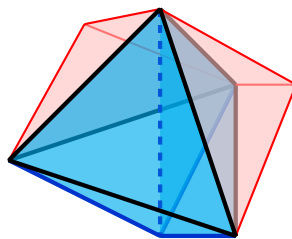# Inscribable polytopes
via Delaunay triangulations

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Introduction

The mathematical field of discrete geometry is a link between combinatorics and geometry. It investigates the shape and structure of finite, discrete objects such as polytopes, triangulations and graphs. The analogue to this area in computer science is computational geometry, which deals with algorithms that solve computational problems like optimization, scheduling and data-mining. Such algorithms often use objects from discrete geometry as the data structures. High dimensional polytopes are used in the simplex method to solve linear optimization problems, for example, to find an optimal schedule for the subway trains of Berlin [34]. Delaunay triangulations are used as grids for CAD (computer aided design) where components can be tested and optimized using physical simulations with the Finite Element method. Graphs and especially binary trees are frequently used data structures in all kinds of computer programs. They can speed up search operations, deal with storage data of varying size and provide fast data access. Discrete geometry provides the mathematical basis.

The main structures that appear in this thesis are inscribed simplicial polytopes and in particular their f-vectors. A polytope is called simplicial if all its proper faces are simplices. Two classes of polytopes will be of special interest: stacked polytopes and cyclic polytopes. They play an important role in theorems about f-vectors of simplicial polytopes. Also vital for this thesis are Delaunay triangulations which by Brown [9] can be seen as projections of inscribed polytopes and thereby turn out to be a powerful tool for us.

The f-vector of a complex like a triangulation or polytope is the statistic of how many faces of which dimension the complex possesses. This simple but fundamental property has also applications in computational geometry. For example, the run time and the data space usage of an algorithm that runs a search or even an iteration on such a complex is influenced by the f-vector of the structure.

Many theorems have been found that at least partially characterize f-vectors of polytopes. In dimension one and two f-vectors are too simple to be interesting. In dimension three, Steinitz [45] gave a complete

characterization for all f-vectors of 3-polytopes. In higher dimension, a lot is known for f-vectors of simplicial polytopes. For example, the Dehn-Sommerville equations [12, 42] state that already the first half of an f-vector of a simplicial polytope is enough to complete the whole f-vector. In 1970 McMullen [33] proved the Upper Bound Theorem: The f-vector of any polytope is component-wise smaller or equal to the f-vector of any cyclic polytope of the same dimension and number of vertices. In 1971/1973 Barnette proved the Lower Bound Theorem [1, 2]. He showed that the f-vector of any simplicial polytope is component-wise larger or equal to the f-vector of any stacked polytope of same dimension and number of vertices. Moreover, in dimension larger than three stacked polytopes are characterized by this property. The most impressive theorem is clearly the g-Theorem, which completely characterizes all f-vectors of simplicial polytopes. This theorem has been conjectured by McMullen in 1971. In 1980, the necessity part has been proven by Stanley [43] and the sufficiency part by Billera and Lee [6, 5]. In 2006 Björner [7] gave another comparison theorem which extends Lower and Upper Bound Theorem. It compares the entries of an f-vector of a simplicial polytope with the f-vectors of stacked and cyclic polytopes, that have the same number of k-dimensional faces. For non simplicial polytopes the Upper Bound Theorem still holds. It moreover implies by duality, that for all dimensions and specific number of vertices the lower bounds for polytopes are given by simple polytopes. Something like the g-theorem for all polytopes however, does not even exist for dimension four. See [49] for a survey of the current progress. In this thesis we will investigate the f-vector with respect to inscribable polytopes: We discuss the Lower Bound Theorem, the Upper Bound Theorem, the g-Theorem, f-vectors of three polytopes and characterize all inscribable stacked polytopes (joint work with Günter M. Ziegler).
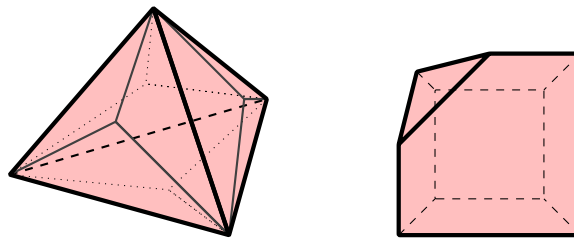

Figure 1: The combinatorial types of the triakis tetrahedron (left) and the one vertex truncated cube (right) are not inscribable. The right example is minimal with respect to the number of faces. [28]

Polytopes are called inscribed if all their vertices are positioned on a

common sphere. The corresponding combinatorial type is named inscribable. The question whether all polytopes are of inscribable type has been first asked by Steiner [44] in 1832. Unfortunately, an argument by Brückner [10], who seemed to prove that the answer is yes, turned out to be wrong. Steinitz [45] in 1928 laid the basis for the whole theory. He revealed the first examples of non inscribable types, including the triakis tetrahedron (see Figure 1) and constructed an infinite family of non inscribable 3-polytopes. He pointed out that the dual polytope of an inscribed polytope is circumscribed, and vice versa (this holds in all dimensions). One of his theorems implies that a simplicial polytope that has at least as many facets as vertices cannot be of inscribable type after all its facets are stacked. In 1967 Jucovič [29] generalized this theorem such that it implies that the d-simplex that is stacked on all facets is of non inscribable type for $d > 2$. In 1974 Grünbaum and Jucovič [24, 30] used this fact to give an asymptotic upper bound for the inscribability exponent. This exponent measures the proportion of vertices that can always be placed on a sphere, while all other vertices lie inside. Although the bound was conjectured to be tight, we managed to improve it drastically in dimension higher than 3. For dimension three Ševec [41] provided some examples in 1982 that underline that the bound is probably tight. In 1985 Schulte [39] extended the discussion about inscribability and circumscribability further to the question how many k-dimensional faces of a d-polytope can be realized to be tangent to the unit sphere. In 1991 and 1997 Jucovič, Ševec and Trenkler [31, 32] investigated quadrangular inscribable 3-polytopes.

In 1978 Brown [9] showed that all $(d-1)$-dimensional Delaunay subdivisions of convex polytopes are images of inscribed d-polytopes under stereographic projection. Due to that, a inscribability of simplicial polytopes corresponds to the Delaunay property of triangulations. Delaunay triangulations (also known as Delaunay tetrahedrizations, or in general Delaunay tessellations or Delaunay subdivision) go back to the Russian mathematician Boris Delaunay (sometimes spelled Delone) [13] in 1934. These triangulations are dual to Voronoi diagrams (also known as Voronoi tessellations or Dirichlet tessellations) which have been known long before [46, 18, 14]. Delaunay triangulations have intensively been studied in computational geometry. More significant for us are the results of Dillencourt in 1990 about realizability of planar Delaunay triangulations [15]. Beside some criteria, this work contains important examples as shown in Figure 2. In 1991 Seidel [40] showed that cyclic polytopes are inscribable and from that he derived exact upper bounds for the number of faces Delaunay triangulations in arbitrary dimensions. The breakthrough

for 3-polytopes clearly came with the characterization of all inscribable 3-polytopes by Hodgson, Rivin and Smith [38, 37, 36, 25, 26]. They identified inscribed 3-polytopes via the Klein Model with ideal polyhedra in hyperbolic space and characterized those by a dihedral angle condition of the edges. On the other hand Dillencourt and Smith [16] presented a purely numerical characterization of simple inscribable polytopes and provided a linear time algorithm to recognize if a certain simple polytope is inscribable. In 1996 they also showed some graph-theoretical conditions [17] for Delaunay triangulations in the plane, these are useful but only sufficient. So, there are characterization of inscribable (simplicial) polytopes, but they do only work for dimension three and even there we still do not have a combinatorial solution. Beside an upper bound theorem from Seidel, there is almost nothing known about f-vectors of inscribable polytopes in higher dimensions. We will change that.
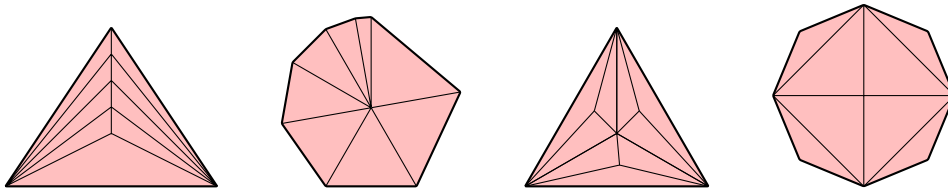
Figure 2: The first two triangulations are Delaunay triangulations, the other two cannot be realized as Delaunay triangulations. [15]

Transformations and projections are fundamental techniques in this thesis. One of them are projective transformations of the unit sphere. They are the first choice to transform inscribed polytopes, because they preserve convexity and inscribability. Brown [9] used the stereographic projection to decomposed the boundary of an inscribed d-polytope into two simplicial complexes, which only overlap in at most $(d-2)$-dimensional cells. One part is visible from the projection point and he calls the image of the projection a furthest point Delaunay triangulation. The other part which is not visible except for lower dimensional faces is projected to what he calls closest point Delaunay triangulation, and which equals the Delaunay triangulations. Brown had chosen the projection point to lie outside the polytope. In contrast, we will pick it as a vertex and call it a vertex projection. This is clearly just a variation of Browns method, but it simplifies the relation between the inscribed polytope and its projected image. It avoids furthest point Delaunay triangulations and so the usual Delaunay triangulation encodes all information about the inscribed polytope. Sphere inversions help us to understand the geometry of certain configurations of affine spaces and spheres. They can provide a clearer

view, especially for parts of Delaunay triangulations. For example, we use a sphere inversion in Chapter 2 to force some parts of a Delaunay triangulation into an affine dependence. This allows us to reduce a d-dimensional problem to a well known one of dimension two.

The name Delaunay polytopes naturally appears if one thinks of Delaunay triangulations and inscribed polytopes. The name "Delaunay polytopes" is also used for lattice polytopes that are inscribed. The main part of research in this topic involves perfect Delaunay polytopes. These have a unique circumscribing ellipsoid. However, this interesting area has little intersection with our work. See [19, 21].

The structure

Chapter 0 gives the preliminaries that we need for this thesis. It starts with the definition of a polytope and the main theorems about f-vectors of simplicial polytopes. It continues with Delaunay triangulations and their relation to inscribed polytopes, and finally discusses extensions of triangulations and how they influence the f-vector of a corresponding polytope.

In Chapter 1, we show by construction that the Lower Bound Theorem for simplicial polytopes, the Upper Bound Theorem for simplicial polytopes, and the characterization of f-vectors of 3-polytopes all hold tight when they are restricted to inscribable polytopes. This is joint work with Günter M. Ziegler [22].

In Chapter 2 we characterize all inscribable stacked polytopes. Since in dimension larger than three stacked polytopes are exactly the polytopes that reach the lower bound of the the simplicial Lower Bound Theorem, we thereby characterize the lower bound for the inscribed simplicial case. This is joint work with Günter M. Ziegler [22].

In Chapter 3 we disprove a conjecture of Grünbaum and Jucovič from 1974 about the inscribability exponent and improve the bounds of the corresponding paper drastically for dimensions higher than three. To do that we strengthen a result of chapter two. We show for all dimensions that the simplex that has four facets stacked cannot have an enclosing sphere that passes through all four apexes of the stackings.

In Chapter 4 we aim for the g-theorem of inscribed simplicial polytopes. Although we cannot solve this question completely in all dimensions, we manage to develop two construction schemes that allow us to prove several results including the following:

- The g-theorem is tight for inscribed simplicial polytopes (at least) up to dimension seven.
- For each simplicial polytope of any dimension there is an inscribed simplicial polytope of the same dimension that has the same number of vertices, edges and 2-faces.
- For any dimension d, any number of vertices $n > d + 1$ and any integer $k < d/2$, there is a k-neighborly inscribed simplicial d-polytope of n vertices, that is not $(k+1)$-neighborly.

The chapter is completed by a discussion why the sufficiency proof of the g-Theorem by Billera and Lee cannot easily be extended to the inscribable case. It includes an example of two cyclic 3-polytopes of 8 vertices that have different face-hyperplane arrangements.

In Chapter 5 we discuss three concepts of how inscribed polytopes can be combined or extended to derive new inscribed polytopes. First we discuss under what conditions an inscribed polytope has a facet that can be stacked such that the result is inscribed. This property is also important for the next two sections: We apply the techniques of chapter three to extend a given inscribed polytope and finally discuss under what conditions two inscribed polytopes can be glued by a common facet, if we previously allow projective transformations of the unit sphere.

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I like to thank my adviser Prof. Günter M. Ziegler for this wonderful topic and for the support and freedom that he gave me. I also thank Karim Adiprasito and once again Prof. Günter M. Ziegler for the work we did together.

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0 Preliminaries

This chapter introduces the basic concepts. For a background about polytopes we recommend [48] and [23]. For triangulations we have [11] and especially for Delaunay triangulations we refer to [20]. The books [3, 4] give the basics for geometry in general, including sphere inversions and transformations. For an introduction to computational geometry, we recommend [27].

0.1 Polytopes and their f-vectors

Polytopes are a fundamental concept in discrete geometry. Instead of giving a detailed survey, we focus on the f-vector of simplicial polytopes and introduce inscribed, stacked and cyclic polytopes.

For $d \geq 0$ the convex hull of finitely many points in $\mathbb{R}^d$ is called a *(convex) polytope*. An equivalent definition is the bounded intersection of finitely many closed halfspaces of $\mathbb{R}^d$. If the dimension of the affine hull of a polytope is d, then it is called a *d-polytope*. The *relative interior* of a set in $\mathbb{R}^d$ is the interior with respect to the affine hull of the set. A *supporting hyperplane* of a d-polytope P is a hyperplane that intersects the polytope but misses the relative interior of P. This intersection, which is again a polytope, is called a *face* of P. Two different hyperplanes may define the same face. By convention, the empty set and P itself are also considered to be faces of P. The empty set is of dimension -1. All faces except the last mentioned two, are called *proper faces*. Faces of dimension $0, 1, d-2$ and $d-1$ are called *vertices, edges, ridges* and *facets*. If P is a d-polytope of $\mathbb{R}^d$, then each facet of P has exactly one supporting hyperplane which divides $\mathbb{R}^d$ into two open halfspaces. The one that intersects P is called the space *beneath the facet*, the other one the space *beyond the facet*. A point $p \in \mathbb{R}^d$ is said to *see a face F of P* if it lies beyond any facet of P that contains F. Two polytopes are said to have the same *combinatorial type* if there exists an inclusion preserving one to one correspondence between the sets of their faces. In reverse, each polytope is a *realization* of its combinatorial type.

Definition 0.1.1 (Inscribed polytopes)**.** A d-polytope is called *inscribed* if its vertices lie on a common $(d-1)$-sphere. A polytope and its combinatorial type are called *inscribable* if the combinatorial type contains a realization that is inscribed.

Definition 0.1.2 (Polytopal complexes and triangulations)**.** Let $d \geq 0$. A finite set Δ of polytopes in $\mathbb{R}^d$ is called a *polytopal complex* if it satisfies the following:

- Each face of each polytope in Δ lies in Δ.
- For each $F, G \in \Delta$ the intersection is a face of F and a face of G.

Each polytope in Δ is called a *face or a cell of Δ*. The union V of the vertices of all polytopes in Δ is called the *set of vertices of Δ*. The *support* $|\Delta|$ of Δ is the union of all polytopes in Δ. The dimension of Δ is the dimensions of $|\Delta|$. A polytopal complex is called *pure* if every face is contained in a d-dimensional face. A polytopal complex is called a *simplicial complex* if all faces are simplices. A simplicial complex is called a *triangulation of V*, if it is pure, has vertex set V and its support is convex. If Δ is pure and d-dimensional, then the *boundary $\partial\Delta$ of Δ* is the polytopal complex that consists of all $(d-1)$-faces (and their faces) that only lie in only one d-face of Δ. All faces of Δ that do not lie in the boundary are called interior faces.

Two important polytopal complexes are: The *face complex of a polytope P* that is the set of all faces of P, and the *boundary complex ∂P of P* that is the face complex of P without P. In case that ∂P is a simplicial complex, P is also called *simplicial.*

Definition 0.1.3 (Star, link, deletion, pyramid)**.** Let C be a polytopal complex and F a face of it.

- The star $star_F(C)$ is the polytopal complex that consists of all faces (and their faces) of C that contain F.
- The link $link_F(C)$ is the polytopal complex that consists of all faces of the star of F that do not intersect with F.
- The deletion $del_F(C)$ is the polytopal complex that consists of all faces of C that do not contain F.
- C is the *pyramid* over an other simplicial complex C', if there exists a vertex $v \notin C'$ that does not lie in the affine hull of any face of C'. Further, C consists of all faces of the form F and $\operatorname{conv}(F, v)$, for F being a face of C'.

The *f-vector* of a $(d-1)$-dimensional polytopal complex Δ is a vector $f = (f_{-1}, f_0, \ldots, f_{d-1})$. For $i = -1, \ldots, d-1$, the entry f_i is the number

of i-dimensional faces of Δ respectively P. The f-vector of a polytope is the f-vector of its boundary complex. Not every sequence of integers defines an f-vector of a polytope. Besides $f_{-1} = 1$, each number of vertices yields finite upper bounds for the number of other dimensional faces. Surprisingly, there are polytopes that reach all upper bounds simultaneously.

We set some notation:
$$d' := \left\lfloor \tfrac{d}{2} \right\rfloor$$

Theorem 0.1.4 (Upper Bound Theorem, McMullen [33])**.**
Let $n > d > 1$. Let P be a d-polytope of n vertices and $\mathcal{Q}$ any simplicial neighborly polytope of same dimension and number of vertices. Then
$$f_i(P) \le f_i(\mathcal{Q}) \qquad for\ i = 1, \ldots, d-1.$$
For $i \le d'$ this is
$$f_{i-1}(P) \le f_{i-1}(\mathcal{Q}) = \binom{f_0}{i}.$$

By the Dehn-Sommerville equations, the f-vector of a simplicial d-polytope is already defined by the first half: $f_{-1}, \ldots, f_{d'-1}$.

Definition 0.1.5 (k-neighborly polytope)**.** For $d \ge 1$ and $0 \le k \le d'$ a d-polytope P is called *k-neighborly* if for all k-subsets M of the vertex set of P, the convex hull of M is a face of P. A polytope is called *neighborly*, if it is d'-neighborly. An equivalent definition to k-neighborly is an f-vector that satisfies:
$$f_{i-1}(P) = f_{i-1}(\mathcal{Q}) = \binom{f_0}{i} \qquad for\ i = 1, \ldots, k.$$

The complete f-vector of a simplicial neighborly d-polytope is:
$$f_{i-1} = \sum_{k=0}^{i} \binom{d-k}{d-j} h_k \qquad for\ \ h_k := \begin{cases} \binom{n-d+k-1}{k} & k = 0, \ldots, d' \\ h_{d-k} & k = d'+1, \ldots, d \end{cases}$$

More about neighborly polytopes can be found in [23]. An important example of neighborly polytopes are the cyclic polytopes:

Definition 0.1.6 (Cyclic polytope)**.** For $d > 1$ the d-dimensional moment curve is given by:
$$c : \mathbb{R} \to \mathbb{R}^d \qquad t \mapsto (t^1, t^2, \ldots, t^d)$$

It is known that the convex hull of any distinct $n > d$ points on the trace of this curve will yield to the same combinatorial type of polytope. This type is called $\mathcal{C}_d(n)$, *the cyclic d-polytope of n vertices*.

All cyclic polytopes are simplicial neighborly polytopes. For a realization of a cyclic polytope whose vertices lie on the moment curve, the parameter t of the curve $c(t)$ induces an order on the vertices. Using this order, the subsets of the vertex set of $C_d(n)$ can be described by a binary vector of length n, where a 1-entry indicates that the vertex is part of the set. *Gale's evenness condition*, which is explained in more detail in [23], states that such a vector with d many 1-entries represents a facet of $C_d(n)$ if and only if between any two 0-entries an even number of 1-entries appear.

This condition defines the cyclic type: The condition completely encodes the incidence relation between facets and vertices. This implies that if the vertices of a given polytope can be ordered in a way such that the polytope satisfies Gale's condition, then it must be of cyclic type. We will use this trick later.

For simplicial polytopes, there is also a lower bound for the f-vector. To understand it, we need the following definition.

Definition 0.1.7 (Stacked polytope)**.** For $d > 1$ a *stacking operation* is performed onto a facet by taking the convex hull of the polytope with a new point that lies beyond the selected facet but beneath all other facets of the polytope. A polytope is called *stacked* if it can be built from a d-simplex by a sequence of stacking operations.

Theorem 0.1.8 (Lower Bound Theorem, Barnette [1] [2])**.** *Let $d > 1$ and $n > d$. Let P be a simplicial d-polytope of n vertices and $\mathcal{S}_d(n)$ any stacked polytope of same dimension and number of vertices. Then*

$$f_i(P) \geq f_i(\mathcal{S}_d(n)) = \binom{d+1}{i+1} + (n - d - 1)\binom{d}{i}$$

for $i = 1, \ldots, d$.

In dimension 3 this is rather boring, because by the Euler characteristic and a double counting argument, the number of vertices already determines the number of edges and facets of a simplicial polytope. By that, every simplicial 3-polytope reaches both upper and lower bound. In dimensions greater than 3, all stacked polytopes reach the lower bound, and these are the only polytopes that do so.

In contrast to the upper bound theorem, the lower bound theorem does not work for non simplicial polytopes. A general lower bound theorem is given by the dual polytopes of simplicial neighborly polytopes. The only problem is, that these bound only hold for specific number of vertices,

namely those that appear as number of facets of simplicial neighborly polytopes (of the same dimension). The dual type of a polytope is the type that has the same numbers of faces but all inclusions are reversed. From that, one gets that there are *simple* polytopes, that provide lower bounds which hold for all polytopes. But this is besides our topic since we are interested in simplicial polytopes. For simplicial polytopes the *g*-Theorem tells exactly which *f*-vectors are possible. Before we get to this theorem, we introduce the *h*-vector and the *g*-vector.

Definition 0.1.9 (*h*-vector and *g*-vector). Let $d \geq 0$ and let $f = (f_{-1}, \ldots, f_{d-1})$ be the *f*-vector of a simplicial *d*-polytopes or the *f*-vector of simplicial complex of dimension $d - 1$. By convention, $f_{-1} := 1$. Following the paper of Billera and Lee [6] we define the polynomials

$$f(t) := \sum_{k=0}^{d} f_{k-1} t^k \qquad \text{and} \qquad h^{(e)}(t) = \sum_{k=0}^{\infty} h_k^{(e)} t^k := (1-t)^e f\left(\frac{t}{1-t}\right).$$

The value e usually equals d or $d + 1$, but in principal it can be any positive integer. The corresponding *h-vector* and *complete g-vector* is given by

$$h = h^{(d)} = (h_0, \ldots, h_d) := (h_0^{(d)}, \ldots, h_d^{(d)})$$

and

$$g = (g_0, \ldots, g_d) := (h_0^{(d+1)}, \ldots, h_d^{(d+1)}).$$

Some notation: The Dehn-Sommerville equations state that for a simplicial polytope the *h*-vector is symmetric ($h_i = h_{d-i}$ for $i = 0, \ldots, d$). Due to that, the complete *g*-vector of a simplicial polytope is determined by its first half. This motivates the definition of the *g-vector* of a simplicial polytope to be

$$g := (g_0, \ldots, g_{d'}).$$

The notation of the *f*-, *g*-, and *h*-vector in the above definitions does not contain a reference to the corresponding polytope (or polytopal complex) *P*. Sometimes it is necessary to denote this relation. Only in this case we will use *P* as additional argument. Example: $g(t)$ and f of a polytope *P* may also be denoted as $g(t, P)$ and $f(P)$.

Remark 0.1.10. Note that $f(t)$ and $h(t)$ are **not** the *f*- and *h-polynomials* that are defined in [11] and [48]. The definition there is:

$$F(t) := \sum_{k=0}^{d} f_{i-1} t^{d-i}, \qquad \text{and} \qquad H(t) = \sum_{k=-\infty}^{\infty} h_i t^{d-i} := F(t-1).$$

Both definitions yield the same h_i, for $i = 0, \ldots, d$. Unfortunately the definition of $H(t)$ and $F(t)$ depends on d. Since in the following we use formulas that include g-vectors and h-vectors with respect to different dimensions, we need a waterproof definition of $h^{(d)}(t)$. That is why we take this "old" definition.

Lemma 0.1.11. *Here is a list of facts that can easily be computed. Let $d \geq 0$.*

- *$f_{-1} = h_0 = g_0 = 1$.*
- *For $i = 0, \ldots, d$, we have*

$$g_i = h_i^{(d+1)} = h_i^{(d)} - h_{i-1}^{(d)} = h_i - h_{i-1}$$

- *An inverse formula exists:*

$$f(t) := (1+t)^e h^{(e)}\left(\frac{t}{1+t}\right).$$

- *By sorting the coefficients we can count:*

$$h_i^{(e)} = \sum_{k=0}^{i} (-1)^{k-i} \binom{e-k}{e-i} f_{k-1}, \qquad g_{i+1} = \sum_{k=-1}^{i} (-1)^{k-i} \binom{d-k}{d-i} f_k,$$

$$f_{i-1} = \sum_{k=0}^{i} \binom{e-k}{e-i} h_k^{(e)}, \qquad f_i := \sum_{k=-1}^{i} \binom{d-k}{d-i} g_{k+1}$$

- *The g-vector of a stacked d-polytope S with $k + d + 1$ vertices is*

$$g(S) = (g_0, \ldots, g_{d'}) = (1, k, 0, 0, \ldots, 0)$$

- *The g-vector of a cyclic d-polytope C with $k + d + 1$ vertices is*

$$g(C) = (g_0, \ldots, g_{d'}) = (1, \binom{k}{1}, \binom{k+1}{2}, \binom{k+3}{3}, \ldots)$$

Remark 0.1.12. Let $d \geq 0$ and $k = 1, \ldots, d$. Then the vectors $(f_0, \ldots, f_{k-1})$ and $(g_1, \ldots, g_k)$ determine each other. For simplicial polytopes, the f-vector is already given by $(g_1, \ldots, g_{d'})$. Since f is a linear function of g, we can express their relation by a matrix. This is called *McMullen's correspondence* (Which was introduced by Björner).
For example, for $d = 3$ we get:

$$(1, f_0, f_1, f_2) = (1, g_1) \begin{pmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

For $d = 4$ we get:

$$(1, f_0, f_1, f_2, f_3) = (1, g_1, g_2) \begin{pmatrix} 1 & 5 & 10 & 10 & 5 \\ 0 & 1 & 4 & 6 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

The first part of each such matrix is a square matrix, whose entries are binomial coefficients. The non 0-entries are one part of an upside down Pascal-triangle. The matrix has upper triangular shape and the diagonal carries only 1-entries. This leads to a very nice affinity that maps $(g_1, \ldots, g_{d'})$ to $(f_0, \ldots, f_{d'-1})$. For $d = 9$ this is

$$(f_0, f_1, f_2, f_3) = (1, 10, \binom{10}{2}, \binom{10}{3}) + (g_1, g_2, g_3, g_4) \begin{pmatrix} 1 & 9 & \binom{9}{2} & \binom{9}{3} \\ 0 & 1 & 8 & \binom{8}{2} \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Back to the main question, which types of vectors are g-vectors of simplicial polytopes.

Theorem 0.1.13 (The g-Theorem, Billera & Lee [6], Stanley [43]). *Let $d \geq 0$. Let $g = (1, g_1, g_2, \ldots, g_d) \in \mathbb{N}_0^{d+1}$. Then g is the complete g-vector of a simplicial d-polytope if and only if*

- *for $k = 1, \ldots, d'$ we have $0 \leq g_k \leq g_{k-1}^{<k-1>}$,*
- *for $k = 1, \ldots, d$ we have: $g_k = -g_{d-k+1}$.*

The last condition of Theorem 0.1.13 is known as the Dehn-Sommerville equations which can also be stated as $h_i = h_{d-i}$, for $i = 0, \ldots, d'$.

The expression $n^{<k>}$ is called the kth pseudo power of n, a definition follows. For $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ there are integers $0 \leq a_1 \leq \cdots \leq a_k$ that satisfy

$$n = \sum_{i=1}^{k} \binom{a_i + i - 1}{i}.$$

This decomposition of n is called the *k-canonical representation of n*. It turns out that it always exists and it is unique. The *kth pseudo power of n* is defined by

$$n^{<i>} := \sum_{i=1}^{k} \binom{a_i + i}{i + 1}.$$

Remark 0.1.14. For $d > 1$ let P be a simplicial d-polytope with g-vector $(g_0, \dots, g_{d'})$. Let $0 < k \le d'$ and let Q be a simplicial polytope that has a minimal number of $(k-1)$-faces while sharing the same g-vector entries $g_0, \dots, g_{k-1}$. Then g_k counts the number of $(k-1)$-faces that P has more than Q.

In the third part of this thesis g_1, g_2 and g_3 are of special interest. For those, the g-Theorem states the following:

Corollary 0.1.15. *Let $d \ge 4$ and P be a simplicial d-polytope, then $g(P) \in \mathbb{N}_0^{d'+1}$ and*

$$ g_0(P) = 1, \qquad 0 \le g_1(P), \qquad 0 \le g_2(P) \le \binom{g_1 + 1}{2}. $$

If $d \ge 6$ we define $0 \le b \le a$ such that $g_2(P) = \binom{a+1}{2} + \binom{b}{1}$. Then

$$ 0 \le g_3(P) \le \binom{a+2}{3} + \binom{b+1}{2}. $$

These conditions are also sufficient, in the sense that for each such integers g_1, g_2, g_3 a corresponding simplicial d-polytope exists. (For example with the remaining g entries set to 0.)

An other useful corollary is:

Corollary 0.1.16 (Pure binomial coefficients in g-vectors)**.** *Let $d > 0$. Let $n_1, \dots, n_{d'} \in \mathbb{N}_0$ be integers and*

$$ g := (1, \binom{n_1}{1}, \binom{n_2 + 1}{2}, \dots, \binom{n_{d'} + d' - 1}{d'}). $$

Then g is the g-vector of a simplicial d-polytope if and only if

$$ n_1 \ge n_2 \ge \cdots \ge n_k \ge 0. $$

0.2 Delaunay triangulations and stellar subdivisions

In this section we repeat the definition of Delaunay subdivision and Delaunay triangulation, which is not consistent throughout the literature. We also gather some facts about Delaunay triangulations that will be useful for us later. A good book about Delaunay triangulations has been written by Edelsbrunner [20].

Definition 0.2.1 (Circumsphere, inside, outside)**.** Let $d \geq k \geq 1$. For a k-simplex δ in $\mathbb{R}^d$ the *circumsphere* is the unique $(k-1)$-sphere that passes through all vertices of δ. For a $(d-1)$-sphere S in $\mathbb{R}^d$, the *interior* of S (or the space inside S) is the open ball whose boundary is S. The *outside* of S is the set of points that neither lie on nor inside S.

Definition 0.2.2 (Delaunay triangulations)**.** Let $d > 1$ and $V \subset \mathbb{R}^{d-1}$ be a finite set of points who affinely span $\mathbb{R}^{d-1}$. Then the *Delaunay subdivision* $\mathcal{D}(V)$ is the polytopal complex that has vertex set V, that has $|D(V)| = \mathrm{conv}(V)$ and that consists of all cells that are defined by the so called *empty circumsphere condition*: For each $F \in D(V)$ exists a $(d-2)$-sphere that passes through all vertices of F and all other points of V lie outside this sphere.

The Delaunay subdivision is unique, pure and it always exists. If the points in V are in sufficiently general position (which is satisfied for example if no $d+1$ points lie on a common sphere), then the Delaunay subdivision of V is a triangulation. In this case the Delaunay subdivision is called *the Delaunay triangulation* of V.

We will employ the following very elegant criterion in order to test whether a given triangulation is the Delaunay triangulation. For this we call a face of a triangulation in $\mathbb{R}^{d-1}$ *Delaunay* if there exists a *supporting sphere* of the face, that is, a $(d-2)$-sphere that passes through the vertices of the face but all other vertices of the triangulation lie outside the sphere. Each interior $(d-2)$-face F is contained in exactly two $(d-1)$-faces, $\mathrm{conv}(F \cup \{v_1\})$ and $\mathrm{conv}(F \cup \{v_2\})$; we call it *locally Delaunay* if there exists a $(d-2)$-sphere passing through the vertices of F such that the vertices v_1 and v_2 lie outside this sphere.

Lemma 0.2.3 (Delaunay Lemma)**.** *Let $d > 1$ and let $V \subset \mathbb{R}^{d-1}$ be a finite, affine spanning set of points. A triangulation T with vertex set V is the Delaunay triangulation if and only if one of the following equivalent statements hold:*

(1) *All $(d-1)$-faces of T are Delaunay.*
(2) *All faces of T are Delaunay.*
(3) *All $(d-2)$-faces of T are Delaunay.*
(4) *All interior $(d-2)$-faces of T are locally Delaunay.*

Proof. The first statement is the definition of a Delaunay triangulation. It implies the second statement: For each face F one can always do a slight change to the supporting sphere of a $(d-1)$-face that contains F

to derive a supporting sphere of F. The second statement implies the third and this in turn the last one. For more details and also for a proof that the last statement implies the first, we refer to Edelsbrunner [20, pp. 7 and 99]. $\qquad\square$

The next lemma is well known and probably the most frequently used criterion to check the Delaunay property.

Lemma 0.2.4 (Angle criterion for Delaunay triangulation in the plane). *Let ABC and BCD be two triangles in the plane that intersect in the edge BC. The edge BC is locally Delaunay if and only if the angle at A in ABC and the angle at D in the BCD sums up to strictly less than π.*

Proof. Two opposite angles of an inscribed quadrilateral always sum up to π. An arbitrary quadrilateral differs from an inscribed one by the position of a single vertex. By moving any vertex of an inscribed quadrilateral straight away/towards the center of the circumcircle, the angle at this vertex decreases/increases, while the opposite angle stays unchanged. $\quad\square$

Lemma 0.2.5 (Sphere pushing). *Let $d \geq 2$ and let S_1, S_2 be $(d-1)$-spheres whose intersection is a $(d-2)$-sphere. Let H^+, H^- be the two open halfspaces that are bounded by the affine hull of $S_1 \cap S_2$. Let $B_1, B_2 \subset \mathbb{R}^d$ be the open balls that are bounded by S_1, S_2. Then*

$$S_1 \cap H^+ \subset B_2 \cap H^+ \quad and \quad S_2 \cap H^- \subset B_1 \cap H^-$$

for the right notation of H^+, H^-.

Proof. Let $S_1, S_2 \subset \mathbb{R}^d$ be two distinct $(d-1)$-spheres with corresponding centers c_1, c_2 and radii r_1, r_2. Following the book [20] the *power* with respect to S_1 is defined by

$$p : \mathbb{R}^d \to \mathbb{R}, \qquad p(x) := \|x - c_1\|^2 - r_1^2.$$

Some properties are

- $p(x) < 0$ if and only if x lies inside S_1,
- $p(x) = 0$ if and only if x lies on S_1.

Let p_2 denote the power with respect to S_2. Then we see that the set $\left\{ p(x) = p_2(x) \,\middle|\, x \in \mathbb{R}^d \right\}$ is a hyperplane H:

$$
\begin{aligned}
p(x) &= p_2(x) \\
\Leftrightarrow \langle x\,,x \rangle - 2\langle x\,,c_1 \rangle + \|c_1\|^2 - r_1^2 &= \langle x\,,x \rangle - 2\langle x\,,c_2 \rangle + \|c_2\|^2 - r_2^2 \\
\Leftrightarrow \langle x\,,2c_2 - 2c_1 \rangle &= \|c_2\|^2 - \|c_1\|^2 + r_1^2 - r_2^2 \quad \text{(constant)}
\end{aligned}
$$

Analogue for $\left\{ p(x) > p_2(x) \,\middle|\, x \in \mathbb{R}^d \right\}$ we get an open halfspace H^+ that is bounded by H. In case that S_1, S_2 intersect, H has to contain this intersection. All this implies that in H^+ each point x on S_1 has $p(x) = 0$ and $p_2(x) < 0$. Hence x lies inside S_2. This implies the lemma. $\qquad\square$

The next lemma applies the "sphere pushing" technique.

Lemma 0.2.6 (Locally Delaunay via enclosing Sphere). *Let $d > 2$ and $a, b, v_1, \ldots, v_{d-1} \in \mathbb{R}^{d-1}$. Let $A = \operatorname{conv}(a, v_1, \ldots, v_{d-1})$ and $B = \operatorname{conv}(b, v_1, \ldots, v_{d-1})$ be two $(d-1)$-simplices. Let $F := \operatorname{conv}(v_1, \ldots, v_{d-1})$ be their intersection and let the line segment (a, b) intersect the relative interior of F. Then F is locally Delaunay if and only if there exists a sphere passing through a and b that contains F inside.*

Proof. Assume F is locally Delaunay. Let $S_F(\alpha)$ be the $(d-3)$-dimensional circumsphere of F scaled by $\alpha \geq 1$ at the circumcenter of F. Let $S(\alpha)$ be the $(d-2)$-sphere that passes through a and $S_F(\alpha)$. Then

- for $\alpha = 1$, $S(\alpha)$ is the empty circumsphere of A (b lies outside),
- for $\alpha > 1$, F lies inside $S(\alpha)$,
- for α large enough, b lies inside $S(\alpha)$ because the hyperplane $\lim_{\alpha \to \infty} S(\alpha)$ is parallel to the affine hull of F (which separates a, b) and passes through a.

Hence there must be a value $\alpha > 1$ such that $S(\alpha)$ passes through a, b and F lies inside.

For the other direction we assume there is a sphere S that passes through a and b while F lies inside. Let S' be a sphere that passes through $a, b, v_1, \ldots, v_k$ such that k is maximal and $v_{k+1}, \ldots, v_{d-1}$ lie inside S'.

We show that $k < d - 1$. For contradiction we assume S' passes through all vertices. Then the affine hull of $S' \cap S$ is a hyperplane that cuts through the relative interior of F. Hence at least one vertex lies on each side. Lemma 0.2.5 then implies that S' contains at least one of these vertices still inside, which is a contradiction.

We now show that $k = d - 2$. For contradiction we assume $k \leq d - 3$. Then there is a hyperplane H that contains the vertices $a, b, v_1, \ldots, v_k$ but no others. Since H contains an interior point of F, it splits the remaining vertices into two nonempty sets. H can be seen as a limit of a homotopy (variable $t \in \mathbb{R}^+$) that starts with S' and contains only spheres that pass

through $a, b, v_1, \ldots, v_k$. We see that at some point $(t > 0)$ a larger k must be possible, which contradicts the maximality of k.

The vertices $a, v_1, \ldots, v_{d-2}$ define a facet G of A, which is also a facet of $\mathrm{conv}(A, B)$. This implies that b and v_{d-1} lie on the same side of G. Hence by Lemma 0.2.5 the sphere that is defined by passing through $a, v_1, \ldots, v_{d-1}$ must have b outside and hence is an empty circumsphere for A. This proves that F is locally Delaunay. $\qquad\square$

Definition 0.2.7 (Stellar subdivision). Let $d > 1$ and p be a point in the interior of a $(d-1)$-simplex σ of a triangulation T in $\mathbb{R}^{d-1}$. A *(single) stellar subdivision* of T at σ by p is the triangulation that replaces σ by p and adds the simplices that are spanned by p and each proper face of σ. We call a triangulation a *multiple stellar subdivision* of T at σ if one or more single stellar subdivisions have been applied, one after an other with points that lie in the relative interior of σ.

Lemma 0.2.8. *Let $d > 1$. Let T be the face complex of a $(d-1)$-simplex in $\mathbb{R}^{d-1}$ and $p \in \mathrm{relint}\,|T|$. Then the stellar subdivision of T by p is a Delaunay triangulation.*

Proof. If the circumsphere of a new full dimensional simplex σ would contain the vertex v that does not lie in σ, then it would contain all points of the original simplex, and hence it would contain the new vertex c in its interior. See Figure 0.1. $\qquad\square$
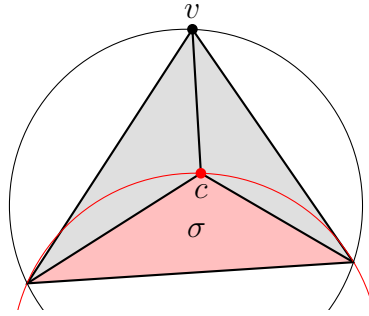

Figure 0.1: The circumsphere of σ cannot contain v.

Lemma 0.2.9 (Any stellar subdivision can be undone). *Let the triangulation T' be obtained from a triangulation T of an affine spanning point set $V \subset \mathbb{R}^{d-1}$ by a single stellar subdivision. If T' is a Delaunay triangulation, then so is T.*

Proof. A stellar subdivision does not destroy a $(d-2)$-face, thus among the supporting spheres for $(d-2)$-faces in T' we have the supporting spheres for all $(d-2)$-faces in T. □

We end this subsection with an example of a triangulation that cannot be realized as a Delaunay triangulation. See Figure 0.2. Later we will see that the three examples of Figure 0.2 all correspond to the same problem: How many stackings can be applied to a simplex, until it looses inscribability.
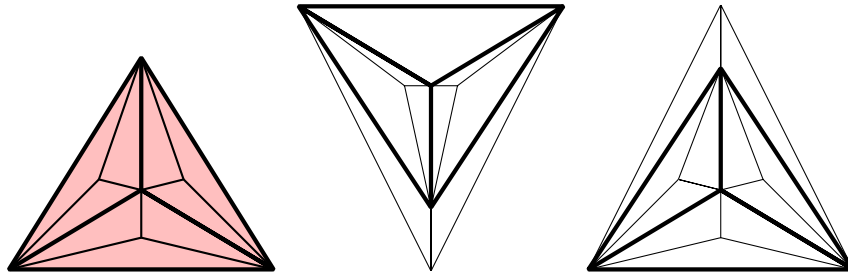

Figure 0.2: Left: A triangulation that cannot be be realized as a Delaunay triangulation. Middle: The same edge graph as in the left figure, but as part of a Delaunay triangulation. Right: An edge graph that does not belong to any Delaunay triangulation.

Steinitz already proved that the triakis tetrahedron (dimension 3) is of non inscribable type. We give an alternative proof that is slightly stronger. The result however is well known.

Lemma 0.2.10. *The triangulation that is described in Figure 0.3 cannot be realized as a Delaunay triangulation. In particular, at least one of the edges $(x, A), (x, B), (x, C)$ must not be locally Delaunay.*

Proof. See Figure 0.3. The nine angles at a, b and c sum up to 6π. The three angles that lie in triangles that contain a boundary edge are each smaller than π, so the remaining six angles must sum to more than 3π. However, each of the three edges Ax, Bx and Cx is locally Delaunay if and only if its two opposite angles sum to less than π. Hence not all three edges can be locally Delaunay. □
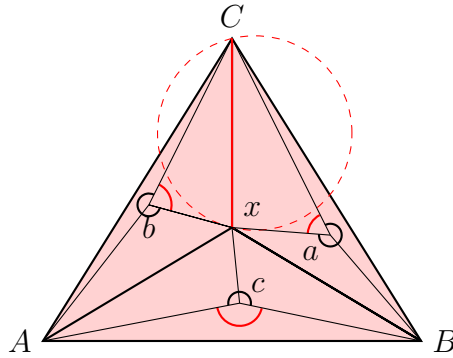
Figure 0.3: In any triangulation of this combinatorial type at least one of the three edges Ax, Bx and Cx is not locally Delaunay.

0.3 Projections and transformations

We will briefly review projective transformations, Möbius transformations and stereographic projections. After this we discuss the relation between inscribed polytopes and Delaunay triangulations. For a broad background we refer to [3, 4].

For $d \geq 1$ let $\mathrm{PGL}(\mathbb{R}^d)$ denote the group of projective transformations in the d-dimensional (real) projective space. We think of the projective space is an extension of $\mathbb{R}^d$ by points at infinity that correspond to all directions in $\mathbb{R}^d$ modulo orientation. Let $\mathrm{M\ddot{o}b}(\mathbb{R}^d)$ denote the group of d-dimensional Möbius transformations in $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$. For $d > 1$ let $\mathrm{PGL}(S^d)$ denote the restrictions to S^d of those transformations in $\mathrm{PGL}(\mathbb{R}^{d+1})$ that keep S^d invariant. Similar, let $\mathrm{M\ddot{o}b}(S^d)$ denote the restrictions to S^d of those Möbius transformations of $\mathrm{M\ddot{o}b}(\mathbb{R}^{d+1})$ that keep S^d invariant. Both are groups and they are known to be the same,

$$\mathrm{PGL}(S^d) = \mathrm{M\ddot{o}b}(S^d).$$

Projective transformations map hyperplanes to hyperplanes. Those that keep $\mathbb{R}^d$ invariant, restrict to affine transformations on $\mathbb{R}^d$. Projective transformations that preserve the unit sphere do also preserve the unit ball.

Möbius transformations are generated by sphere inversions and hence preserve generalized spheres. Generalized means that hyperplanes are spheres that pass through infinity. Moreover Möbius transformations

are conformal, meaning they preserve intersection angles. All Möbius transformations can be decomposed into at most one sphere inversion of radius one and afterward a euclidean transformation.

Both Möbius transformations and projective transformations are isomorphisms in their corresponding space. The following two lemmas illustrate the use of these transformations.

Lemma 0.3.1 (Möbius transformations and Delaunay triangulations)**.**
Let $d > 2$. Let T be a $(d-1)$-dimensional Delaunay subdivision and V its vertex set. Let ψ be a Möbius transformation and $p = \psi^{-1}(\infty)$ be a point outside all circumspheres of $(d-1)$-cells of T. Then the combinatorial type of T is a subcomplex of the combinatorial type of the Delaunay subdivision T' of $\psi(V)$.

In case that $\psi(\infty)$ lies outside all circumspheres of T', both triangulations are combinatorially equivalent.

Proof. The map ψ maps generalized spheres to generalized spheres. The inside of a regular sphere S is mapped to the inside of a regular sphere, if and only if the point that is mapped to ∞ lies outside S. Hence the image of the vertices of a Delaunay $(d-1)$-face have an empty circumsphere if and only p lies outside the original circumsphere. The second part follows by inclusion from the other direction. $\qquad\square$

Lemma 0.3.2 (Möbius transformations and inscribed polytopes)**.** *Let $d > 1$. Let P be a d-dimensional inscribed polytope and V its vertex set. Let ψ be a projective transformation of $\mathrm{PGL}(\mathbb{R}^d)$ that preserves the unit sphere. Then $\psi(P)$ is an inscribed realization of P.*

Proof. The supporting hyperplanes of the facets of P are mapped to supporting hyperplanes of $\psi(P)$. So $\psi(P)$ is a realization of P. Since all vertices stay on the unit sphere, $\psi(P)$ is inscribed. $\qquad\square$

Definition 0.3.3. Let $d \geq 1$ and $N := (0, \ldots, 0, 1) \in \mathbb{R}^d$. The *stereographic projection* is defined by

$$\phi : S^{d-1} \setminus \{N\} \;\longrightarrow\; \mathbb{R}^{d-1} \times \{0\}, \qquad x \mapsto \Big(\frac{x_1}{1 - x_d}, \ldots, \frac{x_{d-1}}{1 - x_d}, 0\Big).$$

The stereographic projection can be extended in two meaningful ways. First, as a radial projection

$$\psi : \mathbb{R}^d \setminus \Big\{x \in \mathbb{R}^d \,\Big|\, x_d < 1\Big\} \to \mathbb{R}^{d-1} \times \{0\}, \qquad x \mapsto \Big(\frac{x_1}{1 - x_d}, \ldots, \frac{x_{d-1}}{1 - x_d}, 0\Big).$$

A second possibility is an inversion at the sphere with center N and radius $\sqrt{2}$

$$\theta : \mathbb{R}^d \setminus \{N\} \to \mathbb{R}^d \setminus \{N\}, \qquad x \mapsto \frac{x - N}{\frac{1}{2}(x_1^2 + \cdots + x_{d-1}^2 + 1 - 2x_d)} + N.$$

The second map is a Möbius transformation, where spheres through N are mapped to affine subspaces. This map is an isomorphism.

Remark 0.3.4. The map ϕ identifies maps of $\text{Möb}(S^d)$ with maps of $\text{Möb}(\mathbb{R}^d)$ since both preserve generalized spheres and ϕ does also.

$$\text{PGL}(S^d) = \text{Möb}(S^d) \simeq \text{Möb}(\mathbb{R}^d)$$

Definition 0.3.5 (Vertex projection)**.** Let $d > 1$ and let P be a d-polytope and V its vertex set. After a euclidean motion we can assume that N is one of its vertices and that all other vertices lie in $\left\{ x \in \mathbb{R}^d \,\middle|\, x_d < 1 \right\}$. The above mentioned radial projection ψ maps $V \setminus \{N\}$ to $\mathbb{R}^{d-1}$ and each face of $del_N \partial P$ to a polytope that is projectively equivalent to the face. We define the *vertex projection* Ψ to be the polytopal map that is induced by ψ on $V \setminus \{N\}$.

Lemma 0.3.6 (Missing star)**.** *Let $d > 1$ and let P be a d-polytope with vertex set $V \cup \{N\}$. Let T be the image of a vertex projection Ψ of P from N and let $U \subset V$. Then $Q := \text{conv}(U \cup \{N\})$ is a k-face of P if and only if $F := \text{conv}(\Psi(U))$ is a $(k-1)$-face of $|T|$ and $F \cap \Psi(V) = \Psi(U)$.*

Proof. Assume Q is a k-face of P. Then it has a supporting hyperplane. The projection of that hyperplane (the part that can be projected) is a supporting hyperplane for F in $|T|$. It contains the vertices $\Psi(F)$ and cannot contain any other. The affine hull of Q contains N and therefore looses one dimension by the projection. Hence F must have dimension $k - 1$.

Assume F is a $(k-1)$-face of $|T|$ and $F \cap \Psi(V) = \Psi(U)$. Then any supporting hyperplane of F for $|T|$ can be extended to a a supporting hyperplane for $\text{conv}(U) \cup \{N\}$ by adding N to the affine hull. This shows that Q is a face of P. By the first argument, it has dimension k. $\qquad\square$

Corollary 0.3.7 (Missing star)**.** *Let $d > 1$ and let P be a d-polytope with vertex set $V \cup \{N\}$. Let T be the image of a vertex projection Ψ of P from N. Then P is simplicial if and only if T is a triangulation and the boundary vertices of T are the vertices of a simplicial polytope that is $|T|$.*

Definition 0.3.8 (Center of stellar subdivision, apex of stacking)**.** Let $d \geq 2$. We call an interior vertex of a pure $(d-1)$-dimensional polytopal complex the *center of a stellar subdivision* if its star is a pyramid over the boundary of a $(d-1)$-simplex. We call a vertex of a d-polytope P the *apex of a stacking* if it is the center of a stellar subdivision in ∂P.

Lemma 0.3.9 (Vertex projection of a stacking)**.** *For $d \geq 2$ let P be a d-polytope that is not a simplex. Let T be any vertex projection of P whose boundary vertices are the vertices of a polytope and let s be any vertex of P. Then s is the apex of a stacking if and only if:*

1. *If s is the projection vertex then the boundary complex of T is the boundary complex of a $(d-1)$-simplex.*

2. *If s has an image s' in the interior of T then s' is the center of a stellar subdivision.*

3. *If s has an image s' in the boundary of T, then s' belongs to exactly one $(d-1)$-cell of T and that cell is a simplex δ. Further, the facets of δ that contain s' are also facets of $|T|$.*

Proof. Assume s is the projection vertex. Then the link of s in ∂P corresponds to the boundary of T. This implies (*1*). Point (*2*) is basically the definition.

Assume s' lies in the boundary of T. Then the link of s contains the projection vertex N. If s is the apex of a stacking, then only one facet in the link of s does not contain N. Hence s' lies in only one $(d-1)$-face δ of T and this face is a simplex. By Lemma 0.3.6 the remaining facets that contain s and N lead to the facets of δ that contain s' and these are also facets of $|T|$.

The other direction: Assume s' lies in only one $(d-1)$-face of T and this face is a simplex δ. Further, let the facets of δ that contain s' be facets of $|T|$. Then by Lemma 0.3.6 each such facet yields a facet of P that contains s and N. In addition, δ corresponds to a facet in P. These are all facets that contain s and they are simplices (d many). Hence s is the apex of a stacking. This shows (*3*). $\qquad\square$

Definition 0.3.10 (Lower convex hull)**.** Let $d > 0$ and P be a d-polytope in $\mathbb{R}^d$. Let F be a facet of P and s the normal vector of the facet that points away from P. Then we call F *down facing* if the last coordinate of s is smaller than 0. The *lower convex hull of P* is defined to be the

polytopal complex that consists of all down facing facets of P and their faces.

Definition 0.3.11 (Regular triangulation)**.** Let $d > 0$ and let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the orthogonal projection that truncates the last coordinate. A triangulation is called *regular* if it is the image under π of a lower convex hull of a $(d + 1)$-polytope.

Remark 0.3.12 (Regular triangulations)**.** Images of a vertex projections are regular subdivisions and vice versa. Both concepts only differ by a projective transformation that fixes the hyperplane $\mathbb{R}^d \times \{0\}$ and maps the projection vertex of the vertex projection to the infinite direction $(0, \dots, 0, 1)$. This transformation directly turns the "projection rays" that emit radially from the projection vertex to parallel rays that go "straight down".

Proposition 0.3.13 (Inscribed polytopes and Delaunay triangulations)**.** *Let $d > 1$. Let $P \subset \mathbb{R}^d$ be a d-polytope and let $V \cup \{N\}$ be its vertex set. Let T be the image of a vertex projection of P from N. If P is inscribed then T is a Delaunay subdivision of dimension $d - 1$. If in addition, P is simplicial then T is a Delaunay triangulation and the boundary vertices of T are vertices of a simplicial polytope $|T'|$.*

If T' is a $(d - 1)$-dimensional Delaunay subdivision, then there exists an inscribed d-polytope P' such that T' is the image of a vertex projection of P'. If in addition, T' is a triangulation where all boundary vertices are vertices of a simplicial polytope that is $|T'|$ then P' is simplicial.

Proof. Assume P is inscribed in the unit sphere S. We can assume that the vertex projection Ψ is based on the stereographic projection ϕ. The image $\Psi(P)$ is clearly a convex set. Let F be a facet of P that does not contain N (there must be at least one) and let S_F be its circumsphere and V_F its vertex set. We know that ϕ maps spheres in $S \setminus \{N\}$ bijectively to spheres in $\mathbb{R}^{d-1}$. So ϕS_F is a $(d - 2)$-sphere that contains $\phi(V_F)$. In reverse, the preimage of the circumsphere of $\phi(V_F)$ must contain S_F. This implies that $\phi(S_F)$ is the circumsphere of $\phi(F)$ and $\phi(F)$ must be a $(d - 1)$-face. Since ϕ is continuous, we see that the inside of S_F which cannot contain any vertex, especially not N, is mapped to the inside of $\phi(S_F)$. Hence $\phi(F)$ is a Delaunay $(d - 1)$-cell. Hence T is a Delaunay subdivision.

Assume T' is a Delaunay subdivision of a point set V'. Then we can use ϕ^{-1} to map V' to the unit sphere S. Let $N = (0, \dots, 0, 1) \in \mathbb{R}^d$. The set $P' := \text{conv}(\phi^{-1}(V) \cup \{N\})$ is an inscribed d-polytope. Its vertex set is

$\phi^{-1}(V) \cup \{N\}$ because every vertex has a supporting hyperplane (tangent to S). Right above we have the arguments that the vertex projection of P' from N is a Delaunay subdivision of V'. Since Delaunay subdivision are unique, this must be T'.

The relation between simplicial polytopes and triangulation with special boundary vertices has been discussed in Corollary 0.3.7. $\qquad\square$

Remark 0.3.14. In 1978 Brown [9] already used the stereographic projection to connect Delaunay subdivisions and inscribed polytopes. He uses a projection from a point on the circumsphere that is not a vertex. This produces two different subdivisions which have the same support. One corresponds to the part of ∂P that is visible from the projection center. Brown calls it the furthest point Delaunay subdivision (usually this is not a Delaunay subdivision). The other part is called a (closest point) Delaunay subdivision. It is the projection image of all facets (and all their faces) that are not visible to the projection center. We prefer the projection from a vertex. This is a well known modification of Browns method and it has several advantages: The combinatorics of a Delaunay triangulation already define the combinatorics of the inscribed polytope, the vertex projection is also defined for polytopes that are not inscribed and the combinatorics of the projection image is independent of the geometric realization.

0.4 Extensions of regular triangulations

This section investigates how the f-vector of a polytope changes if its image under a vertex projection is modified. This will later be a tool to construct polytopes with prescribed f-vectors.

Definition 0.4.1 (Outer f- and g-vector)**.** Let $d > 1$ and let T be a $(d-1)$-dimensional triangulation with boundary complex ∂T. Let $f(T), f(\partial T)$ denote their f-vectors. To simplify notation, we set $f_k := 0$ for all $k < -1$. Then we define the *outer f-vector* by

$$\bar{f}_i(T) := f_i(T) + f_{i-1}(\partial T), \qquad \text{for } i = -1, \dots, d-1.$$

For $e \in \mathbb{N}$ we define the *outer $h^{(e)}$-vector* and *outer g-vector* by

$$\bar{h}_i^{(e)}(T) := \sum_{k=0}^{i} (-1)^{i+k} \binom{e-k}{e-i} \bar{f}_{k-1}, \quad \bar{g}_i(T) := \bar{h}_i^{(d+1)}(T),$$

for $i = 0, \dots, d'$.

The definition of an outer f-vector and g-vector is new, however the concept behind it has been used before [11, sect.2.6].

Remark 0.4.2 (Outer f-vector and vertex projection)**.** Using the polynomials that correspond to the above vectors, we get for $e \in \mathbb{N}$ that

$$
\begin{aligned}
\bar{h}^{(e)}(t,T) &= (1-t)^e \bar{f}(\frac{t}{1-t}, T) \\
&= (1-t)^e \left(f(\frac{t}{1-t}, T) + t \cdot f(\frac{t}{1-t}, \partial T) \right) \\
&= h^{(e)}(t,T) + t \cdot h^{(e)}(t, \partial T).
\end{aligned}
$$

Definition 0.4.3 (Extension and horizon)**.** Let $d > 1$ and let T be a $(d-1)$-dimensional triangulation with vertex set V. Let p a point outside $\mathrm{conv}(V)$. Then we call a polytopal complex T' an *extension of T by p* if

- p does not lie in the affine hull of any boundary $(d-2)$-face of T,
- T' has the vertex set $V \cup \{p\}$,
- T is a subcomplex of T', and
- the support of T' is $\mathrm{conv}(V \cup \{p\})$.

This concept is a special case of a placing triangulation [11, sec. 4.3]. Let H be the combinatorial type of the vertex figure of p in $|T'|$. We call H the *horizon polytope* and its boundary complex ∂H the *horizon of T seen from p*.

Lemma 0.4.4 (Extended triangulation)**.** *Let $d > 1$ and let T be a $(d-1)$-dimensional triangulation in $\mathbb{R}^{d-1}$. Let $p \in \mathbb{R}^{d-1}$ lie outside the support of T and not in the affine hull of any $(d-2)$-face of T. Then there exists a unique extension of T by p, and this is a $(d-1)$-dimensional triangulation.*

Proof. Since such an extension T' must contain T and it only has one vertex more, we only have to argue about the $(d-1)$-faces that contain p. Assume such a "new" face contains $d-1$ vertices of T that do not lie in a common face of the boundary complex of $|T|$. Then it intersects the interior of $|T|$ and hence cannot be a face of T'. So all $(d-1)$-faces of T' that contain p are made from a $(d-2)$-dimensional boundary face and p. $\qquad\square$

Lemma 0.4.5 (Extensions and realizability)**.** *For $d > 1$ let T be the image of the vertex projection of a simplicial d-polytope P from a vertex N. Let T' be an extension of T by a new vertex $p \in \mathbb{R}^{d-1}$. Then there exists a point $p' \in \mathbb{R}^d$ such that N is a vertex of the polytope $P' := \mathrm{conv}(P \cup \{p'\})$ and the vertex projection of P' from N is T'.*

This concept is projectively equivalent to the concept of lifting functions, regular triangulations and pulling triangulations. More can be found in the book [11, 2.2].
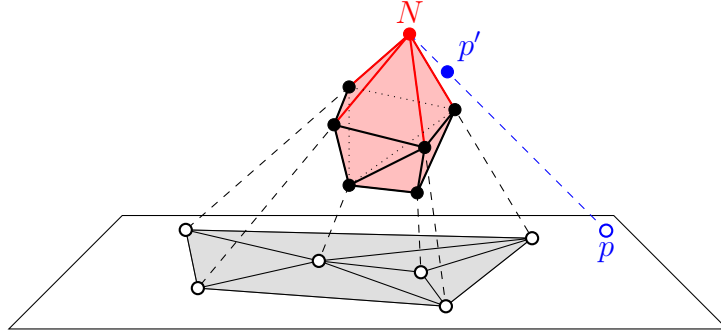

Figure 0.4: The point p can be lifted towards N such that the projection of the convex hull of the polytope with p' is the extension of the projection of the polytope by p.

Proof. Let R be the open ray starting at N that points to p.

$$R := \psi^{-1}(p)$$

Because p lies outside the support of T, we see that R lies completely outside P. See Figure 0.4. Because N is a vertex of P, there exists an open neighborhood U of N, such that all points in U lie beneath all facets of P that do not include N. We define p' to be any point in $R \cap U$. Then $P' := \text{conv}(P \cup \{p'\})$ contains all facets of P such that p' lies beneath. Hence the projection of $\text{conv}(P \cup \{v\})$ contains T as a subcomplex. on the other hand, the image of the projection must be a convex set. Hence the projection of P' is the extension of T by p. $\qquad\square$

The next correspondence has been discovered by the author.

Proposition 0.4.6 (Extensions and outer faces)**.** *For $d > 2$ let T be a $(d-1)$-dimensional triangulation. Let T' be an extension of T by some point p. Let H be the horizon polytope of T seen from p. Then*

$$\bar{f}_i(T') = \bar{f}_i(T) + f_{i-1}(\partial H) + f_{i-2}(\partial H), \qquad for \quad i = -1, \ldots, d-1$$

and

$$\bar{g}_i(T') = \bar{g}_i(T) + g_{i-1}(H), \qquad for \quad i = 0, \ldots, d'$$

where $g_i(T') = h_i^{(d+1)}(T')$ and $g_{i-1}(H) = h_{i-1}^{(d-1)}(H)$.
(Note that H is a $(d-2)$-polytope.)

A simpler way to express this is by using the *index right shift operator* $\mathcal{R}$. For $s \in \mathbb{N}$ this shall be

$$\mathcal{R} : \mathbb{Z}^s \to \mathbb{Z}^{s+1}, \quad \mathcal{R}_0(v) := 0, \quad \mathcal{R}_{i+1}(v) := v_i, \quad \text{for } i = 1, \ldots, s.$$

Then we can formulate

$$\bar{f}(T') = \bar{f}(T) + \mathcal{R}f(\partial H) + \mathcal{R}^2 f(\partial H), \qquad \bar{g}(T') = \bar{g}(T) + \mathcal{R}g(H).$$
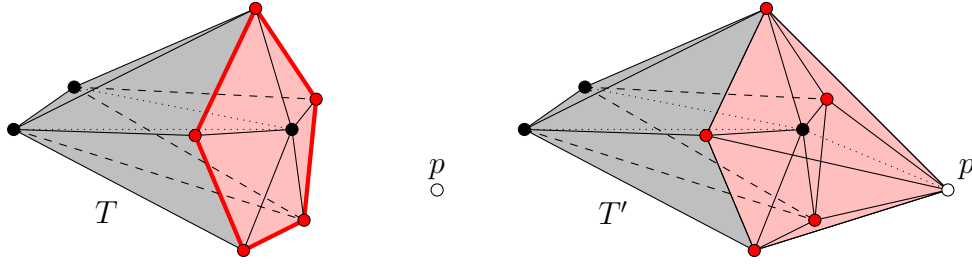

Figure 0.5: Left: The horizon of T seen from p is the red cycle. The corresponding horizon polytope H is a pentagon. Right: To calculate the outer f-vector of T', only the outer f-vector of T and the f-vector of H have to be known.

Proof. Every non empty face of T is one of the following:

- I: An interior face of T. (Also interior in T'.)
- B: A boundary face of T, that is not visible to p. (Also a boundary face of T'.)
- ∂H: A boundary face of T, that corresponds to a face of the horizon of T seen from p. (Also a boundary face of T'.)
- C: A boundary face of T, that is an interior face in T'.

See Figure 0.5. For $j = -1, \ldots, d-1$ let i_j, b_j, h_j, c_j be the number of j-faces of type $I, B, \partial H, C$. Since T' is an extension, all faces of T' that are not in T are the join of p with a face of T that is visible from p. For $j = 0, \ldots, d-1$ we count the j-faces of T'.

$$
\begin{aligned}
\bar{f}_j(T') &= f_j(T') + f_{j-1}(\partial T') \\
&= (i_j + b_j + h_j + c_i + h_{j-1} + c_{j-1}) + (b_{j-1} + h_{j-1} + h_{j-2}) \\
&= (i_j + b_j + h_j + c_j) + (b_{j-1} + h_{j-1} + c_{j-1}) + h_{j-1} + h_{j-2} \\
&= f_j(T) + f_{j-1}(\partial T) + h_{j-1} + h_{j-2} \\
&= \bar{f}_i(T) + f_{i-1}(\partial H) + f_{i-2}(\partial H)
\end{aligned}
$$

We calculate the second part by using the sum formula for the g vector.

$$\bar{g}_i(T')$$

$$= \sum_{k=0}^{i}(-1)^{i+k}\binom{d+1-k}{d+1-i}(\bar{f}_{k-1}(T) + f_{k-2}(\partial H) + f_{k-3}(\partial H))$$

$$= \bar{g}_i(T) + \sum_{k=0}^{i}(-1)^{i+k}\binom{d+1-k}{d+1-i}(f_{k-2}(\partial H) + f_{k-3}(\partial H))$$

$$= \bar{g}_i(T) + \cancel{(-1)^i f_{-2}(\partial H)} + \cancel{(-1)^i f_{-3}(\partial H)} + \cancel{(-1)^{i+1}f_{-2}(\partial H)} + f_{i-2}(\partial H)$$

$$\quad + \sum_{k=1}^{i-1}(-1)^{i+k}\binom{d+1-k}{d+1-i}f_{k-2}(\partial H) + \sum_{k=2}^{i}(-1)^{i+k}\binom{d+1-k}{d+1-i}f_{k-3}(\partial H)$$

$$= \bar{g}_i(T) + f_{i-2}(\partial H)$$

$$\quad + \sum_{k=0}^{i-2}(-1)^{i+k+1}\binom{d-k}{d+1-i}f_{k-1}(\partial H) + \sum_{k=0}^{i-2}(-1)^{i+k}\binom{d-1-k}{d+1-i}f_{k-1}(\partial H)$$

$$= \bar{g}_i(T) + \sum_{k=0}^{i-2}(-1)^{i-1+k}\left[\binom{d-k}{d+1-i} - \binom{d-1-k}{d+1-i}\right]f_{k-1}(\partial H) + f_{i-2}(\partial H)$$

$$= \bar{g}_i(T) + \sum_{k=0}^{i-2}(-1)^{(i-1)+k}\binom{(d-1)-k}{(d-1)-(i-1)}f_{k-1}(\partial H) + f_{i-2}(\partial H)$$

$$= \bar{g}_i(T) + \sum_{k=0}^{i-1}(-1)^{(i-1)+k}\binom{(d-1)-k}{(d-1)-(i-1)}f_{k-1}(\partial H)$$

$$= \bar{g}_i(T) + h_{i-1}^{(d-1)}(\partial H)$$

Since the f-vector of H and ∂H are the same up to the entry $f_{d-2}(H)$, we get that $\bar{g}_i(T') = \bar{g}_i(T) + h_{i-1}^{(d-1)}(H)$ for $i = 0, \ldots, d'$. $\qquad\square$

1 Three theorems about f-vectors of inscribed polytopes

We show that all f-vectors of 3-polytopes, stacked polytopes and cyclic polytopes also occur for inscribed polytopes. This chapter is joint work with Günter M. Ziegler, it is part of the paper [22].

1.1 Construction of all f-vectors of inscribed 3-polytopes

For dimension three all f-vectors of polytopes are known: According to Steinitz [45] (see [23, Sect. 10.3]), the set of all f-vectors of convex 3-polytopes is

$$\{(f_0, f_1, f_2) \in \mathbb{N}^3 : 4 \leq f_2 \leq 2f_0 - 4, \ 4 \leq f_0 \leq 2f_2 - 4, \ f_1 = f_0 + f_2 - 2\}.$$

Theorem 1.1.1. *All f-vectors of 3-polytopes occur for inscribed 3-polytopes.*

Proof. In Figure 1.1 we can see three types of inscribed 3-polytopes. For $n \geq 3$ and $k \geq 0$ these are basically wedges over an n-gon that have been stacked k times.

Type $A_{n,0}$ is constructed as follows. We pick any circle on S^2 that intersects the equator of the unit sphere in two points, and hence cuts the equator into two components. We pick n distinct points on one component such that two points are the intersection points. Then we reflect these n points at the equator hyperplane and take the convex hull of the so derived $2n - 2$ points. This type has two n-gon facets, and $n - 1$ vertical facets, two of which are triangular. It has $2n - 1$ many n-gon edges, and $n - 2$ vertical edges. The number of vertices is $2n - 2$.

Type $B_{n,0}$: We pick any circle in S^2 that intersects the equator of the unit sphere in one point. We pick n points on that circle such that one is the
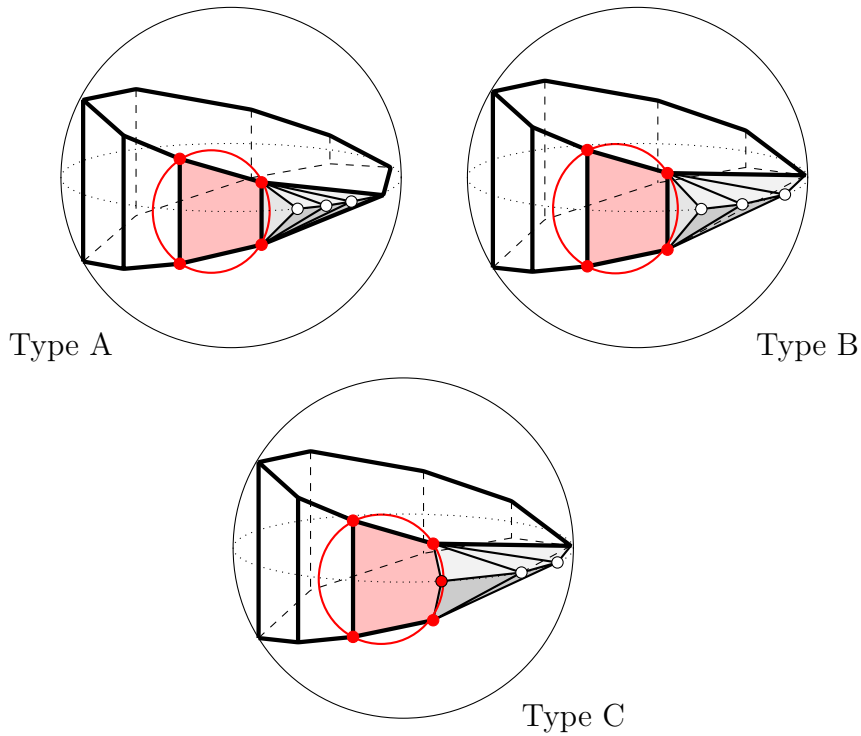
Figure 1.1: Examples of the three inscribed types that realize all *f*-vectors
of 3-polytopes. Type A and B are wedges over an *n*-gon plus *k*
stackings. In type A the upper and lower *n*-gon share an edge,
in type B they only share a vertex. Type C is a modification
of type B where the last stacking is degenerated. The stacking
point is placed in the affine hull of one other face.

intersection point. Then we reflect these points at the equator hyperplane
and take the convex hull of the so derived $2n - 1$ points. This type has
two *n*-gon facets, *n* vertical facets, two of them are triangular. It has $2n$
many *n*-gon edges and $n - 1$ vertical edges. The number of vertices is
$2n - 1$.

The type $A_{n,k}$ and $B_{n,k}$ are derived from $A_{n,0}$ respectively $B_{n,0}$ by adding
k points on the equator, beyond only one facet. This facet shall be one
of the two vertical triangular facets. Then we take the convex hull. It
is not hard to see that these points exist and that they correspond to
a sequence of stackings (in the right order). Hence each of these points
adds two triangular facets, three edges and one vertex.

To get the type $C_{n,k}$ we take $B_{n,k}$ and modify it by adding a vertex on

the equator behind a triangular facet but in affine dependence with a rectangular facet as depicted in Figure 1.1. Then we take the convex hull. This operation adds one vertex, two facets and two edges.

We count the f-vectors of the types. For all three types we have parameters $n \geq 3$ and $k \geq 0$.

$$f(A_{n,k}) = (2n - 2 + k, 3n - 3 + 3k, n + 1 + 2k) = (f_0, f_1, \tfrac{f_0}{2} + 2 + \tfrac{3}{2}k)$$
$$f(B_{n,k}) = (2n - 1 + k, 3n - 1 + 3k, n + 2 + 2k) = (f_0, f_1, \tfrac{f_0}{2} + 2 + \tfrac{3}{2}k + \tfrac{1}{2})$$
$$f(C_{n,k}) = (2n + k, \quad 3n + 1 + 3k, n + 3 + 2k) = (f_0, f_1, \tfrac{f_0}{2} + 2 + \tfrac{3}{2}k + 1)$$

The types $A_{n,0}$ produces simple polytopes while A and B provide all f-vectors with the minimal number of facets for any given number of vertices. $A_{3,k}$ are inscribed stacked polytopes with arbitrary many vertices. These are simplicial and hence give the maximal number of facets for any given number of vertices. It is easy to see that for any number of vertices all permissible numbers of facets can be obtained by choosing the right Type. $\qquad\square$

1.2 The Lower Bound Theorem for inscribed simplicial polytopes

As a corollary to Lemma 0.3.13 we obtain that the Lower Bound Theorem is tight for inscribable polytopes, by applying it to the following type of triangulation: Take a simplex δ and take an oriented edge E from a vertex to any interior point of δ. Then add any number k of distinct points on the relative interior of the edge and order them according to the direction of the E. Introduce the points one after an other by stellar subdivisions of the simplex where they lie. See Figure 1.2.

Proposition 1.2.1. *For all $d \geq 2$, $n \geq 0$, there is an inscribed stacked d-polytope on $d + 1 + n$ vertices.*

Proof. We show by induction that all cells of the above construction are locally Delaunay.

First, by Lemma 0.2.8 the stellar subdivided simplex is a Delaunay triangulation in all dimensions.

For induction we assume that we have already placed k vertices and the result is a Delaunay triangulation T. According to the construction, we
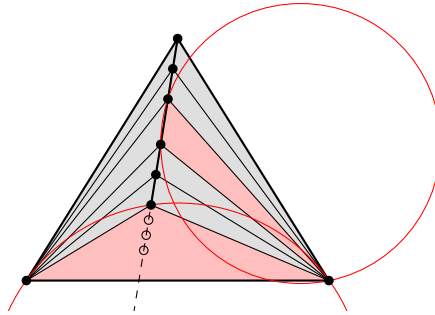
Figure 1.2: This type of triangulation is always Delaunay. The same concept works in higher dimensions.

place the next vertex p further on the ray, inside a simplex and thereby cause a stellar subdivision. The result shall be T'.

To check its Delaunay property, it suffices to show that all $(d-1)$-cells are locally Delaunay. For all $(d-1)$-cells that lie in T and T' it suffices to check that they cannot contain p. Their circumspheres intersect the ray E in exactly two points. In between are all point of E that lie inside the sphere, clearly p is not there. So the new cells remain to be checked, but only against new cells. Since these cells form a stellar subdivided simplex, they are also locally Delaunay. Hence T' is a Delaunay triangulation. □

1.3 The Upper Bound Theorem for inscribed polytopes and Delaunay triangulations

The Upper Bound Theorem for inscribed polytopes is also tight for inscribed (simplicial) polytopes. The straight way to prove this is to show that all cyclic polytopes are of inscribable type. In fact, this has already been done by Seidel [40] when he showed an upper bound theorem for Delaunay triangulations. In fact he proved that every cyclic polytope has an inscribed realization that can be stacked such that the result is still inscribed.

After we repeat what is known, we will use our own technique to show it. This will also reveal something about the structure of a cyclic polytope.

Proposition 1.3.1. *The d-dimensional cyclic polytope $C_d(n)$ with n*

vertices is inscribable for all $d \geq 2$ and $n \geq d + 1$.

Thus all f-vectors of neighborly polytopes occur for inscribable simplicial polytopes and hence the Upper Bound Theorem is tight even for inscribable simplicial polytopes.

We will sketch the two known proofs:

Proof 1 (Seidel [40, p. 521]). The *spherical moment curve* is given by

$$C : \mathbb{R}^+ \to \mathbb{R}^d, \qquad c(t) := \frac{1}{1 + t^2 + t^4 + \cdots + t^{2(d-1)}} (1, t, t^2, \ldots, t^{d-1}).$$

This curve lies on the image of the hyperplane $x_1 = 1$ under inversion in the origin, that is, on the sphere with center $\frac{1}{2} e_1$ and radius $\frac{1}{2}$. Using Descartes' rule of signs one gets that this curve (restricted to the domain $t > 0$!) is of order d, and thus the convex hull of any n distinct points on this curve is an inscribed realization of $C_d(n)$. $\square$

Proof 2 (Grünbaum [23, p. 67]). For even $d \geq 2$, we consider the *trigonometric moment curve*

$$c : (-\pi, \pi] \to \mathbb{R}^d$$

$$c(t) := \left(\sin(t), \cos(t), \ \sin(2t), \cos(2t), \ \ldots, \ \sin(\tfrac{d}{2}t), \cos(\tfrac{d}{2}t) \right).$$

Obviously its image lies on a sphere. We verify that this is a curve of order d using the fact that any nonzero trigonometric polynomial of degree $\frac{d}{2}$ has at most d zeros per period (see e.g. Polya & Szegő [35, pp. 72-73]). Thus we get that the convex hull of any n points on this curve yields an inscribed realization of $C_d(n)$. (Compare [48, pp. 75-76].)

For odd $d \geq 3$, we check using Gale's evenness criterion that any "vertex splitting" on $C_{d-1}(n-1)$ results in a realization of $C_d(n)$; this yields inscribed realizations of $C_d(n)$ where all vertices except for those labeled 1 and n lie on a hyperplane. (See e.g. Seidel [40, p. 528], where the "vertex splitting" is called "pseudo-bipyramid".) $\square$

Our approach creates a Delaunay triangulation of a cyclic $(d-1)$-polytope and shows that it is the vertex projection of an inscribed cyclic d-polytope. This combinatorial approach offers a nice view to cyclic polytopes. The idea behind it is similar to what Seidel did. We will also use a sphere inversion and the moment curve, but we do not restrict to $t > 0$.

Definition 1.3.2 (Induced triangulation of a cyclic polytope)**.** Let C be a cyclic polytope whose vertices lie on the moment curve. Like in Gale's evenness condition, the curve induces an order on the vertices and a triangulation: The first d vertices define a simplex, whose complex is a triangulation. Each following vertex defines an extension of the previous triangulation. After adding the last vertex, we call the result a triangulation that is *induced by the moment curve.*

Lemma 1.3.3 (The induced triangulation type is Delaunay)**.** *For any $n > d > 1$ exists a realization of the cyclic polytope $C_d(n)$ with vertices on the moment curve, such that the induced triangulation is a Delaunay triangulation.*

Proof. We prove this by induction on the number of vertices, according to the construction of the induced triangulation.

The first $d + 1$ vertices are picked arbitrarily on the moment curve, then the face complex of the corresponding simplex is a Delaunay triangulation. For induction we assume that we already have a Delaunay triangulation T that is the induced triangulation of a cyclic polytope. We then define T' to be the Delaunay subdivision that contains all vertices of T and an additional vertex p on the moment curve. Since the moment curve is unbounded, we can place p far enough, such that it lies outside all circumspheres of cells in T. We can also assume that T' is a triangulation. Since T' must contain T we get by Lemma 0.4.4, that there is only one such triangulation: The extension of T by p. □

Proposition 1.3.4 (Cyclic polytopes are inscribable)**.** *For every $n > d \geq 2$, the cyclic d-polytope $C_d(n)$ with n vertices is inscribable.*

Proof. It suffices to show that the moment curve induced triangulation T of a cyclic $(d − 1)$ polytope P of $n − 1$ vertices is the vertex projection of a cyclic polytope. By Lemma 0.3.13 the Delaunay condition corresponds to inscribability.

Let P' be a polytope such that T is the vertex projection from a vertex N. Because the facets of a polytope already define it, it is enough to show that all facets of P' satisfy Gale's evenness condition. To check that we order the vertices according to the order that is given by the moment curve and put N at the end. Let B be a binary vector, that describes a facet of P'.

Assume the last entry of B is 1. Then by the vertex projection the vector of the first $n − 1$ entries must describe a boundary face of T. This is

a facet of P and it satisfies Gale's Condition for $n - 1$ vertices. Hence B must also satisfy Gales condition for n vertices, because appending a 1-entry cannot destroy this condition.

Assume the last entry of B is 0. Let $i + 1$ be the position of the last 1-entry. Then B describes a $(d - 1)$-dimensional cell of T that is created in step $i + 1$ of the construction of T. Hence the $(d - 2)$-face, that is indicated by the first i entries of B, was a boundary face in step i. In step $i + 1$ it became an interior face. This implies that the vector of the first i entries of B ends with a block of 1-entries of odd size. This odd block is even in B because position $i + 1$ is a 1-entry. Hence B must also satisfy Gales condition.

We have shown that all facets of P' belong to a cyclic polytope. Since P' is a polytope, it must be cyclic. $\qquad\square$

Remark 1.3.5. We have seen that moment curve induced triangulations can be realized as a Delaunay triangulation by choosing the points far enough. We can modify this construction by applying a stellar subdivision to the the first simplex directly after it appears. What we then get is the vertex projection of an inscribed cyclic polytope that has been stacked with a point on its circumsphere. By the arguments of Seidel [40] this leads to a Delaunay triangulation with a maximal number of faces. In Chapter 5 we will see that even two stackings are possible.

2 Characterization of inscribable stacked polytopes

In this section we characterize all inscribable stacked d-polytopes. This is of special interest for us, since in dimension larger than two stacked polytopes are known to characterize the lower bound of the Lower Bound Theorem for simplicial polytopes. This chapter is joint work with Günter M. Ziegler, it is part of the paper [22].

Our characterization is given by an easy combinatorial criterion that is necessary and sufficient. First we are going to assign to each combinatorial type of stacked polytopes an induced tree structure, where basically every node represents a stacking. Our criterion will then only refer to the node degree of this tree.

We prove the theorem by solving an equivalent problem for Delaunay triangulations. The sufficiency part will provide an iterative construction that uses stellar subdivisions. The necessity part will be done by reduction to a minimal example of a non inscribable type. This type is a simplex that is stacked on four facets.

We will close this section with two corollaries about combinatorial properties of inscribed stacked polytopes.

2.1 Induced tree of stacked polytopes and stellar subdivisions

A stacking operation can be imagined as gluing a simplex onto a facet, or combinatorially as a stellar subdivision of a facet. It creates a simple vertex.

A concrete stacking construction of a stacked polytope P clearly *induces a triangulation* of P. We see that this triangulation has only interior faces of dimension d and $d-1$. In dimension at least three we can show that it only depends on P and is independent of the specific construction:

Assume we have two such triangulations for the same stacked polytope P. Then P must have a simple vertex, we call it v. Since both triangulations do only have interior faces of dimension d and $d-1$, we see that v must lie in a single d-simplex, which is the same in both triangulations. If we remove this simplex we get a new stacked polytope which has less vertices than P and the corresponding two new triangulations differ if and only if the old ones did. After finitely many repetitions, we end up with a simplex and two identical triangulations.

In dimension two all polytopes are simple and all triangulations of an n-gon without interior vertices are induced triangulation.
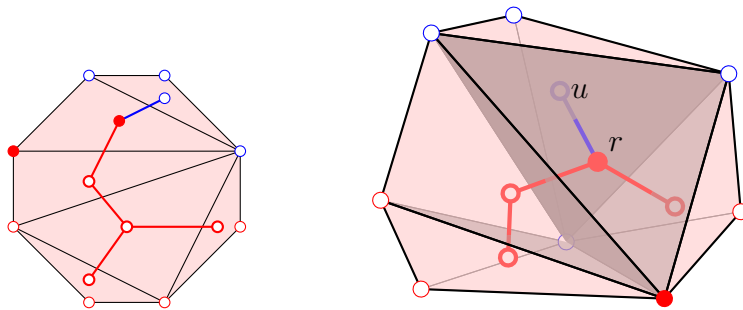

Figure 2.1: Left: A dual tree of an octagon. In dimension two each polytope is a stacked polytope and the dual tree is not unique. Right: The dual tree of a stacked 3-polytope. In dimension at least three the dual tree of a stacked polytope exists and it is unique.

Definition 2.1.1 (Dual tree of a stacked polytope)**.** For $d \geq 3$, the *dual tree* $\mathcal{T}_P$ of a stacked d-polytope P is the dual graph of the *triangulation that is induced by stackings*. Every d-face in the triangulation corresponds to a node and every interior $(d-1)$-face corresponds to an edge of the tree. See Figure 2.1.

The graph $\mathcal{T}_P$ given by Definition 2.1.1 is indeed a tree if P is stacked.

Remark 2.1.2 (Different constructions of a stacked polytope)**.** All stacking constructions of a stacked polytope P have the same induced triangulation and dual tree $\mathcal{T}_P$. They can be identified with all orderings of the nodes of $\mathcal{T}_P$ that satisfy the following: Any node can be first. This node represents the root of $\mathcal{T}_P$ and thereby induces a partial ordering on $\mathcal{T}_P$. Any linear extension of this order represents a stacking construction.
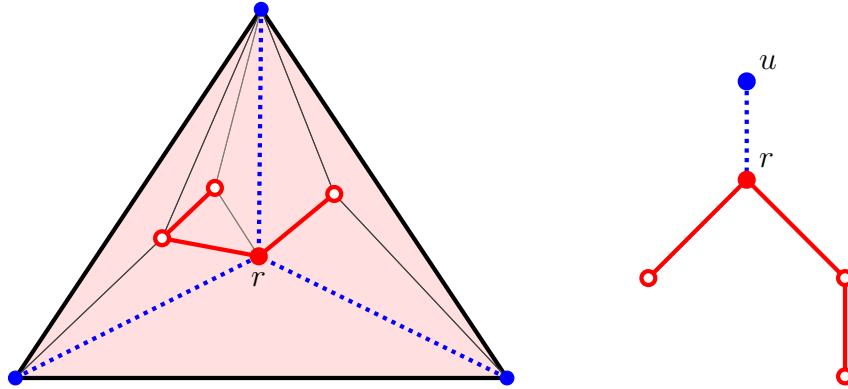
Figure 2.2: A stellar subdivision of a 2-simplex and its dual rooted tree.

By the use of Proposition 0.3.13 we see that the same concept can be applied to triangulations that are created by a sequence of stellar subdivisions started at a simplex. Such triangulations T are exactly the vertex projections of stacked polytopes P where the projection vertex is simple. The concept of a dual tree can be translated: The simplex in the induced triangulation of P that contains the projection vertex is picked as a root of the dual tree. Then every vertex that does not belong to this simplex corresponds to a stacking and hence to a node of the tree. Equivalently in T: The root represents all boundary vertices, and every other vertex a stellar subdivision. See Figure 2.2.

Definition 2.1.3 (Dual tree of a stellar subdivision)**.** Let $d \geq 3$. The *dual tree* $\mathcal{T}_T$ of a multiple stellar subdivision of a $(d-1)$-simplex σ, has one node for every vertex that is inserted by a single stellar subdivision. Only exception, the root node r corresponds to σ and is not a stellar subdivision. A node v' is a child of the node v if it corresponds to a single stellar subdivision of a $(d-1)$-face has been introduced by v. See Figure 2.2.

Every simplicial facet can be stacked if the stacking is just flat enough. So every triangulation that has only interior faces of degree d and $d-1$ corresponds to a stacked polytope type. Its dual tree $\mathcal{T}_P$ has maximal vertex degree

$$\max \deg \mathcal{T}_P \leq d + 1$$

and we see that every tree of maximal vertex degree at most $d+1$ also comes from a stacked polytope. In contrast to the induced triangulation, the dual tree does not contain all information about the combinatorial type of a stacked polytope.

2.2 Characterization of inscribable stacked polytopes

The following theorem will be proven in this section:

Theorem 2.2.1. *Let $d > 1$. A stacked d-polytope is inscribable if and only if all nodes of its dual tree have node degree at most 3.*

Corollary 2.2.2. *Let $d > 1$. A simplex that is stacked on k facets is of inscribable type if and only if $k < 4$.*

By Proposition 0.3.13 the theorem is equivalent to the following if we project the stacked polytope from a simple vertex:

Theorem 2.2.3. *Let $d > 1$. A triangulation that is a multiple stellar subdivision of a $(d-1)$-simplex can be realized as a Delaunay triangulation if and only if at most two of the $(d-1)$-simplices that are generated in each single stellar subdivision are further subdivided.*

Equivalently: the dual tree of the multiple stellar subdivision is a binary tree where the root has at most one child.

Corollary 2.2.4. *Let T be a triangulation that is a single stellar subdivided simplex where k of its full dimensional cells have been subdivided again. Then T is of Delaunay realizable type if and only if $k < 3$.*

2.2.1 Realizing all inscribed stacked polytopes

The following proposition establishes the "if" part of Main Theorem 2.2.1 (and thus also of Main Theorem 2.2.3).

Proposition 2.2.5. *Let $d > 1$ and let T be a Delaunay triangulation in $\mathbb{R}^{d-1}$. Let c be an interior vertex of degree d and let F_1 and F_2 be two $(d-1)$-faces of T that contain c. Then one can perform a single stellar subdivisions of T at F_1 and then at F_2 such that the result is again a Delaunay triangulation.*

Proof. Let $F_1, \ldots, F_d$ be the $(d-1)$-faces of T that contain c, and let R be the set of all other $(d-1)$-faces of T. Let $v_1, \ldots, v_d$ be the vertices of $F_1, \ldots, F_d$ such that v_i is not contained in F_i.

Then the circumspheres of $F_3, \ldots, F_d$ pass through c. The intersection of the tangent hyperplanes of these $d - 2$ spheres at the point c contains a line t through c. This line must be tangent to all those spheres.

Let U be a small open ball around c that, like c, lies outside the circumspheres of all cells in R. Then $U \cap t \setminus \{c\}$ consists of two disjoint, open line segments. We chose any two points, x_1, x_2, one in each line segment, and use them for single stellar subdivisions of F_1 respectively F_2. See Figure 2.3.
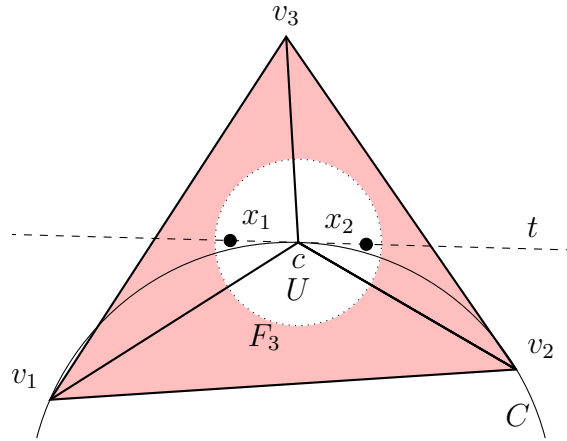

Figure 2.3: How to choose the subdivision points.

We claim that the resulting triangulation T' is again Delaunay. First we check that x_1 and x_2 lie inside F_1 resp. F_2: By the circumsphere construction they lie outside all facets $F_3, \ldots, F_d$ but clearly inside $\text{conv}\{v_1, \ldots, v_d\}$, hence they lie in $F_1 \cup F_2$. Because t contains c, which is a vertex of F_1 and F_2, only one component of $t \setminus \{c\}$ can be contained in F_1 and only one can be contained in F_2. Hence we can assume that x_1 lies in the relative interior of F_1 and x_2 lie in the relative interior of F_2.

Now we need to show that all interior $(d - 2)$-faces of T' are locally Delaunay. The cells in $R \cup \{F_3, \ldots, F_d\}$ lie in both triangulations T and in T'. Their empty circumspheres in T stay empty in T' since x_1, x_2 lie outside.

Let $\mathcal{I}$ be the faces of T' that are not faces of T. It remains to show that all $(d - 2)$-faces in T' that are contained in two facets of $\mathcal{I}$ are locally Delaunay. The first type lies in $(d - 1)$-faces that both contain x_1 or both contain x_2. In this case the locally Delaunay condition is given by Lemma 0.2.8. The second type lies in a $(d - 1)$-face that contains
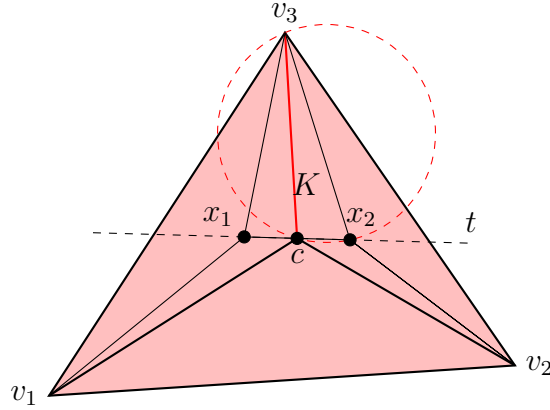
Figure 2.4: The circumsphere does not contain x_1 because x_1, c and x_2 are collinear.

x_1 and an other $(d-1)$-face that contains x_2. There is only one such $(d-2)$-face, namely the intersection of F_1 and F_2. Let's call this face K. The circumsphere of $\text{conv}(K \cup \{x_2\})$ does not contain x_1, because x_1, c, x_2 are collinear and c lies between x_1 and x_2 (see Figure 2.4). Hence also K is locally Delaunay and thus all interior $(d-2)$-faces of T' are locally Delaunay. Hence T' is a Delaunay triangulation. □

Using this result, we also obtain examples of stacked polytopes that go beyond the rather special construction given by Corollary 1.2.1.

2.2.2 Necessity: Three stellar subdivisions are too much

The following section establishes the "only if" part of Main Theorem 2.2.3 (and thus also of Main Theorem 2.2.1): If multiple stellar subdivisions are performed on three facets F_1, F_2 and F_3 at a simple interior vertex of an arbitrary triangulation, then the resulting triangulation is not a Delaunay triangulation. Since one dimensional triangulations have only two facets around a vertex, we restrict to $d > 2$.

For this, it suffices to consider the complex Δ that arises by a single stellar subdivision of a $(d-1)$-simplex $\sigma \subset \mathbb{R}^{d-1}$ using an arbitrary interior point $c \in \sigma$. This complex Δ with $(d-1)$-faces $F_1, \ldots, F_d$ is Delaunay by Lemma 0.2.8. Now we apply single stellar subdivisions to the cells F_1, F_2, F_3 by arbitrary interior points r_1, r_2, r_3. Our claim is that the resulting triangulation T cannot be Delaunay.

In order to prove this claim, we first construct a point x that depends only on Δ. Its position with respect to Δ is established in Lemma 2.2.6. Then Lemma 2.2.7 records the properties of x with respect to the subdivision T. Finally, we establish in Proposition 2.2.8 that T cannot be Delaunay: For that we use an inversion in a sphere centered at x in order to simplify the situation so that a projection argument reduces the claim to the case $d = 3$, which was established in Proposition 2.2.5.

Let $\sigma = \operatorname{conv}\{v_1, \ldots, v_d\}$ be a $(d-1)$-simplex in $\mathbb{R}^{d-1}$, let $c \in \sigma$ be an interior point, and let Δ be the single stellar subdivision of σ by c, with $(d-1)$-faces $F_1, \ldots, F_d$, labeled such that $v_i \notin F_i$.

For some k ($1 \leq k < d$) let $\mathcal{F} := \{F_{k+1}, \ldots, F_d\}$ and $\mathcal{G} := \{F_1, \ldots, F_k\}$. Then $V_{\mathcal{F}} := \{v_1, \ldots, v_k\}$ is the set of vertices of σ that lie in all cells of $\mathcal{F}$, while $V_{\mathcal{G}} := \{v_{k+1}, \ldots, v_d\}$ is the set of vertices of σ that lie in all cells of $\mathcal{G}$.

Now $E_{\mathcal{F}} := \operatorname{aff}(V_{\mathcal{F}} \cup \{c\})$ is an affine subspace of dimension k, while $E_{\mathcal{G}} := \operatorname{aff}(V_{\mathcal{G}} \cup \{c\})$ has dimension $d - k$. The two spaces together affinely span $\mathbb{R}^{d-1}$, so by dimension reasons they intersect in a line ℓ. This line intersects the two complementary faces $\operatorname{conv}(V_{\mathcal{F}})$ and $\operatorname{conv}(V_{\mathcal{G}})$ of σ in relatively interior points $\bar{x}$ resp. $\bar{y}$: Indeed, c can be described uniquely as a convex combination of two points $\bar{x} \in \operatorname{conv}(V_{\mathcal{F}})$ and $\bar{y} \in \operatorname{conv}(V_{\mathcal{G}})$, so in particular $\bar{x}, \bar{y}$ and c lie on a line. As $E_{\mathcal{F}}$ contains c and $\bar{x}$, and $E_{\mathcal{G}}$ contains c and $\bar{y}$, we find that $\ell = E_{\mathcal{G}} \cap E_{\mathcal{F}}$ is the line spanned by $\bar{x}, \bar{y}$, and c.

Let $C_{\mathcal{F}}$ denote the unique $(k-1)$-sphere that contains $V_{\mathcal{F}} \cup \{c\}$, that is, the circumsphere of the k-simplex $\operatorname{conv}(V_{\mathcal{F}} \cup \{c\})$, which is also the intersection of the circumspheres of $F_{k+1}, \ldots, F_d$. The point c lies in the intersection $\ell \cap C_{\mathcal{F}}$. The line ℓ also contains the point $\ell \cap \operatorname{conv}(V_{\mathcal{F}}) = \{\bar{x}\}$, which is a relative interior point of $\operatorname{conv}(V_{\mathcal{F}})$ and thus for $k > 1$ lies in the interior of the circumspheres of $F_{k+1}, \ldots, F_d$ and thus in the interior of the sphere $C_{\mathcal{F}}$ relative to the subspace $E_{\mathcal{F}}$. Thus $C_{\mathcal{F}} \cap \ell = \{c, x\}$, where the second intersection point x is distinct from c, and lies outside σ for $k > 1$.

As for $C_{\mathcal{F}}$ and $x, \bar{x}$, we define $C_{\mathcal{G}}$ and $y, \bar{y}$ for $\mathcal{G}$: See Figure 2.5.

Lemma 2.2.6. *Let $d > 2$. In the situation just described, the point x lies outside the circumspheres of $F_1, \ldots, F_k$ and on the circumspheres of $F_{k+1}, \ldots, F_d$.*

Proof. The five points $x, \bar{x}, c, \bar{y}, y$ lie in this order along the line ℓ, where
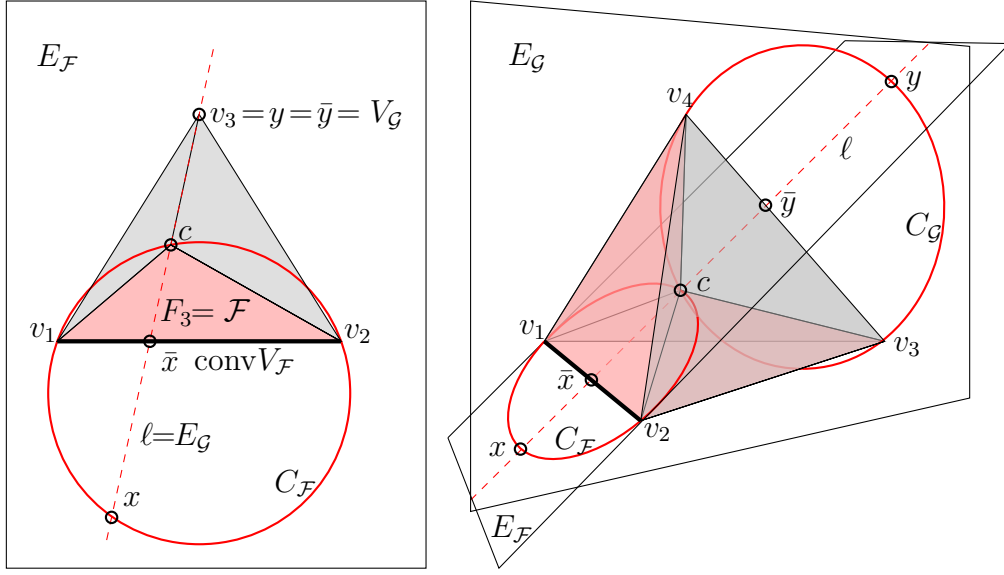
Figure 2.5: The situation of Lemma 2.2.6. The left figure illustrates $d = 3$, $k = 2$, the right one $d = 4$, $k = 2$.

the first two points coincide in the case $k = 1$, while the last two coincide for $d - k = 1$. The circumspheres of $F_1, \ldots, F_k$ intersect the line ℓ in $\{c, y\}$, and thus the point x lies outside these spheres, while the circumspheres of $F_{k+1}, \ldots, F_d$ intersect the line ℓ in $\{c, x\}$. $\qquad\square$

Lemma 2.2.7. *Let $d > 2$. If in the above situation the stellar subdivision of some or all of the facets $F_1, \ldots, F_k$ results in a Delaunay triangulation T, then the point x lies outside all of the circumspheres of the newly created $(d-1)$-faces.*

Proof. Without loss of generality, we assume that T is a single stellar subdivision of Δ at F_1 by a new vertex r inside F_1. This will result in d new $(d-1)$-faces $F_1', \ldots, F_d'$. We label $F_1' := \mathrm{conv}\{r, v_2, \ldots, v_d\}$ and for $i = 2, \ldots, d$ we set $F_i' := \mathrm{conv}(\{r, c, v_2, \ldots, v_d\} \setminus \{v_i\})$. (See Figure 2.6)

For $i = 1, \ldots, k$ we have that F_i contains $V_{\mathcal{G}}$ and hence also $\bar{y}$. On the other side c does not lie inside the circumsphere C_{F_i} of F_i. The order $\bar{x} \to c \to \bar{y}$ that we see on ℓ (Lemma 2.2.6) then implies that $\bar{x}$ lies outside C_{F_i}.

For $i = k+1, \ldots, d$ we have that F_i' and F_i share the $(d-2)$-face $K := \mathrm{conv}(\{c, v_2, \ldots, v_d\} \setminus \{v_i\})$. Their circumspheres intersect in $\mathrm{aff}(K)$. The line ℓ intersects $\mathrm{aff}(K)$ in c, hence x and $\bar{x}$ lie on the same side of
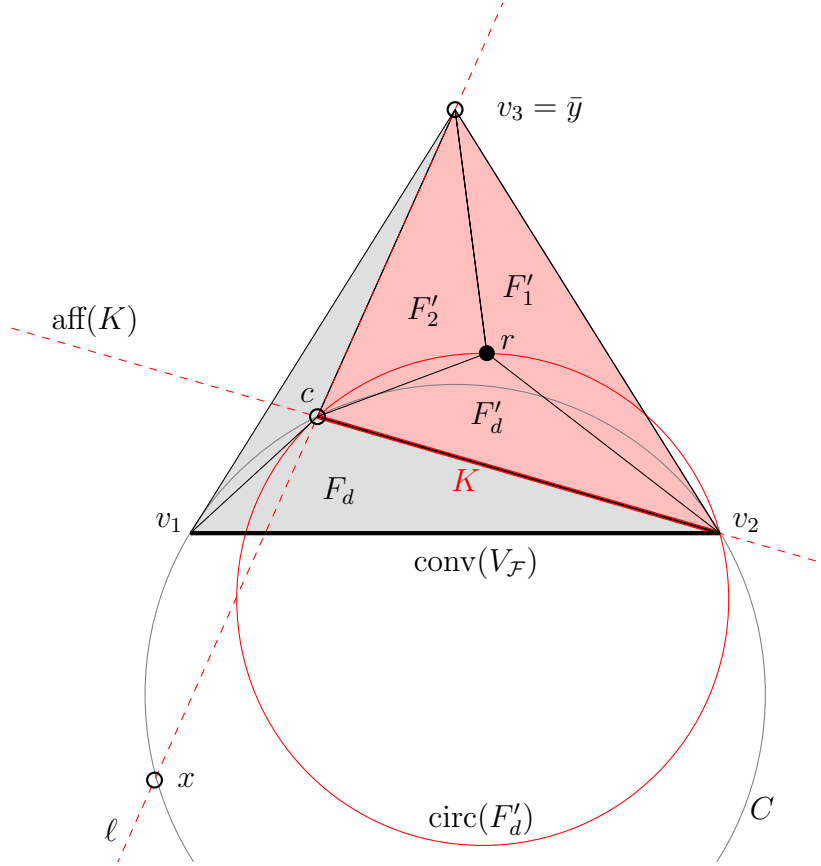
Figure 2.6: The three cases of Lemma 2.2.7, for $d = 3$, $k = 2$.

$\mathrm{aff}(K)$, as well as v_1, because $\bar{x}$ is a convex combination of v_1 and $\mathrm{aff}(K)$. So, the circumsphere of F_j passes through x and v_1 on the same side of $\mathrm{aff}(K)$. Because T is Delaunay, the circumsphere of F_j' does not contain v_1 and hence also not x. □

Proposition 2.2.8. *Let $d > 2$. Let Δ be a single stellar subdivision of a $(d-1)$-simplex $\sigma = \mathrm{conv}\{v_1, \dots, v_d\}$ in $\mathbb{R}^{d-1}$ by an interior point $c \in \sigma$, so the $(d-1)$-cells of Δ are $F_i = \mathrm{conv}(\{c, v_1, \dots, v_d\} \backslash v_i)$ for $i = 1, \dots, d$.*

Let T arise from this Delaunay triangulation Δ by single stellar subdivisions of $F_1, \dots, F_k$ by interior points $r_i \in F_i$ ($1 \leq i \leq k$).

If T is a Delaunay triangulation, then $k < 3$.

Proof. For $d = 3$ this was established in Lemma 0.2.10, so we assume $d > 3$. As a single stellar subdivision can be undone without destroying

the Delaunay property (Corollary 0.2.9), it is enough to show that T cannot be a Delaunay triangulation if $k = 3$.

For the sake of contradiction we assume that such a T is a Delaunay triangulation. Then we are in the situation discussed above, where we find a point x that by Lemmas 2.2.6 and 2.2.7 lies on the circumspheres of the facets $F_4, \ldots, F_d$, but outside the circumspheres of all other facets of T. Let R denote the set of these other facets. The inversion of $\mathbb{R}^{d-1}$ in the unit sphere centered at x sends all the vertices of T to new points in $\mathbb{R}^{d-1}$. The proof of Lemma 0.3.1 shows that this inversion induces a simplicial map

$$\Psi : R \quad \longrightarrow \quad T'$$

and the image T' is a subcomplex of a Delaunay triangulation. As an abbreviation, we denote the images of Ψ by a prime sign $()'$; for example, $\Psi(v_1) = v_1'$. Note that if we apply this to the images of simplices $\sigma, F_1, \ldots, F_3$, then we refer to the simplices obtained by applying Ψ to the vertices.
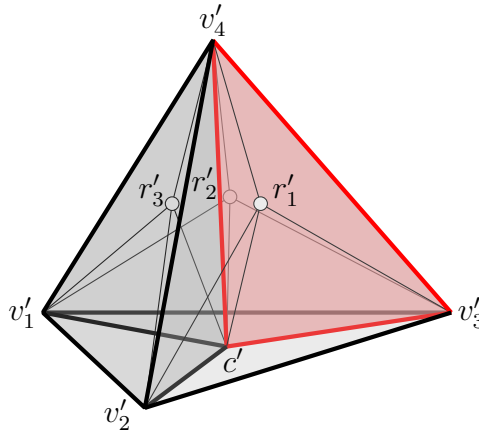

Figure 2.7: An example of T' for $d = 4$ (3-dimensional triangulation). The vertices v_1', v_2', v_3', c' are coplanar. The face conv$\{c', v_3', v_4'\}$ is marked red.

We will now show that T' is a Delaunay triangulation and investigate its properties. Let r_1', r_2', r_3' be the three images of the vertices that where used to perform single stellar subdivisions to F_1, F_2, F_3. Then these vertices are also interior vertices of T' and hence T' is the result of single stellar subdivisions of F_1', F_2' and F_3' by r_1', r_2', r_3'. The inversion centered at x also implies that c', v_1', v_2' and v_3' lie in a common 2-plane, since their preimages lie on a 2-sphere that passes through x. Note that no three

of these four vertices lie in line. By checking some vertex incidences, we figure out that the structure of T' can be described as follows: Take a $(d-1)$-simplex, split it into three simplices by inserting a vertex c' in the relative interior of a 2-face and then apply single stellar subdivisions to each of those three $(d-1)$-simplices by points r'_1, r'_2 and r'_3. In particular, the support of T' is convex, so T' is the Delaunay triangulation of the set $\{c', r'_1, r'_2, r'_3, v'_1, \ldots, v'_d\}$ in $\mathbb{R}^{d-1}$. (See Figure 2.7.)
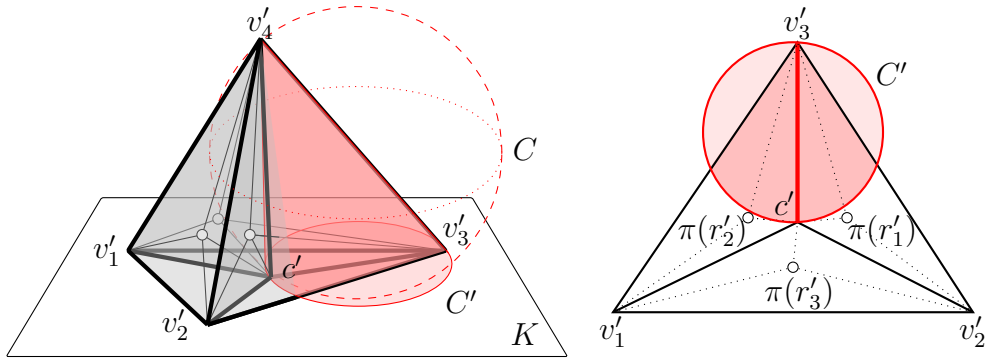

Figure 2.8: Left: A supporting sphere C of $\mathrm{conv}\{c', v'_3, v'_4\}$ intersects the 2-plane K in a circle C'. The pyramid $\mathrm{conv}\{C', v'_4\}$ lies inside C. Right: The image of a projection from v'_4 to K.

Let K be the 2-plane that contains c', v'_1, v'_2 and v'_3 and let T'_K denote the maximal subcomplex of T' that lies in K. We define barycentric coordinates by the affine basis $(v'_1, \ldots, v'_d)$ and define π to be the coordinate projection that sets the weights of $v'_4, \ldots, v'_d$ to zero:

$$\pi : \mathrm{relint}(T') \to \mathrm{relint}(T'_K) = \mathrm{relint}(\mathrm{conv}\{v_1, \ldots, v_3\}).$$

We see that F'_1 and F'_2 share the $(d-2)$-face $\mathrm{conv}\{c', v'_3, \ldots, v'_d\}$ and this face must have a have a supporting sphere. We pick one and call it C. The intersection of C with K is a supporting sphere for the edge (c', v'_3) in T'_K. We call this 1-sphere C' and notice that the preimage $\mathrm{conv}(C') \cap \mathrm{relint}\, |T'_K|$ under π, which is contained in $\mathrm{conv}(C' \cup \{v'_4, \ldots, v'_d\})$, lies completely inside C. See the left picture of Figure 2.8.

This implies that the images of r'_1, r'_2 and r'_3 under the projection π lie outside C', but in the relative interior of $|T'_K|$. From this we derive that we can apply single stellar subdivisions to the three 2-faces of T'_K by the vertices $\pi(r'_1), \pi(r'_2)$ and $\pi(r'_3)$ such that the edges $(c', v'_1), (c', v'_2)$ and (c', v'_3) would still be locally Delaunay. See the right part of Figure 2.8. However, in Lemma 0.2.10 we have already proved that this is impossible.

$\square$

2.2.3 Properties of inscribed stacked polytopes

Proposition 2.2.9 (Inscribed and stacked with bounded vertex degree)**.** *Let $d \geq 2$ and $n \geq d+1$. Then there exists an inscribed stacked d-polytope that has n vertices, and each vertex has edge degree at most $2d$.*

Proof. Since $d = 2$ is obvious, let $d > 2$.

We start with an arbitrary d-simplex, which is inscribed. All its vertices are simple. We label them from 1 to $d+1$. Now, for every $k = d+2, \ldots, n$ we refer to Proposition 2.2.5 in order to stack a new vertex with label k onto the facet $\{k - d, k - d + 1, \ldots, k - 1\}$ that lies at the simple vertex $k - 1$. This in particular destroys the facet $\{k - d, k - d + 1, \ldots, k - 1\}$. It thereby makes the vertex $k - d$ inaccessible for all following stackings and creates the new facet $\{k - d + 1, k - d + 2, \ldots, k\}$ that contains the new simple vertex k.

In the inscribed stacked polytope that is created this way, two vertices i and j are adjacent if and only if $|i - j| \leq d$. $\qquad\square$

Proposition 2.2.10. *Let $d \geq 3$ and P be an inscribable stacked d-polytope. Then less than half of the vertices of P are simple, except if P is a simplex.*

For general stacked d-polytopes roughly $\frac{d-1}{d}$ of the vertices can be simple.

Proof. We show this by a graph theoretical argument: We choose any leaf node r of the dual tree T of the stacked polytope P. The corresponding simplex δ in the induced triangulation has exactly one simple node and d other nodes. We define r to be the root of T. By traversing T from the root r, we identify the vertices of P that are not in δ with the nodes of $T \setminus r$ in the obvious way. We see that the root and the leaves of T correspond to the simple nodes of P. Since P is inscribed, we know that T must be a binary tree and the root, which counts as a leaf, has only one child. This implies that the number of leafs in T is at most one larger than the number of interior nodes in T. Together with the $d > 2$ non simplicial nodes of δ, the proposition follows. $\qquad\square$

3 Enclosing spheres and the inscribability exponent

 In this section we strengthen Corollary 2.2.2 as follows:

Theorem 3.0.11 (Enclosed sphere and 4 stackings). *For $d \geq 2$ and $k \geq 0$ let P be a d-simplex with k facets stacked. Then P has a realization such that there is a sphere S passing through the k stacking vertices and $P \subset \operatorname{conv} S$ if and only if $k < 4$.*

Using this we get an improvement of a theorem of Grünbaum and Jucovič [24] from 1974 about the upper bound of the inscribability exponent. We thereby disprove the conjecture that the old bound is tight.

Definition 3.0.12 (Inscribability exponent). Let $d > 1$. Let $\mathcal{P}_d$ denote the set of all d-polytopes. Let $v(P)$ be the number of vertices of a polytope P and $s(P)$ the maximal number of vertices that can be placed on a sphere while the remaining vertices lie strictly inside the sphere. The *inscribability exponent* is then defined by

$$s^{(d)} := \liminf_{P \in \mathcal{P}_d} \frac{\ln s(P)}{\ln v(P)}.$$

Grünbaum and Jucovič [24] proved the following theorem for dimension three and explained why the argument easily extends to higher dimensions. In addition, they conjectured that equality holds in all dimensions.

Theorem 3.0.13 (Inscribability exponent [24]). *For $d \geq 3$ it holds that*

$$s^{(d)} \leq \frac{\ln(d-1)}{\ln d} = \log_d(d-1).$$

Using Theorem 3.0.11 we will be able to modify the proof of this theorem such that it yields:

Theorem 3.0.14 (Inscribability exponent). *For $d > 1$,*

$$s^{(d)} \leq \log_d 2.$$

The proof of Theorem 3.0.13 uses the following fact:

Lemma 3.0.15 (Enclosed sphere and $d - 1$ stackings [24])**.** *For $d \geq 3$ let P be a d-simplex with all facets stacked. Then there is no sphere S passing through the $d + 1$ stacking vertices such that $P \subset \operatorname{conv} S$.*

Remark 3.0.16. This lemma is not explicitly stated but it is used as a fact in [24]. In 1967 Jucovič [29] improved a theorem by Steinitz which implies the lemma for $d = 3$. Then, in 1974 Grünbaum and Jucovič [24] mentioned that it can be extended to higher dimensions (and used this as a fact in the paper). Here is a hint for extending the 3-dimensional case to arbitrary dimension. The original proof assigns angles to edges and adds them up around a set of facets. In higher dimensions one has to use area angles and assign them to ridges.

Remark 3.0.17. Here is an example that shows that it is not screamingly obvious that Corollary 2.2.2 implies Theorem 3.0.11. See Figure 3.1.
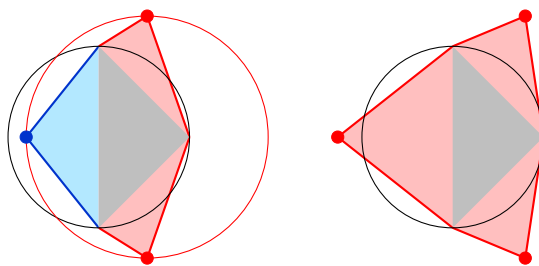

Figure 3.1: The left picture shows a stacked 2-polytope that has an enclosing sphere passing through the thick vertices. The right polytope can easily be made inscribed by flattening the three stackings. This trick also works in higher dimension. It cannot be applied in the left case since only the flattening guarantees that convexity is kept.

Proof of Theorem 3.0.11. The case $k \leq 3$ is easy since three times stacked simplices are inscribable. In this case the circumsphere is the enclosing sphere.

We prove the case $k > 3$ by contradiction, assuming that we have four stackings and S exists. We will investigate the Delaunay triangulation of the vertex set of P and transform it by a sphere inversion. The result will also be a Delaunay triangulation and maintain the Delaunay property after a small modification. The obtained triangulation will then match the

configuration described in Theorem 2.2.4 (with three stellar subdivisions). This will contradict the Delaunay property and imply the theorem.

Some labeling: Let P be a stacked d-polytope that is constructed by stacking four facets of a d-simplex $\delta = \text{conv}(v_0, \ldots, v_d)$ with vertices s_0, s_1, s_2 and s_3. Let U be the circumsphere of δ. Since P is simplicial, we can assume that all vertices lie in general position, especially that $v_0, \ldots, v_d$ lie inside S and none on S. Let F_i be the $(d-1)$-face of δ that does not contain v_i ($i = 0, \ldots, d$), and let s_j be the vertex that stacks F_j ($j = 0, \ldots, 3$).

Let us assume that no vertex $s_0, \ldots, v_d$ lies inside U, then we can deduce that P is of inscribable type: We can move $s_0, \ldots, s_3$ straight towards an interior point of P, until they all lie on U. This will not change the combinatorial type since we only "flatten" the stackings. By Corollary 2.2.2 we know that this type is not inscribable, so at least one vertex of $s_0, \ldots, s_3$ lies inside U.

Let us assume that two vertices s_0 and s_1 lie inside U. The intersection of S with U is a sphere of dimension $d - 2$. Its affine hull is a hyperplane H, which separates s_0 and s_1 from δ. Since P is convex, and (s_0, s_1) is not an edge of P, this is impossible. Hence exactly one vertex of $s_0, \ldots, s_3$ lies inside U.
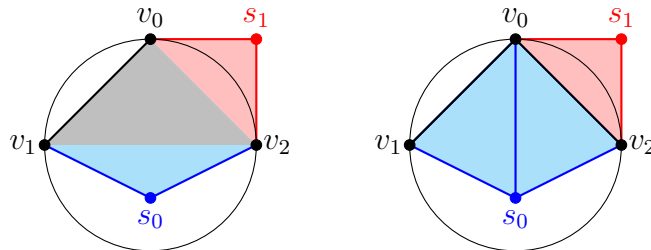

Figure 3.2: Left: A gray 2-simplex stacked on two facets. Right: The corresponding Delaunay triangulation.

Let s_0 lie inside U while s_1, s_2, s_3 lie outside. We will now reconstruct the Delaunay triangulation T of the vertex set of P. We will start with δ and add the vertices $s_0, \ldots, s_3$ to the convex hull. After each vertex we will adjust the current triangulation to make it Delaunay. (This technique is similar to the Bowyer–Watson algorithm [8] [47].) The face complex of δ is a Delaunay triangulation. We add the cell $\text{conv}(F_0, s_0)$. Since s_0 lies in U, this is not Delaunay and we need to flip the face F_0. The result is a Delaunay triangulation of d many d-simplices that all

share the edge (s_0, v_0) and pairwise share a $(d-1)$-cell. We add the cell $\mathrm{conv}(s_1, F_1)$. Since S passes through s_0 and s_1 and encloses F_1, we see by Lemma 0.2.6 that F_1 is locally Delaunay. All other $(d-1)$-faces are still locally Delaunay, hence the current triangulation is Delaunay. The same argument works for s_2 and s_3, so we just add $\mathrm{conv}(F_2, v_2)$ and $\mathrm{conv}(F_3, v_3)$. This implies that the Delaunay triangulation of the vertex set of P equals the induced triangulation of the stacked polytope P, with one interior $(d-1)$-face F_0 being flipped.
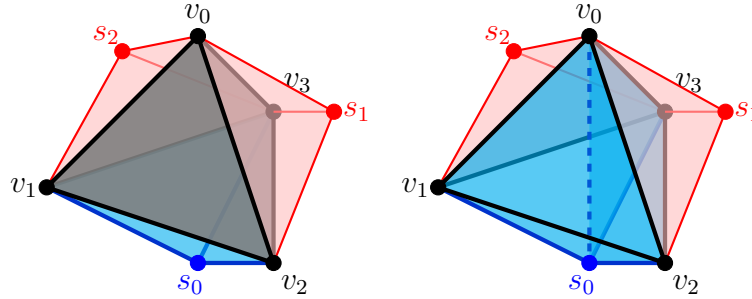

Figure 3.3: Left: A gray 3-simplex stacked with three vertices. Right: The corresponding Delaunay triangulation.

The fact that P is at least 3-dimensional and that it has exactly four stackings is only needed in the last argument of the proof. So we can illustrate most of the concepts by pictures of low-dimensional polytopes with fewer stackings. For T see Figure 3.2 and Figure 3.3.

Since all vertices of T are the vertices of a simplicial polytope, T is the vertex projection of an inscribed simplicial polytope. We take that polytope and use an other vertex projection from the vertex that corresponds to s_0 to derive a new Delaunay triangulation T'. (This is like a sphere inversion of the Delaunay triangulation at s_0. The image of infinity creates a new vertex.) For simplicity, we denote the images of this transformation by prime sign after the name. The new point which is the image of infinity shall be ∞'.

The new Delaunay triangulation T' is a flipped double simplex where three of its d-cells are stellar subdivided. See Figure 3.4 and Figure 3.5. We explain why: All facets of δ' except F_0' are in T'. Since they lie in cells that also contain s_0', they must lie at the boundary of T'. The outside of T corresponds now to the faces in T' that are contained in faces that also contain ∞'. This includes all ridges of δ' and all stacking vertices s_1', s_2' and s_3'. So we get that T' looks like a flipped double simplex, but instead

Figure 3.4: Left: Figure 3.2 after a sphere inversion that is centered at the blue vertex. The blue cells are new. Right: the same, except that v_0' is moved closer to ∞'. This creates a new cell $\mathrm{conv}\{v_0'', v_1'', v_2''\}$.
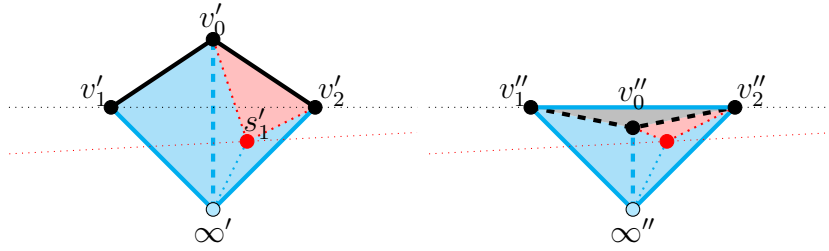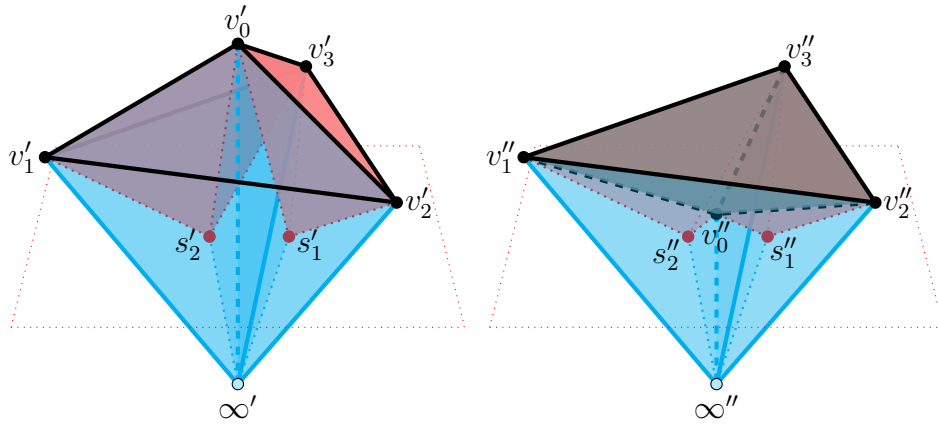

Figure 3.5: Left: Figure 3.3 after a sphere inversion that is centered at the blue vertex. The blue cells are new. Right: the same, except that v_0' is moved closer to ∞'. This creates a new cell $\mathrm{conv}\{v_0'', v_1'', v_2'', v_3''\}$.

of stacking it three times, three of its d-cells are stellar subdivided.

Since S passes through s_0 and separates ∞ from δ, the image S' is a hyperplane and all vertices of δ' lie on one side, while ∞' lies on the other side. We define the hyperplane $H := \mathrm{aff}(v_1', \ldots, v_d')$ and notice that it has v_0' on one side while $\infty', s_1', s_2', s_3'$ are on the other side. We derive a new triangulation T'' by moving v_0' straight towards ∞' unless it has passed H. After that, we fix the position in a sufficiently small distance to H and name the new vertex v_0''. The result is still a pure simplicial complex, but no longer convex. We fix this by adding the cell $\mathrm{conv}(v_0'', v_1'', \ldots, v_d'')$. Then the shape of T'' is a stellar subdivided d-simplex where three of its

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d-cells have been stellar subdivided again. We indicate all objects of T'' by a double prime.

We now prove that T'' is a Delaunay triangulation. It suffices to show that all $(d-1)$-cells are locally Delaunay. Boundary cells can be omitted and also all cells that contain s_1'', s_2'' or s_3'' since they are internal parts of a single stellar subdivision. Everything that is left to check are the $(d-1)$-cells that consist of the convex hull of v_0'' with a ridge of δ'. By construction, all vertices of T'' except $v_1'', \ldots, v_d''$ lie below H. Hence H is a supporting hyperplane of $|T''|$. Since v_0'' is arbitrary close to H, the circumsphere of the cell $\mathrm{conv}(v_0'', v_1'', \ldots, v_d'')$ can be assumed to be empty. Hence this cell and all its $(d-1)$-faces are locally Delaunay.

We have only left to check the $(d-1)$-faces that consist of the edge (∞'', v_0'') and $d-2$ other vertices of δ''. Let $F'' := \mathrm{conv}(v_0'', R'')$ be such a $(d-1)$-face and R'' a $(d-2)$-face. Then R'' is a $(d-2)$-face in the boundary of T' and also in T''. Hence R'' has a supporting hyperplane H_R for $|T'|$ and $|T''|$. The face $F' := \mathrm{conv}(v_0', R'')$ is locally Delaunay in T' and hence there is a supporting sphere S_F for it. The intersection of S_F with H_R is a $(d-2)$-sphere S_R and it contains all vertices of F' and F'' except v_0''. The vertex v_0'' lies inside S_F, so we can now continuously pass S_F through S_R until it also touches, v_0'' (see also Lemma 0.2.5). The result is a supporting sphere for F'' since it is smaller than S_F, at least on the side of H_R that faces $|T''|$. See Figure 3.6. Hence all interior $(d-1)$-cells are locally Delaunay. This implies that T'' is a Delaunay Triangulation.
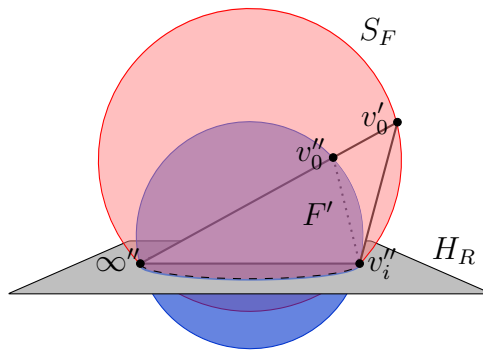

Figure 3.6: The red sphere intersects the gray plane in a circle. The blue sphere is created by pushing the red sphere through this circle. On the side upper side of the plane the blue sphere lies inside the red one.

This is a contradiction to Theorem 2.2.3 which implies that T'' has no Delaunay realization. So S does not exist for $k = 4$ and the theorem follows. $\qquad\square$

We could now copy the proof of Grünbaum and Jucovič and modify it by placing the right values. Instead we give our own version, which is based on the ideas of [29]. In contrast to the known proof, it will contain new results about stacked polytopes that are derived in this thesis.

Lemma 3.0.18 (Enclosed sphere). *For $d > 1$ let P be a d-polytope that lies in the closed unit ball. Let the boundary of P contain the ridges of a d-simplex δ and let V be the vertex set of P without the vertices of δ. Then V decomposes into $d + 1$ (possibly empty) subsets where each subset lies exactly beyond one facet of δ. Further, at most three of these subsets can intersect the unit sphere.*

Proof. If any vertex of P would lie beyond two faces of δ, then the ridge that shares these two faces would not have a supporting hyperplane and hence could not be a ridge of P. Assume we have four vertices of P at the unit sphere, each behind a different facet of δ. Then the convex hull of δ with those four simplices is a subset of P and hence enclosed by the unit sphere S. By Theorem 3.0.11 this is impossible. $\qquad\square$

Proof of Theorem 3.0.14. We will construct an infinite sequence of stacked polytopes that have a very poor inscribability ratio.

Let S be an enclosing sphere of a stacked d-polytope P and let Δ be the induced triangulation and $\mathcal{T}_P$ the induced dual tree. We pick any node to be the root of $\mathcal{T}_P$, every other node then corresponds to a stacking and thereby to a vertex. The root itself represents $d + 1$ vertices. By this identification, we have covered all vertices of P exactly once. By Lemma 3.0.18 at most three branches (including children and the parent) of every node of $\mathcal{T}_P$ can participate in having a vertex on S. This reduces to question of the largest number of inscribable vertices to the question of the largest tree of node degree at most 3 (a rooted binary tree, the root may have 3 children) that can be embedded into $\mathcal{T}_P$ (a rooted tree whose nodes have at most d children, the root may have $d + 1$ children).

By Theorem 2.2.1 every such binary tree corresponds to an induced triangulation of an inscribable stacked d-polytope P'. Moreover, extending this tree to the dual tree of P can be achieved by a sequence of sufficiently flat stackings started at P'. This shows that the maximal number of

inscribable vertices of a stacked polytope is exactly the number of nodes of the largest binary tree that can be embedded into its dual tree, plus d.

We consider a stacked d-polytope P_n where P_0 is the d-simplex, and by recursion for $n > 0$ we define P_n to be P_{n-1} stacked on all facets. Since all stacked polytopes are realizable, every P_n exists.

What is a maximal binary tree D in the dual tree of P_n? For sure it will contain the node R that is represented by P_0, otherwise we reduce the problem by symmetry to P_{n-1}. So if D contains the root, the best choice is clearly to choose a complete tree of length n that is binary except that the root has 3 children.

This implies that the maximal number of inscribable vertices for P_n is $d+1$ for the root, and three times the number of nodes of a perfect binary tree of length $n-1$. For the total number of vertices P_n we get $d+1$ for the root and $(d+1)$ times the number of nodes of a perfect tree, with d children of length $n-1$.

$$\text{inscr. vertices} = d+1+3\sum_{i=0}^{n-1} 2^i = (d-3) + 3 \cdot 2^n$$

$$\text{all vertices} = d+1+(d+1)\sum_{i=0}^{n-1} d^i = (d - \frac{d+1}{d-1}) + \frac{d+1}{d-1}d^n.$$

This implies an upper bound for the inscribability exponent:

$$s^{(d)} \le \liminf_n \frac{\ln((d-3)+3\cdot 2^n)}{\ln((d-\frac{d+1}{d-1})+\frac{d+1}{d-1}d^n)}$$

We simplify nominator and denominator, and divide both by n:

$$nom. = \frac{\ln\left[2^n \cdot (3+(d-3)2^{-n})\right]}{n} = \ln 2 + \frac{\ln(3+(d-3)2^{-n})}{n}$$

$$denom. = \frac{\ln\left[d^n \cdot (\frac{d+1}{d-1}+(d-\frac{d+1}{d-1})d^{-n})\right]}{n} = \ln d + \frac{\ln(\frac{d+1}{d-1}+(d-\frac{d+1}{d-1})d^{-n})}{n}$$

In the limit, the complicated terms vanish, hence

$$s^{(d)} \le \frac{\ln 2}{\ln d} = \log_d 2.$$

$\square$

This result is also nice since it naturally covers the case $d = 2$: All two dimensional polytopes are inscribable hence $s^{(2)} = 1$. In contrast to the conjecture we have shown that the inscribability exponent is getting smaller in higher dimensions, at least if $s^{(3)} = \log_3 2$. There are investigations [41] that at least for some classes of 3-polytopes show that $s^{(3)} = \log_3 2$.

We conjecture that $\log_d 2$ is the inscribability exponent for dimension d.

It would also be interesting to get a non asymptotic inscribability bound that depends on the dimension and the number of vertices. Our proof provides such bounds for special numbers. It should not be too hard to derive a bound for every number of vertices. One could argue that complete binary trees provide the worst possibility for embedding large binary trees. The corresponding binary trees are then probably also complete trees. This would then provide exact bounds for all stacked polytopes.

4 Construction techniques of f-vectors of inscribable simplicial polytopes

In this chapter we develop a construction scheme that allows us to prove that certain classes of f-vectors of simplicial polytopes also appear for inscribed simplicial polytopes.

4.1 A technique to extend triangulations

In this section we present constructions for Delaunay triangulations for which we can control the outer f-vector. Their special property is that they can be iterated in almost any order and they guarantee that the result is a Delaunay triangulation.

Definition 4.1.1 (Equator and ground polytope)**.** Let $d > 2$. We define the *height function h* and projection π by

$$h : \mathbb{R}^{d-1} \to \mathbb{R}, \quad h(x) := x_{d-1} \quad \text{and}$$

$$\pi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-2}, \quad \pi(x) := (x_1, \dots, x_{d-2}).$$

Let $\mathbb{R}^{d-1}$ be the ambient space. We call a hyperplane *vertical* if any normal vector x of it has last coordinate $h(x) = 0$. We call a $(d-2)$-polytope *vertical* in $\mathbb{R}^{d-1}$ if its affine hull is vertical. Let T be a $(d-1)$-dimensional triangulation, a $(d-1)$-polytope or the complex of a $(d-2)$-simplex. If T has no $(d-2)$-dimensional vertical faces in its boundary and does not lie in a vertical hyperplane, then we define the *equator* of T to be the simplicial complex that consists of all faces of $|T|$ that lie in a vertical hyperplane. We will then say that *T has an equator*. We call $\pi(|T|) \subset \mathbb{R}^{d-2}$ the *ground polytope of T*.

The ground polytope is a $(d-2)$-dimensional polytope. If T has an equator, then π is a bijection between the equator and the boundary of $\pi(|T|)$.

Definition 4.1.2 (Hex triangulation)**.** We call a triangulation *T hex (high extendable)* if it has an equator and all vertices in its boundary are vertices of a simplicial polytope that is $|T|$.

Lemma 4.1.3 (High extension)**.** *Let $d > 2$ and T be a hex triangulation in $\mathbb{R}^{d-1}$. Let V be its vertex set and let p be a point in $\mathbb{R}^{d-2} \times \{0\}$ that does not lie in the affine hull of any face of the ground polytope of T. If $\lambda \in \mathbb{R}$ is sufficiently large then $p' := p + \lambda e_{d-1}$ lies outside $|T|$ and the equator of T is the horizon of T seen from p'.*

If T is also Delaunay, or just a $(d-2)$-simplex, then λ can be chosen large enough, such that the extension of T by p' is a hex Delaunay triangulation.

We call such an extension a *high extension* and p the *base point* of the extension. If T has an equator, then an extension T' of T can only have vertical $(d-2)$-faces in its boundary, if p lies in the affine hull of a facet of $\pi(|T|)$. So by avoiding these positions one can iterate such extensions.

Proof. Let F be a $(d-2)$-face in the boundary of T. Note that its affine hull is not vertical, so for large enough λ the point p' lies beyond F if and only if F lies on the upper convex hull of $|T|$. The boundary of the upper convex hull is exactly the equator, so the first statement is true.

In case that T is a Delaunay triangulation, there are only finitely many circumspheres of $(d-1)$-faces of T. If λ is large enough, p' lies outside all of those. In this case the unique Delaunay subdivision of the vertex set $V \cup \{p'\}$ must contain T and it must triangulate the set $V \cup \{p'\}$. This can only be the extension we have created, so it is a Delaunay triangulation.

Since we have only finitely many conditions for λ being large enough, a suitable $\lambda \in \mathbb{R}$ exists. $\qquad\square$

Definition 4.1.4 (Types of high extensions)**.** Let $d > 2$ and T be a hex triangulation in $\mathbb{R}^{d-1}$. Let V be the vertex set of the ground polytope $\pi(|T|)$. We define three types of high extensions. The base point shall be p.

- A high extension of T is called a *stay extension* if p lies in the relative interior of $\pi(|T|)$. (See Figure 4.1)
- A high extension of T is called a *step extension* if $V \cup \{p\}$ are the vertices of a simplicial polytope. (See Figure 4.1)
- A high extension of T is called a *switch extension* if the convex hull of $V \cup \{p\}$ contains exactly one vertex of V in its interior. (See Figure 4.2)
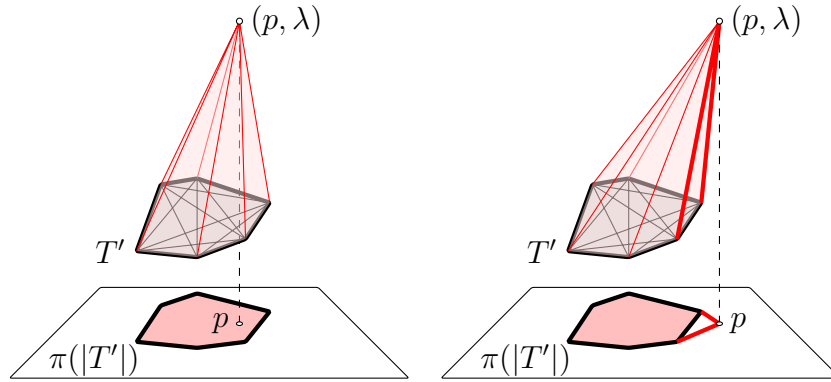
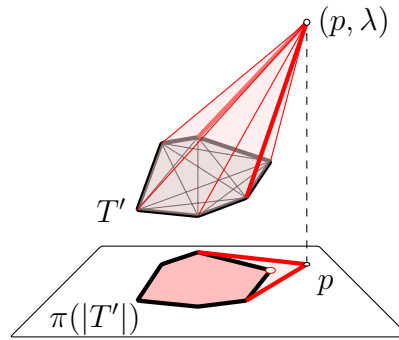Figure 4.1: A stay extensions and a step extension.

Figure 4.2: A switch extension.

Lemma 4.1.5 (Flat extension). *Let $d \geq 2$ and let $P \subset \mathbb{R}^{d-1}$ be a d-polytope that has an equator. Let F be a simplicial facet of P that intersects the equator in a ridge R. (Hence $\pi(R)$ is a facet of $\pi(P)$.) Then there exists a point $p \in \mathbb{R}^d$ that can be used to stack P at F and $\pi(p)$ can be used to stack $\pi(P)$ at $\pi(R)$. See Figure 4.3*

Proof. There are exactly two facets of P that contain R, one is F the other one shall be G. The set of points that lie beyond F and beneath all other facets of P is the interior of a polyhedron K. At its boundary there must be a point q that lies in the supporting hyperplane of G, but on the other side of R than G. Hence its projection must lie outside $\pi(P)$. Any interior point p of K that is close enough to q will be possible to stack F, while its projection stacks $\pi(P)$. $\square$
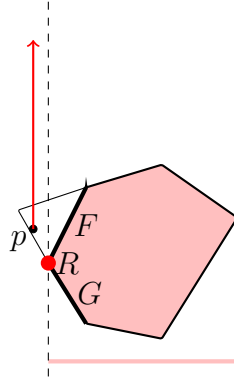
Figure 4.3: How to determine a point for a flat extension, and how the number of visible facets can be increased.

Definition 4.1.6 (Flat extension). Let $d \geq 2$. Let T be a hex triangulation in $\mathbb{R}^{d-1}$ that has n vertices in its equator and let $k \in \{0, \ldots, n-d+1\}$. Then we can apply Lemma 4.1.5 to $P := |T|$. Let p be the new point. Then, by assuming general position for p we can raise its height to gain a new vertex p' such that the horizon of $|T|$ seen from p' has exactly $k + d - 1$ vertices. We call the extension of T by p' a *flat extension of degree k*.

Lemma 4.1.7. *Let T be a triangulation in $\mathbb{R}^{d-1}$ and T' a flat extension of degree k of T. Then the outer g-vector of T' satisfies*

$$\bar{g}_1(T') = \bar{g}_1(T) + 1, \quad \bar{g}_2(T') = \bar{g}_2(T) + k.$$

Proof. This is an application of Lemma 0.4.6. $\qquad\square$

If T is a hex triangulation then a flat extensions of T can be assumed to be hex as well because the new point can be placed in general position. This gives the possibility to iterate flat extensions. Unfortunately, the height of the new point is bounded from above, so there is no guarantee that the Delaunay property can be kept.

Definition 4.1.8 (Push in extension). Let $d \geq 4$ and $m > 0$ be integers. Let T be a hex triangulation in $\mathbb{R}^{d-1}$. Let $p_m \in \mathbb{R}^{d-2}$ be the base point of a step extension T' of T in particular all vertices of $\pi(|T|)$ are also vertices in $\pi(|T'|)$. Let q be a point in the relative interior of a facet of $\pi(|T|)$ that is visible to p_m. Let $p_1, \ldots, p_m$ be distinct points on the open line segment (q, p_m) such that none of them lies in affine hull of any
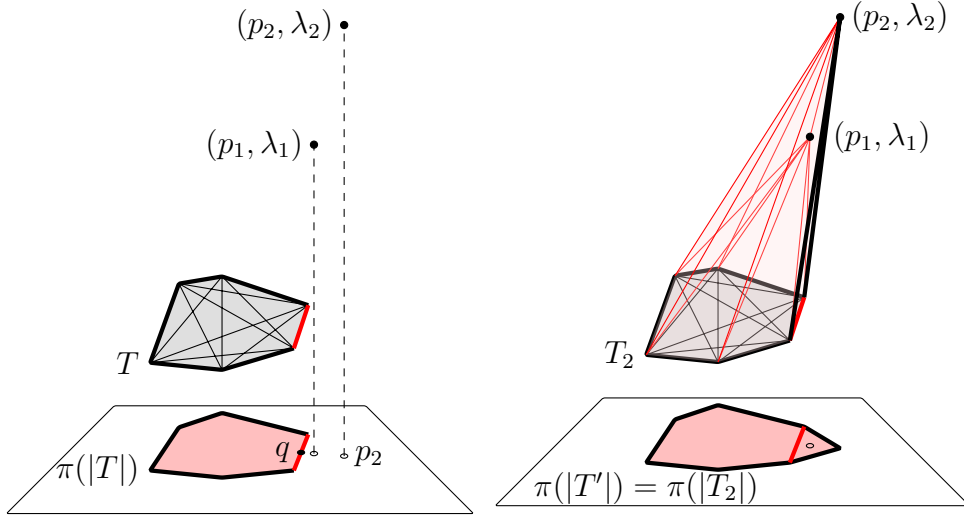
Figure 4.4: An illustration of a push in extension for $d = 4$, $s = 1$ and $n = 2$. For $d < 5$ life is boring, because p_1 and p_2 must exactly see $d - 2$ vertices. In higher dimension this changes.

facet of $\pi(|T|)$. Then we call the sequence $(T = T_0, T_1, \dots, T_m)$ a *push in extension* if for $i = 1, \dots, m$ the triangulation T_i is a high extension of T_{i-1} with base point p_i. See Figure 4.4.

Lemma 4.1.9 (Push in extension). *Let $d \geq 4$ and $m > 0$ be integers. Let T be a hex triangulation in $\mathbb{R}^{d-1}$. Let $p_m \in \mathbb{R}^{d-2}$ be the base point of a step extension T' of T such that all vertices of $\pi(|T|)$ are also vertices in $\pi(|T'|)$. Let n be the number of vertices of $\pi(|T|)$ that are visible from p_m and $0 \leq \lambda_1 \leq \cdots \leq \lambda_m = n - d + 2$ be integers. Then there exists a push in extension $(T = T_0, T_1, \dots, T_s)$ with points $p_1, \dots, p_m$ such that for $i = 1, \dots, m$ we have that p_i sees $\lambda_i + d - 2$ vertices of T_{i-1}.*

For $g(\pi(|T|)) = (1, r, s, *, *, \dots)$ and $i = 1, \dots, m$ we then have

$$g(\pi(|T_i|)) = (1, r + 1, s + \lambda_j, *, *, \dots),$$

$$\bar{g}(T_i) = \bar{g}(T) + (0, i, i(r+1) - 1, is + \sum_{j=1}^{i-1} \lambda_j, *, *, \dots).$$

If T is a hex Delaunay triangulation, then $T_1, \dots, T_m$ can be constructed to be Delaunay as well.

Proof. Since we only use high extensions, the Delaunay property can be preserved.

We will now pick $p_1, \ldots, p_m$ as follows. Let F be a facet of $\pi(|T|)$ that is visible from p_m and let q be an arbitrary point in the relative interior of F. If we imagine we travel from q to p_m, then the visible points of $\pi(|T|)$ will increase one by one at a time starting from the $d - 2$ vertices of F to finally $\lambda_m + d - 2$. So we can pick $p_1, \ldots, p_{m-1}$ with increasing distance to q such that for $i = 1, \ldots, m$ we have that p_i sees exactly $\lambda_i + d - 2$ vertices of $\pi(|T|)$. Further, by the choice of q we can assume that p_i does not lie the affine hull of any facet of $\pi(|T|)$.

We investigate the corresponding high extensions by induction. At the beginning, p_1 lies outside $\pi(|T|)$, not in any affine hull of any facet of $\pi(|T|)$ and it sees $\lambda_1 + d - 2$ vertices. Hence we can define a step extension T_1 of T with base point p_1. For $i = 2, \ldots, m$ we assume that T_{i-1} is high extension of T_{i-2} by the base point p_{i-1}. And we assume that all facets of $\pi(|T_{i-1}|)$ are either facets of $\pi(|T|)$ or they contain p_{i-1}. The induction step follows. Since the line segment (q, p_m) points away from $\pi(|T|)$, p_i must lie outside $\pi(|T_{i-1}|)$ and it sees all the facets that contain p_{i-1}. Hence p_i lies outside all affine hulls of facets of $\pi(|T_{i-1}|)$. This implies that we can define a high extension T_i and $\pi(|T_i|)$ will only contain facets of $\pi(|T|)$ and facets that contain p_i.

We count vertices and edges of $\pi(|T_i|)$ for $i = 1, \ldots, m$. Since all vertices $\pi(|T|)$ are also vertices of $\pi(|T'|)$ they must also be vertices of each $\pi(|T_i|)$. For $k = 2, \ldots, m$ all faces in $\pi(|T_k - 1|)$ that contain p_{k-1} are visible to p_k hence $\pi(|T_i|)$ has one vertex more than $\pi(|T|)$ and $\lambda_i + d - 2$ more edges. We simply count

$$g_1(\pi(|T_i|)) = g_1(\pi(|T|)) + 1 = r + 1,$$

$$f_1(\pi(|T_i|)) = f_1(\pi(|T|)) + (d - 2) + \lambda_i$$

which yields that

$$g_2(\pi(|T_i|)) = g_2(\pi(|T|)) + \lambda_i = s + \lambda_i.$$

To determine the outer g-vector of $T = T_1, \ldots, T_m$ we use Lemma 0.4.6, which for $j = 1, \ldots 3$ yields that

$$\bar{g}_j(T_i) = \bar{g}_j(T) + \sum_{k=0}^{i-1} g_{j-1}(\pi(|T_k|)).$$

Hence

$$\bar{g}_1(T_i) = \bar{g}_1(T) + 1 + (i - 1) = \bar{g}_1(T) + i,$$

$$\bar{g}_2(T_i) = \bar{g}_2(T) + r + (i - 1)(r + 1) = \bar{g}_2(T) + i(r + 1) - 1,$$

$$\bar{g}_3(T_i) = \bar{g}_3(T) + s + (i-1)s + \sum_{j=1}^{i-1} \lambda_j.$$

$\square$

The push in extension is a step extension followed by $m-1$ switch extensions.

4.2 Step and Stay Construction

In this section we use the techniques introduced in the previous section to create some f-vectors. Because the effect of each construction step depends on all previous steps, we develop a scheme that allows us to calculate in which order we have to use which type of construction.

We will define "icicle matrices" which are special 0/1 matrices. We also define what a g-vector of this matrix will be. First we will investigate the properties of these matrices and their g-vectors. Then we prove by construction that the g-vector of any icicle matrix corresponds to a g-vector of an inscribed simplicial polytope. The construction will be a double induction, over dimension and number of vertices. In each step a Delaunay triangulation will be created by using either a step or a stay extension. The icicle matrix encodes when to use which of the two options.

For any $d \in \mathbb{N}$ we define $d' := \lfloor \frac{d}{2} \rfloor$.

Definition 4.2.1 (Icicle matrix). Let $d \geq 2$ and $n > 0$. We call a matrix $I \in \{0,1\}^{d' \times n}$ an *icicle matrix* if for any $1 \leq k < d'$ and $1 \leq m \leq n$,

$$a_{k,m} \geq a_{k+1,m}.$$

We define the *support* $|I_k|$ of the kth row of I by

$$|I_k| := \{m = 1, \dots, n \mid a_{k,m} = 1\}.$$

In terms of support, an icicle matrix is characterized by the *subset property*

$$|I_k| \subseteq |I_{k-1}|, \qquad \text{for } k = 2, \dots, d'.$$

The *g-vector of an icicle matrix* I is given by

$$g_i(I) := \# \left\{ (s_1, \dots, s_i) \in \mathbb{N}^i \,\middle|\, \begin{array}{l} n \geq s_1 \geq \cdots \geq s_i \geq 1 \\ a_{(1,s_1)} = \cdots = a_{(i,s_i)} = 1 \end{array} \right\},$$

for $i = 0, \dots, d'$.

Theorem 4.2.2 (Step and Stay Theorem)**.** *Let $d \geq 2$ and $n > 0$. Let $I := (a_{i,j})$ be a $d' \times n$ icicle matrix. Then there exists an inscribed simplicial d-polytope P and a $(d-1)$-dimensional hex Delaunay triangulation T such that*

$$g_i(P) = \bar{g}_i(T) = g_i(I) \qquad for \ i = 0, \ldots, d'.$$

Example 4.2.3 (Counting monotone sequences in an icicle matrix)**.** By definition $g_i(I)$ counts the number of non increasing sequences of length i. As an example, we give an icicle matrix and two such sequences. We mark the sequences by brackets. On the left example we have $s = (s_1, s_2, s_3) = (5, 4, 1)$, on the right we have: $s = (s_1, s_2, s_3, s_4) = (6, 6, 5, 5)$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & [1] & 1 \\ 1 & 0 & 0 & [1] & 1 & 1 \\ [1] & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & [1] \\ 1 & 0 & 0 & 1 & 1 & [1] \\ 1 & 0 & 0 & 0 & [1] & 0 \\ 0 & 0 & 0 & 0 & [1] & 0 \end{pmatrix}$$

To count all possible sequences one can do the following: We define a matrix Σ by replacing each 1-entry in the second row of I by the sum of all 1-entries in the first row that lie in the same column or to the right. Then we replaces all 1-entries in the third row of I by the sum of all new entries in the second row that lie in the same column or to the right. We proceed with the following rows in the same way. If we sum up all numbers in a row of Σ, we get the number of sequences that start in this row. Example:

$$I = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ \Sigma = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 5 & 0 & 0 & 3 & 2 & 1 \\ 11 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}, \ \begin{matrix} g_1(I) = 5, \\ g_2(I) = 11, \\ g_3(I) = 14, \\ g_4(I) = 3 \end{matrix}$$

Each entry of Σ counts the number of monotone sequences in I starting at the current point.

We see that a 0-column can be deleted without changing $g(I)$. If we eliminate all 0-columns, the first row contains only 1-entries. In this case we have: $n = g_1(I)$.

Definition 4.2.4 (g-vectors of submatrices)**.** Let $d \geq 2$ and $n > 0$. Let I be a $d' \times n$ icicle matrix. Then for $k = 1, \ldots, d'$ and $m = 1, \ldots, n$ we denote by $I^{k,m}$ the lower left submatrix of I that has k rows and m columns. By definition it is an icicle matrix. Let $G(I) \in \mathbb{N}^{d' \times n}$ denote the following matrix of g-vectors of submatrices

$$G_{k,m}(I) := g(I^{k,m}) \qquad for \quad k = 1, \ldots, d' \text{ and } m = 1, \ldots, n.$$

Example 4.2.5. We continue with the icicle matrix from the previous example. Then $G(I)$ is

$$
\begin{pmatrix}
(1,1,1,1,0) & (1,2,2,2,0) & (1,2,2,2,0) & (1,3,4,4,0) & (1,4,7,8,1) & (1,5,11,14,3) \\
(1,1,1,0) & (1,1,1,0) & (1,1,1,0) & (1,2,2,0) & (1,3,4,1) & (1,4,6,2) \\
(1,1,0) & (1,1,0) & (1,1,0) & (1,1,0) & (1,2,1) & (1,2,1) \\
(1,0) & (1,0) & (1,0) & (1,0) & (1,1) & (1,1)
\end{pmatrix}
$$

These entries are related to each other as we will see later.

Definition 4.2.6 (The complete g-vector matrix). Let $d \geq 2$ and $n > 0$. Let I be a $d' \times n$ icicle matrix. Then the *complete g-vector matrix* $\bar{G}(I)$ is obtained by appending a last row and a 0-th column to $G(I)$: For $k = 0, \ldots n$ set $\bar{G}_{0,k} := (1)$ and for $m \in [n]$ set $\bar{G}_{m,0} := (1, 0, \ldots, 0) \in \{0,1\}^{d'-m+1}$.

$$
\bar{G}(I) = \left(
\begin{array}{c|ccccc}
(1,0,0,0,0)| & & & & & \\
(1,0,0,0)| & & & G(I) & & \\
(1,0,0)| & & & & & \\
(1,0)| & & & & & \\
\hline
(1)| & (1) & (1) & (1) & (1) & (1)
\end{array}
\right)
$$

This extends the definition of $G(I)$ in the natural way: In the 0th column of $G(I)$ we are counting sequences in a matrix without entries. There is none except for length 0, where we have the empty sequence. This also explains the first entry of all g-vectors.

Lemma 4.2.7 (Recursive structure). *Let $d \geq 2$ and $n > 0$. Let I be a $d' \times n$ icicle matrix. Then the complete g-vector matrix $\bar{G}(I)$ satisfies for all $m = 1, \ldots, n$ and $k = 1, \ldots, d'$ that*

- *if $I_{k,m} = 1$, then $\bar{G}_{k,m}(I) = \bar{G}_{k,m-1}(I) + \mathcal{R}(\bar{G}_{k+1,m}(I))$,*
- *if $I_{k,m} = 0$, then $\bar{G}_{k,m}(I) = \bar{G}_{k,m-1}(I)$.*

Here $\mathcal{R}$ denotes the index right shift operator.

Proof. Let k, m be given, and let $I_{k,m} = 1$. We recall that each entry in a g-vector relates to counting sequences of particular length that end in the first row of the submatrix. This can be decomposed into two sets of sequences, those that use the upper right 1-entry and those that do not. For $i = 1, \ldots, d'$ we have

$$
\#\left\{
\begin{array}{c}
s_1, \ldots, s_i \\
\in \mathbb{N}
\end{array}
\left|
\begin{array}{c}
1 \leq s_i \leq \ldots \leq s_1 \leq m-1 \\
a_{(1,s_1)} = \ldots = a_{(i,s_i)} = 1
\end{array}
\right.
\right\}
+ \#\left\{
\begin{array}{c}
s_2, \ldots, s_i \\
\in \mathbb{N}
\end{array}
\left|
\begin{array}{c}
1 \leq s_i \leq \ldots \leq s_2 \leq m \\
a_{(1,s_1)} = \ldots = a_{(i,s_i)} = 1
\end{array}
\right.
\right\}.
$$

This implies that $g_i(I^{m,k}) = g_i(I^{m-1,k}) + g_{i-1}(I^{m,k+1})$. In case that $I_{k,m} = 0$, nothing changes except that the second term has to be omitted: $g_i(I^{m,k}) = g_i(I^{m-1,k})$. $\qquad\square$

We set some notation before we start with the proof of the Step and Stay Theorem. For unspecified $x \in \mathbb{N}$ we define the north pole $N := (0, \ldots, 0, 1) \in \mathbb{R}^x$, the projection that truncates the last coordinate $\pi : \mathbb{R}^x \to \mathbb{R}^{x-1}$, and the stereographic projection $\phi : S^{x-1} \setminus \{N\} \to \mathbb{R}^{x-1}$. In addition, for an unspecified polytope that contains N as its highest vertex, we define the vertex projection Ψ that geometrically corresponds to ϕ.

Proof of the Step and Stay Theorem. It suffices to construct a $(d-1)$-dimensional hex Delaunay triangulation T, such that $\bar{g}(T) = g(I)$. Proposition 0.3.13 and Remark 0.4.2 then imply the existence of an inscribed polytope P such that $g(P) = g(I)$.

We can assume that the upper right entry of I is 1. We will create a hex Delaunay triangulation $T_{k,m}$ for each 1-entry $I_{k,m} = 1$, such that $G_{k,m}(I) = \bar{g}(T_{k,m})$ holds. We do this by a double induction over $k = d', \ldots, 1$ and $m = 0, \ldots, n$. We think of

$$\mathfrak{T}_{k,m} := (T_{k,m}) \quad \text{for } k = 1, \ldots, d' \text{ and } m \in I_k$$

as a $d' \times n$ matrix whose entries are hex Delaunay triangulations, except that index (k, m) is "unavailable" if $I_{k,m} = 0$. During the construction, each "available" triangulation $T_{k,m}$ will satisfy the following properties:

- **Each entry:** $T_{k,m}$ is a hex Delaunay triangulation of dimension $d - 2k + 1$ and $\bar{g}(T_{k,m}) = G_{k,m}(I)$.
- **Relation between row entries:** $T_{k,m}$ is a step or a stay extension of the "next available" triangulation "to the left" of $T_{k,m}$ in $\mathfrak{T}$. If there is none, then $T_{k,m}$ is a step or a stay extension of the face complex of a simplex, which we call $T_{k,0}$.
- **The projection:** For $k < d'$ the ground polytope $\pi(|T_{k,m}|)$ is inscribed in the unit sphere and it contains the north pole as a vertex. This ensures the existence of the Delaunay triangulation $\Psi \circ \pi(|T_{k,m}|)$, where Ψ is the vertex projection (stereographic) from the north pole.
- **Outer g-vector of the projection:** For $k < d'$ and $T_{k,m}$ not being "the rightmost" triangulation of the kth row of $\mathfrak{T}$, let T_{k,m^+} be the "next right available" triangulation for $T_{k,m}$. Then the outer g-vector of the projection $\Psi \circ \pi(|T_{k,m}|)$ is $G_{k+1,m^+}(I)$.
- **Projection appears in the row below:** With conditions as in the bullet point above, we have that $\Psi \circ \pi(|T_{k,m}|) = T_{k+1,m^+}$ if this position is available. If not, $\Psi \circ \pi(|T_{k,m}|)$ lies at the "next available" position "to the left" of position $(k + 1, m^+)$. If there is no such position, then $\Psi \circ \pi(|T_{k,m}|) = T_{k+1,0}$.

The idea is to evolve the recursion rule of $G(I)$ in Lemma 4.2.7 to a construction rule for $\mathfrak{T}$. The connection between different rows will be the projection $\Psi \circ \pi(|\cdot|)$ and the connection between triangulations in the same row will be that they are made by step and stay extensions of one another.

Preparation:

We define $T_{d',0}$ to be the face complex of any $(d - 2d' + 1)$-simplex in $\mathbb{R}^{d-2d'+1}$. If d is even, this is a line segment, if d is odd, it is a triangle. This is a hex Delaunay triangulation and its outer g-vector is $(1,0)$. For $k = 1, \ldots, d'$ and $m = 1, \ldots, n$ we define the *next left position*

$$(k, m)^- := (k, \max_{i \in |I_k| \cup \{0\}} \{i < m\}).$$

For $m \in |I_{d'}|$, we define $T_{d',m}$ to be any high extension of $T_{(d',m)^-}$. By Lemma 4.1.3, we can assume that these extensions are hex Delaunay triangulations of dimension $d - 2d' + 1$. The last row of $\mathfrak{T}$ is then

$$\begin{bmatrix} T_{d',0} & \ldots & T_{(d',m)^-} & \text{-}\text{-}\text{-} & T_{d',m} & \ldots \end{bmatrix}.$$

The dots indicate some unknown entries. The hyphens indicate a sequence of unavailable entries.

First induction, on k (every second dimension): We start a "reversed" induction over the rows $k = (d' - 1), \ldots, 1$ in this order. For a fixed row index k, we assume that for all $m \in |I_{k+1}|$, the following holds:

- **The first:** $T_{k+1,0}$ is the face complex of a $(d - 2k - 1)$-simplex that has an equator.
- **Each entry:** $T_{k+1,m}$ is a hex Delaunay triangulations of dimension $d - 2k - 1$ and $\bar{g}(T_{k+1,m}) = G_{k+1,m}(I)$
- **Relation between row entries:** $T_{k+1,m}$ is a high extension of $T_{(k+1,m)^-}$.

We see this holds for $k + 1 = d'$.

First induction, induction step: Let V be the vertex set of $T_{k+1,0}$. Then $\text{conv}(\phi^{-1}(V) \cup \{N\})$ is a $(d - 2k)$-simplex in $\mathbb{R}^{d-2k} \times \{0\}$. Let $T_{k,-1}$ be its face complex in $\mathbb{R}^{d-2k+1}$. Note, that it has an equator and $\Psi \circ \pi(|T_{k,-1}|) = T_{k+1,0}$. If we think of $\mathfrak{T}$ also having a column with index -1 and 0 then we have

$$\begin{bmatrix} T_{k,-1} & * & \ldots \\ T_{k+1,-1} & T_{k+1,0} & \ldots \\ & \ldots & \end{bmatrix}.$$

Second induction, on m (number of vertices): To define the triangulations in the kth row, we start an induction over $m \in |I_k| \cup \{0\}$. For each $m \in |I_k| \cup \{0\}$, we define the *next position* m^+ with respect to k. We also define the position $\tilde{m}$

$$m^+ := \min \left\{ i \in |I_k| \cup \{\infty\} \,|\, i < m \right\},$$

$$\tilde{m} := \max \left\{ i \in |I_{k+1}| \cup \{0\} \,\middle|\, i \le m^+ \right\}.$$

The position $\tilde{m}$ can be described as "the last entry below the next entry". In the same way we define $m^{++} := (m^+)^+$ and $\tilde{m}^+ := \widetilde{(m^+)}$ for $m^+ \in |I_k| \cup \{\infty\}$. The induction assumption is:

- **Each entry:** $T_{k,m}$ is a hex Delaunay triangulation of dimension $d - 2k + 1$ and $\bar{g}(T_{k,m}) = G_{k,m}(I)$.
- **The projection:** $\Psi \circ \pi(|T_{k,m}|) = T_{k+1,\tilde{m}}$.

Second induction, initial step: We create the initial step for $m = 0$. Then m^+ is the position of the first 1-entry in the kth row. Because I is an icicle matrix, $\tilde{0}$ is either 0 or 0^+. If $\tilde{0} = 0$, then we define $T_{k,0}$ to be any stay extension of $T_{k,-1}$. A stay extension does not change the ground polytope of the triangulation, so we get

$$\Psi \circ \pi(|T_{k,0}|) = \Psi \circ \pi(|T_{k,-1}|) = T_{k+1,0} = T_{k+1,\tilde{0}}.$$

In case that $\tilde{0} = 0^+$, the induction assumption tells us that $T_{k+1,\tilde{0}}$ is an extension of $T_{k,0}$ by a point p. Then we define $T_{k,0}$ to be any step extension of $T_{k,-1}$ by the base point $\psi^{-1}(p)$. Again

$$\Psi \circ \pi(|T_{k,0}|) = T_{k+1,\tilde{0}}.$$

By Lemma 4.1.3, we know that in both cases $T_{k,0}$ is the face complex of a $(d - 2k + 1)$-simplex, which is a hex Delaunay triangulation. Its outer g-vector is $\bar{g}(T_{k,0}) = (1, 0, \ldots, 0) = \bar{G}_{k,0}(I)$. So for $m = 0$ this holds. So $\mathfrak{T}$ looks like

$$
\begin{bmatrix}
T_{k,-1} & T_{k,0} & \text{---} & (T_{k,0^+}) & * \ldots \\
& \searrow & \downarrow & & \\
T_{k+1,-1} & T_{k+1,\tilde{0}} & \text{---} & - & * \ldots \\
\ldots & & & &
\end{bmatrix},
$$

or

$$
\begin{bmatrix}
T_{k,-1} & T_{k,0} & \text{---} & (T_{k,0^+}) & * \ldots \\
& \searrow & \searrow \rightarrow \searrow & & \\
T_{k+1,-1} & T_{k+1,0} & \text{---} & T_{k+1,\tilde{0}} & * \ldots \\
\ldots & & & &
\end{bmatrix}.
$$

The arrow indicates the projection $\Psi \circ \pi(|\cdot|)$.

Second induction, induction step: Assume m satisfies the induction condition and $m^+ \neq \infty$, which would end the induction. Then we have two cases:

In the first case we assume that $m^{++} = \infty$, or $I_{k+1,m^{++}} = 0$. Then $\tilde{m}^+ = \tilde{m}$, because I is an icicle matrix. We define T_{k,m^+} to be any stay extension of $T_{k,m}$. This leaves ground polytope unchanged, and we get:

$$\Psi \circ \pi(|T_{k,m^+}|) = \Psi \circ \pi(|T_{k,m}|) = T_{k+1,\tilde{m}} = T_{k+1,\tilde{m}^+}$$

In the second case we have that $m^{++} < \infty$ and $I_{k+1,m^{++}} = 1$. Then, $\tilde{m}^+ = m^{++}$. Because I is an icicle matrix and by the induction assumption, $T_{k+1,\tilde{m}^+}$ is an extension of $T_{k+1,\tilde{m}}$ by a point p. We define T_{k,m^+} to be any step extension of $T_{k,m}$ by the base point $\Psi^{-1}(p)$. Then:

$$\Psi \circ \pi(|T_{k,m^+}|) = T_{k+1,\tilde{m}^+}$$

By Lemma 4.1.3 we can assume in both cases, that T_{k,m^+} is a hex Delaunay triangulation of dimension $d-2k+1$. Using Lemma 0.4.6 and Lemma 4.2.7, we get that

$$
\begin{aligned}
\bar{g}(T_{k,m^+}) &= \bar{g}(T_{k,m}) + \mathcal{R}(\bar{g}(T_{k+1,\tilde{m}})) \\
&= G_{k,m}(I) + \mathcal{R}(G_{k+1,\tilde{m}}(I)) \\
&= G_{k,m^+-1}(I) + \mathcal{R}(G_{k+1,m^+}(I)) \\
&= G_{k,m^+}(I).
\end{aligned}
$$

We illustrate some examples of $\mathfrak{T}$. In the first example $I_{k+1,m^{++}} = 0$, in the second example $I_{k+1,m^{++}} = 1$. Again, the arrows indicate projection.

$$
\begin{bmatrix}
\cdots * & T_{k,m} & \text{---} & T_{k,m^+} & \text{---} & (T_{k,m^{++}}) & * \cdots \\
& \downarrow & \swarrow \leftarrow \nwarrow & & & & \\
\cdots * & T_{k+1,\tilde{m}} & \text{---} & - & \text{---} & - & * \cdots
\end{bmatrix}
$$

$$
\begin{bmatrix}
\cdots * & T_{k,m} & \text{---} & T_{k,m^+} & \text{---} & (T_{k,m^{++}}) & * \cdots \\
& & \searrow \rightarrow \searrow & & \searrow \rightarrow \searrow & & \\
\cdots & * & \cdots * & T_{k+1,\tilde{m}} & \text{---} & T_{k+1,\tilde{m}^+} & * \cdots
\end{bmatrix}
$$

Conclusion: After the induction over m the requirements of the induction step k are satisfied. This implies that $T_{1,n}$ is a hex Delaunay triangulation of dimension $d-1$ and

$$\bar{g}(T_{1,n}) = G_{1,n}(I) = g(I).$$

$\square$

4.3 Applications of the Step and Stay Theorem

In this section we present some result based on the construction techniques
that we have developed in the last two sections.

Theorem 4.3.1 (Pure binomial coefficients)**.** *Let $d \geq 2$ and $n_1 \geq \cdots \geq n_{d'} \geq 0$ be integers. Then there is an inscribed simplicial polytope of dimension d whose g-vector is*

$$g = (1, \binom{n_1}{1}, \binom{n_2 + 1}{2}, \ldots, \binom{n_{d'} + d' - 1}{d'}).$$

Note that by Corollary 0.1.16, these are the only g-vectors of the type
$g = (1, \binom{*}{1}, \ldots, \binom{*}{d'})$.

Proof. By theorem 4.2.2 it suffices to create a $d' \times n$ icicle matrix $I = (a_{k,m})$
such that $g = g(I)$. We define $a_{k,m} = 1$ if and only if $m \geq n - n_k$. Then
for $i = 1, \ldots, d'$ we have

$$g_i(I) = \# \left\{ (s_1, \ldots s_i) \in \mathbb{N}^i \,\middle|\, \begin{array}{c} 1 \leq s_i \leq \cdots \leq s_1 \leq n \\ a_{(1,s_1)} = \cdots = a_{(i,s_i)} = 1 \end{array} \right\}.$$

The numbers $n_1, \ldots, n_{d'}$ are decreasing, and we see that $a_{(i,s_i)} = 1$ exactly
if $s_i > n - n_i$. So we can simplify

$$g_i(I) = \# \left\{ (s_1, \ldots s_i) \in \mathbb{N}^i \,\middle|\, (n - n_i) \leq s_i \leq \cdots \leq s_1 \leq n \right\}.$$

Rephrased,

$$g_i(I) = \# \left\{ (s_1, \ldots, s_i) \in \mathbb{N}^i \,\middle|\, 0 < s_i < \cdots < s_1 \leq (n_i + i - 1) \right\} = \binom{n_i + i - 1}{i}.$$

$\square$

Proposition 4.3.2. *Let $d \geq 0$. For every simplicial d-polytope there
exists an inscribed simplicial d-polytope with the same number of vertices
and edges.*

Proof. Dimension 0 and 1 is trivial, dimension 2 and 3 are covered by
stacked inscribable polytopes. So we assume $d \geq 4$. It suffices to show
that for any simplicial d-polytope P there exists a $(d' \times n)$ icicle matrix
I such that $g(I) = (1, n, g_2, \ldots, g_{d'})$ and $n = g_1(P)$ and $g_2 = g_2(P)$.

Let the first row of I consist of n many 1-entries. The second row shall be $v = (v_1, \ldots, v_n) \in \{0,1\}^n$ and all other entries shall be 0. For any choice of v this is an icicle matrix and $g_1(I) = n$.

We define integers $a \geq b \geq 0$ through

$$g_2(I) = \binom{a+1}{2} + b.$$

Note that this is unique with $a \leq n$ and equality only if $b = 0$. We define for $i = 1, \ldots, n$ that $v_i = 1$ if and only if $i \geq n - a$ and $i \neq n - a + b$. Then

$$g_2(I) = [(a+1) + \cdots + 1] - (a + 1 - b) = [a + \cdots + 1] + b = \binom{a+1}{2} + b.$$

$\square$

Corollary 4.3.3. *Let $d \leq 5$. For every simplicial d-polytope there exists an inscribed simplicial d-polytope with the same f-vector.*

Example 4.3.4. This scheme is incapable of creating the vector $g = (1, 2, 2, 1)$, which is the g-vector of a simplicial polytope. Let us assume that there is such a matrix I. Since $g_1(I) = 2$, it must contain two 1-entries in its first column. We can assume that there are no 0-columns, so I has two columns. Since $g_2 = 2$ we have one choice to set the second row

$$I = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ * & * \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ * & * \end{pmatrix}, \qquad \begin{array}{l} g_1(I) = 2, \\ g_2(I) = 2, \\ g_3(I) = *(even) \end{array} \quad .$$

Since all entries in the second row of Σ are even and $g_3(I)$ is a sum of numbers of the second row, $g_3 = 1$ is impossible.

4.4 Controlling the number of vertices, edges and 2-faces

In the last section we have seen that the Step and Stay construction is not capable of creating all possible (g_1, g_2, g_3) combinations that simplicial d-polytopes have. This section will provide a different scheme that allows us to show the following theorem.

Theorem 4.4.1. *Let $d \geq 2$ and P be a simplicial d-polytope. Then there exists an inscribed simplicial d-polytope with the same number of vertices, edges and 2-faces.*

Corollary 4.4.2. *For $d \leq 7$ and each simplicial d-polytope there is an inscribed simplicial d-polytope with the same f-vector.*

Corollary 4.4.3. *For $d \geq 6$ and each simplicial d-polytope there is an inscribed simplicial d-polytope with the same g_1, g_2, g_3.*

Similar to the last section, this theorem is equivalent to the existence of a hex $(d-1)$-dimensional Delaunay triangulation T such that

$$\bar{g}_1(T) = g_1(P), \quad \bar{g}_2(T) = g_2(P), \quad \bar{g}_3(T) = g_3(P).$$

To prove the theorem we will define the G_3-*matrix* and its *evaluation*, which will be three values (g_1, g_2, g_3). We first show that for any g_1, g_2, g_3 that appear as first g-vector entries of simplicial polytopes a G_3-matrix can be constructed that has them as evaluation. Then we prove by construction the existence of an inscribed simplicial polytope whose g-vector starts with g_1, g_2, g_3. Similar to the icicle matrix the G_3-matrix will serve as a "construction recipe".

Definition 4.4.4 (G_3-matrix). A $(2 \times n)$-matrix $J = \binom{v}{w}$ is a G_3-*matrix of length n*, if it satisfies:

- For $i = 1, \ldots, n$, we have $v_i \in \{0, 1\}$ and $w_i \in \{0, \ldots, n\}$.
- For every $i = 1, \ldots, n$ where $v_i = 1$, let $m \geq i$ be the minimum such that $m = n$ or $a_{1,m+1} = 1$. Then

$$\sum_{k=i}^{m} w_k \leq \sum_{k=1}^{i} v_k.$$

We define the *evaluation* of a vector $x \in \mathbb{N}^n$ by

$$\mathrm{eval} : \mathbb{N}_0^n \to \mathbb{N}_0, \qquad \mathrm{eval}(x) := nx_1 + \cdots + 1x_n = \sum_{i=1}^{n} x_i(n - i + 1).$$

and the *evaluation* of a G_3 matrix $\binom{v}{w}$ of length n by

$$\mathrm{eval}(J) := (n, \mathrm{eval}(v), \mathrm{eval}(w)).$$

Example 4.4.5. Let us check whether the following matrix is G_3.

$$J = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 5 \end{pmatrix}$$

We have to check the sum formula. To do that we derive a vector $\mathfrak{b}$ of upper bounds from the first row

$$\mathfrak{b} = \begin{pmatrix} 1 & 2 & * & 3 & 4 & * & * & * & 5 & 6 \end{pmatrix}.$$

Then we compute a vector based of w that we have to compare to $\mathfrak{b}$,

$$\begin{pmatrix} 0 & (1+0) & * & 0 & (1+2+0+1) & * & * & * & 1 & 6 \end{pmatrix}.$$

Since $\mathfrak{b}$ is component wise greater or equal to the second vector, J is a G_3 matrix.

In the next example J' is not a G_3-matrix. We denote the values that we have to compare as subscript.

$$J' = \begin{pmatrix} 1_1 & 1_2 & 0_* & 1_3 & 1_4 & 0_* & 0_* & 0_* & 1_5 & 0_* \\ 3_3 & 0_{(0+2)} & 2_* & 3_3 & 2_{(2+2+1+0)} & 2_* & 1_* & 0_* & 4_{(4+1)} & 1_* \end{pmatrix}$$

The violation appears in column 1 and 5.

Lemma 4.4.6 (Describing g_1, g_2, g_3 by a G_3-matrix). *Let $d \geq 6$ and P be a simplicial d-polytope. Then there exists a G_3-matrix J such that*

$$\text{eval}(J) = (g_1(P), g_2(P), g_3(P)).$$

Proof. Let $n = g_1(P)$ and let $J = \binom{v}{w}$ be a matrix of size $2 \times n$ with variables $v, w \in \mathbb{N}_0^n$. Similar to the proof of Proposition 4.3.2 we define integers $a \geq b \geq 0$ such that $g_2 = \binom{a+1}{2} + \binom{b}{1}$ and for $i = 1, \ldots, n$ we set $v_i = 1$ if and only if $i \geq n - a$ and $i \neq n - a + b$. Then $g_2 = \text{eval}(v)$ holds.

For the construction of w, we define a function that returns the maximal possible value below each 1 entry of v. Below a 0 it shall return 0,

$$\mathfrak{b} : \{1, \ldots, n\} \to \mathbb{N}_0, \qquad \mathfrak{b}_i := v_i \sum_{k=1}^{i} v_k, \qquad \text{for } i = 1, \ldots, n.$$

By the g-theorem, $0 \leq g_3 \leq \binom{a+2}{3} + \binom{b+1}{2}$.

Case 1: We assume that $g_3 = \binom{a+2}{3} + \binom{b+1}{2}$. For $i = 1, \ldots, n$, we define $w_i := \mathfrak{b}_i$, which, by definition, yields a G_3-matrix.
If $b = 0$, then by construction of v, this is

$$w = (0, \ldots, 0, 1, \ldots, a)$$

and we get

$$\text{eval}(w) = a \cdot 1 + (a-1) \cdot 2 \cdots + 1 \cdot a = \binom{a+2}{3} = g_3.$$

If $b > 0$, we have

$$
\begin{aligned}
\mathrm{eval}(w) &= \mathrm{eval}(0, \ldots, 0, 1, \ldots, b, 0, b+1, \ldots, a) \\
&= \mathrm{eval}(0, \ldots, 0, 0, 1, \ldots, b, b+1, \ldots, a) + (1 + \cdots + b) \\
&= \binom{a+2}{3} + \binom{b+1}{2} = g_3.
\end{aligned}
$$

Case 2: We assume that $g_3 < \binom{a+2}{3} + \binom{b+1}{2}$. We will now, in three steps, increase the entries of a vector of length n, until its evaluation is g_3. For $0 \leq k \leq \mathfrak{b}_n$ we define vectors $w^k \in \mathbb{N}_0^n$ by

$$
w_i^k := \min(\mathfrak{b}_i, k), \qquad \text{for } i = 1, \ldots, n.
$$

These vectors have the shape

$$
w^k = (0, \ldots, 0, \ 1, 2, \ldots, k, \ k, \ldots, k).
$$

There is one exception: At position b a 0 is inserted and all entries up to position b are shifted by one position to the left. Now we need to find the right k.

$$
k := \max_j \{\mathrm{eval}(w^j) \leq g_3\}
$$

In case that $\mathrm{eval}(w^k) = g_3$, we have found a suitable vector $w := w^k$. Then J is the desired G_3-matrix.

Assume $\mathrm{eval}(w^k) < g_3$. We define for $1 \leq s \leq n$ vectors $w^{k,s} \in \mathbb{N}_0^n$ by

$$
w_i^{k,s} := \begin{cases} \min(\mathfrak{b}_i, k+1) & i < s, \\ \min(\mathfrak{b}_i, k) & s \leq i, \end{cases} \qquad \text{for } i = 1, \ldots, n.
$$

These vectors have the shape

$$
w^{k,s} = (0, \ldots, 0, \ 1, \ldots, k{+}1, \ k{+}1, \ldots, k{+}1, \ k, \ldots, k).
$$

Again one exception: At position b a 0 is inserted. Let

$$
s := \max_i \{\mathrm{eval}(w^{k,i}) \leq g_3\}.
$$

In case that $\mathrm{eval}(w^{k,s}) = g_3$, we have found a suitable vector $w := w^{k,s}$ and J is a G_3-matrix.

Assume $\mathrm{eval}(w^{k,s}) < g_3$. Since k is maximal, we have $s < n$. Using the definition of the evaluation and the fact that s is maximal we get

$$
r := g_3 - \mathrm{eval}(w^{k,s}) < \mathrm{eval}(w^{k,s+1}) - \mathrm{eval}(w^{k,s}) = n - s.
$$

We set $t := n - r + 1$ and define w by

$$w_i := \begin{cases} w_i^{k,s} & i \neq t, \\ w_i^{k,s} + 1 & i = t, \end{cases} \qquad \text{for } i = 1, \dots, n.$$

Then $\mathrm{eval}(w) = \mathrm{eval}(w^{k,s}) + r = g_3$.

We have left to check that J is G_3. The only non obvious position is t. We see that $t > s + 1$ hence for $t \neq b$ we have $w_t^{k,s} < \mathfrak{b}_t$. In case that $t = b$, we have to combine the information of positions t and $t - 1$. But since we then have $w_{t-1}^{k,s} < \mathfrak{b}_t$ this is no problem. This shows that J is a G_3-matrix. $\qquad\square$

Lemma 4.4.6 allows us to reduce the proof of Theorem 4.4.1 to the following problem: *Given $d \geq 6$ and a G_3-matrix J, construct a hex $(d-1)$-dimensional Delaunay triangulation T such that*

$$\mathrm{eval}(J) = (\bar{g}_1(T), \bar{g}_2(T), \bar{g}_3(T)).$$

We set some notation: For unspecified $x \in \mathbb{N}$ we define the north pole $N := (0, \dots, 0, 1) \in \mathbb{R}^x$, the projection that truncates the last coordinate $\pi : \mathbb{R}^x \to \mathbb{R}^{x-1}$, and the stereographic projection $\phi : S^x \setminus \{N\} \to \mathbb{R}^x$. In addition, for an unspecified polytope that contains N as its highest vertex, we define the vertex projection Ψ that geometrically corresponds to ϕ. (This vertex projection is also defined in case that not all vertices lie on the sphere. Then we take the radial projection that corresponds to ϕ.)

Proof. Let $J = \binom{v}{w}$ be a matrix of size $2 \times n$ with row vectors $v, w \in \mathbb{N}_0^n$. Let $\mathfrak{b}, \mathfrak{c} \in \mathbb{N}_0^n$ be

$$\mathfrak{b}_i = \sum_{j=1}^{i} v_j, \qquad \mathfrak{c}_i = \sum_{j=1}^{i} w_j.$$

We will create a matrix like structure $\mathfrak{S}$ that consists of three row vectors of hex triangulations.

$$\mathfrak{S} = \begin{bmatrix} & T_{-1} & T_0 & \dots & T_{n-2} & T_{n-1} & T_n \\ A_{-1} & A_0 & A_1 & \dots & A_{n-1} & A_n & \\ B_{-1} & B_0 & B_{\mathfrak{b}_1} & \dots & B_{\mathfrak{b}_{n-1}} & B_{\mathfrak{b}_n} & \end{bmatrix}$$

This will have the following properties:

- The entries of the first row are $(d-1)$-dimensional hex Delaunay triangulations, where T_0 is a simplex and T_n is the desired triangulation that we aim to construct. Each entry is a high extension of its left neighbor.
- In the second row we have hex triangulations of dimension $d-3$. Each is the vertex projection of the ground polytope of the entry above it.
- The third row consist of hex Delaunay triangulations of dimension $d-5$. They will also correspond to some ground polytope vertex projections of entries of the second row.

In a preparation step, we will create $B_{-1}, \ldots, B_{\mathfrak{b}_n}$, A_0 and T_{-1}. Then we use an induction over $i = 1, \ldots, n$ in which we create T_{i-1} and A_i as extensions of T_{i-2} and A_{i-1}. In a final step, we build T_n. Throughout the construction we keep track of the outer g-vector of the components.

Preparation: Let B_{-1} be the face complex of any $(d-5)$-simplex in $\mathbb{R}^{d-5}$. For $m = 0, \ldots, \mathfrak{b}_n$ let B_m be any proper extension of B_{m-1} by a point p_m. We can assume they are all Delaunay triangulations.

$$[B_{-1}, B_0, B_{\mathfrak{b}_1}, B_{\mathfrak{b}_2}, \ldots, B_{\mathfrak{b}_n}]$$

Let $V \subset \mathbb{R}^{d-5}$ be the vertex set of B_{-1}. Then $\mathrm{conv}(\phi^{-1}(V) \cup \{N\})$ is an inscribed $(d-4)$-simplex in $\mathbb{R}^{d-4} \times \{0\}$. Let A_{-1} be its face complex in $\mathbb{R}^{d-3}$. By definition, it has an equator and $\Psi \circ \pi(|A_{-1}|) = B_{-1}$. Let A_0 be any high extension of A_{-1} by the base point $\phi^{-1}(p_0)$. Then A_0 is the face complex of a $(d-3)$-simplex. This is a hex Delaunay triangulation and $\Psi \circ \pi(|A_0|) = B_0$. Let $V' \subset \mathbb{R}^{d-3}$ be the vertex set of A_0. Then $\mathrm{conv}(\phi^{-1}(V') \cup \{N\})$ is a $(d-2)$-simplex in $\mathbb{R}^{d-2} \times \{0\}$. Let T_{-1} be its face complex in $\mathbb{R}^{d-1}$. It has an equator and $\Psi \circ \pi(|T_{-1}|) = A_0$.

$$\mathfrak{S} = \begin{bmatrix} & T_{-1} & * \ldots \\ A_{-1} & A_0 & * \ldots \\ B_{-1} & B_0 & * \ldots \end{bmatrix}$$

(The new part is depicted in red. The blue part has not yet been constructed.)

Induction on the number of vertices:

Our induction will start with an optional step that appears if $v_1 = 0$ and then iterate over every i that has $v_i = 1$. It will define all entries in $\mathfrak{S}$ except T_n.

The induction assumes for a given $i \in \mathbb{N}$ that

- In case $i > 1$ we have that T_{i-2} is a hex Delaunay triangulations of dimension $d - 1$ and $\bar{g}(T_{i-2}) = (1,\, i - 2,\, \sum_{j=1}^{i-2} \mathfrak{b}_j,\, \sum_{j=1}^{i-2} \mathfrak{c}_j,\, \ldots)$.
- For $i = 1$ we have T_{i-2} is a hex triangulations that is a $(d-2)$-simplex in $\mathbb{R}^{d-1}$.
- A_{i-1} is a hex triangulation of dimension $d - 3$ and

$$\Psi \circ \pi(|T_{i-2}|) = A_{i-1}.$$

- $\Psi \circ \pi(|A_{i-1}|) = B_{\mathfrak{b}_{i-1}}$.
- $\bar{g}(A_{i-1}) = (1,\, \mathfrak{b}_{i-1},\, \mathfrak{c}_{i-1},\, \ldots)$.

The case $v_1 = 0$: Let s be the maximal number such that

$$v_1 = \cdots = v_s = 0.$$

For $k = 1, \ldots, s$ we define $A_k := A_0$ and let T_{k-1} be any stay extension of T_{k-2}. By Lemma 4.1.3 we can assume that T_{k-1} is a hex $(d - 1)$-dimensional Delaunay triangulation. Since T_{k-1} is a stay extension, the image of its projection is unchanged. We have

$$\Psi \circ \pi(|T_{s-1}|) = \Psi \circ \pi(|T_{-1}|) = A_0 = A_s$$

$$\Psi \circ \pi(|A_s|) = B_0 = B_{\mathfrak{b}_s}$$

$$\bar{g}(A_s) = \bar{g}(A_0) = (1, 0, 0,\, \ldots).$$

Lemma 0.4.6 shows that $\bar{g}(T_{s-1}) = (1,\, s - 1,\, 0,\, \ldots)$. An illustration:

$$J = \begin{pmatrix} 0 & \cdots & 0 & *\ldots \\ 0 & \cdots & 0 & *\ldots \end{pmatrix}, \mathfrak{S} = \begin{bmatrix} & T_{-1} & T_0 & \cdots & T_{s-1} & *\ldots \\ A_{-1} & A_0 & =A_1 & \cdots & =A_s & *\ldots \\ B_{-1} & B_0 & B_0 & \cdots & B_0 & *\ldots \end{bmatrix}.$$

We now continue with two cases in which $v_i = 1$.

Case 1: Assume that $v_i = 1$. Assume further that $v_{i+1} = 1$ or $i = n$ holds. Illustrated:

$$J = \begin{pmatrix} \cdots & * & 1 & 1 & *\ldots \\ \cdots & * & w_i & * & \ldots \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cdots & * & 1 \\ \cdots & * & w_i \end{pmatrix}.$$

Since $v_i = 1$, we have that $\mathfrak{b}_i = \mathfrak{b}_{i-1} + 1$ and hence $B_{\mathfrak{b}_i}$ is an extension of $B_{\mathfrak{b}_{i-1}}$ by a point p. We define A_i to be any flat extension of degree w_i of A_{i-1} by the base point $\psi^{-1}(p)$. By Lemma 4.1.3 we can assume that A_i is a hex triangulations of dimension $d - 3$ and $\Psi \circ \pi(|A_i|) = B_{\mathfrak{b}_i}$. By Lemma 0.4.6 we have that

$$\bar{g}(A_i) = (1,\, \mathfrak{b}_{i-1} + 1,\, \mathfrak{c}_{i-1} + w_i,\, \ldots) = (1,\, \mathfrak{b}_i,\, \mathfrak{c}_i,\, \ldots).$$

Let p' be the new point in A_i. We use Lemma 0.4.5 to create a new vertex $p'' \in \mathbb{R}^{d-2}$ and define T_{i-1} to be any high extension of T_{i-2} with base point p'' such that $\Psi \circ \pi(|T_{i-1}|) = A_i$. Illustrated:

$$J = \begin{pmatrix} \cdots & * & 1 & 1 & * \cdots \\ \cdots & * & w_i & * & \cdots \end{pmatrix}, \quad \mathfrak{S} = \begin{bmatrix} \cdots * & T_{i-2} & T_{i-1} & * \cdots \\ \cdots * & A_{i-1} & A_i & * \cdots \\ \cdots * & B_{\mathfrak{b}_{i-1}} & B_{\mathfrak{b}_i} & * \cdots \end{bmatrix}.$$

By Lemma 4.1.3 we can assume that T_{i-1} is a hex Delaunay triangulations of dimension $d-1$. In case that $i = 1$ the extension of $T_{1,-1}$ is the face complex of a simplex and

$$\bar{g}(T_{1,i}) = (1, 0, \ldots, 0).$$

For $i > 1$, Lemma 0.4.6 shows that

$$\bar{g}(T_{i-1}) = \bar{g}(T_{i-2}) + \mathcal{R}(\bar{g}(A_{i-1})) = \left(1, i-1, \sum_{j=1}^{i-1} \mathfrak{b}_j, \sum_{j=1}^{i-1} \mathfrak{c}_j, \ldots\right).$$

($\mathcal{R}$ denotes the index right shift operator.)

Case 2: Assume that $v_i = 1$ and $v_{i+1} = 0$. Let t be the maximal number such that $v_{i+1} = \cdots = v_t = 0$. Illustrated:

$$J = \begin{pmatrix} \cdots * & 1 & 0 & \cdots & 0 & * \cdots \\ \cdots * & w_i & w_{i+1} & \cdots & w_t & * \cdots \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cdots * & 1 & 0 & \cdots & 0 \\ \cdots * & w_i & w_{i+1} & \cdots & w_t \end{pmatrix}.$$

Then

$$B_{\mathfrak{b}_i} = \cdots = B_{\mathfrak{b}_t} = B_{\mathfrak{b}_{i-1}+1}$$

and hence $B_{\mathfrak{b}_t}$ is an extension of $B_{\mathfrak{b}_{i-1}}$ by a point p. We define A_t to be any flat extension of A_{i-1} of degree $w_i + \cdots + w_t$ with base point $\psi^{-1}(p)$. The new created point shall be $p' \in \mathbb{R}^{d-3}$. We see that A_t is a hex triangulation of dimension $d-3$ and

$$\bar{g}(A_t) = (1, \sum_{j=1}^{t} v_j + 1, \sum_{j=1}^{t} w_j + (w_i + \cdots + w_t), *) = (1, \mathfrak{b}_t, \mathfrak{c}_t, *).$$

Because the extension is flat, all vertices in the boundary of A_{i-1} also lie in the boundary of A_t. By using Lemma 0.4.5 we create a new point $p'' \in \mathbb{R}^{d-2}$ such that it could be used for a step extension T' of T_{i-2} and such that $\Psi \circ \pi(|T'|) = A_t$. This allows us to apply Lemma 4.1.9 to create a push in extension $(T_{i-1}, \ldots, T_{t-1})$ that has corresponding "visibility numbers"

$$(\lambda_1, \ldots, \lambda_{t-i}) = w_i, (w_i + w_{i+1}), \ldots, (w_i + \cdots + w_t).$$

The last base point shall be p'', then $\Psi \circ \pi(|T_{t-1}|) = A_t$. By Lemma 4.1.9 T_{t-1} is a $(d-1)$-dimensional hex Delaunay triangulation and we have:

$$\bar{g}_1(T_{t-1}) = \bar{g}_1(T_{i-2}) + (t - i + 1) = i - 2 + t - i + 1 = t - 1,$$

$$\bar{g}_2(T_i) = \bar{g}_2(T_{i-2}) + (t - i + 1)(\mathfrak{b}_{i-1} + 1) - 1$$

$$= \sum_{k=1}^{i-2} \mathfrak{b}_k + \mathfrak{b}_{i-1} + (t - i)\mathfrak{b}_i = \sum_{k=1}^{t-1} \mathfrak{b}_k,$$

$$\bar{g}_3(T_i) = \bar{g}_3(T_{i-2}) + (t - i + 1)\mathfrak{c}_{i-1} + \sum_{j=1}^{t-i} \lambda_j$$

$$= \sum_{k=1}^{i-2} \mathfrak{c}_k + \mathfrak{c}_{i-1} + \sum_{j=1}^{t-i}(\mathfrak{c}_{i-1} + \lambda_j) = \sum_{k=1}^{t-1} \mathfrak{c}_k.$$

Illustrated:

$$J = \begin{pmatrix} \cdots * & 1 & 0 & \ldots & 0 & * \cdots \\ \cdots * & w_i & w_{i+1} & \ldots & w_t & * \cdots \end{pmatrix},$$

$$\mathfrak{S} = \begin{bmatrix} \cdots * & T_{i-2} & T_{i-1} & \ldots & T_{t-1} & * \cdots \\ \cdots * & A_{i-1} & A_i & \ldots & A_t & * \cdots \\ \cdots * & B_{\mathfrak{b}_{i-1}} & B_{\mathfrak{b}_i} & \text{---} & B_{\mathfrak{b}_t} & * \cdots \end{bmatrix}.$$

The final step: To complete the construction, let T_n be any proper high extension of T_{n-1}. Illustrated:

$$\mathfrak{S} = \begin{bmatrix} \cdots * & T_{n-1} & T_n \\ \cdots * & A_n & \\ \cdots * & B_{\mathfrak{b}_n} & \end{bmatrix}.$$

Then T_n can be assumed to be a Delaunay triangulation and by Lemma 0.4.6 we get

$$\bar{g}(T_{1,n}) = \left(1, n, \sum_{i=1}^{n} \mathfrak{b}_i, \sum_{i=1}^{n} \mathfrak{c}_i, \ \ldots \right)$$

$$= \left(1, n, \sum_{i=1}^{n}\sum_{j=1}^{i} v_i, \sum_{i=1}^{n}\sum_{j=1}^{i} w_i, \ \ldots \right) = \left(1, n, \mathrm{eval}(v), \mathrm{eval}(w), \ \ldots \right).$$

$\square$

4.5 Discussion about the sufficiency part of the g-Theorem

The last sections invested quite some effort to construct inscribed polytopes with certain f-vectors. But why is this necessary? Billera and Lee already gave a construction for simplicial polytopes in the proof of the necessity part of the g-Theorem. Can this construction be modified to construct inscribed simplicial polytopes as well? We discuss this question and point out where the problems appear.

4.5.1 The proof by Billera and Lee

In 1980 Billera and Lee proved the sufficiency part [6] of the g-Theorem for simplicial polytopes. They first proved the existence of a simplicial sphere for a given g-vector. Then in a second part they realized the construction geometrically to get a polytope.

Theorem 4.5.1 (The sufficiency part of the g-theorem, Billera & Lee [6])**.** *Let $d > 1$ and $h = (1, h_1, \ldots, h_d)$ be an integer vector that satisfies*

- *$h_i = h_{d-i}$, for $i = 0, \ldots, d'$,*
- *$h_{i+1} \geq h_i$ for $i = 0, \ldots, d' - 1$, and*
- *$h_{i+1} - h_i \leq (h_i - h_{i-1})^{<i>}$ for $i = 0, \ldots, d' - 1$.*

Then h is the h-vector of a simplicial d-polytope.
(The corresponding complete g-vector $g = (g_0, \ldots, g_d)$ is given by $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i = 1, \ldots, d$.)

Lemma 4.5.2 (Simplicial $(d-1)$-spheres, Billera & Lee [6])**.** *Let $d > 1$ and $h = (1, h_1, \ldots, h_d)$ be an integer vector that satisfies*

- *$h_i = h_{d-i}$, for $i = 0, \ldots, d'$,*
- *$h_{i+1} \geq h_i$ for $i = 0, \ldots, d' - 1$, and*
- *$h_{i+1} - h_i \leq (h_i - h_{i-1})^{<i>}$ for $i = 0, \ldots, d' - 1$.*

Then h is the h-vector of a simplicial $(d-1)$-sphere.
(The corresponding complete g-vector $g = (g_0, \ldots, g_d)$ is given by $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i = 1, \ldots, d$.)

Sketching the proof of the lemma (full version: Billera & Lee [6]).
We aim to construct a $(d-1)$-sphere for any complete g-vector $g =$

$(g_0, \ldots, g_d)$ that satisfies the above given conditions. The first condition is also known as Dehn-Sommerville equations and it automatically holds for simplicial spheres. Hence it suffices to construct a $(d-1)$-sphere with g-vector $g = (1, g_1, \ldots, g_{d'})$.

Outline: Let $n = d + 1 + g_1$. We choose a set of facets $\mathfrak{B}$ of the cyclic polytope $C_{d+1}(n)$, such that the corresponding simplicial complex is a shellable simplicial d-ball and its boundary complex is a $(d-1)$-sphere with the desired g-vector.

- Let V be the vertex set of a cyclic polytope of type $C_{d+1}(n)$. As described in Gale's evenness criterion, we order its vertices $v_1, \ldots, v_n$ and define for each $F \subset V$ the binary vector $b_F \in \{0, 1\}^n$ by $b_{F,i} = 1 \Leftrightarrow v_i \in F$.
- Let $\mathcal{E}$ be the set of facets of $C_{d+1}(n)$ that contain v_1 and that satisfy, that the number $k \geq 0$, that shall be the minimum such that $v_{n-k}, \ldots, v_n$ are vertices of F, is even. (This is called *an even right end set*). The reverse lexicographic order (read from right to left) of the corresponding $0/1$-vectors induces an order on $\mathcal{E}$.
- We identify each $0/1$-vector of a facet of $\mathcal{E}$ with a monomial in the following way: We decompose all 1-entries (in case d is odd we excluding the first 1, which is at position 0) into consecutive pairs $p_1, \ldots, p_{d'}$ of 1-entries. Then we define a monomial in variables $X_0, X_1, X_2 \ldots$ that contain as many X_i as there are pairs that have exactly i 0-entries to its left, for $i = 1, 2, \ldots$. We define $X_0 := 1$

Example: $(1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1)$ corresponds to

$$1 \cdot X_2 \cdot X_3 \cdot X_3 = X_2 X_3^2.$$

and $(1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0)$ corresponds to

$$1 \cdot 1 \cdot 1 \cdot X_1 \cdot X_2 = X_1 X_2.$$

- Let $\Phi^{(n)}$ be the set of monomials of degree at most n. Then the above described identification $\alpha : \mathcal{E} \to \Phi^n$ is injective and it preserves a reverse lexicographic order,
  which can be introduced to $\Phi^{(n)}$ similar as we did for $\{0, 1\}^n$.
- We define a LOIM (reverse lexicographic order ideal of monomials) $M \subset \Phi^{(n)}$ by the given g-vector, such that we have exactly g_i monomials of degree i, for $i = 1, \ldots, d'$. It is shown that such a LOIM is unique and it exists if and only if $g = (g_0, \ldots, g_{d'})$ satisfies the requirements of the theorem.

Example: Let $d' = 3$, then $g = (1, 3, 4, 2)$ is associated with

$$1, \ x, y, z, \ xx, xy, xz, yy, \ xxx, xxy.$$

This is denoted in reverse lexicographic order by

$$1 < x < xx < xxx < y < xy < xxy < yy < z < xz < yz.$$

- We define the set of facets $\mathfrak{B} := \alpha^{-1}(M)$ and the corresponding simplicial complex Δ.
- The paper shows that the order, that is induced on $\mathfrak{B}$, is a shelling order. It also turns out that a facet whose corresponding monomial degree is p intersects exactly p previous facets in $(d-2)$-faces. Therefore $h(\Delta) = (g_1, \ldots, g_{d'}, 0 \ldots, 0) \in \mathbb{R}^{d+1}$ and thus Δ cannot contain interior faces of dimension smaller than $d - d'$.
- Since Δ is shellable, $\partial \Delta$ is a simplicial sphere.
- By the general formula $h(t) = (1-t)^d f(\frac{t}{1-t})$ and a short calculation we get

$$g_i = h_i(\Delta) = h_i(\partial\Delta) - h_{i-1}(\partial\Delta) = g_i(\partial\Delta), \quad \text{for } i = 0, \ldots, d'.$$

$\square$

After the combinatorial part was done the paper realizes the concept by a construction of a polytope. We will be very brief here since this part is very technical.

Sketching the proof of the theorem, full version: Billera & Lee [6].
Let $n = d + 1 + g_1$. For given $d > 1$ and n a cyclic $(d+1)$-polytope $C_{d+1}(n)$ is constructed, by carefully choosing its vertices as points with increasing parameters on the moment curve $c : \mathbb{R} \to \mathbb{R}^{d+1}$, starting by $v_1 = c(0) = 0$. The construction of this polytope depends on n and d but not on rest of g. Arbitrary close to v_1 a point z is defined that lies in a specific direction from 0. The vertex figure of z in $\text{conv}(\{z\} \cup C)$ then turns out to be a simplicial polytope and its g-vector is g. The facets of C that are visible from z correspond exactly to the set of faces $\mathfrak{B}$ that are chosen in the previous proof. The boundary of the complex Δ of the previous proof corresponds here to the boundary of the vertex figure of z. $\square$

4.5.2 Geometric interpretation

We derive some geometric aspects of the construction that is given above.

Let P be the combinatorial type of the vertex figure of z in $\text{conv}(\{z\} \cup C)$. By Gale's evenness criterion, we see that the vertex figure of w in C has the combinatorial type of a cyclic d-polytope of $n - 1$ vertices (just cut of the first entry of the binary vectors). Since w and z have almost the same position, the only difference between P and the vertex figure of v in C is that P has one additional vertex which corresponds to the edge zw.

This in particular shows that there is a cyclic d-polytope $\tilde{C}$ whose face hyperplane arrangement provides one cell for each g-vector of any simplicial d-polytope of n vertices. To create a simplicial polytope for a specific g-vector, we just have to take the convex hull of C with a vertex in the right cell.

Remark 4.5.3. The paper also notes that the proof can be modified such that C can have arbitrary more vertices (more than n). Using this modification and the fact that the realization of C is independent of g, we can follow that for any d and n there is a realization of $C_d(n)$ whose face hyperplane arrangement provides one cell for each g-vector of any simplicial d-polytope of *at most* n vertices.

4.5.3 Approach for inscribable polytopes

To adapt this concept to inscribable polytopes we need three conditions. First, a cyclic d-polytope of $n - 1$ vertices that is inscribed in the unit sphere. Second, its face hyperplane arrangement has to provides a cell that corresponds to a given g-vector, and third, the interior of this cell has to intersect the unit sphere.

The first part is easy since by Proposition 1.3.4 all cyclic polytopes are inscribable.

The second part is at least non trivial since the face hyperplane arrangement of a cyclic polytope type in dimension greater than two is not unique (we always assume that the face hyperplane arrangement lives in projective space). This holds even under the restriction that the vertices are placed on the moment curve as the following example shows.

Proposition 4.5.4 (The face hyperplane arrangement of a cyclic polytope is not unique)**.** *There are two cyclic 3-polytopes with 8 vertices, whose face hyperplane arrangements (in projective space) differ combinatorially.*
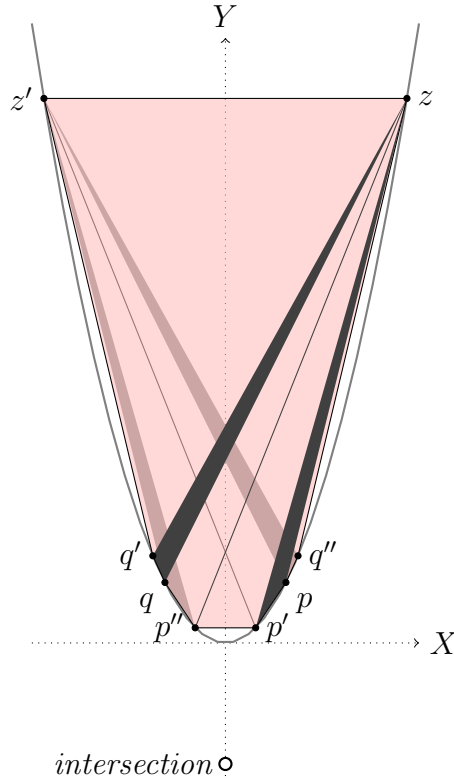
Figure 4.5: View from Z-axis: A cyclic 3-polytope whose supporting hyperplanes of the four gray facets meet in a single point at the Y-axis. A perturbation of p' changes the combinatorics of the face hyperplane arrangement.

Proof. Let c be the moment curve in $\mathbb{R}^3$,

$$c : \mathbb{R} \to \mathbb{R}^3, \qquad c(t) = (t, t^2, t^3).$$

We will construct eight points in symmetric position on $|c|$ such that the corresponding cyclic polytope has four face hyperplanes that meet in one point on the Y-axis. By perturbing one of the points, only one of the hyperplanes will be affected and leave the intersection, which implies the proposition.

We define

$$z := c(3) = (3, 9, 27), \quad p := c(1) = (1, 1, 1), \quad q := c(-1) = (-1, 1, -1).$$

See Figure 4.5. First we create two hyperplanes H_p and H_q that contain

z and that lie tangent to $|c|$ at p respectively q.

$$\frac{\partial c'}{\partial t}(1) = (1, 2, 3), \quad \frac{\partial c'}{\partial t}(-1) = (1, -2, 3),$$

$$v_p := z - p = (2, 8, 26), \quad n_p := (1, 2, 3) \times v_p = 16 \cdot (5, 1, -1)$$

$$H_p := \left\{ x \in \mathbb{R}^3 \,\middle|\, \langle (5, 1, -1), x \rangle = -3 \right\}$$

$$v_q := z - q = (4, 8, 28), \quad n_q := (1, -2, 3) \times v_q = 4 \cdot (7, -5, 1)$$

$$H_q := \left\{ x \in \mathbb{R}^3 \,\middle|\, \langle (7, -5, 1), x \rangle = 3 \right\}$$

It is easy to see that the intersection of H_p and H_q with the Y-axis is at -3 and $-\frac{3}{5}$. We define a point $q' := c(1 + \epsilon)$ with a sufficiently small $\epsilon > 0$ such that the hyperplane $H_q' := \mathrm{aff}(q, q', z)$ intersects the Y-axis strictly between 0 and -3. We define $p' := c(r)$ with an $-1 < r < 0$ such that $H_p' := \mathrm{aff}(p, p', z)$ intersects the Y-axis exactly where H_q' does. This is possible since the intersection value varies continuously from 3 to 0 as we move r from -1 to 0.

We define points z', q'', p'' that lie symmetric to z, q', p' in the sense that

$$z' := c(-3), \qquad q'' := c(-c^{-1}(q')), \qquad q'' := c(-c^{-1}(q')).$$

We now have eight points $z, p', p, q'', q', q, p'', z'$ that appear in this order on the moment curve. The cyclic polytope that is the convex hull of these points, has, amongst others the following face hyperplanes: H_p', H_q' as well as $H_p'' := \mathrm{aff}(p, q'', z')$ and $H_q'' := \mathrm{aff}(q, p'', z')$. By symmetry, they all meet at a common point on the Y-axis.

If we move p' a bit along $|c|$ only H_p' will be affected and leave the intersection point. Hence C with a perturbed vertex p' has a different face hyperplane arrangement than C, although both are cyclic 3-polytopes of eight vertices that lie on the moment curve. $\qquad\square$

Hence, even if it fells unlikely, there might be a g-vector and a corresponding dimension, such that no inscribed cyclic polytope provides the desired cell in its face hyperplane arrangement.

The third part is the hardest. Assume we have given an inscribed cyclic polytope $\tilde{C}$ and a specific cell of its face hyperplane arrangement that does not intersect the circumsphere. How can we change the position of the vertices, such that the cell does intersect the sphere? The first attempt would be to apply a Möbius transformation of the unit sphere to the vertices of $\tilde{C}$. Unfortunately, these Möbius transformation extend to

projective transformations of $\mathbb{R}\mathrm{P}^d$. Since the whole problem is projectively invariant such transformations do not change anything. The next idea would be to move the vertices of $\tilde{C}$ along the sphere until the desired cell intersects the circumsphere. This is hard, since the polytope has to remain a cyclic polytope and we do not even understand under what conditions the desired cell remains to be a part of the face hyperplane arrangement. This problem is in deed serious, since we know examples of inscribable polytope types where even a stacking at a specific facet creates a polytope of non inscribable type (see Theorem 2.2.1). This underlines that there might be g-vectors, for which this approach cannot succeed at all.

Conclusion: We see that modifying the proof of Billera and Lee to an inscribed version contains a lot of difficult problems. What about our approach? We see that the possibility of creating inscribed polytopes are far greater than the two constructions schemes we have shown in this thesis. In fact these schemes did not fail at some dimension, but they became unhandy because the result of each step depends on all previous steps. Maybe two additional dimensions can be proven with such type of schemes, but it will be a lot of work and it does not solve the general question.

We state the following conjecture:

Conjecture 4.5.5. *For any d and any simplicial d-polytope, there exists an inscribed simplicial d-polytope that has the same f-vector.*

5 Geometric composition of inscribable polytopes

In this chapter we discuss how a facet of an inscribed polytope can the changed, such that the result is still inscribed. It turns out that such a facet must at least allow a stacking with a vertex on the circumsphere.

In the first section we call this property inscribed stackable and discuss which polytopes provide inscribed stackable facets. In the next section we show that all inscribed stacked polytopes, and all polytopes that are made by high extensions have realizations that can be glued to inscribed stackable polytopes while inscribability is preserved. In the last part we discuss under what conditions two arbitrary inscribed polytopes can be glued facet to facet to form a new inscribed polytope.

5.1 Inscribed stackable

We discuss under what conditions an inscribed polytope possesses a facet that can be stacked, such that the resulting polytope is inscribed.

Definition 5.1.1 (Inscribed stackable)**.** Let $d > 1$ and let P be an inscribed d-polytope and F a simplicial facet of P. If there is a point v on the circumsphere of P that can be used to stack a facet F of P, then we call F and P *inscribed stackable*. The combinatorial type shall be *stack inscribable*. We call P *inscribed double stackable* if it has two distinct simplicial facets such that both facets can be stacked, one after the other, by points on the circumsphere of P. We call this combinatorial type *double stack inscribable*.

Remark 5.1.2 (Simple vertices)**.** A stacking on a simplicial facet will always create a simple vertex. By Proposition 2.2.5 we see that an inscribed simplicial polytope that contains a simple vertex that lies in two simplicial facets, is automatically inscribed double stackable.

There is an obvious criterion whether a facet of an inscribed polytope is inscribed stackable.

Proposition 5.1.3. *Let $d > 1$ and let F be a simplicial facet of an inscribed d-polytope. Let $H_1, \ldots, H_d$ be the supporting hyperplanes of the facets of P that intersect F in a ridge of P. Then F is inscribed stackable, if and only if one of the following conditions hold.*

- *The intersection of $H_1, \ldots, H_d$ is a point beneath F.*
- *The intersection of $H_1, \ldots, H_d$ is empty.*
- *The intersection of $H_1, \ldots, H_d$ is a point beyond F, but outside the unit sphere.*

This is a corollary of Definition 0.1.7 using the beneath beyond technique. Obviously, the mentioned intersection point changes under transformations of $\mathrm{PGL}(S^{d-1})$. For 3-polytopes, we found a condition that is invariant under such transformations. Before we present it, we need some definitions.

Definition 5.1.4 (The outside of a circumsphere in S^d)**.** Let $d > 1$. Let P be a d-polytope inscribed in the sphere S and let F be a facet of P. Then we call the intersection of the affine hull of F with S the *circumsphere of F*. We call the space beyond F intersected with S the *inside of this circumsphere* and the space beneath F intersected with S the *outside of this circumsphere*.
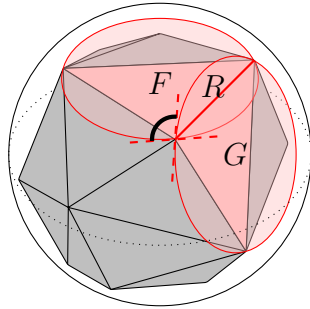

Figure 5.1: Inscribed intersection angle between facets F and G of an inscribed 3-polytope.

Definition 5.1.5 (Inscribed intersection angle)**.** Let $d > 2$ and let P be a d-polytope inscribed in the unit sphere S. Let F and G be two facets of P that intersect in a ridge R. Let S_F and S_G be the circumspheres of F and G in S. Then we call the supplementary angle of the outer intersection angles (measured in tangent space) of S_F with S_G the *inscribed intersection angle* of F and G respective the *intersection angle at R in P*. See Figure 5.1.

Remark 5.1.6. The *inscribed intersection angle* is of greater use than the angle between F and G, since it is invariant under Möbius transformations of the unit sphere. Hence it is also invariant under projective transformations that preserve S.

The same holds for the stereographic projection. We can calculate the inscribed intersection angle of an inscribed d-polytope by measuring the corresponding angle in the Delaunay triangulation that is derived from a stereographic projection of an inscribed polytope. We see that this angle is also well defined for Delaunay triangulations, and invariant under Möbius transformations.

Now we can state the inscribed stackability criterion.

Proposition 5.1.7 (Stackability criterion)**.** *Let P be a 3-polytope and F a simplicial facet of P. Let α, β and γ be the inscribed intersection angles of the three edges of F. Then F is inscribed stackable if and only if*

$$\alpha + \beta + \gamma > \pi.$$

By the stereographic projection, this corresponds to a criterion for a stellar subdivision of a triangle in a planar Delaunay triangulation.

Corollary 5.1.8. *Let P be a 3-polytope and F a simplicial facet of P. Let α', β', γ' be the outer intersection angles between the circumspheres of the three facets that share an edge with F. Then F is inscribed stackable if and only if*

$$\alpha' + \beta' + \gamma' < \pi.$$

Proof of the proposition. We will use a vertex projection from a vertex of F. This projection preserves angles and leads to four cases of configurations in $\mathbb{R}^2$. For each case we will check incidences and angles to proof the corollary.

Let v_1, v_2, v_3 be the vertices of F and let

$$F_1 = \operatorname{conv}(v_2, v_3, w_1), \quad F_2 = \operatorname{conv}(v_1, v_3, w_2), \quad F_3 = \operatorname{conv}(v_1, v_2, w_3).$$

be the neighboring facets of F in P. Let S_F, S_1, S_2, S_3 be the corresponding circumspheres in S. Let $X \subset S$ be the set of points where a stacking on F is possible (maybe empty). See Figure 5.2.

We investigate the image of the stereographic projection of P from v_1. For brevity we denote the image of an object in S by a prime sign $(.)'$. Since
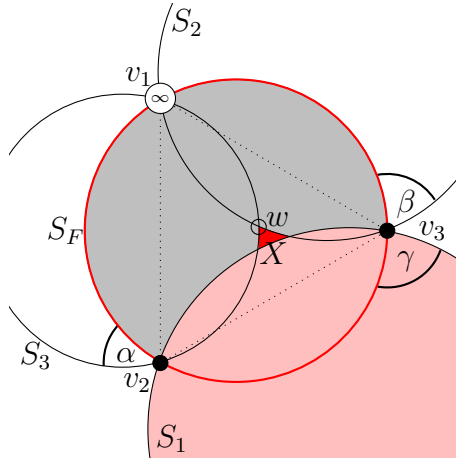
Figure 5.2: The local picture on the sphere.

the projection (from $S \setminus \{v_1\}$ to $\mathbb{R}^2$) is conformal, the intersection angles between the circumspheres are preserved. Since S_F, S_2 and S_3 contain v_1, their images are lines and outsides and insides of S_F, S_2, S_3 are mapped to a halfplanes. Since the outside of S_1 contains v_1, the interior of S_1 is mapped to a finite ball B and S_1' intersects the line S_F' in $\{v_2', v_3'\}$. Let $\{v_1, w\}$ define the intersection of S_2 with S_3. See four cases in Figure 5.3 and Figure 5.4.

Assume w does not lie inside S_F. See Figure 5.3. Then we see that the sum of α and β is larger or equal to π. In this case X' is the complement of a finite ball intersected with an unbounded set, so X is nonempty.
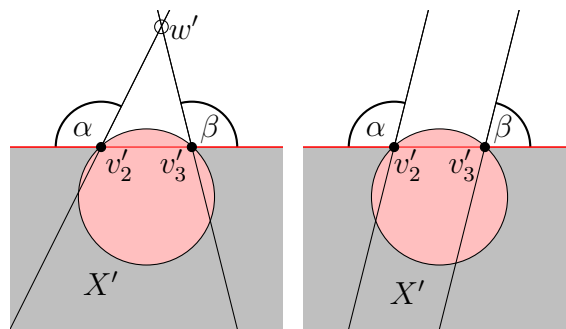

Figure 5.3: The projection from v_1. w lies outside S_F.

Assume w lies inside S_F. By the choice of w we can also assume that all intersections of the spheres S_1, S_2, S_3 lie either on or inside S_F. Then

we have only the two configurations left to check that are depicted in Figure 5.3. We see that X' is nonempty if and only if w' lies outside S_1'. We take a look at the triangle $(S_1' \cap S_2') \cup \{v_3'\}$. Its angle sum is π and two of its angles are α and γ. The third one is larger than β if and only if w' lies outside S_1'.
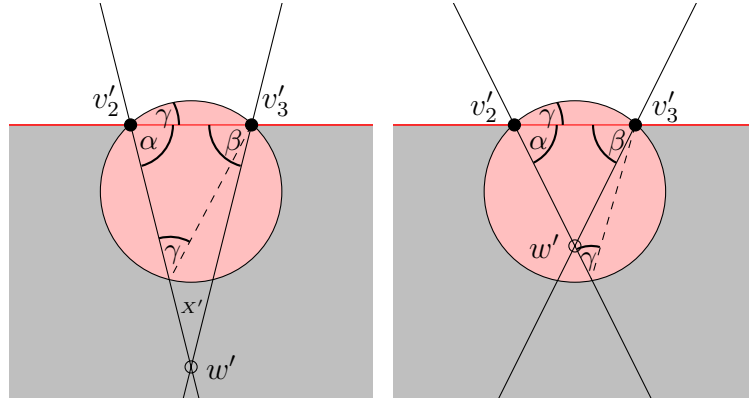

Figure 5.4: The projection from v_1. w lies inside S_F.

These cases show that the angle criterion holds in all possible cases. $\qquad\square$

The following Theorem is joint work with Karim Adiprasito.

Theorem 5.1.9 (Four stackable facets)**.** *Let P be an inscribed simplicial 3-polytope. Then P has at least four inscribed stackable facets and P is double inscribed stackable.*

Proof. We will investigate a vertex projection of P from a vertex v and show that three triangles can be stellar subdivided without ruining the Delaunay property. By referring to the choice of v we will get four stackable facets at P. Having four stackable facets will directly imply the double stackability.

Let v be a vertex of P and let T be the image of the vertex projection from v. By Proposition 0.3.13 we can assume that T is a Delaunay triangulation in $\mathbb{R}^2$. Let N be the set of n triangles in T that cannot be stellar subdivided without breaking the Delaunay condition. Let S be all other triangles in T, let s be their number.

Assume P has no simple vertex. Then each triangle of T can only share at most one edge of the boundary of T. In this case Figure 5.5 illustrates
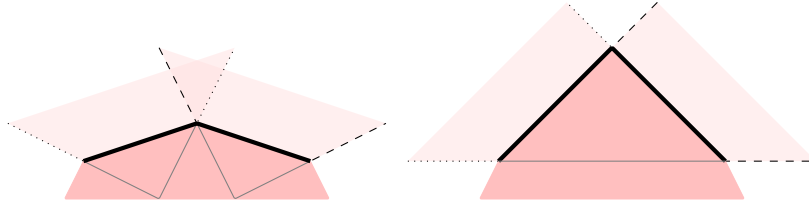
Figure 5.5: Two angle fans of angles that face neighboring boundary edges of a triangulation T. These fans must intersect outside the support of T except they intersect in the relative interior of T. This can only happen if they belong to the same triangle.

that the sum of all angles that face a boundary edge is at least 2π, because the corresponding angle fans cover the exterior of T completely. For any $\delta \in N$ Figure 5.6 and Lemma 0.2.4 illustrates that the angles of the neighboring triangles that face the edges of δ sum up to at least π. Let x be the sum of all angles that face a triangle in S. Then we compare the total number of angles $(n + s)\pi$ to what we have

$$(n + s)\pi \geq n\pi + 2\pi + x.$$

Since x is non negative, s must be a least 2. But in this case x cannot be zero, so we must have at least three stellar subdividable triangles. This argument still holds if we assume that one of the inscribed stackable facets already contained v. Hence we have at least four inscribed stackable facets.
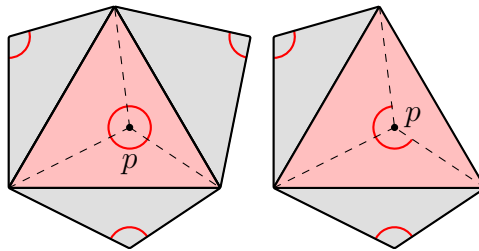
Figure 5.6: Left: The angle criterion implies that this configuration is Delaunay if and only if the three angles in the white triangle sum up to less than π. Right: The same holds for the two angles in the white triangles.

Assume P has a simple vertex w. Then, by Proposition 2.2.5, all three facets that contain w are inscribed stackable. Let T be the image of a vertex projection of P from w. If T has a triangle that contains two

boundary edges of T, then we know that this can be stellar subdivided since the corresponding facet also lies at a simple vertex. If there is no such triangle, we can even find three stellar subdividable facets by referring to the arguments above. Hence, in every inscribed simplicial 3-polytope, we have at least 4 facets that can be stacked such that the result is again an inscribed polytope.

About the double stacking: Stacking a facet only effects the stackability of facets that share a ridge with it. If we have four facets in $\mathbb{R}^3$, where each two share an edge, then P is a simplex. By Proposition 2.2.5 this is double inscribed stackable. If it is not a simplex, then two of the four facets are not adjacent and hence can both be stacked independent of one an other. □

In higher dimensions life is much harder as the following example shows. This is also joint work with Karim Adiprasito.

Proposition 5.1.10 (600 cell)**.** *The regular simplicial inscribed 4-polytope that is called the 600-cell is not inscribed stackable.*

This refers to the regular realization of the 600 cell, the combinatorial type might be stack inscribable.
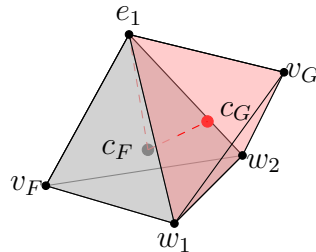

Figure 5.7: These are two 3-faces of the regular inscribed 4-polytope that is called the 600-cell. Both are regular simplices. The distance between the circumcenters c_F, c_G of both cells, measured in S^3, is shorter than the radius $r = \|c_F - e_1\|$ of both circumspheres.

Proof. It suffices to show that each point on S^3 lies inside at least two circumspheres of facets of the 600-cell. By the symmetry of the 600-cell, this can shown by proving the existence of one point inside the circumsphere of one facet F, that lies inside all circumspheres of facet that share a ridge with F. We will show that the normalized center of

mass c_F of the vertices of F is such a point. By symmetry, it suffices to show that there is one facet $G \neq F$ that shares a ridge with F whose circumsphere contains c_F. Note that the center of the circumsphere of G is the center of mass of the vertices of G. So it suffices to prove that the angle (viewed from $(0,0,0,0)$) between the two normalized center of masses c_F, c_G of F and G is smaller than the angle between c_F and a vertex of F.

It is well known that the following vectors, with any even permutations of their entries, and any signs in front of their entries, describe the set of vertices of a regular 600 cell P that is inscribed in the unit sphere.

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \frac{1}{2} \cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \qquad \frac{1}{2} \cdot \begin{pmatrix}\phi\\1\\\frac{1}{\phi}\\0\end{pmatrix}$$

The value $\phi := \frac{1+\sqrt{5}}{2}$ is known as the golden ratio. The following known identities will be useful:

$$\phi^2 = \phi + 1, \quad \tfrac{1}{\phi} = \phi - 1, \quad \tfrac{1}{\phi^2} = 2 - \phi$$

We pick five vertices

$$e_1 := \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, v_F := \begin{pmatrix}\phi\\1\\\frac{1}{\phi}\\0\end{pmatrix}, v_G := \begin{pmatrix}\phi\\-1\\\frac{1}{\phi}\\0\end{pmatrix}, w_1 := \begin{pmatrix}\phi\\0\\1\\\frac{1}{\phi}\end{pmatrix}, w_1 := \begin{pmatrix}\phi\\0\\1\\-\frac{1}{\phi}\end{pmatrix}.$$

It is known that two vertices share an edge in P if they enclose an angle of $\cos^{-1}(\frac{\phi}{2})$. We compute that

$$\langle e_1 , v_F \rangle = \langle e_1 , v_G \rangle = \langle e_1 , w_1 \rangle = \langle e_1 , w_2 \rangle = \frac{\phi}{2},$$

$$\langle v_F , w_1 \rangle = \langle v_F , w_2 \rangle = \langle v_G , w_1 \rangle = \langle v_G , w_2 \rangle = \frac{1}{4}(\phi^2 + \frac{1}{\phi}) = \frac{\phi}{2},$$

$$\langle w_1 , w_2 \rangle = \frac{1}{4}(\phi^2 - 1 + \frac{1}{\phi^2}) = \frac{\phi}{2}.$$

Since there are no 4-cliques in P that do not belong to a facet, this shows that the five vertices belong to two facets $F := \operatorname{conv}(v_F, e_1, w_1, w_2)$ and $G := \operatorname{conv}(v_G, e_1, w_1, w_2)$ of P which share a 2-face. See Figure 5.7.

We compute the normalized center of masses of F and G.

$$\tilde{c}_F := v_F + e_1 + w_1 + w_2 = (1 + 3\phi, 1, \tfrac{1}{\phi} + 2, 0),$$

$$\tilde{c}_G := v_G + e_1 + w_1 + w_2 = (1 + 3\phi, -1, \tfrac{1}{\phi} + 2, 0),$$

$$L := \|c_F\| = \|c_G\|, \quad c_F := \frac{\tilde{c}_F}{L}, \quad c_G := \frac{\tilde{c}_G}{L}.$$

Then we compare

$$
\begin{aligned}
angle(c_F, c_G) \;&<\; angle(c_F, e_1) \\
\Leftrightarrow \langle c_F, c_G \rangle \;&>\; \langle c_F, e_1 \rangle \\
\Leftrightarrow \langle \tilde{c}_F, \tilde{c}_G \rangle \;&>\; \langle \tilde{c}_F, e_1 \rangle L \\
\Leftrightarrow (1 + 3\phi)^2 - 1 + (\tfrac{1}{\phi} + 2)^2 \;&>\; (1 + 3\phi)\sqrt{(1 + 3\phi)^2 + 1 + (\tfrac{1}{\phi} + 2)^2} \\
\Leftrightarrow 11 + 18\phi \;&>\; (1 + 3\phi)\sqrt{13 + 18\phi} \\
\Leftrightarrow 40,12... \;&>\; 37.99...
\end{aligned}
$$

Which is true, so the normalized center of mass of each facet lie in all circumspheres of the neighboring facets. This shows that the 600-cell is not inscribed stackable. $\qquad\square$

5.2 Constructions to extend inscribed polytopes

We will show that all stacked polytopes, cyclic polytopes and all polytopes that are created by high extensions are double stack inscribable. Moreover, these polytopes have realizations for each inscribed stackable polytope such that both can be glued facet to facet to form an inscribed polytope.

Definition 5.2.1 (Gluing polytopes). Let P and Q be two polytopes of same dimension whose intersection is a facet F of both. Let the union $U = P \cup Q$ be a polytope that contains all proper faces of P and Q except F. Then we call U the *glued polytope of P and Q along F*.

Since all stacked polytopes have a simple vertex, Proposition 2.2.5 implies that they are double inscribed stackable.

Proposition 5.2.2 (Stack constructions at inscribed polytopes). *Let $d \geq 2$ and let P be an inscribed stackable polytope and let C be an inscribed stacked polytope. Then there is a realization of C, that can be glued to P such that the result is inscribed.*

Proof. Let F be the inscribed stackable facet of P. Short and simple: Reconstruct the combinatorics of C by a sequence of stackings starting at F. The more precise version: We project P from a vertex and realize a projection image of C inside the image of F.

Since C is inscribed stackable we can stack one of its facets G by a point on the cirumsphere of C. We call the image of the vertex projection from this point T_C. By Proposition 0.3.13 T_C is a Delaunay triangulation whose combinatorics equals the combinatorics of C, except that the interior of G is missing. The boundary of T_C is the image of the boundary of G. Since C was stacked, we see that T_C must be a simplex where a sequence of stellar subdivisions has been applied. By Theorem 2.2.3 we know that the dual tree of these stellar subdivisions is a binary tree.

We use a vertex projection of P from a vertex that does not lie in F. By Proposition 0.3.13 the image T_P is a Delaunay triangulation and the image of F is a $(d-1)$-simplex that can be stellar subdivided without destroying the Delaunay property. By that, Proposition 2.2.5 allows us to realize all subdivisions of T_C in the image of F without loosing the Delaunay property.

Reversing the projection of P leads to an inscribed polytope X whose boundary decomposes into two parts. The intersection of both parts is the boundary of G. One part is the boundary of P without the interior of F and the other part equals the combinatorics of the boundary of Q without the interior of G. Hence X is inscribable and it is a realization of C glued to P. $\qquad\square$

Definition 5.2.3 (High construction)**.** Let $d > 2$. We call a set of points $c = (c_i)_{i=1,\dots,n} \in \mathbb{R}^{(d-1)\times n}$ a *high construction* if it satisfies the following properties:

- The first $d-1$ vertices form a non vertical $(d-2)$-simplex δ. Which we call *the initial simplex.*
- The vertices are ordered by their height (last coordinate strictly increases).
- By starting with δ the vertices $c_d, \dots, c_n$ define a sequence of high extensions starting at δ.

Let P be a polytope that has a vertex projection that has the vertices of a high construction. Then we call P the *result of a high construction.*

Results of high constructions include step and stay constructions and G_3-constructions. They also include cyclic polytopes:

Remark 5.2.4. We see that the construction of a cyclic polytope as we constructed in the proof Proposition 1.3.4 is a high construction: The way to choose the position of the vertices was influenced by the moment curve and not raised straight up, however, the vertices increase in height and form a high extension. Hence cyclic polytopes are results of high constructions.

Theorem 5.2.5 (High constructions at inscribed polytopes)**.** *Let $d \geq 2$. Let P be an inscribed stackable d-polytope and C a d-polytope that is the result of a high construction c. Then C has a realization that can be glued to P such that the result is an inscribed polytope.*

Proof. Let F be an inscribed stackable facet of P. We will project P from a point beyond F on the circumsphere. The boundary of the resulting triangulation will be the boundary of a simplex. We will take this as initial part to start a high construction that is similar to c.

Since F is inscribed stackable we can stack it by a new point p on the circumsphere of P. Let T_P be the image of the vertex projection from p, which by Proposition 0.3.13 is a Delaunay triangulation. Its boundary complex is the boundary of a simplex Δ_P, so it is a hex Delaunay triangulation and the interior of T_P corresponds to the boundary of P without F.

Let $\Delta_d := \operatorname{conv}(c_1, \ldots, c_d)$ denote the initial simplex of c. By an affine transformation ϕ we transform Δ_c to Δ. By the freedom of the alignment of P we can assume that this transformation keeps the direction $(0, \ldots, 0, 1)$ invariant. Then $\phi(\Delta_c)$ is still a high extension triangulation with the same combinatorics as Δ_c, although it may not be Delaunay any more.

The goal now is to replace the initial simplex of $\phi(c)$ by Δ_P and repair the Delaunay property by adjusting the heights of $\phi(c_{d+1}), \ldots, \phi(c_n)$. We will call such a triangulation T. By Lemma 4.1.3 we can raise the height of $\phi(c_{d+1})$ until its extension of Δ_P is a Delaunay triangulation. We continue with this argument with $\phi(c_{d+2}), \ldots, \phi(c_n)$ until we have realized such a T.

Finally, we reverse the vertex projection from p and thereby create an inscribed simplicial polytope X from T. By construction the boundary complex of X decomposes into two simplicial complexes that intersect in the boundary of F. One part equals the boundary of P without F. and the other part corresponds to all faces of C except the one that

corresponds to the initial simplex. Hence X is inscribed and it is a realization of C glued to P. □

Proposition 5.2.6 (High constructions are double stack inscribable)**.**
Let $d > 2$. All d-polytopes that are the result of a high construction are double stack inscribable.

Proof. Let c be a high construction and T the triangulation it creates. We will stellar subdivide one $(d-1)$-face of T and also add a simplex to the boundary such that both operations corresponds to the stackings of the resulting polytope. To guarantee the Delaunay condition of the modification of T we have to adjust the heights of the vertices of c.

We start with the initial simplex $\text{conv}(c_1, \ldots, c_{d-1})$. We pick any point from its relative interior and lower its height by any amount. This point shall be x_1 and the simplex $\text{conv}(x_1, c_0, \ldots, c_{d-2})$ shall be δ. (It corresponds to the first stacking)

We then introduce the point c_d and raise its height (new vertex named c'_d) until the two simplices δ and $\delta_d := \text{conv}(c_0, \ldots, c_{d-2}, c'_d)$ form a Delaunay triangulation and the line segment (x_1, c'_d) intersects the initial simplex in an interior point. The last condition is possible since x_1 lies directly below the relative interior of the initial simplex. We call this triangulation T_0. Since T_0 is Delaunay, we can find a point in the relative interior of δ_d that lies outside the circumsphere of δ. There we introduce x_2 and use it for a stellar subdivision of T_0. (This will correspond to the second stacking) The result is a hex Delaunay triangulation that we call T_d.

We then iterate the following process for $i = d+1, \ldots, n$. The triangulation T_i is the extension of T_{i-1} by a vertex c_i that has a raised height. The height is sufficiently large such that T_i is Delaunay, and the line segment (x_1, c_i) intersects the initial simplex in its relative interior.

By Lemma 0.3.9 the polytope type P' that has T_n as a vertex projection differs from the one that has T by two additional stackings. Since T_d is hex Delaunay, P is inscribable. □

5.3 Gluing inscribed polytopes

We investigate under what conditions two inscribed polytopes can be glued facet to facet to form an inscribed polytope. We will see how

g-vectors are effected by a gluing operation and show the existence of an inscribed simplicial polytope with a non-unimodal f-vector.

Lemma 5.3.1 (g-vector of glued polytopes). *Let $d \geq 2$. Let U be a d-polytope that is glued from simplicial d-polytopes P and Q. Then*

$$g(U) = g(P) + g(Q) + (-1, 1, 0, 0, 0, \ldots, 0).$$

Proof. As we have seen in the first chapter, the g-vector is an affine function of the first half of the f-vector. For $i = 0, \ldots, d' - 1$ we have

$$f_i = \binom{d+1}{i} + \sum_{k=1}^{i+1} \binom{d-k+1}{d-i} g_k.$$

So for the restrictions $f' = (f_0, \ldots, f_{d'-1})$ and $g' = (g_1, \ldots, g_{d'})$ we can write

$$g'(P) = M(f'(P) - f'(\Delta_d)).$$

Where M is an invertible matrix, and $f'(\Delta_d)$ denotes the f'-vector of a d-simplex.

Given P, Q, U as in the lemma, we simply count that

$$f'(U) = f'(P) + f'(Q) - f'(\Delta_{d-1}).$$

So we get

$$
\begin{aligned}
& g'(U) \\
= \ & M(f'(U) - f'(\Delta_d)) \\
= \ & M(f'(P) + f'(Q) - f(\Delta_{d-1}) - f'(\Delta_d)) \\
= \ & M(f'(P) - f'(\Delta_d)) + M(f'(Q) - f'(\Delta_d)) + M(f'(\Delta_d) - f'(\Delta_{d-1})) \\
= \ & g'(P) + g'(Q) + M(f'(\Delta_d) - f'(\Delta_{d-1})).
\end{aligned}
$$

A closer look at the entries $i = 0, \ldots, d'$ shows

$$f_i'(\Delta_d) - f_i'(\Delta_{d-1}) = \binom{d+1}{i+1} - \binom{d}{i+1} = \binom{d}{i}.$$

On the other hand for a g'-vector $(1, 0, \ldots, 0)$ we exactly get

$$f_i = \sum_{k=1}^{i+1} \binom{d-k+1}{d-i} g_k = \binom{d}{i}.$$

So:

$$g(U) = (1, g'(U)) = (1, g'(P)) + (1, g'(Q)) + (-1, 1, 0, 0, \ldots, 0).$$

$\square$

This for example shows that stacking a simplicial polytope, which means gluing a simplex, changes the g-vector only by adding a 1 to the entry g_1.

Proposition 5.3.2 (Nonunimodality). *There are inscribed simplicial polytopes that have nonunimodal f-vector.*

Proof. As mentioned in the book [48], it suffices to create an inscribed simplicial d-polytope, that has the following g-vector

$$g_1 = n - d - 1 + r, \qquad g_i = \binom{n - d - 1}{i} \qquad \text{for } i = 2, \ldots, d'.$$

Where $d = 20$, $n = 169$ and $r = 4303045807457$. Then it turns out that $f_{11} > f_{12} < f_{13}$ (We are not showing the calculation).

If r would be zero, then the g-vector would be the one of a neighborly polytope. By referring to Lemma 5.3.1 we see that the desired g-vector can be obtained by stacking r-times on a neighborly polytope, for example on a cyclic polytope. Since all cyclic polytopes are stack inscribable, we can apply arbitrary many stacking operations such that the combinatorial type is still inscribed. See Proposition 5.2.6 and Proposition 5.2.2. $\qquad\square$

Before we start gluing inscribed polytopes, we consider transformations that transform inscribed polytopes to inscribed polytopes. The projective transformations that keep the unit sphere invariant seem to be the right choice. They keep the unit ball invariant, which implies that the image of an inscribed polytope is again a polytope of the same combinatorial type and it has all its vertices on the unit sphere, hence is an inscribed polytope.

Remark 5.3.3. What about other projective transformations? The projective image of the unit sphere is a non degenerate conic section. Hence our inscribed polytope has to have its vertices on the intersection of a non degenerate conic section with the unit sphere to allow such a transformation. This is very restrictive. It would be interesting to look at polytopes whose vertices lie on such an intersection and what the transformation would cause. A simpler question is which polytopes have several circumscribing ellipsoids. For lattice polytopes there is the study of perfect Delaunay polytopes which have only one circumscribing ellipsoid. See [19, 21]. We do not investigate that further and stick to projective transformations that preserve the unit sphere.

Lemma 5.3.4 (Matching facets). *Let $\{v_1, \ldots, v_3\}, \{w_1, \ldots, w_3\} \subset S^2$ be two sets of 3 points. Then there exist exactly two projective transformations ψ of RP^3 that keep S^2 invariant and $\psi(v_i) = w_i$ for $i = 1, \ldots, 3$. Only one preserves the orientation in S^2.*

For all $d > 0$ and $\{v_1, \ldots, v_4\}, \{w_1, \ldots, w_4\} \subset S^{d-1}$ being two sets of four points in general position, there is no projective map ψ of RP^d such that $\psi(S^{d-1}) = S^{d-1}$ and $\psi(v_i) = w_i$ for $i = 1, \ldots, 4$.

For us the term *general position* means, that the set of vertex vectors $(v_1, \ldots, v_4, w_1, \ldots, w_4)$ for which such a transformation exists, is of dimension less than $(d-1)^8$. The projective space RP^d is non orientable, so projective maps cannot be orientation preserving or reversing. Nevertheless S as a subset of RP^d is orientable. So projective maps that keep S invariant can be judged to be orientation preserving or reversing on S.

Proof. Because rotations are sphere preserving projective transformations, we can assume that $v_1 = w_1$. By a stereographic projection ψ from w_1, we transform our problem into finding a Möbius transformation $t : \hat{\mathbb{R}}^{d-1} \to \hat{\mathbb{R}}^{d-1}$ that satisfies

$$t(\infty) = \infty, \qquad t \circ \pi(v_2) = \pi(w_2), \qquad t \circ \pi(v_3) = \pi(w_3).$$

Note that the group of infinity fixing Möbius transformations is the group of similarities. For $d = 3$, there are exactly two similarities of $\mathbb{R}^2$ that transforms the oriented edge $(\pi(v_2), \pi(v_3))$ to the oriented edge $(\pi(w_2), \pi(w_3))$. One is orientation preserving. Hence there are two projective transformations of RP^d and one preserves the orientation of S. For $d > 3$, there are more transformations. In terms of t, they differ by rotation and reflection around the line $(\pi(w_2), \pi(w_3))$.

For four vertices and any $d > 2$, we see that we need to find a similarity that transform a triangle into an other triangle. Since two arbitrary triangles are usually not similar we only have such a map in special cases. $\qquad \square$

To glue polytopes, we also have to check that the union does not violate convexity. Note that a projective transformation cannot help here, because it maps convex sets to convex sets if no point of the set is mapped to infinity. Note that a projective transformation that keeps S invariant, also keeps the inside of S invariant.

Proposition 5.3.5 (Convexity criterion). *Let $d > 2$. Let P and Q be inscribed d-polytopes and let their intersection F be a facet of both. Let $R_1, \ldots, R_k$ be the $(d-2)$-faces of F and let $\alpha_1, \ldots, \alpha_k$ be the corresponding inscribed intersection angles in P and $\beta_1, \ldots, \beta_k$ the ones in Q. Then P can be glued to Q via F, if and only if*

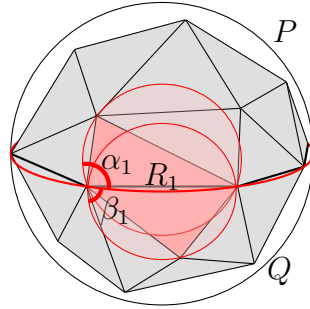$$\alpha_i + \beta_i > \pi \qquad for \quad i = 1, \ldots, k.$$

Figure 5.8: The inscribed 3-polytopes P and Q share a facet that lies at the equator. Both polytopes can be glued if the corresponding inscribed intersection angles at the common ridges sum up to more than π.

Proof. See Figure 5.8. Let F_i^P and F_i^Q be the facets of P and Q that are not F and share R_i. Since all inscribed intersection angles refer to the circumsphere of F (from different sides), we have for $i = 1, \ldots, d$ that $\gamma_i := (\alpha_i + \beta_i) - \pi$ is the inscribed intersection angle between F_i^P and F_i^Q. Since the circumspheres of F_i^P and F_i^Q must define supporting hyperplanes for the faces, we see that F_i^P and F_i^Q are locally convex if and only if $\gamma_i > 0$. Since all other ridges are already locally convex, the proposition follows. $\square$

Corollary 5.3.6 (Convexity criterion in $\mathbb{R}^3$). *Let P and Q be inscribed 3-polytopes and let F, G be simplicial facets of P, Q. Let $\alpha_1, \alpha_2, \alpha_3$ resp. $\beta_1, \beta_2, \beta_3$ be the inscribed intersection angles of the $(d-2)$-faces of F resp. G. Then there exists a projective transformation T such that $T(P) \cup Q$ is an inscribed simplicial polytope that has the combinatorics of P glued to Q via facets F, G, if and only if there is a permutation i_1, i_2, i_3 of $1, 2, 3$ such that*

$$\alpha_{i_1} + \beta_1 < \pi \quad and \quad \alpha_{i_2} + \beta_2 < \pi \quad and \quad \alpha_{i_3} + \beta_3 < \pi.$$

We have seen that the inscribed intersection angles can easily be calculated before we apply any unit sphere preserving projective transformation that would match the desired pair of faces. However, in higher dimensions it is unlikely that such a projective transformation exists. Here is an alternative that works in all dimensions, even though it has its price.

Proposition 5.3.7 (Bridge)**.** *Let $d \geq 2$. Let P and Q be inscribed d-polytopes with inscribed stackable facets F and G. Then there exists a projective transformation ϕ such that $\phi(P) \cup Q$ is an inscribed simplicial polytope, that contains all faces of $\phi(P)$ and Q except $\phi(F)$ and G.*

Of course the resulting polytope can have additional faces. This operation can be seen as gluing $T(P)$ and Q to opposite ends of a simplicial d-polytope of $2d$ vertices. The price of new faces that we have to pay for this *bridge* depends on the shape of faces $T(F)$ and G.
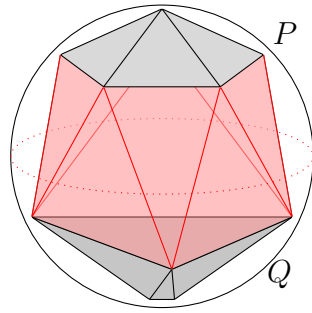

Figure 5.9: The inscribed 3-polytopes P and Q are pulled to opposite corners of the sphere. Then the convex hull is taken. This technique can be generalized to non simplicial polytopes. (P is not simplicial.)

Proof. See Figure 5.9. Since the unit sphere preserving projective transformations form a group, we can apply transformations to P and Q. We will do this in a way such that the equator of the unit sphere lies only beyond one face of the image of P and the same for Q on the other side. Then we take the convex hull.

Since P is inscribed stackable, there is an open set on S where a stacking point can be placed. We take any $(d-2)$-sphere inside this set and map the sphere by a unit sphere preserving projective transformation to the equator of S. We do the same for Q and assure, if necessary by rotation, that the image Q' of Q lies on the other side of the equator than the

image P' of P. The set conv$(P', Q', Equator)$ contains all faces of P' and Q' except the images of F and G. Hence the convex hull of P' and Q' also contains all faces of P' and Q' except the images of F and G.

If there is a non simplicial facet in between P' and Q' (for example because of symmetry), then we can rotate Q' around the north south axis until we reach a sufficient general position of the vertices. □

Notation

$A \dot\cup B$ disjoint union of A and B.

$\pi(\cdot)$ projection that cuts the last coordinate

$\phi(\cdot)$ stereographic projection

$\Psi(\cdot)$ vertex projection

$|T|$ support of a triangulation T

$[n]$ $\{1, \ldots, n\}$

d' $\lfloor \frac{d}{2} \rfloor$ in case that d is a dimension

$del_K(\cdot)$ deletion

$f(\cdot)$ f-vector

$\bar{f}(\cdot)$ outer f-vector

$g(\cdot)$ g-vector

$\bar{g}(\cdot)$ outer g-vector

$h^{(e)}(\cdot)$ general h-vector with respect to e

$\mathrm{M\ddot{o}b}(\mathbb{R}^d)$ group of Möbius transformations of $\hat{\mathbb{R}}^d$

$\mathrm{M\ddot{o}b}(S^d)$ group of Möbius transf. of $S^d \subset \hat{\mathbb{R}}^{d+1}$

$n^{<k>}$ k-th pseudo power of n

∂P boundary complex of P

$\mathrm{PGL}(\mathbb{R}^d)$ group of projective transformations of $\mathbb{R}^d$

$\mathrm{PGL}(S^d)$ group of proj. transf. of $S^d \subset \mathbb{R}\mathrm{P}^{d+1}$

$\hat{\mathbb{R}}^d$ one point compactification of $\mathbb{R}^d$

$\mathbb{R}\mathrm{P}^d$ projective space

S^{d-1} unit sphere

Bibliography

[1] D. W. BARNETTE, *The minimum number of vertices of a simple polytope*, Israel J. Math., 10 (1971), pp. 121–125.

[2] ——, *A proof of the lower bound conjecture for convex polytopes*, Pacific J. Math., 46 (1973), pp. 349–354.

[3] M. BERGER, *Geometry I, corrected edition*, Springer-Verlag, 2009.

[4] ——, *Geometry II, corrected edition*, Springer-Verlag, 2009.

[5] L. J. BILLERA AND C. W. LEE, *Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*, Bull. Amer. Math. Soc. (N.S.), 2 (1980), pp. 181–185.

[6] ——, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Combin. Theory Ser. A, 31 (1981), pp. 237–255.

[7] A. BJÖRNER, *A comparison theorem for f-vectors of simplicial polytopes*, Pure Appl. Math., 1 (2007), pp. 347–356.

[8] A. BOWYER, *Computing Dirichlet tessellations*, Comput. J., 24 (1981), pp. 162–166.

[9] K. Q. BROWN, *Voronoi diagrams from convex hulls*, Inf. Process. Lett., 9 (1979), pp. 223–228.

[10] M. BRÜCKNER, *Vielecke und Vielfache*, Teubner, 1900.

[11] J. A. DE LOERA, J. RAMBAU, AND F. P. SANTOS, *Triangulations*, Springer, 2010.

[12] M. DEHN, *Die Eulersche Formel im Zusammenhang mit dem Inhalt in der Nicht-Euklidischen Geometrie*, Math. Ann., 61 (1906), pp. 561–586.

[13] B. DELAUNAY, *Sur la sphère vide*, Bull. Acad. Sci. URSS, VII. Ser.,

Bibliography

(1934), pp. 793–800.

[14] R. Descartes, *Principia Philosophiae*, Ludovicus Elzevirius, Amsterdam, 1644.

[15] M. B. Dillencourt, *Realizability of Delaunay triangulations*, Inf. Process. Lett., 33 (1990), pp. 283–287.

[16] M. B. Dillencourt and W. D. Smith, *A linear-time algorithm for testing the inscribability of trivalent polyhedra*, Int. J. Comput. Geom. Appl., 5 (1995), pp. 21–36.

[17] ———, *Graph-theoretical conditions for inscribability and Delaunay realizability*, Discrete Math., 161 (1996), pp. 63–77.

[18] L. Dirichlet, *Über die Reduction der positiven quadratischen Formen mit drei unbestimmten Zahlen*, J. Reine Angew. Math., 1 (1850), pp. 209–227.

[19] M. Dutour, *The six-dimensional Delaunay polytopes*, European J. Combin., 25 (2004), pp. 535–548.

[20] H. Edelsbrunner, *Geometry and Topology for Mesh Generation*, Cambridge University Press, 2001.

[21] R. Erdahl, A. Ordine, and K. Rybnikov, *Perfect Delaunay polytopes and perfect quadratic functions on lattices*, in Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics, vol. 452 of Contemp. Math., Amer. Math. Soc., 2008, pp. 93–114.

[22] B. Gonska and G. M. Ziegler, *Inscribable stacked polytopes*, arXiv:1111.5322, to appear in Adv. Geom., (2011).

[23] B. Grünbaum, *Convex Polytopes, second edition*, vol. 221 of Graduate Texts in Math., Springer-Verlag, New York, 2003.

[24] B. Grünbaum and E. Jucovič, *On non-inscribable polytopes*, Czechoslovak Math. J., 24 (1974), pp. 424–429.

[25] C. D. Hodgson and I. Rivin, *A characterization of compact convex polyhedra in hyperbolic 3-space*, Invent. Math., 111 (1993), pp. 77–111.

[26] C. D. Hodgson, I. Rivin, and W. D. Smith, *A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere.*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 246–251.

[27] M. Joswig and T. Theobald, *Algorithmische Geometrie*, Vieweg-Verlag, 2008.

[28] E. Jucovič, *On noninscribable polyhedra*, Mat.-Fys. Časopis Sloven. Akad. Vied, 15 (1965), pp. 90–94.

[29] ——, *Bemerkungen zu einem Satz von Steinitz*, Elem. Math., 22 (1967), pp. 39–39.

[30] ——, *On polyhedral surfaces which are not inscribable in spherical shells*, in Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), vol. 260 of Colloq. Internat. CNRS, CNRS, Paris, 1978, pp. 251–253.

[31] E. Jucovič and S. Ševec, *Note on inscribability of quadrangular polyhedra with restricted number of edge-types*, J. Geom., 42 (1991), pp. 126–131.

[32] E. Jucovič, S. Ševec, and M. Trenkler, *Constructions of triangular and quadrangular polyhedra of inscribable type*, Math. Slovaca, 47 (1997), pp. 313–317.

[33] P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematika, 17 (1970), pp. 179–184.

[34] R. Möhring, M. Grötschel, and R. Borndörfer, *Service Design im öffentlichen Transport*, TU Berlin, Zuse-Institut Berlin (ZIB), DFG-Forschungszentrum Matheon. Research project.

[35] G. Pólya and G. Szegő, Problems and Theorems in Analysis, Vol. II,Second ed., Grundlehren Series Vol. 216, Springer-Verlag, 1976.

[36] I. Rivin, *On geometry of convex ideal polyhedra in hyperbolic 3-space*, Topology, 32 (1993), pp. 87–92.

[37] ——, *Intrinsic geometry of convex ideal polyhedra in hyperbolic 3-space*, in Analysis, algebra, and computers in mathematical research (Luleå, 1992), vol. 156 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1994, pp. 275–291.

[38] ——, *A characterization of ideal polyhedra in hyperbolic* 3-*space*, Ann. Math., 143 (1996), pp. 51–70.

[39] E. Schulte, *Analogues of Steinitz's theorem about noninscribable polytopes*, in Intuitive geometry (Siófok, 1985), vol. 48 of Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, 1987, pp. 503–516.

[40] R. Seidel, *Exact upper bounds for the number of faces in d-dimensional Voronoĭ diagrams*, in Applied geometry and discrete mathematics, vol. 4 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Amer. Math. Soc., Providence, RI, 1991, pp. 517–529.

[41] S. Ševec, *On the noninscribability of some families of polyhedra*, Math. Slovaca, 32 (1982), pp. 23–34.

[42] D. M. Y. Sommerville, *The relations connecting the angle-sums and volume of a polytope in space of n dimensions*, in Proceedings Royal Soc. London, vol. 115 of A, The Royal Society, 1927, pp. 103–119.

[43] R. P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math., 35 (1980), pp. 236–238.

[44] J. Steiner, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander*, Gesammelte Werke Reimer, Berlin (Original:Fincke, Berlin (1832)), 1 (1881), pp. 229–458. in the appendix question 77.

[45] E. Steinitz, *Über isoperimetrische Probleme bei konvexen Polyedern*, J. Reine Angew. Math., 158/159 (1927/1928), pp. 129–153/133–143.

[46] G. Voronoï, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. deuxième mémoire. recherches sur les parallélloèdres primitifs*, J. Reine Angew. Math., 134 (1908), pp. 198–287.

[47] D. F. Watson, *Computing the n-dimensional Delaunay tessellation with application to Voronoĭ polytopes*, Comput. J., 24 (1981), pp. 167–172.

[48] G. M. Ziegler, *Lectures on Polytopes, revised edition*, vol. 152 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.

[49] ——, *Face numbers of 4-polytopes and 3-spheres*, in Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), Beijing, 2002, Higher Ed. Press, pp. 625–634.

Zusammenfassung

In dieser Arbeit geht es um einschreibbare simpliziale Polytope und insbesondere um deren Seitenvektor (f-vector). Die Frage nach der Einschreibbarkeit von Polytopen geht zurück auf Steiner in 1832. Simpliziale Polytope und deren Seitenvektoren gehören zu den grundlegenden Strukturen der diskreten Geometrie.

Seit 1980 gibt es eine vollständige Charakterisierung (das g-Theorem) der Seiten-Vektoren von simplizialen Polytopen und seit 1992 eine vollständige Charakterisierung der einschreibbaren 3-Polytope. Wenig ist jedoch bekannt über den Seitenvektor von eingeschriebenen Polytopen und allgemein über höherdimensionale eingeschriebene Polytope.

Wir zeigen unter anderem folgende Resultate:

- Das g-Theorem gilt uneingeschränkt für simpliziale eingeschriebene Polytope bis mindestens Dimension sieben.
- Genauer: Zu jedem simplizialen Polytop gibt es ein eingeschriebenes simpliziales Polytop gleicher Dimension das gleich viele Ecken, Kanten und 2-Seiten besitzt.
- Zu jeder Dimension d und jeder natürlichen Zahl $k < d/2$ existiert ein simpliziales, eingeschriebenes Polytop, welches k-nachbarschaftlich, jedoch nicht $(k+1)$-nachbarschaftlich ist.
- Wir charakterisieren alle einschreibbaren Stapelpolytope zu beliebiger Dimension mit einem einfachen kombinatorischen Kriterium.
- Wir verschärfen eine obere Schranke für den Einschreibbareitsexponenten in höheren Dimensionen und widerlegen damit eine Vermutung von Grünbaum und Jucovič.
- Wir zeigen das alle simplizialen, eingeschriebenen 3-Polytope mindestens 4 Facetten haben die bestapelt werden können, sodass das Polytope danach eingeschrieben bleibt.

Da per stereographischer Projektionen eingeschriebenen Polytope mit Delaunay Zerlegungen identifiziert werden können, führen wir unsere Konstruktionen hauptsächlich auf Konstruktionen von Delaunay Triangulierungen zurück.