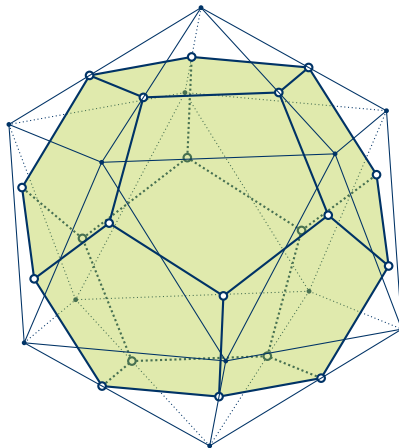


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# Optimization Methods in Discrete Geometry



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וראיתי תשוקתך לחכמות הלמודיות עצומה והנחתיך להתלמד בהם לדעתי מה אחריתך.

Umseitiges Zitat findet sich in den ersten Seiten des *Führer der Unschlüssigen* (דלאִלָּה אלהאירין) von Moses Maimonides (auch Rabbi Moshe ben Maimon מיימון בן משה oder kurz RaMBaM םרמב"ם) aus dem Jahre 1200. Wir zitieren aus der hebräischen Übersetzung, מורה נבוכים, des judäo-arabischen Originals von Samuel ben Jehuda ibn Tibbon aus dem Jahr 1204. Hier einige spätere Übersetzungen des Zitats:

*Tunc autem vidi vehementiam desiderii tui ad scientias disciplinales: et idcirco permisi ut exerceres anima tuam in illis secundum quod percepi de intellectu tuo perfecto.*

—Agostino Giustiniani, *Rabbi Mossei Aegyptii Dux seu Director dubitantium aut perplexorum*, 1520

*Und bemerkte ich auch, daß Dein Eifer für das mathematische Studium etwas zu weit ging, so ließ ich Dich dennoch fortfahren, weil ich wohl wußte, nach welchem Ziele Du strebst.*

—Raphael I. Fürstenthal, *Doctor Perplexorum von Rabbi Moses Maimonides*, 1839

*et, voyant que tu avais un grand amour pour les mathématiques, je te laissais libre de t'y exercer, sachant quel devait être ton avenir.*

—Salomon Munk, *Moise ben Maimoun, Dalalat al hairin, Les guide des égarés*, 1856

*Observing your great fondness for mathematics, I let you study them more deeply, for I felt sure of your ultimate success.*

—Michael Friedländer, *The guide for the perplexed by Moses Maimonides*, 1881

*Ich sah, daß Du Dich zu den mathematischen Wissenschaften sehr hingezogen fühltest, und überließ Dich ihrem Studium, da ich wohl wußte, wohin Du schließlich gelangen wirst.*

—Adolf Weiss, *Moses ben Maimon, Führer der Unschlüssigen*, 1923

ראיתי את גודל תשוקתך אל המתמטיקה. לכן הנחתי לך להתאמן בה מתוך שידעתי למה אתה עתיד להגיע.

—Michael Schwartz, *מורה נבוכים*, 2002





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# Summary

In this thesis we study some problems from *discrete geometry* and develop new methods for solving them. Many such problems can be formulated as optimization problems over spaces defined by systems of polynomial inequalities. Our method consists of two steps, which can be summarized as follows:

1. Model a problem from discrete geometry as a system of polynomial inequalities and solve it numerically.
2. From the numerical solution, derive an exact solution, which may provide additional structural information.

For any specific problem, each of these two steps needs to be adapted. We illustrate this approach in different applications ranging from the classification of polytopes to packing problems.

In the first chapter we address the question: *Is a given simplicial sphere the boundary of a polytope?* There are two basic challenges: if the sphere is polytopal, finding a realization; if it is not polytopal, proving that no realization exists. To address the first of these we provide a method for finding a realization of a simplicial sphere or an oriented matroid, if it is polytopal. We first solve a suitable system of inequalities numerically using non-convex optimization and then convert the solution to rational coordinates, such that it can be checked in exact arithmetic that we have found a realization of the simplicial sphere. This is a heuristic method that works well if the dimension is not too high. If the simplicial sphere is not polytopal, we adopt the known method of finding biquadratic final polynomials to prove non-polytopality. We consider *partial* chirotopes associated to a simplicial sphere and use this to find biquadratic final polynomials that prove non-realizability. In many cases this is a much faster approach than generating all compatible uniform oriented matroids associated to a simplicial sphere, i.e. generating all *complete* chirotopes, and then finding biquadratic final polynomials for each of them. On non-realizable simplicial spheres without biquadratic final polynomials our methods for proving non-realizability fail. However, we were able to combine both parts—finding

realizations and proving non-realizability—in order to completely classify some families of simplicial polytopes.

There are two major classification results presented in this chapter.

- Lutz provided a list of all simplicial 3-spheres with 10 vertices [Lut08]. Our method allows us to classify *all simplicial 4-polytopes with 10 vertices*: There are 162 004 combinatorial types. For 9 vertices, the corresponding classification was obtained by Altshuler, Bokowski and Steinberg in 1980 [ABS80].
- Hiroyuki and Padrol classified uniform oriented matroids of small rank with few elements [MP15]. We use their classification to obtain a complete classification of *simplicial neighborly polytopes* with 11 vertices in dimensions 4, 6 and 7, and other classification results. Also here, the last significant progress had been made more than 25 years ago. We give a survey of known results.

We also adapt our method to search for inscribed realizations, i.e. realizations with all vertices on the unit sphere. For all the simplicial neighborly polytopes that we classified, we did also find inscribed realizations.

The results from this chapter appeared as a preprint: [Fir15b].

The second chapter considers the following problem: *Given two polytopes  $P$  and  $Q$ , we are looking for a polytope  $P'$  of largest volume, such that  $P'$  is similar to  $P$  and  $P'$  is contained in  $Q$ .* We ask this question for all dimensions and are particularly interested in the 3-dimensional case. Croft [Cro80] has considered all 20 pairs of the 5 platonic solids; he obtained optimal inclusions for 14 cases. We offer a solution for the remaining 6 cases. While we obtain numerical solutions first, we then determine inclusions with algebraic numbers as coordinates.

The results from this chapter have been published as [Fir15a].

In the third chapter we look at two further problems. In a first part, we consider a question of W. Kuperberg concerning the *packing of non-intersecting unit cylinders all touching the unit sphere*. We investigate a generalization of this problem varying the radius of the cylinders. The most surprising result here is a configuration of 6 cylinders of radius larger than 1, all touching the unit sphere.

In a second part, we search for *straight line drawings of planar graphs with prescribed face areas*. Primarily, we are interested in 4-connected triangulations of the triangle with all equal areas. We investigate many examples with few vertices and formulate a conjecture based on these results.

# Chapter 1

## Realization of simplicial spheres and oriented matroids

### 1.0 Definitions: Polytopes and Matroids

We quickly review the basics of polytopes, oriented matroids and chirotopes. Our notations mostly coincide with those outlined in [BLVS<sup>+</sup>99], [RGZ04, Sect. 6], [BS89] and the introduction of [MP15].

**Definition 1** (face lattice, combinatorial equivalence, neighborly). The set of faces of a  $d$ -polytope  $P$ , partially ordered by inclusion, is called the *face lattice* of  $P$ . The number  $m$ -dimensional faces of a  $d$ -dimensional polytope  $P$  is denoted by  $f_m(P)$  or  $f_m$  and the vector  $(f_0, f_1, \dots, f_d)$  is the *f-vector* of  $P$ . A  $d$ -polytope is *simplicial* if all of its facets contain exactly  $d$  vertices. It is *simple* if each of its vertices is contained in exactly  $d$  facets. A polytope is *k-neighborly* if any set of  $k$  vertices is a face,  $f_{k-1} = \binom{f_0}{k}$ . A  $d$ -polytope is *neighborly* if it is  $k$ -neighborly for all  $k \leq \lfloor \frac{d}{2} \rfloor$ . Two polytopes are called *combinatorially equivalent* if they have isomorphic face lattices. A  $d$ -polytope is *inscribed* if all its vertices lie on the unit  $(d-1)$ -sphere, i.e. if  $\sum_{i=1}^d v_i^2 = 1$  for each vertex  $v = (v_1, \dots, v_d)$ . If  $P$  is combinatorially equivalent to an inscribed polytope it is *inscribable*.

**Definition 2** (simplicial complex, triangulation, simplicial sphere). A homeomorphism from the geometric realization of a simplicial complex  $\mathcal{C}$  to a topological space  $X$  is a *triangulation* of  $X$ . If  $X$  is a sphere, we call the triangulation *simplicial sphere*.

The boundary of a simplicial polytope gives rise to a simplicial sphere.

**Definition 3** (polytopal). A simplicial sphere  $|\mathcal{C}| \rightarrow S^{d-1}$  is *polytopal* if it arises from the boundary of a simplicial  $d$ -polytope  $P$ , i.e. if  $\mathcal{C}$  is isomorphic to the set of faces in the boundary of  $P$ .

**Definition 4** (prerequisites for covectors). Let  $E$  be a finite set. A *sign vector* is an element  $C \in \{-, 0, +\}^E$ . Given two sign vectors  $C$  and  $D$ , we define their composition as

$$(C \circ D)_e := \begin{cases} C_e & \text{if } C_e \neq 0 \\ D_e & \text{otherwise} \end{cases}$$

for  $e \in E$ . An element  $e \in E$  *separates*  $C$  and  $D$ , if  $0 \neq C_e = -D_e$ . The set of all elements which separate  $C$  and  $D$  is denoted by  $S(C, D)$ .

**Definition 5** (oriented matroid given by covectors). An *oriented matroid* is given by a finite set  $E$  together with a set  $\mathcal{L} \subset \{-, 0, +\}^E$  of *covectors*, which satisfies

- i)  $\mathbf{0} \in \mathcal{L}$ ,
- ii) if  $C \in \mathcal{L}$  then  $-C \in \mathcal{L}$ ,
- iii) if  $C$  and  $D \in \mathcal{L}$ , then  $C \circ D \in \mathcal{L}$ ,
- iv) if  $C, D \in \mathcal{L}$  and  $e \in S(C, D)$ ,  
then there is  $Z \in \mathcal{L}$  such that  $Z_e = 0$  and  $Z_f = (C \circ D)_f$  for all  $f \in E \setminus S(C, D)$ .

**Definition 6** (rank, uniform). The *rank* of the oriented matroid is defined as the rank of the underlying matroid. It is called *uniform* if the underlying matroid is uniform

**Definition 7** (acyclic, face lattice, matroid polytope). An oriented matroid  $\mathcal{M} = (E, \mathcal{L})$  is called *acyclic* if

$$(+, \dots, +) \in \mathcal{L}$$

The set of *faces* of  $\mathcal{M}$  is defined as

$$FL(\mathcal{M}) := \{C^0 \mid C \in \mathcal{L} \cap \{0, +\}^E\},$$

where  $C^0$  denotes the set of elements  $e \in E$  such that  $C_e = 0$ . The set of faces  $FL(\mathcal{M})$  is partially ordered by inclusion and is called the *face lattice* of  $\mathcal{M}$ . An acyclic oriented matroid is *k-neighborly* if any set of  $k$  elements of  $E$  is a face. An acyclic oriented matroid of rank  $r$  is *neighborly* if it is  $k$ -neighborly for all  $k \leq \lfloor \frac{r-1}{2} \rfloor$ . The acyclic oriented matroid  $\mathcal{M}$  is called a *matroid polytope* if for every  $e \in E$ , we have  $\{e\} \in FL(\mathcal{M})$ , that is if it is 1-neighborly.

The face lattice of a uniform matroid polytope induces a simplicial sphere.

**Definition 8** (chirotope). Let  $E$  be a finite set and  $r$  an integer. A *chirotope* of rank  $r$  is a map

$$\chi : E^r \rightarrow \{-1, 0, 1\}$$

such that

- i)  $\chi$  is alternating, i.e.  $\chi \circ \sigma = \text{sign}(\sigma)\chi$  for all permutations  $\sigma \in \Sigma_r$ .

ii) For all  $\lambda \in E^{r-2}$  and  $a, b, d, e \in E \setminus \lambda$  the set

$$\{\chi(\lambda, a, b)\chi(\lambda, c, d), -\chi(\lambda, a, c)\chi(\lambda, b, d), \chi(\lambda, a, d)\chi(\lambda, b, c)\}$$

is either equal to  $\{0\}$  or contains  $\{-1, 1\}$ .

iii) The set of elements of  $E^r$  that are not mapped to zero by  $\chi$  constitutes the basis elements of a matroid, and is in particular non-empty.

A chirotope gives rise to an oriented matroid and vice versa. If 0 is not in the image of  $\chi$ , we obtain a *uniform* oriented matroid.

**Definition 9** (oriented matroid and chirotope of a configuration of vectors). Given a configuration  $X$  of  $n$  vectors  $p_1, \dots, p_n \in \mathbb{R}^d$  that span  $\mathbb{R}^d$ , the *oriented matroid*  $\mathcal{M}_X := ([n], \mathcal{L}_X)$  of rank  $r := d + 1$  is given by

$$\mathcal{L}_X := \{(\text{sign}(q \cdot \bar{p}_1), \dots, \text{sign}(q \cdot \bar{p}_n)) \mid q \in \mathbb{R}^r\},$$

where

$$\bar{p}_m := \begin{pmatrix} p_m \\ 1 \end{pmatrix}.$$

The associated *chirotope*  $\chi_X$  of  $\mathcal{M}_X$  is the map:

$$\begin{aligned} \chi_X : \{1, \dots, n\}^r &\rightarrow \{-1, 0, 1\} \\ (m_1, \dots, m_r) &\mapsto \text{sign det}(\bar{p}_{m_1}, \dots, \bar{p}_{m_r}). \end{aligned}$$

If the points are in *general position*, i.e. if 0 is not in the image of  $\chi_X$ , we obtain a *uniform* oriented matroid.

The oriented matroid and chirotope of a configuration of vectors is indeed an oriented matroid and chirotope; in fact, the property of the former inspire the definition of the latter. If the point configuration is the set of vertices of a simplicial polytope  $P$ , we will obtain a matroid polytope  $\mathcal{M}$ , and the face lattice of  $P$  will be isomorphic to the face lattice of  $\mathcal{M}$ .

**Definition 10.** A chirotope (resp. oriented matroid) is *realizable* if it can be obtained as the chirotope (resp. oriented matroid) of a configuration of vectors.

## 1.1 Introduction

### 1.1.1 Previous results

The classification of polytopes has been a major goal in discrete geometry. Euclid's elements culminates in the last proposition of its last book in which Euclid remarks that there are precisely five regular polyhedra [Euc82, Liber XIII, Propositio 18]. The study of polytopes in higher dimensions was started in the middle of the 19th century, see for example the work by Schläfli

[Sch01], Wiener [Wie64], Stringham [Str80] and Schlegel [Sch86]. The enumeration of polytopes with a fixed number of faces emerged as a question, and Eberhardt [Ebe91] gave some answers, see Brückner [Brü00].

We are interested in the classification of polytopes up to combinatorial type. Two polytopes are combinatorially equivalent if they have isomorphic face lattices. Given an arbitrary lattice  $L$ , we can ask: is  $L$  polytopal, i.e. the face lattice of a polytope? Other than classifying *all* polytopes of dimension  $d$  with  $n$  vertices we will focus on the subfamilies of *simplicial* and *simplicial neighborly* polytopes. These families are of particular interest: a polytope is simplicial if all of its facets are simplices. If all the vertices of a polytope are in general position, then it is simplicial. It is simplicial neighborly if the number of its  $i$ -dimensional faces is maximized for all  $i$  among all polytopes with a fixed number of vertices by McMullen's Upper Bound Theorem [McM70]. The first known neighborly polytopes were the cyclic polytopes; Motzkin conjectured that the cyclic polytopes are the only combinatorial types of neighborly polytopes, which turned out to be false, see [Mot57] and [Gal63, p. 225 and §2]. In even dimensions all neighborly polytopes are simplicial, while in odd dimensions there are neighborly but non-simplicial polytopes. (For example, every 3-polytope is neighborly, but not every 3-polytope is simplicial.)

The classification of combinatorial types of  $d$ -polytopes on  $n$  vertices for  $n \leq d + 3$  was achieved by using Gale diagrams, see [Grü67, Sect. 6.1-3] and [Zie95, Sect. 6.5]. There are formulae for the number of combinatorial types:

- For  $n = d + 2$  vertices there are  $\lfloor d^2/4 \rfloor$  combinatorial types of polytopes,  $\lfloor d/2 \rfloor$  combinatorial types of simplicial polytopes and there is only one neighborly polytope: the cyclic polytope, see [Grü67, Sect. 6.1].
- For  $n = d + 3$  vertices an erroneous formula for the number of  $d$ -polytopes with  $n$  vertices was given by Lloyd [Llo70]; it has been corrected by Fusy [Fus06, Th. 1], see A114289. For simplicial  $d$ -polytopes with  $n$  vertices there is a formula by Perles [Grü67, Sect. 6.2, Th. 6.3.2, p. 113 and p. 424], see Bagchi and Datta, [BD98, Rem. 6 (C)], A000943. There is a formula for the number of neighborly and simplicial neighborly  $d$ -polytopes with  $d + 3$  vertices, see the work by McMullen and Altshuler [McM74], [AM73] and A007147. The number of simplicial neighborly  $(2n - 3)$ -polytopes with  $2n$  vertices coincides with the number of self-dual 2-colored necklaces with  $2n$  beads and it is possible to find a simple bijection between these two combinatorial objects. We explain this in Appendix A. Similarly, there is also a relation between the self-dual 2-colored necklaces on  $2n$  beads and simplicial  $(n - 3)$ -polytopes with  $n$  vertices. This is provided by Montellano-Ballesteros and Strausz [MBS04].

The most important result in the classification of polytopes of dimension 3 is Steinitz's theorem: the face lattices of (simplicial) 3-polytopes are in bijection with the 3-connected (cubic) planar graphs with at least 4 vertices, see [Ste22, Satz 43, p. 77]. The asymptotic behavior of the number of combinatorial types (simplicial) 3-polytopes with  $n$  vertices is known precisely, see [BW88] [Tut80] [RW82] and A000944, A00109.



$d \backslash n$	4	5	6	7	8	9	10	11	12
$d = 3$ all	1	2	7	34	257	2 606	32 300	440 564	6 384 634
simplicial	1	1	2	5	14	50	233	1 249	7 595
+neighborly	1	1	2	5	14	50	233	1 249	7 595
$d = 4$ all		1	4	31	1 294 [AS85]	?	?	?	?
simplicial		1	2	5	37 [GS67]	1 142 [ABS80]	<b>162 004</b>	?	?
neighborly		1	1	1	3 [GS67]	23 [AS73]	431 [Alt77]	<b>13 935</b>	$\geq 556\,061$ $\leq 556\,062$
$d = 5$ all			1	6	116	47 923 [FMM13]	?	?	?
simplicial			1	2	8	322 [FMM13]	?	?	?
+neighborly			1	1	2	126 [FS04]	<b>159 375</b>	?	?
$d = 6$ all				1	9	379	?	?	?
simplicial				1	3	18	?	?	?
neighborly				1	1	1	37 [BS87]	<b>42 099</b>	?
$d = 7$ all					1	12	1 133	?	?
simplicial					1	3	29	?	?
+neighborly					1	1	4	<b>35 993</b>	?
$d = 8$ all						1	16	3 210	?
simplicial						1	4	57	?
neighborly						1	1	1	2 586 [MP15]
$d = 9$ all							1	20	8 803
simplicial							1	4	96
+neighborly							1	1	5
$d = 10$ all								1	25
simplicial								1	5
neighborly								1	1
any $d$ , all	2	4	13	73	1 677	?	?	?	?
simplicial	2	3	6	14	64	1 537	?	?	?
+neighborly	2	3	5	9	22	203	<b>160 083</b>	?	?

Table 1.1: Families of  $d$ -polytopes with  $n$  vertices that have been enumerated. **Boldface** results are new. For  $n \leq d+3$  results are due to [Grü67] [AM73] [McM74] [Fus06]. For the row  $d = 3$ , see [BW88] [Tut80] [RW82].

For higher dimensional polytopes much less is known. In fact, there is no hope to find a criterion in terms of local conditions as in the 3-dimensional case, see [Stu87] and [Kal88]. For an overview of known realization algorithms see [BLVS<sup>+</sup>99, A.5, p. 486]. Already in dimension 4, deciding whether a given face lattice is the face lattice of a polytope is in fact complete for the *existential theory of the reals*, see Richter-Gebert and Ziegler [RGZ95]. This problem is known to be NP-hard, and also the problem of determining whether an orientable matroid is realizable is known to be NP-hard, see the results by Mněv and Shor [Mně88, Sho91], even if restricted to neighborly polytopes, by a result of Adiprasito, Padrol and Theran [APT14].

However, complete enumerations/classifications have been achieved for some pairs  $(d, n)$  with  $n \geq d + 4$  and  $d \geq 4$ . The first attempt was an enumeration of simplicial 4-polytopes with 8-vertices by Brückner [Brü09]. A mistake in his enumeration was fixed by the complete classification of this family by Grünbaum and Sreedharan [GS67], thereby also classifying neighborly 4-polytopes with 8-vertices. It also provided the first examples of non-cyclic neighborly polytopes. This was followed by some results by Altshuler, Bokowski and Steinberg [AS73] [Alt77] [ABS80] [AS85], all for 4-dimensional polytopes, until the classification of neighborly 6-polytopes with 10 vertices by Bokowski and Shemer [BS87]. After that, no significant progress in the classification of these families of polytopes had been made for a long time. Closely related to these classifications are the classifications of oriented matroids. A good summary of results in this area is [Fin01]. Very recently Fukuda, Miyata and Moriyama [FMM13] classified various families of oriented matroids and obtained classification of 5-polytopes with 9 vertices. Miyata and Padrol [MP15] classified neighborly 8-polytopes with 12 vertices.

Table 1.1 summarizes known and new enumeration results of families of  $d$ -polytopes on  $n$  vertices.

### 1.1.2 Our contributions

We propose a new algorithmic approach in order to give complete enumeration results for simplicial 3-spheres with 10 vertices and for various families of neighborly polytopes. We not only provide a complete description in rational coordinates, but also give realizations with all vertices on the unit sphere if possible, thereby proving inscribability for many polytopes. We then present two further applications: The classification of simplicial 3-spheres with small valence and a special realization of the Bokowski–Ewald–Kleinschmidt polytope. We hope that our results might be used as a treasure trove of examples and potential counterexamples for the study of polytopes. By polar duality our enumeration results for families of (inscribable) *simplicial* polytopes imply the results on corresponding families of (circumscribable) *simple* polytopes.

A simplicial polytope with vertices in general position yields a uniform matroid polytope by taking the induced oriented matroid. (Recall that every combinatorial type of simplicial polytope has a realization with its vertices in general position.) We denote this map from simplicial polytopes to uniform matroid polytopes by  $M$ . In turn, a uniform matroid polytope gives rise to a simplicial sphere, by taking the face lattice. We denote this map from uniform matroid polytopes to simplicial spheres by  $S$ . The composition  $S \circ M$  maps a simplicial polytope to the simplicial sphere induced by its boundary. A uniform matroid polytope is realizable if and only if it has a preimage under the map  $M$ . A simplicial sphere is polytopal if and only

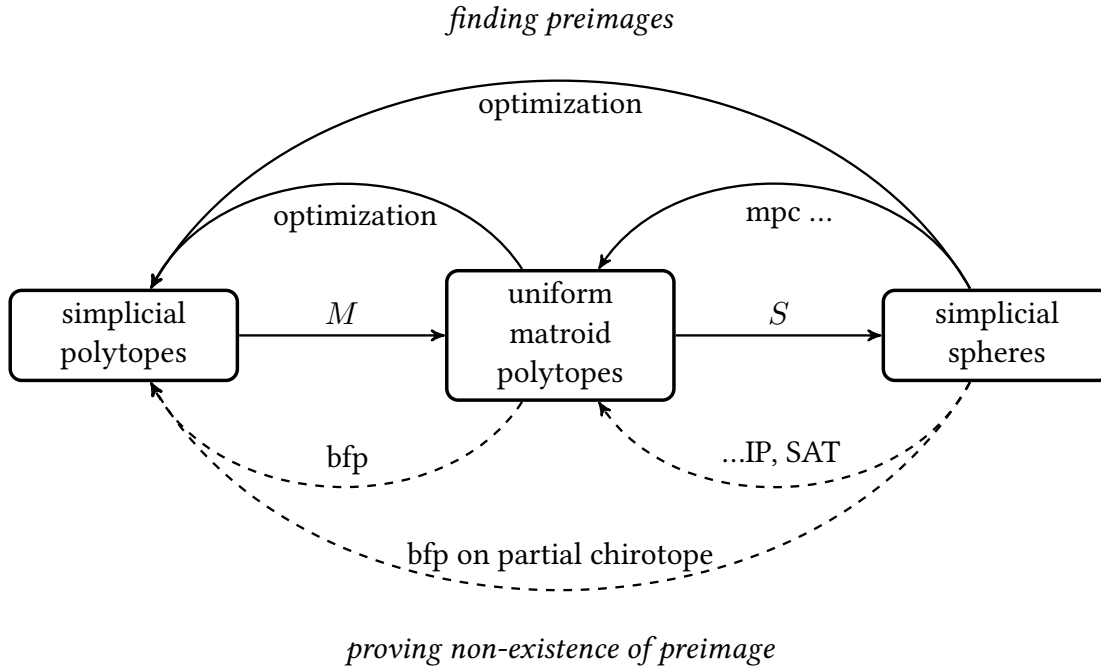


Figure 1.1: Simplicial polytopes, uniform matroid polytopes and simplicial spheres.

if it has a preimage under the map  $S \circ M$ , i.e. if it has a preimage under the map  $S$  that is a realizable uniform matroid polytope. For the map  $S$  there is a technique that provides definite results in both directions: to either prove the non-existence of preimages or to find preimages. This technique checks the consistency of the chirotope axioms, which are binary constraints, and is discussed in Section 1.2.3. For the maps  $M$  and  $S \circ M$  we use different methods for proving the non-existence of preimages and finding preimages under these maps:

**finding preimages:** Here we employ non-linear optimization tools in order to solve systems of non-linear inequalities. This is explained in Section 1.2.1. The realizations are first obtained numerically and then converted to rational realizations, such that the combinatorial type can be checked using exact arithmetic; see Section 1.2.2.

**proving the non-existence of preimages:** For proving non-realizability we rely on classical methods of finding final polynomials such as finding biquadratic final polynomials (bfp), see [BS89, Sect. 7.3, p. 121] and apply this also to partial chirotopes, see Section 1.2.3.

We combine these techniques with previous classification results: For the enumeration of some families of neighborly polytopes we build on the enumeration of corresponding families of neighborly uniform oriented matroids given by Miyata and Padrol [MP15] [Miy]. For the enumeration of simplicial 4-polytopes with 10 vertices and 4-polytopes with small valence we build on the enumeration of corresponding simplicial spheres by Lutz [Lut08] [FLS] [Lut].

The methods presented here can not only be used to realize a combinatorial type of simplicial polytope, but also the combinatorial type of a uniform matroid polytope or even a non-uniform oriented matroid. In the resulting point configurations additional methods would have to be used in order to obtain results in exact arithmetic and not only numerical results, since the methods presented in Section 1.2.2 would fail in this case. It is also possible to consider other (simplicial) manifolds than the sphere and obtain realizations. The optimization approach also allows for additional requirements on the objects being realized; we exemplify this with inscribability: In many cases and without much additional difficulty we could find realizations on the sphere, proving inscribability of the polytopes in question. For all the families of neighborly polytopes we enumerated, there was not a single non-inscribable case, which leads us to the belief that there might be none, see Conjecture 24. Since we are free to choose an objective function, we can also find extremal realizations. We don't need to focus on combinatorial type, but can consider other equivalence classes of polytopes. An example where we optimize over all polytopes similar to a given polytope can be found in [Fir15a].

## 1.2 Methods for finding realizations and proving non-realizability

### 1.2.1 Finding realizations and inscriptions as an optimization problem

Let  $\chi_{\mathcal{M}}$  be a chirotope of a uniform matroid polytope  $\mathcal{M}$  of rank  $r$  with  $n$  elements. To the chirotope we associate the following system of polynomial inequalities:

$$\chi_{\mathcal{M}}(m_1, \dots, m_r) \det(\bar{p}_{m_1}, \dots, \bar{p}_{m_r}) > 0 \text{ for all } m_1, \dots, m_r \in \binom{[n]}{r} \quad (1.1)$$

Here  $p_m := (p_{m,1}, \dots, p_{m,r-1})$  for  $1 \leq m \leq n$  are vectors of real variables, so there are  $nm$  many variables. Recall that

$$\bar{p}_m := \begin{pmatrix} p_m \\ 1 \end{pmatrix}.$$

The system is defined over  $\mathbb{R}[p_{m,i}]$  for  $1 \leq m \leq n$  and  $1 \leq i \leq r-1$  with  $\binom{n}{r}$  homogeneous inequalities of degree  $r$ .

**Proposition 11.** *A uniform matroid polytope  $\mathcal{M}$  is realizable if and only if the system (1.1) has a solution.*

Inequality (1.1) implies

$$\chi(m_1, \dots, m_r) = \text{sign} \det(\bar{p}_{m_1}, \dots, \bar{p}_{m_r}),$$

which is just what is needed in the definition of chirotope of a configurations of vectors.

In addition, we could ask for all vertices to lie on the unit sphere:

$$\sum_{i=1}^{r-1} p_{m,i}^2 = 1 \text{ for } 1 \leq m \leq n. \quad (1.2)$$

**Proposition 12.** *The uniform matroid polytope  $\mathcal{M}$  is realizable as an inscribed polytope if and only if the system (1.1) and (1.2) has a solution.*

We can weaken (1.1) and only consider inequalities that concern faces of  $\mathcal{M}$ :

$$\begin{aligned} \chi_{\mathcal{M}}(m_1, \dots, m_r) \det(\bar{p}_{m_1}, \dots, \bar{p}_{m_r}) > 0 \text{ for all } m_1, \dots, m_r \in \binom{[n]}{r} \\ \text{if for some } j, (m_1, \dots, \widehat{m}_j, \dots, m_r) \text{ is a face of } M \end{aligned} \quad (1.3)$$

**Proposition 13.** *The face lattice of  $\mathcal{M}$  is polytopal if and only if the system (1.3) has a solution.*

**Proposition 14.** *The face lattice of  $\mathcal{M}$  is the face lattice of an inscribable polytope if and only if the system (1.3) and (1.2) has a solution.*

In the last two propositions only the partial information of the chirotope is required, namely the orientation of the simplices that contain a face. It is straightforward to generate this partial information when given a simplicial complex.

In order to solve such systems of inequalities and equations, we need a solver for non-linear programs. For our computations we have used SCIP, which uses branch and bound techniques and linear underestimation in order to find a feasible solution within a certain precision; see [Ach09] and [ABKW08] for details.

For numerical reasons, the solver cannot handle strict inequalities, which is why we adapt inequalities (1.1) by replacing “ $> 0$ ” by “ $\geq \varepsilon$ ” for some small positive  $\varepsilon$ . If we simply replace the strict inequalities by weak inequalities or if  $\varepsilon$  is very close to machine precision, we will obtain trivial solutions. It is important to choose  $\varepsilon$  adequately. We can be certain that for our results no feasible solutions have been discarded, since we prove the infeasibility of the system using different techniques, see Section 1.2.3.

In all cases discussed, a numerical solution with a certain precision can be turned into a rational solution of the system in question, see Section 1.2.2. Once we have a solution with rational coordinates we can prove the validity of the systems of (in)equalities (1.1), (1.2) and (1.3) by calculation in exact arithmetic.

This procedure works reasonably well in practice for *finding* realizations if they exist. If there is no realization and the system of inequalities and equations is therefore infeasible, the optimizer does not terminate in a reasonable amount of time or runs out of memory. See Section 1.2.3 on how to handle potentially non-realizable cases.

### 1.2.2 Finding rational points on the sphere

From the calculations described in the previous section, we obtain *numerical* solutions of the system of inequalities and equations, which are not guaranteed to be correct. The goal is to derive *rational* points from these solutions that satisfy the system of inequalities and equations in exact arithmetic. This is in particular easy if we investigate the realization of *uniform* matroids that are realized as *simplicial* polytopes, whose combinatorial type is unchanged by a small distortion. When we look at inscribed realizations, we start with a point  $x$  given numerically that is very close to the unit sphere, and we are looking for a point  $x'$  with rational

coordinates on the sphere very close to  $x$ . Since the rational points are dense in the unit sphere, the existence of a good rational approximation is guaranteed. Consider a rational line through the sphere that intersects the unit sphere in two points. If one of the points is rational, let's say it is the north pole, then the other intersection point will also be rational. We notice that stereographic projection and its inverse send rational points to rational points and are continuous away from the projection point.

This enables us to find a suitable rational point constructively as follows:

*Construction 15.*

**Step 1** Use stereographic projection to map  $x \in S^d$  to a point  $\tilde{x} \in \mathbb{R}^{d-1}$ .

**Step 2** Find a suitable rational approximation  $\tilde{x}'$  for  $\tilde{x}$ .

**Step 3** Use the inverse stereographic projection to map  $\tilde{x}'$  to a rational point  $x'$  on the sphere.

### 1.2.3 Certificates for non-realizability

#### Biquadratic final polynomials

An oriented matroid is non-realizable if and only if it has a final polynomial, see [BS89]. Given a (uniform) oriented matroid, there is a good algorithm for *showing non-realizability*, which finds *biquadratic final polynomials* that prove non-realizability. This is described by Bokowski, Richter and Sturmfels [BR90], [BS89, Sect. 7.3, p. 121]. There are cases of non-realizable oriented matroids that do not possess a *biquadratic* final polynomial, but do possess a final polynomial (and hence are non-realizable); the first one is given by Richter-Gebert [RG96]. We might have found another such instance, see Theorem 16 iv).

#### From simplicial spheres to uniform oriented matroids

Showing that a simplicial sphere is not realizable is done in two steps:

- i) generate all compatible uniform matroid polytopes (possibly there aren't any!).
- ii) find final polynomials for all of them.

Given a simplicial sphere  $S$ , the values of a compatible chirotope  $\chi$  on tuples that contain a face of  $S$  are already determined, if we fix the sign of one of those tuples. (We can always flip all the signs of a chirotope and obtain a valid chirotope again.) All compatible chirotopes are precisely the ones that satisfy the conditions on the signs derived from the Graßmann-Plücker identities, see condition ii) in Definition 8. These can be formulated as a Boolean satisfiability problem (SAT), compare the work by Schewe [Sch10], and has been implemented by David Bremner, see [BBG09, Sect. 3]. It can also be formulated as an integer program, which has been done by the author. Then an exact solver for integer programs can be used to generate all compatible matroids. It might of course be the case that the system has no solution, which means that the simplicial sphere has no compatible uniform matroid polytopes. In the case of odd-dimensional neighborly simplicial spheres, there is at most one solution. This property is

called rigidity and has been established for polytopes of even dimension in [She82] and, more generally, for neighborly oriented matroids of odd rank in [Stu88].

### Using partial chirotopes

In some cases there will be many compatible chirotopes and there might be too many to find (biquadratic) final polynomials for all of them. Sometimes, however, we are still able to prove non-realizability by using only partial information of the chirotopes. As mentioned above, all compatible chirotopes for a given simplicial sphere have their values on tuples that contain a face of  $S$  in common, if we fix the sign of one of those tuples. In general those chirotopes will have values on even more tuples in common, and these values can be determined by examining the conditions on the signs derived from the Graßmann-Plücker identities. We call this the *partial chirotope* compatible with  $S$ . The question whether a partial chirotope can be completed is NP-complete, see [Tsc01] and [Bai05]. In many cases with few vertices the following approach works reasonably well. The method of finding biquadratic final polynomials by Bokowski and Richter [BR90], consists of setting up a linear program that encodes the 3-term Graßmann-Plücker relations using the signs of the chirotopes. If the program is infeasible, then a biquadratic final polynomial exists. If the complete chirotope is not known, but only the partial chirotope, we can still set up the linear program, only with less constraints. The infeasibility of this program will still prove the existence of a biquadratic final polynomial.

To summarize, another method for proving that a simplicial sphere is not realizable, without generating all compatible matroid polytopes, is the following:

- i) find a partial chirotope (if there are any compatible uniform matroid polytopes)
- ii) find biquadratic final polynomials for it.

### 1.2.4 Computations and hardware

For the calculations of the results presented in Section 1.3 the systems of (in)equalities from the simplicial complex and face lattices that are passed to SCIP were set up with the computer algebra system Sage [S<sup>+</sup>14]. The smaller cases were done on a desktop PC, with 8GB of RAM, the larger cases ran on a cluster on about 300 Xeon CPUs with about 3GB RAM each. The time needed for an individual realization varied depending on the dimension and number of points between less than a second and several minutes. In the largest cases the size of the system of (in)equalities passed to SCIP were several hundred megabytes. Sage was also used to verify the solution in exact arithmetic and to prove non-realizability by an implementation of the biquadratic final polynomial method which runs on the partial chirotope. For the LP that has to be solved for the biquadratic final polynomial method we use the solver GLPK and the exact solver ppl from within Sage.

## 1.3 Results

Data for all our results are available at the author's web page:

<http://page.mi.fu-berlin.de/moritz/>

### 1.3.1 Realizations and inscriptions of neighborly polytopes

Miyata and Padrol [MP15] enumerate simplicial neighborly uniform oriented matroids of various ranks and number of elements. This allows us to apply the methods from Section 1.2 in order to find realizations of those neighborly uniform oriented matroids. A sewing method of Padrol [Pad13] provides many neighborly polytopes, which are also inscribable, see [GP15], and include all simplicial neighborly  $d$ -polytopes with up to  $d + 3$  vertices. We present results on realizability and inscribability for neighborly simplicial  $d$ -polytopes with  $n$  vertices for  $d = 4, 5, 6$  and  $7$  and  $n > d + 3$ .

#### Neighborly 4-polytopes

The neighborly 4-polytopes given by Padrol's sewing construction include the 3 combinatorial types of neighborly 4-polytopes with 8 vertices, described in [GS67]. The number combinatorial types of neighborly 4-polytopes with  $n$  vertices was previously only known for  $n \leq 10$ , compare A133338. The number of combinatorial types of neighborly 4-polytopes with 9 vertices was determined by Altshuler and Steinberg [AS73], for 10 vertices it was determined by Altshuler [Alt77].

**Theorem 16.**

- i) All 23 distinct combinatorial types of neighborly 4-polytopes with 9 vertices are inscribable.
- ii) All 431 distinct combinatorial types of neighborly 4-polytopes with 10 vertices are inscribable.
- iii) There are precisely 13 935 distinct combinatorial types of neighborly 4-polytopes with 11 vertices. All of these are inscribable.
- iv) The number of distinct combinatorial types of neighborly 4-polytopes with 12 vertices is 556 061 or 556 062, and at least 556 061 of those are inscribable.

*Proof.*

- i)-ii) We provide rational inscribed realizations for all known combinatorial types.
- iii) Out of the 13 937 combinatorial types of neighborly oriented matroids, 2 admit a biquadratic final polynomial, see [MP15, Sect. 4.1.1]. For the remaining combinatorial types we provide rational inscribed realizations
- iv) We analyzed the 556 144 combinatorial types of neighborly oriented matroids given by Miyata and Padrol [MP15, p. 3]: Using methods from Section 1.2.1 we realized all but 83 cases. On those we ran the biquadratic final polynomial method and obtained certificates for non-realizability in 82 cases. The only case left is #374225, which has the following facet list:



$$F374225 = \{[0148] [01410] [0189] [01910] [0236] [02311] [0249] [02411] [0267] [0279] [0367] [0379] [03911] [0458] [0459] [041011] [0589] [091011] [1234] [1238] [1245] [1258] [13411] [13811] [1458] [141011] [1679] [16710] [1689] [16810] [17910] [181011] [23411] [2356] [23510] [23810] [2456] [2467] [2479] [25810] [35611] [351011] [36711] [37911] [381011] [4567] [4579] [5678] [56810] [561011] [5789] [6789] [671011] [791011]\}$$

It remains to decide realizability in this case. If one can find a final polynomial, then this would be an example without *biquadratic* final polynomial with a smaller number of vertices than the example provided by Richter-Gebert [RG96], which has 14 elements and is of rank 3.

□

### Simplicial neighborly 5-polytopes

The number of combinatorial types of simplicial neighborly 5-polytopes with 9 vertices was determined by Finbow-Singh [FS04],[Fin14] and also in [FMM13].

#### Theorem 17.

- i) All 126 distinct combinatorial types of simplicial neighborly 5-polytopes with 9 vertices are inscribable.
- ii) There are precisely 159 375 distinct combinatorial types of simplicial neighborly 5-polytopes with 10 vertices. All of these are inscribable.

*Proof.*

- i) We provide rational inscribed realizations for all known combinatorial types.
- ii) Miyata and Padrol give 159 750 neighborly uniform oriented matroids of rank 6 on 10 elements, one for each combinatorial type of face lattice. We realize 159 375 of these face lattices, while not paying attention to realizing the specific matroid, and show that they are all inscribable. We use partial information of the chirotope coming from the faces, together with the biquadratic final polynomial method to find certificates for non-realizability for an additional 189 face lattices. For the remaining 186 cases, in addition to the partial information coming from the faces, we use the information coming from Graßmann–Plücker relations. This allows us to determine sufficiently many signs of the chirotope in order to obtain biquadratic final polynomials, see Section 1.2.3.

□

### Neighborly 6-polytopes

The number of combinatorial types of simplicial neighborly 6-polytopes with 10 vertices was determined by Bokowski and Shemer [BS87].

**Theorem 18.**

- i) All 37 distinct combinatorial types of neighborly 6-polytopes with 10 vertices are inscribable.
- ii) There are precisely 42 099 distinct combinatorial types of neighborly 6-polytopes with 11 vertices. All of these are inscribable.
- iii) There are precisely 4 523 simplicial 2-neighborly 6-polytopes with 10 vertices. All of these are inscribable.

*Proof.* Notice that i) is included in iii). We provide rational inscribed realizations for all combinatorial types.  $\square$

	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$	$n = 11$	$n = 12$
$d = 4$	1	1	1	3	<u>23</u>	<u>431</u>	<b>13 935</b>	$\geq$ <b>556 061</b> $\leq$ <b>556 062</b>
$d = 5$		1	1	2	<u>126</u>	<b>159 375</b>	?	?
$d = 6$			1	1	1	<u>37</u>	<b>42 099</b>	?
$d = 7$				1	1	4	<b>35 993</b>	?

Table 1.2: The numbers of combinatorial types of neighborly simplicial  $d$ -polytopes with  $n$  vertices. **Boldface** results are new. All polytopes in the classes enumerated in this table can be inscribed. Underlined results about inscribability are new

### Simplicial neighborly 7-polytopes

**Theorem 19.** *There are precisely 35 993 distinct combinatorial types of simplicial neighborly 7-polytopes with 11 vertices. All of these are inscribable.*

*Proof.* We provide rational inscribed realizations for all known combinatorial types.  $\square$

### Summary

We summarize the results in Table 1.2, compare [MP15, Table 3].

#### 1.3.2 Simplicial 4-polytopes with 10 vertices.

The number of simplicial 3-polytopes with  $n$  vertices is known for  $n \leq 23$ , see A000109. Because of the connection with planar graphs it is easier to classify simplicial 3-polytopes than simplicial 4-polytopes. The number of simplicial 4-polytopes with  $n$  vertices was previously

known only for  $n \leq 9$ , compare [A222318](#), see [[ABS80](#)] [[GS67](#)] [[FMM13](#)]. The number of triangulations of  $S^3$  is known for  $n \leq 10$ .

vertices	triangulations of $S^3$	non-polytopal	polytopal
5	1	1	0
6	2	2	0
7	5	5	0
8	39	2	37
9	1 296	154	1 142
10	247 882	<b>85 878</b>	<b>162 004</b>

Table 1.3: polytopal and non-polytopal simplicial 3-spheres. **Boldface** results are new

Frank Lutz gives a complete enumeration of all combinatorial 3-manifolds with 10 vertices in [[Lut08](#)]. He finds precisely 247 882 triangulations of  $S^3$  and asked for the number of simplicial polytopes with 10 vertices [[Lut08](#), Prob. 4]:

Classify all simplicial 3-spheres with 10 vertices into polytopal and non-polytopal spheres.

We will give a complete classification.

**Theorem 20.**

- i) *There are precisely 162 004 distinct combinatorial types of simplicial 4-polytopes with 10 vertices.*
- ii) *There are precisely  $161\,978 + D$  distinct combinatorial types of inscribable simplicial 4-polytopes with 10 vertices, for some  $0 \leq D \leq 13$ .*
- iii) *All combinatorial types of simplicial 4-polytopes with up to 8 vertices are inscribable.*
- iv) *Out of the 1142 combinatorial types of 4-polytopes with 9 vertices, precisely 1140 are inscribable.*

*Proof.*

- i) Previously it was known that out of the 247 882 triangulations of  $S^3$ , at least 135 317 are polytopal and at least 85 638 are non-polytopal. The last number is largely due to David Bremner. He used his program “matroid polytope completion” (mpc), see [[BBG09](#), Sect. 3], to find matroids for these triangulations. If there are no compatible matroids for a given triangulation, this triangulation cannot be polytopal. This way he could sort out 85 636 cases. Two additional non-realizable cases are the following: there is one non-realizable neighborly triangulation of  $S^3$  with 10 vertices, see Theorem 16 ii), and there is one non-realizable triangulation of  $S^3$  with  $f$ -vector  $(1, 10, 40, 60, 30)$ , which is discussed under the name  $T2766$  in Section 1.3.3. We realized 162 004 of the triangulations

$f$ -vector	$S^3$	non-polytopal	polytopal	inscribable	p., non-i. <sup>1</sup>
(10, 30, 40, 20)	30	0	30	27	3
(10, 31, 42, 21)	124	0	124	$\leq 119$ and $\geq \underline{118}$	$\geq 5$ and $\leq \underline{6}$
(10, 32, 44, 22)	385	0	385	$\leq 381$ and $\geq \underline{379}$	$\geq 4$ and $\leq \underline{6}$
(10, 33, 46, 23)	952	0	952	$\leq 951$ and $\geq \underline{948}$	$\geq 1$ and $\leq \underline{4}$
(10, 34, 48, 24)	2 142	0	2 142	$\leq 2\ 142$ and $\geq \underline{2\ 139}$	$\geq 0$ and $\leq \underline{3}$
(10, 35, 50, 25)	4 340	28	4 312	$\leq 4\ 312$ and $\geq \underline{4\ 309}$	$\geq 0$ and $\leq \underline{3}$
(10, 36, 52, 26)	8 106	151	7 955	$\leq 7\ 955$ and $\geq \underline{7\ 954}$	$\geq 0$ and $\leq \underline{1}$
(10, 37, 54, 27)	13 853	583	13 270	13 270	0
(10, 38, 56, 28)	21 702	1 862	19 840	19 840	0
(10, 39, 58, 29)	30 526	4 547	25 979	25 979	0
(10, 40, 60, 30)	38 553	9 267	29 286	29 286	0
(10, 41, 62, 31)	42 498	15 680	26 818	26 818	0
(10, 42, 64, 32)	39 299	20 645	18 654	18 654	0
(10, 43, 66, 33)	28 087	19 027	9 060	9 060	0
(10, 44, 68, 34)	13 745	10 979	2 766	2 766	0
(10, 45, 70, 35)	3 540	3 109	431	431	0
(10, *, *, *)	247 882	85 878	162 004	$\geq \underline{161\ 978}$ $\leq 161\ 991$	$\leq \underline{26}$ $\geq 13$

Table 1.4: Simplicial 3-spheres with 10 vertices. Conjectured tight bounds are underlined.

of  $S^3$  with methods described in Section 1.2.1. For the 240 remaining cases, we applied the methods from Section 1.2.3. In all but one case we could prove the existence of a bi-quadratic final polynomial by using only partial chirotopes. In the remaining case, 12418 in Lutz's numbering, we generated all 2 985 compatible chirotopes and found bi-quadratic final polynomials for all of them.

- ii) We expect  $D$  to be zero. We could inscribe all but 26 cases of the 162 004 realizable simplicial spheres, and we use the criterion for the inscribability for stacked polytopes given by Gonska and Ziegler [GZ13, Th. 1] to show 3 of the combinatorial types of polytopes with  $f$ -vector (10, 30, 40, 20) are not inscribable. Hao Chen provides a proof of the non-inscribability of 10 additional cases in [Fir15b, Appendix 2].

In Lutz's numbering, the remaining 13 cases are: 2458, 7037, 8059, 8062, 116369, 116370, 116407, 116434, 116437, 134098, 136359, 136366, 136376.

- iii) We found rational coordinates on the sphere with methods described in Section 1.2.1.
- iv) Here these methods provide rational inscribed realizations for all but 2 out of the 1 142 distinct combinatorial types. We could realize the other two combinatorial types, but not with all vertices on the sphere. One case is the 4-simplex, stacked on 4 of its faces; this is non-inscribable because of the criterion given by Gonska and Ziegler. The other case is constructed as follows: Take the direct sum of two triangles and choose a vertex  $v$  in it.

<sup>1</sup>polytopal, but non-inscribable

Then stack on all three facets that do not contain  $v$ . A proof of the non-inscribability of this polytope with the name (9, 355) is provided by Hao Chen in in [Fir15b, Appendix 2].

We observe that these two polytopes are the only 4-polytopes with 9 vertices whose edge-graphs have an independent set of size 4; all the other 1140 polytopes have maximal independent sets of smaller size.

□

The results are summarized in Tables 1.4 and 1.5.

$f$ -vector	$S^3$	non-polytopal	polytopal	inscribable	p., non-i. <sup>1</sup>
(5, 10, 10, 5)	1	0	1	1	0
(6, 14, 16, 8)	1	0	1	1	0
(6, 15, 18, 9)	1	0	1	1	0
(6, *, *, *)	2	0	2	2	0
(7, 18, 22, 11)	1	0	1	1	0
(7, 19, 24, 12)	2	0	2	2	0
(7, 20, 26, 13)	1	0	1	1	0
(7, 21, 28, 14)	1	0	1	1	0
(7, *, *, *)	5	0	5	5	0
(8, 22, 28, 14)	3	0	3	3	0
(8, 23, 30, 15)	5	0	5	5	0
(8, 24, 32, 16)	8	0	8	8	0
(8, 25, 34, 17)	8	0	8	8	0
(8, 26, 36, 18)	6	0	6	6	0
(8, 27, 38, 19)	5	1	4	4	0
(8, 28, 40, 20)	4	1	3	3	0
(8, *, *, *)	39	2	37	37	0
(9, 26, 34, 17)	7	0	7	6	1
(9, 27, 36, 18)	23	0	23	22	1
(9, 28, 38, 19)	45	0	45	45	0
(9, 29, 40, 20)	84	0	84	84	0
(9, 30, 42, 21)	128	0	128	128	0
(9, 31, 44, 22)	175	3	172	172	0
(9, 32, 46, 23)	223	11	212	212	0
(9, 33, 48, 24)	231	22	209	209	0
(9, 34, 50, 25)	209	46	163	163	0
(9, 35, 52, 26)	121	45	76	76	0
(9, 36, 54, 27)	51	28	23	23	0
(9, *, *, *)	1296	154	1142	1140	2

Table 1.5: Simplicial 3-spheres with  $\leq 9$  vertices

### 1.3.3 Manifolds with small valence

We consider a combinatorial analogue of curvature or angular defect.

**Definition 21** (valence). We call a  $(d-2)$ -face of a  $d$ -dimensional simplicial complex a *subridge*. The *valence* of a subridge is the number facets it is contained in.

Frick, Lutz and Sullivan consider simplicial manifolds with small valence [FLS, Fri15]. One case they study in particular are 3-dimensional manifolds with valence less or equal than 5. The result of a computer enumeration by Lutz [Fri15, Th. 3.10]: Out of the 4787 distinct combinatorial types of 3-dimensional manifolds, there are 4761 triangulations of the 3-sphere  $S^3$ .

Previously little was known about the polytopality of most of these spheres. In contrast to Section 1.3.2 the number of vertices of these triangulation can be larger than 10, it can be in fact be as large as 120. We were able to realize as polytopes, and even better to find inscriptions on the sphere with rational coordinates, for all but 2 of those triangulations of  $S^3$ .

**Theorem 22.** *Out of the 4761 simplicial 3-spheres with small valence at least 4759 are realizable and inscribable.*

The triangulations, for which we could not find realizations, are  $T2766$  and  $T2775$  in the numbering used in [Fri15]. The triangulations are given as follows:

$$T2766 = \{[0123] [0124] [0135] [0146] [0156] [0234] [0347] [0357] [0467] [0568] [0578] [0678] [1239] [1248] [1289] [1356] [1369] [1469] [1489] [2347] [2379] [2478] [2578] [2579] [2589] [3569] [3579] [4678] [4689] [5689]\}$$

$$T2775 = \{[0123] [0124] [0135] [0146] [0156] [0234] [0347] [0357] [0467] [0568] [0578] [0678] [1239] [12410] [12910] [13511] [13911] [14612] [141012] [15612] [151112] [191011] [1101112] [23412] [23912] [241012] [291013] [291213] [2101213] [3467] [34612] [35711] [36711] [36911] [36912] [5689] [56912] [57813] [571113] [58913] [591213] [5111213] [67811] [68911] [781113] [891011] [891013] [8101113] [10111213]\}$$

Triangulation  $T2766$  cannot be realized as a polytope, as explained in [BS95] and [Fri13].

It remains to show that triangulation  $T2775$  is not polytopal, which is what we expect to be the case.

### 1.3.4 A special inscribed realization for the Bokowski–Ewald–Kleinschmidt polytope

Bokowski, Ewald and Kleinschmidt provide a 4-polytope on 10 vertices with disconnected realization space, see [BEK84] and [BGdO90]. While enumerating all simplicial 4-polytopes with 10 vertices, we also realized this one: It has number 6986 in Lutz's numbering. We provide

the following rational coordinates for the vertices  $v_0, v_1 \dots v_9 \in \mathbb{Q}^4$  on the sphere:

$v_0$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$	$v_9$
$-\frac{20}{583}$	$-\frac{2}{17}$	$-\frac{6}{61}$	$-\frac{5}{18}$	$-\frac{4}{237}$	$\frac{10}{59}$	$\frac{4}{79}$	$\frac{4}{27}$	$\frac{28}{79}$	$\frac{48}{221}$
$\frac{2}{53}$	$\frac{16}{51}$	$\frac{20}{61}$	$-\frac{1}{6}$	$\frac{8}{79}$	$\frac{10}{59}$	$-\frac{80}{553}$	$-\frac{40}{189}$	$-\frac{4}{79}$	$\frac{32}{221}$
$\frac{38}{583}$	$-\frac{10}{51}$	$\frac{6}{61}$	$-\frac{1}{18}$	$-\frac{56}{237}$	$\frac{16}{59}$	$\frac{40}{553}$	$-\frac{8}{63}$	$-\frac{20}{79}$	$-\frac{12}{221}$
$\frac{581}{583}$	$\frac{47}{51}$	$\frac{57}{61}$	$\frac{17}{18}$	$\frac{229}{237}$	$\frac{55}{59}$	$\frac{545}{553}$	$\frac{181}{189}$	$\frac{71}{79}$	$\frac{213}{221}$

The polytope has the  $f$ -vector  $(1, 10, 38, 56, 28)$  and its facet list is:  $\{[0123] [0124] [0134] [0235] [0249] [0259] [0346] [0356] [0467] [0479] [0569] [0679] [1238] [1249] [1258] [1259] [1348] [1489] [1589] [2358] [3467] [3478] [3567] [3578] [4789] [5678] [5689] [6789]\}$

We can stack over the facet  $[0259]$  with the point  $(\frac{2}{23}, \frac{5}{23}, \frac{4}{23}, \frac{22}{23})$  and over another facet  $[0356]$  with the point  $(\frac{4}{203}, -\frac{80}{609}, \frac{8}{87}, \frac{601}{609})$ . These two facets lie in the same orbit of the involution given by the permutation  $(1, 7)(2, 6)(3, 9)$ , such that the stacking points are also on the sphere. This gives rise to two configurations of 10 points in  $\mathbb{R}^3$  which have combinatorial equivalent Delaunay triangulations, but lie in distinct components of its realization space.

**Theorem 23** (compare [APT14, Cor. 4.18]). *There is a 3-dimensional configuration of 10 points whose Delaunay triangulation has a disconnected realization space.*

This improves the previously smallest example given by Adiprasito, Padrol and Theran, which was a 25-dimensional configuration of 30 points.

### 1.3.5 Remaining questions and a brave conjecture

Already Steiner [Ste32, Question 77], p. 316] asked whether all polytopes are inscribable. One reason why “inscribable” is an interesting property of a polytope is the close relationship with Delaunay triangulations and Voronoi diagrams, as provided by [Bro79], see also Section 1.3.4. For 3-dimensional polytopes, the situation is well understood. There is a characterization of inscribable 3-polytopes by Hodgson, Rivin and Smith [HRS92], and there are conditions for inscribability on the edge-graph given by Dillencourt and Smith [DS96], which can be checked algorithmically. A criterion for the inscribability of *stacked* polytopes that also works in higher dimensions is given by Gonska and Ziegler [GZ13, Th. 1]. Having a complete edge graph is far away from being stacked. The question whether all (even-dimensional) neighborly polytopes are inscribable has been asked by Gonska and Padrol in [GP15, p. 2]. All the neighborly

polytopes in the families that we have enumerated are inscribable. Although this is all the evidence we have, we propose a conjecture.

**Conjecture 24.** *All 2-neighborly simplicial polytopes are inscribable.*

Out of the simplicial 4-polytopes with 10 vertices, which we have enumerated, we are able to decide inscribability in all but 13 cases. We expect the remaining cases to be non-inscribable.

*Question 25* (see Theorem 20 ii). Are the remaining 13 cases non-inscribable?

*Question 26.* Is there an efficient method for proving non-inscribability of combinatorial types of non-stacked polytopes of dimension greater than 3?

For two families we are able to realize all but a single combinatorial type. We expect the answers to the following questions to be negative:

*Question 27* (see Theorem 16 iv). Is there a 4-dimensional polytope on 12 vertices with facet list  $F374225$ ? Is it inscribable?

*Question 28* (see Theorem 22). Is there a 4-dimensional polytope on 14 vertices with facet list  $T2775$ ? Is it inscribable?



## Chapter 2

# Polytopal inclusions

Given two polytopes  $P$  and  $Q$ , we can ask: *What is a polytope  $P'$  of largest volume such that  $P'$  is similar to  $P$  and contained in  $Q$ ?* By “similar” we understand that  $P'$  can be transformed into  $P$  by a dilation and rigid motions. Instead of “largest volume” we might as well ask for a polyhedron that maximizes the dilation factor between  $P$  and  $P'$ . An equivalent question asks for the smallest polytope  $Q'$  which is similar to  $Q$  and contains  $P$ .

The earliest investigation of this topic may already be found in Kepler’s work, [Kep19, libri V, caput I, p. 181]. One finds descriptions of the largest regular tetrahedron included in a cube and of the largest cube included in a regular dodecahedron, although no claim on maximality is made.

A substantial contribution was made by Croft, [Cro80]. Here the case where  $P$  and  $Q$  are three-dimensional is considered. He notes that apart from exceptional cases *local* maxima must be immobile and must satisfy seven linear constraints, see [Cro80, Theorem, p. 279]. Using this information he calculates all local maxima and obtains *global* maximal configurations, see [Cro80, p. 283–295]. Letting  $P$  and  $Q$  range over the Platonic solids, Croft gives a complete answer for 14 out of the 20 non-trivial cases. This is the problem described by the same author, Falconer and Guy as Problem B3 in [CFG91, p. 52]; see below for a solution to the remaining six cases.

Containment problems for (simple) polygons are discussed for example in [Cha83] and [AAS98], and some algorithms are given. Taking  $P$  to be a regular  $n$ -gon and  $Q$  to be a regular  $m$ -gon, the size of the largest copy of  $P$  inside  $Q$  is known if and only if  $n$  and  $m$  share a common prime factor. If they are co-prime only conjectural results are known; see the article by Dilworth and Mane, [DM10].

More general containment problems are studied by Gritzmann and Klee, [GK94]. They also allow other groups than the group of similarities act on the polyhedra. Gritzmann and Klee state the problem where the group acting is the group of similarities, [GK94, p. 143], but do not discuss a computational approach.

The related problem of finding a largest, not necessarily regular,  $j$ -simplices in  $k$ -cubes is related to Hadamard matrices and discussed in [HKL96]. In some cases, the maximizer is indeed a *regular* simplex; see [MRT09] for details.

A short summary of the results in this chapter has been posted by the author on MathOverflow, [Fir14b].

In Section 2.1 and 2.2, we present a method for finding solutions to this problem in general. In Section 2.3, we apply this method to some special cases and thereby solve the above-mentioned Problem B3 numerically and offer conjectural exact algebraic solutions.

## 2.1 Setting up the optimization problem

Let  $P$  and  $Q$  be polyhedra, let  $p$  be the dimension of  $P$  and  $q$  be the dimension of  $Q$ . We assume  $q \geq p$ ; otherwise, it is not quite clear what it means that  $P$  is included in  $Q$ . Let  $H_1, \dots, H_m$  be the defining half spaces for  $Q$  such that

$$Q = \bigcap_{k=1}^m H_k$$

and let  $w_1, \dots, w_n$  denote the vertices of  $P$ . We formulate the problem of finding the largest polyhedron  $P'$  such that  $P'$  is contained in  $Q$  and similar to  $P$  as a quadratic maximization problem.

*Problem 29.*

<b>Input data:</b>	halfspaces $H_1, \dots, H_m$ of $Q$ , vertices $w_1 \dots w_n$ of $P$
<b>Variables:</b>	$s$ and $v_{ij}$ for $1 \leq i \leq n, 1 \leq j \leq q$
<b>Objective function:</b>	maximize $s$
<b>Linear constraints:</b>	$(v_{i1}, \dots, v_{iq}) \in H_k$ for $1 \leq i \leq n, 1 \leq k \leq m$
<b>Quadratic constraints:</b>	$\sum_{l=1}^q (v_{il} - v_{jl})^2 = s \ w_i - w_j\ _2^2$ for $1 \leq i < j \leq n$

In this formulation the variable  $s$  can be thought of as the square of the dilation factor between  $P$  and  $P'$ . The other variables are the coordinates of the vertices of  $P'$ . The linear constraints consist of  $nm$  weak inequalities. They ensure that  $P' \subset Q$ . The quadratic constraints assert that the distances between vertices of  $P'$  agree with those of  $P$  up to a dilation factor  $\sqrt{s}$ , which is the same for all pairs of vertices. Hence the quadratic equalities ensure that  $P'$  is similar to  $P$ .

A global optimum of the optimization problem gives us a largest polyhedron  $P'$ , as desired. It might happen that there are combinatorially different optimal solutions to our problem. The goal in Section 2.3 is to identify *one* of the optimal solutions. From that we can deduce the optimal dilation factor and hence answer the question: How large is the largest polyhedron  $P'$  similar to  $P$  and contained in  $Q$ . We do not explain in what combinatorially different ways  $P'$  can be contained in  $Q$ , but rather describe one possible inclusion.

### 2.1.1 Improved formulation

The above formulation for Problem 29 is particularly simple and straightforward. However an equivalent formulation using fewer variables and fewer quadratic constraints can be obtained as follows.

Choose an affine basis from the set of vertices of  $P$ . For the optimization problem we can then take only those variables  $v_{ij}$ , such that  $w_i$  belong to that affine basis, and substitute all occurrences of other variables by linear combinations of the former. These linear combinations can be obtained from the vertices of  $P$ , using the fact that we have chosen an affine basis. Using this substitution, we have  $(p+1)q+1$  variables in total, and this number only depends on the dimensions of  $P$  and  $Q$  and not on the number of vertices of  $P$ .

In order to obtain fewer quadratic constraints we also focus on the chosen affine basis: It suffices to ensure that all the distances between all pairs of two vectors in the affine basis are all scaled by the same factor  $\sqrt{s}$ . Because there are  $q+1$  vectors in the affine basis, we obtain

$$\binom{q+1}{2} = \frac{1}{2}(q+1)(q+2)$$

quadratic equations. Counting the number of linear equations we see that there are  $nm$  of them, independent of the dimension of  $Q$ .

An axis aligned bounding box for  $Q$  gives bounds on the variables  $v_{ij}$ . We can trivially include a copy of  $P$ , whose circumsphere coincides with the in-sphere of  $Q$ , so a lower bound for  $s$  would be the square of the Keplerian ratio:

$$s \geq \left( \frac{\text{inradius of } Q}{\text{circumradius of } P} \right)^2.$$

In a similar way we could give an upper bound for  $s$ , but in view of the objective function this does not seem necessary.

The equations used in setting up Problem 29 depend on the position of  $Q$ . If many of the defining hyperplanes for  $Q$  are parallel to many coordinate axes, then fewer variables are used in the linear equations. Also the choice of an affine basis of  $P$  might influence the number of variables used in the equations.

In solving Problem 29 with a numerical solver, the precision for the input of the polyhedron should be higher than the desired precision.

If  $P$  and  $Q$  possess symmetry one can use this symmetry to obtain additional constraints. For example if  $P$  and  $Q$  are centrally symmetric, then it suffices to search a maximal  $P'$  among

those copies of  $P$  which are concentric with  $Q$ . See [Cro80, OBSERVATION p. 288] for a simple proof.

If  $P$  and  $Q$  are regular polyhedra, one can say without loss of generality that one vertex of  $P'$  must lie in one face of  $Q$ .

### 2.1.2 Solving the optimization problem numerically

In order to solve Problem 29 numerically we can use SCIP, which is a solver for mixed integer non-linear programming. This solver uses branch and bound techniques in order to find a *global* optimum within a certain precision; see [Ach09] and [ABKW08] for details. We don't use SCIP's capability to handle integer variables, because all of our variables are continuous.

## 2.2 From numerical to symbolic solutions of systems of polynomial equations

**Definition 30.** Let  $f_1, \dots, f_k \in \mathbb{Q}[X_1, \dots, X_n]$  be polynomials in several variables and consider the system of polynomial equations

$$f_1 = 0, f_2 = 0, \dots, f_k = 0.$$

A *numerical solution* with error  $\varepsilon \in \mathbb{R}$  to this system is an element  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , such that

$$|f_i(x)| \leq \varepsilon \text{ for all } 1 \leq i \leq k.$$

A numerical solution with error 0 is a solution of the polynomial system. The goal of this section is to show a heuristic how to obtain a solution of a polynomial system when a numerical solution is given. We explain a 3-step method.

### Method 31.

*Step 1 Improve the numerical solution.*

*Step 2 For each variable guess an algebraic number close to the approximation.*

*Step 3 Verify the solution by exact calculation in the field of real algebraic numbers.*

The purpose of Step 1 is to start with a numerical solution with error  $\varepsilon$  and sharpen the solution to obtain another numerical solution with smaller error  $\varepsilon'$ . This can for example be done by using multi-dimensional Newton's method. The second step can be done using integer relation algorithms such as LLL ([LLL82]) and PSLQ ([FBA99]). Step 3 is possible because calculations in the field of real algebraic numbers can be done effectively on a computer, see the monograph [BPR06] for an comprehensive overview.

In general we cannot expect Method 31 to work, it might even happen that a system has a numerical solution which do not lead to any solution. On the other hand, if the numerical solutions obtained in Step 1 converge to a solution of the system, which can sometimes be certified (for example using Smale's alpha theory [Sma86]), then Step 2 will eventually find a

solution, although in practice the integer relation algorithms might be too slow, especially if the solutions are algebraic numbers of high degree.

We can expect to find solutions if they consist of algebraic numbers with minimal polynomials of low degree and small coefficients. See Section 2.3 and 3.2 for two successful applications of this method.

So far we presented this method to find solutions for polynomial systems defined over  $F[X_1, \dots, X_n]$  with  $F = \mathbb{Q}$ . This is because in Step 2, we expect solutions to be algebraic numbers. The method also works if we replace  $\mathbb{Q}$  by an algebraic extension of  $\mathbb{Q}$  and can be easily adapted to other settings, as long as we are able to identify solutions in Step 2 correctly.

### 2.2.1 Limitations of the method

The solver SCIP, which can be used for solving Problem 29 finds a *global* optimum, but the calculations are done only with a certain prescribed precision. In general it might be the case that there exists a maximizer  $P'$  that attains the maximal dilation factor  $\sqrt{s}$  and a second locally maximal feasible solution  $P''$ , with dilation factor  $\sqrt{s - \varepsilon}$ , for a small  $\varepsilon > 0$ . Indeed, it is possible to construct examples of  $P$  and  $Q$  where that is the case for arbitrarily small  $\varepsilon$ . Take for example  $P$  and  $Q$  to both be similar to the same rectangle with almost equal side length. Hence in order to make sure that we have indeed found an optimal solution to Problem 29, we make the following assumptions.

*Assumption 32.* The solution  $\tilde{P}$  to Problem 29 has sufficient precision such that there is only one local maximum  $P'$  near  $\tilde{P}$ .

*Assumption 33.* Problem 29 has been solved with sufficient precision such that the dilation factor  $\sqrt{s}$  of the local maximum  $P'$  near  $\tilde{P}$  is the *global* maximum.

*Assumption 34.* Problem 29 has been solved with sufficient precision such that  $\tilde{P}$  and the local maximum  $P'$  near  $\tilde{P}$  satisfy the same vertex-face incidences with  $Q$ .

The precision necessary for the solution to satisfy these properties depends on  $P$  and  $Q$  and because there exist examples where the global maximum and the second largest local maximum are arbitrarily close, it is in general not possible to prescribe the precision necessary for Assumptions 32-34 to hold.

Assumptions 32-34 also deal with possible numerical mistakes or bugs of a solver for Problem 29.

If Assumptions 32 and 33 hold and we can, because of Assumption 34, identify an exact algebraic solution near  $P'$ , and this will be a maximizer of the problem. In any case, even if the assumptions do not hold, we obtain a lower bound if we can solve the system derived from the approximate solution  $P'$ .

In the calculations in Section 2.3 we do not attempt to prove that Assumptions 32-34 hold, but we state the precision which was used to solve the problems. In this sense our calculations below do not prove optimality but provide putatively optimal results.

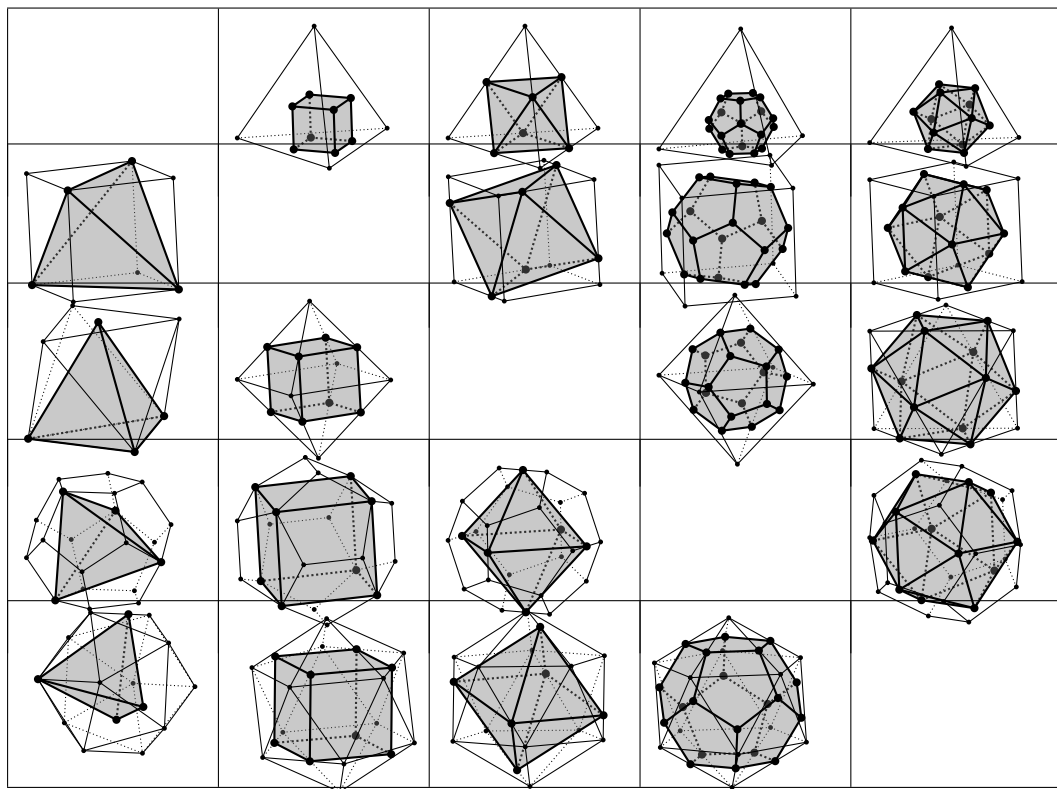


Table 2.1: Maximal Platonic solids included in a Platonic solid

$Q \backslash P$	$T$	$C$	$O$	$D$	$I$
$T$		0.29590654	0.50000000	★ 0.16263158	0.27009076
$C$	1.4142136		1.0606602	0.39428348	0.61803399
$O$	1.0000000	0.58578644		★ 0.31340182	0.54018151
$D$	2.2882456	1.6180340	1.8512296		★ 1.3090170
$I$	★ 1.3474429	★ 0.93874890	1.1810180	★ 0.58017873	

Table 2.2: Numerical values.

### 2.3 Inclusions of Platonic solids

When each of  $P$  and  $Q$  is taken to be one of the 5 Platonic solids, that is, one of the regular three-dimensional polyhedra, we can consider 20 non-trivial inclusions. Croft found optimal pairs in 14 out of these 20 cases and proved optimality in [Cro80]. In the following we assume that the regular three-dimensional polyhedron  $Q$  has side length 1. We abbreviate tetrahedron, cube, octahedron, dodecahedron and icosahedron by  $T, C, O, D$  and  $I$  and denote the golden ratio by  $\phi$ . With the methods described above we are able to confirm all the known cases and

answer all six unknown cases. The solver used was SCIP version 3.1.0 with a precision set to  $10^{-10}$ . With the improved formulation described above the calculations for all 20 inclusions took a few hours on a single core of a Xeon CPU running at 3 GHz, using fewer than 8GB of RAM. Some cases were solved in less than a second.

$Q \backslash P$	$T$	$C$	$O$	$D$	$I$
$T$		$\frac{1}{1+\frac{2}{3}\sqrt{3}+\frac{1}{2}\sqrt{6}}$	$\frac{1}{2}$	$\star d$	$\frac{1}{\phi^2\sqrt{2}}$
$C$	$\sqrt{2}$		$\frac{3}{4}\sqrt{2}$	$\frac{1}{\sqrt{2}\phi^3}(1-\frac{1}{2}\sqrt{10}+\frac{1}{2}\sqrt{2}+\sqrt{5})$	$\frac{1}{\phi}$
$O$	1	$2-\sqrt{2}$		$\star\frac{(25\sqrt{2})-(9\sqrt{10})}{22}$	$\frac{\sqrt{2}}{\phi^2}$
$D$	$\phi\sqrt{2}$	$\phi$	$\frac{\phi^2}{\sqrt{2}}$		$\star\frac{1}{2\phi}+1$
$I$	$\star t$	$\star\frac{5+7\sqrt{5}}{22}$	$\frac{1}{2}(1-\frac{1}{2}\sqrt{10}+\frac{1}{2}\sqrt{2}+\sqrt{5})$	$\star\frac{15-\sqrt{5}}{22}$	

$\phi =$  golden ratio

$$t = \text{zero near } 1.3 \text{ of } 5041x^{32} - 1318386x^{30} + 60348584x^{28} - 924552262x^{26} + 5246771058x^{24} - 15736320636x^{22} + 29448527368x^{20} - 37805732980x^{18} + 35173457839x^{16} - 24298372458x^{14} + 12495147544x^{12} - 4717349124x^{10} + 1256858478x^8 - 217962112x^6 + 21904868x^4 - 1536272x^2 + 160801$$

$$d = \text{zero near } 0.16 \text{ of } 4096x^{16} - 3701760x^{14} + 809622720x^{12} - 17054118000x^{10} + 79233311025x^8 - 94166084250x^6 + 31024053000x^4 - 3236760000x^2 + 65610000$$

Table 2.3: Exact values.

Table 2.2 and 2.3 give decimal approximations and symbolic values of the side length of a largest copy of  $P$  inside  $Q$ , where  $P$  and  $Q$  range over the Platonic solids. For completeness we restate the results of Croft; he gives a similar but incomplete table: [Cro80, p. 295]. We correct three typos in his table, the corresponding cells are *emphasized*; new results are marked with a star.

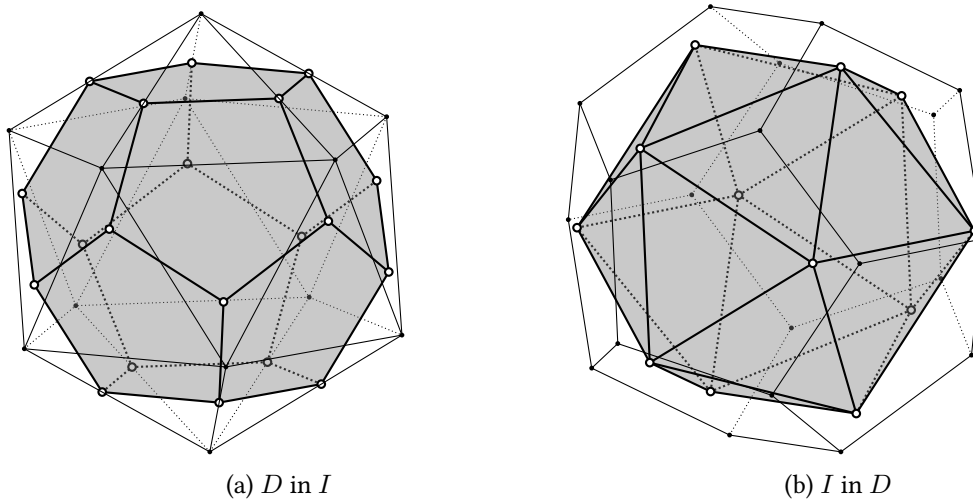


Figure 2.1: Self reciprocal cases

For the 6 previously unknown cases we provide below a description of an optimal position.

### 2.3.1 Dodecahedron in Icosahedron

For  $D$  in  $I$ , we are in a concentric situation. The five vertices of one face of  $D$  lie on the five edges of  $I$  incident to a common vertex, one on each. The five vertices of the opposite face of that face of  $D$  also lie on five edges of  $I$  incident to a common vertex, namely the vertex of  $I$  antipodal to the one mentioned before. The other ten vertices of  $D$  lie in the interior of faces of  $I$ . The side length is

$$\frac{15 - \sqrt{5}}{22} \approx 0.58017873.$$

### 2.3.2 Icosahedron in Dodecahedron

For  $I$  in  $D$  we are also in a concentric situation; each of the 12 vertices of  $I$  lies in the interior of one of the 12 faces of  $D$ , and in each face of  $D$  there is one vertex of  $I$ . Let's position  $D$  in the usual fashion such that 6 of its edges are parallel to the 3 coordinate axes. To each of the 12 vertices on these edges of  $D$  we associate the unique face that contains one but not the other vertex of the edge in its boundary. This gives us pairs  $v, f$  of vertices and faces of  $D$ . For each pair  $v, f$  a vertex of  $I$  lies on the bisector of  $f$  which goes through  $v$ , and its position on the bisector is the point where the bisector is divided in two parts, such that the larger part has



$\frac{\phi}{2}$  times the length of the whole bisector. The position of the vertex of  $I$  is closer to  $v$  and the absolute distance to  $v$  is

$$\left(1 - \frac{\phi}{2}\right) \cdot \frac{1}{2} \sqrt[4]{5} \phi^{\frac{3}{2}} = \frac{\sqrt[4]{5}}{4\sqrt{\phi}}.$$

Recall we assume that  $D$  has side length 1 which results in a bisector of length

$$\frac{1}{2} \sqrt[4]{5} \phi^{\frac{3}{2}}.$$

The edge length of  $I$  obtained in this way is

$$\frac{1}{2\phi} + 1 \approx 1.3090170.$$

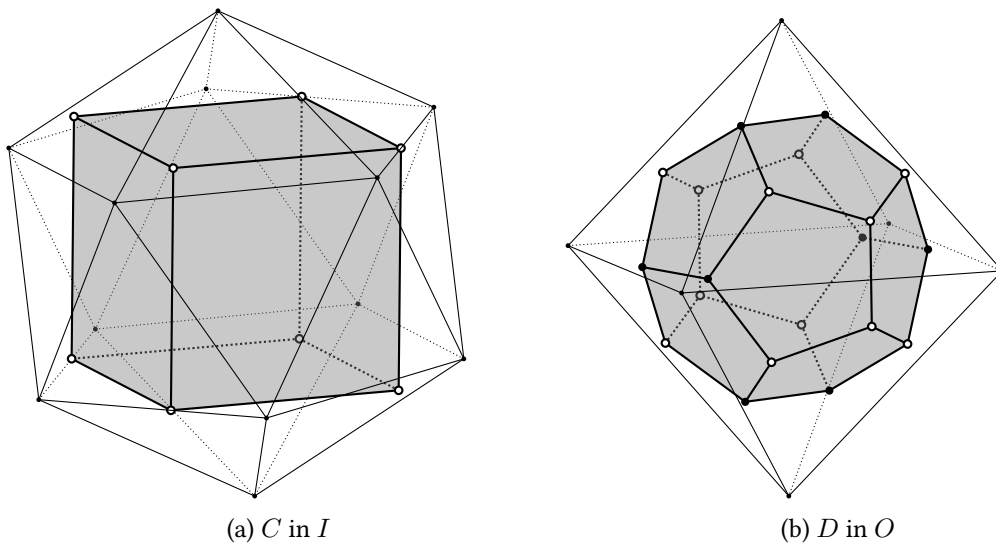


Figure 2.2: Two reciprocal cases

### 2.3.3 Cube in Icosahedron

This is also a concentric situation. For  $C$  in  $I$ , two vertices of one edge of  $C$  lie in the interior of two adjacent edges in  $I$ , which are not contained in the same face. And the vertices of the antipodal edge of this edge in  $C$  lie in the interior of the corresponding antipodal edges in  $I$ . The other 4 edges of  $C$  lie in the interior of faces of  $I$ . The side length is

$$\frac{5 + 7\sqrt{5}}{22} \approx 0.93874890.$$

### 2.3.4 Dodecahedron in Octahedron

Again this is a concentric situation. Put two opposite edges of  $D$  in a hyperplane spanned by 4 vertices of  $O$ . Four faces of  $O$  each contain an edge of  $D$  and the other four faces of  $O$  each

contain only one vertex of  $D$ . The incidences can be seen in Figure 2.2b; vertices of  $D$  that lie in the interior of a face of  $O$  are marked white. See the considerations about reciprocity below. For  $D$  in  $O$  the maximum is

$$\frac{(25\sqrt{2}) - (9\sqrt{10})}{22} \approx 0.31340182.$$

### Reciprocity of $C \subset I$ and $D \subset O$

If  $P \subset Q$  are concentric and  $P$  is maximal in  $Q$  we can take polar reciprocals and obtain  $Q^\circ \subset P^\circ$ , such that  $Q^\circ$  is maximal in  $P^\circ$ . Because  $C^\circ = O$  and  $I^\circ = D$ , we can check that the two previous cases are reciprocal:

$$\frac{(25\sqrt{2}) - (9\sqrt{10})}{22} \left( \frac{\phi^3}{\sqrt{2}} \right) = \frac{5 + 7\sqrt{5}}{22}.$$

Concentric  $C$  and  $D$ , which are reciprocals with respect to the unit sphere, have the product of their edge lengths constant, namely  $2\sqrt{2}$ . Similarly for concentric, reciprocal  $I$  and  $D$  this product equals  $\frac{4}{\phi^3}$ . The factor  $\frac{\phi^3}{\sqrt{2}}$  is the quotient of these two numbers.

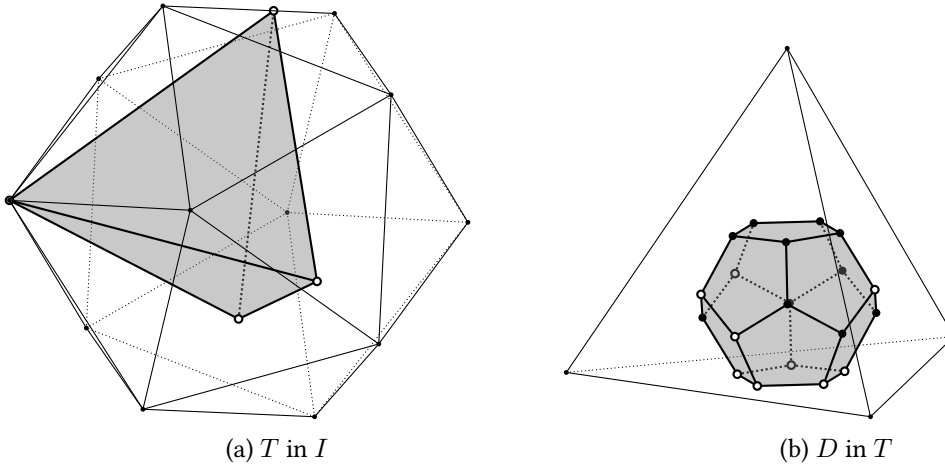


Figure 2.3: Two cases with more involved solutions

### 2.3.5 Tetrahedron in Icosahedron

The incidences of the  $T$  in  $I$  are best seen in Figure 2.3a: one vertex of  $T$  coincides with one vertex  $v$  of  $I$ , another vertex of  $T$  lies on an edge of  $I$ , which is neither incident to the vertex  $v$  nor to its antipode, and the two remaining vertices lie in the interior of faces of  $I$ .

While in this case the resulting system can be solved somewhat automatically by the computer algebra system Mathematica 9 (while version 8 was not able to perform the calculation), we use the methods described in Section 2.2. We choose two variables each for the barycentric coordinates for the two vertices in the interior of faces of  $I$  and one variable for barycentric

coordinates for the vertex in the interior of an edge of  $I$ . Together with a variable  $t$  for the side length of  $T$ , i.e. the dilation factor, this results in a system of 6 quadratic equations in 6 variables. The 6 equations confirm that all 6 edges are of length  $t$ . We use the open source computer algebra system *sage*, [S<sup>+</sup>14]. For the Newton method, i.e. Step 1 we use *mpmath*, [J<sup>+</sup>13], and for the integer relation, i.e. Step 2 PARI, [Par14] is used. It is sufficient to obtain 800 decimal digits in Step 1 of Method 31 described in Section 2.2 in order to obtain the exact values for the variables in Step 2. The exact edge length is the zero near 1.3474429 of the following polynomial:

$$\begin{aligned} &5041t^{32} - 1318386t^{30} + 60348584t^{28} - 924552262t^{26} + 5246771058t^{24} \\ &- 15736320636t^{22} + 29448527368t^{20} - 37805732980t^{18} + 35173457839t^{16} \\ &- 24298372458t^{14} + 12495147544t^{12} - 4717349124t^{10} + 1256858478t^8 \\ &- 217962112t^6 + 21904868t^4 - 1536272t^2 + 160801. \end{aligned}$$

### 2.3.6 Dodecahedron in Tetrahedron

The incidences are best seen in Figure 2.3b: a complete face of  $D$  is contained in one face of  $T$ , two vertices of  $D$  lie in another face of  $T$  and the two other faces of  $T$  contain one vertex of  $D$  each. We choose a variable  $d$  for the side length of  $D$  and four additional variables that describe the position of the vertices of  $D$  that lie in a face of  $T$ , which is not the face that contains a complete face of  $D$ . Making sure that the edges between these four vertices have the correct length results again in a system of 6 quadratic equations with 5 variables, which can be successfully solved as in the previous case. In this case 350 decimal digits suffice to find solutions in the field of real algebraic numbers. The exact edge length is the zero near 0.16263158 of the following polynomial:

$$\begin{aligned} &4096d^{16} - 3701760d^{14} + 809622720d^{12} - 17054118000d^{10} + 79233311025d^8 - \\ &94166084250d^6 + 31024053000d^4 - 3236760000d^2 + 65610000. \end{aligned}$$

## 2.4 Further applications

Possibly interesting situations where the method of this chapter could be applied include the following cases.

- a) Take  $P$  and  $Q$  to be (regular) polygons.
- b) Take  $P$  and  $Q$  to be regular polyhedra of dimension greater than 3.
- c) Take  $P$  to be a  $n$ -cube and  $Q$  an  $m$ -cube with  $n < m$ .
- d) Take  $P$  to be a regular  $n$ -simplex and  $Q$  an  $m$ -cube with  $n \leq m$ .
- e) Take  $Q$  to be any polyhedron and  $P$  some projection of  $Q$ .

For the first case, i.e., finding the largest regular  $n$ -gon in a regular  $m$ -gon, the author has checked the conjecture of Dilworth and Mane [DM10, Section 9] for co-prime  $m$  and  $n$  up to a precision of  $10^{-10}$  for all pairs  $m, n$  with  $m, n \leq 120$ . It is possible to modify Problem 29 in order to solve similar packing problems.

## Chapter 3

# Miscellaneous Results

### 3.1 Cylinders touching a unit sphere

#### 3.1.1 Introduction

A well known packing problem is to determine the number of spheres of radius 1 that can be arranged around the unit sphere, such that they touch the unit sphere and do not intersect each other. This number is called the *kissing number* and we can ask this question for any dimension. In dimension 3, i.e. packing 2-spheres, it is possible to fit 12 spheres. For example by kissing the 12 vertices of a regular icosahedron, which is circumscribed by the unit sphere; see Figure 3.1. It turns out that it is impossible to pack 13 spheres, which was proved by Schütte and van der Waerden [SvdW53]. An arguably simpler proof was given by Leech a little bit later [Lee56].

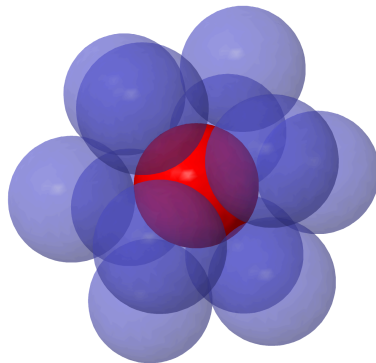


Figure 3.1: The unit sphere with 12 kissing spheres of radius 1.

In dimension 4 the question was only recently settled by Musin [Mus08]: here the kissing number is 24. Pfender and Ziegler [PZ04] provide a concise survey and the history of the problem.

Instead of attaching spheres of radius 1 to the unit 2-sphere  $S^2$ , we can also attach a number of cylinders. Here a (solid) *cylinder* of radius  $r$  is a set of points isometric to  $D_r^2 \times \mathbb{R}$ , where  $D_r^2$  denotes the disc around 0 of radius  $r$ . A *unit cylinder* is a cylinder of radius 1. A cylinder is *touching* (or *kissing*) the unit sphere  $S^2$  if it intersects it in exactly one point, i.e. if it is tangent to the sphere. The following question goes back to W. Kuperberg [Kup90].

*Question 35.* How many unit cylinders can be arranged such that they all touch the unit sphere and have pair-wisely disjoint interiors?

### 3.1.2 Setting up the problem

To generalize *Question 35* we define for each  $n \in \mathbb{N}$  and  $r \in \mathbb{R}$  the following statement, whose truth value depends on  $n$  and  $r$ .

$S(n, r)$ . *There is a configuration of  $n$  cylinders of radius  $r$  with pair-wisely disjoint interiors, such that they all touch the unit sphere.*

Furthermore we define  $\nu(r)$  to be the largest  $n \in \mathbb{N}$  such that  $S(n, r)$  is true. Also we define  $\rho(n)$  to be the supremum of radii  $r \in \mathbb{R}$  such that  $S(n, r)$  is true. In fact, the supremum is always attained, as we can deduce from the compactness of the configuration space that we define later in this section. With this terminology, *Question 35* asks for the value of  $\nu(1)$ .

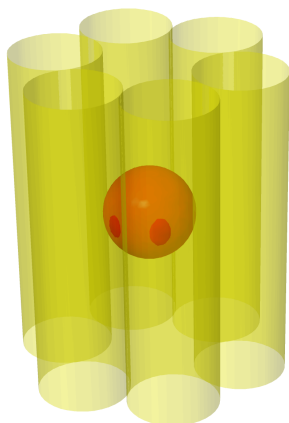
Given a pair  $(n, r)$ , the following three statements are equivalent:

1.  $S(n, r)$
2.  $\nu(r) \geq n$
3.  $\rho(n) \geq r$

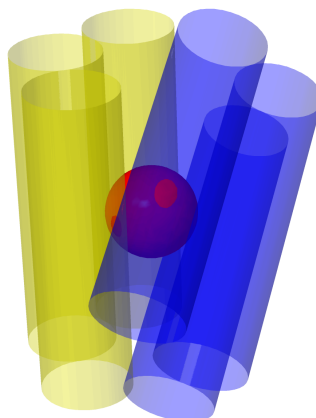
Therefore, if we find a configuration such that  $S(n, r)$  is true, we immediately obtain lower bounds for  $\nu(r)$  and  $\rho(n)$ . We keep using  $\nu$  and  $\rho$  because it is convenient to have both at our disposal and different authors use one or the other concept to describe bounds. The following monotonicity conditions hold:

1. If  $r' \leq r$ , then  $\nu(r') \geq \nu(r)$ .
2. If  $n' \leq n$ , then  $\rho(n') \geq \rho(n)$ .

One can arrange 6 cylinders around a unit sphere in a way such that they are all parallel and touch at 6 points forming a regular hexagon on an equator of the sphere, see Figure 3.2a. In general, we can arrange  $n$  cylinders around a unit sphere, such that they are all parallel and touch the equator at the vertices of a regular  $n$ -gon. We call this arrangement the *circular arrangement*. With the circular arrangement,  $S(6, 1)$  is true and we obtain  $\nu(1) \geq 6$  and  $\rho(6) \geq 1$ . There is also another configuration with 6 cylinders, see Figure 3.3. Here the six cylinders touch at the vertices of the octahedron, defined as the convex hull of the standard basis of  $\mathbb{R}^3$  and their antipodes, in such a way that two of the cylinders are parallel to each coordinate axis.



(a) Circular arrangement.



(b) Twisted circular arrangement.

Figure 3.2

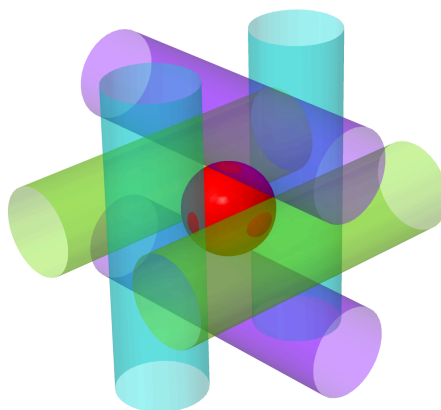


Figure 3.3: Octahedral arrangement.

For upper bounds it suffices to find a pair  $(n, r)$  such that  $S(n, r)$  is false. In this case we obtain immediately for  $\nu(r)$  and  $\rho(n)$ :

$$\begin{aligned}\nu(r) &< n \\ \rho(n) &< r\end{aligned}$$

Concerning upper bounds, Heppes and Szabó [HS91] proved  $\nu(1) < 9$  using the methods of “shadows”. Braß and Wenk [BW00] improved this to  $\nu(1) < 8$ . They consider the area of the intersection between the cylinders and a sphere of radius larger than 1 concentric to the unit sphere. We will adapt their arguments to various radii in order to obtain more upper bounds below.

Another configuration with 6 cylinders can be obtained from the circular arrangement by rotating three of them, see Figure 3.2b. We want to investigate which other such configurations are possible. In order to study *all* all valid configurations of  $n$  cylinders of radius  $r$ , we define an appropriate configuration space

$$\tilde{\mathcal{C}}_{(n,r)} := \left\{ \left( \left( \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}, \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \right)_{1 \leq k \leq n} \in (\mathbb{R}^{(3 \times 2)})^n \mid \text{such that Conditions 1.-4. are satisfied} \right\}.$$

Conditions:

1.  $x_k^2 + y_k^2 + z_k^2 = 1$
2.  $\begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \cdot \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = 0$
3.  $a_k^2 + b_k^2 + c_k^2 = 1$
4.  $d(l_i, l_j) \geq 2r$  for  $1 \leq i < j \leq n$ , where we define the line

$$l_k := \left\{ (1+r) \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} + \alpha \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$$

and denote with  $d(\cdot, \cdot)$  the distance between two lines.

We set

$$w_k := (1+r) \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \quad \text{and} \quad v_k := \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix},$$

and the distance between the two lines  $l_i$  and  $l_j$  is given as follows:

$$d(l_i, l_j) = \begin{cases} \frac{|(w_i - w_j) \cdot (v_i \times v_j)|}{|v_i \times v_j|} & \text{if } l_i \text{ and } l_j \text{ are skew} \\ |v_i \times (w_j - w_i)| & \text{if } l_i \text{ and } l_j \text{ are parallel} \end{cases}$$

Therefore we can reformulate Condition 4 as:



$$4.' \quad 4r^2|v_i \times v_j|^2 \leq |(w_i - w_j) \cdot (v_i \times v_j)|^2 \quad \text{and} \quad 4r^2 \leq |v_i \times (w_j - w_i)|^2.$$

With this reformulation we see that all constraints defining  $\tilde{\mathcal{C}}_{(n,r)}$  are either polynomial equalities or polynomials inequalities.

We have some group actions on  $\tilde{\mathcal{C}}_{(n,r)}$ :

i) An action of  $(\mathbb{Z}/2\mathbb{Z})^n$  is induced by the action of  $\mathbb{Z}/2\mathbb{Z}$  in each factor of  $(\mathbb{R}^{(3 \times 2)})^n$  by

$$\left( \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}, \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}, \begin{pmatrix} -a_k \\ -b_k \\ -c_k \end{pmatrix} \right).$$

This changes the direction of a cylinder attached to the opposite direction.

ii) An action of the orthogonal group  $O(3)$  in all  $2n$  factors of  $(\mathbb{R}^{(3 \times 2)})^n \cong (\mathbb{R}^3)^{2n}$  simultaneously, which corresponds to an orthogonal transformation of the whole configuration.

iii) An action of the symmetric group  $\Sigma_n$ , interchanging the  $n$  factors of  $(\mathbb{R}^{(3 \times 2)})^n$ , i.e interchanging the  $n$  cylinders.

In fact, we can define an actions of  $G := (\mathbb{Z}/2\mathbb{Z})^n \times O(3)$  and consider the ordered configuration space

$$\hat{\mathcal{C}}_{(n,r)} := \tilde{\mathcal{C}}_{(n,r)} / G.$$

Further dividing out the induced action by  $\Sigma_n$  gives the corresponding unordered configuration space:

$$\mathcal{C}_{(n,r)} := \hat{\mathcal{C}}_{(n,r)} / \Sigma_n.$$

With this definition, Question 35 can be rephrased as follows: What is the largest  $n$  such that  $\mathcal{C}_{(n,r)}$  (or equivalently  $\tilde{\mathcal{C}}_{(n,r)}$ ) is non-empty? Similarly, the statement  $S(n, r)$  is equivalent to the non-emptiness of  $\mathcal{C}_{(n,r)}$  (or of  $\tilde{\mathcal{C}}_{(n,r)}$ ).

Kuperberg repeated Question 35 on MathOverflow [Kup14] and also asked about some properties of the configuration space  $\mathcal{C}_{(6,1)}$ . He asked if it  $\mathcal{C}_{(6,1)}$  connected, more specifically, if the octahedral configuration is an isolated point and what more can be said about  $\mathcal{C}_{(6,1)}$ . Parts of the next sections have been given as an answer by the author to this MathOverflow post [Fir14a].

### 3.1.3 New lower bounds

In order to obtain lower bounds for  $\nu(r)$  and  $\rho(n)$  we strive to find a valid configuration such that  $S(n, r)$  is true. For each  $n$ , the circular configuration provides the lower bound

$$\rho(n) \geq \frac{\sin(\frac{\pi}{n})}{1 - \sin(\frac{\pi}{n})}.$$

The beginning of the decimal expansion of these values is displayed in Table 3.1. Asymptotically this provides a lower bound

$$\rho(n) \geq \frac{\pi}{n} + \frac{\pi^2}{n^2} + \frac{5}{6} \frac{\pi^3}{n^3} + \frac{2}{3} \frac{\pi^4}{n^4} + \Omega(n^{-5}).$$

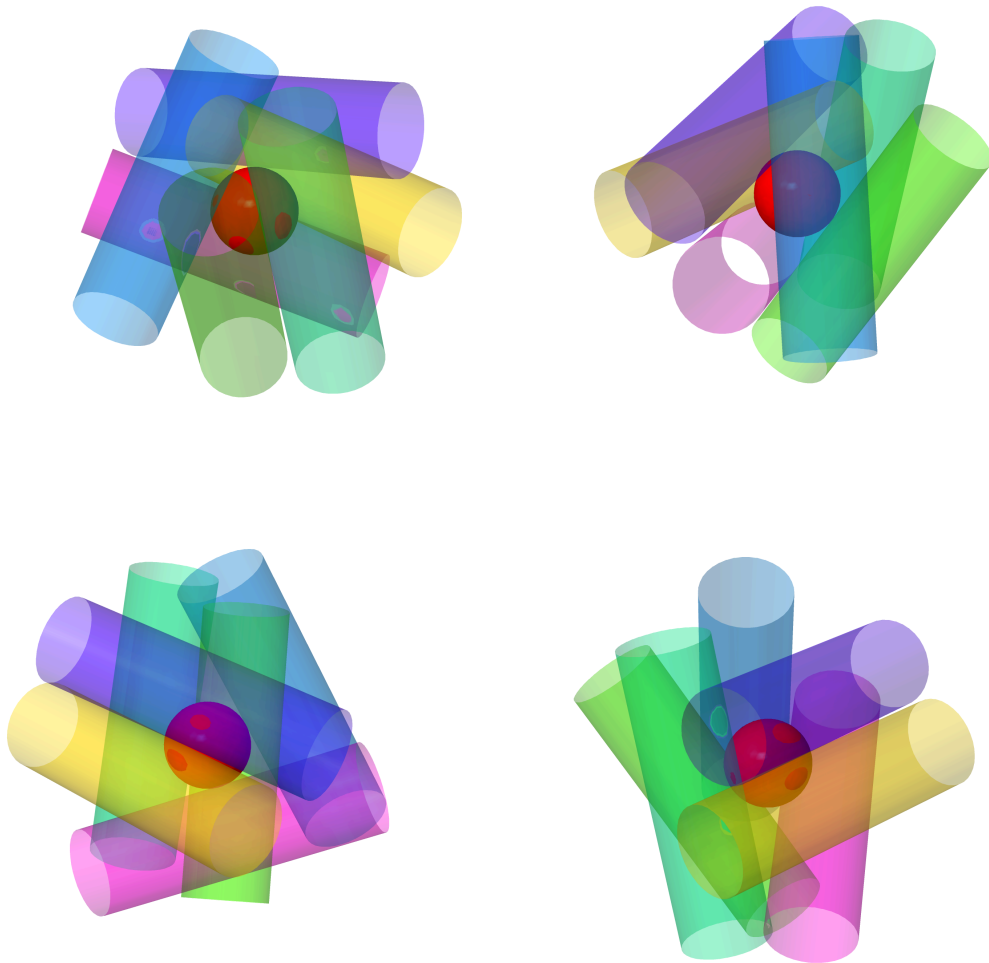


Figure 3.4: An arrangement of 6 cylinders with  $r \geq 1.0496594$ .

This was the best known bound. In particular, for  $n = 6$ , we obtain  $\rho(6) \geq 1$  by the circular configuration. The octahedral configuration gives the same lower bound. Therefore it might be tempting to conjecture  $\rho(6) = 1$ . In this context, Heppes and Szabó observe that both the circular and the octahedral configuration

“are ‘tight’ in the sense that no single cylinder can be replaced by a different one.”  
 [HS91, p. 112]

One result of this section is a disproof of this conjecture. We will show that  $\rho(6) > 1$  and that there is a configuration of 6 cylinders of radius 1 such that *every* cylinder can be replaced by a different one and can be moreover replaced by one with larger radius.

$n$	3	4	5	6	7	8
$\frac{\sin(\frac{\pi}{n})}{1-\sin(\frac{\pi}{n})}$	6.464101	2.414213	1.425919	1	0.766421	0.619914
new lower bound			1.45289	1.04965	0.846934	

Table 3.1: Lower bounds for  $\rho(n)$ .

The set  $\tilde{\mathcal{C}}_{(n,r)}$  is a semi-algebraic set, hence for each  $n$ , we can set up a non-linear program that maximizes  $r$  as an objective function and tries to find feasible points in  $\tilde{\mathcal{C}}_{(n,r)}$ . We used the non-linear solver SCIP, [Ach09], in order to find configurations for  $n = 5, 6$  and  $7$ . The results are purely heuristic; they only provide lower bounds and no claim is made that these lower bounds are tight. See Table 3.1 for a summary of the results.

In particular, we obtain  $\rho(6) > 1.049659$ : We can pack 6 cylinders of radius slightly larger than one. The results obtained from the optimization are numerical, but we can find exact coordinates for the cylinders: For the points  $(x_i, y_i, z_i)$ , we use the method from Section 1.2.2 to find rational coordinates. For the vectors tangent to them we use similar methods to find exact coordinates close to the given floating point coordinates. The coordinates for such a configuration are provided in Appendix B. It can be shown in exact arithmetic that the corresponding configuration really lies in  $\tilde{\mathcal{C}}_{(6,r)}$ . See Figure 3.4 for a drawing of this configuration. In this configuration we can shrink the radius from  $r$  to 1, while the cylinders stay tangent to the sphere. This gives us the desired configuration of 6 unit cylinders touching a sphere, such that we have room to move each cylinder a little bit. Hence  $\tilde{\mathcal{C}}_{(6,1)}$  contains a point that has a neighborhood homeomorphic to a ball of dimension 18.

### 3.1.4 New upper bounds

The best known method to obtain upper bounds is presented by Braß and Wenk [BW00]; the idea is to intersect a cylinder attached to the unit sphere with a larger concentric sphere, see Figure 3.5a. We generalize their method by considering not only the unit cylinder, but a cylinder  $C_r$  of radius  $r$ . Also, for the larger sphere we want to be able to vary the radius, which we choose to be such that its square is  $s$ :

$$S_{\sqrt{s}}^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = s\}$$

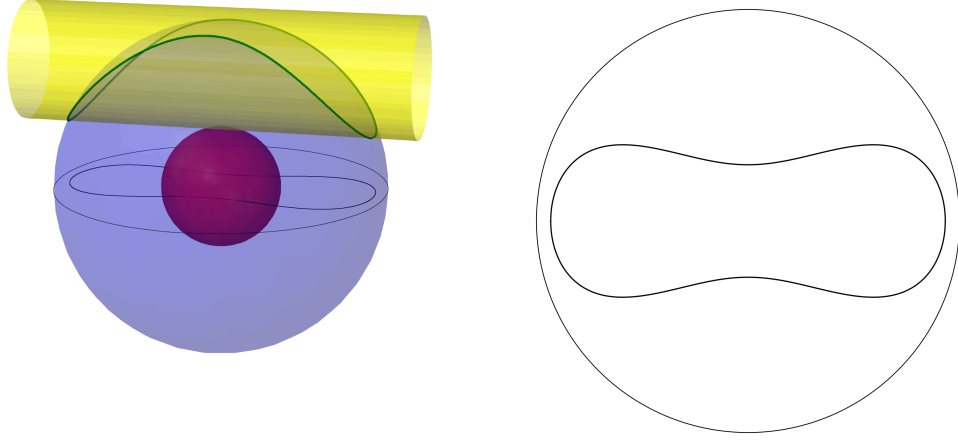
$$C_r := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + (z - (1 + r))^2 \leq r^2\}$$

The intersection of  $C_r$  and  $S_{\sqrt{s}}^2$  lies completely in the upper hemisphere of  $S_{\sqrt{s}}^2$ . To calculate the area of the intersection  $S_{\sqrt{s}}^2 \cap C_r$  we proceed analogously to Braß and Wenk [BW00, p. 282f] and define a region

$$D_{(s,r)} := \{(x, y) \in \mathbb{R}^2 \mid (x, y, \sqrt{s - x^2 - y^2}) \in S_{\sqrt{s}}^2 \cap C_r\}$$

as the projection of this intersection to the disc of radius  $\sqrt{s}$  in  $\mathbb{R}^2$ , see Figure 3.5. If

$$1 \leq s \leq (2r + 1)^2,$$



(a) Intersection of a cylinder and a sphere... (b) ...projected to the region  $D_{(s,r)}$ .

Figure 3.5

then  $D_{(s,r)}$  has one connected component and this region can be characterized as

$$D_{(s,r)} = \{(x, y) \in \mathbb{R}^2 \mid y^2 + (\sqrt{s - x^2 - y^2} - (1 + r))^2 \leq r^2\} \quad (3.1)$$

$$= \left\{ (x, y) \in \mathbb{R}^2 \mid |x| \leq \sqrt{s-1} \text{ and } |y| \leq \frac{\sqrt{4(s-x^2)(r+1)^2 - (x^2 - 2r - s - 1)^2}}{2(r+1)} \right\}. \quad (3.2)$$

To calculate the area we consider the following integral:

$$\text{area}(S_{\sqrt{s}}^2 \cap C_r) = \int_{D_{(s,r)}} \frac{\sqrt{s}}{\sqrt{s - x^2 - y^2}} dy dx \quad (3.3)$$

$$= \int_{-\sqrt{s-1}}^{\sqrt{s-1}} \int_{-\frac{\sqrt{4(s-x^2)(r+1)^2 - (x^2 - 2r - s - 1)^2}}{2(r+1)}}^{\frac{\sqrt{4(s-x^2)(r+1)^2 - (x^2 - 2r - s - 1)^2}}{2(r+1)}} \frac{\sqrt{s}}{\sqrt{s - x^2 - y^2}} dy dx \quad (3.4)$$

$$= 2\sqrt{s} \int_{-\sqrt{s-1}}^{\sqrt{s-1}} \arcsin \left( \frac{\sqrt{4(s-x^2)(r+1)^2 - (x^2 - 2r - s - 1)^2}}{2(r+1)\sqrt{s-x^2}} \right) dx \quad (3.5)$$

The equality in line (3.4) is only valid if  $D_{(s,r)}$  has exactly one connected component this is the case if  $1 \leq s \leq (2r+1)^2$ . For the surface area of the sphere of radius  $\sqrt{s}$  we have  $\text{area}(S_{\sqrt{2}}^2) = 4\pi s$ . For each radius  $r$  of a touching cylinder  $C_r$  there is a radius  $\sqrt{s}$  such that the ratio of the intersection  $S_{\sqrt{s}}^2 \cap C_r$  over the surface area of  $S_{\sqrt{s}}^2$  is maximized. We define

$n$	$r$ , s.t. $R(r) \approx 1/n$	$s$ where maximum is attained
3	8.123015726697261129873583	33.03418955859792227769316
4	3.119690083242860621653928	13.07503514026454367498966
5	1.893940144132469649262296	8.216388387022702088808793
6	1.362728791829127036542209	6.126706244255249463805262
7	1.069484644843172117577150	4.981942051842078249194959
8	0.8842320082596736518347155	4.263994114455319289020014
9	0.7566957511004313621344271	3.773047212617967033981508
10	0.6635122014473178737120738	3.416553700137037614676634
11	0.5923978489139096427663741	3.146028266886454600947278

Table 3.2

$R(r)$  to be this ratio:

$$R(r) := \max_{1 \leq s \leq (2r+1)^2} \frac{\text{area}(S_{\sqrt{s}}^2 \cap C_r)}{\text{area}(S_{\sqrt{s}}^2)}.$$

It is possible to evaluate  $R(r)$  for various  $r$  numerically to any desired accuracy. For example for  $r = 1$  the maximum is obtained for  $s \approx 4.71207860139335456983$  and we get  $R(1) \approx 1/7.32863173838663276115$ . Since  $R(1) > 1/8$ , we see that  $S(8, 1)$  is false: It is impossible to attach 8 non-intersecting cylinders of radius 1 to the unit sphere. In other words we get  $\nu(1) < 8$  and  $\rho(8) < 1$ . This was shown by Braß and Wenk; for simplicity, they choose  $s = 4.7$ .

Given a natural number  $n$ , we can ask for the radius  $r$  such that  $R(r) = 1/n$ . In Table 3.2 we provide numerical values such that  $R(r) \approx 1/n$  for the first few integers  $n \geq 3$ . From this we can read off upper bounds for  $\nu(n)$  and  $\rho(n)$ . For example for 8 cylinders, the radius  $r$  cannot be larger than 0.88423201 in order to obtain a valid configuration:  $\rho(8) < 0.88423201$  and  $\nu(0.88423201) < 8$ .

Braß and Wenk observe that from these upper bounds we obtain  $\nu(r) = O(r^{-\frac{3}{2}})$  and  $\rho(n) = O(n^{-\frac{2}{3}})$  as asymptotic behavior. From the circular configuration above, we obtain  $\rho(n) \geq \pi/n$ , so  $\rho(n) = \Omega(n^{-1})$  asymptotically. We conclude with a Question.

*Question 36.* How does  $\rho(n)$  grow asymptotically as  $n$  goes to infinity?

## 3.2 Embedding of planar graphs with prescribed face areas

### 3.2.1 Introduction

We introduce a few definitions and basic concepts from the theory of planar graphs. Standard references include [Die97, Chapter 4] and [Har69, Chapter 11].

We only consider finite simple graphs.

**Definition 37.** A graph  $G$  is *planar* if there exists a topological embedding from  $G$ , viewed as 1-dimensional  $CW$ -complex, to  $\mathbb{R}^2$ . A *plane graph* is a graph together with such an embedding. Given a plane graph  $G$ , the connected components of the complement of the image of  $G$  in  $\mathbb{R}^2$  are called *faces* of  $G$ . Also the set of vertices contained in the boundary of such a region is called a *face*, if there is no danger in confusing these two notions of faces. If the region associated to a face is bounded, the face is called *inner face* and otherwise *outer face*. Faces consisting of three vertices are called *triangles*. A plane graph is a *triangulation* if all its faces are triangles.

**Definition 38.** Two plane graphs are called *combinatorially equivalent* if there is an isomorphism of graphs between them, such that the outer face is mapped to the outer faces and the inner faces are mapped to the inner faces.

**Definition 39.** Given a planar graph  $G$ , a *straight line embedding* of  $G$  is a topological embedding, such that all edges of  $G$  are mapped to line segments.

**Definition 40.** A graph is called  *$k$ -vertex-connected* or  *$k$ -connected* for short, if the graph remains connected after removing any set of  $k - 1$  vertices.

**Definition 41.** Let  $G$  be a plane graph and  $f$  an inner face of  $G$ . We define the *stellation* on  $f$  of  $G$  to be a plane graph that is obtained by inserting a vertex in the interior of  $f$ , which is then connected by edges to every vertex of  $f$ .

Planar graphs are characterized by the Kuratowski–Wagner criterion: A graph  $G$  is planar if and only if  $G$  does not contain  $K_5$  or  $K_{3,3}$  as a minor. We are interested in straight line embeddings of planar graphs. Fáry’s Theorem asserts that every Graph  $G$  can be drawn without crossings, such that all edges are straight line segments, see [Wag36] and [Fá48]. By Steinitz’s Theorem [Ste22, Satz 43, p. 77], graphs of 3-polytopes are exactly 3-connected planar graphs with more than 4 vertices. For drawing 3-connected planar graphs there is the “spring embedding theorem” by Tutte, [Tut63]: If we choose an outer face of a 3-connected planar graph  $G$  and fix a convex polygon as coordinates for the outer face, we can find a straight line drawing of  $G$  such that all inner vertices are the respective barycenters of the neighboring vertices. Then all the inner faces are realized as convex polygons. The coordinates of the inner vertices can be found by solving a system of linear equations. We can set up different system of equations in order to find straight line embeddings with different properties, and then solve these systems. In this chapter we investigate straight line drawings with prescribed face areas. A similar approach can be used for optimizing other features of straight line drawings of graphs.

### 3.2.2 Embeddings with prescribed areas

Let  $G$  be a plane graph on  $n$  vertices  $v_1, \dots, v_n$  with an distinguished outer face  $f_0$  and  $k$  inner faces  $f_1, \dots, f_k$ . The faces are given as ordered lists of vertices, such that they are consistently oriented; if the face  $f_i$  has  $n_i$  vertices, it is given as  $f_i = (v_{i,1}, \dots, v_{i,n_i})$ . The vertices of the outer face  $f_0 = (v_{0,0}, \dots, v_{0,n_0})$  are fixed as a convex polygon:  $c(v_{0,j}) = (v_{0,j}^x, v_{0,j}^y) \in \mathbb{R}^2$ . For each inner face  $f$ , we are given a positive weight  $w(f)$ .

**Definition 42.** An *embedding with prescribed areas* is a straight line drawing of  $G$  without crossings, such that

$$\text{area}(f_i) = w(f_i) \text{ for all } 1 \leq i \leq k.$$

If all the weights  $w(f_i)$  are equal, we call this drawing an *equiareal embedding*.

Consider the following system of equations with real variables  $v = (v^x, v^y)$  for all vertices  $v$  that are not contained in the outer face.

$$\begin{aligned} \sum_{j=1}^{n_i-1} \det \begin{pmatrix} v_{i,j}^x & v_{i,j+1}^x \\ v_{i,j}^y & v_{i,j+1}^y \end{pmatrix} &= w(f_i) \text{ for all } 1 \leq i \leq k \\ \sum_{j=1}^{n_i-1} \det \begin{pmatrix} v_{0,j+1}^x & v_{0,j}^x \\ v_{0,j+1}^y & v_{0,j}^y \end{pmatrix} &= \sum_{i=1}^k w(f_i) \\ (v_i^x, v_i^y) &\neq (v_j^x, v_j^y) \text{ for } i \neq j. \end{aligned} \tag{3.6}$$

**Proposition 43.** *There is a straight line drawing of  $G$  with prescribed areas if and only if the System 3.6 has a solution.*

This system ensures that all inner faces  $f_i$  are polygons with area  $w(f_i)$ . We can add the corresponding equations to ensure that the inner faces are *convex* polygons. Assume that we construct  $G'$  from  $G$  by subdividing a face  $f$  of  $G$  into two faces  $f_1$  and  $f_2$  of  $G'$  and assume instead of  $w(f)$  we are given two weights  $w(f_1)$  and  $w(f_2)$  that sum up to  $w(f)$ . The corresponding system of equations for  $G'$  will have the same number of variables, but one equation for the face  $f$  is replaced by two equations for  $f_1$  and  $f_2$  that, when added, yield the original constraint. Therefore, for a fixed number of variables, the number of constraints is maximized in triangular graphs, i.e. triangulations of the triangle. If we have  $n$  vertices and fix the coordinates of the outer face, i.e. the triangle, there are  $2(n-3) = 2n-6$  variables for the coordinates of the inner variables. In this situation we have  $2n-5$  inner faces by double counting and Euler's formula. Hence the system has  $2n-5$  constraints, but one of them is just the sum of the other  $2n-6$  and hence the degrees of freedom equals the number of constraints. Therefore we consider the particularly interesting case of *triangulations of the triangle* in the next sections.

### 3.2.3 Finding graph drawings with prescribed areas

Given a planar graph with few vertices together with weights for its faces, we propose the following method to find a straight line embeddings with prescribed areas, i.e. solving the corresponding non-linear system of equations. This method is a slight variation of Method 31 from Chapter 2.

#### Method 44.

*Step 1 Find a numerical solution of system 3.6, using non-linear optimization.*

*Step 2 Sharpen solution to a higher precision using Newton's method.*

*Step 3 Find an exact algebraic solution by using integer relation algorithms.*

*Step 4 Verify the solution by exact calculation in the field of real algebraic numbers.*

For Step 1 we use the non-linear optimizer to get a numerical result, e.g. with a precision set to  $10^{-9}$ . We can use for example the optimizer SCIP [Ach09]. In general we expect a running time, which is exponential in the number of vertices of planar graph. Depending on the expected algebraic complexity of the solution we sharpen the solution using multi-dimensional Newton's method to a precision somewhere between  $10^{-20}$  and  $10^{-6000}$ . We can use the arbitrary precision library mpmath [J<sup>+</sup>13]. For the third step, we use the integer relation algorithm ([LLL82]) and PSLQ ([FBA99]) as they are implemented in Pari/GP [Par14]. Step 4 can be done, because it is possible to add and multiply in the field of real algebraic number effectively on a computer, see [BPR06]. We use the implementation in sage [S<sup>+</sup>14].

We apply this method to different kinds of planar graphs. As examples we explore embeddings of triangulations of the triangle in the next sections. It turns out that for many triangulations of the triangle with few vertices, Step 4 seems to be the bottleneck in terms of running time.

This method does not provide a way of proving that a given graph with prescribed weights is *not* embeddable with prescribed areas. However it can be proved that not all graphs are realizable with arbitrary areas, see the next section.

### 3.2.4 A counterexample

**Proposition 45.** *Let  $G$  be a triangulation of the triangle  $ABC$  into 7 triangles  $ABF$ ,  $BCE$ ,  $CAD$ ,  $EDF$ ,  $DEC$ ,  $AFD$  and  $BEF$  with a weight function  $w$ , defined as*

$$w(T) := \begin{cases} 1 & \text{if } T \in \{ABF, BCE, CAD, EDF, DEC\} \\ 3 & \text{if } T \in \{AFD, BEF\}, \end{cases}$$

*see Figure 3.6. There is no straight line embedding of  $G$  such that the areas of the faces equal the weights of the faces.*



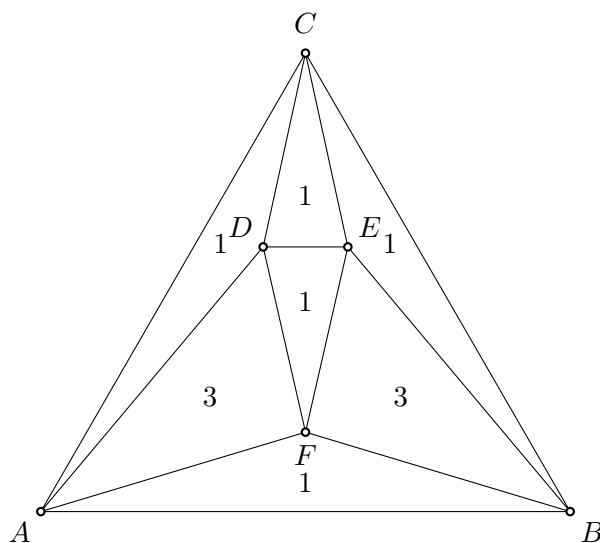


Figure 3.6: It is impossible to find a straight line embedding of the octahedron graph with prescribed face areas as indicated.

*Proof.* We show in an elementary way that the corresponding system of equations, see Proposition 43, has no real solution. Since an affine linear transformation does not change the proportion of the areas in the triangulation, we can choose coordinates for the outer triangle as  $A = (-1, 0)$ ,  $B = (1, 0)$  and  $C = (0, 1)$ . Then we obtain the following system of equations.

$$\frac{2}{11} = 2F_y \quad (3.7)$$

$$\frac{2}{11} = -E_x - E_y + 1 \quad (3.8)$$

$$\frac{2}{11} = D_x - D_y + 1 \quad (3.9)$$

$$\frac{2}{11} = D_y E_x - D_x E_y - (D_y - E_y)F_x + (D_x - E_x)F_y \quad (3.10)$$

$$\frac{2}{11} = -(D_y - 1)E_x + D_x E_y - D_x \quad (3.11)$$

$$\frac{6}{11} = D_y F_x - (D_x + 1)F_y + D_y \quad (3.12)$$

$$\frac{6}{11} = -E_y F_x + (E_x - 1)F_y + E_y \quad (3.13)$$

Equations (3.7), (3.8) and (3.9) assert that the points  $F$ ,  $E$  and  $D$  lie on lines parallel to the sides of  $ABC$  with a certain distance from them. If we substitute those equations, the system

simplifies to

$$\frac{2}{11} = -\frac{2}{11} (11 D_y - 5) E_y - (D_y - E_y) F_x + \frac{10}{11} D_y - \frac{18}{121} \quad (3.14)$$

$$\frac{2}{11} = \frac{2}{11} (11 D_y - 10) E_y - \frac{20}{11} D_y + \frac{18}{11} \quad (3.15)$$

$$\frac{6}{11} = D_y F_x + \frac{10}{11} D_y - \frac{2}{121} \quad (3.16)$$

$$\frac{6}{11} = -E_y F_x + \frac{10}{11} E_y - \frac{2}{121} \quad (3.17)$$

We use equation (3.16) and (3.17) to express  $D_y$  and  $E_y$  in terms of  $F_x$ . If we substitute these expressions in equation 3.15 or 3.14, we obtain  $(11F_x)^2 + 2 = 0$ , which has no real solution.  $\square$

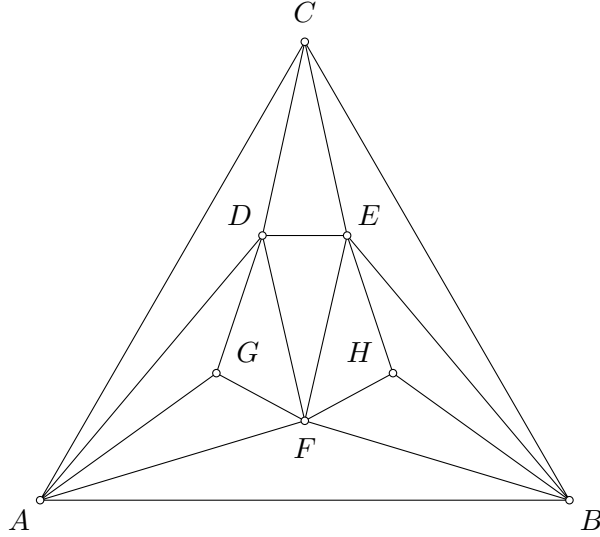


Figure 3.7: There is no equiareal embedding of this graph.

### 3.2.5 Embeddings with all equal areas

**Corollary 46.** *Let  $G$  be a triangulation of the triangle  $ABC$  into 11 triangles  $ABF$ ,  $BCE$ ,  $CAD$ ,  $EDF$ ,  $DEC$ ,  $AFG$ ,  $FDG$ ,  $AGD$ ,  $BEH$ ,  $HEF$  and  $BHF$ , see Figure 3.7. There is no equiareal straight line embedding of  $G$ .*

*Proof.* The existence of an equiareal embedding of  $G$  would contradict Proposition 45.  $\square$

**Proposition 47.** *Let  $G$  be a graph that is obtained from successive stellation of the triangle, which is its outer face. For every weight function  $w$ , there is an embedding with prescribed area of  $G$ .*

*Proof.* By an inductive argument this can be reduced to the case of getting an embedding with prescribed areas for the triangle stellated once. This embedding is given by barycentric coordinates.  $\square$

**Proposition 48.** *Given a 3-connected planar graph  $G$  with odd positive integers  $w(f_i)$  as weights for all inner faces  $f_i$ . Let  $H$  be a graph that is formed by some stellation on all inner faces  $f_i$ , repeated  $\frac{w(f_i)-1}{2}$  times. Then there is a straight line drawing of  $G$  with the area corresponding to  $w$  if and only if there is an equiareal embedding of  $H$ .*

*Proof.* Given an embedding with prescribed areas  $w$  of  $G$ , we can use Proposition 47 to find an equiareal embedding within each inner face of  $G$ . This gives an equiareal embedding of  $H$ . Conversely, every equiareal embedding of  $H$  already contains an embedding with prescribed areas  $w$  of  $G$ .  $\square$

In this sense, allowing stellation corresponds to allowing arbitrary odd integers as weights. Notice that all 4-connected triangulations of the triangle do not contain any stellated faces. This is what makes the question whether this family of graphs can always be embedded with all equal areas interesting.

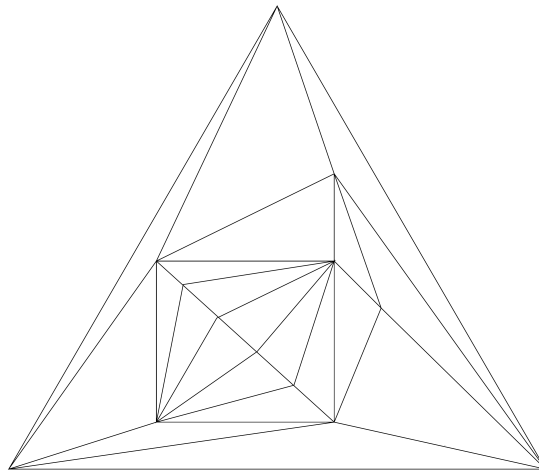


Figure 3.8: A triangulations with 13 vertices, image taken from [SS06, Fig 3].

### 3.2.6 Equiareal embeddings of 4-connected triangulations of the triangle

The following theorem shows that the counterexample to equiareal embeddability in Corollary 46 is indeed the one with the minimal number of vertices.

**Theorem 49.** *All triangulations of the triangle with  $\leq 8$  vertices possess equiareal embeddings, except for the triangulation described in Corollary 46.*

*Proof.* The triangulations have been generated by using plantri [BM11]; in this case there are 99 relevant triangulations. We use Method 44, described in Section 3.2.3. All embeddings and their exact coordinates can be found in Appendix C. We fix the outer triangle to have coordinates  $(-1, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ . Then all coordinates of the embeddings are algebraic numbers of degree less or equal than 2, so they can be expressed as square roots.  $\square$

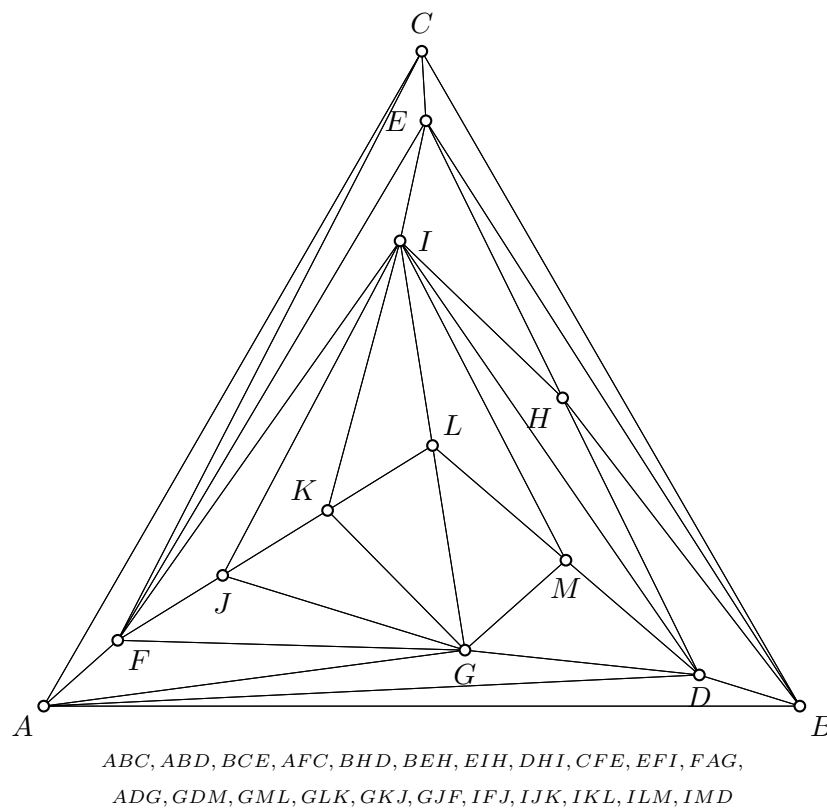


Figure 3.9: An equiareal embedding of the graph in Figure 3.8 and a list of its faces.

In an unpublished note by Sabariego and Stump [SS06], they give a counter example of a triangulation of the triangle that does not possess an equiareal embedding similar to the one discussed in Corollary 46, Figure 3.7, but with one more vertex. Also they try to find equiareal embeddings of 4-connected triangulations of the triangles, by using matlab, up to a precision of 0.01. The first example, where this fails is given in Figure 3.8.

Attempts to prove that this “possible counterexample” is in fact not embeddable with equal areas are made using singular, maple and mathematica. These attempts are bound to fail, since the graph in question is in fact embeddable with all equal areas, see Figure 3.9 and Table 3.3.

All data produced during the proofs of the following theorems can be found on the author’s web page:

<http://page.mi.fu-berlin.de/moritz/>

$A_x$	-1.0000000...	$x + 1$
$A_y$	0.000000000...	$x$
$B_x$	1.0000000...	$x - 1$
$B_y$	0.000000000...	$x$
$C_x$	-0.000000000...	$x$
$C_y$	1.0000000...	$x - 1$
$D_x$	0.73420368...	$551353635x^7 - 290554614x^6 - 185013171x^5 - 368624550x^4 + 296923109x^3 + 207875510x^2 - 159866746x + 16493844$
$D_y$	0.047619048...	$-21x + 1$
$E_x$	0.010941323...	$-162002673x^6 + 144629037x^5 + 107810388x^4 - 146237364x^3 + 46949273x^2 - 5579073x + 55612$
$E_y$	0.89382058...	$-54000891x^8 + 34077393x^7 + 37739604x^6 + 13615973x^5 - 25211753x^4 - 39250208x^3 + 39539954x^2 - 9710604x + 665640$
$F_x$	-0.80426458...	$-1967175315x^5 - 2085095628x^4 - 454236615x^3 - 41278629x^2 - 1048040x - 7425$
$F_y$	0.10049732...	$-655725105x^{12} + 304168284x^{11} + 531619473x^{10} + 251047384x^9 - 290333673x^8 + 104228801x^7 + 79925280x^6 - 566745567x^5 - 253169362x^4 + 288211695x^3 + 549374326x^2 - 386215542x + 33004116$
$G_x$	0.11421795...	$1966725674928x^8 + 1489309937640x^7 - 1563546503925x^6 + 271739154708x^5 + 1247071121223x^4 - 1527718943351x^3 - 1677169185380x^2 - 260056151257x + 53644895414$
$G_y$	0.085512500...	$-72841691664x^5 + 50612846760x^4 - 11127966003x^3 + 1094008545x^2 - 50202336x + 878180$
$H_x$	0.37257250...	$-1049777321040x^6 + 745206072408x^5 + 624992192496x^4 - 817700863770x^3 + 306874278907x^2 - 43938578500x + 1477479775$
$H_y$	0.47071981...	$-7776128304x^6 + 3379301856x^5 + 5154079896x^4 - 4524312492x^3 + 1316942277x^2 - 154048163x + 6047332$
$I_x$	-0.05743043...	$-1625280245253x^7 + 270964348389x^6 + 1153047543921x^5 - 632627551395x^4 + 108502722692x^3 - 2712577950x^2 - 645802316x$
$I_y$	0.71045709...	$-10173050423991x^8 + 176090293197x^7 + 4867116206967x^6 + 5085063961362x^5 - 443009725977x^4 - 6115450605811x^3 + 3913232625685x^2 - 907834404164x + 73580561900$
$J_x$	-0.52671180...	$42127263956957760x^5 + 40935840967537488x^4 + 12582634079208753x^3 + 1544062690559592x^2 + 61146695367345x - 393408208106$
$J_y$	0.19965981...	$-263685466989846720x^8 + 200735183349018864x^7 + 145801868821409871x^6 + 32086033108443027x^5 - 125967873690136800x^4 - 161496808913829811x^3 + 224713629650903725x^2 - 81653113889382032x + 8809135861919972$
$K_x$	-0.24915902...	$10531815989239440x^6 - 1227030436362024x^5 - 6299255112723279x^4 - 2599285558838031x^3 - 390787756054206x^2 - 16083593673527x + 626468411627$
$K_y$	0.29882231...	$-65921366747461680x^8 + 3867574133267208x^7 + 28673493195799431x^6 + 28649748039431841x^5 - 5217866315880267x^4 - 3834135505511157x^3 + 28324136407631741x^2 - 7597190117785039x + 720380009313628$
$L_x$	0.028393759...	$-312053807088576x^5 - 248166247818384x^4 - 59599906630047x^3 - 2398222359540x^2 + 432014796087x - 8801691112$
$L_y$	0.39798480...	$-1953225681406272x^6 + 695487975607152x^5 + 1243965988557873x^4 - 1040566332018192x^3 + 318022341367577x^2 - 43868569812002x + 2290425257780$
$M_x$	0.38129872...	$74892913701258240x^5 - 27400195484500608x^4 - 6962340558846204x^3 + 2984341863454434x^2 - 135112325788377x - 20846810904601$
$M_y$	0.22280192...	$93754832707501056x^5 - 74730417376358016x^4 + 23442664647403764x^3 - 3618033495115536x^2 + 274779852139107x - 8220093808982$

Table 3.3: Approximate values and minimal polynomials of exact values of the coordinates of the embedding displayed in Figure 3.9.

$n$	6	7	8	9	10	11	12	13	14	15	16	17
	1	1	3	13	47	217	1041	5288	27844	150608	831229	4660535

Table 3.4: The number of 4-connected triangulations of the triangle with  $n$  vertices (and  $2n - 5$  inner faces).

**Theorem 50.** *For  $n \leq 11$  there are equiareal embeddings of all 4-connected triangulations of the triangle with  $n$  vertices.*

*Proof.* Here a proof can be obtained in exactly the same manner as for Theorem 49; we find exact algebraic solutions using Method 44. The number of triangulations to be considered is  $1 + 1 + 3 + 13 + 47 + 217 = 282$ , see Table 3.4. We fix the outer triangle to be have coordinates  $(-1, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ . Then all coordinates of the embeddings are algebraic numbers of degree less or equal than 13.  $\square$

**Theorem 51.** *For  $n = 12$  there are embeddings of all 4-connected triangulations of the triangle with  $n$  vertices (and  $f := 2n - 5$  inner faces) such that for all inner triangles  $T$ , we have*

$$\left| \text{area}(T) - \frac{1}{f} \right| < 10^{-100\,000},$$

while the area of the outer face is normalized to 1.

*Proof.* We again generate the 1 041 triangulations in question by using plantri [BM11], and we apply Method 44. For 966 triangulations we obtain exact solutions. In the remaining 75 cases, Step 4 of Method 44 is not completed: The verification of the results in exact algebraic arithmetic does not finish in a reasonable amount of time (we let it run for more than a day on a modern desktop PC). However, we can take the algebraic solution for every coordinate, which we obtained in Step 3, and obtain a bit more 100 000 digits very quickly, using one-dimensional Newton's method. With these very good approximations we can verify that the error for the faces is as small as claimed.  $\square$

It seems highly plausible that the algebraic numbers that we found are in fact the correct solutions yielding an embedding with all equal areas; in Step 2 of Method 44 we only used 2 000 digits from the multi-dimensional Newton's method to determine the algebraic numbers, which then give a much higher precision.

**Theorem 52.** *For all  $13 \leq n \leq 15$  there are embeddings of all 4-connected triangulations of the triangle with  $n$  vertices (and  $f := 2n - 5$  inner faces) such that for all inner triangles  $T$ , we have*

$$\left| \text{area}(T) - \frac{1}{f} \right| < 10^{-9},$$

while the area of the outer face is normalized to 1.

*Proof.* Again, plantri is used to generate the  $5\,288 + 27\,844 + 150\,608 = 183\,740$ , see Table 3.4, relevant triangulations of the triangle. Because of the large number of cases we only apply the first step of Method 44 and solve the corresponding non-linear system numerically.  $\square$

**Conjecture 53.** *There are equiareal embeddings for all 4-connected triangulations of the triangle.*

Every 4-connected triangulation of the triangle gives rise to a 4-connected simplicial 3-polytope. By a result from Dillencourt and Smith [DS96], 4-connected simplicial 3-polytopes are always inscribable. In view of Conjecture 53 we ask if a similar result is valid for 4-connected simplicial 3-polytopes.

*Question 54.* Can every combinatorial type of 4-connected simplicial 3-polytope be inscribed in the sphere, such that all of its faces have equal area?





## Appendix A

# Simplicial neighborly polytopes and self-dual 2-colored necklaces

The number of simplicial neighborly  $(2m-3)$ -polytopes with  $2m$  vertices has been determined by Altshuler and McMullen [AM73, Th. 1, p. 263] as

$$2^{\lfloor \frac{m-3}{2} \rfloor} + \frac{1}{4m} \sum_{\substack{h|m \\ h \text{ odd}}} \phi(h) 2^{\frac{m}{h}} \quad (\text{A.1})$$

where  $\phi$  denotes Euler's function. Eleven years later Palmer and Robinson counted the number of self-dual 2-colored necklaces on  $2m$  beads and obtained the same formula [PR84, p. 209, 2nd display]. They seem not to have been aware of the work by Altshuler and McMullen. In both proofs the Pólya enumeration theorem is used to count the combinatorial objects. We provide a simple bijection of the two sets of combinatorial objects.

**Definition 55** (2-colored necklace, self-dual, defect, balanced). For a natural number  $m$ , a 2-colored string on  $m$  beads is an element  $a \in \{0, 1\}^m$ , written as  $a = (a_0, a_1, \dots, a_{m-1})$ . A 2-colored necklace on  $m$  beads is an equivalence class of those strings, where rotations are taken as equivalent. For  $m = 2n$ , a 2-colored necklace on  $2n$  beads is *self-dual* if for all  $i$

$$a_i = 1 - a_{i+n}.$$

Here indices are understood modulo  $2n$ . Given 2-colored necklace  $a$  on  $2n$  beads the *discrepancy* at an index  $i$  is

$$\text{disc}_i(a) := \left( \sum_{j=1}^{n-1} a_{i+j} \right) - \left( \sum_{j=1}^{n-1} a_{i+n+j} \right)$$

A 2-colored necklace on  $2n$  beads is *balanced* if for all  $0 \leq i < 2n$

$$|\text{disc}_i(a)| \leq 1$$

**Lemma 56.** *For each  $n$ , there is a bijection between self-dual 2-colored necklaces on  $2n$  beads and balanced self-dual 2-colored necklaces on  $4n$  beads.*

*Proof.* We give a map  $f$  from the set of self-dual 2-colored necklaces on  $2n$  beads to the set of balanced self-dual 2-colored necklaces on  $4n$  beads and an inverse map  $g$ .

Let  $a = (a_0, a_1, \dots, a_{2n-1})$  be a self-dual 2-colored necklace. We define a self-dual, balanced 2-colored necklace  $f(a) := b := (b_0, b_1, \dots, b_{4n})$  on  $4n$  beads by setting for  $0 \leq i < 2n$

$$\begin{aligned} b_{2i} &:= 1 - a_i \\ b_{2i+1} &:= a_{i+1} \end{aligned}$$

It can be checked that  $b$  is well-defined, self-dual and balanced.

Given a self-dual balanced 2-colored necklace  $b = (b_1, \dots, b_{4n-1})$  on  $4n$ -vertices, we define a self-dual 2-colored necklace  $g(b) := a := (a_1, \dots, a_{2n-1})$  on  $2n$  beads by setting for  $0 \leq i < 2n$

$$a_i := b_{2i}$$

It can be checked that  $a$  is well-defined and self-dual and that  $f$  and  $g$  are inverses of each other.  $\square$

**Proposition 57.** *For each  $n$  there is a bijection between self-dual 2-colored necklaces on  $2n$  beads and simplicial neighborly  $(2n - 3)$ -polytopes with  $2n$  vertices.*

*Proof.* The standard distended Gale diagrams of simplicial neighborly  $(2n - 3)$ -polytopes with  $2n$  vertices are in bijection with self-dual balanced necklaces on  $4n$  beads, see [Grü67, Sect. 6.3], [AM73]. This together with Lemma 56 completes the proof.  $\square$

There is also a relation between the self-dual 2-colored necklaces on  $2n$  beads and simplicial  $(n - 3)$ -polytopes with  $n$  vertices. This is provided by Montellano-Ballesteros and Strausz [MBS04].

## Appendix B

### A configuration of 6 kissing cylinders

$x_{0,0} \approx 1.000000000000000$	$a_{0,0} \approx 0.000000000000000$
$x_{0,1} \approx 0.000000000000000$	$a_{0,1} \approx 1.000000000000000$
$x_{0,2} \approx 0.000000000000000$	$a_{0,2} \approx 0.000000000000000$
$x_{1,0} \approx -0.761080103914984$	$a_{1,0} \approx -0.428579900666850$
$x_{1,1} \approx -0.369312591779669$	$a_{1,1} \approx -0.330791186836092$
$x_{1,2} \approx -0.533259116169372$	$a_{1,2} \approx 0.840771347903794$
$x_{2,0} \approx -0.0504003690576108$	$a_{2,0} \approx -0.240282261154000$
$x_{2,1} \approx -0.998545390551329$	$a_{2,1} \approx -0.00649232221585465$
$x_{2,2} \approx -0.0191547854999806$	$a_{2,2} \approx 0.970681350767061$
$x_{3,0} \approx -0.819169072511517$	$a_{3,0} \approx 0.326994670196851$
$x_{3,1} \approx 0.407965684647994$	$a_{3,1} \approx -0.245273732162278$
$x_{3,2} \approx 0.403145173343691$	$a_{3,2} \approx 0.912641924291252$
$x_{4,0} \approx 0.440170351845229$	$a_{4,0} \approx -0.255443986900137$
$x_{4,1} \approx 0.265623603413540$	$a_{4,1} \approx 0.952815991063124$
$x_{4,2} \approx 0.857726158319807$	$a_{4,2} \approx -0.163981879276214$
$x_{5,0} \approx 0.259868767440186$	$a_{5,0} \approx -0.605407049117424$
$x_{5,1} \approx 0.608174646997684$	$a_{5,1} \approx 0.707745934478224$
$x_{5,2} \approx -0.750061212474263$	$a_{5,2} \approx 0.364112615969947$

$r \approx 1.04965943344990874522205944077$

Table B.1: Approximate coordinates for 6 cylinders of radius  $r$  touching the unit sphere, see Section 3.1

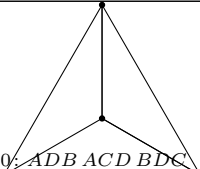
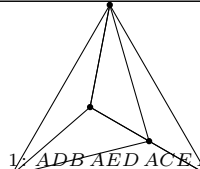
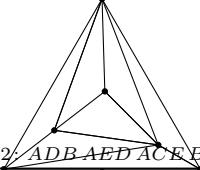
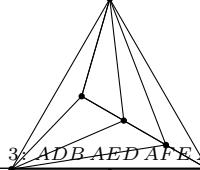
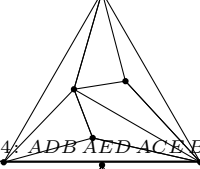
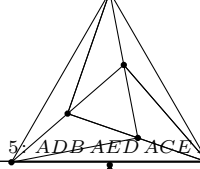
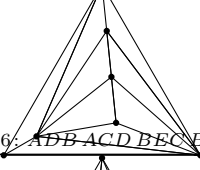
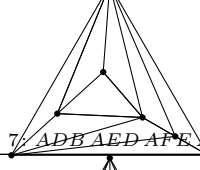
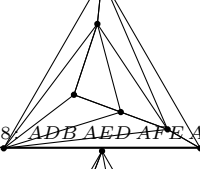
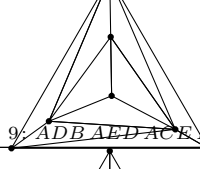
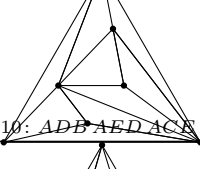
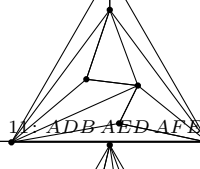
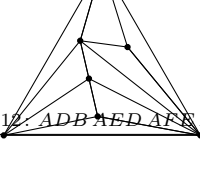
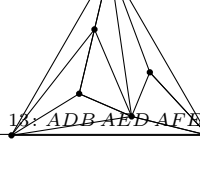
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$x_{0,1} = 0$	$a_{0,1} = 1$
$x_{0,2} = 0$	$a_{0,2} = 0$
$x_{1,0} = -$	$a_{1,0} = -$
$x_{1,1} = -$	$a_{1,1} = -$
$x_{1,2} = -$	$a_{1,2} = -$
$x_{2,0} = -$	$a_{2,0} = -$
$x_{2,1} = -$	$a_{2,1} = -$
$x_{2,2} = -$	$a_{2,2} = -$
$x_{3,0} = -$	$a_{3,0} = -$
$x_{3,1} = -$	$a_{3,1} = -$
$x_{3,2} = -$	$a_{3,2} = -$
$x_{4,0} = -$	$a_{4,0} = -$
$x_{4,1} = -$	$a_{4,1} = -$
$x_{4,2} = -$	$a_{4,2} = -$
$x_{5,0} = -$	$a_{5,0} = -$
$x_{5,1} = -$	$a_{5,1} = -$
$x_{5,2} = -$	$a_{5,2} = -$
$r =$	

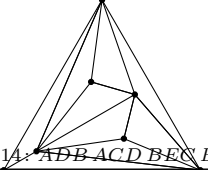
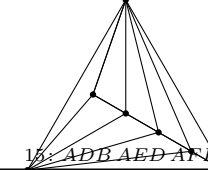
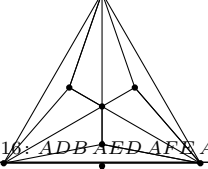
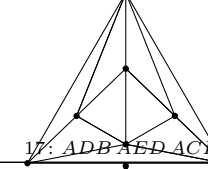
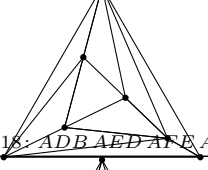
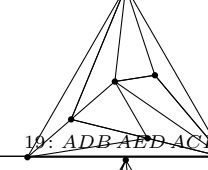
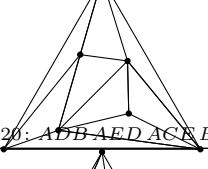
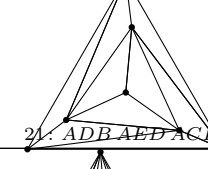

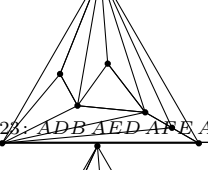
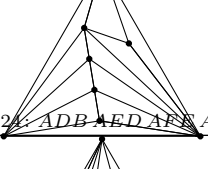
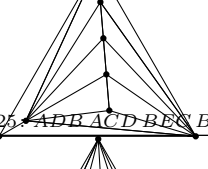
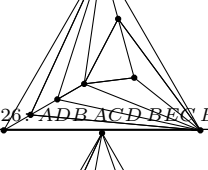
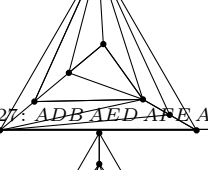
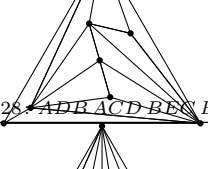
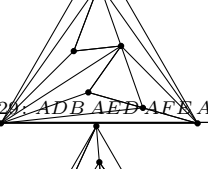
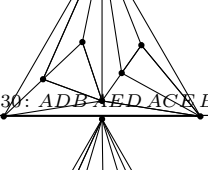
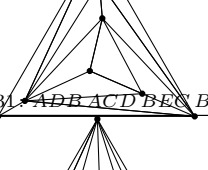
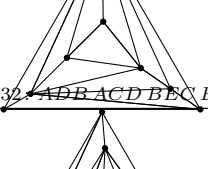
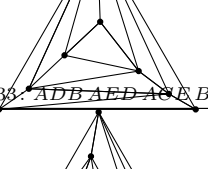


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& 7534380169949578 \\
& 20401092022458243 \\
& 10879068300786118 \\
& 20401092022458243 \\
& 26163659140944 \\
& 519116419783301 \\
& 518361308134124 \\
& 519116419783301 \\
& 9943563670467 \\
& 519116419783301 \\
& 1413529642053792 \\
& 1725565197084413 \\
& 3519856935166464 \\
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& 8627825985422065 \\
& 1126358859981600 \\
& 2558915781719089 \\
& 679708430772000 \\
& 2558915781719089 \\
& 2194849002917839 \\
& 2558915781719089 \\
& 4252090266167328 \\
& 16362452125556137 \\
& 9951228545476608 \\
& 16362452125556137 \\
& 12272840680346711 \\
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& 1902238\sqrt{1327315815707530680659628354174757774043553044885422505656704702021580022924502201536929502679634653965385476950248886032824075292309527221} \\
& 3157098477857677599587638959976066140315335670815141255563816220015201151027 \\
& + 1122325150431484919937450764435113541125677235708503375634944682907321762209 \\
& + 3157098477857677599587638959976066140315335670815141255563816220015201151027
\end{aligned}$$
Table B.2: Exact coordinates for 6 cylinders of radius  $r$  touching the unit sphere, see Section 3.1.

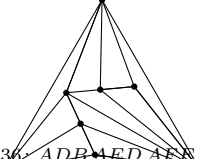
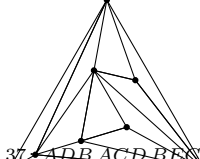

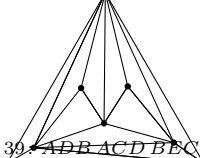
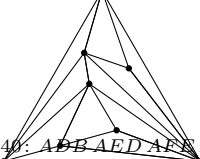
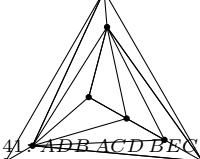
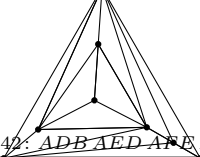
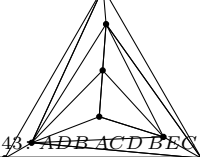
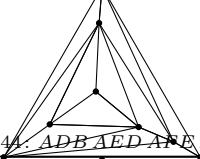
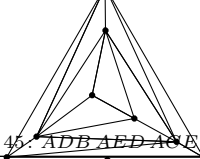
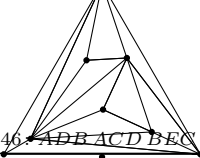
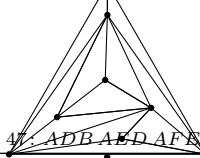
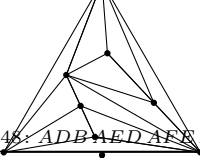
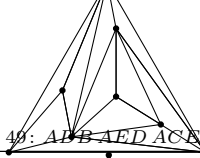
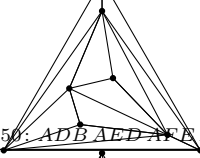
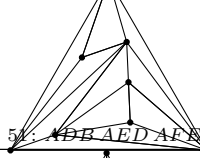
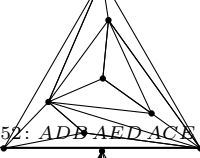
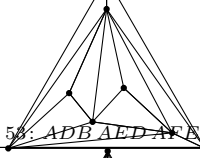
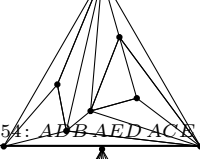
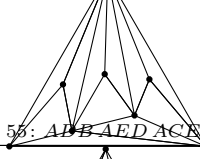
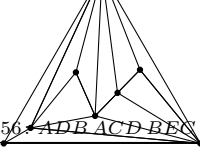
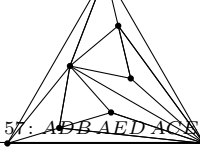
# Appendix C

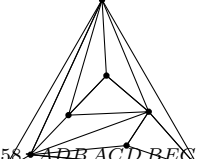
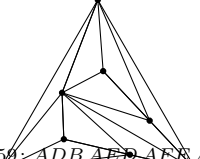
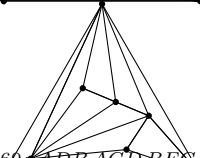
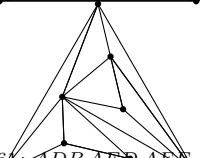
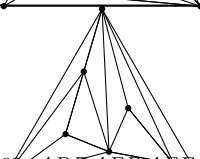
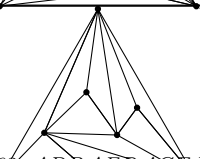
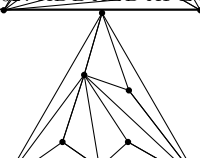
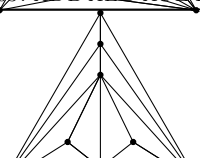
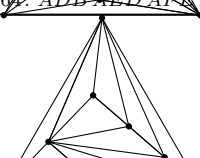
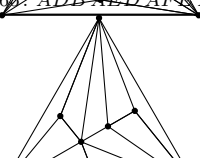
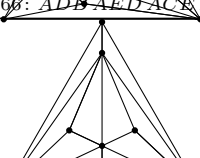
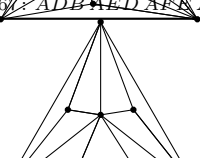
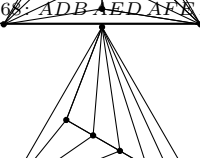

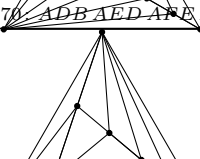
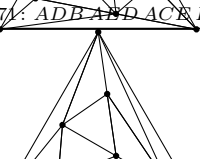
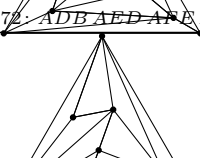
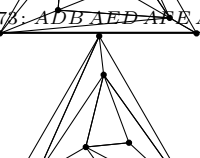
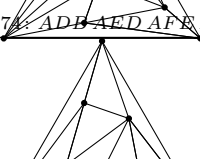
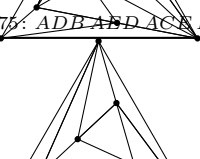
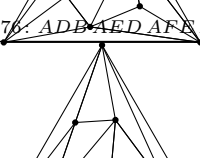

## Triangulations of the triangle

In the following table, we display equiareal straight line drawings of all triangulations of the triangle with up to 8 vertices, that can be drawn equiareally. There is one triangulation with 8 vertices that cannot be drawn equiareally, see Corollary 46 in Chapter 3.2. All triangles have the vertices  $A = (-1, 0)$ ,  $B = (1, 0)$  and  $C = (0, 1)$ . The aspect ratio of the pictures displays the outer triangle as equilateral.

 <p>0: <math>ADB ACD BDC</math></p> $D = \left(0, \frac{1}{3}\right)$	 <p>1: <math>ADB AED ACE BEC BDE</math></p> $D = \left(-\frac{1}{5}, \frac{2}{5}\right)$ $E = \left(\frac{2}{5}, \frac{1}{5}\right)$
 <p>2: <math>ADB AED ACE BEC BFE BDF DEF</math></p> $D = \left(-\frac{17}{35}, \frac{8}{35}\right)$ $E = \left(\frac{4}{7}, \frac{1}{7}\right)$ $F = \left(\frac{1}{35}, \frac{16}{35}\right)$	 <p>3: <math>ADB AED AFE ACF BFC BEF BDE</math></p> $D = \left(-\frac{2}{7}, \frac{3}{7}\right)$ $E = \left(\frac{1}{7}, \frac{2}{7}\right)$ $F = \left(\frac{4}{7}, \frac{1}{7}\right)$
 <p>4: <math>ADB AED ACE BFC BDF DCF DEC</math></p> $D = \left(-\frac{2}{7}, \frac{3}{7}\right)$ $E = \left(-\frac{2}{21}, \frac{1}{7}\right)$ $F = \left(\frac{5}{21}, \frac{10}{21}\right)$	 <p>5: <math>ADB AED ACE BFC BDF DEF ECF</math></p> $D = \left(-\frac{3}{7}, \frac{2}{7}\right)$ $E = \left(\frac{2}{7}, \frac{1}{7}\right)$ $F = \left(\frac{1}{7}, \frac{4}{7}\right)$
 <p>6: <math>ADB ACDBEC BDE DFEDGF DCG CFG CEF</math></p> $D = \left(-\frac{2}{3}, \frac{1}{9}\right)$ $E = \left(\frac{1}{21}, \frac{46}{63}\right)$ $F = \left(\frac{2}{21}, \frac{29}{63}\right)$ $G = \left(\frac{1}{7}, \frac{4}{21}\right)$	 <p>7: <math>ADB AED AFE ACF BFC BEF BGE BDG DEG</math></p> $D = \left(-\frac{8}{15}, \frac{11}{45}\right)$ $E = \left(\frac{1}{3}, \frac{2}{9}\right)$ $F = \left(\frac{2}{3}, \frac{1}{9}\right)$ $G = \left(-\frac{1}{15}, \frac{22}{45}\right)$
 <p>8: <math>ADB AED AFE AGF ACG BGC BDG DFG DEF</math></p> $D = \left(-\frac{1}{21}, \frac{46}{63}\right)$ $E = \left(-\frac{2}{7}, \frac{11}{35}\right)$ $F = \left(\frac{4}{21}, \frac{67}{315}\right)$ $G = \left(\frac{2}{3}, \frac{1}{9}\right)$	 <p>9: <math>ADB AED ACE BEC BFE BDF DGF DEG EFG</math></p> $D = \left(-\frac{13}{21}, \frac{10}{63}\right)$ $E = \left(\frac{2}{3}, \frac{1}{9}\right)$ $F = \left(\frac{1}{105}, \frac{206}{315}\right)$ $G = \left(\frac{2}{105}, \frac{97}{315}\right)$
 <p>10: <math>ADB AED ACE BFC BDF DGF DCG DEC CFG</math></p> $D = \left(-\frac{4}{9}, \frac{1}{3}\right)$ $E = \left(-\frac{4}{27}, \frac{1}{9}\right)$ $F = \left(\frac{1}{9}, \frac{2}{3}\right)$ $G = \left(\frac{2}{9}, \frac{1}{3}\right)$	 <p>11: <math>ADB AED AFE AGF ACG BDC DFC DEF FGC</math></p> $D = \left(0, \frac{7}{9}\right)$ $E = \left(-\frac{5}{21}, \frac{10}{27}\right)$ $F = \left(\frac{2}{7}, \frac{1}{3}\right)$ $G = \left(\frac{2}{21}, \frac{1}{9}\right)$
 <p>12: <math>ADB AED AFE ACF BGC BDG DCG DEC EFC</math></p> $D = \left(-\frac{2}{9}, \frac{5}{9}\right)$ $E = \left(-\frac{1}{15}, \frac{1}{3}\right)$ $F = \left(-\frac{2}{45}, \frac{1}{9}\right)$ $G = \left(\frac{7}{27}, \frac{14}{27}\right)$	 <p>13: <math>ADB AED AFE ACF BGC BFG BDF DEF FCG</math></p> $D = \left(-\frac{7}{45}, \frac{28}{45}\right)$ $E = \left(-\frac{14}{45}, \frac{11}{45}\right)$ $F = \left(\frac{2}{9}, \frac{1}{9}\right)$ $G = \left(\frac{11}{27}, \frac{10}{27}\right)$

 <p>14: <math>ADBACD BEC BFE BDF DEF DGE DCG CEG</math></p> $D = \left(-\frac{2}{3}, \frac{1}{9}\right)$ $E = \left(\frac{1}{3}, \frac{4}{9}\right)$ $F = \left(-\frac{1}{9}, \frac{14}{27}\right)$ $G = \left(\frac{2}{9}, \frac{5}{27}\right)$	 <p>15: <math>ADB AED AFE AGF ACG BGC BFG BEF BDE</math></p> $D = \left(-\frac{1}{3}, \frac{4}{9}\right)$ $E = \left(0, \frac{1}{3}\right)$ $F = \left(\frac{1}{3}, \frac{2}{9}\right)$ $G = \left(\frac{2}{3}, \frac{1}{9}\right)$
 <p>16: <math>ADB AED AFE ACF BGC BEG BDE ECG EFC</math></p> $D = \left(-\frac{1}{3}, \frac{4}{9}\right)$ $E = \left(0, \frac{1}{3}\right)$ $F = \left(0, \frac{1}{9}\right)$ $G = \left(\frac{1}{3}, \frac{4}{9}\right)$	 <p>17: <math>ADB AED ACE BFC BGF BDG DEG EFG ECF</math></p> $D = \left(-\frac{1}{2}, \frac{5}{18}\right)$ $E = \left(0, \frac{1}{9}\right)$ $F = \left(\frac{1}{2}, \frac{5}{18}\right)$ $G = \left(0, \frac{5}{9}\right)$
 <p>18: <math>ADB AED AFE ACF BFC BGF BDG DEG EFG</math></p> $D = \left(-\frac{4}{21}, \frac{37}{63}\right)$ $E = \left(-\frac{8}{21}, \frac{11}{63}\right)$ $F = \left(\frac{2}{3}, \frac{1}{9}\right)$ $G = \left(\frac{5}{21}, \frac{22}{63}\right)$	 <p>19: <math>ADB AED ACE BFC BGF BDG DEG ECG CFG</math></p> $D = \left(-\frac{5}{9}, \frac{2}{9}\right)$ $E = \left(\frac{2}{9}, \frac{1}{9}\right)$ $F = \left(\frac{8}{27}, \frac{13}{27}\right)$ $G = \left(-\frac{1}{9}, \frac{9}{9}\right)$
 <p>20: <math>ADB AED ACE BFC BDF DEF EGF ECG CFG</math></p> $D = \left(-\frac{2}{9}, \frac{5}{9}\right)$ $E = \left(-\frac{4}{9}, \frac{1}{9}\right)$ $F = \left(\frac{7}{27}, \frac{14}{27}\right)$ $G = \left(\frac{22}{81}, \frac{17}{81}\right)$	 <p>21: <math>ADB AED ACE BFC BDF DGF DEG EFG ECF</math></p> $D = \left(-\frac{1}{9}\sqrt{6} - \frac{1}{3}, -\frac{1}{9}\sqrt{6} + \frac{4}{9}\right)$ $E = \left(\frac{2}{9}\sqrt{6}, \frac{1}{9}\right)$ $F = \left(-\frac{1}{9}\sqrt{6} + \frac{1}{3}, \frac{1}{9}\sqrt{6} + \frac{4}{9}\right)$ $G = \left(0, \frac{1}{3}\right)$
 <p>22: <math>ADB AED ACE BFC BDF DGF DHG DCH DEC CGH CFG</math></p> $D = \left(-\frac{6}{11}, \frac{3}{11}\right)$ $E = \left(-\frac{2}{11}, \frac{1}{11}\right)$ $F = \left(\frac{5}{77}, \frac{58}{77}\right)$ $G = \left(\frac{10}{77}, \frac{39}{77}\right)$ $H = \left(\frac{15}{77}, \frac{20}{77}\right)$	 <p>23: <math>ADB AED AFE AGF ACG BGC BFG BHF BEH BDE EFH</math></p> $D = \left(-\frac{95}{231}, \frac{94}{231}\right)$ $E = \left(-\frac{18}{77}, \frac{17}{77}\right)$ $F = \left(\frac{5}{11}, \frac{2}{11}\right)$ $G = \left(\frac{8}{11}, \frac{1}{11}\right)$ $H = \left(\frac{17}{36}, \frac{1}{36}\right)$
 <p>24: <math>ADB AED AFE AGF ACG BHC BDH DCH DEC EFC FGC</math></p> $D = \left(-\frac{2}{11}, \frac{7}{11}\right)$ $E = \left(-\frac{10}{77}, \frac{5}{11}\right)$ $F = \left(-\frac{6}{77}, \frac{3}{11}\right)$ $G = \left(-\frac{2}{77}, \frac{1}{11}\right)$ $H = \left(\frac{3}{11}, \frac{6}{11}\right)$	 <p>25: <math>ADB ACD BEC BDE DFE DGF DHG DCH CGH CFG CEF</math></p> $D = \left(-\frac{8}{11}, \frac{1}{11}\right)$ $E = \left(\frac{1}{33}, \frac{26}{33}\right)$ $F = \left(\frac{2}{33}, \frac{19}{33}\right)$ $G = \left(\frac{1}{11}, \frac{4}{11}\right)$ $H = \left(\frac{4}{33}, \frac{5}{33}\right)$
 <p>26: <math>ADB ACD BEC BFE BGF BDG DCG CFG CHF CEH EFH</math></p> $D = \left(-\frac{8}{11}, \frac{1}{11}\right)$ $E = \left(\frac{9}{55}, \frac{36}{55}\right)$ $F = \left(-\frac{2}{11}, \frac{3}{11}\right)$ $G = \left(-\frac{5}{11}, \frac{2}{11}\right)$ $H = \left(\frac{18}{55}, \frac{17}{55}\right)$	 <p>27: <math>ADB AED AFE ACF BFC BEF BGC BHG BDH DEH EGH</math></p> $D = \left(-\frac{50}{77}, \frac{13}{77}\right)$ $E = \left(\frac{5}{11}, \frac{2}{11}\right)$ $F = \left(\frac{8}{11}, \frac{1}{11}\right)$ $G = \left(\frac{4}{77}, \frac{39}{77}\right)$ $H = \left(-\frac{23}{77}, \frac{26}{77}\right)$
 <p>28: <math>ADB ACD BEC BFE BDF DGF DHG DCH CGH CFG CEF</math></p> $D = \left(-\frac{8}{11}, \frac{1}{11}\right)$ $E = \left(\frac{86}{297}, \frac{157}{297}\right)$ $F = \left(\frac{13}{99}, \frac{58}{99}\right)$ $G = \left(-\frac{4}{165}, \frac{61}{165}\right)$ $H = \left(\frac{41}{495}, \frac{76}{495}\right)$	 <p>29: <math>ADB AED AFE AGF AHG ACH BDC DFC DEF FHC FGH</math></p> $D = \left(0, \frac{9}{11}\right)$ $E = \left(-\frac{7}{27}, \frac{14}{33}\right)$ $F = \left(\frac{2}{9}, \frac{5}{11}\right)$ $G = \left(-\frac{1}{9}, \frac{1}{11}\right)$ $H = \left(\frac{4}{9}, \frac{1}{11}\right)$
 <p>30: <math>ADB AED ACE BFC BGF BEG BHE BDH DEH ECG CFG</math></p> $D = \left(-\frac{3}{5}, \frac{12}{55}\right)$ $E = \left(0, \frac{1}{11}\right)$ $F = \left(\frac{2}{5}, \frac{23}{55}\right)$ $G = \left(\frac{1}{5}, \frac{14}{55}\right)$ $H = \left(-\frac{1}{5}, \frac{24}{55}\right)$	 <p>31: <math>ADB ACD BEC BDE DFE DGF DHG DCH CFH CEF FGH</math></p> $D = \left(-\frac{8}{11}, \frac{1}{11}\right)$ $E = \left(\frac{1}{33}, \frac{26}{33}\right)$ $F = \left(\frac{2}{33}, \frac{19}{33}\right)$ $G = \left(-\frac{1}{15}, \frac{4}{15}\right)$ $H = \left(\frac{7}{15}, \frac{2}{15}\right)$
 <p>32: <math>ADB ACD BEC BFE BGF BHG BDH DFH DEF DCE FGH</math></p> $D = \left(-\frac{8}{11}, \frac{1}{11}\right)$ $E = \left(\frac{23}{33}, \frac{4}{33}\right)$ $F = \left(\frac{13}{33}, \frac{8}{33}\right)$ $G = \left(\frac{2}{165}, \frac{17}{33}\right)$ $H = \left(-\frac{59}{165}, \frac{10}{33}\right)$	 <p>33: <math>ADB AED ACE BFC BFE BGF BHG BDH DFH DEF FGH</math></p> $D = \left(-\frac{23}{33}, \frac{4}{33}\right)$ $E = \left(\frac{8}{11}, \frac{1}{11}\right)$ $F = \left(\frac{7}{33}, \frac{52}{33}\right)$ $G = \left(\frac{1}{35}, \frac{18}{35}\right)$ $H = \left(-\frac{386}{385}, \frac{367}{385}\right)$
 <p>34: <math>ADB ACD BEC BDE DFE DCF CGF CHG CEH EGH EFG</math></p> $D = \left(-\frac{8}{11}, \frac{1}{11}\right)$ $E = \left(\frac{1}{33}, \frac{26}{33}\right)$ $F = \left(-\frac{86}{231}, \frac{41}{231}\right)$ $G = \left(-\frac{4}{231}, \frac{61}{231}\right)$ $H = \left(\frac{29}{231}, \frac{25}{231}\right)$	 <p>35: <math>ADB AED AFE AGF AHG ACH BHC BHG BDG DFG DEF</math></p> $D = \left(-\frac{6}{77}, \frac{57}{77}\right)$ $E = \left(-\frac{131}{385}, \frac{128}{385}\right)$ $F = \left(\frac{2}{35}, \frac{9}{35}\right)$ $G = \left(\frac{5}{11}, \frac{2}{11}\right)$ $H = \left(\frac{8}{11}, \frac{1}{11}\right)$

 <p>36: <math>ADB AED AFE ACF BGC BHG BDH DCH DEC EFC CGH</math></p> $D = \begin{pmatrix} -4 & 5 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -12 & 3 \\ -55 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 4 & 1 \\ -55 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 18 & 27 \\ 55 & 55 \end{pmatrix}$ $H = \begin{pmatrix} -1 & 26 \\ -11 & 11 \end{pmatrix}$	 <p>37: <math>ADB ACD BEC BFE BDF DGF DCG CHG CFH CEF FGH</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 86 & 157 \\ 297 & 297 \end{pmatrix}$ $F = \begin{pmatrix} 13 & 58 \\ -99 & 99 \end{pmatrix}$ $G = \begin{pmatrix} -26 & 17 \\ -99 & 99 \end{pmatrix}$ $H = \begin{pmatrix} 20 & 25 \\ 99 & 99 \end{pmatrix}$
 <p>38: <math>ADB AED ACE BFC BDF DGF DCG DEC CHG CFH FGH</math></p> $D = \begin{pmatrix} -6 & 3 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -2 & 1 \\ -11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 5 & 58 \\ 77 & 77 \end{pmatrix}$ $G = \begin{pmatrix} -4 & 11 \\ -35 & 35 \end{pmatrix}$ $H = \begin{pmatrix} 122 & 132 \\ 385 & 385 \end{pmatrix}$	 <p>39: <math>ADB ACD BEC BFE BGF BHG BDH DGH DEG DCE EFG</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 23 & 4 \\ 33 & 33 \end{pmatrix}$ $F = \begin{pmatrix} 188 & 13 \\ 693 & 693 \end{pmatrix}$ $G = \begin{pmatrix} -1 & 18 \\ 77 & 77 \end{pmatrix}$ $H = \begin{pmatrix} -19 & 34 \\ -27 & 27 \end{pmatrix}$
 <p>40: <math>ADB AED AFE ACF BGC BDG DCG DEC EHC EFH FCH</math></p> $D = \begin{pmatrix} -2 & 7 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -10 & 5 \\ -77 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -164 & 1 \\ 385 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 6 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 57 & 2 \\ 385 & 11 \end{pmatrix}$	 <p>41: <math>ADB ACD BEC BDE DFE DGF DHG DCH CEH EGH EFG</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 26 \\ 33 & 33 \end{pmatrix}$ $F = \begin{pmatrix} -12 & 29 \\ 77 & 77 \end{pmatrix}$ $G = \begin{pmatrix} 53 & 58 \\ 231 & 231 \end{pmatrix}$ $H = \begin{pmatrix} 142 & 29 \\ 231 & 231 \end{pmatrix}$
 <p>42: <math>ADB AED AFE ACF BFC BEF BGE BDG DHG DEH EGH</math></p> $D = \begin{pmatrix} -50 & 13 \\ 77 & 77 \end{pmatrix}$ $E = \begin{pmatrix} 5 & 2 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 258 \\ -77 & 385 \end{pmatrix}$ $H = \begin{pmatrix} -6 & 131 \\ -11 & 11 \end{pmatrix}$	 <p>43: <math>ADB ACD BEC BDE DFE DGF DHG DCH CEH EFH FGH</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 26 \\ 33 & 33 \end{pmatrix}$ $F = \begin{pmatrix} -1 & 596 \\ 231 & 1155 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 94 \\ -77 & 385 \end{pmatrix}$ $H = \begin{pmatrix} 142 & 29 \\ 231 & 231 \end{pmatrix}$
 <p>44: <math>ADB AED AFE AGF ACG BGC BDC DFG DHF DEH EFH</math></p> $D = \begin{pmatrix} -1 & 26 \\ 33 & 33 \end{pmatrix}$ $E = \begin{pmatrix} -614 & 223 \\ 1155 & 1155 \end{pmatrix}$ $F = \begin{pmatrix} 86 & 41 \\ 231 & 231 \end{pmatrix}$ $G = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} -73 & 446 \\ -11 & 11 \end{pmatrix}$	 <p>45: <math>ADB AED ACE BEC BFE BDF DGF DHG DEH EFH FGH</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 26 \\ 33 & 33 \end{pmatrix}$ $F = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 1 & 172 \\ 231 & 231 \end{pmatrix}$ $H = \begin{pmatrix} -152 & 421 \\ 1155 & 1155 \end{pmatrix}$
 <p>46: <math>ADB ACD BEC BFE BDF DFE DGE DHG DCH CEH EGH</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 25 & 56 \\ 99 & 99 \end{pmatrix}$ $F = \begin{pmatrix} 47 & 164 \\ -297 & 297 \end{pmatrix}$ $G = \begin{pmatrix} 1 & 26 \\ 99 & 99 \end{pmatrix}$ $H = \begin{pmatrix} 50 & 13 \\ 99 & 99 \end{pmatrix}$	 <p>47: <math>ADB AED AFE AGF ACG BDC DFC DHF DEH EFH FGC</math></p> $D = \begin{pmatrix} 0 & 9 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -23 & 12 \\ 45 & 55 \end{pmatrix}$ $F = \begin{pmatrix} 4 & 3 \\ 9 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 4 & 1 \\ 27 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 24 \\ -45 & 24 \end{pmatrix}$
 <p>48: <math>ADB AED AFE ACF BGC BHG BDH DGH DCG DEC EFC</math></p> $D = \begin{pmatrix} -4 & 5 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -12 & 3 \\ -55 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 4 & 1 \\ -55 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 29 & 16 \\ 55 & 55 \end{pmatrix}$ $H = \begin{pmatrix} 2 & 32 \\ 55 & 55 \end{pmatrix}$	 <p>49: <math>ADB AED ACE BFC BEF BDE EGF EHG ECH CFH FGH</math></p> $D = \begin{pmatrix} -5 & 4 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -4 & 1 \\ -11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 1 & 8 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 1 & 18 \\ 11 & 55 \end{pmatrix}$ $H = \begin{pmatrix} 6 & 9 \\ 11 & 55 \end{pmatrix}$
 <p>50: <math>ADB AED AFE AGF ACG BDC DGC DHG DEH EGH EFG</math></p> $D = \begin{pmatrix} 0 & 9 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 4 \\ -3 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -2 & 5 \\ 9 & 33 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 1 \\ 3 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 14 \\ 9 & 23 \end{pmatrix}$	 <p>51: <math>ADB AED AFE ACF BEC BDE EGC EFG FHG FCH CGH</math></p> $D = \begin{pmatrix} -3 & 6 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 2 & 7 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -6 & 1 \\ -11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 5 & 5 \\ 3 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 12 & 9 \\ 11 & 55 \end{pmatrix}$
 <p>52: <math>ADB AED ACE BFC BDF DGF DHG DCH DEC CFH FGH</math></p> $D = \begin{pmatrix} -6 & 3 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -2 & 1 \\ -11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 5 & 58 \\ 77 & 77 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 158 \\ 385 & 385 \end{pmatrix}$ $H = \begin{pmatrix} 194 & 79 \\ 385 & 385 \end{pmatrix}$	 <p>53: <math>ADB AED AFE AGF ACG BDC DGC DHG DFH DEF FGH</math></p> $D = \begin{pmatrix} 0 & 9 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -8 & 25 \\ -21 & 77 \end{pmatrix}$ $F = \begin{pmatrix} -1 & 12 \\ -1 & 77 \end{pmatrix}$ $G = \begin{pmatrix} 2 & 13 \\ 3 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 11 & 82 \\ 33 & 231 \end{pmatrix}$
 <p>54: <math>ADB AED ACE BFC BGF BEG BDE ECG CHG CFH FGH</math></p> $D = \begin{pmatrix} -5 & 4 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -4 & 1 \\ -11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 68 & 247 \\ 385 & 385 \end{pmatrix}$ $G = \begin{pmatrix} -9 & 16 \\ -77 & 77 \end{pmatrix}$ $H = \begin{pmatrix} 136 & 109 \\ -77 & 77 \end{pmatrix}$	 <p>55: <math>ADB AED ACE BFC BGF BHG BEH BDE EGH ECG CFG</math></p> $D = \begin{pmatrix} -5 & 4 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -4 & 1 \\ -11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 14 & 13 \\ 33 & 33 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 2 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} -3 & 14 \\ -11 & 11 \end{pmatrix}$
 <p>56: <math>ADB ACD BEC BFE BGF BHG BDH DGH DCG CFG CEF</math></p> $D = \begin{pmatrix} -8 & 1 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 191 & 214 \\ 495 & 495 \end{pmatrix}$ $F = \begin{pmatrix} 29 & 49 \\ 165 & 165 \end{pmatrix}$ $G = \begin{pmatrix} -7 & 16 \\ -99 & 99 \end{pmatrix}$ $H = \begin{pmatrix} -79 & 124 \\ -297 & 297 \end{pmatrix}$	 <p>57: <math>ADB AED ACE BFC BDF DGF DCG DHC DEH ECH CFG</math></p> $D = \begin{pmatrix} -4 & 5 \\ -11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -26 & 1 \\ -55 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 7 & 38 \\ 55 & 55 \end{pmatrix}$ $G = \begin{pmatrix} 14 & 21 \\ 55 & 55 \end{pmatrix}$ $H = \begin{pmatrix} 3 & 2 \\ 55 & 55 \end{pmatrix}$

 $D = \begin{pmatrix} -8 & 1 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 47 & 34 \\ 99 & 99 \end{pmatrix}$ $F = \begin{pmatrix} 2 & 5 \\ 45 & 0 \end{pmatrix}$ $G = \begin{pmatrix} -169 & 32 \\ 495 & 99 \end{pmatrix}$ $H = \begin{pmatrix} 74 & 43 \\ 297 & 297 \end{pmatrix}$ <p>58: <del>ADB ACD BCE</del> BFE BGF BDG DEG DHE DCH CEH EFG</p>	 $D = \begin{pmatrix} -4 & 5 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -19 & 2 \\ 55 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 18 & 1 \\ 55 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 29 & 16 \\ 55 & 55 \end{pmatrix}$ $H = \begin{pmatrix} 3 & 32 \\ 55 & 55 \end{pmatrix}$ <p>59: ADB AED AFE ACF BGC BHG BDH DGH DCG DFC DEF</p>
 $D = \begin{pmatrix} -8 & 1 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 47 & 34 \\ 99 & 99 \end{pmatrix}$ $F = \begin{pmatrix} 23 & 14 \\ 165 & 33 \end{pmatrix}$ $G = \begin{pmatrix} -97 & 50 \\ 495 & 99 \end{pmatrix}$ $H = \begin{pmatrix} 74 & 43 \\ 297 & 297 \end{pmatrix}$ <p>60: <del>ADB ACD BCE</del> BFE BGF BDG DFG DEF DHE DCH CEH</p>	 $D = \begin{pmatrix} -4 & 5 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -19 & 2 \\ 55 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 18 & 1 \\ 55 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 7 & 38 \\ 55 & 55 \end{pmatrix}$ $H = \begin{pmatrix} 14 & 21 \\ 55 & 55 \end{pmatrix}$ <p>61: ADB AED AFE ACF BGC BDG DHG DCH DFC DEF CGH</p>
 $D = \begin{pmatrix} -92 & 313 \\ 495 & 495 \end{pmatrix}$ $E = \begin{pmatrix} -182 & 131 \\ 495 & 495 \end{pmatrix}$ $F = \begin{pmatrix} 7 & 16 \\ 99 & 99 \end{pmatrix}$ $G = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 79 & 124 \\ 297 & 297 \end{pmatrix}$ <p>62: <del>ADB AED AFE</del> AGF ACG BGC BHG BFH BDF DEF FGH</p>	 $D = \begin{pmatrix} -6 & 3 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -2 & 1 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 92 & 97 \\ 231 & 231 \end{pmatrix}$ $G = \begin{pmatrix} 19 & 20 \\ 77 & 77 \end{pmatrix}$ $H = \begin{pmatrix} -9 & 118 \\ 77 & 231 \end{pmatrix}$ <p>63: ADB AED ACE BFC BGF BHG BDH DGH DCG DEC CFG</p>
 $D = \begin{pmatrix} -2 & 7 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 31 & 8 \\ 77 & 33 \end{pmatrix}$ $F = \begin{pmatrix} -2 & 1 \\ 77 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 6 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 61 & 8 \\ 33 & 33 \end{pmatrix}$ <p>64: <del>ADB AED AFE</del> ACF BGC BDG DCG DHC DFH DEF FCH</p>	 $D = \begin{pmatrix} 0 & 9 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 0 & 7 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -1 & 8 \\ 3 & 33 \end{pmatrix}$ $G = \begin{pmatrix} 0 & 1 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 8 \\ 3 & 33 \end{pmatrix}$ <p>65: ADB AED AFE AGF ACG BDC DEC EHC EGH EFG GCH</p>
 $D = \begin{pmatrix} -6 & 3 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -2 & 1 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 7 & 2 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 4 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} -1 & 6 \\ 11 & 11 \end{pmatrix}$ <p>66: <del>ADB AED ACE</del> BFC BGF BHG BDH DGH DFG DCF DEC</p>	 $D = \begin{pmatrix} -13 & 14 \\ 33 & 33 \end{pmatrix}$ $E = \begin{pmatrix} -2 & 3 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -2 & 1 \\ 33 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 4 & 5 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 4 \\ 11 & 11 \end{pmatrix}$ <p>67: ADB AED AFE ACF BGC BHG BEH BDE ECH EFC CGH</p>
 $D = \begin{pmatrix} 0 & 9 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -1 & 4 \\ 3 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 3 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 0 & 1 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 4 \\ 3 & 11 \end{pmatrix}$ <p>68: <del>ADB AED AFE</del> AGF ACG BDC DHC DFH DEF FCH FGC</p>	 $D = \begin{pmatrix} -1 & 16 \\ 3 & 33 \end{pmatrix}$ $E = \begin{pmatrix} 0 & 5 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -1 & 2 \\ 5 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 2 & 1 \\ 5 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 16 \\ 5 & 33 \end{pmatrix}$ <p>69: ADB AED AFE AGF ACG BHC BEH BDE ECH EGC EFG</p>
 $D = \begin{pmatrix} -4 & 5 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -1 & 4 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 2 & 3 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 5 & 2 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 4 \\ 11 & 11 \end{pmatrix}$ <p>70: <del>ADB AED AFE</del> AGF AHG ACH BHC BGH BFG BEF BDE</p>	 $D = \begin{pmatrix} -7 & 2 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 2 & 1 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 6 & 3 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 1 & 6 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} -3 & 4 \\ 11 & 11 \end{pmatrix}$ <p>71: ADB AED ACE BFC BGF BHG BDH DEH EGH EFG ECF</p>
 $D = \begin{pmatrix} -25 & 56 \\ 99 & 99 \end{pmatrix}$ $E = \begin{pmatrix} -50 & 13 \\ 99 & 99 \end{pmatrix}$ $F = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 119 & 74 \\ 397 & 297 \end{pmatrix}$ $H = \begin{pmatrix} 2 & 11 \\ 27 & 26 \end{pmatrix}$ <p>72: <del>ADB AED AFE</del> ACF BFC BGF BHG BDH DEH EGH EFG</p>	 $D = \begin{pmatrix} -4 & 5 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 9 & 3 \\ -22 & 22 \end{pmatrix}$ $F = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 1 & 7 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 2 & 3 \\ 11 & 11 \end{pmatrix}$ <p>73: ADB AED AFE ACF BFC BGF BDG DHG DEH EFH FGH</p>
 $D = \begin{pmatrix} -13 & 23 \\ 44 & 44 \end{pmatrix}$ $E = \begin{pmatrix} 5 & 25 \\ 44 & 44 \end{pmatrix}$ $F = \begin{pmatrix} -3 & 29 \\ 88 & 88 \end{pmatrix}$ $G = \begin{pmatrix} -1 & 1 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 7 & 2 \\ 11 & 11 \end{pmatrix}$ <p>74: <del>ADB AED AFE</del> AGF ACG BHC BEH BDE EFH FGH GCH</p>	 $D = \begin{pmatrix} -7 & 2 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} 2 & 1 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 1 & 17 \\ 23 & 27 \end{pmatrix}$ $G = \begin{pmatrix} -3 & 23 \\ -22 & 66 \end{pmatrix}$ $H = \begin{pmatrix} 10 & 37 \\ -2 & 99 \end{pmatrix}$ <p>75: ADB AED ACE BFC BDF DG DEG ECG CHG CFH FGH</p>
 $D = \begin{pmatrix} -2 & 7 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -4 & 3 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} -4 & 1 \\ 33 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 6 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} 38 & 7 \\ 33 & 33 \end{pmatrix}$ <p>76: <del>ADB AED AFE</del> ACF BGC BDG DEG EFG FGH FCH CGH</p>	 $D = \begin{pmatrix} -20 & 7 \\ 33 & 33 \end{pmatrix}$ $E = \begin{pmatrix} 4 & 3 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 4 & 1 \\ 33 & 11 \end{pmatrix}$ $G = \begin{pmatrix} 2 & 7 \\ 11 & 11 \end{pmatrix}$ $H = \begin{pmatrix} -7 & 14 \\ 3 & 14 \end{pmatrix}$ <p>77: ADB AED AFE ACF BGC BHG BDH DEH EGH EGC EFC</p>
 $D = \begin{pmatrix} -3 & 6 \\ 11 & 11 \end{pmatrix}$ $E = \begin{pmatrix} -6 & 1 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 15 & 3 \\ 22 & 22 \end{pmatrix}$ $G = \begin{pmatrix} 3 & 37 \\ 22 & 66 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 26 \\ 11 & 99 \end{pmatrix}$ <p>78: <del>ADB AED ACE</del> BFC BGF BDG DEG EHG EFH ECF FGH</p>	 $D = \begin{pmatrix} -23 & 4 \\ 33 & 33 \end{pmatrix}$ $E = \begin{pmatrix} 8 & 1 \\ 11 & 11 \end{pmatrix}$ $F = \begin{pmatrix} 13 & 82 \\ 231 & 231 \end{pmatrix}$ $G = \begin{pmatrix} -2 & 13 \\ 4 & 21 \end{pmatrix}$ $H = \begin{pmatrix} -21 & 5 \\ 4 & 21 \end{pmatrix}$ <p>79: ADB AED ACE BFC BFE BGF BDG DHG DEH EFH FGH</p>



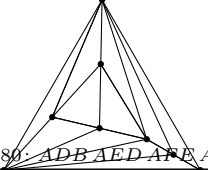
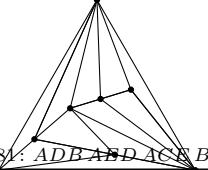
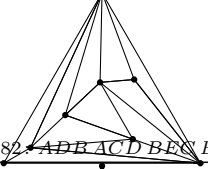
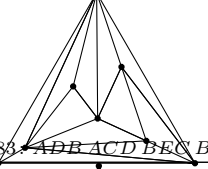
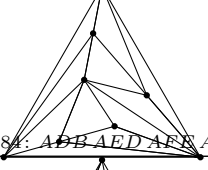
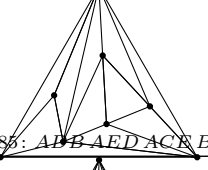
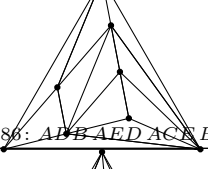
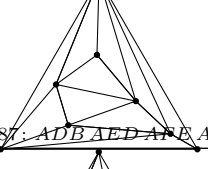
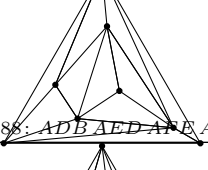
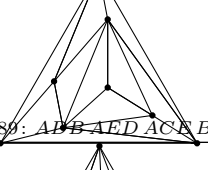
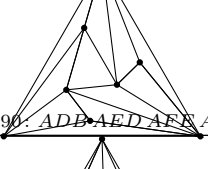
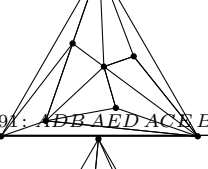
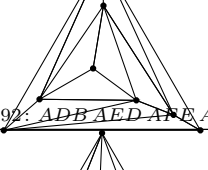
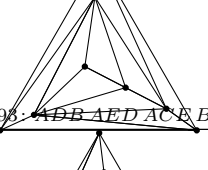
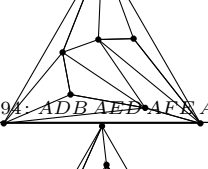
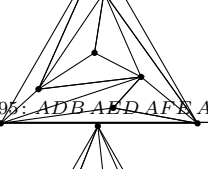
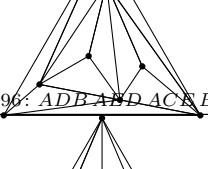
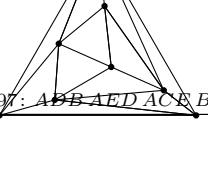
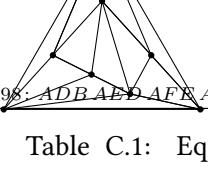
 <p>80: <math>ADB AED AFE ACF BGC BGF BFG BHF BDH DEH EFH</math></p> $D = \begin{pmatrix} -\frac{39}{77}, \frac{24}{77} \\ -\frac{2}{77}, \frac{19}{77} \\ \frac{5}{11}, \frac{2}{11} \\ \frac{8}{11}, \frac{1}{11} \\ -\frac{1}{77}, \frac{48}{77} \end{pmatrix}$	 <p>81: <math>ADB AED ACE BFC BGF BHG BDH DEH ECH CGH CFG</math></p> $D = \begin{pmatrix} -\frac{7}{11}, \frac{2}{11} \\ \frac{2}{11}, \frac{1}{11} \\ \frac{19}{55}, \frac{26}{55} \\ \frac{2}{55}, \frac{23}{55} \\ -\frac{3}{11}, \frac{4}{11} \end{pmatrix}$
 <p>82: <math>ADB ACD BEF BFE BGF BDG DHG DCH CFH CEF FGH</math></p> $D = \begin{pmatrix} -\frac{8}{11}, \frac{1}{11} \\ \frac{97}{297}, \frac{146}{297} \\ -\frac{2}{99}, \frac{47}{99} \\ -\frac{37}{99}, \frac{28}{99} \\ \frac{31}{99}, \frac{14}{99} \end{pmatrix}$	 <p>83: <math>ADB ACD BEF BFE BGF BDG DFG DHF DCH CEH EFH</math></p> $D = \begin{pmatrix} -\frac{8}{11}, \frac{1}{11} \\ \frac{25}{99}, \frac{56}{99} \\ \frac{1}{99}, \frac{26}{99} \\ -\frac{71}{297}, \frac{134}{297} \\ \frac{50}{99}, \frac{13}{99} \end{pmatrix}$
 <p>84: <math>ADB AED AFE ACF BGC BDG DEG ECG EHC EFH FCH</math></p> $D = \begin{pmatrix} -\frac{1}{11}, \frac{8}{11} \\ -\frac{2}{11}, \frac{5}{11} \\ \frac{24}{55}, \frac{1}{11} \\ \frac{5}{11}, \frac{4}{11} \\ \frac{7}{55}, \frac{2}{11} \end{pmatrix}$	 <p>85: <math>ADB AED ACE BFC BGF BEG BDE EHG ECH CFH FGH</math></p> $D = \begin{pmatrix} -\frac{5}{11}, \frac{4}{11} \\ -\frac{4}{11}, \frac{1}{11} \\ \frac{40}{77}, \frac{23}{77} \\ \frac{3}{77}, \frac{46}{77} \\ \frac{6}{77}, \frac{15}{77} \end{pmatrix}$
 <p>86: <math>ADB AED ACE BFC BDF DEF EGF EHG ECH CGH CFG</math></p> $D = \begin{pmatrix} -\frac{5}{11}, \frac{4}{11} \\ -\frac{4}{11}, \frac{1}{11} \\ \frac{1}{11}, \frac{8}{11} \\ \frac{2}{11}, \frac{5}{11} \\ \frac{3}{11}, \frac{2}{11} \end{pmatrix}$	 <p>87: <math>ADB AED AFE ACF BFC BGF BHG BDH DGH DEG EFG</math></p> $D = \begin{pmatrix} -\frac{130}{297}, \frac{113}{297} \\ -\frac{31}{99}, \frac{14}{99} \\ \frac{8}{11}, \frac{1}{11} \\ \frac{37}{99}, \frac{28}{99} \\ -\frac{19}{99}, \frac{494}{99} \end{pmatrix}$
 <p>88: <math>ADB AED AFE ACF BFC BGF BDG DEG EHG EFH FGH</math></p> $D = \begin{pmatrix} -\frac{47}{99}, \frac{34}{99} \\ -\frac{74}{297}, \frac{43}{297} \\ \frac{8}{11}, \frac{1}{11} \\ \frac{5}{99}, \frac{68}{99} \\ \frac{157}{274}, \frac{274}{99} \end{pmatrix}$	 <p>89: <math>ADB AED ACE BFC BDF DEF EGF EHG ECH CFH FGH</math></p> $D = \begin{pmatrix} -\frac{5}{11}, \frac{4}{11} \\ -\frac{4}{11}, \frac{1}{11} \\ \frac{1}{11}, \frac{8}{11} \\ \frac{1}{11}, \frac{18}{55} \\ \frac{6}{9}, \frac{9}{9} \end{pmatrix}$
 <p>90: <math>ADB AED AFE ACF BGC BHG BDH DEH ECH EFC CGH</math></p> $D = \begin{pmatrix} -\frac{2}{11}, \frac{7}{11} \\ -\frac{4}{11}, \frac{3}{11} \\ \frac{4}{11}, \frac{1}{11} \\ -\frac{38}{99}, \frac{43}{99} \\ \frac{5}{99}, \frac{10}{99} \end{pmatrix}$	 <p>91: <math>ADB AED ACE BFC BGF BDG DEG EHG ECH CGH CFG</math></p> $D = \begin{pmatrix} -\frac{3}{11}, \frac{6}{11} \\ -\frac{6}{11}, \frac{1}{11} \\ \frac{23}{66}, \frac{31}{66} \\ \frac{1}{22}, \frac{9}{22} \\ \frac{1}{2}, \frac{2}{2} \end{pmatrix}$
 <p>92: <math>ADB AED AFE ACF BFC BGF BDG DHG DEH EGH EFG</math></p> $D = \begin{pmatrix} -\frac{1}{9}\sqrt{6} - \frac{4}{11}, -\frac{1}{9}\sqrt{6} + \frac{5}{11} \\ \frac{19}{99}\sqrt{6} - \frac{4}{33}, \frac{1}{99}\sqrt{6} + \frac{5}{33} \\ \frac{8}{11}, \frac{1}{11} \\ -\frac{8}{99}\sqrt{6} + \frac{7}{33}, \frac{10}{99}\sqrt{6} + \frac{16}{33} \end{pmatrix}$	 <p>93: <math>ADB AED ACE BFC BDF BDR DCR DFG DGH EGH ECF</math></p> $D = \begin{pmatrix} \frac{1}{11}\sqrt{13} - \frac{4}{11}, \frac{1}{11}\sqrt{13} + \frac{5}{11} \\ -\frac{2}{11}\sqrt{13}, \frac{1}{11} \\ \frac{1}{11}\sqrt{13} + \frac{4}{11}, -\frac{1}{11}\sqrt{13} + \frac{5}{11} \\ \frac{2}{55}\sqrt{13} + \frac{8}{55}, -\frac{2}{55}\sqrt{13} + \frac{21}{55} \end{pmatrix}$
 <p>94: <math>ADB AED AFE ACF BGC BHF BDH DFH DEF ECH CGH</math></p> $D = \begin{pmatrix} -\frac{1}{11}\sqrt{2} - \frac{3}{11}, -\frac{1}{11}\sqrt{2} + \frac{6}{11} \\ \frac{1}{33}\sqrt{2} - \frac{4}{11}, -\frac{1}{33}\sqrt{2} + \frac{7}{33} \\ \frac{2}{11}\sqrt{2} + \frac{2}{11}, \frac{1}{11} \\ -\frac{1}{33}\sqrt{2} + \frac{4}{11}, \frac{1}{33}\sqrt{2} + \frac{5}{11} \end{pmatrix}$	 <p>95: <math>ADB AED AFE ACF BGC BDG DHC DGH EHC EFC</math></p> $D = \begin{pmatrix} -\frac{2}{33}\sqrt{22} - \frac{1}{3}, -\frac{2}{33}\sqrt{22} + \frac{16}{33} \\ \frac{1}{11}\sqrt{22}, \frac{3}{11} \\ \frac{1}{33}\sqrt{22}, \frac{1}{11} \\ -\frac{2}{33}\sqrt{22} + \frac{1}{3}, \frac{2}{33}\sqrt{22} + \frac{16}{33} \end{pmatrix}$
 <p>96: <math>ADB AED ACE BFC BDF DGF DEG EFG EHF ECH CFH</math></p> $D = \begin{pmatrix} -\frac{7}{11}, \frac{2}{11} \\ \frac{2}{11}, \frac{1}{11} \\ \frac{1}{27}, \frac{17}{27} \\ -\frac{3}{22}, \frac{23}{66} \\ \frac{9}{19}, \frac{19}{19} \end{pmatrix}$	 <p>97: <math>ADB AED ACE BFC BGF BDG DHC DGH EHC EFC</math></p> $D = \begin{pmatrix} \frac{1}{88}\sqrt{145} - \frac{47}{88}, \frac{1}{88}\sqrt{145} + \frac{25}{88} \\ -\frac{2}{55}\sqrt{29}\sqrt{5}, \frac{1}{11} \\ \frac{1}{88}\sqrt{145} + \frac{47}{88}, -\frac{1}{88}\sqrt{145} + \frac{25}{88} \\ \frac{1}{176}\sqrt{145}, \frac{113}{176} \end{pmatrix}$
 <p>98: <math>ADB AED AFE ACF BGC BHG BDH DEH EFH FGH FCG</math></p> $D = \begin{pmatrix} -\frac{1}{2}, \frac{7}{22} \\ -\frac{3}{28}, \frac{9}{44} \\ \frac{2}{2}, \frac{1}{11} \\ \frac{1}{2}, \frac{7}{22} \\ 0, \frac{7}{2} \end{pmatrix}$	

Table C.1: Equiareal triangulations of the triangle with up to 8 vertices. In all examples we have  $A = (-1, 0), B = (1, 0)$  and  $C = (0, 1)$ .

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All other figures have been created by the author using Tikz and Sage. Many thanks to Jean-Philippe Labbé for writing the tikz-function for the polytopes class from sage; the figures in Chapter 2 were made with this feature.

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All references of the form A..... are the corresponding sequences in [Slo].





# Zusammenfassung

In der vorliegenden Arbeit werden verschiedene *Probleme der diskreten Geometrie* untersucht und *Methoden* entwickelt um diese zu lösen. Die Methoden machen sich zunutze, dass sich viele Probleme der diskreten Geometrie als *Optimierungsproblem* über einem polynomiellen Ungleichungssystem darstellen lassen. In einigen Fällen gelingt es, eine *numerische* Lösung eines solchen Systems zu bestimmen und dann in eine *exakte* Lösung zu überführen, die wiederum etwas über das betrachtete Problem aussagt. Wir geben einige Beispiele von solchen Anwendungen.

Die Frage welche simplizialen Sphären als Polytop realisiert werden können wird im ersten Kapitel behandelt. Hierbei entwickeln wir eine Möglichkeit für eine gegebene simpliziale Sphäre eine Realisierung zu finden, falls diese existiert. Außerdem werden die bekannten Beweismethoden für die Nichtrealisierbarkeit derart verbessert, dass sie effizient auf große Familien von simplizialen Sphären und uniformen orientierten Matroiden angewandt werden können. Die Realisierungsmethode besteht darin, ein nicht-lineares Gleichungssystem, welches die Realisierbarkeit der simplizialen Sphäre beschreibt, in einem ersten Schritt numerisch zu lösen und diese numerische Lösung in einem zweiten Schritt in rationale Koordinaten umzuwandeln, für die man dann beweisen kann, dass es sich tatsächlich um eine Realisierung der simplizialen Sphäre handelt. Mit diesen Methoden gelingt die vollständige Klassifizierung der kombinatorischen Typen von simplizialen 4-Polytopen mit 10 Ecken: es gibt genau 162 004. Für simpliziale 4-Polytope mit 9 Ecken wurde deren Anzahl, nämlich 1 142, von Altshuler, Bokowski und Steinberg bereits 1980 bestimmt. Außerdem erhalten wir die vollständige Klassifizierung von simplizialen nachbarschaftlichen Polytope mit 10 Ecken in Dimension 5 und mit 11 Ecken in den Dimensionen 4, 6 und 7.

Wir behandeln in diesem Kapitel nicht nur die Polytopalität von simplizialen Sphären, sondern beschäftigen uns auch mit der *Einschreibbarkeit* derselben, das heißt wir suchen Realisierungen, so dass alle Koordinaten auf der Einheitssphäre liegen. Hierzu wird zu unserer Überraschung festgestellt, dass alle untersuchten 2-nachbarschaftlichen simplizialen Polytope einschreibbar sind.

Das zweite Kapitel behandelt eine Fragestellung aus dem Gebiet der Packungen. Seien zwei Polytope  $P$  und  $Q$  gegeben. Wir suchen ein Polytop  $P'$  von maximalem Volumen, so dass  $P'$  ähnlich zu  $P$  ist und außerdem  $P'$  in  $Q$  enthalten ist. Diese Frage wurde bereits verschiedentlich untersucht, insbesondere in der Dimension 2, wo  $P$  und  $Q$  also Polygone sind. In Dimension 3 hat Croft 1980 alle 20 Paare von 5 platonischen Körpern untersucht und konnte für 14 die optimalen Inklusionen ermitteln. Wir beschäftigen uns mit den verbleibenden 6 Fällen und wenden unsere Methoden darauf an. Zwar erhalten wir hier zunächst nur numerische Ergebnisse, können daraus jedoch Inklusionen bestimmen, deren Koordinaten algebraische Zahlen sind.

Im dritten Kapitel werden zwei weitere Probleme untersucht: Zylinderpackungen an der Sphäre und das Zeichnen von planaren Graphen mit vorgeschriebenen Flächeninhalten.



## **Erklärung**

Gemäß §7 (4), der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin vom 8. Januar 2007 versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die vorliegende Arbeit selbständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht bereits zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

Moritz Firsching