin and some small oscillation (observe that we display only a region close to the desired value of four), increasing with decreasing step size. Both effects are due to the representation in the plot where a division is involved. We conclude that the independent constraint solver converges in the continuum limit.

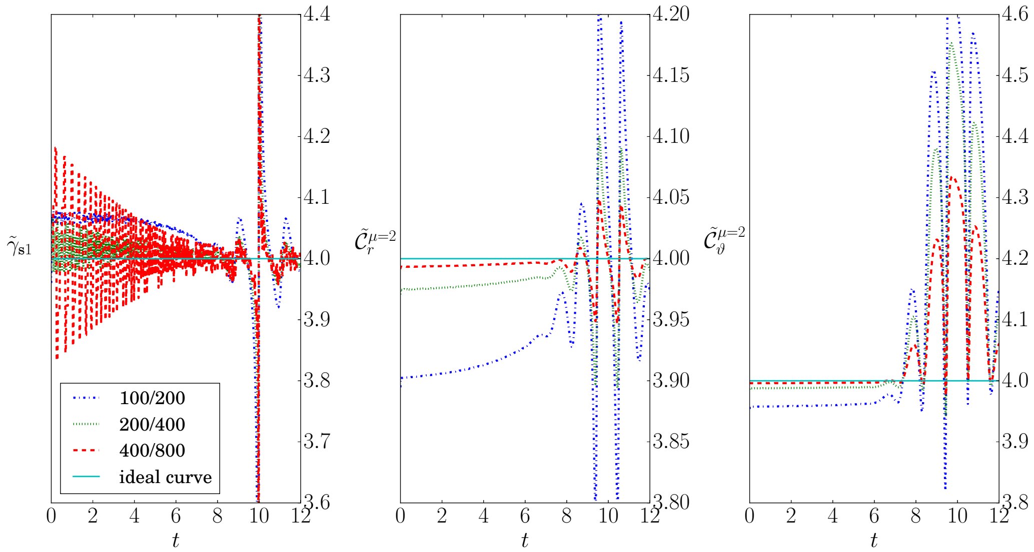


Figure 5.12.: We show the result after we solve the constraints at each time instance for $\{\tilde{\gamma}_{\rm s1}, \tilde{K}_{\rm s1}, \tilde{K}_{\rm v}\}$. Since the violations in the Hamiltonian constraint are indistinguishable from zero (Newton-Raphson solver) we display again the convergence factors of the variable $\tilde{\gamma}_{\rm s1}$. The spectral cutoff is again at $L = 12$, the tolerance of the `ODEint` solver is set to 10^{-10}.

Summary

For the constraints regarded as parabolic-strongly hyperbolic system we can state that several versions of the solver work fine. The solvers are fed with the exact linear solution for the freely specifiable source terms and solve the constraints correctly everywhere. This statement is true for both sets of variables $\{\tilde{\gamma}_{\rm s1}, \tilde{K}_{\rm s2}, \tilde{K}_{\rm v}\}$ and $\{\tilde{\gamma}_{\rm s1}, \tilde{K}_{\rm s1}, \tilde{K}_{\rm v}\}$. We discussed two possible options for the solver of the Hamiltonian constraint as parabolic equation, the `ODEint` solver and the application of the Newton-Raphson method. For the momentum constraint as strongly hyperbolic set of equations we discussed several successful options to obtain initial values close to the origin.

5.4.2. The fully constrained evolution

With the experience of the 1+1-dimensional simulations in section 5.3 we intend to implement a fully constrained evolution. The analysis in section 4.7.2 suggests to apply the evolution equations for the set $\left\{\tilde{\gamma}_\varphi, \tilde{\gamma}_{\mathrm{v}}, \tilde{\gamma}_{\mathrm{t}}, \tilde{K}_\varphi, \tilde{K}_{\mathrm{t}}\right\}$. The variables $\{\tilde{\gamma}_{\mathrm{s}2}, \tilde{K}_{\mathrm{s}2}\}$ can be obtained by algebraic assignments. The constraints should therefore be solved as a parabolic-strongly hyperbolic set of equations for $\{\tilde{\gamma}_{\mathrm{s}1}, \tilde{K}_{\mathrm{s}1}, \tilde{K}_{\mathrm{v}}\}$.

We try to benefit from earlier results for the simulation of the wave equation, see the discussion in section 2.6. There we derived some “filter” properties for grid points at or in the neighborhood of the origin, which were justified by the knowledge that the ℓ^{th} mode has a behavior to $\mathcal{O}(r^\ell)$ close to the origin. In section 2.5 we derived a similar behavior for the eigenfunctions of the Laplace operator on the sphere (which applies for our variables by assumption). The ℓ^{th} mode has contributions $\mathcal{O}(r^\ell)$ and $\mathcal{O}(r^{\ell\pm 2})$ close to the origin with corresponding spin-weight. Since we want to reproduce the exact solution (with spin-weight 2 since it is a gravitational perturbation) we know that the leading contribution close to the origin from the modes is $\mathcal{O}(r^{\ell-2})$ for $\ell \geq 2$ (we expect for $\ell = 0$ and $\ell = 1$ close to the origin only contributions $\mathcal{O}(r^0)$ and $\mathcal{O}(r^1)$ from different spin-weight).

The fully regular evolution equations for the components of the extrinsic curvature tensor have a coordinate singularity at the origin, even though they are fully regular. It is recommendable to take care of it explicitly. The structure of the evolution equations is (compare with the equations in theorem 4.5.2)

$$\partial_t \tilde{\gamma} \sim \tilde{K}, \tag{5.30a}$$

$$\partial_t \tilde{K} = \frac{f}{r^2} + \frac{g}{r} + h. \tag{5.30b}$$

We made good experience with the use of the following lemma.

Lemma 5.4.1. Close to the origin we write the evolution equation for the extrinsic curvature as

$$\partial_t \tilde{K} = \frac{1}{2}\partial_r^2 f + \partial_r g + h. \tag{5.31}$$

Proof. Since the equation is regular we know that f approaches zero in the limit $r \to 0$ (multiply equation (5.30b) with r^2 and set $r = 0$). Therefore we can apply L'Hôpital's rule and get

$$\partial_t \tilde{K} = \frac{\frac{1}{2}\partial_r f + g}{r} + h \tag{5.32}$$

close to the origin. Again using the same regularity argument implies that we can apply L'Hôpital's rule once more and we obtain the claim. □

Remark 5.4.1. With the exact solution (see section 4.6) we can confirm that $f = \mathcal{O}(r^2)$ and $g = \mathcal{O}(r)$.

All variables u are arranged such that they have the following Taylor expansion close to the origin (see section 4.4 and proposition 4.4.1)

$$u = a + br + cr^2 + dr^3 + \dots \tag{5.33}$$

with r-independent coefficients a, b, c, d, ... Evaluating the variable u at the first grid points at $r = 0$, Δr, $2\Delta r$ and $3\Delta r$ allows us to determine the coefficients for all variables. The derivatives have the following form,

$$\partial_r u = b + 2cr + 3dr^2 + \dots, \tag{5.34a}$$
$$\partial_r^2 u = 2c + 6dr + \dots. \tag{5.34b}$$

With lemma 5.4.1 we write the right-hand sides of the evolution equations for the extrinsic curvature close to the origin as (on the linear level the right-hand side only contains quantities of the spatial metric, hence we can unambiguously label the coefficients for $\tilde{\gamma}_{\mathrm{s1}}$ as a_{s1}, b_{s1}, ... and the same for the other variables)

$$\partial_t \tilde{K}_{\mathrm{s1}}|_0 = -c_{\mathrm{s1}} + 3c_{\mathrm{s2}} - \frac{\cos\vartheta}{2\sin\vartheta}c_{\mathrm{v}} + \frac{\cos\vartheta}{4\sin\vartheta}\partial_\vartheta c_{\mathrm{s2}} - \frac{3}{2}\partial_\vartheta c_{\mathrm{v}} + \frac{1}{4}\partial_\vartheta^2 c_{\mathrm{s1}} \tag{5.35a}$$

$$\begin{aligned}\partial_t \tilde{K}_{\mathrm{s2}}|_0 = {}& 2c_{\mathrm{s1}} - 6c_{\mathrm{s2}} - c_{\mathrm{t}} + \frac{5\cos\vartheta}{2\sin\vartheta}c_{\mathrm{v}} - \frac{\cos\vartheta}{4\sin\vartheta}\partial_\vartheta c_{\mathrm{s1}} - \frac{\cos\vartheta}{2\sin\vartheta}\partial_\vartheta c_{\mathrm{s2}} \\ &+ \frac{3\cos\vartheta}{2\sin\vartheta}\partial_\vartheta c_{\mathrm{t}} + \frac{5}{2}\partial_\vartheta c_{\mathrm{v}} - \frac{1}{4}\partial_\vartheta^2 c_{\mathrm{s1}} - \frac{1}{2}\partial_\vartheta^2 c_{\mathrm{s2}} + \frac{1}{2}\partial_\vartheta^2 c_{\mathrm{t}}\end{aligned} \tag{5.35b}$$

$$\partial_t \tilde{K}_{\mathrm{v}}|_0 = \frac{2\cos\vartheta}{\sin\vartheta}c_{\mathrm{v}} - c_{\mathrm{v}} + \frac{1}{2}\partial_\vartheta c_{\mathrm{s1}} - \partial_\vartheta c_{\mathrm{s2}} + \partial_\vartheta c_{\mathrm{t}} \tag{5.35c}$$

$$\partial_t \tilde{K}_{\mathrm{t}}|_0 = -3c_{\mathrm{t}} - \frac{3\cos\vartheta}{2\sin\vartheta}c_{\mathrm{v}} + \frac{\cos\vartheta}{4\sin\vartheta}\partial_\vartheta c_{\mathrm{s1}} + \frac{3}{2}\partial_\vartheta c_{\mathrm{v}} - \frac{1}{4}\partial_\vartheta^2 c_{\mathrm{s1}} \tag{5.35d}$$

for the zeroth grid point. For the first grid point we perform the same expansion and get equations of the general form

$$\partial_t \tilde{K}|_1 = C + D\Delta r \tag{5.36}$$

with terms C and D to be determined. We calculated the contribution C already in equation (5.35). We intend to calculate D in the same manner (here for simplicity just for $\tilde{K}_{\mathrm{s1}}$ and $\tilde{K}_{\mathrm{t}}$, say D_{s1} and D_{t}),

$$\frac{1}{6}D_{\mathrm{s1}} = d_{\mathrm{s1}} - 3d_{\mathrm{s2}} - \frac{1}{2}d_{\mathrm{t}} + \frac{5\cos\vartheta}{4\sin\vartheta}d_{\mathrm{v}} - \frac{\cos\vartheta}{8\sin\vartheta}\partial_\vartheta d_{\mathrm{s1}} - \frac{\cos\vartheta}{4\sin\vartheta}\partial_\vartheta d_{\mathrm{s2}}$$

$$+\frac{3\cos\vartheta}{4\sin\vartheta}\partial_\vartheta d_{\mathrm{t}}+\frac{5}{4}\partial_\vartheta d_{\mathrm{v}}-\frac{1}{8}\partial_\vartheta^2 d_{\mathrm{s1}}-\frac{1}{4}\partial_\vartheta^2 d_{\mathrm{s2}}+\frac{1}{4}d_{\mathrm{t}} \tag{5.37a}$$

$$\frac{1}{6}D_{\mathrm{t}}=-\frac{\cos\vartheta}{\sin\vartheta}d_{\mathrm{v}}-\frac{3}{2}d_{\mathrm{t}}+\frac{\cos\vartheta}{4\sin\vartheta}\partial_\vartheta d_{\mathrm{s1}}+\partial_\vartheta d_{\mathrm{v}}-\frac{1}{4}\partial_\vartheta^2 d_{\mathrm{s1}}. \tag{5.37b}$$

Th quantities $\partial_t\tilde{K}_{\mathrm{s2}}|_1$ and $\partial_t\tilde{K}_{\mathrm{s2}}|_1$ follow then with equation (5.36).

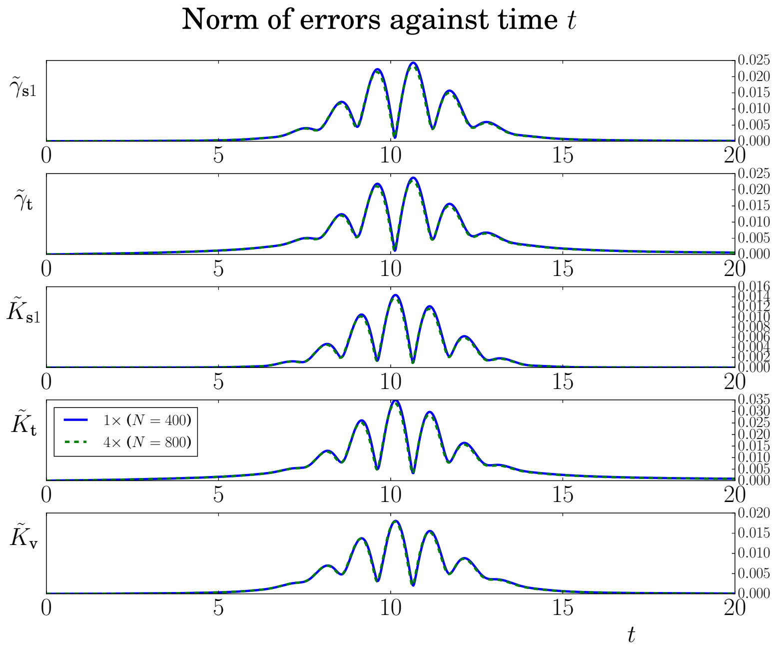


Figure 5.13.: We show the convergence results for the fully constrained scheme. The constraints are solved for $\{\tilde{\gamma}_{\mathrm{s1}}, \tilde{K}_{\mathrm{s1}}, \tilde{K}_{\mathrm{v}}\}$ at each time step. In addition the numerical evolution of the tensor components is used. We take the remaining variables from the exact solution. The spectral cutoff is again at $L = 12$, the tolerance of the `ODEint` solver is set to 10^{-10}.

As demonstration we show the results in figures 5.13 and 5.14. This is the fully constrained scheme, so the constraints (with `ODEint` solver for $\tilde{K}_{\mathrm{s1}}$ and $\tilde{K}_{\mathrm{v}}$ and Newton-Raphson solver for $\tilde{\gamma}_{\mathrm{s1}}$) are solved for on each time step. In addition we evolve the tensor components numerically. The plots show that the numerical solution converges to the exact one.

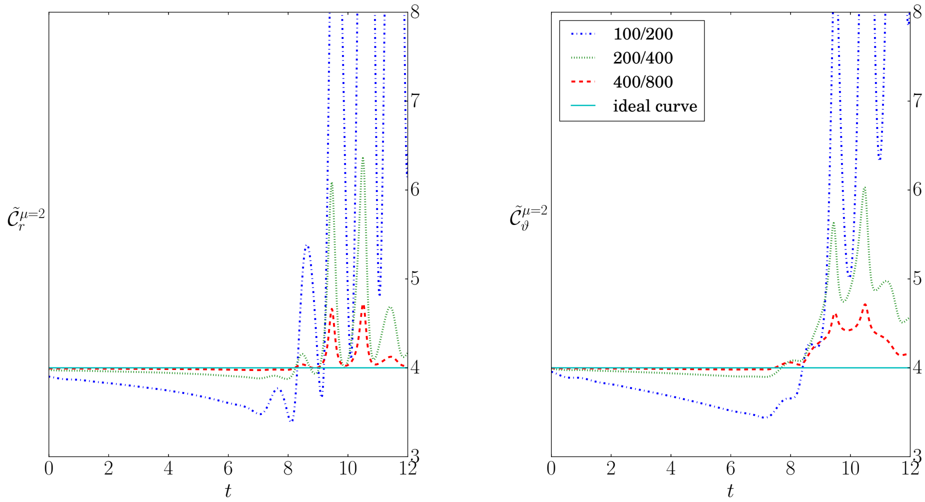


Figure 5.14.: We demonstrate the convergence of the constraints (for $\{\tilde{\gamma}_{s1}, \tilde{K}_{s1}, \tilde{K}_v\}$) in the fully constrained scheme, corresponding to the situation in figure 5.13. Since the violations in the Hamiltonian constraint are indistinguishable from zero (Newton-Raphson solver) we skip the corresponding convergence plot.

5.5. The nonlinear level

On the full nonlinear level we demonstrate that the evolutionary constraint solver can be applied in a similar way as for the linear level. For the purpose of illustration we restrict ourselves to the set $\{\gamma_{s1}, K_{s2}, K_v\}$. The remaining variables are taken from the exact **linear** analytical solution.

For the following simulations we excite again the $\ell = 2$-mode for the perturbation which has an amplitude $A = 10^{-5}$, see section 3.2.3.

5.5.1. The constraint solver

5.5.1.1. Initial data for the constraint solver

For the nonlinear constraint solver we aim to use exactly the same procedure as for the linear situation. A very straightforward generalization of the linear scheme is to use one of the methods described in section 5.4. We use the Taylor expansion in r at the origin for the variables, insert them in the momentum constraint and obtain, for each of the two components of the constraint, two relations which allow us to determine the required initial data.

The form of the relations for the momentum constraint is exactly the same as in equation (5.29), just the coefficients are nonlinear in the remaining variables and slightly more complicated. The corresponding coefficients $a_1, \ldots, e_4$ are listed in appendix A.5.

As already mentioned in section 5.4 the evolution equation for γ_{s1} is completely regular and we shall therefore use it to obtain initial values for γ_{s1} with the free evolution of the first few grid points.

5.5.1.2. Numerical results

The basic task in obtaining the initial data for the solver is to solve a linear system

$$Au = b \qquad (5.28 \text{ rev.})$$

for u. We use the same Python routine (`bicgstab`, see its discussion in section 5.4) as for the linear equations. It turns out that the routine works well if the wave package is somewhere close to the origin and therefore the absolute values of the entries of A and b are significantly different from zero. Also if the perturbation is far away the procedure produces reliable results. Far away the absolute values of the entries of A and b are very close to zero. In fact they are usually below the machine precision of the order 10^{-16}. Actually the routine gives the value 0 for the components of u, which is correct up to that threshold.

There is an intermediate regime though. In that regime the absolute values in A and b are already very small (usually somewhere below 10^{-10}) but in fact the zero-solution for u is not an appropriate solution. Nevertheless the routine seems to favor a vanishing u quite often. That problem can be solved by the simple observation that

$$Au = b \Leftrightarrow \alpha A u = \alpha b \qquad (5.38)$$

for some **amplification factor**[11] $\alpha \in \mathbb{R}$. The statement remains true of course if this α is different for the applications of the solver at different points in r-direction. Numerical experiments show that a proper choice is an r-dependent amplification factor α with values

[11] Here we choose the letter α but remark that there is no relation to the lapse function, which we also denoted with α.

- $\alpha = 10^5$ for the initial value solver at the origin (for the 0^{th} grid point),
- $\alpha = 10^2$ for the initial value solver at $r = \epsilon$ (for the 1^{st} grid point) if the perturbation is not too close ($< r = 3$) to the origin,
- $\alpha = 10^0$ for the initial value solver at $r = \epsilon$ (for the 1^{st} grid point) if the perturbation is close ($< r = 3$) to the origin[12].

In the numerics we fix the values for the amplification factor α by hand. An improvement would be to implement a routine that determines α dynamically.

After we obtain the initial values, we use the `ODEint` solver to obtain the solution of the ordinary differential equations as described in section 5.3 and already used for the 2+1-dimensional linear situation in section 5.4 as well.

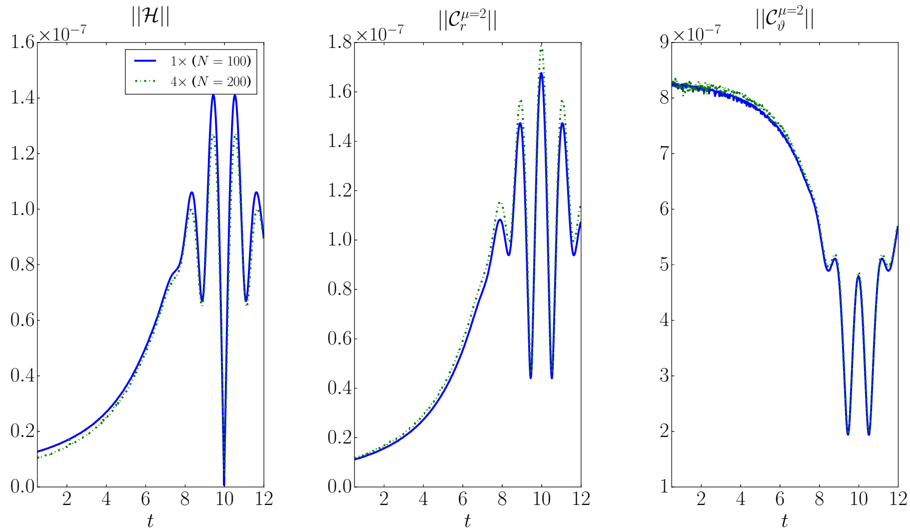


Figure 5.15.: The constraints are solved at each time instance, the remaining variables are taken from the exact **linear** solution. The solid blue curve represents the norm of the residual for spatial resolution in r of $N = 100$, the dash-dotted green curve four times the residual of $N = 200$. The spectral cutoff is again $L = 12$, the tolerance of the solver 10^{-10}. The amplitude of the perturbation is $A = 10^{-5}$.

As result we show in figure 5.15 the residuals of the constraints and observe that they converge. As illustration for the convergence properties we shown figure 5.15. We observe that the numbers are rather small (notice that the values on the vertical axis are multiplied with 10^{-7}). The representation of the values is

[12] The experiments show that $\alpha = 10^0$ is in fact a good choice there. Keeping the value of 10^2 leads to improper results.

slightly inaccurate, for the nonlinear constraints the effect turns out to be larger than on the linear level. In the considered case it appeared that the division of two minor inaccuracies increased the effect. Therefore we decided to show the convergence in the representation of figure 5.15 and not with the quotient.

We already see that for a resolution of $N = 100$ and $N = 200$ the convergent regime (corresponding to coinciding curves) is basically reached already. Hence with some minor modifications there seems to be no essential difficulty in the application of the constraint solver in the parabolic-hyperbolic formulation.

6. Conclusion and outlook

In the thesis we examined certain aspects concerned with the analytical and numerical understanding of Einstein's vacuum equations in spherical polar coordinates with a regular origin.

The choice of coordinates comes with many benefits but a major difficulty is the occurrence of coordinate singularities. The use of spherical harmonics is of computational advantage and regularizes the singularities almost everywhere. The origin remains formally singular. General relativity is a tensor theory and hence we need to employ spin-weighted spherical harmonics as basis functions. After the introduction of some necessary analytical and numerical tools we derived the explicit expressions of our basis functions in chapter 2. The elaboration of the eigenfunctions of the Laplace operator in spherical polar coordinates for quantities with spin-weight allowed us to understand the mode structure and to draw conclusions for the behavior at the origin. Furthermore we studied the scalar wave equation in spherical coordinates and derived a scheme to numerically regularize it at the origin.

We want to apply our achievements to the theory of general relativity. Therefore we discussed several aspects of the theory in chapter 3. We focused on the Cauchy formulation that is well suited for the application of techniques of the theory of partial differential equations. The equations constrain the choice of initial data. We discussed several interpretations including a recent approach to the constraints as evolutionary system. Because of its importance for the presented project we also considered the issue of gauge choices and symmetry reduction.

Our main interest lies in Einstein's vacuum equations in spherical polar coordinates. For the actual computations we restricted the problem to axisymmetry. The situation is of interest as such and we face the same problems as in the situation without symmetry assumption. This makes it a good setting to test new techniques and formulations. We started out in chapter 4 by understanding particular consequences in axisymmetry. These together with the intention to expand quantities in spin-weighted harmonics motivated our choice of variables. Further we derived the nonlinear equations in our setting, linearized them about the flat background, and derived the mode equations, arising after explicit expansion in harmonics. Our formulation allows to solve the linear problem analytically and we presented the derivation of our solution. In a main part of chapter 4 we analyzed the equations from the point of view of partial differential equations. We discussed several variations. We introduced a modification that allows us to achieve a formulation consisting of the constraints as parabolic-strongly hyperbolic set of equations in the time coordinate r and the evolution equations as strongly hyperbolic set of equations in the time coordinate t. Strong hyperbolicity is of significant importance because it is required for (local) well-posedness if appropriate initial data are provided. Our formulation allows us to investigate a recent approach to the constraints as evolutionary problem as well.

In chapter 5 we tackled the numerical implementation of the resulting equations. We implemented the code from scratch and documented validation tests. For the evolutionary approach we need to obtain initial values at the regular origin. We discussed several options how to derive them on the individual levels. One aim was to reproduce the exact solution numerically for the linearized equations. We started with the 1+1-dimensional mode level and showed that the constraint solver provides us with the correct initial data for the spacetime evolution at any initial slice. Together with the evolution equations we showed a successful implementation of the constrained evolution. Also on the full linear level we demonstrated the working of the constraint solver, as well as the essentials of the fully constrained evolution. Further we implemented the nonlinear constraint solver and showed that it solves the nonlinear constraints at any initial slice in the spacetime with initial data obtained from a regular origin.

There are several issues that deserve further investigations in the future, some of analytical and some of numerical nature.

We examined the scalar wave equation in spherical polar coordinates. It provides us with insights how to regularize the origin. With our knowledge gained for the Laplace operator it would also be of interest to complete and proper document the analysis of the homogeneous wave equations for spin-weighted quantities, including their numerical implementation.

We discussed several possibilities for the gauge in the thesis. For the derivation, analysis and implementation we chose the geodesic gauge. It might be interesting to examine different gauge choices. This includes a non-vanishing shift (even though it might not be possible any more to write down the exact solution in closed form) and the polar slicing which restricts the number of components of the extrinsic curvature. Since the evolution equations for the extrinsic curvature contain coordinate singularities in the current approach it might be beneficial for the implementation.

For the numerical implementation we desire to complete the full linear scheme and to continue to work on the nonlinear level. For the linear level we considered the constraint solver for two sets of variables and applied different numerical techniques. On the nonlinear level we demonstrated successfully one possibility of the constraint solver. Further options require to derive, in an analogous way, the initial data at the origin and to implement the solver. The inclusion of the spacetime evolution is of considerable interest in physics. It would be desirable to investigate gravitational collapse scenarios numerically, which seems to be possible in principle in our formulation. The presented framework permits subcritical evolutions in a straightforward way. For supercritical evolutions we expect the formation of a black hole. Therefore the implementation of an apparent horizon finder would be helpful, see Thornburg (2007).

We restricted our studies to hypersurface-orthogonal axisymmetry. Another interesting modification is the inclusion of twist in axisymmetry, see Rinne (2005). It allows to study rotating spacetimes with its numerous applications. Having mentioned departure from non-twisting spacetimes a significant generalization of our results concerns the removal of the axisymmetry assumption. The motivation to assume it was computational simplification, not a conceptual limitation. Similar techniques should be working on the full 3+1-dimensional level without symmetries at all. For the implementation further improvements seem to be necessary including the parallelization of the code.

A. Appendix

A.1. Some supplementary mathematical material

A.1.1. Some curvature quantities

For general relativity we need several special tensor fields. In particular in section 3.2.2 we make use of them (we also refer to the literature cited at the beginning of section 3.2 for additional material). They can be derived from the fundamental physical field, i.e. the metric tensor g with components $g_{\mu\nu}$. As always we employ Einstein's summation convention, see section 3.1.

We use a special affine connection (covariant derivative ∇), the Levi-Civita connection (see section 3.2.1) with components

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\lambda\nu} \right), \tag{A.1}$$

called **Christoffel symbols of second kind** (The Christoffel symbols of first kind are $\Gamma_{\mu\nu\lambda} = g_{\mu\rho}\Gamma^{\rho}{}_{\nu\lambda}$.

Definition A.1.1. The **Riemannian curvature tensor** Riem has components

$$R^{\mu}{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^{\mu}{}_{\nu\rho} - \partial_\rho \Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\nu\rho} - \Gamma^{\mu}{}_{\sigma\rho}\Gamma^{\sigma}{}_{\nu\lambda}. \tag{A.2}$$

The **Ricci tensor** Ric is the contraction of Riem and has components

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{A.3}$$

The **Ricci scalar** R is the further contraction of Ric and reads in components

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.4}$$

Remarks A.1.1.
- It is important to note the structure of the quantities. In the Riemannian curvature tensor second derivatives of the metric, the physical field, only occur in a quasilinear way (see definition 2.2.2). It is nonlinear only in lower derivatives. Contraction does not change this property.

- There are many different conventions in the literature, see for example the very beginning of Misner et al. (1973) for a collection. We follow them ("MTW conventions").

A.1.2. Some coordinate systems

We list some special coordinate system. They are defined for arbitrary dimensions of course but usually we are just interested in the case that they cover a 3-dimensional space or 3+1-dimensional spacetime (which just means that a fourth coordinate $-\partial_t$ is "added"). Generalization to higher dimensions is straightforward.

Cartesian coordinates Quite intuitive are **Cartesian coordinates** $(x, y, z) \in \mathbb{R}^3$. They are defined globally in $\mathbb{R}^3$ and are everywhere orthonormal[1] for the basis $(\partial_x, \partial_y, \partial_z)$. They are named after René Descartes. For the Lorentzian index $(3, 1)$ an orthonormal Cartesian basis is $(-\partial_t, \partial_x, \partial_y, \partial_z)$, see Nakahara (2003).

Cylindrical coordinates Even though one of the aims of the thesis is to formulate Einstein's equations in spherical coordinates we list **cylindrical coordinates** as well. The usual basis is given by $(-\partial_t, \partial_\rho, \partial_\varphi, \partial_z)$ and the coordinate functions are defined on the following domains

- timelike coordinate $t \in [t_0, t_{\text{end}}] \subset \mathbb{R}$,
- radial (in the x-y-plane) coordinate $\rho \in [0, R] \in \mathbb{R}_{\geq 0}$,
- $\varphi \in [0, 2\pi)$,
- $z \in \mathbb{R}$ which defines (as in Cartesian coordinates) the axis of symmetry.

The transformation between Cartesian coordinates and these cylindrical ones is given as

$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}\right\} \leftrightarrow \left\{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \varphi &= \arctan y/x \\ z &= z \end{aligned}\right. . \tag{A.5}$$

[1] That means that the vectors are perpendicular to each other (vanishing inner product) and normalized to 1.

Spherical polar coordinates Mostly in this thesis we are interested in **spherical polar coordinates** for the spatial part. A very common choice and used by us is a basis $(-\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi)$[2]. That means the coordinates are $(t, r, \vartheta, \varphi)$ and they are defined on the following domains

- timelike coordinate $t \in [t_0, t_{\text{end}}] \subset \mathbb{R}$,
- radial coordinate $r \in [0, R] \in \mathbb{R}_{\geq 0}$,
- $\vartheta \in [0, \pi]$,
- $\varphi \in [0, 2\pi)$.

These coordinates are used for example on the flat background spacetime of topology $\mathbb{R} \times \Sigma = \mathbb{R} \times \mathbb{R} \times S^2$ (the flat Minkowski spacetime). The transformation between Cartesian coordinates (restricted to the spatial part) $(x, y, z) \in \mathbb{R}^3$ and those coordinates is given as

$$\left.\begin{array}{l} x = r\cos\varphi\sin\vartheta \\ y = r\sin\varphi\sin\vartheta \\ z = r\cos\vartheta \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} r = \sqrt{x^2+y^2+z^2} \\ \vartheta = \arccos\frac{z}{r} = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan\frac{y}{x} \end{array}\right. .^{3} \tag{A.6}$$

Examples A.1.1. As an example we calculate for S^2 and $\mathbb{R} \times S^2$ the Christoffel symbols because we make use of them in the main body of the thesis.

- We need in section 2.2.5 some quantities on the sphere S^2, so for the metric (we denote these quantities for the sphere with a hat $\hat{}$) $\hat{g} = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2 = \mathrm{diag}(1, \sin^2\vartheta)$ and its inverse $\hat{g}^{-1} = \mathrm{diag}(1, (\sin^2\vartheta)^{-1})$. The non-vanishing Christoffel symbols are

$$\hat{\Gamma}^{\vartheta}{}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{A.7a}$$

$$\hat{\Gamma}^{\varphi}{}_{\varphi\vartheta} = \hat{\Gamma}^{\varphi}{}_{\vartheta\varphi} = \frac{\cos\vartheta}{\sin\vartheta}. \tag{A.7b}$$

[2] Even though this basis is probably the most used one for spherical coordinates it is only orthogonal. One could also use the orthonormal basis $(-\partial_t, \partial_r, r^{-1}\partial_\vartheta, (r\sin\vartheta)^{-1}\partial_\varphi)$ as favored by the group in Meudon, see Bonazzola et al. (2004), Grandclément and Novak (2009). In principle the calculations are similar but hard to compare with those in the current thesis because of the different basis choices.

[3] One should take more care for the transformation from Cartesian in spherical coordinates. At the origin ($x = y = z = r = 0$) the angle ϑ is not defined and the expression for φ is only correct for $x > 0$. For $x = 0$ we set $\varphi = \pm\pi/2$, for $x < 0$ one has to add or subtract π (depending on the sign of y) to obtain a value in the desired domain. Those considerations will usually not play a role for us and are consequently ignored in the remainder of the thesis.

The non-vanishing entries of the two-dimensional Levi-Civita tensor on the sphere are

$$\hat{\epsilon}_{\vartheta\varphi} = -\hat{\epsilon}_{\varphi\vartheta} = \sin\vartheta, \tag{A.8a}$$
$$\Rightarrow \hat{\epsilon}^{\vartheta}{}_{\varphi} = \hat{g}^{\vartheta\vartheta}\hat{\epsilon}_{\vartheta\varphi} = \sin\vartheta, \tag{A.8b}$$
$$\hat{\epsilon}^{\vartheta}{}_{\vartheta} = \hat{g}^{\vartheta\vartheta}\hat{\epsilon}_{\vartheta\vartheta} = 0 = \hat{\epsilon}^{\varphi}{}_{\varphi}, \tag{A.8c}$$
$$\hat{\epsilon}^{\varphi}{}_{\vartheta} = \hat{g}^{\varphi\varphi}\hat{\epsilon}_{\varphi\vartheta} = -\frac{1}{\sin\vartheta}. \tag{A.8d}$$

- For $\mathbb{R}^3$ in spherical polar coordinates as are introduced on page 211 the non-vanishing Christoffel symbols are

$$\Gamma^{r}{}_{\vartheta\vartheta} = -r,\ \Gamma^{r}{}_{\varphi\varphi} = -r\sin^2\vartheta, \tag{A.9a}$$
$$\Gamma^{\vartheta}{}_{r\vartheta} = \Gamma^{\vartheta}{}_{\vartheta r} = r^{-1},\ \Gamma^{\vartheta}{}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{A.9b}$$
$$\Gamma^{\varphi}{}_{r\varphi} = \Gamma^{\varphi}{}_{\varphi r} = r^{-1},\ \Gamma^{\varphi}{}_{\vartheta\varphi} = \Gamma^{\varphi}{}_{\varphi\vartheta} = \frac{\cos\vartheta}{\sin\vartheta}. \tag{A.9c}$$

A.2. Nonlinear evolution equations

A.2.1. Evolution equations for the extrinsic curvature components

In section 4.5.1 the nonlinear evolution equations for the components of the extrinsic curvature where just mentioned but, because of their length, not explicitly written down. We do it in the current appendix. In theorem 4.5.1 the evolution equations for a component K of the extrinsic curvature have the form

$$\partial_t K = \frac{\mathcal{K}}{\kappa} \tag{A.10}$$

and the part $\mathcal{K}$ is listed here for the single components.

$$\begin{aligned}\kappa\partial_t K_{\mathrm{s1}} = {} & -2\,r^2\,K_{\mathrm{s1}}^2\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4\ +2\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{s2}}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4\\ & -\frac{1}{2}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\ +4\,r^2\,K_{\mathrm{s1}}^2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2\\ & -4\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{s2}}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2\ +\frac{1}{2}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\\ & -2\,r^2\,K_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^4\ +2\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{s2}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^4\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{4}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\\
&-\frac{1}{4}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\ -\frac{1}{8}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{s1}}^2\\
&+\frac{1}{4}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\ -\frac{1}{8}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\ -\frac{1}{4}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{t}}\\
&-\frac{1}{4}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{8}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{t}}\\
&-\frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}}\ +\frac{1}{4}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{v}}\\
&+\frac{1}{4}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta^2\gamma_{\mathrm{s1}}\ -\frac{1}{4}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{s1}}\\
&-\frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s1}}\ +\frac{1}{4}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s1}}\\
&-\frac{1}{4}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{s1}}\ +\frac{1}{4}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s1}}\\
&+r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s2}}\ -\frac{3}{2}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{4}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{8}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{4}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{s2}}\ -\frac{1}{8}\,r\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{s2}}\\
&-\frac{1}{8}\,r\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{s2}}\ -\frac{1}{4}\,r^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{s2}}\\
&-\frac{1}{4}\,r^2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s2}}^2\ +\frac{1}{2}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{t}}\\
&-\frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{t}}\ -\frac{1}{4}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{8}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{t}}\ +\frac{1}{4}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{t}}\\
&-\frac{1}{8}\,r\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{t}}\ -\frac{1}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{3}{8}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{t}}\ -\frac{1}{4}\,r^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{4}\,r^2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^2\,\partial_r\gamma_{\mathrm{t}}^2\ -\frac{1}{4}\,r^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{t}}^2\\
&-\frac{1}{2}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_r\gamma_{\mathrm{v}}\ +r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{v}}\\
&+\frac{1}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{v}}\ -\frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{v}}
\end{aligned}
$$

$$
\begin{gathered}
+\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{v}}\ -\frac{1}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}} \\
+\frac{1}{4}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}} \\
-\frac{1}{4}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}} \\
+\frac{1}{2}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}}\ +\frac{1}{2}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^4\ \partial_r^2\gamma_{\mathrm{s2}} \\
-\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_r^2\gamma_{\mathrm{s2}}.
\end{gathered}
\tag{A.11}
$$

$$
\begin{gathered}
\kappa\partial_t K_{\mathrm{s2}} = \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ -\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4 \\
-2\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4\ +2\ r^2\ K_{\mathrm{s2}}^2\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4 \\
+\frac{3}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ -\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2 \\
+\gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ +4\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2 \\
-4\ r^2\ K_{\mathrm{s2}}^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ -\frac{3}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3 \\
-2\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^4\ +2\ r^2\ K_{\mathrm{s2}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^4 \\
-\frac{1}{4}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ -\frac{3}{4}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}} \\
+\frac{3}{4}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{8}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}^2 \\
-\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}} \\
-\frac{3}{4}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{3}{8}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{s2}} \\
+\frac{1}{2}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}^2\ -\frac{1}{4}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}^2 \\
+\frac{3}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ -\frac{3}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}} \\
-\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{3}{4}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}} \\
+\frac{1}{4}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ -\frac{3}{8}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{t}} \\
+\frac{1}{2}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ -\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_\vartheta\gamma_{\mathrm{t}} \\
-\gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \partial_\vartheta\gamma_{\mathrm{t}}^2\ +\frac{3}{4}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}^2
\end{gathered}
$$

$$
\begin{aligned}
&+\frac{3}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}}\;-\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{v}}\\
&-\frac{1}{4}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{v}}\;-\frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_\vartheta\gamma_{\mathrm{v}}\\
&+\frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_\vartheta\gamma_{\mathrm{v}}\;-\frac{1}{4}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta^2\gamma_{\mathrm{s1}}\\
&+\frac{1}{4}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{s1}}\;-\frac{1}{2}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta^2\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{s2}}\;+\frac{1}{2}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta^2\gamma_{\mathrm{t}}\\
&-\frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{t}}\;+\frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s1}}\\
&-\frac{1}{4}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s1}}\;+\frac{1}{4}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{s1}}\\
&-\frac{1}{4}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s1}}\;-2\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s2}}\;+\frac{5}{2}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{s2}}\\
&-\frac{3}{4}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_r\gamma_{\mathrm{s2}}\;-\frac{3}{8}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{s2}}\\
&-\frac{3}{4}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{s2}}\;+\frac{3}{8}\,r\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{s2}}\\
&-\frac{1}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{s2}}\;+\frac{3}{8}\,r\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s2}}\;+\frac{1}{4}\,r^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{4}\,r^2\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{s2}}^2\;-r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{t}}\;+\frac{3}{4}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{8}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{t}}\;-\frac{1}{4}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{3}{8}\,r\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{t}}\;+\frac{3}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{t}}\\
&-\frac{9}{8}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{t}}\;-\frac{1}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{4}\,r^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{t}}\;+\frac{1}{2}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_r\gamma_{\mathrm{v}}\\
&-r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{v}}\;-\frac{1}{2}\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{v}}\\
&+\frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{v}}\;-\frac{1}{2}\,r^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s2}}\,\partial_r\gamma_{\mathrm{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\ -\frac{3}{4}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\\
&-\frac{3}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{3}{4}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\\
&+\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}}\\
&-\frac{1}{2}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^4\ \partial_r^2\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_r^2\gamma_{\mathrm{s2}}.
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
\kappa\partial_t K_{\mathrm{v}} = &-2\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4\ +2\ r^2\ K_{\mathrm{s2}}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4\\
&-\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ +4\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\\
&-4\ r^2\ K_{\mathrm{s2}}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ +\gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\\
&-2\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^4\ +2\ r^2\ K_{\mathrm{s2}}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^4\\
&+\frac{1}{2}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4\ \partial_\vartheta\gamma_{\mathrm{s1}}\ -\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\\
&+\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&-\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ -\frac{1}{4}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&-\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}^2\ +\frac{1}{4}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}^2\\
&-\frac{3}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{2}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&+\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{4}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&-\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{2}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&+\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}^2\ -\frac{3}{4}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}^2\\
&-\gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ +\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\\
&+\frac{1}{2}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ -\frac{1}{2}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_\vartheta\gamma_{\mathrm{v}}\\
&+\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta^2\gamma_{\mathrm{s2}}\ -\frac{1}{2}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta^2\gamma_{\mathrm{s2}}\\
&-\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta^2\gamma_{\mathrm{t}}\ +\frac{1}{2}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta^2\gamma_{\mathrm{t}}\\
&+\frac{1}{4}\ r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^4\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^4\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}\, r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{4}\, r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{s2}}\\
&-\frac{1}{2}\, r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s2}}\ +r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{t}}\\
&-r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\ -\frac{1}{4}\, r\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{2}\, r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\ -\frac{3}{4}\, r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\\
&-r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{t}}\ +\frac{3}{4}\, r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{2}\, r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}}\ -\frac{1}{2}\, r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^4\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\, r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{2}\, r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\\
&-\frac{1}{2}\, r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}.
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
\kappa\partial_t K_{\mathrm{t}} = &-2\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4\ +2\ r^2\ K_{\mathrm{s2}}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^4\\
&-\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ +4\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\\
&-4\ r^2\ K_{\mathrm{s2}}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ +\frac{1}{2}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\\
&-2\ r^2\ K_{\mathrm{s1}}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^4\ +2\ r^2\ K_{\mathrm{s2}}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^4\\
&+\frac{1}{4}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ -\frac{1}{4}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\\
&-\frac{1}{4}\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{8}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}^2\\
&-\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{4}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&+\frac{1}{4}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ -\frac{1}{8}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&-\frac{1}{4}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{8}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&+\frac{1}{2}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{v}}\ -\frac{1}{4}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_\vartheta\gamma_{\mathrm{v}}\\
&-\frac{1}{4}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta^2\gamma_{\mathrm{s1}}\ +\frac{1}{4}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta^2\gamma_{\mathrm{s1}}\\
&+\frac{1}{4}\, r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s1}}\ +\frac{1}{4}\, r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s1}}\\
&-\frac{1}{4}\, r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s1}}\ +\frac{1}{4}\, r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_r\gamma_{\mathrm{s2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s2}} \\
&-\frac{1}{8}\ r\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s2}}\ -\frac{1}{8}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{s2}} \\
&-r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{t}}\ +\frac{1}{2}\ r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}} \\
&+r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\ -\frac{1}{4}\ r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_r\gamma_{\mathrm{t}} \\
&-\frac{1}{8}\ r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ \partial_r\gamma_{\mathrm{t}}\ +\frac{1}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}} \\
&-\frac{1}{8}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{t}} \\
&+\frac{3}{8}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{t}}\ +\frac{1}{4}\ r^2\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s1}}\ \partial_r\gamma_{\mathrm{t}} \\
&-\frac{1}{2}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\ +\frac{3}{4}\ r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}} \\
&+\frac{1}{2}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^2\ \partial_r\gamma_{\mathrm{t}}^2\ -\frac{1}{4}\ r^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}^2 \\
&-\frac{1}{2}\ r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{v}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{v}} \\
&+\frac{1}{2}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{v}}\ -\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{v}} \\
&-\frac{1}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{4}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}} \\
&+\frac{1}{4}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\ -\frac{1}{4}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}} \\
&+\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}} \\
&-\frac{1}{2}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^3\ \partial_r^2\gamma_{\mathrm{t}}\ +\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r^2\gamma_{\mathrm{t}}.
\end{aligned}
\tag{A.14}
$$

A.3. Nonlinear constraints

Similar as for some evolution equations the nonlinear constraints are just mentioned in theorem 4.5.1 in section 4.5.1, but actually listed only here in the appendix.

A.3.1. Hamiltonian constraint $\mathcal{H}$

$$
\begin{aligned}
\mathcal{H} = {} & \gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3 - r^2\,K_{\mathrm{v}}^2\,\gamma_{\mathrm{s1}}^3\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3 - \gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4 - 4\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{s2}}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4 \\
& + r^2\,K_{\mathrm{s2}}^2\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4 - r^2\,K_{\mathrm{t}}^2\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}} \\
& - 2\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}} - r^2\,K_{\mathrm{s2}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}} - r^2\,K_{\mathrm{t}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}} \\
& - \gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2 + 2\,r^2\,K_{\mathrm{v}}^2\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2 + \gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2 \\
& + 8\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{s2}}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2 - 2\,r^2\,K_{\mathrm{s2}}^2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2 + 2\,r^2\,K_{\mathrm{t}}^2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2 \\
& - 2\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3 + 4\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3 + 2\,r^2\,K_{\mathrm{s2}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3 \\
& + 2\,r^2\,K_{\mathrm{t}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3 - r^2\,K_{\mathrm{v}}^2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^4 - 4\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{s2}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^4 \\
& + r^2\,K_{\mathrm{s2}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^4 - r^2\,K_{\mathrm{t}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^4 - 2\,r^2\,K_{\mathrm{s1}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^5 \\
& - r^2\,K_{\mathrm{s2}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^5 - r^2\,K_{\mathrm{t}}\,K_{\mathrm{v}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^5 - \frac{1}{2}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{s1}} \\
& - \gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}} + \frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}} + \frac{1}{4}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{s1}}^2 \\
& - \frac{1}{2}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}} + \frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s2}} - \gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}} \\
& + \frac{1}{2}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{s2}} + \frac{1}{2}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{s2}}^2 - \frac{1}{4}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s2}}^2 \\
& + \frac{3}{2}\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{t}} - 2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}} - \frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{t}} \\
& + \gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^3\,\partial_\vartheta\gamma_{\mathrm{t}} + \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{t}} - \frac{1}{2}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{t}} \\
& + \frac{1}{2}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_\vartheta\gamma_{\mathrm{t}} - \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_\vartheta\gamma_{\mathrm{t}} - \gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}\,\partial_\vartheta\gamma_{\mathrm{t}}^2 \\
& + \frac{3}{4}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta\gamma_{\mathrm{t}}^2 + 2\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}} - \frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{v}} \\
& - \frac{1}{2}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_\vartheta\gamma_{\mathrm{v}} - \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s2}}\,\partial_\vartheta\gamma_{\mathrm{v}} + \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_\vartheta\gamma_{\mathrm{v}} \\
& - \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta^2\gamma_{\mathrm{s1}} + \frac{1}{2}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{s1}} - \frac{1}{2}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta^2\gamma_{\mathrm{s2}} \\
& + \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{s2}} + \frac{1}{2}\,\gamma_{\mathrm{s1}}^2\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\partial_\vartheta^2\gamma_{\mathrm{t}} - \frac{1}{2}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}\,\gamma_{\mathrm{v}}^2\,\partial_\vartheta^2\gamma_{\mathrm{t}} \\
& + r\,\gamma_{\mathrm{s2}}^4\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s1}} - \frac{1}{2}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s1}} + \frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{t}}\,\partial_r\gamma_{\mathrm{s1}} \\
& - \frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^3\,\partial_\vartheta\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s1}} - 3\,r\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^3\,\gamma_{\mathrm{t}}^4\,\partial_r\gamma_{\mathrm{s2}} + \frac{1}{2}\,r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_r\gamma_{\mathrm{s2}} \\
& + 4\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}^2\,\partial_r\gamma_{\mathrm{s2}} - r\,\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}\,\gamma_{\mathrm{t}}^2\,\gamma_{\mathrm{v}}^3\,\partial_r\gamma_{\mathrm{s2}} - \frac{1}{2}\,r\,\gamma_{\mathrm{s2}}^2\,\gamma_{\mathrm{t}}^3\,\gamma_{\mathrm{v}}\,\partial_\vartheta\gamma_{\mathrm{s1}}\,\partial_r\gamma_{\mathrm{s2}}
\end{aligned}
$$

$$
\begin{aligned}
&-r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ r\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{s2}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^4\ \partial_r\gamma_{\mathrm{s1}}\ \partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{4}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^4\ \partial_r\gamma_{\mathrm{s2}}^2\ +\frac{1}{4}\ r^2\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}^2\ -\frac{3}{2}\ r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}}\\
&+r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\ +r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_r\gamma_{\mathrm{t}}\ -\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\\
&+\frac{1}{2}\ r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\ +2\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{t}}\ -\frac{3}{2}\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{t}}\\
&-\frac{1}{2}\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}}\ +\frac{1}{2}\ r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{t}}\ -\frac{1}{4}\ r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^4\ \gamma_{\mathrm{t}}^2\ \partial_r\gamma_{\mathrm{t}}^2\\
&+\frac{1}{4}\ r^2\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}^2\ +r\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{v}}\ -2\ r\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{v}}\\
&-r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ \partial_r\gamma_{\mathrm{v}}\ +r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{v}}\ -r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s2}}\ \partial_r\gamma_{\mathrm{v}}\\
&+r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\ -r\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{s2}}\ -r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\\
&+r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{t}}\ +r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}}\ -r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\partial_\vartheta\gamma_{\mathrm{v}}\\
&-r^2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^4\ \partial_r^2\gamma_{\mathrm{s2}}\ +r^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \gamma_{\mathrm{v}}^2\ \partial_r^2\gamma_{\mathrm{s2}} = 0. \qquad \text{(A.15)}
\end{aligned}
$$

A.3.2. Modified momentum constraint $\mathcal{C}_r^{\mu=2}$

As discussed in theorem 4.7.3 solve the modified momentum constraint with an additional inhomogeneity on the right-hand side. Our choice for the parameter is $\mu = 2$.

$$
\begin{aligned}
\mathcal{C}_r^{\mu=2} = &\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ -4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ -2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\\
&+2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ +\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ +\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\\
&-\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ +4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ +2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\\
&-2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ -\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ -\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\\
&+2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta K_{\mathrm{s1}}\ -2\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta K_{\mathrm{s1}}\ +\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta K_{\mathrm{s2}}\\
&-\gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta K_{\mathrm{s2}}\ +\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta K_{\mathrm{t}}\ -\gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta K_{\mathrm{t}}\\
&+\gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta K_{\mathrm{v}}\ -\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{v}}\ +\frac{3}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\\
&+K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}
\end{aligned}
$$

$$
\begin{aligned}
&-K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +K_{\mathrm{s1}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ -K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&-2\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ -K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ -K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&+\frac{3}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +3\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{3}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&+\frac{3}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ +2\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ +K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\\
&+K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ -4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\\
&-2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -2\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -4\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r K_{\mathrm{s1}}\\
&+4\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r K_{\mathrm{s1}}\ +2\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r K_{\mathrm{s2}}\ -2\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r K_{\mathrm{s2}}\\
&+\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s1}}\ -2\ r\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}}\ -r\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s2}}\ +2\ r\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}\ +r\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}\\
&-r\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_r\gamma_{\mathrm{t}}\ +\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}}\ +r\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\\
&-r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_r\gamma_{\mathrm{v}} = 2\partial_r \mathrm{tr} K. \qquad \text{(A.16)}
\end{aligned}
$$

A.3.3. Modified momentum constraint $\mathcal{C}_\vartheta^{\mu=2}$

Again we consider the modified constraints and refer to theorem 4.7.3 for the corresponding discussion. We choose $\mu = 2$.

$$
\begin{aligned}
\mathcal{C}_\vartheta^{\mu=2} = {}&2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ +3\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ -2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\\
&-3\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ -2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta K_{\mathrm{s1}}\ +2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{s1}}\\
&+3\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta K_{\mathrm{s2}}\ -3\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{s2}}\ +\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta K_{\mathrm{t}}\\
&-\gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{t}}\ +K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\\
&+\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&+\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\\
&-\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ -K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&+K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{3}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\\
&-K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -2\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ -K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}
\end{aligned}
$$

$$
\begin{aligned}
&-K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ +r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r K_{\mathrm{v}}\ -r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r K_{\mathrm{v}} \\
&+\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s1}}\ +r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}}\ -\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}} \\
&+\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\ -r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{v}} = 2\partial_\vartheta \mathrm{tr}K.
\end{aligned}
\tag{A.17}
$$

A.3.4. Original momentum constraint $\mathcal{C}_r^{\mu=0}$

For completeness we also give the original momentum constraint (corresponds to $\mu = 0$ in theorem 4.7.3).

$$
\begin{aligned}
\mathcal{C}_r^{\mu=0} = &\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ -4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ -2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3 \\
&+2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ +\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ +\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}} \\
&-\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ +4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ +2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2 \\
&-2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ -\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ -\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3 \\
&+2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta K_{\mathrm{s1}}\ -2\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta K_{\mathrm{s1}}\ +\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta K_{\mathrm{s2}} \\
&-\gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta K_{\mathrm{s2}}\ +\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta K_{\mathrm{t}}\ -\gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta K_{\mathrm{t}} \\
&+\gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta K_{\mathrm{v}}\ -\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{v}}\ +\frac{3}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}} \\
&+K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}} \\
&-K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +K_{\mathrm{s1}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}} \\
&+\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\ -K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}^2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \partial_\vartheta\gamma_{\mathrm{t}} \\
&-2\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ -K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}}\ -K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}} \\
&+\frac{3}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +3\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{3}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}} \\
&+\frac{3}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{v}}^3\ \partial_\vartheta\gamma_{\mathrm{t}}\ +2\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ +K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}} \\
&+K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ -4\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}} \\
&-2\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -2\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -2\ r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r K_{\mathrm{s2}} \\
&+2\ r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r K_{\mathrm{s2}}\ +\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s1}}\ -2\ r\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}} \\
&-r\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{s2}}\ +2\ r\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}
\end{aligned}
$$

$$+r\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}}\ -r\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_r\gamma_{\mathrm{t}}\ +\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{t}}$$
$$+r\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\ -r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_r\gamma_{\mathrm{v}} = 0. \tag{A.18}$$

A.3.5. Original momentum constraint $\mathcal{C}_\vartheta^{\mu=0}$

And the same for the other non-vanishing component,

$$\begin{aligned}
\mathcal{C}_\vartheta^{\mu=0} = {} & 2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ +3\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ -2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2 \\
& -3\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ +2\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta K_{\mathrm{s1}}\ -2\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{s1}} \\
& -\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta K_{\mathrm{s2}}\ +\gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{s2}}\ +\gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta K_{\mathrm{t}} \\
& -\gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta K_{\mathrm{t}}\ +K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}} \\
& +\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s1}}\ +K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_\vartheta\gamma_{\mathrm{s2}} \\
& +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}} \\
& -\frac{1}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\ -K_{\mathrm{t}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{2}\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{t}} \\
& +K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{1}{2}\ K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}}\ +\frac{3}{2}\ K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_\vartheta\gamma_{\mathrm{t}} \\
& -K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \partial_\vartheta\gamma_{\mathrm{v}}\ -2\ K_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ -K_{\mathrm{s2}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}} \\
& -K_{\mathrm{t}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_\vartheta\gamma_{\mathrm{v}}\ +r\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r K_{\mathrm{v}}\ -r\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r K_{\mathrm{v}} \\
& +\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^3\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s1}}\ +r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s1}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^3\ \partial_r\gamma_{\mathrm{s2}}\ -\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{s2}} \\
& +\frac{1}{2}\ r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}\ \gamma_{\mathrm{v}}^2\ \partial_r\gamma_{\mathrm{t}}\ -r\ K_{\mathrm{v}}\ \gamma_{\mathrm{s2}}^2\ \gamma_{\mathrm{t}}^2\ \gamma_{\mathrm{v}}\ \partial_r\gamma_{\mathrm{v}} = 0.
\end{aligned} \tag{A.19}$$

A.4. Explicit form of the exact regular solution for $\ell = 2$

In section 4.6 we calculated explicitly a solution for the mode $\ell = 2$. Here we continue that example.

Example A.4.1. For $\ell = 2$ we give a concrete example with a generating function $G = A\exp(-x^2/2)$ for $x = r \pm t$ (ingoing package for positive sign, outgoing for negative one). G is a Gaussian with uniform variance $\sigma = 1$ and amplitude A (which is assumed to be 1 in the following). The regular solution then reads (see footnote 5 on page 174 for a remark on the ingoing and outgoing properties of the regular exact solution).

$$\begin{aligned}
\hat{\gamma}_{\mathrm{s1}}|_{\ell=2} = &-\frac{72}{r^5}e^{-(r-t)^2/2} + \frac{72}{r^5}e^{-(r+t)^2/2} - \frac{48}{r^3}e^{-(r-t)^2/2} + \frac{48}{r^3}e^{-(r+t)^2/2} \\
&-\frac{24}{r}e^{-(r-t)^2/2} + \frac{24}{r}e^{-(r+t)^2/2} + \frac{72t}{r^4}e^{-(r-t)^2/2} + \frac{72t}{r^4}e^{-(r+t)^2/2} \\
&+\frac{48t}{r^2}e^{-(r-t)^2/2} + \frac{48t}{r^2}e^{-(r+t)^2/2} - \frac{24t^2}{r^3}e^{-(r-t)^2/2} + \frac{24t^2}{r^3}e^{-(r+t)^2/2},
\end{aligned} \tag{A.20a}$$

$$\begin{aligned}
\hat{\gamma}_{\mathrm{s2}}|_{\ell=2} = &\frac{36}{r^5}e^{-(r-t)^2/2} - \frac{36}{r^5}e^{-(r+t)^2/2} + \frac{24}{r^3}e^{-(r-t)^2/2} - \frac{24}{r^3}e^{-(r+t)^2/2} \\
&+\frac{12}{r}e^{-(r-t)^2/2} - \frac{12}{r}e^{-(r+t)^2/2} - \frac{36t}{r^4}e^{-(r-t)^2/2} - \frac{36t}{r^4}e^{-(r+t)^2/2} \\
&-\frac{24t}{r^2}e^{-(r-t)^2/2} - \frac{24t}{r^2}e^{-(r+t)^2/2} + \frac{12t^2}{r^3}e^{-(r-t)^2/2} - \frac{12t^2}{r^3}e^{-(r+t)^2/2},
\end{aligned} \tag{A.20b}$$

$$\begin{aligned}
\hat{\gamma}_{\mathrm{v}}|_{\ell=2} = &\frac{24}{r^5}e^{-(r-t)^2/2} - \frac{24}{r^5}e^{-(r+t)^2/2} + \frac{12}{r^3}e^{-(r-t)^2/2} - \frac{12}{r^3}e^{-(r+t)^2/2} \\
&+4re^{-(r-t)^2/2} - 4re^{-(r+t)^2/2} - 12te^{-(r-t)^2/2} - 12te^{-(r+t)^2/2} \\
&-\frac{24t}{r^4}e^{-(r-t)^2/2} - \frac{24t}{r^4}e^{-(r+t)^2/2} - \frac{12t}{r^2}e^{-(r-t)^2/2} - \frac{12t}{r^2}e^{-(r+t)^2/2} \\
&+\frac{12t^2}{r^3}e^{-(r-t)^2/2} - \frac{12t^2}{r^3}e^{-(r+t)^2/2} + \frac{12t^2}{r}e^{-(r-t)^2/2} - \frac{12t^2}{r}e^{-(r+t)^2/2} \\
&-\frac{4t^3}{r^2}e^{-(r-t)^2/2} - \frac{4t^3}{r^2}e^{-(r+t)^2/2},
\end{aligned} \tag{A.20c}$$

$$\begin{aligned}
\hat{\gamma}_{\mathrm{t}}|_{\ell=2} = &-\frac{6}{r^5}e^{-(r-t)^2/2} + \frac{6}{r^5}e^{-(r+t)^2/2} + 8re^{-(r-t)^2/2} - 8re^{-(r+t)^2/2} \\
&-2r^3e^{-(r-t)^2/2} + 2r^3e^{-(r+t)^2/2} \\
&-12te^{-(r-t)^2/2} - 12te^{-(r+t)^2/2} + \frac{6t}{r^4}e^{-(r-t)^2/2} + \frac{6t}{r^4}e^{-(r+t)^2/2} \\
&+8r^2te^{-(r-t)^2/2} + 8r^2te^{-(r+t)^2/2} - \frac{6t^2}{r^3}e^{-(r-t)^2/2} + \frac{6t^2}{r^3}e^{-(r+t)^2/2} \\
&-12rt^2e^{-(r-t)^2/2} + 12rt^2e^{-(r+t)^2/2} + 8t^3e^{-(r-t)^2/2} + 8t^3e^{-(r+t)^2/2} \\
&+\frac{4t^3}{r^2}e^{-(r-t)^2/2} + \frac{4t^3}{r^2}e^{-(r+t)^2/2} - \frac{2t^4}{r}e^{-(r-t)^2/2} + \frac{2t^4}{r}e^{-(r+t)^2/2},
\end{aligned} \tag{A.20d}$$

$$\begin{aligned}
\hat{K}_{\mathrm{s1}}|_{\ell=2} = &-6e^{-(r-t)^2/2} - 6e^{-(r+t)^2/2} + \frac{18t}{r^5}e^{-(r-t)^2/2} - \frac{18t}{r^5}e^{-(r+t)^2/2} \\
&-\frac{18t}{r^3}e^{-(r-t)^2/2} - \frac{18t}{r^3}e^{-(r+t)^2/2} + \frac{18t}{r}e^{-(r-t)^2/2} - \frac{18t}{r}e^{-(r+t)^2/2} \\
&-\frac{18t^2}{r^4}e^{-(r-t)^2/2} - \frac{18t^2}{r^4}e^{-(r+t)^2/2} - \frac{18t^2}{r^2}e^{-(r-t)^2/2} - \frac{18t^2}{r^2}e^{-(r+t)^2/2}
\end{aligned}$$

$$+\frac{6t^3}{r^3}e^{-(r-t)^2/2} - \frac{6t^3}{r^3}e^{-(r+t)^2/2}, \tag{A.20e}$$

$$\begin{aligned}\hat{K}_{\mathrm{s}2}|_{\ell=2} = {} & -6e^{-(r-t)^2/2} - 6e^{-(r+t)^2/2} + \frac{18t}{r^5}e^{-(r-t)^2/2} - \frac{18t}{r^5}e^{-(r+t)^2/2} \\ & -\frac{18t}{r^3}e^{-(r-t)^2/2} - \frac{18t}{r^3}e^{-(r+t)^2/2} + \frac{18t}{r}e^{-(r-t)^2/2} - \frac{18t}{r}e^{-(r+t)^2/2} \\ & -\frac{18t^2}{r^4}e^{-(r-t)^2/2} - \frac{18t^2}{r^4}e^{-(r+t)^2/2} - \frac{18t^2}{r^2}e^{-(r-t)^2/2} - \frac{18t^2}{r^2}e^{-(r+t)^2/2} \\ & +\frac{6t^3}{r^3}e^{-(r-t)^2/2} - \frac{6t^3}{r^3}e^{-(r+t)^2/2},\end{aligned} \tag{A.20f}$$

$$\begin{aligned}\hat{K}_{\mathrm{v}}|_{\ell=2} = {} & 6e^{-(r-t)^2/2} + 6e^{-(r+t)^2/2} - 2r^2e^{-(r-t)^2/2} - 2r^2e^{-(r+t)^2/2} \\ & +\frac{12t}{r^5}e^{-(r-t)^2/2} - \frac{12t}{r^5}e^{-(r+t)^2/2} + \frac{6t}{r^3}e^{-(r-t)^2/2} - \frac{6t}{r^3}e^{-(r+t)^2/2} \\ & -\frac{6t}{r}e^{-(r-t)^2/2} + \frac{6t}{r}e^{-(r+t)^2/2} + 8rte^{-(r-t)^2/2} - 8rte^{-(r+t)^2/2} \\ & -12t^2e^{-(r-t)^2/2} - 12t^2e^{-(r+t)^2/2} - \frac{12t^2}{r^4}e^{-(r-t)^2/2} - \frac{12t^2}{r^4}e^{-(r+t)^2/2} \\ & -\frac{6t^2}{r^2}e^{-(r-t)^2/2} - \frac{6t^2}{r^2}e^{-(r+t)^2/2} + \frac{6t^3}{r^3}e^{-(r-t)^2/2} - \frac{6t^3}{r^3}e^{-(r+t)^2/2} \\ & +\frac{8t^3}{r}e^{-(r-t)^2/2} - \frac{8t^3}{r}e^{-(r+t)^2/2} - \frac{2t^4}{r^2}e^{-(r-t)^2/2} - \frac{2t^4}{r^2}e^{-(r+t)^2/2},\end{aligned} \tag{A.20g}$$

$$\begin{aligned}\hat{K}_{\mathrm{t}}|_{\ell=2} = {} & 6e^{-(r-t)^2/2} + 6e^{-(r+t)^2/2} - 8r^2e^{-(r-t)^2/2} - 8r^2e^{-(r+t)^2/2} \\ & +r^4e^{-(r-t)^2/2} + r^4e^{-(r+t)^2/2} - \frac{3t}{r^5}e^{-(r-t)^2/2} + \frac{3t}{r^5}e^{-(r+t)^2/2} \\ & +\frac{3t}{r^3}e^{-(r-t)^2/2} - \frac{3t}{r^3}e^{-(r+t)^2/2} + 22rte^{-(r-t)^2/2} - 22rte^{-(r+t)^2/2} \\ & -5rt^3e^{-(r-t)^2/2} + 5rt^3e^{-(r+t)^2/2} - 18t^2e^{-(r-t)^2/2} - 18t^2e^{-(r+t)^2/2} \\ & +\frac{3t^2}{r^4}e^{-(r-t)^2/2} + \frac{3t^2}{r^4}e^{-(r+t)^2/2} - \frac{3t^2}{r^2}e^{-(r-t)^2/2} - \frac{3t^2}{r^2}e^{-(r+t)^2/2} \\ & +10r^2t^2e^{-(r-t)^2/2} + 10r^2t^2e^{-(r+t)^2/2} - \frac{3t^3}{r^3}e^{-(r-t)^2/2} + \frac{3t^3}{r^3}e^{-(r+t)^2/2} \\ & +\frac{2t^3}{r}e^{-(r-t)^2/2} - \frac{2t^3}{r}e^{-(r+t)^2/2} - 10rt^3e^{-(r-t)^2/2} + 10rt^3e^{-(r+t)^2/2} \\ & +5t^4e^{-(r-t)^2/2} + 5t^4e^{-(r+t)^2/2} + \frac{2t^4}{r^2}e^{-(r-t)^2/2} + \frac{2t^4}{r^2}e^{-(r+t)^2/2} \\ & -\frac{t^5}{r}e^{-(r-t)^2/2} + \frac{t^5}{r}e^{-(r+t)^2/2}.\end{aligned} \tag{A.20h}$$

One immediately sees that the quantities in equation (A.20) are singular at the origin. In fact they are just formally singular as will be shown with the following lines. Calculating the Taylor expansion at the origin shows (here $\overset{0}{=}$ explicitly refers to equality in a neighborhood of the origin)

$$\hat{\gamma}_{\mathrm{s1}}|_{\ell=2} \overset{0}{=} -\frac{16}{5}a_\gamma(t) - \frac{8}{35}b_\gamma(t)r^2, \tag{A.21a}$$
$$\hat{\gamma}_{\mathrm{s2}}|_{\ell=2} \overset{0}{=} \frac{8}{5}a_\gamma(t) + \frac{4}{35}b_\gamma(t)r^2, \tag{A.21b}$$
$$\hat{\gamma}_{\mathrm{v}}|_{\ell=2} \overset{0}{=} -\frac{8}{5}a_\gamma(t) - \frac{44}{105}b_\gamma(t)r^2, \tag{A.21c}$$
$$\hat{\gamma}_{\mathrm{t}}|_{\ell=2} \overset{0}{=} -\frac{8}{5}a_\gamma(t) - \frac{4}{21}b_\gamma(t)r^2, \tag{A.21d}$$
$$\hat{K}_{\mathrm{s1}}|_{\ell=2} \overset{0}{=} \frac{4}{5}a_K(t) + \frac{2}{35}b_K(t)r^2, \tag{A.21e}$$
$$\hat{K}_{\mathrm{s2}}|_{\ell=2} \overset{0}{=} \frac{4}{5}a_K(t) + \frac{2}{35}b_K(t)r^2, \tag{A.21f}$$
$$\hat{K}_{\mathrm{v}}|_{\ell=2} \overset{0}{=} -\frac{4}{5}a_K(t) - \frac{2}{21}b_K(t)r^2, \tag{A.21g}$$
$$\hat{K}_{\mathrm{t}}|_{\ell=2} \overset{0}{=} -\frac{4}{5}a_K(t) - \frac{22}{105}b_K(t)r^2 \tag{A.21h}$$

with

$$a_\gamma(t) = t\left(15 - 10t^2 + t^4\right)e^{-t^2/2}, \tag{A.22a}$$
$$b_\gamma(t) = t\left(-105 + 105t^2 - 21t^4 + t^6\right)e^{-t^2/2}, \tag{A.22b}$$
$$a_K(t) = \left(-15 + 45t^2 - 15t^4 + t^6\right)e^{-t^2/2}, \tag{A.22c}$$
$$b_K(t) = \left(105 - 420t^2 + 210t^4 - 28t^6 + t^8\right)e^{-t^2/2}. \tag{A.22d}$$

This shows regularity, parity of the modes and the behavior of $\mathcal{O}(r^0) = \mathcal{O}(r^{\ell-2})$. Note in equations (A.20) and (A.21) the relations $\hat{K}_{\mathrm{s1}} = \hat{K}_{\mathrm{s2}}$ and $\hat{\gamma}_{\mathrm{s1}} = -2\hat{\gamma}_{\mathrm{s2}}$ and compare with the discussions at the end of section 4.7.2.

A.5. Nonlinear initial data for the momentum constraint

Here we list coefficients of the initial data for the solver of the nonlinear constraints as was missing in section 5.5.1.1. The lower index of the variables (to the right of the vertical bar) denotes the position in the Taylor expansion. If there is an upper index as well it denotes the power of the quantity.

$$\begin{aligned}
a_1 = &-4\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^3\ \gamma_{\mathrm{t}}\big|_0^3\ +2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0 \\
&+\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ +4\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0^2 \\
&-2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ -\frac{\cos\vartheta}{\sin\vartheta}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3 \\
&+2\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{s}1}\big|_0\ -2\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{s}1}\big|_0 \\
&+\gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0\ -\gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0 \\
&+K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s}1}\big|_0\ +\frac{1}{2}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s}1}\big|_0 \\
&+K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{s}2}\big|_0\ +\frac{1}{2}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{s}2}\big|_0 \\
&-2\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0\ -K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
&+3\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0\ +\frac{3}{2}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
&+2\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0\ +K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0 \\
&-4\ K_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0\ -2\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0,
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
b_1 = &-2\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^3\ \gamma_{\mathrm{t}}\big|_0^3\ +\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0 \\
&+2\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0^2\ -\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3 \\
&+\frac{1}{2}\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s}1}\big|_0\ +\frac{1}{2}\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{s}2}\big|_0 \\
&-\gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0\ +\frac{3}{2}\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
&+\gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0\ -2\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0,
\end{aligned} \tag{A.24}$$

$$\begin{aligned}
c_1 = &\ \frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s}1}\big|_0^2\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ -\frac{\cos\vartheta}{\sin\vartheta}\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2 \\
&+\frac{3}{2}\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s}1}\big|_0\ -\gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s}1}\big|_0 \\
&+\frac{1}{2}\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s}2}\big|_0\ -\gamma_{\mathrm{s}1}\big|_0^2\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
&+\frac{3}{2}\ \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{v}}\big|_0\ -\gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{t}}\big|_0,
\end{aligned} \tag{A.25}$$

$$d_1 = \gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ -\gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3, \tag{A.26}$$

$$e_1 = \gamma_{\mathrm{s}1}\big|_0^2\ \gamma_{\mathrm{s}2}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ -\gamma_{\mathrm{s}1}\big|_0\ \gamma_{\mathrm{s}2}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2, \tag{A.27}$$

$$\begin{aligned}
a_2 = 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{t}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3 - 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{t}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2 \\
-2\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3\, \partial_\vartheta\, K_\mathrm{s1}\big|_0 + 2\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\, K_\mathrm{s1}\big|_0 \\
+\gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3\, \partial_\vartheta\, K_\mathrm{t}\big|_0 - \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\, K_\mathrm{t}\big|_0 \\
+K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3\, \partial_\vartheta\gamma_\mathrm{s1}\big|_0 + \frac{1}{2}\, K_\mathrm{t}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3\, \partial_\vartheta\gamma_\mathrm{s1}\big|_0 \\
+K_\mathrm{t}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^3\, \partial_\vartheta\gamma_\mathrm{s2}\big|_0 + K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{s2}\big|_0 \\
-\frac{1}{2}\, K_\mathrm{t}\big|_0\, \gamma_\mathrm{s2}\big|_0\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{s2}\big|_0 - K_\mathrm{t}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{v}\big|_0 \\
+K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{v}\big|_0 + \frac{3}{2}\, K_\mathrm{t}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{v}\big|_0 \\
-2\, K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0\, \partial_\vartheta\gamma_\mathrm{t}\big|_0 - K_\mathrm{t}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0\, \partial_\vartheta\gamma_\mathrm{t}\big|_0,
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
b_2 = \frac{1}{2}\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3\, \partial_\vartheta\gamma_\mathrm{s1}\big|_0 + \frac{1}{2}\, \gamma_\mathrm{s2}\big|_0\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{s2}\big|_0 \\
+\frac{1}{2}\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0\, \gamma_\mathrm{v}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{v}\big|_0 - \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0\, \partial_\vartheta\gamma_\mathrm{t}\big|_0,
\end{aligned} \tag{A.29}$$

$$\begin{aligned}
c_2 = 3\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3 - 3\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2 \\
+\frac{1}{2}\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0\, \partial_\vartheta\gamma_\mathrm{s1}\big|_0 + \frac{1}{2}\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0\, \partial_\vartheta\gamma_\mathrm{s2}\big|_0 \\
+\frac{1}{2}\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0\, \gamma_\mathrm{v}\big|_0\, \partial_\vartheta\gamma_\mathrm{v}\big|_0 - \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \partial_\vartheta\gamma_\mathrm{t}\big|_0,
\end{aligned} \tag{A.30}$$

$$d_2 = 3\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3 - 3\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0^2, \tag{A.31}$$

$$e_2 = 0, \tag{A.32}$$

$$\begin{aligned}
a_3 = 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{v}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s1}\big|_1\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{v}\big|_0\, \gamma_\mathrm{s1}\big|_0^2\, \gamma_\mathrm{s2}\big|_0\, \gamma_\mathrm{s2}\big|_1\, \gamma_\mathrm{t}\big|_0^2 \\
-8\, K_\mathrm{s1}\big|_1\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3 - 4\, K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s1}\big|_1\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3 \\
-2\, K_\mathrm{s2}\big|_0\, \gamma_\mathrm{s1}\big|_1\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^3 - 14\, K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{s2}\big|_1\, \gamma_\mathrm{t}\big|_0^3 \\
-7\, K_\mathrm{s2}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{s2}\big|_1\, \gamma_\mathrm{t}\big|_0^3 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{v}\big|_0\, \gamma_\mathrm{s1}\big|_0^2\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0\, \gamma_\mathrm{t}\big|_1 \\
-12\, K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{t}\big|_1 - 6\, K_\mathrm{s2}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{t}\big|_1 \\
-K_\mathrm{t}\big|_0\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^3\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{t}\big|_1 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{s1}\big|_1\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0 \\
+\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{t}\big|_1\, \gamma_\mathrm{s1}\big|_0\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{s1}\big|_0\, \gamma_\mathrm{s1}\big|_1\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0 \\
+\frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{s2}\big|_0\, \gamma_\mathrm{s1}\big|_1\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0 + \frac{\cos\vartheta}{\sin\vartheta}\, K_\mathrm{t}\big|_0\, \gamma_\mathrm{s1}\big|_1\, \gamma_\mathrm{s2}\big|_0^2\, \gamma_\mathrm{t}\big|_0^2\, \gamma_\mathrm{v}\big|_0
\end{aligned}$$

$$+\frac{1}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0 + 4\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0$$

$$+2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0$$

$$+\frac{1}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0 + 4\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0$$

$$+2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0 + 2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0$$

$$+\frac{1}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0 - \frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2$$

$$-\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2 + 8\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0^2$$

$$+10\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0^2 + 5\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0^2$$

$$-\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2 + 8\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2$$

$$+4\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2 + K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2$$

$$-2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3 - \frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3$$

$$-2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3 - \frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3$$

$$-\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3 - 2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3$$

$$-\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3 - \frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3$$

$$+2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1 + \frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1$$

$$+\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1 - K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1$$

$$-2\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1 + 8\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1$$

$$+4\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1 - 6\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1$$

$$-3\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1 - 3\,\frac{\cos\vartheta}{\sin\vartheta}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1$$

$$+2\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0 + 4\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0$$

$$+4\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0 - 2\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0$$

$$-2\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0 + 2\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0$$

$$-6\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_0 + 2\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\,K_{\mathrm{s1}}\big|_1$$

$$
\begin{gathered}
-2\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{s1}}\big|_1\ \ +\gamma_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0 \\
+2\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0\ \ +2\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0 \\
-\gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0\ \ -\gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0 \\
+\gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0\ \ -3\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\ K_{\mathrm{s2}}\big|_0 \\
+\gamma_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0\ \ +2\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0 \\
+2\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0\ \ -\gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0 \\
-\gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0\ \ +\gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0 \\
-3\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\ K_{\mathrm{t}}\big|_0\ \ +\gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\ K_{\mathrm{t}}\big|_1 \\
-\gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\ K_{\mathrm{t}}\big|_1\ \ +2\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0 \\
+2\ \gamma_{\mathrm{s1}}\big|_0^2\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0\ \ +2\ \gamma_{\mathrm{s1}}\big|_0^2\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0 \\
-\gamma_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0\ \ -\gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0 \\
-\gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0\ \ -2\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\ K_{\mathrm{v}}\big|_0 \\
+\frac{3}{2}\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +3\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
+3\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +K_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
+\frac{1}{2}\ K_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +2\ K_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
+K_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
+2\ K_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +K_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
+K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ -K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s2}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
-K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +K_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
+\frac{1}{2}\ K_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +\frac{1}{2}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 \\
-2\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{v}}\big|_1\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\ \ +\frac{3}{2}\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_1 \\
+K_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_1\ \ +\frac{1}{2}\ K_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_1 \\
+\frac{1}{2}\ K_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0^2\ \gamma_{\mathrm{t}}\big|_0^2\ \gamma_{\mathrm{v}}\big|_0\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_1\ \ -K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s2}}\big|_0\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s1}}\big|_1 \\
+\frac{1}{2}\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\big|_0\ \ +\frac{1}{2}\ K_{\mathrm{v}}\big|_0\ \gamma_{\mathrm{s1}}\big|_0\ \gamma_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{v}}\big|_0^2\ \partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 \\
+K_{\mathrm{s1}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\big|_0\ \ +\frac{1}{2}\ K_{\mathrm{t}}\big|_1\ \gamma_{\mathrm{t}}\big|_0\ \gamma_{\mathrm{v}}\big|_0^3\ \partial_\vartheta\gamma_{\mathrm{s2}}\big|_0
\end{gathered}
$$

$$
\begin{gathered}
+K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 + \frac{1}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 \\
+\frac{1}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 + K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 \\
+3\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 + \frac{3}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 \\
+\frac{3}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0 + \frac{1}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_1 \\
+K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_1 + \frac{1}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_1 \\
+\frac{1}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_1 - 2\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-2\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0^2\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0^2\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-2\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - K_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - 4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - 2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + \frac{3}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
+\frac{3}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + 3\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
+\frac{3}{2}\,K_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + 3\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
+\frac{3}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + \frac{3}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + 3\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
+9\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + \frac{9}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 \\
+\frac{9}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0^2\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 \\
-2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 - K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 \\
-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 + \frac{3}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 \\
+3\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 + \frac{3}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1
\end{gathered}
$$

$$
\begin{aligned}
&+\frac{3}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1\; +2\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&+K_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; +2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&+K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; +K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&+4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; +2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&+2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; +4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&+2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; +2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -4\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-8\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -4\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-4\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; +2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\\
&+K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\; +K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\\
&-K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\; -4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\\
&-2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\; -2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1, \qquad \text{(A.33)}
\end{aligned}
$$

$$
\begin{aligned}
b_3 =&\ \frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\; -\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\\
&+\frac{1}{2}\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\; +\frac{1}{2}\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0\\
&-\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0\; +\frac{3}{2}\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0\\
&+\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\; -2\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0, \qquad \text{(A.34)}
\end{aligned}
$$

$$
\begin{aligned}
c_3 =&\ \frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\big|_0^2\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\; -\frac{\cos\vartheta}{\sin\vartheta}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\\
&+\frac{3}{2}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\; -\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{s1}}\big|_0\\
&+\frac{1}{2}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0\; -\gamma_{\mathrm{s1}}\big|_0^2\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0\\
&+\frac{3}{2}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0\; -\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0, \qquad \text{(A.35)}
\end{aligned}
$$

$$
d_3 = \gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\; -\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^3, \qquad \text{(A.36)}
$$

$$
\begin{aligned}
e_3 &= \gamma_{\rm s1}\big|_0^2\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ -\gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm t}\big|_0\ \gamma_{\rm v}\big|_0^2, \qquad\qquad \text{(A.37)}\\
a_4 &= 2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_1\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ +2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_0\ \gamma_{\rm s1}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\\
&+\frac{7}{2}\ K_{\rm v}\big|_0\ \gamma_{\rm s1}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ +6\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_0\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\\
&+10\ K_{\rm v}\big|_0\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ +6\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_0\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\\
&+9\ K_{\rm v}\big|_0\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ -2\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_1\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\\
&-4\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_0\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\ -\frac{13}{2}\ K_{\rm v}\big|_0\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\\
&-4\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0\ \gamma_{\rm t}\big|_1\ \gamma_{\rm v}\big|_0^2\ -\frac{11}{2}\ K_{\rm v}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0\ \gamma_{\rm t}\big|_1\ \gamma_{\rm v}\big|_0^2\\
&-4\ \frac{\cos\vartheta}{\sin\vartheta}\ K_{\rm t}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0\ \gamma_{\rm v}\big|_1\ -7\ K_{\rm v}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0\ \gamma_{\rm v}\big|_1\\
&-2\ \gamma_{\rm s1}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm s1}\big|_0\ -6\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm s1}\big|_0\\
&-6\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ \partial_\vartheta\ K_{\rm s1}\big|_0\ +4\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm s1}\big|_0\\
&+4\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0\ \gamma_{\rm t}\big|_1\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm s1}\big|_0\ +4\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0\ \gamma_{\rm v}\big|_1\ \partial_\vartheta\ K_{\rm s1}\big|_0\\
&-2\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm s1}\big|_1\ +2\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm s1}\big|_1\\
&+3\ \gamma_{\rm s1}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm s2}\big|_0\ +9\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm s2}\big|_0\\
&+9\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ \partial_\vartheta\ K_{\rm s2}\big|_0\ -6\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm s2}\big|_0\\
&-6\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0\ \gamma_{\rm t}\big|_1\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm s2}\big|_0\ -6\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0\ \gamma_{\rm v}\big|_1\ \partial_\vartheta\ K_{\rm s2}\big|_0\\
&+\gamma_{\rm s1}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm t}\big|_0\ +3\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm t}\big|_0\\
&+3\ \gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ \partial_\vartheta\ K_{\rm t}\big|_0\ -2\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm t}\big|_0\\
&-2\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0\ \gamma_{\rm t}\big|_1\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm t}\big|_0\ -2\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0\ \gamma_{\rm v}\big|_1\ \partial_\vartheta\ K_{\rm t}\big|_0\\
&+\gamma_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\ K_{\rm t}\big|_1\ -\gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0^2\ \partial_\vartheta\ K_{\rm t}\big|_1\\
&+K_{\rm s1}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\gamma_{\rm s1}\big|_0\ +\frac{1}{2}\ K_{\rm t}\big|_1\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\gamma_{\rm s1}\big|_0\\
&+3\ K_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\gamma_{\rm s1}\big|_0\ +\frac{3}{2}\ K_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\gamma_{\rm s1}\big|_0\\
&+\frac{3}{2}\ K_{\rm t}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^3\ \partial_\vartheta\gamma_{\rm s1}\big|_0\ +3\ K_{\rm s1}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ \partial_\vartheta\gamma_{\rm s1}\big|_0\\
&+\frac{3}{2}\ K_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ \partial_\vartheta\gamma_{\rm s1}\big|_0\ +\frac{3}{2}\ K_{\rm t}\big|_0\ \gamma_{\rm s2}\big|_0^3\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm t}\big|_1\ \partial_\vartheta\gamma_{\rm s1}\big|_0\\
&+K_{\rm v}\big|_0\ \gamma_{\rm s2}\big|_0\ \gamma_{\rm s2}\big|_1\ \gamma_{\rm t}\big|_0^2\ \gamma_{\rm v}\big|_0\ \partial_\vartheta\gamma_{\rm s1}\big|_0\ +K_{\rm v}\big|_0\ \gamma_{\rm s2}\big|_0^2\ \gamma_{\rm t}\big|_0\ \gamma_{\rm t}\big|_1\ \gamma_{\rm v}\big|_0\ \partial_\vartheta\gamma_{\rm s1}\big|_0
\end{aligned}
$$

$$
\begin{gathered}
+\frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{s1}}\big|_{0} \;+ K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{3}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s1}}\big|_{1} \\
+\frac{1}{2}\, K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{3}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s1}}\big|_{1} \;+ \frac{1}{2}\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{3}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s1}}\big|_{1} \\
+\frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{s1}}\big|_{1} \;+ K_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;+ 2\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+3\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;+ \frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+\frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;+ K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+K_{\mathrm{s1}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;- \frac{1}{2}\, K_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;+ \frac{1}{2}\, K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
-\frac{1}{2}\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;+ 2\, K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;- K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+\frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;+ 2\, K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \;- K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{0} \\
+K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}^{3}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{1} \;+ \frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{1} \\
+K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{1} \;+ \frac{1}{2}\, K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{1} \\
-\frac{1}{2}\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{s2}}\big|_{1} \;- K_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{3}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
-K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}^{3}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;- 3\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
-2\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{3}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{t}}\big|_{1}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;+ \frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
+K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;+ \frac{1}{2}\, K_{\mathrm{v}}\big|_{0}\, \gamma_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
+K_{\mathrm{s1}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;+ \frac{3}{2}\, K_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
+2\, K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;+ K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
+3\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{1}\, \gamma_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;+ K_{\mathrm{s1}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \\
+\frac{1}{2}\, K_{\mathrm{s2}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0} \;+ \frac{3}{2}\, K_{\mathrm{t}}\big|_{0}\, \gamma_{\mathrm{s2}}\big|_{0}^{2}\, \gamma_{\mathrm{t}}\big|_{1}\, \gamma_{\mathrm{v}}\big|_{0}^{2}\, \partial_{\vartheta}\gamma_{\mathrm{v}}\big|_{0}
\end{gathered}
$$

$$
\begin{aligned}
&+\frac{1}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + 2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0\\
&+K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 + 3\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0\\
&-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^3\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 + \frac{1}{2}\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1\\
&+K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 + \frac{1}{2}\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1\\
&+\frac{3}{2}\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_1 - K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - 2\,K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{s1}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - K_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - 2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_1\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - 4\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - 2\,K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{t}}\big|_1\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0\\
&-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_1\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0 - K_{\mathrm{v}}\big|_0\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\\
&-2\,K_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1 - K_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1\\
&-K_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_1, \qquad \text{(A.38)}
\end{aligned}
$$

$$
\begin{aligned}
b_4 &= \frac{1}{2}\,\gamma_{\mathrm{s2}}\big|_0^3\,\gamma_{\mathrm{t}}\big|_0^3\,\partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 + \frac{1}{2}\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0\\
&+\frac{1}{2}\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - \gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0, \qquad \text{(A.39)}\\
c_4 &= 4\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^3\,\gamma_{\mathrm{t}}\big|_0^3 - 4\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0^2\\
&+\frac{1}{2}\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{s1}}\big|_0 + \frac{1}{2}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{s2}}\big|_0\\
&+\frac{1}{2}\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0\,\gamma_{\mathrm{v}}\big|_0\,\partial_\vartheta\gamma_{\mathrm{v}}\big|_0 - \gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\partial_\vartheta\gamma_{\mathrm{t}}\big|_0, \qquad \text{(A.40)}\\
d_4 &= 3\,\gamma_{\mathrm{s1}}\big|_0\,\gamma_{\mathrm{s2}}\big|_0^3\,\gamma_{\mathrm{t}}\big|_0^3 - 3\,\gamma_{\mathrm{s2}}\big|_0^2\,\gamma_{\mathrm{t}}\big|_0^2\,\gamma_{\mathrm{v}}\big|_0^2, \qquad \text{(A.41)}\\
e_4 &= 0. \qquad \text{(A.42)}
\end{aligned}
$$

Acknowledgments

It is my pleasure to express my gratitude to the many people who have assisted me during my time as a PhD student and contributed to my scientific achievements documented in the thesis.

First and foremost, I am extremely grateful to my supervisor Oliver Rinne. His encouragement and advice throughout the course of my PhD, the numerous discussions and suggestions during all the time and the continuous support in the project were of considerable value. He shared his most helpful ideas and insights with me. I highly appreciate his support in the transition to the scripting language Python.

My warmest appreciation for many conversations and advice from various people. I highly benefited from discussions over the years that improved my scientific comprehension. Those involved include in addition Lars Andersson, Giorgios Doulis, Helmut Friedrich, András László, Maciej Maliborski, István Rácz and Jeffrey Winicour.

I gratefully acknowledge Lars Andersson and Ralf Kornhuber for being my mentors in the graduate schools.

In the preparation of the thesis I highly benefited from advice, hints and suggestions for improvements from Alexander Kegeles, Maciej Maliborski, Claudio Paganini, Markus Strehlau, and in particular Oliver Rinne.

I am grateful to the referees Oliver Rinne and Jörg Frauendiener. The published version of the thesis benefited from numerous suggestions and remarks by Jörg Frauendiener. I am very thankful to Ye Sle Cha, Lars Andersson, Jörg Frauendiener, Ralf Kornhuber and Oliver Rinne for joining the thesis committee and their assistance in the process of the defense.

Financial support from the German Research Foundation DFG (through grant RI 2246/2) and the International Max Planck Research School for Mathematical and Physical Aspects of Gravitation, Cosmology and Quantum Field Theory is gratefully acknowledged.

There are surely worthy supporters that are not explicitly mentioned here. I ask your forgiveness. I am most grateful to all of you.

Zusammenfassung

Die vorliegende Dissertation dokumentiert unsere Studien von Einsteins Vakuumgleichungen in sphärischen Polarkoordinaten. Offene Fragen dieser Situation betreffen sowohl Anwendungen wie gravitativer Kollaps als auch konzeptionelle Belange wie die Handhabung der auftretenden Koordinatensingularität. Wir beantworten die konzeptionellen Aspekte und zeigen, wie diese numerisch implementiert werden können.

Unsere Koordinatenwahl erlaubt einen spektralen Ansatz. Als Basisfunktionen verwenden wir Kugelflächenfunktionen mit Spin-Gewichtung. Für die meisten Ableitungen und Anwendungen nehmen wir hyperflächen-orthogonale Axialsymmetrie an. Die Annahmen führen zu rechnerischen Vereinfachungen, stellen aber keine konzeptionelle Limitierung dar.

Wir untersuchen die Eigenfunktionen des Laplace-Operators in sphärischen Koordinaten für Größen mit verschiedenen Spin-Gewichtungen und leiten die Konsequenzen ab. Eine systematische Erforschung der skalaren Wellengleichung in diesen Koordinaten führt zu hilfreichen Einsichten zur Regularisierung der Koordinatensingularität am Ursprung, und wir bestätigen diese numerisch.

Wir zeigen, weshalb eine gängige Eichwahl in Axialsymmetrie ungeeignet für die Entwicklung in spin-gewichteten Kugelflächenfunktionen ist und diskutieren Alternativen. Wir leiten die axialsymmetrischen Einsteingleichungen in geeigneter Eichung her und lösen die linearisierten Gleichungen analytisch.

Eine neuartige Formulierung von Einsteins Zwangsbedingungen stellt diese als evolutionäres System dar. Wir analysieren das gesamten System an Gleichungen und führen Modifikationen ein, die uns erlauben, zwei Sätze an lokal wohlgestellten Problemen zu formulieren.

Unsere numerische Implementierung benutzt eine hybride Diskretisierung bestehend aus Techniken der finiten Differenzen und der Pseudo-Spektralmethode. Wir simulieren die hergeleiteten Gleichungen und präsentieren eine erfolgreiche Implementierung der parabolisch-hyperbolischen Formulierung der nichtlinearen Zwangsbedingungen. Dafür leiten wir mehrere Möglichkeiten her, um die Anfangswerte am regulären Ursprung zu erhalten. Wir demonstrieren weiterhin, dass unsere Implementierung in der Lage ist, die exakte lineare Lösung unter Berücksichtigung aller Zwangsbedingungen zu reproduzieren.

Die in dieser Dissertation erhaltenen Resultate weisen eine mögliche Lösung auf, wie Einsteins Vakuumgleichungen numerisch in sphärischen Polarkoordinaten mit regulärem Ursprung simuliert werden können. Wir präsentieren eine der ersten numerischen Studien eines evolutionären Lösers der Zwangsbedingungen.

Selbständigkeitserklärung

Hiermit versichere ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage diese Arbeit selbständig verfasst zu haben, außerdem, dass ich meine Arbeit nicht schon einmal in einem früheren Promotionsverfahren eingereicht habe.

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Education

- Since October 2012 employed as a PhD student (enrolled at Freie Universität Berlin) at the Max Planck Institute for Gravitational Physics in Potsdam, supervised by Prof. Dr. Oliver Rinne, financed by the German Research Foundation (DFG) and the International Max Planck Research School.
- February 2012: Diploma in physics (thesis “Decoherence in Loop Quantum Cosmology” supervised by Prof. Dr. Claus Kiefer).
- 2005 – 2012: Undergraduate studies of physics and mathematics at the University of Cologne.

Teaching experience

- 2015: Teaching problem classes in mathematical relativity, Mathematical Institute, Freie Universität Berlin.
- April – July 2012: Scientific assistant with teaching duties, Institute of Theoretical Physics in Cologne.
- 2008 – 2012: Tutor for teaching freshmen courses in calculus and assistant for leading problem classes in mathematical methods, classical theoretical physics, quantum mechanics and computational physics, University of Cologne, eight semesters.

Relevant memberships

- Member of two graduate schools, “International Max Planck Research School for Mathematical and Physical Aspects of Gravitation, Cosmology and Quantum Field Theory” (formally “International Max Planck Research School for Geometric Analysis, Gravitation and String Theory”) and “Berlin Mathematical School” (BMS).
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Publications

1. Together with Maciej Maliborski: Maliborski and Schell (2016).
2. Together with Oliver Rinne: Schell and Rinne (2015).
3. Together with Claus Kiefer: Kiefer and Schell (2013).

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