Die Dissertation mit dem Titel

Hysteresis-Delay Differential Equations

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vorgelegt.

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Berlin, den 15. Juni 2015 Eyal Ron

Abstract

In this dissertation we study differential equations with both discontinuous hysteresis of non-ideal relay type and delay terms. We study general properties (existence and uniqueness of solutions) and focus on the stability of periodic solutions. We give an application for control theory.

In Chapter I we study hysteresis-delay ordinary differential equations. We show existence and uniqueness of solutions for such equations. In order to study stability of periodic solutions, we create a Poincaré map and show that the stability is determined by the spectrum of its formal linearization. This last step turned out to be especially challenging. We reduce the stability analysis of the formal linearization to an equivalent finite-dimensional problem.

In Chapter II we study hysteresis-delay parabolic partial differential equation. The hysteresis and delay terms are in the boundary condition of the equation. This can be seen as applying an additional controller (delay) to a thermostat model with hysteresis. Applying nonlocal and semigroup theory we prove existence and uniqueness of solutions for such equations. We decompose the problem into a system of infinitely many ordinary differential equations via the Fourier decomposition. Under a certain assumption we show that stability of periodic solutions is determined by finitely many equations. In the last section we give examples in which there is a periodic solution that can be stabilized by using the methods of this dissertation.

Zusammenfassung

In dieser Doktorarbeit untersuchen wir die allgemeinen Eigenschaften, wie das Dasein und die Eindeutigkeit von Lösungen, von Differentialgleichungen mit sowohl unstetiger Hysterese der ,,nichtidealen Relais-Art", als auch Verzögerungstermen. Wir besorgen eine kontrolltheoretische Anwendung.

In Kapitel I untersuchen wir gewöhnliche Hysterese-Verzögerungsgleichungen, indem wir die Existenz und Eindeutigkeit von Lösungen derartiger Gleichungen beweisen. Um die Stabilität von regelmäßigen Lösungen zu bestimmen, konstruieren wir eine geeignete Poincaré-Abbildung und zeigen, daß sich die Stabilität mittels des Spektrums von deren formalen Linearisation bestimmen läßt; dieser letzte Schritt erwies sich als besonders anspruchsvoll. Wir führen die Stabilitätsanalyse der formalen Linearisation auf ein äquivalentes endlich-dimensionales Problem zurück.

In Kapitel II betrachten wir parabolische Hysterese-Verzögerungsdifferentialgleichungen, wobei sich die Hysterese- und Verzögerungsterme in den Randbedingung ergeben. Solche Systeme folgen aus einem Thermostatmodell mit Hystere, auf das ein Kontroller (Verzögerungsoperator) wirkt. Unter Verwendung der Halbgruppen und nicht-lokalen Theorie beweisen wir das Dasein und die Eindeutigkeit von Lösungen derartiger Gleichungen. Durch Fourieranalyse zerlegen wir dieses Problem in ein System unendlich vieler gewöhnlicher Differentialgleichungen. Unter einer gewissen Annahme zeigen wir, daß sich die Stabilität von regelmäßigen Lösungen durch endlich viele Gleichungen bestimmen läßt. In dem letzten Abschnitt führen wir Beispiele an, in denen es eine regelmäßige Lösung, die unter Verwendung der Methoden dieser Dissertation stabilisiert werden kann, gibt.

Acknowledgement

I do not mean to be a cliché, alas it seems impossible to acknowledge the end of a multi-year project without being one.

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Appendix

Introduction

In this dissertation we develop a general theory of hysteresis-delay differential equations, i.e., differential equations with both a discontinuous hysteresis operator and delay terms. Our focus is on the stability of periodic solutions of such equations. We concentrate on the situation where the period equals the delay.

This introduction is structured as follows. We begin by introducing the main problems to be studied in the dissertation (problems (1) and (3)). Then we list the main difficulties in studying these problems and the main results that are achieved in the dissertation. Next we give a survey of literature, state the layout of the dissertation, and conclude the introduction with some preliminary notation.

1. Hysteresis-delay ordinary differential equations

We consider a linear delay term and a non-ideal relay hysteresis operator (see below), which is a discontinuous nonlinear operator. The main problem that we study takes then the form

$$\begin{aligned}
&u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{\Lambda}u + \mathbf{B}u(t-2T), \quad t > 0,\\
&u(t) = \varphi(t), \quad t \in (-2T, 0),\\
&u(0+) = x.
\end{aligned} \tag{1}$$

Here

- $u(t), k \in \mathbb{R}^N$,
- $\mathbf{\Lambda}, \mathbf{B} \in \mathbb{R}^{N\times N}$,
- $\mathbf{M}$ is a linear functional on $\mathbb{R}^N$,

and $\mathcal{H}(\mathbf{M}u)(t)$ is a discontinuous hysteresis operator of non-ideal relay type defined as follows (see Figure 1 and the accurate definition in Section 1.2): Fix two thresholds α and β such that $\alpha < \beta$. If $\mathbf{M}u(t) \le \alpha$ or $\mathbf{M}u(t) \ge \beta$, then $\mathcal{H}(\mathbf{M}u)(t) = 1$ or $\mathcal{H}(\mathbf{M}u)(t) = -1$ respectively. If $\mathbf{M}u(t) \in (\alpha, \beta)$, then $\mathcal{H}(\mathbf{M}u)(t)$ takes the same value as "just before". We say that the hysteresis **switches** when it jumps from 1 to -1 (or vice versa). The corresponding time moment is called a **switching time**.

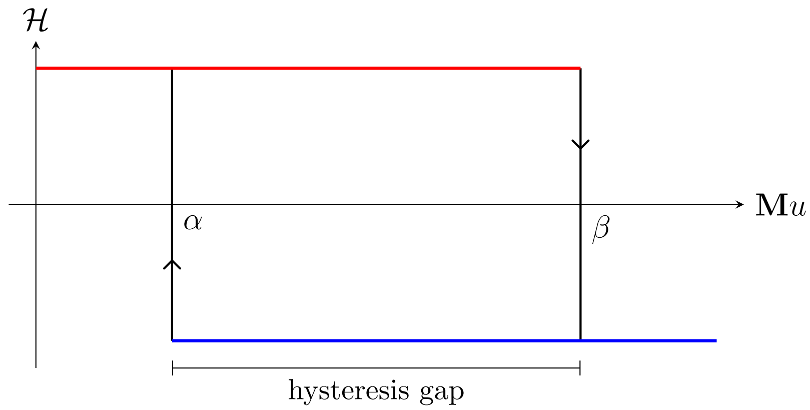

Figure 1: Hysteresis operator of non-ideal relay type

Note that problem (1) is an infinite-dimensional problem (due to delay) with a discontinuous right hand side (due to the hysteresis).

We remark that the delay is taken to be $2T$ due to technical reasons: it makes the (lengthy) calculations in later sections more elegant. The $2T$-periodic solution that we study is symmetric around its mid-point (see Assumption 2.12). If we denoted its period by T, then the mid-point would be $\frac{T}{2}$. But we want to avoid fractions as much as possible, and hence denote the period by $2T$ (and hence also the delay). This makes the mid-point T.

We study this problem in Chapter I (see problem (1.1)–(1.3)). We prove general properties (such as existence and uniqueness of solutions) and focus mainly on studying stability of $2T$-periodic solutions. See a summary of results in Section 3 of the introduction.

In Chapter II we use the theory developed in Chapter I in order to study stability of $2T$-periodic solutions of parabolic hysteresis-delay partial differential equations. This is motivated by an application, described in Section 2 of the introduction.

2. Hysteresis-delay partial differential equations and a model application

The main PDE problem that is studied in Chapter II is a parabolic PDE with non-ideal relay operator and a delay term on the boundary (problem (3) defined below). We show here the motivation for considering this problem and its relation to problem (1).

Hysteresis operators arise naturally in mathematical models of physical systems. In certain cases they are used as controllers. In the last twenty years delay equations have also been used as "controllers" via the application of Pyragas control (see below). We describe next a control scheme that has both hysteresis and delay terms. This scheme is partly the motivation for developing the general hysteresis-delay differential equations theory in this dissertation.

Thermal control model (hysteresis only). As an example of a system controlled by hysteresis, consider a model of thermal control in a bounded domain $Q \subset \mathbb{R}^n$. It was originally proposed by Glashoff and Sprekels [19, 20]). In this model, the temperature in the domain is regulated via heating (or cooling) elements on the boundary of the domain. The heating elements operate based on information received from thermal sensors inside the domain and obey hysteresis laws. Denoting the temperature at a point $x \in Q$ at time t by $u(x,t)$, the equations governing the system take the form

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t), \qquad x \in Q,\ t > 0, \\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} &= k(x)\mathcal{H}(\hat{u})(t), \qquad t > 0, \end{aligned} \tag{2}$$

where ν is the outward normal to the boundary, $k(x)$ is the density of the heating elements, and

$$\hat{u}(t) := \int_Q m(x)u(x,t)dx$$

is the mean temperature in Q at time t with some weight function $m(x)$. The hysteresis operator $\mathcal{H}$ plays the role of controlling feedback, which means that it determines when the system cools down or heats up. Problem (2) contains no reaction term for simplicity.

Modified thermal control model (hysteresis and delay). Gurevich [22] and Gurevich and Tikhomirov [25] showed the existence of both stable and unstable periodic solutions for problem (2) (both co-existing in the same system and separately for different $k(x)$ and $m(x)$).

It may be desirable to change these stability properties. For example, in some cases the thermal control model has two periodic solutions. One of them, with a small period, is stable (but not desired, as it causes rapid switchings of the controller which may be physically damaged as a result), and another, with a larger period, is unstable (but desired). Hence the following question arises: Is it possible to change the stability of such solutions in a way that the solutions themselves are unchanged (especially for desired periodic solutions)?

We attempt to change the stability by employing a method known as **Pyragas control** as follows: Assume that $u_p(x,t)$ is a $2T$-periodic solution to problem (2). Due to periodicity, expressions of the form $u(x,t-2T) - u(x,t)$ vanish for all $t \geq 2T$ when evaluated for u_p. Such terms are called **Pyragas terms**. We modify problem (2) by inserting such a term into the boundary condition (since the heating elements are on the domain of the boundary), thus obtaining the system

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \qquad x \in Q,\ t > 0, \\ &\left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} = k(x)\mathcal{H}(\hat{u})(t) + b(x)(\hat{u}(t) - \hat{u}(t-2T)), \qquad t > 0, \end{aligned} \tag{3}$$

where $b(x)$ is some smooth function on ∂Q called the **Pyragas coefficient**. Since the periodic solution u_p vanishes on the new added term, it is still a solution to problem (3). The question now is whether $b(x)$ can be chosen such that the stability of u_p is changed? We answer this question in Chapter II.

Reduction to a finite-number of ODEs. As we already mentioned, the bulk of this dissertation (Chapter I) is devoted to the study of the ODE problem (1). The question is: how can one reduce the PDE problem (3) to a problem of the form (1)? A complete account of how such a reduction is possible is given in Sections 6.4 and 7, but we point out the main idea here. It is composed of two steps.

In the first step we do a Fourier decomposition with respect to the eingenfunctions of the Laplacian with homogeneous Neumann boundary conditions. This results in a system of infinitely many ODEs.

In the second step we include an additional assumption: the weight function $m(x)$ has a finite number of non-zero Fourier modes. Then we show that the stability of a periodic solution to the system of infinitely many ODEs depends on the stability of a corresponding periodic solution to a problem of the form

$$\begin{aligned} &u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{\Lambda}u + \mathbf{A}[u(t-2T) - u(t)], \quad t > 0, \\ &u(t) = \varphi(t), \quad t \in (-2T, 0), \\ &u(0+) = x. \end{aligned}$$

The last problem is of the form of problem (1).

A similar process was carried out by Gurevich [22] for problem (2).

3. Difficulties and summary of results

Difficulties. We list here the main difficulties in studying stability of solutions of problem (1).

1. The main problem is the interaction between discontinuity (caused by hysteresis) and infinite-dimensionality (caused by delay). Specifically, to which space do the initial data belong?

   We study the stability of a periodic solution via a Poincaré map (see Section 3) and calculate a formal linearization[1] of the map in the process (see

[1] By a formal linearization we mean formally creating an expression for the Fréchet derivative of the Poincaré map, without showing that it is indeed its derivative.

Section 4). This Poincaré map, $\mathbf{P}(\varphi, x)$, involves the term $u(\varphi, x; t_{switch}(\varphi, x))$, where $u(\varphi, x; t)$ is a solution to problem (1) and $t_{switch}(\varphi, x)$ is its switching time. Formally differentiating $\mathbf{P}$ at the initial data of the periodic solution (φ_p, x_p), yields that the formal linearization contains the time derivative of the periodic solution u_p'. Due to periodicity this equals the derivative of the initial data φ_p'.

The derivative φ_p is not in $C[-2T, 0]$, and hence the formal linearization is not a well-defined operator on $C[-2T, 0]$ or $W_p^1(-2T, 0)$ (Sobolev spaces), see Discussions 1.6 and 4.19. The derivative φ_p' is, however, a well-defined operator on L_p spaces, but these spaces do not fit for linearization either: the formal remainder, $\mathbf{P}(\varphi_p + \nu, x_p + y) - \mathbf{P}(\varphi_p, x_p) - D\mathbf{P}(\varphi_p, x_p)[\nu, y]$, is not $o(\nu)$ in L_p spaces. A hybrid space must be taken as the space of initial data, which is a Lebesgue space on the interval $(-2T, 0)$ and a fractional Sobolev space on its subinterval $(-T - \sigma, 0)$ with a small $\sigma > 0$. This space is denoted by $\mathbb{B}_p^s$, see Section 2.1 for its exact definition. See Discussion 4.19 for a detailed explanation of this choice.

We stress that the derivative of the initial data does not exist when one uses a Poincaré map to study delay equations without hysteresis (see [14, Chapter XIV.3]). However, an analogous derivative does occur when one uses a Poincaré map to study hysteresis equations without delay (see [22, 25]). But then the initial data belongs to the space $\mathbb{R}^N$ and not to a function space, hence this difficulty is not present.

2. There is another difficulty related to the interaction between discontinuity and delay. It occurs when trying to calculate the total derivative of the Poincaré map. The periodic solution itself has an extra regularity (see Lemma 2.13), hence the Poincaré map appears to have partial derivatives on it, but not everywhere on the neighbourhood of it. Hence, the Poincaré map does not have a total derivative on the periodic solution, and it is not straightforward that the formal linearization determines stability.

3. The formal linerization that we find is an infinite-dimensional operator (due to delay), hence analysis of its spectrum is not straightforward.

Summary of results. The following lists the main results in this dissertation.

Chapter I.

- Existence and uniqueness of solutions to the hysteresis-delay ODE problem (1) is proved (Theorem 1.12).

- Poincaré map is defined for a given periodic solution (Section 3). It is shown that the stability of the periodic solution depends on the stability of a fixed point of the Poincaré map (Lemma 3.18).

- The main theorem of Chapter I shows that the stability of a fixed point of the Poincaré map depends on the spectral radius of the formal linearization of the map (Theorem 4.18). An explicit expression is given for the latter.

- The spectral radius of the linear operator that determines the stability of a fixed point of the Poincaré map is studied (Section 5). This problem is reduced to a finite-dimensional problem, which leads to finding the roots of an explicitly given scalar characteristic equation (Lemma 5.13).

Chapter II.

- Existence and uniqueness of solutions to a hysteresis-delay parabolic PDE problem (3) is proved (Theorem 6.22).

- Reduction of the stability problem of a given periodic solution to a hysteresis-delay parabolic PDE problem (3) to an ODE problem (1) (Theorem 7.6) is shown.

- Application: examples in which there exists a periodic solution that can be stabilized using the methods developed in this dissertation are given (Theorem 8.5).

4. Survey of literature

Non-ideal relay hysteresis. The monograph [33] prompted a great number of mathematical works on general hysteresis. Considerable amount of models of hysteresis with ordinary differential equations were studied. Periodic solutions for ordinary differential equations with a hysteresis of non-ideal relay type were studied e.g. in [6, 46]. Non-ideal relay and partial differential equations were handled in [7, 53], where the focus was on issues such as existence, uniqueness or regularity of solutions. Periodicity was studied mostly in the case of the thermal control problem.

Thermal control model. A model similar to problem (2) was suggested in [19, 20]. For a one-dimensional spatial domain ($n = 1$) periodicity was studied in [18, 21, 42]. The case of a multidimensional domain ($n \geq 2$) turned out to be much harder for discontinuous hysteresis. One possible solution was to consider a continuous model of hysteresis [23].

The first results for periodic solutions in a multidimensional domain with discontinuous hysteresis on the boundary were established in [24] using a fixed-point method. A new approach, which decomposed the equation into a system of infinitely many ODEs, was suggested in [22] and further investigated in [25]. In the latter paper, an algorithm for finding periodic solutions was given. In addition, the existence of stable and unstable periodc solutions was shown, but the question of stabilizing/destabilizing them was not discussed.

Delay differential equations. Delay equations were thoroughly studied in the last few decades. The reader is referred to [14, 27] for a general introduction on the topic. Periodic solutions were also well studied for delay equations, including stability analysis via a Poincaré map [14, Chapter XIV.3]. Pyragas control, which add a non-invasive control in the form of a delay term to an equation, was suggested in [43], and has since then become very popular. A summary of the vast amount of results following the original publication can be found in [44].

Hysteresis-Delay differential equations. Systems with delay and hysteresis are a relatively new topic. It was studied mostly for ordinary differential equations with delay only inside the hysteresis operator [10, 17, 36, 56], namely $\mathcal{H}(u(\cdot - \tau))(t) = \mathcal{H}(u)(t - \tau)$.

Questions regarding periodic solutions for such problems were studied for very specific models and equations [2, 4, 29, 31, 34].

Problems where the delay is in the hysteresis are simpler to study compared with problem (1), since the delay then adds only a finite number of dimensions to the system [11, 48]. In [48] a rather general form of such equations was studied, and it was shown that the Poincaré map is smooth in a neighbourhood of a periodic orbit under some limitations. Possible bifurcations, such as grazing or corner collision bifurcations, were shown to arise from violations of those limitations. Those bifurcation situations may render the Poincaré map discontinuous. Questions of existence, uniqueness or stability analysis of periodic orbits in general settings were not discussed.

There are very few papers about differential equations with continuous hysteresis and delay outside the hysteresis operator. Existence of oscillating solutions to the problem

$$\begin{aligned} &u'(t) = C - h\omega(t-\tau)u(t-\tau), \quad t > 0, \\ &\omega(t) = \mathcal{H}_\beta(u(t)), \quad t > -\tau, \end{aligned}$$

where $\mathcal{H}_\beta$ is the generalized play model (see e.g. [53]) was shown in [9,57]. Periodic solutions or stability analysis was not studied there.

To the best knowledge of the author, no papers on discontinuous hysteresis of non-ideal relay type with delay outside the hysteresis exist.

5. Layout

This dissertation consists of two chapters. Chapter I studies a system of finitely many ordinary differential equations with hysteresis and delay terms. Chapter II studies parabolic partial differential equations with hysteresis and delay terms on the boundary.

Chapter I consists of Sections 1–5. The main goal of this chapter is to analyse the stability of periodic solutions to problem (1).

In Section 1 we define the system of ODEs to be studied and establish its fundamental properties (such as existence and uniqueness of solutions).

In Sections 2–5 (the remaining sections in Chapter I), we study stability of periodic solutions.

In Section 2 we define what stability is and state the main problem of Chapter I: studying the stability of a periodic solutions.

In Section 3 we introduce the main tools for studying stability: the Poincaré and hit maps. We show that stability of the Poincaré map implies stability of the periodic solution.

In Section 4 we study linearization (i.e., Fréchet derivative) the Poincaré map. This step, which is normally a straightforward one, becomes the technical heart of Chapter I. This is due to the fact that the Poincaré map has a discontinuity that comes from the hysteresis.

In Section 5 we study the spectrum of the formal linearization of the Poincaré map from the previous section. It is an infinite-dimensional operator due to the delay. We reduce the analysis of it to a spectral problem for a finite-dimensional operator, namely, studying the root of a scalar characteristic function.

Chapter II consists of Sections 6–8. The main goal of this chapter is to analyse the stability of periodic solutions for problem (3). Specifically, we show that unstable periodic solutions to problem (2) become a stable periodic solutions to problem (3) under an appropriate choice of $b(x)$.

In Section 6 we define the PDE to be studied and establish its fundamental properties (such as existence and uniqueness of solutions). In Section 6.4 and 6.5 we convert the problem to a system of infinitely many ODEs.

In Section 7 we show that the stability of the periodic solution depends on finitely many ODEs.

Finally, in Section 8, we give an application of our theory. We give examples in which there is a periodic solution that can be stabilized using the methods of this dissertation.

6. Preliminary notation

The following notation is used throughout this dissertation.

1. Operators are denoted by bold letters: $\mathbf{P}$, $\mathbf{\Pi}$, et cetera. Matrices are also denoted by bold letters since they are linear operators: $\mathbf{A}$, $\mathbf{B}$ et cetera. There is one exception to the rule: The hysteresis operator $\mathcal{H}$ is denoted by a mathematical calligraphic letter to empathize its importance.

2. Arguments of linear operators are put between square bracket: $\mathbf{L}[\nu, y]$. Arguments of nonlinear operators are put between standard round brackets: $\mathbf{P}(\varphi, x)$. If the linear operator depends only on one argument, then the brackets may be omitted: $\mathbf{M}u$.

3. Spaces are usually denoted using a blackboard font: $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), et cetera.

4. For a product of two normed vector spaces, we use the product norm given by the sum of the norms of each space, i.e, if $\mathbb{X}$, $\mathbb{Y}$ are normed spaces, then the norm of $\mathbb{X} \times \mathbb{Y}$ is $\|\cdot\|_{\mathbb{X}} + \|\cdot\|_{\mathbb{Y}}$.

5. When evaluating the norm of a vector, we omit its brackets, i.e. norm of (ν, y) is $\|\nu, y\|$.

6. There are two different types of derivatives in this dissertation. The notation is as follows.

   6.1. A weak derivative [45] is denoted by putting an apostrophe, i.e.,

   $$\text{The weak derivative of } f(t) \text{ with respect to } t : f'(t).$$

   6.2. A Fréchet derivative [16] is denoted by D. We also refer to it a **linearization** in this dissertation (as a Fréchet derivative is the act of finding the linear part of an operator). There is a distinction between total and partial derivatives: if $\mathbb{X}$, $\mathbb{Y}$ are Banach space, and $\mathbf{P} : \mathbb{X} \times \mathbb{R} \to \mathbb{Y}$ is a nonlinear operator, then we denote the

   - Total Fréchet derivative (or: linearization) of $\mathbf{P}$ applied to (ν, y) by $D\mathbf{P}(\varphi, x)[\nu, y]$.
   - Partial Fréchet derivative of $\mathbf{P}$ with respect to φ or x by $D_\varphi\mathbf{P}(\varphi, x)[\nu]$ or $D_x\mathbf{P}(\varphi, x)[\nu]$, respectively.

Chapter I

Ordinary Hysteresis-Delay Differential Equations

1. Setting of the problem. Existence and uniqueness of solutions

In this section we establish the setting for Chapter I. The main equation is presented. The operators that compose it are defined. In the bulk of the section we prove fundamental properties of the problem, specifically, existence and uniqueness of solutions. We close the section by showing that solutions are uniformly Lipschitz continuously dependent on their initial data (when the hysteresis has a fixed value).

1.1 General hysteresis-delay ordinary differential equation

We call a differential equation a **hysteresis-delay differential equation** if it has both hysteresis and delay terms. Consider the N-dimensional hysteresis-delay ordinary differential equations (the specific form in which it is written is motivated in Section 2 in the introduction)

$$u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{\Lambda}u + \mathbf{A}[u(t-2T) - u(t)], \quad t > 0, \tag{1.1}$$

with initial conditions

$$u(t) = \varphi(t), \quad t \in (-2T, 0), \tag{1.2}$$

$$u(0+) = x. \tag{1.3}$$

where $u(0+)$ is in the sense of traces from the right. Here

- $u(t) \in \mathbb{R}^N$, $k \in \mathbb{R}^N$,
- $\mathbf{M} = (m_0, \dots, m_{N-1})$, $m_0 \neq 0$ is a linear functional on $\mathbb{R}^N$, called **average**[2],
- $\mathbf{\Lambda}, \mathbf{A} \in \mathbb{R}^{N\times N}$,
- $T \in \mathbb{R}^+$ (positive real numbers),

are all fixed, and

- the hysteresis operator $\mathcal{H}$ is defined in Section 1.2.

Discussion 1.1. The natural (and shorter) way to write equation (1.1) would be

$$u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{B}u(t) + \mathbf{C}u(t-2T), \tag{1.4}$$

where $\mathbf{B}, \mathbf{C}$ are $n \times n$ matrices[3]. This is just a rewrite of problem (1.1) with $\mathbf{B} := (\mathbf{\Lambda} + \mathbf{A})$ and $\mathbf{C} := \mathbf{A}$. There are two reasons for why we chose our specific form.

[2]Such a functional with non-zero entries in its vector representation is called a non-trivial functional.

[3]There is a minus sign before $\mathbf{B}$ in order to be consistent with Notation 1.7 later on.

1. In Chapter II we study a hysteresis-delay PDE and reduce it to a system of ODEs $\big($see Section 7, especially problem (7.1)$\big)$. This system has the form of problem (1.1)–(1.3) (the letter $\mathbf{\Lambda}$ also comes from this reduction). See also Section 2 in introduction.

2. In the introduction we presented Pyragas control. It is a method to control the stability of periodic solutions. If we consider the simple linear hysteresis ODE

$$u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{\Lambda}u,$$

then equation (1.1) is a Pyragas controlled version of it for a $2T$-periodic solution.

Another thing that may raise an eyebrow is the choice of the delay; why is it $2T$ and not just T? The reason is technical. See the explanation in the introduction.

1.2 Hysteresis operator

The term hysteresis applies to a wide number of operators; the specific one that we use is called a **non ideal relay** operator. We first define it rigorously, and then try to explain it in an informal way. The following definition is standard. It is taken from Visintin [53, chapter VI]. See also Krasnosel'skiĭ [33].

Definition 1.2 (Hysteresis)**.** Fix $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$. For any $g \in C[0, T_1]$ (the space of continuous functions on $[0, T_1]$), $T_1 \in \mathbb{R}^+$, the hysteresis operator (or a non ideal relay operator)

$$z = \mathcal{H}(g) : [0, T_1] \to \{-1, 1\},$$

is defined as follows (see Figure 2). Let $X_t = \{t' \in (0, t] : g(t') = \alpha \text{ or } \beta\}$. Then

$$z(0) := \begin{cases} 1 & \text{if } g(0) < \beta, \\ -1 & \text{if } g(0) \geq \beta, \end{cases}$$
$$z(t) := \begin{cases} z(0) & \text{if } X_t = \emptyset, \\ 1 & \text{if } X_t \neq \emptyset \text{ and } g(maxX_t) = \alpha, \\ -1 & \text{if } X_t \neq \emptyset \text{ and } g(maxX_t) = \beta. \end{cases} \tag{1.5}$$

Definition 1.3 (Switching time)**.** A time t_1 is called a **switching time** (or just a **switching**) for a function $g \in C[0, T_1]$, if the function $\mathcal{H}(g)(t)$ is discontinuous at t_1.

Informal explanation. The hysteresis operator $\mathcal{H}$ is a **discontinuous** operator. For any continuous function g the hysteresis is a function from $[0, T_1]$ to $\{-1, 1\}$. Since the range has only two discrete elements, there is a jump when the hysteresis changes its value.

The hysteresis includes two constants in its definition: $\alpha < \beta$. The interval $[\alpha, \beta]$

is called **the hysteresis gap**.

If $g(t)$ is not in the hysteresis gap for a given t, then the output of the hysteresis operator is very simple. It equals 1 if $g(t) < \alpha$, and -1 if $g(t) > \beta$.

Things get interesting if $g(t)$ is in the hysteresis gap. Then the value of the hysteresis depends on the history of $g(t)$ (hence the name: hysteresis means "lagging behind" in ancient Greek). Informally, we can say that the hysteresis stays with the value that it had "just before".

Another explanation is that the hysteresis does not change its value as long as $g(t)$ is in the hysteresis gap. Intuitively, we can think of it as a "lazy" operator, since it is too "lazy" to change its value in the hysteresis gap. The value in the gap depends on whether $g(t)$ entered the gap through α (then the hysteresis will be 1 in the gap) or β (hysteresis will be -1 in the gap). Once $g(t)$ leaves the gap, i.e., equals α or β, the hysteresis may be obliged to change its values.

Remark 1.4. The definition in Visintin [53] has an extra parameter $\xi \in \{-1, 1\}$. It denotes the value of the hysteresis in the case $g(0) \in (\alpha, \beta)$. Here we assume, without loss of generality, that $\xi = 1$.

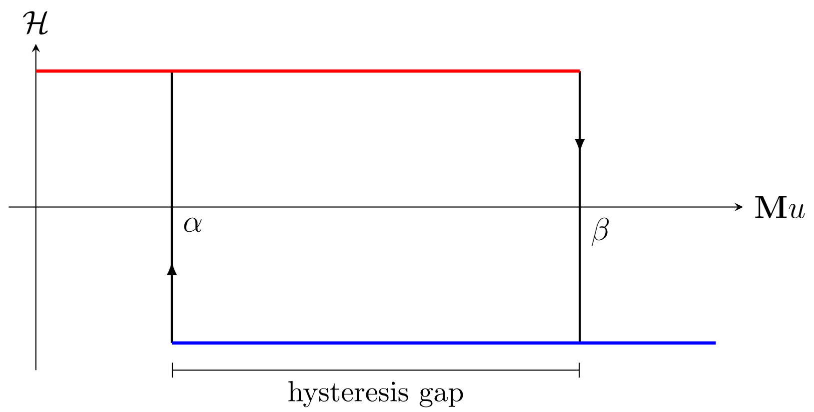

Figure 2: Hysteresis operator of non-ideal relay type

1.3 Definitions: spaces, solutions and switching points

To define a solution for problem (1.1)–(1.3), we first need to define the appropriate Lebesgue and Sobolev spaces. The following definitions are standard.

Let $L_p(a, b)$, $1 < p < \infty$, be the Lebesgue space on the real line with the norm

$$\|\varphi\|_{L_p(a,b)} = \left(\int_a^b |\varphi(s)|^p ds \right)^{\frac{1}{p}}.$$

Let L_∞ be the space of essentially bounded measurable functions with the norm

$$\|\varphi\|_{L_\infty(a,b)} = ess\ sup_{t \in (a,b)} |\varphi(t)|.$$

Denote N copies of the space L_p, $1 < p \leq \infty$ by $\mathbb{L}_p$:

$$\mathbb{L}_p(a,b) = (L_p(a,b))^N,$$

with the norm of $\varphi = (\varphi_1, \ldots, \varphi_N)$, $\varphi_1, \ldots, \varphi_N \in L_p(a,b)$, given by

$$\|\varphi\|_{\mathbb{L}_p(a,b)} = \left(\int_a^b \|\varphi(s)\|_{\mathbb{R}^N}^p ds \right)^{\frac{1}{p}}, \quad 1 < p < \infty,$$
$$\|\varphi\|_{\mathbb{L}_\infty(a,b)} = ess\, sup_{t \in (a,b)} \|\varphi(s)\|_{\mathbb{R}^N}.$$

If the interval (a,b) is not specified, then

$$\mathbb{L}_p := \mathbb{L}_p(-2T, 0).$$

Let $W_p^1(a,b)$ be the standard Sobolev space of L_p functions whose weak derivative belongs to $L_p(a,b)$, with the norm

$$\|\varphi\|_{W_p^1(a,b)} = \|\varphi\|_{L_p(a,b)} + \|\varphi'\|_{L_p(a,b)}.$$

Denote N copies of $W_p^1(a,b)$ by $\mathbb{W}_p^1(a,b)$.

Define similarly for $k \in \mathbb{N}$ the space $W_p^k(a,b)$ as the standard Sobolev space of $L_p(a,b)$ functions which are k times differentiable, where each derivative belongs to $L_p(a,b)$. Denote N copies of $W_p^k(a,b)$ by $\mathbb{W}_p^k(a,b)$.

Definition 1.5 (solution to problem (1.1)–(1.3))**.** Given $T_1 > 0$, a function $u \in \mathbb{L}_p(-2T, T_1) \cap \mathbb{W}_p^1(0, T_1)$ (i.e. $u \in \mathbb{L}_p(-2T, T_1)$ and $u\big|_{(0,T_1)} \in \mathbb{W}_p^1(0,T_1)$) is called a solution to problem (1.1)–(1.3) on $[-2T, T_1]$ with initial data $(\varphi, x) \in \mathbb{L}_p \times \mathbb{R}^N$, if u satisfies equation (1.1) for a.e. $t > 0$, condition (1.2) for a.e. $t \in (-2T, 0)$ and condition (1.3) in the sense of traces.

A function u is called a solution on $[-2T, \infty)$ if it is a solution on $[-2T, T_1]$ for every $T_1 > 0$.

We write

$$u(\varphi, x; t)$$

for the solution to problem (1.1) with initial conditions (1.2)–(1.3).

Discussion 1.6. Readers experienced with delay equations may wonder about the choice of the space $\mathbb{L}_p$ for initial data (and not the more standard space $C[-2T, 0]$ of continuous functions). The space $C[-2T, 0]$ has an obvious advantage that the extra initial condition, $u(0+) = x$, is not needed for it. We try to motivate here why did we choose the space $\mathbb{L}_p$ nevertheless.

First note that solutions of problem (1.1)–(1.3) do not become smoother with time (unlike in classical delay differential equations). This happens since a change of value of the hysteresis causes a discontinuity in the derivative. Whether the initial data are in $C[-2T,0]$ or in $\mathbb{L}_p$, we end up with the same regularity for $t>0$ (which is $\mathbb{W}_p^1(0,\infty)$, see Theorem 1.12). If we choose the space $C[-2T,0]$ for initial data, we would lose generality (less possible initial data) and gain no extra regularity.

The real problem arises when considering the stability of a periodic solution. In the method that we choose to study stability, some process of linearization is needed (see Section 4). The linearization depends on the switching time of the periodic solution, which in turn depends on its initial data. Using the chain rule eventually yields a derivative of the initial data of the periodic solution around which we try to linearize (specifically, of φ from condition (1.2), see Section 4.2 for details).

Due to periodicity the initial data has the same regularity of the solution (see Lemma 2.13), and hence it does have a weak derivative. However, this weak derivative is in the $\mathbb{L}_p$ space. Hence the linearization belongs to the space $\mathbb{L}_p$, and not to to the space of continuous functions. Which is why we end up working with the $\mathbb{L}_p$ space.

1.4 Existence and uniqueness of solutions

In this section we prove existence and uniqueness results for problem (1.1)–(1.3).

Notation 1.7. For brevity, we use the following notation throughout the rest of this chapter.

$$\mathbf{B} = (\mathbf{\Lambda} + \mathbf{A}). \tag{1.6}$$

This is the same $\mathbf{B}$ that was used in Discussion 1.1. Equation (1.1) rewritten with $\mathbf{B}$ is:

$$u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{B}u(t) + \mathbf{A}u(t-2T), \quad t>0. \tag{1.7}$$

The value of the hysteresis $\mathcal{H}(\mathbf{M}u)(t)$ in equation (1.7) can be 1 or -1. Hence we define two versions of this equation. In the one where $\mathcal{H}(\mathbf{M}u)(t) = +1$ we denote the unknown function as $u_+(t)$, and in the other, where $\mathcal{H}(\mathbf{M}u)(t) = -1$, as $u_-(t)$.

$$\begin{aligned}
&u'_+(t) = k - \mathbf{B}u_+(t) + \mathbf{A}u_+(t-2T), && t>0, && (1.8)\\
&u_+(s) = \varphi_+(s), && s\in(-2T,0), && (1.9)\\
&u_+(0+) = x_+ && && (1.10)
\end{aligned}$$

and

$$\begin{aligned}
&u'_-(t) = -k - \mathbf{B}u_-(t) + \mathbf{A}u_-(t-2T), && t>0, && (1.11)\\
&u_-(s) = \varphi_-(s), && s\in(-2T,0), && (1.12)\\
&u_-(0+) = x_-. && && (1.13)
\end{aligned}$$

Solutions to these problems are defined in a similar fashion as in Definition 1.5. We denote by $u_\pm(\varphi_\pm, x_\pm; t)$ a solution to problem (1.8)–(1.10) or problem (1.11)–(1.13) respectively with initial data $(\varphi_\pm, x_\pm)$.

Remark 1.8. In the rest of this section we treat only the case in which the initial data x in (1.3) is such that $\mathbf{M}x < \beta$. This means that $\mathcal{H}(\mathbf{M}u)(0) = 1$. The proofs for the other case are similar.

The proofs in this section use the equivalent integral form of problems (1.8)–(1.10) and (1.11)–(1.13).

$$u_+(t) = e^{-\mathbf{B}t}x_+ + \int_0^t e^{\mathbf{B}(s-t)}\mathbf{A}u_+(s-2T)ds + \int_0^t e^{\mathbf{B}(s-t)}kds, \tag{1.14}$$
$$u_+(s) = \varphi_+(s), \quad s \in (-2T, 0). \tag{1.15}$$

$$u_-(t) = e^{-\mathbf{B}t}x_- + \int_0^t e^{\mathbf{B}(s-t)}\mathbf{A}u_-(s-2T)ds - \int_0^t e^{\mathbf{B}(s-t)}kds, \tag{1.16}$$
$$u_-(s) = \varphi_-(s), \quad s \in (-2T, 0). \tag{1.17}$$

The next lemma shows existence and uniqueness of solutions for the previous two integral equations.

Lemma 1.9. *For any $T_1 > 0$, there exists a unique solution $u_+(t) \in \mathbb{W}_p^1(0, T_1)\Big($or $u_-(t)\Big)$ of problem (1.14)–(1.15) $\Big($or (1.16)–(1.17)$\Big)$ with initial data $(\varphi_+, x_+)\Big($or $(\varphi_-, x_-)\Big) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N$.*

Proof. We prove for u_+. The proof is similar for u_-.

The proof uses the method of steps[4]. For $t \in (0, 2T)$, u_+ on the right hand side of equation (1.14) can be replaced by the initial data φ_+.

$$u_+(t) = e^{-\mathbf{B}t}x_+ + \int_0^t e^{\mathbf{B}(s-t)}\mathbf{A}\varphi_+(s-2T)ds + \int_0^t e^{\mathbf{B}(s-t)}kds, \quad t \in (0, 2T).$$

Then $u_+ \in \mathbb{W}_p^1(0, 2T)$, since $\varphi_+ \in \mathbb{L}_p(-2T, 0)$.

Do the same for $t \in (2T, 4T)$, using $(u_+(s+2T)_{s\in(-2T,0)}, u_+(2T))$ as the new history (i.e., initial data). Continue doing so using the method of steps (see Footnote 4) for the intervals $(4T, 6T), (6T, 8T)$ et cetera, until an interval that contains T_1 is reached. □

The next lemma shows that a solution has finitely many switching points in a finite time interval.

Lemma 1.10 (switchings do not accumulate)**.** *For every $(\varphi, x) \in \mathbb{L}_p \times \mathbb{R}^N$ and $T_1 > 0$, there exists a positive integer*

$$\bar{N} := \bar{N}(\varphi, x, T_1) > 0$$

[4]This is a standard method for dealing with delay equations. See [50, Chapter 2].

such that for a time sequence $0 < t_1, t_2, \ldots, t_{\bar{N}}$*, if* $u(t)$ *is defined on* $[0, t_{\bar{N}}]$*,* $t_1, t_2, \ldots, t_{\bar{N}}$ *are switching times of* u*, and* $u(t) = u(\varphi, x; t)$ *is a solution to problem (1.1)–(1.3) on* $[-2T, t_{\bar{N}}]$*, then*

$$t_{\bar{N}} > T_1$$

and $u(t)$ *is a solution to problem (1.1)–(1.3) on* $[-2T, T_1]$*.*

Proof. We focus on proving that $t_{\bar{N}} > T_1$. This immediately implies that $u(t)$ is a solution to problem (1.1)–(1.3) on $[-2T, T_1]$.

Let t_0, $0 < t_0 < T_1$, be the smallest time such that $Mu(t_0) = \alpha$. If there is no such t_0, then there is no more than one switch, and the proof is done. Without loss of generality assume that $t_0 = 0$.

To prove the existence of $\bar{N}$ it is sufficient to bound from below the difference between two adjacent switchings times (for times smaller than T_1). We do so in two steps.

Step I. Denote

$$G := \min\{1, 2T\}, \tag{1.18}$$

where G stands for "Gap", as in "the gap between intersections"[5].

If the difference between switchings is bigger than or equal to G, then it is bounded from below by G. If it is smaller than G, then we bound it from below in Step I.I.

Step I.I. Let $i \geq 1$ be odd such that $\mathbf{M}u(t_i) = \beta$. Assume that $t_i - t_{i-1} < G$, and found a lower bound under this assumption.

In the interval $[t_{i-1}, t_i]$, $u(t)$ equals the solution $u_+(t - t_{i-1})$ of (1.14)–(1.15) with initial data[6]

$$\begin{aligned}
&(\varphi^{(1)}, x^{(1)}) := (\varphi, x) \text{ if } i = 1, \text{or} \\
&(\varphi^{(i)}, x^{(i)}) := (u(s + t_{i-1})\big|_{s \in (-2T, 0)}, u(t_{i-1})) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N \text{ if } i \neq 1.
\end{aligned}$$

Since t_i is a switching time, then $\mathbf{M}u_+(t_i - t_{i-1}) = \beta$. Hence integral equation (1.14) implies that (after reordering the terms on the right hand side)

$$\begin{aligned}
\beta &= \mathbf{M}u_+(t_i - t_{i-1}) \\
&= \mathbf{M}\left[e^{-\mathbf{B}(t_i - t_{i-1})}x^{(i)} + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s - t_i)}k\,ds + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s - t_i)}\mathbf{A}u_+(s - t_{i-1} - 2T)\,ds\right].
\end{aligned}$$

[5] This constant was chosen because the estimate in step I.I is true only for differences smaller than or equal to $\min\{1, 2T\}$. See specifically equation (1.21) for the choice of 1. See the paragraph after equation (1.19) for the choice of $2T$.

[6] The initial data of the case $i = 1$ is defined differently since it is the original initial data of $u(\varphi, x; t)$.

The first term inside the square brackets on the right hand side can be written by [39, Chapter 1, Theorem 2.4]

$$e^{-\mathbf{B}(t_i-t_{i-1})}x^{(i)} = x^{(i)} - \mathbf{B}\int_{t_{i-1}}^{t_i} e^{\mathbf{B}(t_{i-1}-s)}x^{(i)}ds.$$

Hence

$$\begin{aligned}\beta &= \mathbf{M}u_+(t_i-t_{i-1})\\ &= \mathbf{M}\Bigg[x^{(i)} - \mathbf{B}\int_{t_{i-1}}^{t_i} e^{\mathbf{B}(t_{i-1}-s)}x^{(i)}ds + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s-t_i)}kds\\ &\quad + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s-t_i)}\mathbf{A}u_+(s-t_{i-1}-2T)ds\Bigg].\end{aligned} \tag{1.19}$$

Since we handle the case where $t_i - t_{i-1} < G \leq 2T$, the function u_+ in the integral in the preceding equation can be replaced by the initial data $\varphi^{(i)}$. The operator $\mathbf{M}$ is linear. Then the previous equation yields

$$\begin{aligned}\beta &= \mathbf{M}u_+(t_i-t_{i-1})\\ &= \underbrace{\mathbf{M}x^{(i)}}_{=\alpha} + \mathbf{M}\Bigg[-\mathbf{B}\int_{t_{i-1}}^{t_i} e^{\mathbf{B}(t_{i-1}-s)}x^{(i)}ds + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s-t_i)}kds\\ &\quad + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s-t_i)}\mathbf{A}\varphi^{(i)}(s-t_{i-1}-2T)ds\Bigg].\end{aligned}$$

Move α to the left hand side, take absolute value on both sides and use the fact that $\mathbf{M}$ is a bounded linear operator.

$$\begin{aligned}|\beta-\alpha| &\leq \|\mathbf{M}\|\left\|-\mathbf{B}\int_{t_{i-1}}^{t_i} e^{\mathbf{B}(t_{i-1}-s)}x^{(i)}ds + \int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s-t_i)}kds\right\|_{\mathbb{R}^N}\\ &\quad + \|\mathbf{M}\|\left\|\int_{t_{i-1}}^{t_i} e^{\mathbf{B}(s-t_i)}\mathbf{A}\varphi^{(i)}(s-t_{i-1}-2T)ds\right\|_{\mathbb{R}^N}.\end{aligned} \tag{1.20}$$

Set

$$Q := \max_{t\in[0,T_1]}\{\|e^{-\mathbf{B}t}\|\}.$$

Then equation (1.20) and Lemma 9.9 (for the second term) imply that

$$\begin{aligned}&|\beta-\alpha|\\ &\leq (t_i-t_{i-1})\|\mathbf{M}\|Q\Bigg(\|\mathbf{B}\|\|x^{(i)}\|_{\mathbb{R}^N} + \|k\|_{\mathbb{R}^N}\Bigg) + (t_i-t_{i-1}))^{\frac{p-1}{p}}Q\|\mathbf{A}\|\|\mathbf{M}\|\|\varphi^{(i)}\|_{\mathbb{L}_p(t_{i-1},t_i)}.\end{aligned} \tag{1.21}$$

Note that $(t_i-t_{i-1}) < (t_i-t_{i-1})^{\frac{p-1}{p}}$ (since we assumed in Step I.I that the gap is smaller than $G \leq 1$), and that $\|\varphi^{(i)}\|_{\mathbb{L}_p(-2T,-2T+t_i-t_{i-1})} \leq \|\varphi^{(i)}\|_{\mathbb{L}_p(-2T,0)}$. Inequality (1.21) then becomes

$$|\beta-\alpha| \leq (t_i-t_{i-1})^{\frac{p-1}{p}}\|\mathbf{M}\|Q\Bigg(\|\mathbf{B}\|\|x^{(i)}\|_{\mathbb{R}^N} + \|k\|_{\mathbb{R}^N} + \|\mathbf{A}\|\|\varphi^{(i)}\|_{\mathbb{L}_p(-2T,0)}\Bigg).$$

Isolate $(t_i - t_{i-1})$.

$$t_i - t_{i-1} \geq \left(\frac{|\beta - \alpha|}{\|\mathbf{M}\| Q \Big(\|\mathbf{B}\| \|x^{(i)}\|_{\mathbb{R}^N} + \|k\|_{\mathbb{R}^N} + \|\mathbf{A}\| \|\varphi^{(i)}\|_{\mathbb{L}_p(-2T,0)} \Big)} \right)^{\frac{p}{p-1}}.$$

Step I.II. Recall that the last inequality in Step I.I was achieved under the assumption that $t_i - t_{i-1} < G$, and conclude that

$$t_i - t_{i-1} \geq \min \left\{ G, \left(\frac{|\beta - \alpha|}{\|\mathbf{M}\| Q \Big(\|\mathbf{B}\| \|x^{(i)}\|_{\mathbb{R}^N} + \|k\|_{\mathbb{R}^N} + \|\mathbf{A}\| \|\varphi^{(i)}\|_{\mathbb{L}_p(-2T,0)} \Big)} \right)^{\frac{p}{p-1}} \right\}. \tag{1.22}$$

A similar calculation can be done for even i. Hence the bound (1.22) is true for $i \in \mathbb{N} \geq 1$.

Step II. In order to bound away from zero in $[0, T_1]$ the right hand side of (1.22), we need to bound $x^{(i)}$ and $\varphi^{(i)}$ uniformly for each i for which $t_i < T_1$. We look for a bound on $\|u(t)\|_{\mathbb{R}^N}$ for $t \in [0, T_1]$. This gives, naturally, a bound on $x^{(i)}$, but also a bound on $\varphi^{(i)}$ since

$$\|\varphi^{(i)}\|_{\mathbb{L}_p(-2T,0)} \leq \|\varphi\|_{\mathbb{L}_p(-2T,0)} + T_1^{\frac{1}{p}} \|u(t)\|_{\mathbb{L}_\infty(0,T_1)}.$$

Write the general hysteresis-delay problem (1.1)–(1.3) in an integral form

$$u(t) = e^{-\mathbf{B}t} x + \int_0^t e^{\mathbf{B}(s-t)} \mathbf{A} u(s - 2T) ds + \int_0^t e^{\mathbf{B}(s-t)} k \mathcal{H}(\mathbf{M}u)(t) ds, \tag{1.23}$$

$$u(s) = \varphi(s), \quad s \in (-2T, 0). \tag{1.24}$$

The integral involving $\mathbf{A}u$ in equation (1.23) can be divided into two parts. One where $t \in [0, 2T]$, and u equals the initial data φ. The other[7] for $t > 2T$. Equation (1.23) takes the form

$$\begin{aligned} u(t) = e^{-\mathbf{B}t} x &+ \int_0^{2T} e^{\mathbf{B}(s-t)} \mathbf{A} \varphi(s - 2T) ds \\ &+ \int_{2T}^t e^{\mathbf{B}(s-t)} \mathbf{A} u(s - 2T) ds + \int_0^t e^{\mathbf{B}(s-t)} k \mathcal{H}(\mathbf{M}u)(t) ds. \end{aligned}$$

Take $\mathbb{R}^N$ norm on both sides. The following inequality takes place for $t \in [0, T_1]$, regardless of the sign of the hysteresis $\mathcal{H}$

$$\|u(t)\|_{\mathbb{R}^N} \leq Q \|x\|_{\mathbb{R}^N} + (2T)^{\frac{p-1}{p}} Q \|\mathbf{A}\| \|\varphi\|_{\mathbb{L}_p} + T_1 Q \|k\|_{\mathbb{R}^N} + \int_0^t Q \|\mathbf{A}\| \|u(s)\|_{\mathbb{R}^N} ds.$$

[7] We implicitly assumed here that $T_1 > 2T$. If it is not then the integral for $t > 2T$ does not exist and the evaluation continues in the same manner.

If we set

$$\alpha := Q\|x\| + (2T)^{\frac{p-1}{p}} Q\|\mathbf{A}\|\|\varphi\|_{\mathbb{L}_p} + T_1 Q\|k\|,$$
$$\beta := Q\|\mathbf{A}\|,$$

then the inequality can be written as

$$\|u(t)\|_{\mathbb{R}^N} \leq \alpha + \int_0^t \beta\|u(s)\|_{\mathbb{R}^N} ds.$$

Gronwall's Lemma ([26], section 1.3, equations (3.1)–(3.2)) implies that

$$\|u(t)\| \leq \alpha + \int_0^t \beta\alpha e^{(t-s)\beta} ds.$$

The right hand side is continuous for $t \in [0, T_1]$, and hence there is a bound for $u(t)$ for $t \in [0, T_1]$. □

Consider an alternative equivalent definition (see discussion after the definition) of a solution to the general problem (1.1)–(1.3).

Definition 1.11 (Solution)**.** Given $T_1 > 0$, a function $u \in \mathbb{L}_p(-2T, T_1) \cap \mathbb{W}_p^1(0, T_1)$, is a solution to problem (1.1)–(1.3) with initial data $(\varphi, x) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N$ if

1. u has finitely many switching times $t_1 < t_2 < \cdots < t_j$ in the interval $[0, T_1]$ (or possibly no switching times at all).
2. $u(t)$ equals the solution $u_+^{(1)}(t)$ of problem (1.8)–(1.10) with initial data $(\varphi, x) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N$, for $t \in [-2T, t_1]$ (or $t \in [-2T, T_1]$ if there are no switching times).
3. If there is at least one switching time, define $t_{j+1} := T_1$ (if $t_j < T_1$). Then for every $2 \leq i \leq j+1$ (or every $2 \leq i \leq j$ if $t_j = T_1$) the following hold, if $t \in [t_{i-1}, t_i]$,

   3.1. For even i: the solution satisfies

   $$u(t) = u_-^{(i)}(t - t_{i-1}),$$

   where $u_-^{(i)}$ is the solution to problem (1.11)–(1.13) with initial data

   $$(u(s + t_{i-1})_{s \in (-2T, 0)}, u(t_{i-1})) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N.$$

   3.2. For odd $i > 1$: the solution satisfies

   $$u(t) = u_+^{(i)}(t - t_{i-1}),$$

   where $u_+^{(i)}$ is the solution to problem (1.8)–(1.10) with initial data

   $$(u(s + t_{i-1})_{s \in (-2T, 0)}, u(t_{i-1})) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N.$$

Due to Lemma 1.10, if a solution exists on $[0, T_1]$, then it has finitely many switching times. Then items 2 and 3 of Definition 1.11 are just a more detailed description of Definition 1.5, and the two definitions are equivalent.

Theorem 1.12. *For every $(\varphi, x) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N$ there exists a unique solution to problem (1.1)–(1.3) in $[-2T, \infty)$.*

Proof. We prove only for the case where $\mathbf{M}x < \beta$. The proof for the other case is similar.

Choose $T_1 > 0$ and let $\bar{N}(\varphi, x, T_1)$ be given by Lemma 1.10.

We use Definition 1.11 of a solution: define $u^{(1)}(t)$ to be the solution to problem (1.8)–(1.10) with initial data (φ, x). By Lemma 1.9, $u^{(1)}(t)$ exists uniquely. Denote by $t_1 > 0$ the smallest time such that $u^{(1)}(t_1) = \beta$ or $t_1 = T_1$ if no such time exists. By Definition 1.11 this means that $u(t)$ exists uniquely on $[-2T, t_1]$ and equals $u^{(1)}(t)$ for a.e. $t \in [-2T, t_1]$.

If $t_1 < T_1$, define $u^{(2)}(t)$ to be the solution to problem (1.11)–(1.13) with initial data $(u(s + t_1)_{s\in(-2T,0)}, u(t_1)) \in \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N$. Denote by $t_2 > t_1$ the smallest time such that $u^{(2)}(t_2 - t_1) = \alpha$, or $t_2 = T_1$ if no such time exists. Setting $u(t) := u^{(2)}(t - t_1)$ and repeating the same arguments as in the previous paragraph, show that there exists a unique solution on $[-2T, t_2]$.

If $t_2 < T_1$, continue and define times $t_3, t_4 \ldots$ in the same manner. By Lemma 1.10, there exists t_i such that $i \le \bar{N}(\varphi, x, T_1)$ and $t_i > T_1$. This shows that the solutions exists uniquely on $[-2T, T_1]$.

Since $T_1 > 0$ was chosen arbitrarily, by definition the solutions exist uniquely on $[-2T, \infty)$. □

To finish the section we prove an auxiliary result on u_+ and u_- from equation (1.8) and equation (1.11), respectively. It is used in Section 3 (in the proofs of Lemma 3.12 and Lemma 3.18).

Lemma 1.13. *The solution $u_+(\varphi, x; t)$ ($u_-(\varphi, x; t)$) of problem (1.8)–(1.10) ((1.11)–(1.13)) is Lipschitz continuously dependent on its initial data uniformly with respect to t in bounded intervals, i.e, for every $\varepsilon > 0$ and $T_1 > 0$, there exists $\delta > 0$ and $L > 0$ such that if*

$$\|\nu, y\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N} \le \delta$$

then

$$\|u_\pm(\varphi + \nu, x + y; t) - u_\pm(\varphi, x; t)\|_{\mathbb{R}^N} \le L\|\nu, y\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N}$$

for every $t \in [0, T_1]$.

Proof. We prove only for u_+. The proof is given via the method of steps (see Footnote 4).

In the first step, assume that $t \in [0, 2T]$. The integral formula for u_+ (equation (1.14)) shows that $u_+(\varphi, x; t)$ is affine linear in the first two variables. Hence

$$\begin{aligned}
&\|u_+(\varphi + \nu, x + y; t) - u_+(\varphi, x; t)\|_{\mathbb{R}^N} \\
&\leq \left\|e^{-\mathbf{B}t} y\right\|_{\mathbb{R}^N} + \left\|\int_0^t e^{\mathbf{B}(s-t)} \mathbf{A} \nu(s - 2T) ds\right\|_{\mathbb{R}^N} \\
&\leq C\|\nu, y\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N},
\end{aligned}$$

for every $t \in [0, 2T]$, where $C > 0$ is some constant and the integral of ν is bounded by Lemma 9.9.

If $T_1 < 2T$, then choose $L := C$ and $\delta \leq \frac{\varepsilon}{L}$ and the proof is complete. If not, then continue developing in steps until there is a step that contains T_1. The task of choosing L and δ in the case where there is more than one step is left to the reader. $\square$

2. Stability. Statement of the problem

In this section we state the main problem of Chapter I: studying the stability of a periodic solution. For this purpose we first define what stability is and in which space stability is being checked.

The main problem of Chapter I can be phrased informally as follows.

Problem statement (informal): Let u_p be a periodic solution to problem (1.1)–(1.3). Determine the stability of u_p.

Before stating the problem formally (Section 2.4), we have to define to which space perturbations belong (Section 2.1) and what stability is (Section 2.3).

2.1 Definitions: spaces

When studying the stability of a solution, one asks "in which space is it stable?". This technical question is of extreme importance in Chapter I. The reason is that in order to study stability, we try to create (Section 4) a linear version of the problem (in some sense which is defined there). Due to the discontinuous nature of the hysteresis, linearization is a big challenge. So instead of the question "in which space is the solution stable?" we find ourselves asking "in which space is the problem linearizable?"

It will turn out that a Lebesgue space ($\mathbb{L}_p$) is not regular enough for this task, while a Sobolev space ($\mathbb{W}_p^1$) is a bit too regular. A space "in between" those two is needed: a fractional Sobolev space. For further discussion, see Section 4.3.

A function φ is defined to be in the **fractional Sobolev space**, $W_p^s(a,b)$, $p>1$, $0<s<1$, if the following norm is finite:

$$\|\varphi\|_{W_p^s(a,b)} = \|\varphi\|_{L_p(a,b)} + \left(\int_a^b \int_a^b \frac{|\varphi(t)-\varphi(s)|^p}{|t-s|^{1+sp}} ds dt\right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

We call the second term in relation (2.1), the **seminorm of** $W_p^s(a,b)$.

Denote N copies of $W_p^s(a,b)$ by $\mathbb{W}_p^s(a,b)$.

The following condition on p and s is essential for the properties of $\mathbb{W}_p^s$ spaces that we use. See Appendix 9.4. It is assumed, without further mention, throughout the rest of the chapter.

Condition 2.1. The constants p, s satisfy the following condition:

$$p>1, \quad 0<s<1, \quad ps<1. \tag{2.2}$$

We note that the concept of a trace at $t \in [a,b]$ is not defined[8] for $\mathbb{W}_p^s$ space under Condition 2.1. See [51, Chapter 4] for details. This means that the initial condition $u(0+) = x$ in equation (1.3) is still needed[9].

The fractional Sobolev space is used to define the space $\mathbb{B}_p^s(a,b)$. It is a space that contains functions that "begin" as L_p function and "turn into" a fractional Sobolev function at some time point.

Let $a, b \in \mathbb{R}$ be constants such that $-2T < a < b$ (T is the fixed constant from the general ordinary hysteresis-delay equation (1.1)). Define

$$\mathbb{B}_p^s(a,b) = \mathbb{L}_p(-2T,b) \cap \mathbb{W}_p^s(a,b) \tag{2.3}$$

to be the space of functions $\varphi \in \mathbb{L}_p(-2T,b)$ such that φ restricted to the interval $[a,b]$ is in the space $\mathbb{W}_p^s(a,b)$. The norm of $\mathbb{B}_p^s(a,b)$ is defined as

$$\|\varphi\|_{\mathbb{B}_p^s(a,b)} = \|\varphi\|_{\mathbb{L}_p(-2T,b)} + \|\varphi\|_{\mathbb{W}_p^s(a,b)}.$$

Finally, the following space $\mathbb{B}_p^s$ is used in the sequel to study the stability of a periodic solution. Fix a constant σ such that $0 < \sigma \leq \frac{T}{3}$. Define the space

$$\mathbb{B}_p^s := \mathbb{B}_p^s(-T-\sigma, 0) = \mathbb{L}_p(-2T,0) \cap \mathbb{W}_p^s(-T-\sigma,0).$$

The choice of σ is flexible, and stability can be shown for every $0 < \sigma \leq \frac{T}{3}$. The only difference would be the size of allowed perturbations, see Remark (4.24). As for the reason for the bound $\frac{T}{3}$, see the proof of Lemma 4.30 Step II(1).

From now on, the space $\mathbb{B}_p^s$ will be our main working space. Definition 1.11 of a solution remains the same, since $\mathbb{B}_p^s \subset \mathbb{L}_p(-2T,0)$.

We remind that the spaces defined in this document are summarized in Table 1. The reader may refer back to that table whenever needed.

2.2 Hysteresis-delay ordinary differential equations in fractional Sobolev spaces

As mentioned in the previous subsection, stability will be shown for perturbations in the space $\mathbb{B}_p^s \times \mathbb{R}^N$. For this we define the operators $\mathbf{\Psi}_\pm$ and $\boldsymbol{\psi}_\pm$. The properties of $\mathbf{\Psi}_\pm$ and $\boldsymbol{\psi}_\pm$ and the fact that they are well-defined operators are stated and proved in Lemmas 2.3 and 2.6.

Definition 2.2. Define the operator

$$\boldsymbol{\psi}_+ : \mathbb{B}_p^s \times \mathbb{R}^N \times (0,2T) \to \mathbb{B}_p^s,$$

[8]However, a trace is defined when $ps > 1$.

[9]Note that the initial condition $u(0+) = x$ from the right is well-defined, since $u \in \mathbb{W}_p^1(0,\infty)$ by Definition 1.5. The fact that no trace is defined for $\mathbb{W}_p^s$ means that $u(0-)$ is not defined.

as

$$
\boldsymbol{\psi}_{+}(\varphi,x,t)(\theta) :=
\begin{cases}
\varphi(\theta+t), & \theta\in[-2T,-t),\\
\underbrace{e^{-\mathbf{B}(\theta+t)}x+\int_0^{\theta+t} e^{\mathbf{B}(s-t-\theta)}\mathbf{A}\varphi(s-2T)ds+\int_0^{\theta+t} e^{\mathbf{B}(s-t-\theta)}k ds}_{=u_{+}(\varphi,x;\theta+t)}, & \theta\in[-t,0],
\end{cases}
\tag{2.4}
$$

where $0<t<2T$ (which is why we can use φ in the expression). Note that $\boldsymbol{\psi}_+$ was defined using a solution to problem (1.8)–(1.10) (see (1.14)–(1.15)).

We show in the proof of Lemma 2.3 that $\psi_+(\varphi,x,t)|_{(-t,0)}\in\mathbb{W}_p^1(-t,0)$, hence $\psi_+(\varphi,x,t)(0)$ is well-defined and we can introduce the operator

$$
\begin{aligned}
&\mathbf{\Psi}_+ : \mathbb{B}_p^s\times\mathbb{R}^N\times(0,2T)\to\mathbb{B}_p^s\times\mathbb{R}^N,\\
&\mathbf{\Psi}_+(\varphi,x,t) = (\boldsymbol{\psi}_+(\varphi,x,t),\boldsymbol{\psi}_+(\varphi,x,t)(0)).
\end{aligned}
\tag{2.5}
$$

The operator $\mathbf{\Psi}_+$ can be treated as the flow of problem (1.8)–(1.10) in the space $\mathbb{B}_p^s\times\mathbb{R}^N$ for $t\in(0,2T)$ (though we do not prove the flow properties for it, as they are not used in this dissertation).

Define the operators $\boldsymbol{\psi}_-$ and $\mathbf{\Psi}_-$ in a similar way.

Lemma 2.3. *The operators $\mathbf{\Psi}_\pm$ and $\boldsymbol{\psi}_\pm$ are well-defined operators continuous with respect to $t\in(0,2T)$.*

Proof. We prove the well-definiteness claim only for the operators $\boldsymbol{\psi}_+$ and $\mathbf{\Psi}_+$. For this we show that the operators indeed go from the domain-range specified in Definition 2.2.

The function u_+ belongs to the space $\mathbb{W}_p^1(0,t)$ for each $0<t<2T$ (Lemma 1.9), and hence $\boldsymbol{\psi}_+\big|_{(-t,0)}\in\mathbb{W}_p^1(-t,0)$ and specifically $\boldsymbol{\psi}_+\big|_{(-t,0)}\in\mathbb{W}_p^s(-t,0)$. By its definition, the function $\varphi(\cdot+t)\big|_{(-T-\sigma,-t)}$ belongs to the space $\mathbb{W}_p^s(-T-\sigma,-t)$ for $t<T+\sigma$. Hence Lemma 9.5 implies that $\boldsymbol{\psi}_+\in\mathbb{W}_p^s(-T-\sigma,0)$. It is straightforward that $\boldsymbol{\psi}_+\in\mathbb{L}_p$ and hence $\boldsymbol{\psi}_+\in\mathbb{B}_p^s$.

As stated before, for a fixed $t\in(0,2T)$, the operator $\boldsymbol{\psi}_+(\varphi,x,t)$ is in the space $\mathbb{W}_p^1(-t,0)$. Hence $\boldsymbol{\psi}_+(\varphi,x,t)(0)$ exists in the sense of traces, and the operator $\mathbf{\Psi}_+$ is well-defined.

We show the continuity only for the operator $\boldsymbol{\psi}_+$. Define an extension to $\boldsymbol{\psi}_+$ to $[-2T,2T]$ as

$$
\tilde{\boldsymbol{\psi}}_+(\varphi,x,t)(\theta) := \begin{cases}\varphi(\theta+t) & \theta\in[-2T,-t),\\ u_+(\varphi,x;\theta+t) & \theta\in[-t,2T].\end{cases}
$$

For every $0<t<2T$, the operator $\tilde{\boldsymbol{\psi}}_+(\varphi,x,t)$ belongs to the space $\mathbb{W}_p^s(-T-\sigma-t,2T-t)$ if $t<T-\sigma$ or to the space $\mathbb{W}_p^s(-2T,2T-t)$ otherwise, using

the same argument as for $\boldsymbol{\psi}_+$. Hence, by Lemma 9.4 (with $Q = [-T - \sigma, 0]$ and $Q' = [\max\{-T - \sigma - t, -2T\}, 2T - t]$) for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $|\delta| \leq \delta_1$ then

$$\|\tilde{\boldsymbol{\psi}}_+(\varphi, x, t + \delta) - \tilde{\boldsymbol{\psi}}_+(\varphi, x, t)\|_{\mathbb{W}_p^s(-T-\sigma,0)} \leq \varepsilon.$$

Note that $\boldsymbol{\psi}_+(\varphi, x, t)\big|_{(-2T,0)} = \tilde{\boldsymbol{\psi}}_+(\varphi, x, t)\big|_{(-2T,0)}$, and hence the $\mathbb{W}_p^s(-T - \sigma, 0)$ norm of $\boldsymbol{\psi}_+$ is continuous with respect to t. The continuity of $\boldsymbol{\psi}_+$ with respect to $t \in (0, 2T)$ in the $\mathbb{B}_p^s$ norm follows since the continuity of the $\mathbb{L}_p$ norm (which composes the $\mathbb{B}_p^s$ norm) is straightforward. □

Remark 2.4. Lemma 2.3 and Lemma 9.5 imply that $u \in W_p^s(0, T_1)$ for every $T_1 \geq 0$.

Remark 2.5. The continuity in the Lemma 2.3 was given for $t \in (0, 2T)$. We remark that

$$\|\boldsymbol{\Psi}(\varphi, x, t) - (\varphi, u_+(\varphi, x; 0))\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \to 0 \text{ as } t \to 0.$$

Lemma 2.6. *The operators $\boldsymbol{\Psi}_\pm$ are continuously dependent on the "initial data" in the following sense. For every $\varepsilon > 0$ and $0 < T_1 < T + \sigma$, there exists $\delta = \delta(\varepsilon, T_1) > 0$ such that if*

$$\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta,$$

then

$$\begin{aligned}
&\|\boldsymbol{\Psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\Psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \varepsilon, \\
&\|\boldsymbol{\Psi}_-(\varphi^\beta + \nu, x^\beta + y, t) - \boldsymbol{\Psi}_+(\varphi^\beta, x^\beta, t)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \varepsilon,
\end{aligned}$$

for every $t \in (0, T_1]$.

Proof. We prove the claim only for $\boldsymbol{\Psi}_+ = (\boldsymbol{\psi}_+, \boldsymbol{\psi}_+(0))$. Since $\boldsymbol{\psi}_+(\varphi, x, t)(0) = u_+(\varphi, x; t)$, then the result for $\boldsymbol{\psi}_+(\varphi, x, t)(0)$ was proved in Lemma 1.13. We prove it now for $\boldsymbol{\psi}_+$. The $\mathbb{B}_p^s$ norm is composed as a sum of the $\mathbb{L}_p$ norm and $\mathbb{W}_p^s(-T - \sigma, 0)$ norm. We bound only the latter and leave the $\mathbb{L}_p$ bound to the reader.

Fix $\varepsilon > 0$ and $T_1 < T + \sigma$. Assume without loss of generality that $\varepsilon < T$.

Step I. Let $t \in [\varepsilon, T_1]$. By expression (2.4) for $\boldsymbol{\psi}_+$:

$$\begin{aligned}
&\left\|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\right\|_{\mathbb{W}_p^s(-T-\sigma,0)} \\
&= \left\| \begin{cases} \nu(\theta + t), & \theta \in [-2T, -t), \\ u_+(\varphi^\alpha + \nu, x^\alpha + y; \theta + t) - u_+(\varphi^\alpha, x^\alpha; \theta + t)), & \theta \in [-t, 0]. \end{cases} \right\|_{\mathbb{W}_p^s(-T-\sigma,0)}
\end{aligned}$$

Since the lengths of the intervals $(-T - \sigma, -t)$ and $(-t, 0)$ are bounded from below by $\min\{T + \sigma - T_1, \varepsilon\}$ (for fixed ε and T_1), Lemma 9.6 implies that there is

$C_1 := C_1(\varepsilon, T_1) > 0$ and $\tilde{C}_1 = \tilde{C}_1(\varepsilon, T_1) > 0$ such that

$$
\begin{aligned}
&\|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-T-\sigma,0)} \\
&\le C_1\Big(\|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-T-\sigma,-t)} \\
&\quad + \|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-t,0)}\Big), \\
&= C_1\left(\|\nu(\cdot + t)\|_{\mathbb{W}_p^s(-T-\sigma,-t)} + \left\| e^{-\mathbf{B}(\cdot+t)}y + \int_0^{\cdot+t} e^{\mathbf{B}(s-\cdot-t)}\mathbf{A}\nu(s-2T)ds \right\|_{\mathbb{W}_p^s(-t,0)} \right) \\
&\le C_1 \cdot \tilde{C}_1 \|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N},
\end{aligned}
$$

where we used formula (1.14) for u_+ and $\tilde{C}_1$ is some positive constant. Choose $\delta_1 \le \frac{\varepsilon}{C_1 \cdot \tilde{C}_1}$ to obtain the desired result for $t \in [\varepsilon, T_1]$.

Step II. Consider $0 < t < \varepsilon$, and extend the norm to the interval $[-T-\sigma, \varepsilon]$.

$$
\begin{aligned}
&\|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-T-\sigma,0)} \\
&\le \|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-T-\sigma,\varepsilon)} \\
&= \left\| \begin{cases} \nu(\theta + t) & \theta \in [-2T, -t) \\ u_+(\varphi^\alpha + \nu, x^\alpha + y; \theta + t) - u_+(\varphi^\alpha, x^\alpha; \theta + t)) & \theta \in [-t, \varepsilon]. \end{cases} \right\|_{\mathbb{W}_p^s(-T-\sigma,\varepsilon)}
\end{aligned}
$$

Since the lengths of the intervals $(-T-\sigma, -t)$ and $(-t, \varepsilon)$ are bounded from below by $\min\{T + \sigma - T_1, \varepsilon\}$ (for fixed ε and T_1), then Lemma 9.6 implies that there is $C_2 := C_2(\varepsilon, T_1)$ and $\tilde{C}_2 > 0$ such that

$$
\begin{aligned}
&\|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-T-\sigma,\varepsilon)} \\
&\le C_2\Big(\|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-T-\sigma,-t)} \\
&\quad + \|\boldsymbol{\psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, t)\|_{\mathbb{W}_p^s(-t,\varepsilon)}\Big) \\
&= C_2\left(\|\nu(\cdot + t)\|_{\mathbb{W}_p^s(-T-\sigma,-t)} + \|e^{-\mathbf{B}(\cdot+t)y} + \int_0^{\cdot+t} e^{\mathbf{B}(s-\cdot-t)}\mathbf{A}\nu(s-2T)ds \|_{\mathbb{W}_p^s(-t,\varepsilon)} \right) \\
&\le C_2 \cdot \tilde{C}_2 \|\nu, y\|_{\mathbb{W}_p^s(-T-\sigma,0) \times \mathbb{R}^N},
\end{aligned}
$$

where we used again formula (1.14) for u_+ and $\tilde{C}_2$ is some positive constant. Choose $\delta_2 \le \frac{\varepsilon}{C_2 \cdot \tilde{C}_2}$ to get the desired result for $t \in [0, \varepsilon]$.

Taking $\delta = \min\{\delta_1, \delta_2\}$ proves the lemma. $\square$

2.3 Periodic solutions and stability

Definition 2.7 (Periodic solution)**.** A solution $u(t)$ of problem (1.1)–(1.3) on $[-2T, \infty)$ is called a τ-periodic solution to problem (1.1)–(1.3) if $\tau > 0$ and

$$\begin{aligned}
&u(\tau) = x, \\
&u(\tau + s) = \varphi(s), \quad s \in (-2T, 0), \\
&\mathcal{H}(\mathbf{M}u)(\tau) = \mathcal{H}(\mathbf{M}u)(0).
\end{aligned}$$

Definition 2.7 uses implicitly the uniqueness result from Theorem 1.12: if a solution returns to the exact same initial point and history (and the same value of hysteresis) after time τ, then it will do so after every $n\tau$ for $n \in \mathbb{N}$.

Definition 2.8 (Orbit)**.** Let u be a solution to problem (1.8)–(1.10) on $[-2T, \infty)$ with initial data (φ, x). Denote the **orbit** of $u(\varphi, x; t)$ as $\gamma(\varphi, x) \subset \mathbb{B}^s_p \times \mathbb{R}^N$ and define it as

$$\gamma(\varphi, x) = \{\big(u(\varphi, x; t + s)|_{s\in(-2T,0)}, u(\varphi, x; t)\big)|t \geq 0\}.$$

The orbits Γ_1, $\Gamma_2 \subset \mathbb{B}^s_p \times \mathbb{R}^N$ are defined, using u_p from Assumption 2.12 below, as

$$\begin{aligned}
\Gamma_1 &:= \{\big(u_p(s + t)|_{s\in(-2T,0)}, u_p(t)\big)|t \in [0, T]\}, \\
\Gamma_2 &:= \{\big(u_p(s + t)|_{s\in(-2T,0)}, u_p(t)\big)|t \in [T, 2T]\}.
\end{aligned}$$

The **orbit of the periodic solution** u_p then equals

$$\Gamma = \Gamma_1 \cup \Gamma_2.$$

Definition 2.9 (Stability)**.** The periodic solution u_p is called **stable** (or **orbitally stable**) if for every neighbourhood Ω of Γ, there exists neighbourhoods Ω_1 of Γ_1 and Ω_2 of Γ_2 such that if

$$(\varphi, x) \in \Omega_1, \mathbf{M}x < \beta \text{ or } (\varphi, x) \in \Omega_2, \mathbf{M}x \geq \beta,$$

then $\gamma(u) \subset \Omega$.

The periodic solution u_p is called **asymptotically stable** (or **orbitally asymptotically stable**), if in addition to the previous requirements, there exists neighbourhoods Θ_1 of Γ_1 and Θ_2 of Γ_2 such that if

$$(\varphi, x) \in \Theta_1, \mathbf{M}x < \beta \text{ or } (\varphi, x) \in \Theta_2, \mathbf{M}x \geq \beta,$$

then

$$dist((u(\varphi, x; t + s)_{s\in(-2T,0)}, u(\varphi, x; t)), \Gamma) \to 0 \text{ as } t \to \infty,$$

where distance is taken in the $\mathbb{B}^s_p \times \mathbb{R}^N$ space.

The periodic solution u_p is called **unstable** if it is not stable.

Remark 2.10. In this dissertation we prove asymptotic stability of periodic solutions. However, it is possible to slightly modify the proofs to show **asymptotic exponential stability**, i.e., that in addition to asymptotic stability, there exist $0 < q < 1$, $k > 0$ and neighbourhoods Υ_1 of Γ_1 and Υ_2 of Γ_2 such that if

$$(\varphi, x) \in \Upsilon_1, \mathbf{M}x < \beta \text{ or } (\varphi, x) \in \Upsilon_2, \mathbf{M}x \geq \beta,$$

then

$$dist((u(\varphi, x; t+s)_{s\in(-2T,0)}, u(\varphi, x; t)), \Gamma) \leq kq^t$$

for all $t \geq 0$.

Remark 2.11. For delay equations, the space in which the solution lies is often different from the phase space. A solution $u(t)$ of (1.1)–(1.3) belongs to $\mathbb{R}^N$ for every $t \geq 0$, but the phase space is $\mathbb{B}_p^s \times \mathbb{R}^N$.

It is possible to define a flow on this phase space. However, we will not do so as it is not necessary for the stability definition that we use. Note that in order to define a flow properly, the hysteresis, $\mathcal{H}$, needs to have an additional parameter. This parameter will hold the value of the hysteresis in the case that the initial value of the function is between (α, β) (as mentioned in Remark 1.4).

2.4 Problem statement

The following assumption is valid for the rest of Chapter I.

The assumption is inspired from a periodic solution to the heat equation with hysteresis on the boundary [22, 25]. We consider this periodic solution and problem in Section 8 in Chapter II of the dissertation.

Assumption 2.12. Assume that problem (1.1)–(1.3) has a $2T$-periodic solution $u_p := u_p(\varphi^\alpha, x^\alpha; t)$ such that

1. The initial data x^α satisfies[10] $\mathbf{M}x^\alpha = \alpha$.

2. The periodic solution u_p has exactly two switching times along its period: one at $t = T$ (where $\mathbf{M}u_p(T) = \beta$) and one at $t = 2T$ (where $\mathbf{M}u_p(2T) = \alpha$).

3. The derivative of φ^α is anti-symmetric with respect to the point $t = -T$, in the sense that

$$\varphi^{\alpha\prime}(\theta) = -\varphi^{\alpha\prime}(\theta + T) \quad \theta \in [-2T, -T].$$

   These derivatives exist in light of Lemma 2.13 below.

4. The switching is transverse in the sense that

$$\frac{d\mathbf{M}u_p(T-)}{dt}, \frac{d\mathbf{M}u_p(2T-)}{dt} \neq 0. \tag{2.6}$$

[10]We could also study stability if $\mathbf{M}x^\alpha = \beta$. But we choose $\mathbf{M}x^\alpha = \alpha$, due to our choice of initial state of the hysteresis to be 1 for $\alpha < x < \beta$, see Remarks 1.4 and 3.8 for more details.

The next lemma shows that items (1),(2) in Assumption 2.12 imply that u_p is piecewise smooth.

Lemma 2.13. *If items (1),(2) in Assumption 2.12 take place, then φ^α is in the space $C^\infty[-2T,-T] \cap C^\infty[-T,0]$.*

Proof. Since u_p satisfies problem (1.1)–(1.3), then its expression for $t \in [0,T]$ is:

$$u_p(t) = e^{-\mathbf{B}t} x^\alpha + \int_0^t e^{\mathbf{B}(s-t)} \mathbf{A} \varphi^\alpha(s-2T) ds + \int_0^t e^{\mathbf{B}(s-t)} k ds.$$

Since $\varphi^\alpha \in \mathbb{L}_p(-2T,-T)$, then $u_p \in \mathbb{W}_p^1(-T,0)$. Then periodicity shows that $\varphi^\alpha \in \mathbb{W}_p^1(-2T,-T)$, which in turn implies that $u_p \in \mathbb{W}_p^2(-T,0)$ and hence $\varphi^\alpha \in \mathbb{W}_p^2(-2T,-T)$. Continuing with this argument shows that $\varphi^\alpha \in \mathbb{W}_p^k(-2T,-T)$ for every $k \in \mathbb{N}$, and hence $\varphi^\alpha \in C^\infty[-2T,-T]$.

Consider the expression for u_p for $t \in [T,2T]$:

$$u_p(t) = e^{-\mathbf{B}t} x^\alpha + \int_0^T e^{\mathbf{B}(s-t)} \mathbf{A} \varphi^\alpha(s-2T) ds + \int_T^t e^{\mathbf{B}(s-t)} \mathbf{A} \varphi^\alpha(s-2T) ds + \int_0^t e^{\mathbf{B}(s-t)} k ds.$$

Repeating the same argument as before shows that $\varphi^\alpha \in C^\infty[-T,0]$. $\square$

We study the following problem.

Problem 2.14. Assume u_p is a $2T$-periodic solution to (1.1)–(1.3) that satisfies Assumption 2.12. Determine the stability of u_p.

3. Poincaré and hit maps

In this section we define the main tool for studying stability: the Poincaré map. Due to the discontinuous nature of the problem, the Poincaré map is a composition of two maps. We call those maps "hit maps", and define them in this section as well.

In Section 3.1, the Poincaré and hit maps are defined.

In Section 3.2, the continuity and differentiability of the hit operator is proved.

In Section 3.3, basic properties of the Poincaré and hit maps are proven. This section ends with a result that motivates the study of the Poincaré map. Namely that the stability of a periodic solution depends on that of its corresponding Poincaré map. In addition, we show in Lemma 3.19 that solutions that begin "close" to the initial data of the periodic solution are "almost" periodic themselves.

3.1 Definition of the Poincaré and hit maps

Discussion. The general idea behind a Poincaré map is a shift of study: from a differential equation to an associated operator (called a Poincaré map).

This idea was first suggested by Poincaré in 1899 [41] in order to study the three body problem in celestial mechanics. Nowadays a Poincaré map is a common tool to study the stability of periodic solutions.

Take some initial data for a periodic solution. Let us call it $(\varphi^\alpha, x^\alpha)$ as in our case (see Assumption 2.12). Choose a hyperspace (called a cross-section) that contains the initial data. In our case it is $\mathbb{T}_\alpha$ (see Definition 3.1). The cross-section must have the following property: there is an open neighbourhood of $(\varphi^\alpha, x^\alpha)$ in the cross-section such that solutions that begin in this open neighbourhood, reach the cross-section in some finite time. The Poincaré map takes a solution from this neighbourhood to the point at which it reaches the cross-section.

Due to periodicity, $(\varphi^\alpha, x^\alpha)$ is a fixed point of the Poincaré map. It turns out (Lemma 3.18) that the stability of the periodic solution depends on the stability of the fixed point $(\varphi^\alpha, x^\alpha)$ of the Poincaré map. See, e.g., Wiggins [54] for a detailed explanation of a Poincaré map.

For reasons explained below, we need two cross-sections (hyperspaces) for our usage of the Poincaré map. We define them now.

Definition 3.1 (cross-sections)**.** Consider the following cross-sections (subspaces of co-dimension one[11]) of $\mathbb{B}_p^s \times \mathbb{R}^N$:

$$\mathbb{T}_\alpha = \{(\varphi, x) \in \mathbb{B}_p^s \times \mathbb{R}^N | \mathbf{M}x = \alpha\}, \text{ and}$$
$$\mathbb{T}_\beta = \{(\varphi, x) \in \mathbb{B}_p^s \times \mathbb{R}^N | \mathbf{M}x = \beta\}.$$

The point x is needed since, as was already mentioned in Section 2.1, a trace is not defined for functions in the space $\mathbb{B}_p^s$ when $0 < ps < 1$.

We build a Poincaré map as a map from $\mathbb{T}_\alpha$ to itself.

Due to the discontinuity of the system (caused by the hysteresis operator), the Poincaré map is a composition of two maps: one from $\mathbb{T}_\alpha$ to $\mathbb{T}_\beta$ (called $\mathbf{P}_\beta$, since it goes to the hyperspace $\mathbb{T}_\beta$), and the second the other way around, from $\mathbb{T}_\beta$ to $\mathbb{T}_\alpha$ (called $\mathbf{P}_\alpha$).

We call each of those maps a **hit map** since they take a solution until it "hits" one of the cross sections. Note that the "hit" time of a solution is also its switching time.

Remark 3.2. It may be confusing that all the definitions below focus on operators with a subscript β. The reason is that the Poincaré map first takes the periodic solution $(\varphi^\alpha, x^\alpha)$ to $\mathbb{T}_\beta$, and this is done via the hit map $\mathbf{P}_\beta$. It may be counter intuitive that the map $\mathbf{P}_\beta$ is applied before the map $\mathbf{P}_\alpha$, but we use that notation nevertheless since it is easier to remember that the map $\mathbf{P}_\beta$ hits the hyperspace $\mathbb{T}_\beta$.

Before defining the hit maps, we need to define the time moment at which a solution "hits" the subspaces $\mathbb{T}_\beta$ or $\mathbb{T}_\alpha$. We remind the reader that u_+ is a solution to the general problem (1.1)–(1.3) when $\mathcal{H} = 1$ (it was defined in problem (1.8)–(1.10)).

Definition 3.3 (Hit time operator)**.** Let $(\varphi, x) \in \mathbb{L}_p \times \mathbb{R}^N$, $\mathbf{M}x < \beta$. We call an operator

$$\mathbf{t}_\beta : \mathbb{L}_p \times \mathbb{R}^N \to \mathbb{R},$$

a **hit time** operator (or simply a **hit** operator) if

$$\mathbf{M}u_+(\varphi, x; \mathbf{t}_\beta(\varphi, x)) = \beta,$$

and

$$\mathbf{M}u_+(\varphi, x; t) \neq \beta \text{ for } t \in [0, \mathbf{t}_\beta(\varphi, x)).$$

If no such time exists, then

$$\mathbf{t}_\beta(\varphi, x) = \infty.$$

Define $\mathbf{t}_\alpha(\varphi, x)$ in a similar way with $\mathbf{M}x > \alpha$.

[11]These are subspaces of co-dimension one since $\mathbf{M}$ is a non-trivial linear functional (see the definition after equation (1.3)).

Remark 3.4. It would be consistent with the definition of stability (Definition 2.9) to define $\mathbf{t}_\beta$ on the space $\mathbb{B}_p^s$ (as the first component in the product space). We choose the space $\mathbb{L}_p$ for two reasons. The first is that the properties of the hit operator (Lemma 3.12 and Lemma 3.15) can be proved on the space $\mathbb{L}_p$ (unlike the stability result). The second is that in later sections we need results for $\mathbf{t}_\beta$ on the space $\mathbb{L}_p$.

It is important to remark that the results for the hit operator are also true if it is treated as an operator from $\mathbb{B}_p^s \times \mathbb{R}^N \to \mathbb{R}$, since $\mathbb{B}_p^s$ is continuously embedded in $\mathbb{L}_p$ [13].

By the definition of u_p (Assumption 2.12), it is straightforward that $\mathbf{t}_\beta(\varphi^\alpha, x^\alpha) = T$. In Lemma 3.12 we show that the hit time $\mathbf{t}_\beta$ is finite for functions in some neighbourhood $(\varphi^\alpha, x^\alpha)$.

Definition 3.5 (Hit map)**.** Consider the following nonlinear map

$$\mathbf{P}_\beta : \mathcal{D}om(\mathbf{P}_\beta) \to \mathbb{T}_\beta, \quad \mathcal{D}om(\mathbf{P}_\beta) := \{(\varphi, x) \in \mathbb{T}_\alpha \big| \mathbf{t}_\beta(\varphi, x) < \infty\},$$

defined as

$$\mathbf{P}_\beta(\varphi, x) = (\mathbf{P}_\beta^{\mathbb{B}}(\varphi, x), \underbrace{\mathbf{P}_\beta^{\mathbb{R}}(\varphi, x)}_{=\mathbf{P}_\beta^{\mathbb{B}}(\varphi,x)(0)}) = \big(u_+(\varphi, x; \mathbf{t}_\beta(\varphi, x) + s)|_{s\in(-2T,0)}, u_+(\varphi, x; \mathbf{t}_\beta(\varphi, x))\big). \tag{3.1}$$

Define $\mathbf{P}_\alpha$ in a similar way.

The maps $\mathbf{P}_\beta$ and $\mathbf{P}_\alpha$ are called **hit maps**. We say that $\mathbf{P}_\beta$ ($\mathbf{P}_\alpha$) **hits** $\mathbb{T}_\beta$ ($\mathbb{T}_\alpha$) at time $\mathbf{t}_\beta$ ($\mathbf{t}_\alpha$).

Remark 3.6. In what follows we limit the initial data to be such that the hit time $\mathbf{t}_\beta(\varphi, x)$ is less than one delay step ($2T$). In this case $\mathbf{P}_\beta$ has the explicit expression

$$\mathbf{P}_\beta(\varphi, x) := \mathbf{\Psi}_+(\varphi, x, \mathbf{t}_\beta(\varphi, x)), \tag{3.2}$$

which uses $\mathbf{\Psi}_+$ (relation (2.5)) from Definition 2.2. In the same way we have an explicit expression for $\mathbf{P}_\alpha$ which uses $\mathbf{\Psi}_-$ if $\mathbf{t}_\alpha(\varphi, x) < 2T$. We use these expressions in the rest of Chapter I.

Now we can finally define the Poincaré map $\mathbf{P}$.

Definition 3.7 (Poincaré map)**.** The nonlinear map

$$\begin{aligned}&\mathbf{P} : \mathcal{D}om(\mathbf{P}) \to \mathbb{T}_\alpha,\\ &\mathcal{D}om(\mathbf{P}) := \{(\varphi, x) \in \mathbb{T}_\alpha \ \big|(\varphi, x) \in \mathcal{D}om(\mathbf{P}_\beta), \mathbf{P}_\beta(\varphi, x) \in \mathcal{D}om(\mathbf{P}_\alpha)\},\end{aligned}$$

defined as

$$\mathbf{P} = \mathbf{P}_\alpha \circ \mathbf{P}_\beta$$

is called the **Poincaré map**.

Remark 3.8. Now we can see that the assumption on initial conditions of the hysteresis (Remark 1.4) limits the study of stability of a period solution with initial condition $x^\alpha = \alpha$.

The domain of the map $\mathbf{P}_\beta$ is defined via the hit operator $\mathbf{t}_\beta$. The hit operator $\mathbf{t}_\beta$, in turn, is defined using the solution u_+ of the case $\mathcal{H} = 1$. The result is that the map $\mathbf{P}_\beta$ implicitly assumes that $\mathcal{H} = 1$.

Now, the fact that the Poincaré map applies $\mathbf{P}_\beta$ first means that we implicitly assume that if $\alpha < \mathbf{M}u(0) < \beta$ then $\mathcal{H}(\mathbf{M}u)(0) = 1$. We could, in theory, define the Poincaré map as $\mathbf{P}_\beta \circ \mathbf{P}_\alpha$ (to study stability of u_p^β). This would implicitly assume the opposite (if $\alpha < \mathbf{M}u(0) < \beta$, then $\mathcal{H}(\mathbf{M}u)(0) = -1$), and would require us to re-define the hysteresis in Subsection 1.2 accordingly.

Finally, we define what we mean by the stability of a fixed point of a Poincaré map.

Definition 3.9. A fixed point $(\varphi, x) \in \mathbb{T}_\alpha$ of $\mathbf{P}$ is called **stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} < \delta, \ (\varphi + \nu, x + y) \in \mathbb{T}_\alpha,$$

then

$$\mathbf{P}^n(\varphi + \nu, x + y) \in \mathcal{D}om(\mathbf{P}),$$

and

$$\|\mathbf{P}^n(\varphi + \nu, x + y) - (\varphi, x)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \le \varepsilon,$$

for all $n \in \mathbb{N} \cup \{0\}$.

A fixed point $(\varphi, x) \in \mathbb{T}_\alpha$ is called **asymptotically stable**, if in addition to the previous requirements:

$$\|\mathbf{P}^n(\varphi + \nu, x + y) - (\varphi, x)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \to 0 \text{ as } n \to \infty.$$

A fixed point of $\mathbf{P}$ is called **unstable** if it is not stable.

The next subsection studies properties of the hit operator and the Poincaré and hit maps at $(\varphi^\alpha, x^\alpha)$ (the initial data of the periodic solution u_p). For that we now introduce some notation related to the periodic solution.

Notation 3.10. Let u_p be the periodic solution from Assumption 2.12.

1. The initial data for the periodic solution u_p is $(\varphi^\alpha, x^\alpha)$. Due to periodicity

$$\varphi^\alpha(t) = u_p(t + 2T), \quad t \in (-2T, 0).$$

2. Assumption 2.12 implies that $(\varphi^\alpha, x^\alpha) \in \mathcal{D}om(\mathbf{P}_\alpha)$. Denote

$$(\varphi^\beta, x^\beta) := \mathbf{P}_\beta(\varphi^\alpha, x^\alpha).$$

Then the switching times in Assumption 2.12 show that

$$\mathbf{t}_\beta(\varphi^\alpha, x^\alpha), \mathbf{t}_\alpha(\varphi^\beta, x^\beta) = T.$$

Periodicity shows that

$$(\varphi^\alpha, x^\alpha) = \mathbf{P}_\alpha(\varphi^\beta, x^\beta).$$

In addition $(\varphi^\alpha, x^\alpha) \in \mathcal{D}om(\mathbf{P})$, and $(\varphi^\alpha, x^\alpha)$ is a fixed point of $\mathbf{P}$, i.e.,

$$\mathbf{P}(\varphi^\alpha, x^\alpha) = (\varphi^\alpha, x^\alpha).$$

Remark 3.11. From this point on, most results in Chapter I can be phrased in two analogous ways. One related to the hit map $\mathbf{P}_\beta$ (or components of it, like $\mathbf{t}_\beta, \mathbf{\Psi}_+$ from this section, and $\mathbf{L}_\beta, \mathbf{h}_\beta$ from the next section). The other is related to $\mathbf{P}_\alpha$.

We only state results related to the hit map $\mathbf{P}_\beta$, but use each result as it was proved also for $\mathbf{P}_\alpha$. The "translation" of a result from one case to the other is a matter of changing letters. Each β is changed to α (and vice versa). Each $+$ is changed to $-$ (and vice versa). However, in some important results we state both versions.

3.2 Properties of the hit operator

To study problem (2.14) we study the stability of the fixed point $(\varphi^\alpha, x^\alpha)$. To do this we want $\mathbf{P}$ to be differentiable at $(\varphi^\alpha, x^\alpha)$. $\mathbf{P}$ is composed of the map $\mathbf{P}_\beta$, which in turn uses the hit time operator $\mathbf{t}_\beta$. The next lemma shows that $\mathbf{t}_\beta$ is locally Lipschitz continuous at $(\varphi^\alpha, x^\alpha)$, while Lemma 3.15 shows that it is also Fréchet differentiable. See Remark 3.4 as to why we use the space $\mathbb{L}_p$.

Lemma 3.12. *There exists some $\delta > 0$ such that if*

$$\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \delta,$$

then

$$\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) < \infty.$$

In addition, $\mathbf{t}_\beta : \mathbb{L}_p \times \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous at $(\varphi^\alpha, x^\alpha)$ in the following sense:

$$|\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)| \leq C\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N},$$

where $C = C(\delta) > 0$ is independent of (ν, y).

Proof. **Existence and continuity.** Existence is shown in the process of showing continuity. For continuity, choose $\varepsilon > 0$.

Step I. According to equation (2.6) in Assumption 2.12, $\mathbf{M}u_p$ intersects the hyperspace $\mathbf{M}x = \alpha$ transversally. Thus, there are some

$$
\begin{aligned}
&0 < \varepsilon_1 \leq \varepsilon, \\
&0 < \rho_1,
\end{aligned}
$$

such that

$$\mathbf{M}u_+(\varphi^\alpha, x^\alpha; T + \varepsilon_1) = \beta + \rho_1.$$

The average operator $\mathbf{M}$ is a continuous operator, u_+ is uniformly continuously dependent on initial data at $(\varphi^\alpha, x^\alpha)$ by Lemma 1.13. Hence there exists some $\delta_1 = \delta_1(\varepsilon) > 0$ not depending on ε_1 such that if

$$\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \delta_1,$$

then

$$|\mathbf{M}u_+(\varphi^\alpha + \nu, x^\alpha + y; T + \varepsilon_1) - \mathbf{M}u_+(\varphi^\alpha, x^\alpha; T + \varepsilon_1)| \leq \rho_1.$$

This shows that $\mathbf{t}_\beta(\varphi + \nu, x + y) < \infty$ and gives an upper bound

$$\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) \leq T + \varepsilon_1 \leq T + \varepsilon.$$

Step II. Choose some $0 < \varepsilon_2 \leq \min\{\varepsilon, T\}$. Define ρ_2 as

$$\rho_2 := \min_{t \in [0, T - \varepsilon_2]} \beta - \mathbf{M}u(\varphi^\alpha, x^\alpha, t).$$

Note that $\rho_2 > 0$ since T is the first switching time of $\mathbf{M}u_p$ by Assumption 2.12 (i.e, T is the first time that $\mathbf{M}u_p = \beta$). Repeating the process from Step I shows the existence of some $\delta_2 > 0$ such that if

$$\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \delta_2,$$

then

$$|\mathbf{M}u_+(\varphi^\alpha + \nu, x^\alpha + y; t) - \mathbf{M}u_+(\varphi^\alpha, x^\alpha; t)| \leq \rho_2$$

for every $t \in [0, T - \varepsilon_2]$. This shows that

$$\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) \geq T - \varepsilon_2 \geq T - \varepsilon.$$

Step III. Take $\delta := min\{\delta_1, \delta_2\}$. Then if

$$\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \delta,$$

then

$$|\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)| \leq \varepsilon.$$

locally Lipschitz. Take $0 < \tilde{\delta}$ that satisfies the continuity, and $\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \tilde{\delta}$ (we change to $\tilde{\delta}$ since now we need to find $0 < \delta \leq \tilde{\delta}$ such that locally Lipschitz continuity holds).

The solution $u_+(\varphi^\alpha + \nu, x^\alpha + y; t)$ can cross either before (Figure 3 left) or after (Figure 3 right) the periodic solution. If it crosses at the same time, then any Lipschitz constant will do. Unlike in the figure, the two graphs can intersect one another on the way, but this is irrelevant for our argument.

The periodic solution is transverse at the crossing time by Assumption 2.12, and since $\mathbf{t}_\beta$ is continuous at $(\varphi^\alpha, x^\alpha)$, we can assume that $\tilde{\delta}$ is small enough such that the dynamics is as in one of the two images in Figure 3, i.e, $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)$ is in a neighbourhood $\mathcal{N}$ of $\mathbf{t}_\beta(\varphi^\alpha, x^\alpha)$, in which $\mathbf{M}u_+(\varphi^\alpha, x^\alpha; t)$ is monotone.

Solutions of problem (1.1)–(1.3) are continuous. Hence, a vertical line, C_1 (see figure) taken from the point $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)$ intersects the graph of $\mathbf{M}u_+(\varphi_p^\alpha, x_p^\alpha; t)$.

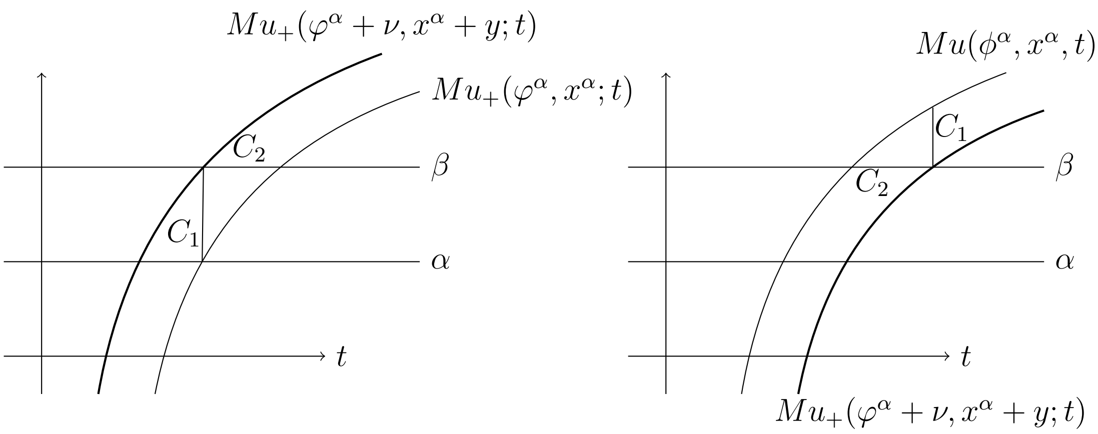

Figure 3: Crossing of functions before/after the periodic solution.

By the notation in Figure 3.

$$C_2 := |\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)| \leq \frac{C_1}{\min_{t \in \mathcal{N}}\{\frac{d\mathbf{M}u_+(\varphi^\alpha, x^\alpha; t)}{dt}\}} \leq Const\, C_1, \tag{3.3}$$

where the last inequality follows since $\frac{d\mathbf{M}u_+(\varphi^\alpha, x^\alpha; t)}{dt}$ is bounded away from zero for $t \in \mathcal{N}$.

We are left with bounding C_1. By its definition.

$$C_1 = |\mathbf{M}u_+(\varphi^\alpha + \nu, x^\alpha + y; \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)) - \mathbf{M}u_+(\varphi^\alpha, x^\alpha; \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y))|.$$

By Step I, $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) < T + \varepsilon$. The average operator $\mathbf{M}$ is a linear bounded operator, u_+ is Lipschitz continuously dependent on its initial data (Lemma 1.13) for a finite t (here, $t < T+\varepsilon$). Hence there is $0 < \delta \leq \tilde{\delta}$ such that if $\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} < \delta$, then the previous equation becomes

$$C_1 \leq Const\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}.$$

Combine this with equation (3.3) to obtain locally Lipschitz continuity. $\square$

From this point on we only consider, without further mention, (ν, y) small enough such that $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) < 2T$. Such (ν, y) exists due to to Lemma 3.12 and the fact that $\mathbf{t}_\beta(\varphi^\beta, x^\alpha) = T$. In this case, expression (3.2) for $\mathbf{P}_\beta$ holds.

The following notation shortens the proofs in the rest of this subsection.

$$\kappa := \kappa(\nu, y) := \mathbf{t}_\beta(\varphi^\alpha, x^\alpha) - \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y). \tag{3.4}$$

The next lemma shows a connection between the magnitudes of κ and (ν, y). We do not specify which norm of f we use, so that the lemma will be as general as possible. To evaluate κ we take its absolute value, since $\kappa \in \mathbb{R}$.

Lemma 3.13. *Let $\mathbb{B}$ be a Banach space. If a function $f : \mathbb{R} \to \mathbb{B}$ is such that*

$$\|f(\kappa)\|_{\mathbb{B}} = O(|\kappa|^G),$$

for some $G > 0$, then $F(\nu, y) := f(\kappa(\nu, y))$ satisfies

$$\|F(\nu, y)\|_{\mathbb{B}} = O(\|\nu, y\|^G_{\mathbb{L}_p \times \mathbb{R}^N}).$$

Proof. By the big-O definition, there exist C, $\varepsilon_1 > 0$ such that if $|\kappa| \leq \varepsilon_1$, then

$$\|f(\kappa)\|_{\mathbb{B}} \leq C|\kappa|^G.$$

By Lemma 3.12, $\mathbf{t}_\beta$ is locally Lipschitz continuous at $(\varphi^\alpha, x^\alpha)$. Hence, there is $\varepsilon \leq \varepsilon_1$ such that if $\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \varepsilon$, then

$$\begin{aligned}
&|\kappa(\nu, y)| \leq \varepsilon_1 \text{ (continuity)},\\
&|\kappa(\nu, y)| \leq L\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \text{ (locally Lipschitz continuity)},
\end{aligned}$$

where L is the Lipschitz constant of $\mathbf{t}_\beta$. Hence if $\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N} \leq \varepsilon$, then

$$\|f(\kappa)\|_{\mathbb{B}} \leq C|\kappa(\nu, y)|^G \leq CL^G\|\nu, y\|^G_{\mathbb{L}_p \times \mathbb{R}^N}.$$

This shows that $F(\nu, y) = O(\|\nu, y\|^G_{\mathbb{L}_p \times \mathbb{R}^N})$. $\square$

The hit time operator is defined via $\mathbf{M}u_+$. Hence we define the following notation.

Notation 3.14. Let $u_+(\varphi, x; t)$ be the solution to problem (1.8)–(1.10). Denote the operator

$$\mathbf{u} : \mathbb{L}_p \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R},$$

as

$$\mathbf{u}(\varphi, x, t) := \mathbf{M}u_+(\varphi, x; t).$$

Lemma 3.15. *The hit time operator* $\mathbf{t}_\beta$ *is Fréchet differential at* $(\varphi^\alpha, x^\alpha)$. *Moreover, it can be written as*

$$\mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y) = \mathbf{t}_\beta(\varphi^\alpha, x^\alpha) + D\mathbf{t}_\beta(\varphi^\alpha, x^\alpha)[\nu, y] + O(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{\min\{2-\frac{1}{p}, 1+\frac{1}{p}\}}),$$

where the linear operator

$$D\mathbf{t}_\beta := D\mathbf{t}_\beta(\varphi^\alpha, x^\alpha) : \mathbb{L}_p \times \mathbb{R}^N \to \mathbb{R}$$

is given by

$$D\mathbf{t}_\beta[\nu, y] = -(D_t\mathbf{u})^{-1} D_{(\varphi,x)}\mathbf{u}[\nu, y], \tag{3.5}$$

where

$$\begin{aligned} &D_t\mathbf{u} := D_t\mathbf{u}(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)) : \mathbb{R} \to \mathbb{R}, \\ &D_{(\varphi,x)}\mathbf{u} := D_{(\varphi,x)}\mathbf{u}(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)) : \mathbb{L}_p \times \mathbb{R}^N \to \mathbb{R}. \end{aligned}$$

$\big($*We prove in the course of the proof that* $\mathbf{u}$ *has partial derivatives by* t *and* (φ, x) *at the point* $(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))$.$\big)$

Proof. By the definition of $\mathbf{t}_\beta$ and u_+, it is straightforward that $\mathbf{t}_\beta(\varphi, x) > 0$ is the first positive time such that $\mathbf{u}(\varphi, x, \mathbf{t}_\beta(\varphi, x)) = \beta$.

If $0 < t < 2T$, then the method of steps (see Footnote 4) and the integral form of u_+ in relation (1.14) show that

$$\mathbf{u}(\varphi, x, t) = \underbrace{\mathbf{M}\Big[e^{-\mathbf{B}t}x + \int_0^t e^{\mathbf{B}(s-t)}k ds)}_{:=\mathbf{u}_1(x,t)} + \underbrace{\mathbf{M}\int_0^t e^{\mathbf{B}(s-t)}\mathbf{A}\varphi(s-2T)ds\Big]}_{:=\mathbf{u}_2(\varphi,t)}. \tag{3.6}$$

We can use expression (3.6) since we assumed (after the proof of Lemma 3.12) that (ν, y) is small enough such that

$$\mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y) < 2T.$$

We calculate the linear parts of $\mathbf{u}_1, \mathbf{u}_2$ from relation (3.6) separately in Steps I and II. In Step III we merge them together, and use it to get the Fréchet derivative of $\mathbf{t}_\beta$.

For brevity, we write in the rest of the proof T instead of $\mathbf{t}_\beta(\varphi^\alpha, x^\alpha)$ and $T-\kappa(\nu, y)$ instead of $\mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y)$. These equalities are known from Assumption 2.12 and Definition (3.4) of κ.

Step I. The function $\mathbf{u}_1$ is smooth in x and t. It can be expanded in a Taylor series[12]

$$\begin{aligned} &\mathbf{u}_1(x^\alpha+y, T-\kappa(\nu, y)) = \\ &\mathbf{u}_1(x^\alpha, T) + D_x\mathbf{u}_1(x^\alpha, T)y - D_t\mathbf{u}_1(x^\alpha, T)\kappa(\nu, y) + O(\|\nu, y\|^2_{\mathbb{L}_p\times\mathbb{R}^N}), \end{aligned} \tag{3.7}$$

[12]For Taylor series for Fréchet derivative see [55] (Chapter 4.6).

where the big-O follows Lemma 3.13, and the fact that $\|y\|_{\mathbb{R}^N} \leq Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}$.

Step II. The function $\mathbf{u}_2$ is linear in φ, hence

$$\mathbf{u}_2(\varphi^\alpha + \nu, T - \kappa(\nu, y)) = \mathbf{u}_2(\varphi^\alpha, T - \kappa(\nu, y)) + \mathbf{u}_2(\nu, T - \kappa(\nu, y)). \tag{3.8}$$

Step II.I. By the expression for $\mathbf{u}_2$ in equation (3.6):

$$\mathbf{u}_2(\varphi^\alpha, T - \kappa(\nu, y)) = \mathbf{M}\left[e^{-\mathbf{B}(T-\kappa(\nu,y))}\int_0^{T-\kappa(\nu,y)} e^{\mathbf{B}s}\mathbf{A}\varphi^\alpha(s-2T)ds\right].$$

By Lemma 2.13 φ^α belongs to the spaces $\mathbb{W}_p^1(-2T, 0)$, $C^\infty[-2T, -T]$ and $C^\infty[-T, 0]$. Hence $\mathbf{u}_2(\varphi^\alpha, \cdot)$ belongs to the spaces $W_p^2(0, 2T)$, $C^\infty[0, T]$ and $C^\infty[T, 2T]$. It follows from Sobolev's inequality that

$$\begin{aligned} &|\mathbf{u}_2(\varphi^\alpha, T - \kappa(\nu, y)) - \mathbf{u}_2(\varphi^\alpha, T) + \mathbf{u}_2'(\varphi^\alpha, T)\kappa(\nu, y)| \\ &\leq Const\|\mathbf{u}_2(\varphi^\alpha, \cdot - \kappa(\nu, y)) - \mathbf{u}_2(\varphi^\alpha, \cdot) + \mathbf{u}_2'(\varphi^\alpha, \cdot)\kappa(\nu, y)\|_{W_p^1(T-\delta,T)}, \end{aligned} \tag{3.9}$$

where $\delta > 0$ is a fixed constant between 0 and T.

Apply the finite difference Lemma (Lemma 9.2) with $Q = [T - \delta, T]$ and $Q' = [0, T + 2\delta]$ on both $\mathbf{u}_2$ and $\mathbf{u}_2'$ to establish that

$$\|\mathbf{u}_2(\varphi^\alpha, \cdot - \kappa(\nu, y)) - \mathbf{u}_2(\varphi^\alpha, \cdot) + \mathbf{u}_2'(\varphi^\alpha, \cdot)\kappa(\nu, y)\|_{W_p^1(T-\delta,T)} \leq |\kappa|^{1+\frac{1}{p}}.$$

The previous inequality, inequality (3.9), and Lemma 3.13 imply that

$$|\mathbf{u}_2(\varphi^\alpha, T - \kappa(\nu, y)) - \mathbf{u}_2(\varphi^\alpha, T) + \underbrace{\mathbf{u}_2'(\varphi^\alpha, T)\kappa(\nu, y)}_{D_t\mathbf{u}_2(\varphi^\alpha,T)\kappa(\nu,y)}| = O(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{1+\frac{1}{p}}). \tag{3.10}$$

Step II.II. By the expression for $\mathbf{u}_2$ in equation (3.6):

$$\mathbf{u}_2(\nu, T - \kappa(\nu, y)) = \mathbf{M}[e^{-\mathbf{B}(T-\kappa(\nu,y))}\int_0^{T-\kappa(\nu,y)} e^{\mathbf{B}s}\mathbf{A}\nu(s-2T)ds].$$

Inside the average function $\mathbf{M}$ there is a composition of two terms. The first is smooth with respect to t in the operator norm[13]:

$$e^{-\mathbf{B}(T-\kappa(\nu,y))} = e^{-\mathbf{B}T} + O(\|\nu, y\|) + O(|\nu, y\|^2), \tag{3.11}$$

where the big-O notation follows from Lemma 3.13.

In the second term the integral can be divided into two parts:

$$\int_0^{T-\kappa(\nu,y)} e^{\mathbf{B}s}\mathbf{A}\nu(s-2T)ds = \int_0^T e^{\mathbf{B}s}\mathbf{A}\nu(s-2T)ds + \int_T^{T-\kappa(\nu,y)} e^{\mathbf{B}s}\mathbf{A}\nu(s-2T)ds. \tag{3.12}$$

[13] I.e, $\|O(\|\nu, y\|)\| \leq Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}$.

The first integral on the right hand side is linear in ν. On the second integral we can apply Lemma 9.9:

$$\left\| \int_T^{T-\kappa(\nu,y)} e^{\mathbf{B}s}\mathbf{A}\nu(s-2T)ds \right\|_{\mathbb{R}^N} \leq Const\,\kappa^{\frac{p-1}{p}} \|\nu(\cdot - 2T)\|_{\mathbb{L}_p(T,T+\kappa(\nu,y))} \leq \|\nu, y\|^{2-\frac{1}{p}}, \tag{3.13}$$

where the last inequality follows Lemma 3.12.

Combining relations (3.11)–(3.13) yields

$$|\mathbf{u}_2(\nu, T-\kappa(\nu,y)) - \underbrace{\mathbf{u}_2(\nu,T)}_{=D_\varphi \mathbf{u}_2(\varphi^\alpha, y^\alpha, T)\nu}| = O(\|\nu,y\|^{2-\frac{1}{p}}). \tag{3.14}$$

Step III. Combining together relations (3.7), (3.8), (3.10), and (3.14) yields

$$\begin{aligned}
\beta &= \mathbf{u}(\varphi^\alpha + \nu, x^\alpha + y, T - \kappa(\nu,y)) \\
&= \underbrace{\mathbf{u}_1(x^\alpha, T) + \mathbf{u}_2(\varphi^\alpha, T)}_{\mathbf{u}(\varphi^\alpha, x^\alpha, T) = \beta} + \underbrace{D_x\mathbf{u}_1(x^\alpha,T)y + D_\varphi \mathbf{u}_2(\varphi^\alpha, x^\alpha, T)\nu}_{= D_{(\varphi,x)}\mathbf{u}(\varphi^\alpha, x^\alpha, T)[\nu,y]} \\
&\underbrace{-D_t\mathbf{u}_1(x^\alpha,T)\kappa(\nu,y) - D_t\mathbf{u}_2(\varphi^\alpha,x^\alpha)\kappa(\nu,y)}_{-D_t\mathbf{u}(\varphi^\alpha,x^\alpha,T)\kappa(\nu,y)} + O(\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{\min\{2-\frac{1}{p},1+\frac{1}{p}\}}).
\end{aligned}$$

Recall that $\kappa(\nu, y) = \mathbf{t}_\beta(\varphi^\alpha, x^\alpha) - \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)$ (relation (3.4)). Then the previous equality becomes

$$\begin{aligned}
&- D_{(\varphi,x)}\mathbf{u}(\varphi^\alpha, x^\alpha, T)[\nu,y] + O(\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{\min\{2-\frac{1}{p},1+\frac{1}{p}\}}) \\
&= D_t\mathbf{u}(\varphi^\alpha, x^\alpha, T)[\mathbf{t}_\beta(\varphi^\alpha, x^\alpha) - \mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y)].
\end{aligned} \tag{3.15}$$

Note that $\mathbf{u}(\varphi^\alpha, x^\alpha, T) = \mathbf{M}u_p(T)$ (since $(\varphi^\alpha, x^\alpha)$ generates the periodic solution). Then
$D_t\mathbf{u}(\varphi^\alpha, x^\alpha, T) : \mathbb{R} \to \mathbb{R}$ is invertible, since $\frac{d\mathbf{M}u_p(T)}{dt} \neq 0$ (Assumption 2.12). This means that relation (3.15) implies that

$$\begin{aligned}
&\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) \\
&= \mathbf{t}_\beta(\varphi^\alpha, x^\alpha) - \Big(D_t\mathbf{u}\big(\varphi^\alpha, x^\alpha, T\big)\Big)^{-1} D_{(\varphi,x)}\mathbf{u}(\varphi^\alpha, x^\alpha, T)[\nu, y] + O(\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{\min\{2-\frac{1}{p},1+\frac{1}{p}\}}).
\end{aligned}$$

Taking the linear part of it yields the Fréchet derivative of $\mathbf{t}_\beta$. □

3.3 Properties of the Poincaré and hit maps

In Lemma 3.12 we ensured that $\mathbf{P}_\beta$ is defined in a neighbourhood of $(\varphi^\alpha, x^\alpha)$. The next lemma shows that it is also continuous at $(\varphi^\alpha, x^\alpha)$.

Lemma 3.16. *The operator $\mathbf{P}_\beta$ is continuous at $(\varphi^\alpha, x^\alpha)$, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that if*

$$\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta,$$

then

$$(\varphi^\alpha + \nu, x^\alpha + y) \in \mathcal{D}om(\mathbf{P}_\beta), \tag{3.16}$$

and

$$\|\mathbf{P}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - \mathbf{P}_\beta(\varphi^\beta, x^\beta)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \varepsilon. \tag{3.17}$$

Proof. The proof is in terms of $\mathbf{\Psi}_+$ from formula (3.2) for $\mathbf{P}_\beta$. Choose $\varepsilon > 0$. Assume without loss of generality that $\varepsilon < \sigma$.

By Lemma 2.3, the operator $\mathbf{\Psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) : \mathbb{B}^s_p \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{B}^s_p \times \mathbb{R}^N$ is continuous with respect to t. Hence there exists $0 < \delta_1 \leq \varepsilon$ such that if

$$|t - \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)| \leq \delta_1,$$

then

$$\|\mathbf{\Psi}_+(\varphi^\alpha, x^\alpha, t) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \frac{\varepsilon}{2}. \tag{3.18}$$

By Lemma 3.12, the operator $\mathbf{t}_\beta$ is continuous in $(\varphi^\alpha, x^\alpha)$. Since the $\mathbb{B}^s_p$ norm is embedded in the $\mathbb{L}_p$ norm (which is used in Lemma 3.12), there exists $\delta_2 > 0$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \delta_2,$$

then

$$|\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)| \leq \delta_1.$$

Combining this with inequality (3.18) implies that

$$\|\mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \frac{\varepsilon}{2}. \tag{3.19}$$

Since $\mathbf{\Psi}_+$ is continuously dependent on initial for $t \in (0, T + \sigma)$ (Lemma 2.6) and $|\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)| < T + \sigma$ (since $\delta_1 \leq \varepsilon < \sigma$), there exists $\delta \leq \delta_2$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \delta,$$

then

$$\|\mathbf{\Psi}_+\big(\varphi^\alpha + \nu, x^\alpha + y, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \frac{\varepsilon}{2}. \tag{3.20}$$

Hence if $\|\nu, y\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \delta$ then

$$\begin{aligned}
&\|\mathbf{\Psi}_+\big(\varphi^\alpha + \nu, x^\alpha + y, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \\
&\leq \underbrace{\|\mathbf{\Psi}_+\big(\varphi^\alpha + \nu, x^\alpha + y, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N}}_{\leq \varepsilon/2 \text{ by inequality (3.20)}} \\
&+ \underbrace{\|\mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)\big) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N}}_{\leq \varepsilon/2 \text{ by inequality (3.19)}} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means that if

$$\|\nu, y\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \delta,$$

then

$$\|\mathbf{P}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - \mathbf{P}_\beta(\varphi^\alpha, x^\alpha)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \varepsilon.$$

□

Remark 3.17. If $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) > \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)$, then the proof in Lemma 3.16 can be repeated for every $t \in [\mathbf{t}_\beta(\varphi^\alpha, x^\alpha), \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)]$.

Hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$(\nu, y) \leq \delta,$$

then

$$\|\mathbf{\Psi}_+(\varphi^\alpha + \nu, x^\alpha + y, t) - \mathbf{\Psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \varepsilon$$

for every

$$t \in [\mathbf{t}_\beta(\varphi^\alpha, x^\alpha), \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)].$$

This remark is used in the proof of the next lemma.

Lemma 3.18. *The periodic solution u_p is asymptotically stable (or stable or unstable) if and only if the fixed point $(\varphi^\alpha, x^\alpha)$ of the Poincaré map $\mathbf{P}$ is asymptotically stable (or stable or unstable respectively).*

Proof. It is straightforward that if $(\varphi^\alpha, x^\alpha)$ is an unstable fixed point of $\mathbf{P}$, then u_p is unstable. It is also clear that if u_p is stable or asymptotically stable, then $(\varphi^\alpha, x^\alpha)$ is a stable or asymptotically stable fixed point of $\mathbf{P}$ respectively. To finish the proof we have to show that stability and asymptotically stability of $(\varphi^\alpha, x^\alpha)$ imply the same for u_p.

By Lemma 3.16, $\mathbf{P}_\beta$ and $\mathbf{P}_\alpha$ are continuous at $(\varphi^\alpha, x^\alpha)$ and (φ^β, x^β) respectively. Remember that $\mathbf{P}_\beta(\varphi^\alpha, x^\alpha) = (\varphi^\beta, x^\beta)$. Then the chain rule implies that $\mathbf{P}$ is continuous at $(\varphi^\alpha, x^\alpha)$. Hence, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \delta,$$

then

$$\|\mathbf{P}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\beta, x^\beta)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \varepsilon,$$
$$\|\mathbf{P}(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\alpha, x^\alpha)\|_{\mathbb{B}^s_p \times \mathbb{R}^N} \leq \varepsilon.$$

Choose $\varepsilon > 0$.

For each $(\nu, y) \in \mathbb{B}^s_p \times \mathbb{R}^N$ denote by $\mathbf{t}_1(\nu, y)$ and $\mathbf{t}_2(\nu, y)$ the first and second switchings of the solution $u(\varphi^\alpha+\nu, x^\alpha+y;t)$ (we take (ν, y) small enough such that two switching times exist).

Step I. We show in this step that for every $\varepsilon>0$ there is a $0<\bar{\delta}\leq\varepsilon$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}\leq\bar{\delta},$$

then

$$dist((u(\varphi^\alpha+\nu, x^\alpha+y;t+s)|_{s\in(-2T,0)}, u(\varphi^\alpha+\nu, x^\alpha+y;t)), \Gamma)\leq\varepsilon,\ \text{for every } 0\leq t\leq\mathbf{t}_2(\nu,y). \tag{3.21}$$

For $t=0$ the claim follows from the definition of $u(\varphi^\alpha+\nu, x^\alpha+y;t)$ and the fact that $\|\nu, y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}\leq\bar{\delta}\leq\varepsilon$. We focus next on $t\in(0,\mathbf{t}_2(\nu,y)]$.

Fix some $0<\gamma<\sigma$. Due to the continuity of $\mathbf{t}_\alpha$ and $\mathbf{t}_\beta$ in Lemma 3.12, there exists $\delta_1>0$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}\leq\delta_1,$$

then both $\mathbf{t}_1(\nu,y)$ and $\mathbf{t}_2(\nu,y)-\mathbf{t}_1(\nu,y)$ are less than or equal to $T+\sigma-\gamma$ (such $\mathbf{t}_1$ and $\mathbf{t}_2$ exist due to the continuity of $\mathbf{P}_\beta$ and the hit time operators).

By continuous dependence on initial data from Lemma 2.6, there exists $\delta_2>0$ such if

$$\|\nu, y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}\leq\delta_2,$$

then

$$\|\mathbf{\Psi}_-(\varphi^\beta+\nu, x^\beta+y,t)-\mathbf{\Psi}_-(\varphi^\beta, x^\beta,t)\|\leq\varepsilon \text{ for every } t\in(0,T+\sigma-\gamma].$$

This shows that if $\mathbf{t}_2(\nu,y)-\mathbf{t}_1(\nu,y)\leq\mathbf{t}_\alpha(\varphi^\beta,x^\beta)$, then

$$dist(\mathbf{\Psi}_-(\varphi^\beta+\nu, x^\beta+y,t),\Gamma_2)\leq\varepsilon \text{ for every } t\in(0,\mathbf{t}_2(\nu,y)-\mathbf{t}_1(\nu,y)]. \tag{3.22}$$

Otherwise, use Remark 3.17 to show that there exists $\delta_3\leq\delta_2$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}\leq\delta_3,$$

then inequality (3.22) for holds for every $t\in[\mathbf{t}_\alpha(\varphi^\alpha,x^\alpha),\mathbf{t}_2(\nu,y)-\mathbf{t}_1(\nu,y)]$.

Use the same argument on $\mathbf{\Psi}_+$. Hence there exists $\delta_4>0$ such that if

$$\|\nu, y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}\leq\delta_4,$$

then

$$dist(\mathbf{\Psi}_+(\varphi^\alpha+\nu, x^\alpha+y;t),\Gamma_1)\leq\varepsilon,\ \text{for every } t\in[0,\mathbf{t}_1(\nu,y)].$$

Choose $\delta_5 > 0$ such that if

$$\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta_5,$$

then

$$\|\mathbf{P}_\beta(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\beta, x^\beta)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta_3.$$

Taking $\bar{\delta} = \min\{\delta_1, \delta_4, \delta_5\}$ shows inequality (3.21).

Step II. Assume, without loss of generality[14], that $\mathbf{M}[x^\alpha + y] = \alpha$. Denote $t_0 := 0$ and the return times of $\mathbf{P}(\varphi^\alpha + \nu, x^\alpha + y)$ to $\mathbb{T}_\alpha$ by $t_2, t_4, \ldots$.

If $(\varphi^\alpha, x^\alpha)$ is a stable fixed point of $\mathbf{P}$, then there exists $\delta \leq \bar{\delta}$ such that if

$$\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta$$

then

$$\begin{aligned} &\mathbf{P}^n(\varphi^\alpha + \nu, x^\alpha + y) \in \mathcal{D}om(\mathbf{P}), \\ &\|\mathbf{P}^n(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\alpha, x^\alpha)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \bar{\delta} \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$.

By Step I, for each t_i, $i = 0, 2, 4, \ldots$ (i is even)

$$dist(u(\varphi^\alpha + \nu, x^\alpha + y; t), \Gamma) \leq \varepsilon, \text{ for every } t \in [t_i, t_{i+2}].$$

Hence $dist(u(\varphi^\alpha + \nu, x^\alpha + y; t), \Gamma) \leq \varepsilon$ for $t \geq 0$.

Now assume that $(\varphi^\alpha, x^\alpha)$ is an asymptotically stable fixed point of $\mathbf{P}$. Choose arbitrary $\varepsilon_2 < \varepsilon$ and $\bar{\delta}_2$ such that Step I holds (with ε_2 playing the role of ε there). Due to asymptotic stability there exists $n \in \mathbb{Z} \cup \{0\}$ such that

$$\|\mathbf{P}^n(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\alpha, x^\alpha)\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \bar{\delta}_2.$$

This implies that $dist(u(\varphi^\alpha + \nu, x^\alpha + y; t), \Gamma) \leq \varepsilon_2$ for $t \geq t_{2n}$ (t_{2n} is the time that $\mathbf{P}^n$ returns to $\mathbb{T}_\alpha$). □

Lemma 3.19 (Almost periodicity of perturbed solutions)**.** *If $(\varphi^\alpha, x^\alpha)$ is an asymptotically stable fixed point of the Poincaré map $\mathbf{P}$, then for every $\varepsilon > 0$ there exists neighbourhoods Ω_1 of Γ_1 and Ω_2 of Γ_2 such that if*

$$(\varphi, x) \in \Omega_1, \mathbf{M}x < \beta \text{ or } (\varphi, x) \in \Omega_2, \mathbf{M}x \geq \beta,$$

then

$$\|u(\varphi, x; t + 2T) - u(\varphi, x; t)\|_{\mathbb{R}^N} \leq \varepsilon,$$

for all $t \geq 0$ and

$$\|u(\varphi, x; t + 2T) - u(\varphi, x; t)\|_{\mathbb{R}^N} \to 0 \text{ as } t \to \infty.$$

[14]The proof when $\mathbf{M}[x^\alpha + y] \neq \alpha$ follows by continuous dependency on initial conditions of $\mathbf{\Psi}_\pm$(Lemma 2.6) and continuity of $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$ (Lemma 3.16), using similar methods that were used in this subsection.

Proof. Choose $\varepsilon > 0$.

By Theorem 3.18 u_p is an asymptotically stable periodic solution. Hence we can choose δ_1 such that if $\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta_1$, then $dist((u(\varphi, x; t+s)_{s\in(-2T,0)}, u(\varphi, x; t)), \Gamma) \leq \frac{\varepsilon}{4}$ for all $t \geq 0$ (where the distance is taken in the $\mathbb{B}_p^s \times \mathbb{R}^N$ norm). Using this and the fact that u satisfies equation 1.4 we can bound the $\mathbb{L}_p$-norm of the derivative of u for all $\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta_1$ and every $t \geq 0$ in the following way:

$$\begin{aligned}\|u'(\varphi^\alpha + \nu, x^\alpha + y; t + \cdot)\|_{\mathbb{L}_p} &\leq \|k\|_{\mathbb{L}_p} + \|\mathbf{B}\|\|u(\varphi^\alpha + \nu, x^\alpha + y; t + \cdot)\|_{\mathbb{L}_p} \\ &\quad + \|\mathbf{C}\|\|u(\varphi^\alpha + \nu, x^\alpha + y; t - 2T + \cdot)\|_{\mathbb{L}_p} \leq C_1,\end{aligned} \tag{3.23}$$

for some constant $C_1 > 0$, where the last inequality follows from the fact that the distance of u from Γ is bounded.

Choose $\delta_2 > 0$ such that $(\varphi^\alpha, x^\alpha)$ is an asymptotically stable fixed point of $\mathbf{P}$ with δ_1 playing the role ε (in Definition 3.9).

For each $(\nu, y) \in \mathbb{B}_p^s \times \mathbb{R}^N$ denote by $\mathbf{t}_1(\nu, y)$ and $\mathbf{t}_2(\nu, y)$ the first and second switchings of the solution $u(\varphi^\alpha + \nu, x^\alpha + y; t)$ (we take (ν, y) small enough such that two switching times exist).

The operator $\mathbf{t}_2(\nu, y)$ is continuous at $(\nu, y) = (0, 0)$ since $\mathbf{t}_\beta$ and $\mathbf{t}_\alpha$ are continuous at $(\varphi^\alpha, x^\alpha)$ and (φ^β, x^β) respectively, $\mathbf{P}^\beta$ is continuous at $(\varphi^\alpha, x^\alpha)$, and $\mathbf{P}^\beta(\varphi^\alpha, x^\alpha) = (\varphi^\beta, x^\beta)$ (see Notation 3.10). Choose $\delta_3 > 0$ such that if

$$\|\nu, y\| \leq \delta_3,$$

then

$$|\mathbf{t}_2(\nu, y) - 2T| \leq \min\{\left(\frac{\varepsilon}{2C_1}\right)^{\frac{p}{p-1}}, \frac{T}{2}\}. \tag{3.24}$$

Let $\delta := \min\{\delta_1, \delta_2, \delta_3\}$. Choose $\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \leq \delta$, and let $t_1, t_2, ..$ be the switching times of $u(\varphi^\alpha + \nu, x^\alpha + y; t)$ in $[0, \infty)$. Assume, without loss of generality (see Footnote 14), $\mathbf{M}[x^\alpha + y] = \alpha$ and denote $t_0 = 0$.

Choose an even $i \in \mathbb{Z} \cup \{0\}$, and let $t \in [t_i, t_{i+2}]$. Then

$$\begin{aligned}&\|u(\varphi^\alpha + \nu, x^\alpha + y; t + 2T) - u(\varphi^\alpha + \nu, x^\alpha + y; t)\|_{\mathbb{R}^N} \\ &\leq \underbrace{\|u(\varphi^\alpha + \nu, x^\alpha + y; t + t_{i+2} - t_i) - u(\varphi^\alpha + \nu, x^\alpha + y; t + 2T)\|_{\mathbb{R}^N}}_{(A)} \\ &+ \underbrace{\|u(\varphi^\alpha + \nu, x^\alpha + y; t + t_{i+2} - t_i) - u(\varphi^\alpha + \nu, x^\alpha + y; t)\|_{\mathbb{R}^N}}_{(B)}.\end{aligned}$$

To evaluate (A), assume without loss of generality that $t_{i+2} - t_i - 2T < 0$, then

$$\begin{aligned}(A) &= \left\|\int_{t_{i+2}-t_i-2T}^{0} u'(t + 2T + r)dr\right\|_{\mathbb{R}^N} \leq \|u'\|_{\mathbb{L}_1(t_{i+2}-t_i-2T,0)} \\ &\leq |t_{i+2} - t_i - 2T|^{1-\frac{1}{p}}\|u'\|_{\mathbb{L}_p(t_{i+2}-t_i-2T,0)} \leq C_1|t_{i+2} - t_i - 2T|^{\frac{p-1}{p}} \leq \frac{\varepsilon}{2},\end{aligned}$$

where the last two inequalities follow relations (3.23) and (3.24).

As for (B) write t as $t = t_i + s$, and note that $s < 3T$ since $t_{i+2} - t_i < 3T$ by relation (3.24). Recall that i is even, and then

$$\begin{aligned}
(B) &= \|u(\varphi^\alpha + \nu, x^\alpha + y; t_{i+2} + s) - u(\varphi^\alpha + \nu, x^\alpha + y; t_i + s)\|_{\mathbb{R}^N} \\
&= \|u(\mathbf{P}^{\frac{i}{2}+1}(\varphi^\alpha + \nu, x^\alpha + y); s) - u(\mathbf{P}^{\frac{i}{2}}(\varphi^\alpha + \nu, x^\alpha + y); s)\|_{\mathbb{R}^N} \\
&\leq \|u(\mathbf{P}^{\frac{i}{2}+1}(\varphi^\alpha + \nu, x^\alpha + y); s) - u(\varphi^\alpha, x^\alpha; s)\|_{\mathbb{R}^N} \\
&\quad + \|u(\mathbf{P}^{\frac{i}{2}}(\varphi^\alpha + \nu, x^\alpha + y); s) - u(\varphi^\alpha, x^\alpha; s)\|_{\mathbb{R}^N} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\end{aligned}$$

where the inequalities follow from the fact that both $\|\mathbf{P}^{\frac{i}{2}}(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\alpha, x^\alpha)\|_{\mathbb{B}_p^s \times \mathbb{R}^N}$ and $\|\mathbf{P}^{\frac{i}{2}+1}(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\alpha, x^\alpha)\|_{\mathbb{B}_p^s \times \mathbb{R}^N}$ are less than or equal to δ_1 and u_p is asymptotically stable. This proves the first claim of the theorem.

Finally, to see the second claim we repeat the proof using the additional two facts:

1. Since u_p is asymptotically stable, then if $\|\nu, y\| \leq \delta$ then for every $\varepsilon > 0$ there exists $T_1 > 0$ such that $dist((u(\varphi, x; t + s)_{s \in (-2T, 0)}, u(\varphi, x; t)), \Gamma) \leq \frac{\varepsilon}{4}$ for all $t \geq T_1$.
2. The expression $|t_{i+2} - t_i - 2T| \to 0$ as $i \to \infty$ since $(\varphi^\alpha, x^\alpha)$ is an asymptotically stable fixed point of $\mathbf{P}$ and Lemma 3.12.

$\square$

4. Stability analysis of the Poincaré map

In the previous section we showed that stability of a periodic solution follows from that of the associated fixed point of the Poincaré map $\mathbf{P}$ (Lemma 3.18). In this section and the next one, we analyse the stability of this fixed point.

The ideal way to do that is to show that the stability of $\mathbf{P}$ follows from that of its linearization, i.e., its Fréchet derivative. We calculate this linearization formally[15], but unfortunately do not give a rigorous proof that it is indeed a linearization.

What can be done rigorously is calculating the linearization of three compositions of the hit maps: $\mathbf{P}_\beta \mathbf{P}_\alpha \mathbf{P}_\beta$. The reason lays in the fact that though we begin with initial data in the space $\mathbb{B}^s_p \times \mathbb{R}^N$, for $t > 2T$ the new initial data for a solution belongs to the space $\mathbb{W}^1_p(-2T, 0) \times \mathbb{R}^N$ (see Definition 1.5). After three iterations, a big enough portion of the first component of the initial data is in $\mathbb{W}^1_p$ so that rigorous linearization can be achieved.

The main result of this section shows the connection between the spectrum of the formal linearization of $\mathbf{P}$ at a fixed point, and the stability of the fixed point.

The section is organized as follows:

In Section 4.1 we create a projected versions of the Poincaré and hit maps.

In Section 4.2 we calculate, formally, the linearization of the projections of the hit maps at a fixed point. This formal calculation gives us a candidate for the linearization to work with in this and the next section.

In Section 4.3 we state the main result: if the spectrum of the formal linearization is less (greater) than one, then the fixed point is stable (unstable). We prove this result in the last two subsections.

In Section 4.4 we give a rigorous calculation of the linearization of the projection of $\mathbf{P}_\beta \mathbf{P}_\alpha \mathbf{P}_\beta$. Finally, in Section 4.5 we give the proof of the theorem from Section 4.3.

4.1 Projections

The Poincaré map $\mathbf{P}$ was defined in Section 3.1 (Definition 3.7) as acting from the cross-section $\mathbb{T}_\alpha$ to itself. The projection which we introduce in this subsection reparametrize $\mathbb{T}_\alpha$ to be the space $\mathbb{B}^s_p \times \mathbb{R}^{N-1}$.

Notation 4.1. Due to its ubiquity we define the constant

$$N_1 := N - 1. \tag{4.1}$$

[15] i.e, using some mathematical tools without checking their validity.

Projection. Let $x = \{x_j\}_{j=1}^{N} \in \mathbb{R}^N$ and $w = \{w_j\}_{j=1}^{N_1} \in \mathbb{R}^{N_1}$. Define the orthogonal projection $\mathbf{E}^{\mathbb{R}} : \mathbb{R}^N \to \mathbb{R}^{N_1}$ as

$$\mathbf{E}^{\mathbb{R}} x := \{x_{j+1}\}_{j=1}^{N_1}.$$

Define the lift operators $\mathbf{R}_\alpha : \mathbb{R}^{N_1} \to \{x \in \mathbb{R}^N | \mathbf{M}x = \alpha\} \subset \mathbb{R}^N$ and $\mathbf{R}_\beta : \mathbb{R}^{N_1} \to \{x \in \mathbb{R}^N | \mathbf{M}x = \beta\} \subset \mathbb{R}^N$ as

$$\begin{aligned} \mathbf{R}_\alpha w &= \left(\frac{\alpha}{m_0} - \frac{1}{m_0} \sum_{j=1}^{N-1} m_j w_j, \{w_j\}_{j=1}^{N-1} \right), \\ \mathbf{R}_\beta w &= \left(\frac{\beta}{m_0} - \frac{1}{m_0} \sum_{j=1}^{N-1} m_j w_j, \{w_j\}_{j=1}^{N-1} \right), \end{aligned} \tag{4.2}$$

where $m_0 \neq 0$ by its definition in Section 1.

The following relations hold by definition for every $w \in \mathbb{R}^{N_1}$.

$$\mathbf{E}^{\mathbb{R}} \mathbf{R}_\alpha w = \mathbf{E}^{\mathbb{R}} \mathbf{R}_\beta w = w.$$

Define a projection on the space $\mathbb{B}_p^s \times \mathbb{R}^N$ as

$$\mathbf{E} : \mathbb{B}_p^s \times \mathbb{R}^N \to \mathbb{B}_p^s \times \mathbb{R}^{N_1} : \quad \mathbf{E}[\varphi, x] = (\varphi, \mathbf{E}^{\mathbb{R}} x). \tag{4.3}$$

The projections $\mathbf{\Pi}_\alpha$ of the hit map $\mathbf{P}_\alpha$ and $\mathbf{\Pi}_\beta$ of $\mathbf{P}_\beta$ are

$$\begin{aligned} \mathbf{\Pi}_\alpha &: \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \quad \mathcal{D}om(\mathbf{\Pi}_\alpha) = \{(\varphi, w) | (\varphi, \mathbf{R}_\alpha w) \in \mathcal{D}om(\mathbf{P}_\alpha)\}, \\ \mathbf{\Pi}_\beta &: \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \quad \mathcal{D}om(\mathbf{\Pi}_\beta) = \{(\varphi, w) | (\varphi, \mathbf{R}_\beta w) \in \mathcal{D}om(\mathbf{P}_\beta)\}, \end{aligned}$$

defined as

$$\begin{aligned} \mathbf{\Pi}_\alpha &: \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \quad \mathbf{\Pi}_\alpha(\varphi, w) = \mathbf{E}\mathbf{P}_\alpha(\varphi, \mathbf{R}_\beta w), \\ \mathbf{\Pi}_\beta &: \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \quad \mathbf{\Pi}_\beta(\varphi, w) = \mathbf{E}\mathbf{P}_\beta(\varphi, \mathbf{R}_\alpha w). \end{aligned} \tag{4.4}$$

The projection of the Poincaré map $\mathbf{P}$

$$\mathbf{\Pi} : \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \qquad \mathcal{D}om(\mathbf{\Pi}) = \{(\varphi, w) | (\varphi, \mathbf{R}_\alpha w) \in \mathcal{D}om(\mathbf{P})\},$$

is defined as

$$\mathbf{\Pi}(\varphi, w) = \mathbf{E}\mathbf{P}(\varphi, \mathbf{R}_\alpha w) \text{ or equivalently as } \mathbf{\Pi}(\varphi, w) = \mathbf{\Pi}_\alpha \mathbf{\Pi}_\beta(\varphi, w). \tag{4.5}$$

Notation 4.2. Denote the projections of x^α, x^β from Notation 3.10 to $\mathbb{R}^{N_1}$ by

$$w^\alpha = \mathbf{E}^{\mathbb{R}} x^\alpha, w^\beta = \mathbf{E}^{\mathbb{R}} x^\beta, \quad w^\alpha, w^\beta \in \mathbb{R}^{N_1}.$$

Remark 4.3. The maps $\mathbf{\Pi}_\alpha$ and $\mathbf{\Pi}_\beta$ are continuous at $(\varphi^\alpha, w^\alpha)$ and (φ^β, w^β) respectively. This follows from the fact that $\mathbf{P}_\alpha, \mathbf{P}_\beta$ are continuous at these points (respectively) by Lemma 3.16.

Notation 4.4. Later we compose the projections of the hit maps. It is easier to denote such compositions by concatenation of indices; i.e, $\mathbf{\Pi}_{\alpha\beta}(\varphi, w) := \mathbf{\Pi}_\alpha \mathbf{\Pi}_\beta(\varphi, w)$, $\mathbf{\Pi}_{\beta\alpha\beta}(\varphi, w) := \mathbf{\Pi}_\beta \mathbf{\Pi}_\alpha \mathbf{\Pi}_\beta(\varphi, w)$, et cetera. We use similar notation also for the composition of the operators $\mathbf{h}_\alpha$ and $\mathbf{h}_\beta$ later in the section (see Section 4.4).

4.2 Formal linearization

The Poincaré map is defined in Definition 3.7 as a composition of two hit maps,

$$\mathbf{P} = \mathbf{P}_\alpha \mathbf{P}_\beta.$$

If the hit maps had a linearization (a Fréchet derivative), then the composition of their linearizations would be the linearization of the Poincaré map.

We cannot prove that the hit maps are linearizable (see discussion in Section 4.3). But if we **assume** that they are linearizable, then we can calculate the linearization **formally**, i.e, using mathematical tools without worrying whether this is allowed or not.

It turns out in the main theorem of this section (Theorem 4.18) that the stability of the fixed point $(\varphi^\alpha, x^\alpha)$ of $\mathbf{P}$ depends on the spectral radius of this "formal" linearization.

We remind that by relations (3.2) and (2.5) $\mathbf{P}_\beta$ is written as

$$\mathbf{P}_\beta(\varphi, x) = \big(\boldsymbol{\psi}_+(\varphi, x, \mathbf{t}_\beta(\varphi, x)), \boldsymbol{\psi}_+(\varphi, x, \mathbf{t}_\beta(\varphi, x))(0)\big),$$

where the operator

$$\boldsymbol{\psi}_+ : \mathbb{B}^s_p \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{B}^s_p,$$

was defined in (2.4) (recall that $\boldsymbol{\psi}_+(\varphi, x, t)(\theta)$ has an explicit value at $\theta = 0$). The map $\mathbf{P}_\alpha$ is written in a similar way, where $\boldsymbol{\psi}_-$ plays the role of $\boldsymbol{\psi}_+$. The next lemma shows that $\boldsymbol{\psi}_+$ has partial derivatives with respect to (φ, x) and t at $\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)$.

Lemma 4.5. *The operator $\boldsymbol{\psi}_+$ has partial derivatives*

$$\begin{aligned} &D_{(\varphi,x)}\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big) : \mathbb{B}^s_p \times \mathbb{R}^N \to \mathbb{B}^s_p, \\ &D_t\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big) : \mathbb{R} \to \mathbb{B}^s_p \end{aligned}$$

such that

$$\begin{aligned} &\boldsymbol{\psi}_+\big(\varphi^\alpha + \nu, x^\alpha + y, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big) = \boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big) \\ &\qquad\qquad\qquad + D_{(\varphi,x)}\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)[\nu, y], \\ &\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha) - \kappa\big) = \boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big) - D_t\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\kappa \\ &\quad + O\left(|\kappa|^{1-s+\frac{1}{p}}\right). \end{aligned} \tag{4.6}$$

These derivatives are given by

$$\begin{aligned} &D_{(\varphi,x)}\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)[\nu, y] = \begin{cases} \nu(\theta + t), & \theta \in [-2T, -T), \\ \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu(s - T)ds + e^{-\mathbf{B}(\theta+T)}y, & \theta \in [-T, 0], \end{cases} \\ &D_t\boldsymbol{\psi}_+\big(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)\kappa = \begin{cases} \varphi^{\alpha\prime}(\theta + T)\kappa, & \theta \in [-2T, -T), \\ u_+'(\varphi^\alpha, x^\alpha; \theta + T)\kappa, & \theta \in [-T, 0]. \end{cases} \end{aligned} \tag{4.7}$$

Proof. The result for $D_{(\varphi,x)}\boldsymbol{\psi}_+$ follows from the formula of $\boldsymbol{\psi}_+$ (relation 2.4) and the fact that $\boldsymbol{\psi}_+$ is affine linear in φ and x.

For the t derivative assume for simplicity that κ is positive (this is also the case on which we focus in later subsections). Recall that $\mathbf{t}_\beta(\varphi^\alpha, x^\alpha) = T$ by Assumption 2.12(2). To prove the claim we show that if $D_t\boldsymbol{\psi}_+$ is given by (4.7) then

$$\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, T-\kappa) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, T) + D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, T)\kappa = O\left(|\kappa|^{1-s+\frac{1}{p}}\right). \quad (4.8)$$

We evaluate the $\mathbb{L}_p$ norm in Step I.I and the $\mathbb{W}_p^s(-T-\sigma, 0)$ norm in Step I.II.

Step I.I. By relation (4.7) and formula (2.4) for $\boldsymbol{\psi}_+$, the left hand side of relation (4.8) equals

$$\begin{cases} \varphi^\alpha(\theta+T-\kappa), & \theta \in [-2T, -T+\kappa), \\ u_+(\varphi^\alpha, x^\alpha; \theta+T-\kappa), & \theta \in [-T+\kappa, 0], \end{cases}$$
$$-\begin{cases} \varphi^\alpha(\theta+T) - \varphi^{\alpha\prime}(\theta+T)\kappa, & \theta \in [-2T, -T), \\ u_+(\varphi^\alpha, x^\alpha; \theta+T) - u_+'(\varphi^\alpha, x^\alpha; \theta+T)\kappa, & \theta \in [-T, 0]. \end{cases}$$

Consider φ^α extended in a $2T$-periodic way for $t \geq 0$ and recall that it is the initial data for the periodic solution u_p (Assumption 2.12). Hence:

$$u_+(\varphi^\alpha, x^\alpha; \theta) = u_p(\theta) = \varphi^\alpha(\theta) \text{ for } \theta \in [0, T].$$

Create an extension $U(\theta)$ of $\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha; T)(\theta)$ to $[-3T, T]$:

$$U(\theta) := \begin{cases} \varphi^\alpha(\theta+T) & \theta, \in [-3T, 0), \\ u_+(\varphi^\alpha, x^\alpha; \theta+T), & \theta \in [0, T]. \end{cases} \quad (4.9)$$

The restriction of $U(\theta-\kappa)$ to $\theta \in [-2T, 0]$ equals $\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, T-\kappa)(\theta)$ since

$$\begin{aligned} U(\theta-\kappa)\Big|_{\theta\in[-2T,0]} &= \begin{cases} \varphi^\alpha(\theta+T-\kappa), & \theta \in [-3T+\kappa, \kappa) \\ u_+(\varphi^\alpha, x^\alpha; \theta+T-\kappa), & \theta \in [\kappa, T+\kappa] \end{cases}\Bigg|_{\theta\in[-2T,0]} \\ &= \varphi^\alpha(\theta+T-\kappa)\big|_{\theta\in[-2T,0]} \\ &= \begin{cases} \varphi^\alpha(\theta+T-\kappa), & \theta \in [-2T, -T+\kappa), \\ u_+(\varphi^\alpha, x^\alpha; \theta+T-\kappa), & \theta \in [-T+\kappa, 0]. \end{cases} \end{aligned}$$

Hence the left hand side of relation (4.8) equals

$$\Big(U(\theta-\kappa) - U(\theta) + \kappa U'(\theta)\Big)\Big|_{\theta\in[-2T,0]}.$$

Apply Lemma 9.2(2) to $U(\theta)$ with $Q = [-2T, 0]$ and $Q' = [-3T, T]$, where U belongs in the spaces $\mathbb{W}_p^2(-2T, -T)$ and $\mathbb{W}_p^2(-T, 0)$ so that

$$\|U(\theta-\kappa) - U(\theta) + \kappa U'(\theta)\|_{\mathbb{L}_p} = O(\kappa^{1+\frac{1}{p}}) = O(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{1+\frac{1}{p}}). \quad (4.10)$$

Step I.II. By Lemma 9.6, the estimate of the $\mathbb{W}_p^s(-T-\sigma,0)$ norm can be divided into the intervals $[-T-\sigma,-T]$ and $[-T,0]$.

The $\mathbb{W}_p^1(-T-\sigma,-T)$ estimate is straightforward since $U'(\theta)$ belongs to $C^\infty[-T-2\sigma,-T]$ by Lemma 2.13.

In the interval $[-T,0]$ there is a complication: $U'(\theta)$ has a jump at $\theta=-T$: it is equal to $\varphi^{\alpha\prime}(0+)-\varphi^{\alpha\prime}(0-)$ at this point. Hence U' is not $\mathbb{W}_p^1(-T-\sigma,0)$ and we cannot apply Lemma 9.2 to U' (as we did for U in Step I.I). We overcome this difficulty using an auxiliary function f:

$$f(\theta) := \begin{cases} a\theta, & \theta\in[-\sigma,0], \\ 0, & \theta\in(0,\infty), \end{cases} \tag{4.11}$$

where $a=\varphi^{\alpha\prime}(0+)-\varphi^{\alpha\prime}(0-)$.

The function f has two properties which are relevant to us. The first is that $\|f\|_{\mathbb{W}_p^s(-\kappa,T-\kappa)}$ is nonlinear[16] of order $O(\kappa)^{1-s+\frac{1}{p}}$ by Lemma 9.11. The second is that if f is added to φ^α, then the jump in the derivative is eliminated. The nonlinearity f is the main reason that we use $\mathbb{W}_p^s$ norm and not $\mathbb{W}_p^1$.

We add and subtract f from the right hand side of (4.8), and estimate the result in the interval $\theta\in[-T,0]$. It is less than or equal to

$$\begin{aligned} &\underbrace{\|\varphi^\alpha(\theta+T-\kappa)-\varphi^\alpha(\theta+T)+\varphi^{\alpha\prime}(\theta+T)\kappa+f(\theta+T-\kappa)\|_{\mathbb{W}_p^s(-T,0)}}_{(i)} \\ &+\underbrace{\|f(\theta+T-\kappa)\|_{\mathbb{W}_p^s(-T,0)}}_{(ii)} \end{aligned} \tag{4.12}$$

By Lemma 9.11, the term (ii) from relation (4.12) is of order $O(|\kappa|^{1-s+\frac{1}{p}})$ (as we already mentioned).

We focus now on term (i). The $\mathbb{W}_p^s(-T,0)$ norm of this term is bounded by the $\mathbb{W}_p^1(-T,0)$ norm. The only difficulty is evaluating the $\mathbb{L}_p(-T,0)$ norm of the derivative.

The expression $\varphi^{\alpha\prime}(\theta+T)+f'(\theta+T)$ belongs to the space $\mathbb{W}_p^1(-T-\sigma,\sigma)$ since the addition of f' eliminated the jump in the derivative of $\varphi^{\beta\prime}$. This expression belongs also to the spaces $\mathbb{W}_p^2(-T-\sigma,0)$ and $\mathbb{W}_p^2(0,\sigma)$. Hence Lemma 9.2(2) can be applied on it with $Q=(0,T)$ and $Q'=(-T-\sigma,\sigma)$.

Before applying Lemma 9.2, let us calculate the weak derivative of $\varphi^{\alpha\prime}(\theta+T)+f'(\theta+T)$ in $[-T,0]$. The derivatives f',f'' vanish in $[0,T]$ (since the derivative is

[16] Note that $1-s+\frac{1}{p}>1$ if and only if $ps<1$, which is satisfied by Condition 2.1.

taken from the right at 0). Hence the weak derivative of $\varphi^{\alpha\prime}(\theta+T)+f'(\theta+T)$ in $[0,T]$ equals φ''^{α}. Lemma 9.2(2) yields now that

$$\begin{aligned}\|(i)'\|_{\mathbb{L}_p(-T,0)} &= \|\varphi^{\alpha\prime}(\theta+T-\kappa)+f'(\theta+T-\kappa)-\varphi^{\alpha\prime}(\theta+T)+\varphi''^{\alpha}(\theta+t)\kappa\|_{\mathbb{L}_p(-T,0)} \\ &= O(|\kappa|^{1+\frac{1}{p}}).\end{aligned}$$

□

A similar proof as in Lemma 4.5 yields the following result.

Lemma 4.6. *The operator $\boldsymbol{\psi}_-$ has partial derivatives by (φ,x) and t at the point $\big(\varphi^\beta,x^\beta,\mathbf{t}_\alpha(\varphi^\beta,x^\beta)\big)$, with their expressions given by*

$$\begin{aligned}&D_{(\varphi,x)}\boldsymbol{\psi}_-\big(\varphi^\beta,x^\beta,\mathbf{t}_\alpha(\varphi^\beta,x^\beta)\big)[\nu,y] \\ &=\begin{cases}\nu(\theta+t), & \theta\in[-2T,-T),\\ \int_{-T}^{\theta}e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu(s-T)ds+e^{-\mathbf{B}(\theta+T)}y, & \theta\in[-T,0],\end{cases} \\ &D_t\boldsymbol{\psi}_-\big(\varphi^\beta,x^\beta,\mathbf{t}_\alpha(\varphi^\beta,x^\beta)\big)\kappa \\ &=\begin{cases}\varphi^{\beta\prime}(\theta+T)\kappa, & \theta\in[-2T,-T),\\ u_-'(\varphi^\beta,x^\beta;\theta+T)\kappa, & \theta\in[-T,0].\end{cases}\end{aligned}\tag{4.13}$$

Finally we comment about the $\mathbb{W}_p^1(-\sigma,0)$ norm of the partial t-derivative of $\boldsymbol{\psi}_\pm$. It is used in the proofs of Lemmas 4.29, and 4.30.

Lemma 4.7. *The operator $D_t\boldsymbol{\psi}_+:\mathbb{R}\to\mathbb{L}_p\cap\mathbb{W}_p^1(-\sigma,0)$ in 4.7 is bounded, and the following estimate takes place*

$$\|\boldsymbol{\psi}_+(\varphi_\alpha,x_\alpha,T-\delta)-\boldsymbol{\psi}_+(\varphi_\alpha,x_\alpha,T)+D_t\boldsymbol{\psi}_+(\varphi_\alpha,x_\alpha,T)\delta\|_{\mathbb{W}_p^1(-\sigma,0)}=O\left(|\delta|^2\right).$$

A similar estimate takes place for $\boldsymbol{\psi}_-$.

Proof. The proof follows from formula (4.7) and the fact that $u_+(\varphi^\alpha,x^\alpha;\theta)$ and $u_-(\varphi^\beta,x^\beta;\theta)$ are in the space $C^\infty(0,T)$ by Assumption 2.12 and Lemma 2.13. □

We next define the linear operator $\mathbf{L}$ as the formal linearization of $\boldsymbol{\psi}_+$ at $\big(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)\big)$ (or equivalently of $\boldsymbol{\psi}_-$ at $\big((\varphi^\beta,x^\beta),\mathbf{t}_\alpha(\varphi^\beta,x^\beta)\big)$, see Remark 4.10 below).

Definition 4.8. The **formal linearization** of $\boldsymbol{\psi}_+$ is a linear operator

$$\mathbf{L}:\mathbb{B}_p^s\times\mathbb{R}^N\to\mathbb{B}_p^s$$

defined as

$$\begin{aligned}\mathbf{L}[\nu,y]=D_t\boldsymbol{\psi}_+&\big(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)\big)D_{(\varphi,x)}\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)[\nu,y] \\ &+D_{(\varphi,x)}\boldsymbol{\psi}_+\big(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)\big)[\nu,y],\end{aligned}\tag{4.14}$$

where $D_{(\varphi,x)}\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)$ is given in Lemma 3.15 and $D_t\boldsymbol{\psi}_+\big(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)\big)$ and $D_{(\varphi,x)}\boldsymbol{\psi}_+\big(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)\big)$ are given in Lemma 4.5. Note that $\mathbf{L}$ is a sum of the partial derivatives of $\boldsymbol{\psi}_+$ at $(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha))$ (however, since we did **not** prove that $\boldsymbol{\psi}_+$ has partial derivatives in a neighbourhood of $(\varphi^\alpha,x^\alpha,\mathbf{t}_\beta(\varphi^\alpha,x^\alpha))$, it is not its derivative).

The proof of the following lemma follows from Definition 4.8 of $\mathbf{L}$, Lemmas 4.5 and 3.15, Lemma 4.6, and Assumption 2.12(4).

Lemma 4.9. *The operator* $\mathbf{L}$ *equals*

$$\mathbf{L}[\nu, y] = \begin{cases} -\frac{\varphi^{\alpha\prime}(\theta+T)}{\mathbf{M}\left(\varphi^{\alpha\prime}(-T-)\right)} \cdot \mathbf{M}\left(\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu(s-T)ds + e^{-\mathbf{B}T}y\right) + \nu(\theta+T), & \theta \in [-2T, -T), \\ -\frac{\varphi^{\alpha\prime}(\theta-T)}{\mathbf{M}\left(\varphi^{\alpha\prime}(-T-)\right)} \cdot \mathbf{M}\left(\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu(s-T)ds + e^{-\mathbf{B}T}y\right) + & \\ \qquad + \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu(s-T)ds + e^{-\mathbf{B}(\theta+T)}y, & \theta \in [-T, 0]. \end{cases} \tag{4.15}$$

Remark 4.10. Due to Lemma 4.6 and Assumption 2.12(3) we can define $\mathbf{L}$ equivalently as:

$$\mathbf{L}[\nu, y] = D_t\boldsymbol{\psi}_-\big(\varphi^\beta, x^\beta, \mathbf{t}_\alpha(\varphi^\beta, x^\beta)\big)D_{(\varphi,x)}\mathbf{t}_\alpha(\varphi^\beta, x^\beta)[\nu, y] + D_{(\varphi,x)}\boldsymbol{\psi}_-\big(\varphi^\beta, x^\beta, \mathbf{t}_\alpha(\varphi^\beta, x^\beta)\big)[\nu, y].$$

The following result is a direct consequence from the structure of $\mathbf{L}$ in formula (4.15) and Lemma 9.5.

Lemma 4.11. *The operator* $\mathbf{L}$ *is a bounded linear operator both as a map*

$$\mathbf{L} : \mathbb{L}_p(-2T, 0) \times \mathbb{R}^N \to \mathbb{L}_p(-2T, 0) \cap \mathbb{W}^1_p(-T, 0),$$

and as a map

$$\mathbf{L} : \mathbb{B}^s_p(-\sigma, 0) \times \mathbb{R}^N \to \mathbb{B}^s_p(-T-\sigma, 0).$$

Definition 4.12. The formal linearizations of $\mathbf{P}_\beta$ and $\mathbf{P}_\alpha$ via $\mathbf{L}$ are an operator from the space $\mathbb{B}^s_p \times \mathbb{R}^N$ to itself, given by

$$\Big(\big(\mathbf{L}[\nu, y]\big)(\theta), \big(\mathbf{L}[\nu, y]\big)(0)\Big), \tag{4.16}$$

where $\mathbf{L}[\nu, y]$ is well-defined at $\theta = 0$ (in the sense of trace) by Lemma 4.11.

Before we define the formal linearization of the projections of the hit maps, we need to calculate the linearization of the lift operators. The formula for $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$ (relation (4.2)) implies that they have the same linearization

$$D\mathbf{R}z := D\mathbf{R}_\alpha z = D\mathbf{R}_\beta z = \left(-\frac{1}{m_0}\sum_{j=1}^{N-1} m_j z_j, \{z_j\}_{j=1}^{N-1}\right), \tag{4.17}$$

where $z \in \mathbb{R}^{N_1}$.

Motivation 4.13. Definition 4.14, follows the following calculation.

$$\begin{aligned}\mathbf{L}_\Pi[\nu, z] &= D\boldsymbol{\Pi}_\beta(\varphi^\alpha, w^\alpha)[\nu, z] = D(\mathbf{E}\mathbf{P}_\alpha(\varphi, \mathbf{R}_\beta w)) = \mathbf{E}D(\mathbf{P}_\alpha(\varphi, \mathbf{E}^{\mathbb{R}}\mathbf{R}_\beta w)) \\ &= \mathbf{E}(D\boldsymbol{\psi}_+(\varphi, \mathbf{E}^{\mathbb{R}}\mathbf{R}_\beta w), D\boldsymbol{\psi}_+(\varphi, \mathbf{E}^{\mathbb{R}}\mathbf{R}_\beta w)(0)) = \big(\mathbf{L}[\nu, D\mathbf{R}z], \mathbf{E}^{\mathbb{R}}\mathbf{L}[\nu, D\mathbf{R}z](0)\big).\end{aligned}$$

Definition 4.14. Denote the formal linearization $\mathbf{\Pi}_\alpha$ at $(\varphi^\beta, \mathbf{E}^{\mathbb{R}} x^\beta)$ and $\mathbf{\Pi}_\beta$ at $(\varphi^\alpha, \mathbf{E}^{\mathbb{R}} x^\alpha)$ by $\mathbf{L}_\Pi$. The chain rule and relation (4.16) imply (see Remark 4.13) that it should be defined as

$$\begin{aligned} &\mathbf{L}_\Pi : \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \\ &\mathbf{L}_\Pi[\nu, z] := \big(\mathbf{L}[\nu, D\mathbf{R}z], \mathbf{E}^{\mathbb{R}}\mathbf{L}[\nu, D\mathbf{R}z](0)\big). \end{aligned} \tag{4.18}$$

The following result is a direct consequence from Lemma 4.11 and formula (4.18).

Lemma 4.15. *The operator $\mathbf{L}_\Pi$ is a bounded linear operator both as a map*

$$\mathbf{L}_\Pi : \mathbb{L}_p(-2T, 0) \times \mathbb{R}^{N_1} \to \big(\mathbb{L}_p(-2T, 0) \cap \mathbb{W}_p^1(-T, 0)\big) \times \mathbb{R}^{N_1},$$

and as a map

$$\mathbf{L}_\Pi : \mathbb{B}_p^s(-\sigma, 0) \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s(-T - \sigma, 0) \times \mathbb{R}^{N_1}.$$

We denote the norms of these linear operators in their respective spaces by $\|\mathbf{L}_\Pi\|_{(1)}$ and $\|\mathbf{L}_\Pi\|_{(2)}$.

4.3 Theorem: stability of the Poincaré map

The following condition is valid throughout the rest of the chapter. It is an addition to Condition 2.1 (namely, $ps < 1$).

Condition 4.16. The constants p, s satisfy the following condition:

$$1 < p < 2, \quad \frac{1}{p} + s > 1. \tag{4.19}$$

Remark 4.17. Condition 4.16 comes up while looking for spaces in which the problem is rigorously linearizable. See Lemmas 4.29 and 4.30.

Conditions 2.1 and 4.19 are equivalent to

$$0 < s < 1 \text{ and } 1 < p < \min\left\{\frac{1}{s}, \frac{1}{1-s}\right\}.$$

Hence for every choice of $0 < s < 1$ there exists p such that s and p satisfy these conditions.

We state now the main result of the section: stability of the Poincaré map.

Theorem 4.18. *Let Conditions 2.1 and 4.16 hold. If the spectral radius $r(\mathbf{L}_\Pi)$ is such that*

$$r(\mathbf{L}_\Pi) < 1, \tag{4.20}$$

then $(\varphi^\alpha, x^\alpha)$ is an asymptotically stable fixed point of the Poincaré map $\mathbf{P} = \mathbf{P}_{\alpha\beta}$. If

$$r(\mathbf{L}_\Pi) > 1, \tag{4.21}$$

then $(\varphi^\alpha, x^\alpha)$ is an unstable fixed point of the Poincaré map.

The proof is given at the end of the section. It combines results which are proved in the next subsections.

Discussion 4.19. The technical settings in this paper can look strange without an explanation. They are mostly dictated by the proof of Theorem 4.18, so now is a good point to discuss them. The interesting questions are:

1. Why do we use the space $\mathbb{W}_p^s$ (in the definition of $\mathbb{B}_p^s$)?
2. Why do we need the constant σ (also in the definition of $\mathbb{B}_p^s$)?
3. Why do we differentiate three iterations of the hit maps (in Theorem 4.20 below)?

If the reader understands the choices, then going through the proofs in this section becomes a much easier task.

Section 4.2 shows that the main ingredient of the formal linearization of the hit map is the operator $\mathbf{L}$, which is the formal linearization of the operator $\boldsymbol{\psi}_+$. For the brevity of this discussion we ignore for the moment the argument x (so everything depends only on φ). Denote a perturbation of φ_α by ν, and recall that $\mathbf{t}_\beta(\varphi_\alpha) = T$. The resulting perturbation of $\mathbf{t}_\beta$ is then $\mathbf{t}_\beta(\varphi_\alpha + \nu) = T - \kappa$ for $\kappa = O(\|\nu\|_{\mathbb{L}_p})$ by Lemma 3.12. Assume for the moment $\kappa > 0$ (this is the case where difficulties are encountered).

If $\mathbf{L}$ was the Fréchet derivative of $\boldsymbol{\psi}_+$, then we would have

$$\|\boldsymbol{\psi}_+(\varphi_\alpha + \nu, T - \kappa) - \boldsymbol{\psi}_+(\varphi_\alpha, T) - \mathbf{L}\nu\| = o(\|\nu\|) \tag{4.22}$$

for an appropriate norm (same on both sides). We will answer questions 1–3 by trying to prove (4.22).

If we examine the expressions for $\boldsymbol{\psi}_+$ (given by (2.4)) and $\mathbf{L}$ (given by (4.15)), we see that both operators are defined in a piecewise way. One can see the main difficulties looking first at the interval $\theta \in (-2T, -T)$. On this interval, the expression in the norm in the left hand side of (4.22) contains, in particular, the term

$$B(\theta) := \nu(\theta + T - \kappa) - \nu(\theta + T), \quad \theta \in (-2T, -T).$$

Space for ν. Due to Definition 1.5 of a solution, the first natural choice would be $\nu \in \mathbb{L}_p$. However, without additional regularity of ν, (B) is not $o\left(\|\nu\|_{\mathbb{L}_p}\right)$.

Another option is a Sobolev space. Since the arguments of the functions in (B) are at least $-T - \kappa$, then if we choose a small $\sigma > 0$, and $\nu \in \mathbb{L}_p \cap \mathbb{W}_p^1(-T - \sigma, 0)$, then for all $0 < \kappa < \sigma$

$$\|B\|_{\mathbb{L}_p(-2T,-T)} \le Const\,\|\nu\|_{\mathbb{W}_p^1(-T-\sigma,0)}\kappa = O\left(\|\nu\|^2_{Lp \cap \mathbb{W}_p^1(-T-\sigma,0)}\right), \tag{4.23}$$

where the last inequality follows from Lemma 3.12. To get the same norm on both sides in (4.22), we now have to estimate, in particular, $\|B\|_{\mathbb{W}_p^1(-T-\sigma,-T)}$. This cannot be done directly without extra regularity of ν, but iterations of the Poincaré map help to gain regularity (see discussion below).

However, the Fréchet derivative of $\boldsymbol{\psi}_+(\varphi, \mathbf{t}_\beta(\varphi))$ at φ_α would then involve the Fréchet derivative with respect to time of the function $\boldsymbol{\psi}_+(\varphi_\alpha, \cdot) : \mathbb{R} \to \mathbb{L}_p \cap \mathbb{W}_p^1(-T-\sigma, 0)$. To show its differentiability at $t = T$, we would have to estimate the $\mathbb{W}_p^1(-T-\sigma, 0)$ norm of

$$u_p(\theta + T - \delta) - u_p(\theta + T) + u_p'(\theta + T)\delta, \quad \theta \in (-T-\sigma, 0), \tag{4.24}$$

where u_p is the periodic solution. But the function in (4.24) does not belong to $\mathbb{W}_p^1(-T-\sigma, 0)$ because $u_p'(\theta+T)$ in general has a jump at $\theta = -T$. This difficulty cannot be solve by iterating the Poincaré map. The remedy is to take $\mathbb{W}_p^s(-T-\sigma, 0)$ instead of $\mathbb{W}_p^1(-T-\sigma, 0)$, see the proof of Lemma 4.5 and specifically the usage of the auxiliary function f there. Estimate (4.23) still holds for $\nu \in \mathbb{L}_p \times \mathbb{W}_p^s(-T-\sigma, 0)$ due to Besov's inequality (Lemma 9.3)

$$\|B\|_{\mathbb{L}_p(-2T,-T)} \le Const \|\nu\|_{\mathbb{W}_p^s(-T-\sigma,0)} \kappa^s \le Const\, \|\nu\|_{\mathbb{W}_p^s(-T-\sigma,0)} \|\nu\|_{\mathbb{L}_p}^s. \tag{4.25}$$

However, to estimate $\|B\|_{\mathbb{W}_p^s(-T-\sigma,-T)}$, the iteration is still needed.

Iterations. First consider two iterations of the hit maps: $\mathbf{P}_\alpha \mathbf{P}_\beta$ (in the proof we iterate the reparametriztions of the hit maps, $\mathbf{\Pi}_\alpha$ and $\mathbf{\Pi}_\beta$, but for this discussion the hit maps themselves will do). Denote the new perturbation for $\mathbf{P}_\alpha$ by ν_1 (we still omit the argument x):

$$\nu_1 := \mathbf{P}_\beta(\varphi_\alpha + \nu) - \mathbf{P}_\beta(\varphi_\alpha) = \boldsymbol{\psi}_+(\varphi_\alpha + \nu, T - \kappa) - \varphi_\beta.$$

Set $\kappa_1 := \mathbf{t}_\alpha(\varphi_\beta) - \mathbf{t}_\alpha(\varphi_\beta + \nu_1) = T - \mathbf{t}_\alpha(\varphi_\beta + \nu_1)$, and assume that $\kappa_1 > 0$ (this is again the most difficult case). We need to estimate the term equivalent to $B(\theta)$ for two iterations, in the $\mathbb{W}_p^s(-T-\sigma, -T)$ norm, i.e.,

$$\|\nu_1(\theta + T - \kappa_1) - \nu_1(\theta + T)\|_{\mathbb{W}_p^s(-T-\sigma,-T)}.$$

We pass to the $\mathbb{W}_p^1(-T-\sigma, -T)$ norm, and try to estimate

$$\|\nu_1'(\theta + T - \kappa_1) - \nu_1'(\theta + T)\|_{\mathbb{L}_p(-T-\sigma,-T)}. \tag{4.26}$$

We note that ν_1 satisfies a delay differential equation (given in (4.41) further on). Examining this delay differential equation shows that (4.26) includes $\|\nu(\cdot - \kappa_1) - \nu\|_{\mathbb{L}_p(-T-\sigma-\kappa,-T)}$. If ν belonged to $\mathbb{W}_p^s(-T-2\sigma, -T)$, then an estimate similar to (4.25) would work. But ν belongs only to $\mathbb{W}_p^s(-T-\sigma, -T)$.

However, when we take three iterations and define κ_2 (> 0 to be definite) similarly to κ_1, we end up with $\|\nu(\cdot - \kappa_2) - \nu\|_{\mathbb{L}_p(-\sigma-\kappa_1-\kappa,0)}$ (see (4.41) and (4.42)). Since $\nu \in \mathbb{W}_p^s(-T-\sigma, 0)$, this is estimated analogously to (4.25).

4.4 Linearization of a composition of three projections of the hit map

"The above argument is somewhat long, but each step consists in proving a rather simple inequality" - Timothy Gowers (The Princeton companion to mathematics)

In this subsection we find the linearization of $\mathbf{\Pi}_{\beta\alpha\beta} = \mathbf{\Pi}_\beta \mathbf{\Pi}_\alpha \mathbf{\Pi}_\beta$. Our candidate for the linearization is $\mathbf{L}_\Pi^3$, where $\mathbf{L}_\Pi$ is given in Definition 4.14. While $\mathbf{L}_\Pi$ was never proved to be the linearization of $\mathbf{\Pi}_\alpha$ or $\mathbf{\Pi}_\beta$, we do prove in this section that $\mathbf{L}_\Pi^3$ is the linearization of $\mathbf{\Pi}_{\beta\alpha\beta}$.

The main result of the section is Theorem 4.20. To state this theorem, we define the operators $\mathbf{h}_\beta^\Pi, \mathbf{h}_\alpha^\Pi : \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}$ as

$$\begin{aligned} \mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z) &= \mathbf{\Pi}_\beta(\varphi^\alpha, w^\alpha) + \mathbf{L}_\Pi[\nu, z] + \mathbf{h}_\beta^\Pi(\nu, z), \\ \mathbf{\Pi}_\alpha(\varphi^\beta + \nu, w^\beta + z) &= \mathbf{\Pi}_\alpha(\varphi^\beta, w^\beta) + \mathbf{L}_\Pi[\nu, z] + \mathbf{h}_\alpha^\Pi(\nu, z), \end{aligned} \tag{4.27}$$

where $\mathbf{\Pi}_\beta, \mathbf{\Pi}_\alpha$ are defined in (4.4) and $\mathbf{L}_\Pi$ in (4.18).

Recall $\mathbf{P}_\beta^{\mathbb{B}}$ from the definition of $\mathbf{P}_\beta$ in relation (3.1), and define the operator $\mathbf{h}_\beta : \mathbb{B}_p^s \times \mathbb{R}^N \to \mathbb{B}_p^s$ as

$$\mathbf{h}_\beta(\nu, D\mathbf{R}z) := \mathbf{P}_\beta^{\mathbb{B}}(\varphi^\alpha + \nu, x^\alpha + D\mathbf{R}z) - \mathbf{P}_\beta^{\mathbb{B}}(\varphi^\alpha, x^\alpha) - \mathbf{L}[\nu, D\mathbf{R}z]. \tag{4.28}$$

Then relations (4.4) (for $\mathbf{\Pi}_\beta$), (4.18) (for $\mathbf{L}_\Pi$), (4.17) (for $\mathbf{R}_\alpha$), (4.27) and (4.28) imply that the term $\mathbf{h}_\beta^\Pi$ (or $\mathbf{h}_\alpha^\Pi$, see Remark 3.11) can be written[17] as

$$\mathbf{h}_\beta^\Pi(\nu, z) = (\mathbf{h}_\beta(\nu, D\mathbf{R}z), \mathbf{E}^{\mathbb{R}}\mathbf{h}_\beta(\nu, D\mathbf{R}z)(0)). \tag{4.29}$$

Theorem 4.20. *The map*

$$\mathbf{\Pi}_{\beta\alpha\beta} = (\mathbf{\Pi}_{\beta\alpha\beta}^{\mathbb{B}}, \mathbf{\Pi}_{\beta\alpha\beta}^{\mathbb{R}}) : \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}, \tag{4.30}$$

is Fréchet differentiable (i.e., linearizable) at $(\varphi^\alpha, x^\alpha)$. Its derivative equals $(\mathbf{L}_\Pi)^3$. In particular $\mathbf{\Pi}_{\beta\alpha\beta}$ can be written as

$$\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) = \mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha, w^\alpha) + (\mathbf{L}_\Pi)^3[\nu, z] + \mathbf{h}_{\beta\alpha\beta}^\Pi(\nu, z). \tag{4.31}$$

Here

1. *$\mathbf{L}_\Pi$ is a linear bounded operator defined in equation (4.18), with $\mathbf{L}$ given by (4.14) or equivalently by (4.15),*

2. *$h_{\beta\alpha\beta}^\Pi : \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}$ is a nonlinear term of order $O(\|\nu, z\|_{\mathbb{B}_p^s \times \mathbb{R}^{N_1}}^\gamma)$,*

where $\gamma = \min\{2 - \frac{1}{p}, \frac{1}{p} + s, 1 - s + \frac{1}{p}\}$.

Note that $\gamma > 1$ due to Conditions 4.16 and 2.1.

[17] The evaluation of $\mathbf{h}_\beta$ is at $D\mathbf{R}z$ since $\mathbf{R}_\alpha$ is affine linear, and hence $\mathbf{R}_\alpha(w^\alpha + z) = \mathbf{R}_\alpha w^\alpha + D\mathbf{R}z = x^\alpha + D\mathbf{R}z$.

Proof of Theorem 4.20. By Lemma 3.16, the hit maps are continuous at $(\varphi^\alpha, x^\alpha)$. Hence, without further mention, we assume that (ν, z) is sufficiently small such that the solution $u(\varphi^\alpha + \nu, \mathbf{E}^{\mathbb{R}}[w^\alpha + z]; t)$ has at least three switchings in $(0, \infty)$.[18]

Part I. Calculating the nonlinear part $\mathbf{h}^{\Pi}_{\beta\alpha\beta}(\nu, z)$.

We express $\mathbf{h}^{\Pi}_{\beta\alpha\beta}$ (from relation (4.31)) in terms of $\mathbf{L}, \mathbf{h}^{\Pi}_{\beta}$ and $\mathbf{h}^{\Pi}_{\alpha}$. This follows from the next (long but simple) calculation. It repeatedly uses representations (4.27) for[19] $\mathbf{\Pi}_\alpha$ and $\mathbf{\Pi}_\beta$:

$$\begin{aligned}
&\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha, w^\alpha) \\
&= \underbrace{\mathbf{L}_\Pi \left[\mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha, w^\alpha)\right]}_{(1)} \\
&\quad \underbrace{+ \mathbf{h}^{\Pi}_{\beta} \left(\overbrace{\mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha, w^\alpha)}^{=:(\nu_2, z_2)} \right)}_{(2)} .
\end{aligned} \tag{4.32}$$

Expand (1) in order to extract the term $(\mathbf{L}_\Pi)^3$ (the candidate for the Fréchet derivative) from it:

$$\begin{aligned}
&(1) \\
&= \mathbf{L}_\Pi \big[\mathbf{L}_\Pi \big[\mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_\beta(\varphi^\alpha, w^\alpha)\big] + \mathbf{h}^{\Pi}_{\alpha}\big(\mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z) \\
&- \mathbf{\Pi}_\beta(\varphi^\alpha, w^\alpha)\big)\big] \\
&= \mathbf{L}_\Pi \left[\mathbf{L}_\Pi \big[\mathbf{L}_\Pi[\nu, z] + \mathbf{h}_\beta(\nu, z)\big] + \mathbf{h}^{\Pi}_{\alpha}\big(\mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_\beta(\varphi^\alpha, w^\alpha)\big)\right] \\
&= (\mathbf{L}_\Pi)^3[\nu, z] + \underbrace{\mathbf{L}_\Pi \left[\mathbf{L}_\Pi \mathbf{h}^{\Pi}_{\beta}(\nu, z) + \mathbf{h}^{\Pi}_{\alpha}\big(\overbrace{\mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_\beta(\varphi^\alpha, w^\alpha)}^{=:(\nu_1, z_1)}\big)\right]}_{(3)} .
\end{aligned} \tag{4.33}$$

To make those "monstrous" expressions more readable, we define:

$$\begin{aligned}
(\nu_1, z_1) &:= \mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_\beta(\varphi^\alpha, w^\alpha) \text{ (for (3) in equation (4.33))}, \\
(\nu_2, z_2) &:= \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha, w^\alpha) \text{ (for (2) in equation (4.32))}.
\end{aligned} \tag{4.34}$$

The nonlinear part, $\mathbf{h}^{\Pi}_{\beta\alpha\beta}(\nu, z)$, is (2) in equation (4.32) plus (3) from the last line in (4.33). Using the notation above, it can be written as

$$\mathbf{h}^{\Pi}_{\beta\alpha\beta}(\nu, z) = \mathbf{L}_\Pi \left[\mathbf{L}_\Pi \mathbf{h}^{\Pi}_{\beta}(\nu, z) + \mathbf{h}^{\Pi}_{\alpha}(\nu_1, z_1)\right] + \mathbf{h}^{\Pi}_{\beta}(\nu_2, z_2). \tag{4.35}$$

[18] The existence of such (ν, z) follows from the fact that $\mathbf{P}_\beta, \mathbf{P}_\alpha$ are continuous at $(\varphi^\alpha, x^\alpha)$ by Lemma 3.16, and hence the composition of $\mathbf{P}_\alpha \mathbf{P}_\beta \mathbf{P}_\alpha$ is continuous as well.

[19] In the usage of representation (4.27), we look at expressions of the kind of $\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z)$ as
$\mathbf{\Pi}_\beta(\mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha, w^\alpha) + \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha, w^\alpha))$.

To prove the lemma we need to show that equation (4.35) is of order γ in the $\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}$ norm[20]. Apply this norm on the previous equation, and use the triangle inequality.

$$\begin{aligned}\|\mathbf{h}^\Pi_{\beta\alpha\beta}(\nu,z)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} &= \left\|\mathbf{L}_\Pi\left[\mathbf{L}_\Pi\mathbf{h}^\Pi_\beta(\nu,z)+\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right]+\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}\\ &\le \left\|\mathbf{L}_\Pi\left[\mathbf{L}_\Pi\mathbf{h}^\Pi_\beta(\nu,z)+\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right]\right\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}+\left\|\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}.\end{aligned}$$

By Lemma 4.15, $\mathbf{L}_\Pi:\mathbb{B}^s_p(-\sigma,0)\times\mathbb{R}^{N_1}\to\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}$ is bounded. Using the norm $\|\cdot\|_{(1)}$ from this lemma on the first term on the right hand side yields

$$\begin{aligned}&\left\|\mathbf{L}_\Pi\left[\mathbf{L}_\Pi\mathbf{h}^\Pi_\beta(\nu,z)+\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right]+\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}}\\ &\le \|\mathbf{L}_\Pi\|_{(2)}\left\|\mathbf{L}_\Pi\mathbf{h}^\Pi_\beta(\nu,z)+\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right\|_{\mathbb{B}^s_p(-\sigma,0)\times\mathbb{R}^{N_1}}+\left\|\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}}\\ &\le \|\mathbf{L}_\Pi\|_{(2)}\left\|\mathbf{L}_\Pi\mathbf{h}^\Pi_\beta(\nu,z)\right\|_{\mathbb{B}^s_p(-\sigma,0)\times\mathbb{R}^{N_1}}\\ &\quad+\|\mathbf{L}_\Pi\|_{(2)}\left\|\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right\|_{\mathbb{B}^s_p(-\sigma,0)\times\mathbb{R}^{N_1}}+\left\|\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}}.\end{aligned}$$

Use Lemma 4.15 again, this time with the property $\mathbf{L}_\Pi:\mathbb{L}_p(-2T,0)\times\mathbb{R}^{N_1}\to\mathbb{B}^s_p(-\sigma,0)\times\mathbb{R}^{N_1}$ and the $\|\cdot\|_{(2)}$ norm.

$$\begin{aligned}&\left\|\mathbf{L}_\Pi\left[\mathbf{L}_\Pi\mathbf{h}^\Pi_\beta(\nu,z)+\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right]+\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}}\\ &\le \|\mathbf{L}_\Pi\|_{(2)}\|\mathbf{L}_\Pi\|_{(1)}\left\|\mathbf{h}^\Pi_\beta(\nu,z)\right\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^{N_1}}+\|\mathbf{L}_\Pi\|_{(2)}\left\|\mathbf{h}^\Pi_\alpha(\nu_1,z_1)\right\|_{\mathbb{B}^s_p(-\sigma,0)\times\mathbb{R}^{N_1}}\\ &\quad+\left\|\mathbf{h}^\Pi_\beta(\nu_2,z_2)\right\|_{\mathbb{B}^s_p(-T-\sigma,0)\times\mathbb{R}^{N_1}}\\ &\le Const\Big(\underbrace{\left\|\mathbf{h}^\Pi_\beta(\nu,z)\right\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^{N_1}}}_{(1)}+\underbrace{\|\mathbf{h}_\alpha(\nu_1,z_1)\|_{\mathbb{W}^1_p(-\sigma,0)}}_{(2)}\\ &+\underbrace{\|\mathbf{h}_\beta(\nu_2,z_2)\|_{\mathbb{B}^s_p(-T-\sigma,0)\cap\mathbb{W}^1_p(-\sigma,0)}}_{(3)}\Big).\end{aligned}\tag{4.36}$$

Lemmas 4.28-4.30 show that the right hand side of (4.36) is $O(\|\nu,z\|^\gamma_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}})$ and conclude the proof.

Part II. A proof that equation (4.36) is $O(\|\nu,z\|^\gamma_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}})$

Part II.I: Preliminaries for the estimate

Notation 4.21. Denote

$$\begin{aligned}y &:= D\mathbf{R}z,\\ y_1 &:= D\mathbf{R}z_1,\\ y_2 &:= D\mathbf{R}z_2.\end{aligned}\tag{4.37}$$

Notation 4.22. Using the previous notation, Notation 4.2 (for w^α, w^β), and the fact that $\mathbf{R}_\alpha,\mathbf{R}_\beta$ from relation (4.2) are affine linear operators, the following liftings

[20] In Section 2.1 we abbreviate $\mathbb{B}^s_p=\mathbb{B}^s_p(-T-\sigma,0)$. We sometimes omit this abbreviation in the proof for clarity; The norm $\mathbb{B}^s_p$ is being evaluated in different intervals in the same equation, and it may be confusing if only one of those intervals is abbreviated.

to $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ satisfy

$$\begin{aligned}
\mathbf{R}_\alpha(w^\alpha + z) &= \underbrace{\mathbf{R}_\alpha(w^\alpha)}_{x^\alpha} + D\mathbf{R}z = x^\alpha + y,\\
\mathbf{R}_\beta(w^\beta + z_1) &= \underbrace{\mathbf{R}_\beta(w^\beta)}_{x^\beta} + D\mathbf{R}z_1 = x^\beta + y_1,\\
\mathbf{R}_\alpha(w^\alpha + z_2) &= \underbrace{\mathbf{R}_\alpha(w^\alpha)}_{x^\alpha} + D\mathbf{R}z_2 = x^\alpha + y_2,
\end{aligned}$$

Notation 4.23. The methods for estimating $\mathbf{h}^{\Pi}_{\beta\alpha\beta}$ differ slightly for different switching times. We give a proof here only for the case where the first three switching times are less than T, i.e.

$$\begin{aligned}
\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) &= T - \kappa,\\
\mathbf{t}_\alpha(\varphi^\beta + \nu_1, x^\beta + y_1) &= T - \kappa_1,\\
\mathbf{t}_\beta(\varphi^\alpha + \nu_2, x^\alpha + y_2) &= T - \kappa_2,
\end{aligned} \tag{4.38}$$

where

$$\kappa, \kappa_1, \kappa_2 > 0.$$

This case is the hardest, as it leaves the "largest chunk" of history corresponding to the perturbed initial data to deal with in the analysis.

Using this notation, ν_1, ν_2 (defined in equation (4.34)) can be written as[21]

$$\nu_1(\theta) = \begin{cases} \nu(\theta + T - \kappa) + \varphi^\alpha(\theta + T - \kappa) - \varphi^\alpha(\theta + T), & \theta \in [-2T, -T],\\ \nu(\theta + T - \kappa) + \varphi^\alpha(\theta + T - \kappa) - \underbrace{u_+(\varphi^\alpha, x^\alpha; \theta + T)}_{=\varphi^\alpha(\theta - T)}, & \theta \in (-T, -T + \kappa],\\ u_+(\varphi^\alpha + \nu, x^\alpha + y; \theta + T - \kappa) - u_+(\varphi^\alpha, x^\alpha; \theta + T), & \theta \in (-T + \kappa, 0], \end{cases} \tag{4.39}$$

and

$$\nu_2(\theta) = \begin{cases} \nu_1(\theta + T - \kappa_1) + \varphi^\beta(\theta + T - \kappa_1) - \varphi^\beta(\theta + T), & \theta \in [-2T, -T],\\ \nu_1(\theta + T - \kappa_1) + \varphi^\beta(\theta + T - \kappa_1) - \underbrace{u_-(\varphi^\beta, x^\beta; \theta + T)}_{=\varphi^\beta(\theta - T)}, & \theta \in [-T, -T + \kappa_1),\\ u_-(\varphi^\beta + \nu_1, \underbrace{x^\beta + y_1}_{=\mathbf{R}_\beta(w^\beta + z_1)}; \theta + T - \kappa_1) - \underbrace{u_-(\varphi^\beta, x^\beta; \theta + T)}_{=\varphi^\beta(\theta - T)}, & \theta \in [-T + \kappa_1, 0]. \end{cases} \tag{4.40}$$

[21]Note that ν_1, ν_2 were defined in relation (4.34) as the $\mathbb{B}^s_p$ components of $\mathbf{\Pi}_\beta, \mathbf{\Pi}_\alpha$. Those components are the same as in $\mathbf{P}_\beta, \mathbf{P}_\alpha$ (see the definition in equation (4.4)). The maps $\mathbf{P}_\beta, \mathbf{P}_\alpha$ are defined via $\boldsymbol{\psi}_+, \boldsymbol{\psi}_-$ (see the definition in equation (2.4)), and the formulas for ν_1, ν_2 are calculated by $\boldsymbol{\psi}_+, \boldsymbol{\psi}_-$.

Since u_+ satisfies equations (1.8)–(1.10), it follows that $\nu_1(\theta)$ satisfies, for $\theta \in (-T+\kappa, 0)$, the following equation:

$$\begin{aligned}
&\dot{\nu}_1(\theta) = -\mathbf{B}\nu_1(\theta) + \mathbf{A}\nu_1(\theta - 2T), && \theta \in (-T+\kappa, 0),\\
&\nu_1(\theta) = \nu(\theta + T - \kappa) + \varphi^\alpha(\theta + T - \kappa) - \underbrace{\varphi^\alpha(\theta+T)}_{=u_p(\theta) \text{ for } \theta \geq -T}, && \theta \in [-3T+\kappa, -T+\kappa],\\
&\nu_1(-T+\kappa+0) = x^\alpha + y - \varphi^\alpha(-2T+\kappa).
\end{aligned} \tag{4.41}$$

In the same manner, $\nu_2(\theta)$ satisfies, for $\theta \in [-T+\kappa_1, 0]$, the equation

$$\begin{aligned}
&\dot{\nu}_2(\theta) = -\mathbf{B}\nu_2(\theta) + \mathbf{A}\nu_2(\theta - 2T), && \theta \in (-T+\kappa_1, 0),\\
&\nu_2(\theta) = \nu_1(\theta + T - \kappa_1) + \varphi^\beta(\theta + T - \kappa_1) &&\\
&\qquad - \underbrace{\varphi^\beta(\theta+T)}_{=u_p(T+\theta) \text{ for } \theta \geq -T}, && \theta \in [-3T+\kappa_1, -T+\kappa_1],\\
&\nu_2(-T+\kappa+0) = x^\beta + y_1 - \varphi^\beta(-2T+\kappa_1).
\end{aligned} \tag{4.42}$$

Remark 4.24. Without further mention we assume in the rest of the proof that (ν, z) is small enough such that $\kappa+\kappa_1+\kappa_2 < \sigma$ and $\sigma+\kappa+\kappa_1+\kappa_2 < T$ (remember that σ is in the definition of $\mathbb{B}_p^s$ in Section 2.1).

The next (very) technical lemma establishes a number of estimates on ν_1, ν_2 and the different κ's. For clarity, we note next to each estimate at least one place in which it is used.

Recall from Notation 4.23 that we study the case of $\kappa, \kappa_1, \kappa_2 > 0$. If some of them are negative, then some adjustments to the estimates in the following lemma are needed, specifically, some intervals will be splitted differently (e.g., estimate (7)).

Recall also that according to Lemma 9.6 it is possible to divide an estimate in the $\mathbb{W}_p^1$ norm into two intervals, as long as each of the intervals is bounded from bellow (i.e., the length of the intervals cannot depend on κ, since κ goes to zero as $\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^n}$ does).

Lemma 4.25. *The following estimates take place with constants independent of κ, κ_1, κ_2.*

(1) $\|\nu_1\|_{\mathbb{L}_p(-2T,0)} \leq Const\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}$. *Used in Step II of Lemma 4.29.*

(2) $\|\nu_1\|_{\mathbb{W}_p^1(-T+\kappa,0)} \leq Const\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}$. *Used in Lemma 4.29 to estimate the $\mathbb{L}_p(-2T, 0)$ norm.*

(3) $\|\nu_1\|_{\mathbb{W}_p^s(-2T+\kappa,-T)} \leq Const\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N}$. *Used in Step II of Lemma 4.29.*

(4) $\|\nu_1\|_{\mathbb{W}_p^s(-2T,0)} \leq Const\|\nu, y\|^{\frac{1}{p}}_{\mathbb{B}_p^s \times \mathbb{R}^N}$. *Used in Lemma 4.29 to show the $\mathbb{L}_p(-2T, 0)$ estimate.*

(5) $\|\nu_2\|_{\mathbb{L}_p(-2T,0)} \leq Const\|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}$. *Used in Step II of Lemma 4.30.*

(6) $\|\nu_2\|_{\mathbb{W}_p^1(-T+\kappa_1,0)} \le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}$. *Used in Step II of Lemma 4.30, interval (2).*

(7) $\|\nu_2\|_{\mathbb{W}_p^s(-T-\sigma,-T+\kappa_1)} \le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{\frac{1}{p}}$. *Used in Lemma 4.30 to estimate the* $\mathbb{L}_p(-2T,0)$ *norm.*

(8) $\|\nu_2\|_{\mathbb{W}_p^s(-2T,-T)} \le Const\|\nu, y\|_{\mathbb{B}_p^s\times\mathbb{R}^N}^{\frac{1}{p}}$. *Used in Step II of Lemma 4.30, interval (3).*

(9) $\|y_1\|, \|y_2\| \le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}$.

(10) $\kappa, \kappa_1, \kappa_2 \le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}$.

Proof. Estimate (10) for κ is a direct consequence of Lemma 3.12 on locally Lipschitz continuity of the hit operator for (ν, y) small enough (in the $\mathbb{B}_p^s \times \mathbb{R}^N$ norm). We mention it here, since it is used to prove estimates (1)–(3).

(1) Relation (4.39) is a piecewise expression for ν_1. We estimate each interval separately.

- $\underline{\theta \in (-2T, -T)}$: The initial data φ^α is in the space $\mathbb{W}_p^1(-2T,0)$ (since $\varphi^\alpha(\theta-2T) = u_p(\theta)$ for $\theta \in [0,2T]$), and belongs to $C^\infty[-2T,-T]$ and $C^\infty[-T,0]$ (Lemma 2.13). We use this and estimate (10) for κ to bound the expression given by equation (4.39):

$$\begin{aligned}\|\nu_1\|_{\mathbb{L}_p(-2T,-T)} &= \|\nu(\cdot+T-\kappa) + \varphi^\alpha(\cdot+T-\kappa) - \varphi^\alpha(\cdot+T)\|_{\mathbb{L}_p(-2T,-T)}\\ &\le \|\nu\|_{\mathbb{L}_p(-T-\kappa,-\kappa)} + \underbrace{\left\|\int_{T-\kappa}^{T} \varphi^{\alpha\prime}(\cdot+s)ds\right\|_{\mathbb{L}_p(-2T,-T)}}_{\le Const\,\kappa\|\varphi^{\alpha\prime}\|_{\mathbb{L}_\infty(-2T,0)}}\\ &\le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}.\end{aligned}$$

.

- $\underline{\theta \in (-T, -T+\kappa)}$: By Assumption 2.12, $u_+(\varphi^\alpha, x^\alpha; \theta+T)$ equals $u_p(\theta+T)$. If we extend φ^α in a $2T$-periodic way, then $u_+(\varphi^\alpha, x^\alpha; \theta+T) = \varphi^\alpha(\theta+T)$ for $\theta \ge -T$ (since u_p is a $2T$-periodic solution). This makes relation (4.39) very similar to the previous interval:

$$\begin{aligned}\|\nu_1\|_{\mathbb{L}_p(-T,-T+\kappa)} &= \big\|\nu(\cdot+T-\kappa) + \varphi^\alpha(\cdot+T-\kappa)\\ &\quad - u_+(\varphi^\alpha, x^\alpha; \cdot+T)\big\|_{\mathbb{L}_p(-T,-T+\kappa)}\\ &= \|\nu(\cdot+T-\kappa) + \varphi^\alpha(\cdot+T-\kappa) - \varphi^\alpha(\cdot+T)\|_{\mathbb{L}_p(-T,-T+\kappa)}\\ &\le \|\nu\|_{\mathbb{L}_p(-\kappa,0)} + \left\|\int_{T-\kappa}^{T} \varphi^{\alpha\prime}(\theta+s)ds\right\|_{\mathbb{L}_p(-T,-T+\kappa)}\\ &\le Const\,\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N},\end{aligned}$$

where the constant is independent of κ (since we didn't use the length of the interval).

- $\underline{\theta \in (-T+\kappa, 0)}$. By relation (4.39) for ν_1 and integral representation (1.14) for u_+

$$\begin{aligned}\|\nu_1\|_{\mathbb{L}_p(-T+\kappa,0)} &= \|u_+(\varphi^\alpha+\nu, x^\alpha+y; \cdot+T-\kappa) - u_+(\varphi^\alpha, x^\alpha; \cdot+T)\|_{\mathbb{L}_p(-T+\kappa,0)} \\ &\le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N},\end{aligned}$$

where the last equality follows from the fact that u_+ is Lipschitz continuous in all variables[22] and estimate (10) for κ.

(2) The $\mathbb{L}_p(-T+\kappa, 0)$ norm was already estimated in (1). It is sufficient to estimate the norm of the weak derivative. By equation (4.41) for ν_1' and its initial condition

$$\begin{aligned}\|\nu_1'\|_{\mathbb{L}_p(-T+\kappa,0)} &= \|-\mathbf{B}\nu_1(\cdot) + \mathbf{A}\nu_1(\cdot-2T)\|_{\mathbb{L}_p(-T+\kappa,0)} \\ &= \big\| -\mathbf{B}\nu_1(\cdot) + \mathbf{A}\big[\nu(\cdot-T-\kappa) \\ &\quad + \varphi^\alpha(\cdot-T-\kappa) - \varphi^\alpha(\cdot-T)\big]\big\|_{\mathbb{L}_p(-T+\kappa,0)} \\ &\le \|\mathbf{B}\|\|\nu_1\|_{\mathbb{L}_p(-T+\kappa,0)} \\ &\quad + \|\mathbf{A}\|\left(\|\nu\|_{\mathbb{L}_p(-2T,-T-\kappa)} + \|\int_{-T-\kappa}^{-T}\varphi^{\alpha\prime}(\cdot+s)ds\|_{\mathbb{L}_p(-T+\kappa,0)}\right) \\ &= Const\left(\|\nu_1\|_{\mathbb{L}_p(-T+\kappa,0)} + \|\nu\|_{\mathbb{L}_p(-2T,-T-\kappa)} + \kappa\right) \\ &\le Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N},\end{aligned}$$

where the last inequality follows from estimates (1) and (10) in this lemma. Note that even though the length of the interval depends on κ (and hence on ν), the final estimate is independent of κ.

(3) The curly brackets in the next equations show to which interval the argument of a function belongs to. We use curly brackets in a similar way in the rest

[22] The only somehow difficult part in the calculations may be the term $\|\int_0^{\theta+T} e^{-\mathbf{B}(s-\theta-T)}\mathbf{A}\nu(s-2T)ds\|_{\mathbb{L}_p(-T+\kappa,0)}$. This is done by writing the definition of the $\mathbb{L}_p$ norm. By Lemma 9.9

$$\begin{aligned}&\left(\int_{-T+\kappa}^0 \left\|\int_0^{\theta+T} e^{-\mathbf{B}(s-\theta-T)}\mathbf{A}\nu(s-2T)ds\right\|_{\mathbb{R}^N}^p d\theta\right)^{\frac{1}{p}} \\ &\le Const\left(\int_{-T+\kappa}^0\left(\int_0^T \|\nu(s-2T)\|_{\mathbb{R}^N}\,ds\right)^p d\theta\right)^{\frac{1}{p}} \\ &\le Const\left(\int_{-T+\kappa}^0 \|\nu\|_{\mathbb{L}_p(-2T,0)}^p d\theta\right)^{\frac{1}{p}} \le Const\|\nu\|_{\mathbb{L}_p(-2T,0)}.\end{aligned}$$

of this chapter. By relation (4.39)

$$\begin{aligned}
\|\nu_1\|_{\mathbb{W}^s_p(-2T+\kappa,-T)} &= \|\nu(\cdot+T-\kappa)+\varphi^\alpha(\cdot+T-\kappa)-\varphi^\alpha(\cdot+T)\|_{\mathbb{W}^s_p(-2T+\kappa,-T)} \\
&\leq Const\Big(\|\nu(\cdot)\|_{\mathbb{W}^s_p(-T,-\kappa)}+\|\varphi^\alpha\underbrace{(\cdot-\kappa)}_{\in[-T,-\kappa]} \\
&\quad -\varphi^\alpha\underbrace{(\cdot)}_{\in[-T+\kappa,0]}\|_{\mathbb{W}^1_p(-T+\kappa,0)}\Big) \\
&\leq Const\left(\|\nu\|_{\mathbb{W}^s_p(-T,-\kappa)}+\kappa\|\varphi^{\alpha\prime}\|_{C(-T,0)}\right) \\
&\leq Const\|\nu,y\|_{\mathbb{B}^s_p\times\mathbb{R}^N},
\end{aligned}$$

where we used estimate (10) and the mean value theorem since φ^α is $C^\infty[-T,0]$ (Lemma 2.13). Note that even though the length of the interval depends on κ (and hence on ν), the final estimate is independent of κ.

(4) The $\mathbb{L}_p(-2T,0)$ norm was already estimated in (1), the $\mathbb{W}^s_p(-T+\kappa,0)$ in (2) (it is bounded by the $\mathbb{W}^1_p(-T+\kappa,0)$ norm), and the $\mathbb{W}^s_p(-2T+\kappa,-T)$ norms in (3).

We are left with estimating the $\mathbb{W}^s_p(-2T,-2T+\sigma),\mathbb{W}^s_p(-T-\sigma,-T+\kappa)$ norms[23].

- $\underline{(-2T,-2T+\sigma)}$: By relation (4.39)[24].

$$\begin{aligned}
\|\nu_1\|_{\mathbb{W}^s_p(-2T,-2T+\sigma)} &= \|\nu(\theta+T-\kappa)+\varphi^\alpha(\theta+T-\kappa)-\varphi^\alpha(\theta+T)\|_{\mathbb{W}^s_p(-2T,-2T+\sigma)} \\
&\leq Const\left(\|\nu\|_{\mathbb{W}^s_p(-T-\kappa,-T)}+\|\varphi^\alpha(\theta-\kappa)-\varphi^\alpha(\theta)\|_{\mathbb{W}^1_p(-T,-T+\sigma)}\right) \\
&\leq Const\Big(\|\nu\|_{\mathbb{W}^s_p(-T-\kappa,-T)}+ \\
&\quad +\|\varphi^\alpha(\theta-\kappa)-\varphi^\alpha(\theta)\|_{\mathbb{W}^1_p(-T,-T+\kappa)} \\
&\quad +\|\varphi^\alpha(\theta-\kappa)-\varphi^\alpha(\theta)\|_{\mathbb{W}^1_p(-T+\kappa,-T+\sigma)}\Big).
\end{aligned}$$

The first term is obviously bounded by $\|\nu,y\|_{\mathbb{B}^s_p\times\mathbb{R}^N}$, and the third term is bounded in the same way as in estimate (3). However, this method does not work for the second term, since φ^α has a jump in the derivative at T. We bound it as follows.

$$\begin{aligned}
\|\varphi^\alpha(\theta-\kappa)-\varphi^\alpha(\theta)\|_{\mathbb{W}^1_p(-T,-T+\kappa)} &\leq Const\Big(\|\varphi^\alpha(\theta-\kappa)-\varphi^\alpha(\theta)\|_{\mathbb{L}_p(-T,-T+\kappa)} \\
&\quad +\|\varphi^{\alpha\prime}(\theta-\kappa)-\varphi^{\alpha\prime}(\theta)\|_{\mathbb{L}_p(-T,-T+\kappa)}\Big) \\
&\leq 2\kappa^{\frac{1}{p}}(\|\varphi^\alpha\|_{\mathbb{L}_\infty(-2T,0)}+\|\varphi^{\alpha\prime}\|_{\mathbb{L}_\infty(-2T,0)}),
\end{aligned}$$

where the last inequality follows Jensen's inequality, the fact that $\varphi^{\alpha\prime}$ is bounded by Assumption 2.12(3), and estimate (10) from this lemma.

[23] Actually, we need to estimate the $\mathbb{W}^s_p(-2T,-2T+\kappa),\mathbb{W}^s_p(-T,-T+\kappa)$ norms. But since $\mathbb{W}^s_p$ norm cannot be divided to subintervals which are arbitrarily close to zero (in this case, as κ goes to zero.), we replace κ by σ to have a finite size of the interval, and then we can decompose by Lemma 9.6 into intervals which are bounded away from zero.

[24] Note that ν is indeed in $\mathbb{W}^s_p$ in the regions in the calculations, since $\kappa<\sigma$, and $\nu\in\mathbb{W}^s_p(-T-\sigma,0)$ by definition of the space $\mathbb{B}^s_p$.

- $\underline{(-T-\sigma,-T+\kappa)}$: By Assumption 2.12, $u_+(\varphi^\alpha, x^\alpha; \theta+T)$ equals $u_p(\theta+T)$. If we extend φ^α in a $2T$-periodic way, then $u_+(\varphi^\alpha, x^\alpha; \theta+T) = \varphi^\alpha(\theta+T)$ for $\theta \geq -T$ (since u_p is a $2T$-periodic solution). Then by relation (4.39)

$$\begin{aligned}\|\nu_1\|_{\mathbb{W}_p^s(-T-\sigma,-T+\kappa)} &= \|\nu(\cdot+T-\kappa)+\varphi^\alpha(\cdot+T-\kappa)\\ &\quad -\varphi^\alpha(\cdot+T)\|_{\mathbb{W}_p^s(-T-\sigma,-T+\kappa)}\\ &\leq \|\nu(\cdot+T-\kappa)\|_{\mathbb{W}_p^s(-T-\sigma,-T+\kappa)}\\ &\quad+\|\varphi^\alpha(\cdot+T-\kappa)-\varphi^\alpha(\cdot+T)\|_{\mathbb{W}_p^1(-T-\sigma,-T)}\\ &\quad+\|\varphi^\alpha(\cdot+T-\kappa)-\varphi^\alpha(\cdot+T)\|_{\mathbb{W}_p^1(-T,-T+\kappa)}.\end{aligned}$$

  The ν term is bounded naturally, the first φ^α term is bounded in the same way as in estimate (3), and the second φ^α term is bounded in the same way as in the interval $(-2T,-2T+\sigma)$.

For estimates (5)–(8), we need estimate (9) for y_1 and estimate (10) for κ_1. For the first one, we use Sobolev's inequality.

$$\|y_1\|_{\mathbb{R}^N} = \|\nu_1(0)\|_{\mathbb{R}^N} \leq Const\|\nu_1\|_{\mathbb{W}_p^1(-T+\kappa,0)} \leq Const\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N},$$

where the last inequality is by estimate (2) in this lemma. Estimate (10) for κ_1 holds by the definition of κ_1 in (4.38), Lemma 3.12 on locally Lipschitz continuity of the hit operator and estimate (1) in this lemma.

(5) Using a similar proof as in (1), with obvious replacements of κ by κ_1, and α by β, and estimate (10) for κ_1, yields

$$\|\nu_2\|_{\mathbb{L}_p(-2T,0)} \leq \|\nu_1, y_1\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N} \leq \|\nu, y\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N},$$

where the last inequality follows estimates (1) and (9) in this lemma.

(6) Using a similar proof as in (2), with the same adjustments as in (5).

(7) $\underline{\theta \in (-T-\sigma,-T+\kappa_1)}$: Extend φ^β in a $2T$-periodic way (as we did for φ^α in estimate (1)). By equation (4.40) for ν_2:

$$\begin{aligned}\|\nu_2\|_{\mathbb{W}_p^s(-T-\sigma,-T+\kappa_1)} &= \|\nu_1(\cdot+T-\kappa_1)+\varphi^\beta(\cdot+T-\kappa_1)\\ &\quad -\varphi^\beta(\cdot+T)\|_{\mathbb{W}_p^s(-T-\sigma,-T+\kappa_1)}\\ &\leq \|\nu_1\|_{\mathbb{W}_p^s(-\sigma-\kappa_1,0)}+\|\varphi^\beta(\cdot-\kappa_1)-\varphi^\beta(\cdot)\|_{\mathbb{W}_p^s(-\sigma,\kappa_1)}\\ &\leq \|\nu_1\|_{\mathbb{W}_p^s(-\sigma-\kappa_1,0)}+Const\|\varphi^\beta(\cdot-\kappa_1)-\varphi^\beta(\cdot)\|_{\mathbb{W}_p^1(-\sigma,0)}+\\ &\quad+Const\|\varphi^\beta(\cdot-\kappa_1)-\varphi^\beta(\cdot)\|_{\mathbb{W}_p^1(0,\kappa_1)},\end{aligned}$$

where the last inequality is obtained by the triangle inequality and the fact that φ^β is $\mathbb{W}_p^1$. The function ν_1 was already bounded in estimate (2). The second term (with the $\mathbb{W}_p^1(-\sigma,0)$ norm) is bounded as in estimate (3) in this lemma. For the last two terms use the triangle inequality to bound each of the functions separately.

$$\begin{aligned}\|\varphi^\beta\|_{\mathbb{W}^1_p(-\kappa_1,0)} &= \|\varphi^\beta\|_{\mathbb{L}_p(-\kappa_1,0)} + \|\varphi^{\beta\prime}\|_{\mathbb{L}_p(-\kappa_1,0)} \\ &\le \kappa_1^{\frac{1}{p}}(\|\varphi^\beta\|_{\mathbb{L}_\infty(-2T,0)} + \|\varphi^{\beta\prime}\|_{\mathbb{L}_\infty(-2T,0)}) \le Const\|\nu, y\|^{\frac{1}{p}}_{\mathbb{B}^s_p\times\mathbb{R}^N}.\end{aligned}$$

Evaluate $\|\varphi^\beta\|_{\mathbb{W}^s_p(0,\kappa_1)}$ in the same way.

(8) $\underline{\theta \in (-2T,-T)}$: By equation (4.40) for ν_2, the triangle inequality, and the fact that $\mathbb{W}^1_p$ is embedded in $\mathbb{W}^s_p$

$$\begin{aligned}\|\nu_2\|_{\mathbb{W}^s_p(-2T,-T)} &= \big\|\nu_1(\theta+T-\kappa_1) + \varphi^\beta(\theta+T-\kappa_1) - \varphi^\beta(\theta+T)\big\|_{\mathbb{W}^s_p(-2T,-T)} \\ &\le \|\nu_1(\cdot)\|_{\mathbb{W}^s_p(-T-\kappa_1,-\kappa_1)} + Const\|\varphi^\beta(\cdot-\kappa_1) - \varphi^\beta(\cdot)\|_{\mathbb{W}^1_p(-T,-T+\kappa_1)} \\ &+ \|\varphi^\beta(\cdot-\kappa_1) - \varphi^\beta(\cdot)\|_{\mathbb{W}^1_p(-T+\kappa_1,0)}.\end{aligned}$$

The norm of $\|\nu_1\|$ was bounded in estimate (4), and the second term (involving φ^β) is bounded as in the proof of (7). The third term (also involving φ^β) is estimated by $Const\|\varphi^\beta\|_{\mathbb{W}^2_p(0,T)}\kappa_1$ followed by estimate (10) for κ_1.

(9) y_1 was already bounded after (4). We bound y_2 in exactly the same way, using estimate (6).

(10) This was shown for κ before the proof of (1), and for κ_1 before the proof of (4). For κ_2 the inequality holds due to estimate (5) in this lemma and Lemma 3.12.

□

We need two final remarks before estimating $h^\Pi_{\beta\alpha\beta}$. They show that the estimates of $\mathbf{h}_\alpha, \mathbf{h}_\beta$ can be divided into two complementary parts.

Remark 4.26. The operator $\mathbf{h}_\beta$ (equation (4.28)) is defined via $\mathbf{P}^{\mathbb{B}}_\beta$ (the first component of $\mathbf{P}_\beta$, see equation (3.1)) and $\mathbf{L}$ (Definition (4.8)).

The operator $\mathbf{P}^{\mathbb{B}}_\beta(\varphi, x)$ is defined as $\boldsymbol{\psi}_+(\varphi, x, \mathbf{t}_\beta(\varphi, x))$ (Remark 3.6). The operator $\boldsymbol{\psi}_+$ is affine linear in (φ, x) by relation (2.4). Hence it can be written as

$$\begin{aligned}\boldsymbol{\psi}_+(\varphi^\alpha+\nu, x^\alpha+y, \mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y)) &= \underbrace{\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y))}_{\boldsymbol{\psi}_+^{(A)}} \\ &+ \underbrace{D_{(\varphi,x)}\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y))[\nu, y]}_{\boldsymbol{\psi}_+^{(B)}}.\end{aligned} \tag{4.43}$$

In a similar way, formula (4.14) shows that the operator $\mathbf{L}$ can also be written as a sum of two terms:

$$\mathbf{L}[\nu, y] = \underbrace{D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))\big(D_{(\varphi,x)}\mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)[\nu, y]}_{\mathbf{L}^{(A)}} + \underbrace{D_{(\varphi,x)}\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, T)[\nu, y]}_{\mathbf{L}^{(B)}}. \tag{4.44}$$

Remark 4.27. Define the following operators using the notation from Remark 4.26:

$$\begin{aligned}\mathbf{h}_\beta^{(A)}(\nu, y) &:= \boldsymbol{\psi}_+^{(A)} - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, T) - \mathbf{L}^{(A)},\\ \mathbf{h}_\beta^{(B)}(\nu, y) &:= \boldsymbol{\psi}_+^{(B)} - \mathbf{L}^{(B)}.\end{aligned} \tag{4.45}$$

Then $\mathbf{h}_\beta$ (equation (4.28)) can be written as

$$\mathbf{h}_\beta(\nu, y) = \mathbf{h}_\beta^{(A)}(\nu, y) + \mathbf{h}_\beta^{(B)}(\nu, y). \tag{4.46}$$

Similarly we could write

$$\mathbf{h}_\beta^{\Pi}(\nu, y) = \mathbf{h}_\beta^{\Pi,(A)}(\nu, y) + \mathbf{h}_\beta^{\Pi,(B)}(\nu, y). \tag{4.47}$$

Part II.II: Estimating $h^{\Pi}_{\beta\alpha\beta}$ (equation (4.36))

We estimate in Lemmas 4.28, 4.29 and 4.30 the (1), (2), (3) terms in equation (4.36) respectively. We show that each of those estimates is of order higher than or equal to γ (and at least the estimate in Lemma 4.30 is of order γ). This proves Theorem 4.20.

In what follows we use frequently estimates (1), (4), (7) and (8) from Lemma 4.25.

$$\|\nu_1\|_{\mathbb{L}_p(-2T,0)}, \|\nu_2\|_{\mathbb{L}_p(-2T,0)}, \|y_1\|_{\mathbb{R}^N}, \|y_2\|_{\mathbb{R}^N}, \kappa, \kappa_1, \kappa_2 \leq Const\|\nu, y\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N}.$$

These oft-used inequalities will be repeatedly applied in the sequel without further mention.

We remind the reader that $\mathbb{L}_p = \mathbb{L}_p(-2T, 0)$ (Section 1.3), and $\mathbb{B}_p^s = \mathbb{B}_p^s(-T-\sigma, 0)$ (Section 2.1).

Lemma 4.28 (corresponds to (1) in equation (4.36)). *The operator* $\mathbf{h}_\beta^{\Pi}$ *(equation (4.29)) satisfies*

$$\|\mathbf{h}_\beta^{\Pi}(\nu, z)\|_{\mathbb{L}_p\times\mathbb{R}^N} = O\left(\|\nu, z\|_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}}^{\min\{2-\frac{1}{p}, 1+\frac{1}{p}, 1+s\}}\right),$$

where $(\nu, z) \in \mathbb{B}_p^s \times \mathbb{R}^{N_1}$.

Proof. Following Remark 4.27, we carry out the calculations in two steps.

<u>**Step I. $\mathbf{h}_\beta^{\Pi,(A)}$.**</u> It is enough to bound $\|\mathbf{h}_\beta^{(A)}(\nu, D\mathbf{R}z)\|_{\mathbb{L}_p\cap\mathbb{W}_p^1(-\sigma,0)}$. Let $y := D\mathbf{R}z$ (as in equation (4.37)). We carry out the estimate in terms of y. This will imply the estimate in z, since $\|y\|_{\mathbb{R}^N} \leq \|D\mathbf{R}\|\|z\|_{\mathbb{R}^{N_1}}$ due to (4.37).

Adding and subtracting $D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))\kappa$ to $\mathbf{h}_\beta^{(A)}$ yields

$$\begin{aligned}\mathbf{h}_\beta^{(A)}(\nu, y) = &\underbrace{\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha+\nu, x^\alpha+y)) - \boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha)) + D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))\kappa}_{(I)}\\ &\underbrace{-D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))\kappa - D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))\big(D_{(\varphi,x)}\mathbf{t}_\beta(\varphi^\alpha, x^\alpha)\big)[\nu, y]}_{(II)},\end{aligned}$$

where $\kappa = T - \mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y)$ (see Notation 4.23). The estimate for (I) follows once noticing that in the proof of Lemma 4.5, in (4.10), the $\mathbb{L}_p$ estimate is $O\left(|\kappa|^{1+\frac{1}{p}}\right)$, and using Lemmas 4.7 and 4.25(10).

Write (II) using the distributive law:

$$(II) = D_t\boldsymbol{\psi}_+(\varphi^\alpha, x^\alpha, \mathbf{t}_\beta(\varphi^\alpha, x^\alpha))\big[-\kappa - D_{(\varphi,x)}\mathbf{t}_\beta(\varphi^\alpha, x^\alpha)[\nu, y]\big].$$

The operator $D_t\boldsymbol{\psi}_+ : \mathbb{R} \to \mathbb{L}_p \cap \mathbb{W}_p^1(-\sigma, 0)$ is linear and bounded by Lemma 4.7. The absolute value of the term inside the brackets is $O(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{\min\{2-\frac{1}{p}, 1+\frac{1}{p}\}})$ (Lemma 3.15), which implies that (II) is of the same magnitude.

<u>**Step II. $\mathbf{h}_\beta^{\Pi,(B)}$.**</u> We estimate the $\mathbb{L}_p$ and the $\mathbb{R}^{N_1}$ norms of $\|\mathbf{h}_\beta^\Pi(\nu, z)\|_{\mathbb{L}_p\times\mathbb{R}^{N_1}}$ separately. To estimate the $\mathbb{L}_p$ norm, we estimate $\|\mathbf{h}_\beta^\Pi(\nu, D\mathbf{R}z)\|_{\mathbb{L}_p}$. Write $\mathbf{h}_\beta^{(B)}$ (equation (4.45)) using $\mathbf{t}_\beta(\varphi^\alpha, x^\alpha) = T$, $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) = T - \kappa$ and the expression for $\boldsymbol{\psi}_+$ from equation (2.4).

$$\begin{aligned}\mathbf{h}_\beta^{(B)}(\nu, y)(\theta) = &\begin{cases} \nu(\theta + T - \kappa), & \theta \in [-2T, -T + \kappa), \\ e^{-\mathbf{B}(\theta+T-\kappa)}y + \int_0^{\theta+T-\kappa} e^{\mathbf{B}(s-\theta-T+\kappa)}\mathbf{A}\nu(s - 2T)ds, & \theta \in [-T + \kappa, 0], \end{cases} \\ &- \begin{cases} \nu(\theta + T), & \theta \in [-2T, -T), \\ e^{-\mathbf{B}(\theta+T)}y + \int_0^{\theta+T} e^{\mathbf{B}(s-\theta-T)}\mathbf{A}\nu(s - 2T)ds, & \theta \in [-T, 0]. \end{cases}\end{aligned} \tag{4.48}$$

Define

$$\rho(\theta) := e^{-\mathbf{B}\theta}y + \int_0^\theta e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu(s - 2T)ds. \tag{4.49}$$

Then

$$\begin{aligned}\mathbf{h}_\beta^{(B)}(\nu, y)(\theta) = &\begin{cases} \nu(\theta + T - \kappa), & \theta \in [-2T, -T + \kappa), \\ \rho(\theta + T - \kappa), & \theta \in [-T + \kappa, 0], \end{cases} \\ &- \begin{cases} \nu(\theta + T), & \theta \in [-2T, -T), \\ \rho(\theta + T), & \theta \in [-T, 0]. \end{cases}\end{aligned} \tag{4.50}$$

The function ρ satisfies the equation

$$\rho'(\theta) = -\mathbf{B}\rho + \mathbf{A}\nu(\theta - 2T), \quad \theta > 0, \tag{4.51}$$
$$\rho(0) = y. \tag{4.52}$$

The previous equation and the definition of ρ (relation (4.49)) imply that

$$\|\rho'\|_{\mathbb{L}_p(0,T)}, \|\rho\|_{\mathbb{L}_p(0,T)} \le Const(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}). \tag{4.53}$$

Since $\mathbb{W}_p^1(-T, 0)$ is embedded in $\mathbb{W}_p^s(-T, 0)$, then

$$\|\rho\|_{\mathbb{W}_p^s(0,T)} \le Const\|\rho\|_{\mathbb{W}_p^1(0,T)} \le Const(\|\nu, y\|_{\mathbb{L}_p(-2T,0)\times\mathbb{R}^N}). \tag{4.54}$$

We divide the interval $[-2T, 0]$ into three non-intersecting intervals and estimate the $\mathbb{L}_p$ norm of $\mathbf{h}_\beta^{(B)}$ in each of those intervals.

(1) $\underline{\theta \in [-2T, -T]}$: By equation (4.50)

$$\begin{aligned}\|\mathbf{h}_\beta^{(B)}(\nu, y)\|_{\mathbb{L}_p(-2T,-T)} &= \|\nu(\theta + T - \kappa) - \nu(\theta + T)\|_{\mathbb{L}_p(-2T,-T)} \\ &= \|\nu(\theta - \kappa) - \nu(\theta)\|_{\mathbb{L}_p(-T,0)}.\end{aligned}$$

The function ν is $\mathbb{W}_p^s$ in this interval since $-T - \kappa > -T - \sigma$ (Remark 4.24). Then Besov's inequality (Lemma 9.3) shows that

$$\|\mathbf{h}_\beta^{(B)}(\nu, y)\|_{\mathbb{L}_p(-2T,-T)} \leq \kappa^s \|\nu\|_{\mathbb{W}_p^s(-T-\sigma,0)} \leq Const\|\nu, y\|^{1+s}_{\mathbb{B}_p^s \times \mathbb{R}^N}. \tag{4.55}$$

(2) $\underline{\theta \in [-T, -T + \kappa]}$: By equation (4.50)

$$\begin{aligned}\|\mathbf{h}_\beta^{(B)}(\nu, y)\|_{\mathbb{L}_p(-T,-T+\kappa)} &= \|\nu(\theta + T - \kappa) - \rho(\theta + T)\|_{\mathbb{L}_p(-T,-T+\kappa)} \\ &= \|\nu(\theta - \kappa) - \rho(\theta)\|_{\mathbb{L}_p(0,\kappa)}.\end{aligned}$$

Each function is in $\mathbb{W}_p^s(-T, -T + \kappa)$. The space $\mathbb{W}_p^s(-\sigma, 0)$ is continuously embedded in the space $\mathbb{L}_{\frac{p}{1-sp}}(-\sigma, 0)$ [13, Theorem 6.5] and hence

$$\begin{aligned}\|\nu(\theta - \kappa)\|_{\mathbb{L}_p(0,\kappa)} &\leq \kappa^{\frac{1}{p} - \frac{1-sp}{p}} \|\nu\|_{\mathbb{L}_{\frac{p}{1-sp}}(-\kappa,0)} \leq \kappa^{\frac{1}{p} - \frac{1-sp}{p}} \|\nu\|_{\mathbb{L}_{\frac{p}{1-sp}}(-\sigma,0)} \\ &\leq \kappa^s \|\nu\|_{\mathbb{W}_p^s(-\sigma,0)} \leq \|\nu, y\|^{1+s}_{\mathbb{B}_p^s \times \mathbb{R}^N}.\end{aligned} \tag{4.56}$$

The estimate for ρ is similar.

(3) $\underline{\theta \in [-T + \kappa, 0]}$: By equation (4.50)

$$\|\mathbf{h}_\beta^{(B)}(\nu, y)\|_{\mathbb{L}_p(-T+\kappa,0)} = \|\rho(\theta + T - \kappa) - \rho(\theta + T)\|_{\mathbb{L}_p(-T+\kappa,0)}.$$

Then ρ is in $\mathbb{W}_p^1(-T, 0)$ (4.53). The estimate is identical to region (1), by additionally applying equation (4.54).

The $\mathbb{R}^{N_1}$ norm is bounded by $Const\|\mathbf{h}_\beta^{(B)}(\nu, y)(0)\|_{\mathbb{R}^N}$. We note that by 4.48

$$\begin{aligned}\|\mathbf{h}_\beta^{(B)}(\nu, y)(0)\|_{\mathbb{R}^N} &= \Big\|e^{-\mathbf{B}(T-\kappa)}y + \int_0^{T-\kappa} e^{\mathbf{B}(s-(T-\kappa))}\mathbf{A}\nu(s - 2T)\, ds - e^{-\mathbf{B}T}y \\ &\quad + \int_0^T e^{\mathbf{B}(s-T)}\mathbf{A}\nu(s - 2T)\, ds\Big\|_{\mathbb{R}^N} \\ &\leq \left\|e^{-\mathbf{B}T}\left(e^{\mathbf{B}\kappa} - \mathbf{I}\right)\right\| \|y\|_{\mathbb{R}^N} + \int_0^{T-\kappa} \left\|e^{\mathbf{B}(s-T)}\right\| \left\|e^{\mathbf{B}\kappa} - \mathbf{I}\right\| \|\mathbf{A}\nu(s - 2T)\|_{\mathbb{R}^N}\, ds \\ &\quad + \int_{T-\kappa}^T \left\|e^{\mathbf{B}(s-T)}\right\| \|\mathbf{A}\nu(s - 2T)\|_{\mathbb{R}^N}\, ds.\end{aligned} \tag{4.57}$$

It is easy to see that the first term in the right hand side of (4.57) is $O\left(\|\nu, y\|^2_{\mathbb{B}_p^s \times \mathbb{R}^N}\right)$. The second term in the right hand side of (4.57) is estimated as

$$\begin{aligned}\int_0^{T-\kappa} \left\|e^{\mathbf{B}(s-T)}\right\| \left\|e^{\mathbf{B}\kappa} - \mathbf{I}\right\| \|\mathbf{A}\nu(s - 2T)\|_{\mathbb{R}^N}\, ds &\leq Const \int_0^T |\kappa|\, \|\nu(s - 2T)\|_{\mathbb{R}^N}\, ds \\ &\leq Const|\kappa|\|\nu\|_{\mathbb{L}_{\frac{p}{1-sp}}} \leq Const|\kappa|\|\nu\|_{\mathbb{W}_p^s} \\ &= O\left(\|\nu, y\|^2_{\mathbb{B}_p^s \times \mathbb{R}^N}\right).\end{aligned}$$

The third term in the right hand side of (4.57) is estimated as

$$\begin{aligned}\int_{T-\kappa}^{T} \left\|e^{\mathbf{B}(s-T)}\right\| \|\mathbf{A}\nu(s-2T)\|_{\mathbb{R}^N}\, ds &\leq Const \int_{T-\kappa}^{T} \|\nu(S-2T)\|_{\mathbb{R}^N}\, ds \\ &\leq Const\, \kappa^{1-\frac{1-sp}{p}} \|\nu\|_{\mathbb{L}_{\frac{p}{1-sp}}} \leq Const\, \kappa^{1-\frac{1-sp}{p}} \|\nu\|_{\mathbb{W}_p^s} \\ &= O\left(\|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N}^{2-\frac{1-sp}{p}}\right) = O\left(\|\nu, y\|^{\min\{2-\frac{1}{p}, \frac{1}{p}+s\}}\right)\end{aligned}$$

$\square$

Lemma 4.29 (**corresponds to (2) in equation (4.36)**). *The operator* $\mathbf{h}_\alpha$ *(equation (4.28)) satisfies*

$$\|\mathbf{h}_\alpha(\nu_1, D\mathbf{R}z_1)\|_{\mathbb{L}_p \cap \mathbb{W}_p^1(-\sigma,0)} = O\left(\|\nu, z\|_{\mathbb{B}_p^s \times \mathbb{R}^{N_1}}^{\min\{2-\frac{1}{p}, \frac{1}{p}+s\}}\right), \tag{4.58}$$

where $(\nu_1, z_1) \in \mathbb{B}_p^s \times \mathbb{R}^{N_1}$ *are defined in (4.34).*

Proof. As in the previous lemma we carry out the calculations in terms of $y := D\mathbf{R}z$ and $y_1 := D\mathbf{R}z_1$ (defined in (4.37)). The final estimate is in terms of y. This implies the estimate in z, since $\|y\|_{\mathbb{R}^N} \leq \|D\mathbf{R}\|\|z\|_{\mathbb{R}^{N_1}}$ due to (4.37).

For the $\mathbb{L}_p$ norm estimate, repeat the proof of Lemma 4.28, with $(\nu_1, z_1) \in \mathbb{B}_p^s \times \mathbb{R}^{N_1}$ instead of (ν, z) and κ_1 instead of κ. Use the following estimates (Lemma 4.25(1),(4)):

$$\begin{aligned}\|\nu_1\|_{\mathbb{L}_p} &\leq Const \|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}^{\frac{1}{p}}, \\ \|\nu_1\|_{\mathbb{W}_p^1(-T,0)} &\leq Const \|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N}^{\frac{1}{p}}.\end{aligned}$$

This changes the final estimate in relation (4.55) in Step II(1), where the last inequality changes to

$$\kappa_1^s \|\nu_1\|_{\mathbb{W}_p^s(-T-\sigma,0)} \leq \|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}^{\frac{1}{p}+s},$$

and in relation (4.56) in Step II(2), where the last inequality changes to

$$\kappa_1^s \|\nu_1\|_{\mathbb{W}_p^s(-\sigma,0)} \leq \|\nu, y\|_{\mathbb{L}_p \times \mathbb{R}^N}^{\frac{1}{p}+s}.$$

Hence the $\mathbb{L}_p$ norm estimate becomes

$$\|\mathbf{h}_\alpha(\nu_1, D\mathbf{R}z_1)\|_{\mathbb{L}_p} = O\left(\|\nu, z\|_{\mathbb{B}_p^s \times \mathbb{R}^{N_1}}^{\min\{2-\frac{1}{p}, \frac{1}{p}+s\}}\right).$$

This is still of nonlinear magnitude since $\frac{1}{p} + s > 1$ (Condition 2.1). Note that the terms in the "minimum" are slightly different than in Lemma 4.28, since $\frac{1}{p} + s < 1 + \frac{1}{p}, 1 + s$.

To complete the proof, the norm of $\mathbb{W}_p^1(-\sigma, 0)$ needs to be estimated. To do so, we need to estimate the weak derivative in the interval $(-\sigma, 0)$.

Following Remark 4.27, we carry out the calculations in two steps.

Step I. $\mathbf{h}_\alpha^{(A)}$. The estimate follows under similar arguments as in Step I in Lemma 4.28 with additional usage of Lemmas 4.7 and 4.25(1),(9),(10).

Step II. $\mathbf{h}_\alpha^{(B)}$. Write $\mathbf{h}_\beta^{(B)}(\nu_1, y_1)$ (equation (4.45)) for $\theta \in [-\sigma, 0]$, using $\mathbf{t}_\beta(\varphi^\alpha, x^\alpha) = T$, $\mathbf{t}_\beta(\varphi^\alpha + \nu, x^\alpha + y) = T - \kappa_1$ and the expression for $\boldsymbol{\psi}_-$ from equation (2.4) (recall Remark 3.11).

$$\begin{aligned}\mathbf{h}_\alpha^{(B)}(\nu_1, y_1) =& e^{-\mathbf{B}(\theta+T-\kappa_1)} y_1 + \int_0^{\theta+T-\kappa_1} e^{\mathbf{B}(s-\theta-T+\kappa_1)} \mathbf{A}\nu_1(s-2T)ds - \\ & - e^{-\mathbf{B}(\theta+T)} y_1 - \int_0^{\theta+T} e^{\mathbf{B}(s-\theta-T)} \mathbf{A}\nu_1(s-2T)ds.\end{aligned}$$

Define

$$\rho_1(\theta) := e^{-\mathbf{B}\theta} y_1 + \int_0^\theta e^{\mathbf{B}(s-\theta)} \mathbf{A}\nu_1(s-2T)ds, \quad \theta > 0. \tag{4.59}$$

Then

$$\mathbf{h}_\alpha^{(B)}(\nu, y) = \rho_1(\theta + T - \kappa_1) - \rho_1(\theta + T). \tag{4.60}$$

The function $\rho_1(\theta)$ satisfies the equation

$$\rho_1'(\theta) = -\mathbf{B}\rho_1(\theta) + \mathbf{A}\nu_1(\theta - 2T), \quad \theta > 0. \tag{4.61}$$

The definition of ρ_1 (relation (4.59)) shows that it belongs to the space $\mathbb{W}_p^1(T - 2\sigma, T)$. We estimate its norm

$$\|\rho_1\|_{\mathbb{L}_p(T-2\sigma,T)} \leq Const\|y_1\|_{\mathbb{R}^N} + \left(\int_{T-2\sigma}^T \left\|\int_0^\theta e^{\mathbf{B}(s-\theta)} \mathbf{A}\nu_1(s-2T)ds\right\|_{\mathbb{R}^N}^p d\theta\right)^{\frac{1}{p}}.$$

The second term in the preceding equation is less than or equal to

$$\begin{aligned}Const\left(\int_{T-2\sigma}^T \left(\int_0^\theta \|\nu_1(s-2T)\|_{\mathbb{R}^N} ds\right)^p d\theta\right)^{\frac{1}{p}} &\leq Const\left(\int_{T-2\sigma}^T \|\nu_1\|_{\mathbb{L}_p(-2T,-T)}^p\right)^{\frac{1}{p}} \\ &\leq Const\|\nu_1\|_{\mathbb{L}_p(-2T,-T)}.\end{aligned}$$

By the previous two inequalities and the equation for ρ' (equation (4.61))

$$\begin{aligned}&\|\rho_1\|_{\mathbb{L}_p(T-2\sigma,T)} \leq Const(\|y_1\|_{\mathbb{R}^N} + \|\nu_1\|_{\mathbb{L}_p(-2T,-T)}) \leq Const(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}), \\ &\|\rho_1'\|_{\mathbb{L}_p(T-2\sigma,T)} \leq \|\mathbf{B}\|\|\rho_1\|_{\mathbb{L}_p(T-2\sigma,T)} + \|\mathbf{A}\|\|\nu_1\|_{\mathbb{L}_p(-T-2\sigma,-T)} \leq Const(\|\nu, y\|_{\mathbb{L}_p\times\mathbb{R}^N}).\end{aligned} \tag{4.62}$$

By relation (4.60)

$$
\begin{aligned}
\|\mathbf{h}_\beta^{(B)'}(\nu_1,y_1)\|_{\mathbb{L}_p(-\sigma,0)} &= \|\rho_1'(\theta+T-\kappa_1)-\rho_1'(\theta+T)\|_{\mathbb{L}_p(-\sigma,0)}\\
&= \Bigg\| -\mathbf{B}\big[\rho_1(\theta+T-\kappa_1)-\rho_1(\theta+T)\big]\\
&\qquad + \mathbf{A}\big[\underbrace{\nu_1(\theta-T-\kappa_1}_{\in[-T-\sigma-\kappa_1,-T-\kappa_1]})-\underbrace{\nu_1(\theta-T)}_{\in[-T-\sigma,-T]}\big]\Bigg\|_{\mathbb{L}_p(-\sigma,0)},
\end{aligned}
$$

where the last estimate follows from equation (4.61).

The terms involving ν_1 and ρ_1 are estimated, as in Step II,(1) in the previous lemma, by Lemma 9.3(1).
The term involving ν_1 is estimated as follows (recall that $\sigma \le \frac{T}{3}$ by its definition in Section 2.1).

$$
\|\mathbf{A}\big[\nu_1(\theta-\kappa_1-T)-\nu_1(\theta-T)\big]\|_{\mathbb{L}_p(-\sigma,0)} \le \kappa_1^s\|\mathbf{A}\|\|\nu_1\|_{\mathbb{W}_p^s(-T-2\sigma,-T)} \le Const\|\nu,y\|^{1+s}_{\mathbb{B}_p^s\times\mathbb{R}^N},
$$

where the last inequality follows from Lemma 4.25(3).

The term involving ρ_1 is estimated as follows.

$$
\big\|-\mathbf{B}\big[\rho_1(\theta+T-\kappa_1)-\rho_1(\theta+T)\big]\big\|_{\mathbb{L}_p(-\sigma,0)} \le \kappa_1^s\|\mathbf{B}\|\|\rho_1\|_{\mathbb{W}_p^s(T-2\sigma,T)} \le Const\|\nu,y\|^{1+s}_{\mathbb{L}_p\times\mathbb{R}^N},
$$

where the last inequality follows equation (4.62). □

Lemma 4.30 (**corresponds to (3) in equation (4.36)**)**.** *The operator* $\mathbf{h}_\beta$ *(equation (4.28)) satisfies*

$$
\|\mathbf{h}_\beta(\nu_2,D\mathbf{R}z_2)\|_{\mathbb{B}_p^s(-T-\sigma,0)} = O\left(\|\nu,z\|^{\min\{2-\frac{1}{p},\frac{1}{p}+s,1-s+\frac{1}{p}\}}_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}}\right), \tag{4.63}
$$

and

$$
\|\mathbf{h}_\beta(\nu_2,D\mathbf{R}z_2)\|_{\mathbb{W}_p^1(-\sigma,0)} = O\left(\|\nu,z\|^{\min\{2-\frac{1}{p},\frac{1}{p}+s,1-s+\frac{1}{p}\}}_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}}\right), \tag{4.64}
$$

where $(\nu_2,z_2)\in\mathbb{B}_p^s\times\mathbb{R}^{N_1}$ *are defined in equation (4.34).*

Proof. As in the previous lemma we carry out the calculations in terms of $y := D\mathbf{R}z$, $y_1 := D\mathbf{R}z_1$ and $y_2 := D\mathbf{R}z_2$ (defined in relation (4.37)). The final estimate is in terms of y. This implies the estimate in z, since $\|y\|_{\mathbb{R}^N} \le \|D\mathbf{R}\|\|z\|_{\mathbb{R}^{N_1}}$.

The $\mathbb{L}_p(-2T,0)$ norm estimate is similar to that in the previous lemma since $(\nu_2,z_2)\in\mathbb{B}_p^s\times\mathbb{R}^N$ (here we use Lemma 4.25(6), (7) and (8)). Hence

$$
\|\mathbf{h}_\beta(\nu_2,D\mathbf{R}z_2)\|_{\mathbb{L}_p(-2T,0)} = O(\|\nu,z\|^{\min\{2-\frac{1}{p},\frac{1}{p}+s\}}_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}}).
$$

To complete the proof, the norm of $\mathbb{W}_p^s(-T-\sigma,0)$ needs to be estimated.

Following Remark 4.27, we carry out the calculations in two steps.

Step I. $\mathbf{h}_\beta^{(A)}$. Estimate (4.63) follows under similar arguments as in Step I in Lemma 4.28 with additional usage of Lemma 4.25(5),(9),(10) and Lemma 4.7 for estimate (4.64).

Step II. $\mathbf{h}_\beta^{(B)}$. We estimate the $\mathbb{W}_p^s(-T-\sigma,0)$ norm via the $\mathbb{W}_p^1(-T-\sigma,0)$ norm. Write $\mathbf{h}_\beta^{(B)}(\nu_2,y_2)$ (relation (4.45)) for $\theta\in[-T-\sigma,0]$, using $\mathbf{t}_\beta(\varphi^\alpha,x^\alpha)=T$ (Assumption 2.12), $\mathbf{t}_\beta(\varphi^\alpha+\nu,x^\alpha+y)=T-\kappa_2$ (relation (4.38)) and the expression for $\boldsymbol{\psi}_+$ from formula (2.4) (cf. (4.48)):

$$
\begin{aligned}
&\mathbf{h}_\beta^{(B)}(\nu_2,y_2)\\
&=\begin{cases}\nu_2(\theta+T-\kappa_2), & \theta\in[-2T,-T+\kappa_2),\\ e^{-\mathbf{B}(\theta+T-\kappa_2)}y_2+\int_0^{\theta+T-\kappa_2}e^{\mathbf{B}(s-\theta-T+\kappa_2)}\mathbf{A}\nu_2(s-2T)ds, & \theta\in[-T+\kappa_2,0],\end{cases}\\
&-\begin{cases}\nu_2(\theta+T), & \theta\in[-2T,-T),\\ e^{-\mathbf{B}(\theta+T)}y_2+\int_0^{\theta+T}e^{\mathbf{B}(s-\theta-T)}\mathbf{A}\nu_2(s-2T)ds, & \theta\in[-T,0].\end{cases}
\end{aligned}
\tag{4.65}
$$

Define (cf. (4.49))

$$
\rho_2(\theta):=e^{-\mathbf{B}\theta}y_2+\int_0^\theta e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu_2(s-2T)ds. \tag{4.66}
$$

Then (cf. (4.50))

$$
\begin{aligned}
\mathbf{h}_\alpha^{(B)}(\nu_2,y_2)=&\begin{cases}\nu_2(\theta+T-\kappa_2), & \theta\in[-2T,-T+\kappa_2),\\ \rho_2(\theta+T-\kappa_2), & \theta\in[-T+\kappa_2,0],\end{cases}\\
&-\begin{cases}\nu_2(\theta+T), & \theta\in[-2T,-T),\\ \rho_2(\theta+T), & \theta\in[-T,0].\end{cases}
\end{aligned}
\tag{4.67}
$$

The function ρ_2 satisfies the equation

$$
\rho_2'(\theta)=-\mathbf{B}\rho_2(\theta)+\mathbf{A}\nu_2(\theta-2T),\quad \theta>0. \tag{4.68}
$$

The previous equation and the definition of ρ_2 show that $\rho_2\in\mathbb{W}_p^1(0,T)$ and satisfies (use Lemma 4.25(5), (9))

$$
\begin{aligned}
\|\rho_2\|_{\mathbb{L}_p(0,T)}&\le\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N},\\
\|\rho_2'\|_{\mathbb{L}_p(0,T)}&\le\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}.
\end{aligned}
\tag{4.69}
$$

The previous equation and the definition of ν_2 yield that $\mathbf{h}_\alpha^{(B)}(\nu_2,y_2)\in\mathbb{W}_p^1(-T-\sigma,0)$ (by relation (4.65), Lemma 4.25(6) and $\nu_2(0)=y_2$ by (4.34), (4.37) and (4.40)).

By (4.69)

$$
\|\rho_2\|_{\mathbb{W}_p^1(0,T)}\le Const(\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}). \tag{4.70}
$$

We divide the interval $[-2T,0]$ into three non-intersecting intervals, and estimate the $\mathbb{L}_p$ norm of $\mathbf{h}_\beta^{(B)'}$ in each of those intervals. This gives an estimate of the $\mathbb{L}_p(-T-\sigma,0)$ of $\mathbf{h}_\beta^{(B)'}$.

(1) $\underline{\theta \in [-T-\sigma, -T]}$: By relation (4.65),

$$\begin{aligned}
\|\mathbf{h}_\alpha^{(B)'}(\nu_2, y_2)\|_{\mathbb{L}_p(-T-\sigma,-T)} &= \|\nu_2'(\theta+T-\kappa_2) - \nu_2'(\theta+T)\|_{\mathbb{L}_p(-T-\sigma,-T)} \\
&= \|\nu_2'(\theta-\kappa_2) - \nu_2'(\theta)\|_{\mathbb{L}_p(-\sigma,0)} \\
&= \| -\mathbf{B}[\nu_2(\theta-\kappa_2) - \nu_2(\theta)] \\
&\quad + \mathbf{A}[\underbrace{\nu_2(\theta-2T-\kappa_2)}_{[-2T-\sigma-\kappa_2,-2T-\kappa_2]} - \underbrace{\nu_2(\theta-2T)}_{[-2T-\sigma,-2T]}]\|_{\mathbb{L}_p(-\sigma,0)},
\end{aligned}$$

where the last equality is given by equation (4.42) (for derivative of ν_2) since $\sigma \leq \frac{T}{3}$ by its definition (see Section 2.1). The last line in the previous relation equals by relation (4.42) (initial conditions of ν_2) to:

$$\begin{aligned}
&\Big\| -\mathbf{B}\big[\nu_2(\theta-\kappa_2) - \nu_2(\theta)\big] + \\
&+ \mathbf{A}\big[\underbrace{\varphi^\beta(\theta-\kappa_1-\kappa_2-T)}_{[-T-\sigma-\kappa_1-\kappa_2,-T-\kappa_1-\kappa_2]} - \underbrace{\varphi^\beta(\theta-\kappa_1-T)}_{[-T-\kappa_1-\sigma,-T-\kappa_1]} - \underbrace{\varphi^\beta(\theta-\kappa_2-T)}_{[-T-\sigma-\kappa_2,-T-\kappa_2]} + \underbrace{\varphi^\beta(\theta-T)}_{[-T-\sigma,-T]} + \\
&+ \underbrace{\nu_1(\theta-\kappa_1-\kappa_2-T)}_{[-T-\sigma-\kappa_1-\kappa_2,-T-\kappa_1-\kappa_2]} - \underbrace{\nu_1(\theta-\kappa_1-T)}_{[-T-\sigma-\kappa_1,-T-\kappa_1]}\big]\Big\|_{\mathbb{L}_p(-\sigma,0)}.
\end{aligned}$$

The ν_2 terms are bounded by Besov's inequality (Lemma 9.3).

$$\| -\mathbf{B}\big[\nu_2(\theta-\kappa_2) - \nu_2(\theta)\big]\|_{\mathbb{L}_p(-\sigma,0)} \leq Const\,\kappa^s\|\nu_2\|_{\mathbb{W}_p^s(-2\sigma,0)} \leq Const\|\nu, y\|^{1+s}_{\mathbb{L}_p\times\mathbb{R}^N},$$

where the last inequality follows from Lemma 4.25(6) once noticing that $-T+\kappa_1 < -2\sigma$ since $\sigma \leq \frac{T}{3}$ by its definition (see Section 2.1).

The φ^β terms can be written as (since their argument is between $[-2T, -T]$)

$$\int_{-\kappa_1}^0 \int_{-\kappa_2}^0 \varphi^{\beta''}(\theta - T + s + r)dsdr.$$

Hence

$$\left\| \int_{-\kappa_1}^0 \int_{-\kappa_2}^0 \varphi^{\beta''}(\theta - T + s + r)dsdr \right\|_{\mathbb{L}_p(-\sigma,0)} \leq Const\,\kappa_1\kappa_2 \leq Const\|\nu, y\|^2_{\mathbb{L}_p\times\mathbb{R}^N}.$$

The ν_1 terms are bounded by Besov's inequality (Lemma 9.3) and Lemma 4.25, (3):

$$\begin{aligned}
\|\nu_1(\theta-\kappa_1-\kappa_2-T) - \nu_1(\theta-\kappa_2-T)\|_{\mathbb{L}_p(-\sigma,0)} &\leq \kappa_1^s\|\nu_1\|_{\mathbb{W}_p^s(-T-\kappa_2-\sigma,-T-\kappa_2)} \\
&\leq Const\|\nu, y\|^{1+s}_{\mathbb{B}_p^s\times\mathbb{R}^N}.
\end{aligned}$$

(2) $\underline{\theta \in [-T, -T+\kappa_2]}$: By equation (4.67).

$$\begin{aligned}
\|\mathbf{h}_\alpha'^{(B)}(\nu_2, y_2)\|_{\mathbb{L}_p(-T,-T+\kappa_2)} &= \|\nu_2'(\theta+T-\kappa_2) - \rho'(\theta+T)\|_{\mathbb{L}_p(-T,-T+\kappa_2)} \\
&= \|\nu_2'(\theta-\kappa_2) - \rho_2'(\theta)\|_{\mathbb{L}_p(0,\kappa_2)} \\
&= \| -\mathbf{B}\nu_2(\theta-\kappa_2) + \mathbf{B}\rho_2(\theta) \\
&\quad + \mathbf{A}\big[\nu_2(\theta-2T-\kappa_2) - \nu_2(\theta-2T)\big]\|_{\mathbb{L}_p(0,\kappa_2)},
\end{aligned} \tag{4.71}$$

where the last equality follows from equation (4.42) (derivative of ν_2), and equation (4.68) (derivative of ρ_2).

The first two terms in the last line of the previous equation are bounded in the same way as in interval (2) in Step II in Lemma 4.28. The term involving ν_2 is bounded as follows:

$$\|-\mathbf{B}\nu_2(\theta-\kappa_2)\|_{\mathbb{L}_p[0,\kappa_2]} \le Const\,\kappa_2^{\frac{1}{p}}\|\nu_2(\theta-\kappa_2)\|_{\mathbb{W}_p^1[0,\kappa_2]} \le Const\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{1+\frac{1}{p}},$$

where the last inequality follows from Lemma 4.25(6), (10). The term involving ρ_2 is bounded as follows:

$$\|\rho_2\|_{\mathbb{L}_p[0,\kappa_2]} \le Const\,\kappa_2^{\frac{1}{p}}\|\rho_2\|_{\mathbb{W}_p^1[0,\kappa_2]} \le Const\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{1+\frac{1}{p}},$$

where the last inequality follows from inequality (4.69).

For the last two terms in the last line in equation (4.71) we write

$$\begin{aligned}&\big\|\mathbf{A}[\nu_2(\cdot-2T-\kappa_2)-\nu_2(\cdot-2T)]\big\|_{\mathbb{L}_p(0,\kappa_2)}\\ &\le Const\big(\underbrace{\|\nu_2\|_{\mathbb{L}_p(-2T-\kappa_2,-2T)}}_{(I)}+\underbrace{\|\nu_2\|_{\mathbb{L}_p(-2T,-2T+\kappa_2)}}_{(II)}\big).\end{aligned}\tag{4.72}$$

We bound (I) in (4.72) in a similar method as in (4.56) in Lemma 4.28, using (4.42) and Lemma 4.25(3),(10).

$$\begin{aligned}(I) &\le \|\nu_1\|_{\mathbb{L}_p(-T-\kappa_1-\kappa_2,-T-\kappa_2)}+\|\varphi^\beta(\theta-T-\kappa_1)-\varphi^\beta(\theta-T)\|_{\mathbb{L}_p(-\kappa_2,0)}\\ &\le Const\left(\kappa_2^s\|\nu_1\|_{\mathbb{W}_p^s(-T-\sigma,-T)}+\kappa_2^s\|\varphi^\beta(\theta-T-\kappa_1)-\varphi^\beta(\theta-T)\|_{\mathbb{W}_p^1(-\sigma,0)}\right)\\ &\le Const\left(\kappa_2^s\|\nu_1\|_{\mathbb{W}_p^s(-T-\sigma,-T)}+\kappa_2^s\kappa_1\|\varphi^\beta(\theta-T)\|_{\mathbb{W}_p^2(-T,0)}\right)\le Const\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{1+s}.\end{aligned}$$

We bound (II) in (4.72) using Lemma 9.3 and Lemma 4.25(8),(10).

$$(II)\le\kappa_2^s\|\nu_2\|_{\mathbb{W}_p^s(-2T,-2T+\sigma)}\le Const\|\nu,y\|_{\mathbb{L}_p\times\mathbb{R}^N}^{s+\frac{1}{p}}.$$

(3) $\underline{\theta\in[-T+\kappa_2,0]}$: By equation (4.67) and equation (4.68).

$$\begin{aligned}\|\mathbf{h}_\alpha'^{(B)}(\nu_2,y_2)\|_{\mathbb{L}_p(-T+\kappa_2,0)} &= \|\rho_2'(\theta+T-\kappa_2)-\rho_2'(\theta+T)\|_{\mathbb{L}_p(-T+\kappa_2,0)}\\ &= \|\rho_2'(\theta-\kappa_2)-\rho_2'(\theta)\|_{\mathbb{L}_p(\kappa_2,T)}\\ &= \Bigg\|-\mathbf{B}\Big[\rho_2(\theta-\kappa_2)-\rho_2(\theta)\Big)\\ &\qquad+\mathbf{A}\big[\nu_2(\theta-2T-\kappa_2)-\nu_2(\theta-2T)\big]\Bigg\|_{\mathbb{L}_p(\kappa_2,T)}\\ &\le \|\mathbf{B}\big(\rho_2(\theta-\kappa_2)-\rho_2(\theta)\Big]\|_{\mathbb{L}_p(\kappa_2,T)}\\ &\qquad+\|\mathbf{A}\big[\underbrace{\nu_2(\theta-2T-\kappa_2)}_{\in[-2T,-T-\kappa_2]}-\underbrace{\nu_2(\theta-2T)}_{\in[-2T+\kappa_2,-T]}\big]\|_{\mathbb{L}_p(\kappa_2,T)}.\end{aligned}$$

Estimate both terms in the preceding equation by Besov's inequality (Lemma 9.3). The first term is estimated as

$$\| -\mathbf{B}\big[\rho_2(\theta-\kappa_2)-\rho_2(\theta)\big]\|_{\mathbb{L}_p(\kappa_2,T)} \le \|\mathbf{B}\|\kappa_2^s\|\rho_2\|_{W_p^s(0,T)} \le Const\|\nu,y\|^{1+s}_{\mathbb{L}_p\times\mathbb{R}^N},$$

where the last inequality follows from equation (4.70).

The second term is estimated as

$$\begin{aligned}\|\mathbf{A}\big[\nu_2(\theta-2T-\kappa_2)-\nu_2(\theta-2T)\big]\|_{\mathbb{L}_p(\kappa_2,T)} &\le Const\,\kappa_2^s\|\nu_2\|_{\mathbb{W}_p^s(-2T,-T)}\\ &\le Const\|\nu,y\|^{\frac{1}{p}+s}_{\mathbb{B}_p^s\times\mathbb{R}^N},\end{aligned}$$

where the last inequality follows Lemma 4.25(8).

We note that estimate (3) in Step II proves relation (4.64). □

□

(end of the proof of Theorem 4.20)

4.5 Proof of Theorem 4.18

We first show that $(\varphi^\alpha, w^\alpha)$ is an asymptotically stable point of the operator $\mathbf{\Pi}_{\beta\alpha\beta}$ (in a sense defined in the following lemma).

Lemma 4.31. *If the spectral radius $r(\mathbf{L}_\Pi)$ of $\mathbf{L}_\Pi$ (see (4.18)) satisfies*

$$r(\mathbf{L}_\Pi) < 1,$$

then $(\varphi^\alpha, w^\alpha)$ is an asymptotically stable point of $\mathbf{\Pi}_{\beta\alpha\beta}$ in the sense that there is some $\delta > 0$ and some constants $E > 0$, C with $(r(\mathbf{L}_\Pi))^3 < C < 1$ such that if $(\nu, z) \in \mathbb{B}_p^s \times \mathbb{R}^{N_1}$ satisfies

$$\|\nu, z\|_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}} \le \delta,$$

then[25] for every odd $n \in \mathbb{Z}$, $n \ge 1$,

$$\left\|\Big(\mathbf{\Pi}_{\beta\alpha\beta}\circ\mathbf{\Pi}_{\alpha\beta\alpha}\Big)^{\frac{n-1}{2}}\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha+\nu, w^\alpha+z)-(\varphi^\beta, w^\beta)\right\|_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}} \le EC^n\|\nu,z\|_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}}, \tag{4.73}$$

and for every even $n \in \mathbb{Z}$, $n \ge 2$,

$$\left\|\Big(\mathbf{\Pi}_{\alpha\beta\alpha}\circ\mathbf{\Pi}_{\beta\alpha\beta}\Big)^{\frac{n}{2}}(\varphi^\alpha+\nu, w^\alpha+z)-(\varphi^\alpha, w^\alpha)\right\|_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}} \le EC^n\|\nu,z\|_{\mathbb{B}_p^s\times\mathbb{R}^{N_1}}. \tag{4.74}$$

[25] The cumbersome technical phrasing of the result basically says that the chains of the type $\mathbf{\Pi}_{\beta\alpha\beta}\mathbf{\Pi}_{\alpha\beta\alpha}\mathbf{\Pi}_{\beta\alpha\beta}\dots$ et cetera are stable.

The same result is true also for $\mathbf{\Pi}_{\alpha\beta\alpha}$.

Proof. We prove the result only for $\mathbf{\Pi}_{\beta\alpha\beta}$ (see Remark 3.11).

By Lemma 9.8, there exists an equivalent norm $\|\cdot\|^*_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}$ such that $\|\mathbf{L}_\Pi\|^* < 1$ and
$\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq \|\nu, z\|^*_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}$ for every $(\nu, z) \in \mathbb{B}^s_p \times \mathbb{R}^{N_1}$. It is straightforward that if the result is true for $\|\cdot\|^*$, then it is also true for $\|\cdot\|$. Hence we assume, without loss of generality[26], that $\|\mathbf{L}_\Pi\| < 1$ (naturally, nonlinear terms remain nonlinear also in the equivalent norm).

Step I. Following Notation 3.10

$$(\varphi^\beta, w^\beta) = \mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha, w^\alpha).$$

Then Theorem 4.20 shows that

$$\|\mathbf{\Pi}_{\alpha\beta\alpha}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\beta, w^\beta)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} = \|(\mathbf{L}_\Pi)^3[\nu, z] + \mathbf{h}^\Pi_{\beta\alpha\beta}(\nu, z)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}, \tag{4.75}$$

where $(\nu, z) \in \mathbb{B}^s_p \times \mathbb{R}^{N_1}$, $\mathbf{L}_\Pi$ is linear and $\mathbf{h}^\Pi_{\beta\alpha\beta}(\nu, z)$ is $o\left(\|\nu, z\|^{\min\{2-\frac{1}{p}, \frac{1}{p}+s, 1-s+\frac{1}{p}\}}_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}\right)$.

Choose $G > 0$ such that $\|\mathbf{L}_\Pi\|^3 + G < 1$. Due to Theorem 4.20 there exists some $\delta > 0$ such that if

$$\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq \delta,$$

then

$$\|h^\Pi_{\beta\alpha\beta}(\nu, z)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq G\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}$$

Let $\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq \delta$. Then the right-hand side of equality (4.75) is estimated as

$$\begin{aligned}
&\|(\mathbf{L}_\Pi)^3[\nu, z] + \mathbf{h}^\Pi_{\beta\alpha\beta}(\nu, z)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \\
&\leq \|(\mathbf{L}_\Pi)^3[\nu, z]\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} + \|\mathbf{h}^\Pi_{\beta\alpha\beta}(\nu, z)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \\
&\leq (\|\mathbf{L}_\Pi\|^3 + G)\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}}.
\end{aligned}$$

Step II. We prove the result only for even n. The proof for odd n is similar. The proof is by induction.

First show for $n = 2$. Choose $\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq \delta$, where δ is such that Step I holds both[27] for $\mathbf{\Pi}_{\beta\alpha\beta}$ and $\mathbf{\Pi}_{\alpha\beta\alpha}$ at the points $(\varphi^\alpha, w^\alpha)$ and (φ^β, w^β), respectively. Then

$$\begin{aligned}
&\|\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\beta, w^\beta)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq C\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq \delta, \\
&\|\mathbf{\Pi}_{\alpha\beta\alpha}(\varphi^\beta + \nu, w^\beta + z) - (\varphi^\alpha, w^\alpha)\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq C\|\nu, z\|_{\mathbb{B}^s_p\times\mathbb{R}^{N_1}} \leq \delta.
\end{aligned} \tag{4.76}$$

[26] We show that if $\|\mathbf{L}_\Pi\| < 1$, then the result holds for $E = 1$. But this is true only for the equivalent norm $\|\cdot\|^*$. For the original norm $\|\cdot\|$ an extra constant is needed (which comes from the equivalence of the two norms).

[27] The result from Step I holds also for $\mathbf{\Pi}_{\alpha\beta\alpha}$ (see Remark 3.11).

where $C < 1$. Consider estimate (4.74) with $n = 2$:

$$\begin{aligned}
&\|\mathbf{\Pi}_{\alpha\beta\alpha} \circ \mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\alpha, w^\alpha)\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}} \\
&= \|\mathbf{\Pi}_{\alpha\beta\alpha}\big((\varphi^\beta, w^\beta) + \underbrace{\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\beta, w^\beta)}_{\leq\delta \text{ in the } \mathbb{B}^s_p\times\mathbb{R}^{N_1} \text{ norm by (4.76)}}\big) - (\varphi^\alpha, w^\alpha)\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}} \\
&\leq C\|\mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\beta, w^\beta)\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}} \\
&\leq C^2\|\nu, z\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}} \leq \delta.
\end{aligned} \tag{4.77}$$

Assume that the lemma is true for some even n. Then for $n + 2$

$$\begin{aligned}
&\left\|(\mathbf{\Pi}_{\alpha\beta\alpha} \circ \mathbf{\Pi}_{\beta\alpha\beta})\left(\left(\mathbf{\Pi}_{\alpha\beta\alpha} \circ \mathbf{\Pi}_{\beta\alpha\beta}\right)^{\frac{n}{2}}(\varphi^\alpha + \nu, w^\alpha + z)\right) - (\varphi^\alpha, w^\alpha)\right\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}} \\
&= \Bigg\|(\mathbf{\Pi}_{\alpha\beta\alpha} \circ \mathbf{\Pi}_{\beta\alpha\beta})\Bigg((\varphi^\alpha, w^\alpha) + \underbrace{\left(\mathbf{\Pi}_{\alpha\beta\alpha} \circ \mathbf{\Pi}_{\beta\alpha\beta}\right)^{\frac{n}{2}}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\alpha, w^\alpha)}_{\leq\delta \text{ by induction assumption in the } \mathbb{B}^s_p\times\mathbb{R}^{N_1} \text{ norm}}\Bigg) \\
&\quad - (\varphi^\alpha, w^\alpha)\Bigg\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}}.
\end{aligned}$$

By estimate (4.76) the preceding norm is less than or equal to

$$C^2\|\left(\mathbf{\Pi}_{\alpha\beta\alpha} \circ \mathbf{\Pi}_{\beta\alpha\beta}\right)^{\frac{n}{2}}(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\alpha, w^\alpha)\|_{\mathbb{B}^s_p \times \mathbb{R}^{N_1}} \leq C^{n+2}\|\nu, z\|_{\mathbb{B}^s_p \times \mathbb{R}^{N}},$$

where the last inequality follows by the induction assumption. This concludes the proof for even n. □

Proof of Theorem 4.18.
Step I. Stability. We begin by showing that $(\varphi^\alpha, w^\alpha)$ is an asymptotically stable fixed point of $\mathbf{\Pi}$ (understood in a similar sense as in Definition 3.9). Then we use it to prove the asymptotic stability of $(\varphi^\alpha, x^\alpha)$ as a fixed point of the Poincaré map $\mathbf{P}$.

Step I.I. Stability of $\mathbf{\Pi}$. Enumerate the iterations of the projected hit maps as follows.

$$\begin{aligned}
I_0 &:= (\varphi^\alpha + \nu, w^\alpha + z), \\
I_1 &:= \mathbf{\Pi}_\beta(\varphi^\alpha + \nu, w^\alpha + z), \\
I_2 &:= \mathbf{\Pi}_{\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z), \\
I_3 &:= \mathbf{\Pi}_{\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z),
\end{aligned}$$

and so on for every $n \in \mathbb{N} \cup \{0\}$.

Choose $\varepsilon > 0$. Choose constants $E \geq 1, C < 1$ that satisfy Lemma 4.31 with

$\delta_1 \leq \frac{\varepsilon}{E}$.

From relation 4.5 (definition of $\mathbf{\Pi}$) it is straightforward that

$$\|\mathbf{\Pi}(\varphi, w)\|_{\mathbb{B}_p^s \times \mathbb{R}^{N_1}} \leq Const\|\mathbf{P}(\varphi, \mathbf{R}_\alpha w)\|_{\mathbb{B}_p^s \times \mathbb{R}^N}$$

for every $(\varphi, w) \in \mathbb{B}_p^s \times \mathbb{R}^N$. This and Lemma 3.16 show that there exists δ, $0 < \delta < \delta_1$ such that if

$$\|\nu, z\| \leq \delta,$$

then

$$\|I_1 - (\varphi^\beta, w^\beta)\|, \|I_2 - (\varphi^\alpha, w^\alpha)\| \leq \delta_1.$$

Let $\|\nu, z\|$ be less than δ. By Lemma 4.31, $(\varphi^\alpha, w^\alpha)$ is an asymptotically stable fixed point of $\Pi_{\beta\alpha\beta}(I_0)$ and $\Pi_{\beta\alpha\beta}(I_2)$ (in the sense defined in Lemma 4.31), and (φ^β, w^β) is an asymptotically stable fixed point of $\Pi_{\alpha\beta\alpha}(I_1)$.

Choose $n \in \mathbb{N}$, $n \geq 3$. All the norms in the following paragraph are the $\mathbb{B}_p^s \times \mathbb{R}^{N_1}$ norm. We omit them for brevity.

1. Let $n \ mod \ 3 = 0$. If n is odd, then $\|I_n - (\varphi^\beta, w^\beta)\| \leq EC^n\|\nu, z\| \leq EC^n\delta_1 \leq C^n\varepsilon < \varepsilon$. If n is even, then $\|I_n - (\varphi^\alpha, w^\alpha)\| \leq EC^n\|\nu, z\| < \varepsilon$.

2. Let $n \ mod \ 3 = 1$. If n is odd, then $\|I_n - (\varphi^\alpha, w^\alpha)\| \leq EC^n\|I_1 - (\varphi^\beta, w^\beta)\| < \varepsilon$. If n is even, then $\|I_n - (\varphi^\beta, w^\beta)\| \leq EC^n\|I_1 - (\varphi^\beta, w^\beta)\| < \varepsilon$.

3. Let $n \ mod \ 3 = 2$. If n is odd, then $\|I_n - (\varphi^\beta, w^\beta)\| \leq EC^n\|I_2 - (\varphi^\alpha, w^\alpha)\| < \varepsilon$. If n is even, then $\|I_n - (\varphi^\alpha, w^\alpha)\| \leq EC^n\|I_2 - (\varphi^\alpha, w^\alpha)\| < \varepsilon$.

This shows that $(\varphi^\alpha, w^\alpha)$ is an asymptotically stable fixed point of $\mathbf{\Pi}$.

Step I.II. Stability of the Poincaré map P. Define the operator $\mathbf{R}^P$: $\mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^N$ as

$$\mathbf{R}^P(\varphi, w) = (\varphi, \mathbf{R}_\alpha w).$$

This operator is bounded, and satisfies

$$\mathbf{R}^P\mathbf{E} = \mathbf{I}. \tag{4.78}$$

By the definition of $\mathbf{\Pi}$ in equation (4.5), the Poincaré map $\mathbf{P}$ can be written as

$$\begin{aligned} \mathbf{P}(\varphi^\alpha + \nu, x^\alpha + y) &= \mathbf{R}^P\mathbf{\Pi}\mathbf{E}(\varphi^\alpha + \nu, x^\alpha + y), \\ \mathbf{P}^n &= \mathbf{R}^P\mathbf{\Pi}^n\mathbf{E}(\varphi^\alpha + \nu, x^\alpha + y), \end{aligned} \tag{4.79}$$

whenever $(\varphi^\alpha + \nu, x^\alpha + y) \in \mathbb{T}_\alpha$. Choose $\varepsilon > 0$. Choose $\delta > 0$ such that the Step I.I holds with $\frac{\varepsilon}{\|D\mathbf{R}^P\|}$ instead of ε.

Note that $\mathbf{R}^P$ (like $\mathbf{R}_\alpha$ from (4.2)) is an affine linear operator. Hence, $\mathbf{R}^P(\varphi_1, w_1) - \mathbf{R}^P(\varphi_2, w_2) = D\mathbf{R}^P(\varphi_1 - \varphi_2, w_1 - w_2)$. By this and the stability of the fixed point $(\varphi^\alpha, w^\alpha)$ of $\mathbf{\Pi}$ from Step I.I, if $\|\nu, y\| \le \delta$, then (we omit the $\mathbb{B}_p^s \times \mathbb{R}^N$ norm for brevity)

$$
\begin{aligned}
&\|\mathbf{P}^n(\varphi^\alpha + \nu, x^\alpha + y) - (\varphi^\alpha, x^\alpha)\| \\
&= \|\mathbf{R}^P \mathbf{\Pi}^n \mathbf{E}[\varphi^\alpha + \nu, x^\alpha + y] - \mathbf{R}^P(\varphi^\alpha, w^\alpha)\| \\
&= \|D\mathbf{R}^P[\mathbf{\Pi}^n(\varphi^\alpha + \nu, w^\alpha + \mathbf{E}^{\mathbb{R}} y) - (\varphi^\alpha, w^\alpha]\| \\
&\le \|D\mathbf{R}^P\| \|\mathbf{\Pi}^n(\varphi^\alpha + \nu, w^\alpha + \mathbf{E}^{\mathbb{R}} y) - (\varphi^\alpha, w^\alpha)\| \le \|D\mathbf{R}^P\| \frac{\varepsilon}{\|D\mathbf{R}^P\|} \le \varepsilon,
\end{aligned}
$$

where the last line follows from that fact that $\|\nu, \mathbf{E}^{\mathbb{R}} y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \le \|\nu, y\|_{\mathbb{B}_p^s \times \mathbb{R}^N} \le \delta$.

Step II. Instability. Consider the operator

$$
\mathbf{\Pi}_{\alpha\beta\alpha\beta\alpha\beta} : \mathbb{B}_p^s \times \mathbb{R}^{N_1} \to \mathbb{B}_p^s \times \mathbb{R}^{N_1}.
$$

It is a composition of the operator $\mathbf{\Pi}_{\alpha\beta\alpha}$ and $\mathbf{\Pi}_{\beta\alpha\beta}$. Using Theorem 4.20 on both operators, we can write their composition at the point $(\varphi^\alpha + \nu, w^\alpha + z)$ as

$$
\mathbf{\Pi}_{\alpha\beta\alpha\beta\alpha\beta}(\varphi^\alpha + \nu, w^\alpha + z) = \mathbf{\Pi}_{\alpha\beta\alpha\beta\alpha\beta}(\varphi^\alpha, w^\alpha) + \mathbf{L}_\Pi^6[\nu, z] + \tilde{h}(\nu, z),
$$

where $\|\tilde{h}(\nu, z)\|_{\mathbb{B}_p^s \times \mathbb{R}^{N_1}} = O(\|\nu, z\|^{\gamma^2})$, where $\gamma = min\{2 - \frac{1}{p}, \frac{1}{p} + s, 1 - s + \frac{1}{p}\} > 1$.

Since the spectral radius of $\mathbf{L}_\Pi$ is bigger than one, then the spectral radius of $\mathbf{L}_\Pi^6$ is also[28] bigger than 1. By [28, Theorem 5.1.5], since the nonlinear part $\tilde{h}$ is of big-O order $\gamma^2 > 1$, then $(\varphi^\alpha, w^\alpha)$ is an unstable fixed point of $\mathbf{\Pi}_{\alpha\beta\alpha\beta\alpha\beta}$. This shows that $\mathbf{\Pi} = \mathbf{\Pi}_{\alpha\beta}$ is also unstable.

Finally, the instability of $\mathbf{P}$ follows since by (4.5) and relations (4.78) and (4.79):

$$
\|\mathbf{\Pi}^n(\varphi^\alpha + \nu, w^\alpha + z) - (\varphi^\alpha, w^\alpha)\| \le \|\mathbf{E}\| \|\mathbf{P}^n(\varphi^\alpha + \nu, x^\alpha + \mathbf{R}_\alpha z) - (\varphi^\alpha, x^\alpha)\|.
$$

□

[28] This follows from Gelfand's formula for spectral radius, which says that $r(\mathbf{L}_\Pi) = \lim_{n\to\infty} \|\mathbf{L}_\Pi^n\|^{\frac{1}{n}}$. Applying it on $\mathbf{L}_\Pi^6$ shows that $r(\mathbf{L}_\Pi^6) = \lim_{n\to\infty} \|(\mathbf{L}_\Pi)^{6n}\|^{\frac{1}{n}}$. Define $m = 6n$ transfers the last terms to $\lim_{m\to\infty} \|(\mathbf{L}_\Pi)^m\|^{\frac{6}{m}} = r(\mathbf{L}_\Pi)^6 > 1$.

5. Spectral analysis of the Poincaré map

In the previous section we showed that the stability of the Poincaré map depends on the spectral radius of its linearization (the "formal" linearization of its projection, to be exact). Calculating this spectral radius is not easy, since it is an operator on an infinite-dimensional space. In this section we simplify this calculation by reducing it to an equivalent finite-dimensional problem.

In this section we study the operator $\mathbf{L}_\Pi$ defined in equation (4.18) in the previous section. This is important since the stability of of the periodic solution u_p as a fixed point of the Poincaré map depends on the spectral radius of $\mathbf{L}_\Pi$ (Theorem 4.18).

In Section 5.1 we show (Lemma 5.6) that the operator $\mathbf{L}_\Pi$ can be written as a sum of two operators: a Volterra-type operator, and a finite-rank operator (Definition 5.5). In Lemma 5.7 we give the basis for the range of the finite-rank operator.

In Section 5.2 we note (Lemma 5.8) that $\mathbf{L}_\Pi$ is a power-compact operator. This means that its spectral radius depends only on its eigenvalues. In Theorem 5.13 we reduce the problem of calculating eigenvalues of $\mathbb{L}_p$ to an equivalent finite-dimensional problem.

Finally, in Section 5.3, we give a formula for the matrix of the finite-dimensional problem.

5.1 Writing $\mathbf{L}_\Pi$ as a matrix-valued operator

The main component of $\mathbf{L}_\Pi$ is the operator $\mathbf{L}$ (Definition 4.14). This operator was calculated, formally, as a linearization at the initial data that generates the periodic solution, $(\varphi^\alpha, x^\alpha)$ (Definition 4.8). These initial data are symmetric around $t = T$ (Assumption 2.12). This symmetry gives the structure of $\mathbf{L}$ two interesting properties.

1. In each of the expressions that composes it, $\mathbf{L}$ uses either $\varphi(\theta)|_{\theta\in[-2T,-T]}$ or $\varphi(\theta)|_{\theta\in[-T,0]}$ (or none of them) - but never both!

2. $\mathbf{L}$ is written as a piecewise function, which is defined separately on the intervals $[-2T, -T]$ and $[-T, 0]$.

See equation (4.15) for the structure of $\mathbf{L}$.

These properties motivate the transformation from the space $\mathbb{B}_p^s$ to the direct product space

$$\big(\mathbb{L}_p(-T, 0) \cap \mathbb{W}_p^s(-\sigma, 0)\big) \times \mathbb{W}_p^s(-T, 0).$$

We introduce notation for the product space in the previous line.

Notation 5.1. Denote

$$\mathbb{B}_p^{sw} := \mathbb{L}_p(-T,0) \cap \mathbb{W}_p^s(-\sigma,0), \tag{5.1}$$

where the w stands for a "working space".

Definition 5.2. Let $\nu \in \mathbb{B}_p^s$. Define a linear operator

$$\mathbf{U} : \mathbb{B}_p^s \to \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0)$$

as

$$\mathbf{U}[\nu] = \begin{pmatrix} \nu(\theta - T) \\ \nu(\theta) \end{pmatrix},$$

where $\theta \in [-T,0]$.

The **inverse of U**

$$\mathbf{U}^{-1} : \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \to \mathbb{B}_p^s$$

is defined as

$$\mathbf{U}^{-1}\begin{pmatrix} \nu_1(\theta) \\ \nu_2(\theta) \end{pmatrix} = \begin{cases} \nu_1(\theta+T), & \theta \in [-2T,-T), \\ \nu_2(\theta), & \theta \in [-T,0], \end{cases}$$

where $\nu_1 \in \mathbb{B}_p^{sw}$, $\nu_2 \in \mathbb{W}_p^s(-T,0)$.

The next definition transforms $\mathbf{L}$ and $\mathbf{L}_\Pi$ to the direct product space.

Definition 5.3. The linear operator

$$\tilde{\mathbf{L}} : \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^N \to \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0), \tag{5.2}$$

is defined as

$$\tilde{\mathbf{L}}[\nu_1,\nu_2,y] := \mathbf{U}\Big[\mathbf{L}\big[\mathbf{U}^{-1}[\nu_1,\nu_2],y\big]\Big].$$

The linear operator

$$\widetilde{\mathbf{L}_\Pi} : \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1} \to \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1},$$

is defined as (cf. Definition 4.14))

$$\widetilde{\mathbf{L}_\Pi}(\nu_1,\nu_2,z) = \Big(\tilde{\mathbf{L}}[\nu_1,\nu_2,D\mathbf{R}z], \mathbf{E}^{\mathbb{R}}\big[\tilde{\mathbf{L}}[\nu_1,\nu_2,D\mathbf{R}z](0)\big]\Big). \tag{5.3}$$

Lemma 5.4. *A complex number $\lambda \in \sigma(\mathbf{L}_\Pi)$ if and only if $\lambda \in \sigma(\widetilde{\mathbf{L}_\Pi})$.*

Proof. Define the operator $\widetilde{\mathbf{U}} : \mathbb{B}^s_p \times \mathbb{R}^N \to \mathbb{B}^{sw}_p \times \mathbb{W}^s_p(-T,0) \times \mathbb{R}^N$ as follows:

$$\widetilde{\mathbf{U}} \begin{bmatrix} \nu \\ y \end{bmatrix} = \begin{pmatrix} \mathbf{U}(\nu) \\ y \end{pmatrix}.$$

It is straightforward that $\widetilde{\mathbf{U}}$ is linear and invertible matrix. Let $\lambda \in \mathbb{C}$, and consider the operator $(\lambda\mathbf{I} - \mathbf{L})$ and $(\lambda\mathbf{I} - \widetilde{\mathbf{L}})$ (note that $\mathbf{I}$ denotes the appropriate identity in each case). Using $\widetilde{\mathbf{U}}$ we have the following connections:

$$\begin{aligned}(\lambda\mathbf{I} - \widetilde{\mathbf{L}}) &= \widetilde{\mathbf{U}}(\lambda\mathbf{I} - \mathbf{L})\widetilde{\mathbf{U}}^{-1}, \\ (\lambda\mathbf{I} - \mathbf{L}) &= \widetilde{\mathbf{U}}^{-1}(\lambda\mathbf{I} - \widetilde{\mathbf{L}})\widetilde{\mathbf{U}},\end{aligned}$$

which shows that $(\lambda\mathbf{I} - \mathbf{L})$ has a bounded inverse if and only if $(\lambda\mathbf{I} - \widetilde{\mathbf{L}})$ has. □

The next definition is used to describe the structure of $\mathbf{L}_\Pi$.

Definition 5.5. A linear operator $\mathbf{Q}$ in a Banach space is called a **finite-rank operator** if

$$dim\,Range(\mathbf{Q}) < \infty.$$

In the next lemma we show that $\widetilde{\mathbf{L}_\Pi}$ is a composition of two operators, one of them is a finite-rank operator.

Lemma 5.6. *The linear operator $\widetilde{\mathbf{L}_\Pi}$ can be written as*

$$\widetilde{\mathbf{L}_\Pi} \begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} = \begin{pmatrix} \mathcal{F} + \mathcal{V} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix}. \tag{5.4}$$

Here $\mathcal{F}$ (see equation (5.7)) is a finite-rank operator and $\mathcal{V}$ is of the form

$$\mathcal{V} := \begin{pmatrix} 0 & \mathbf{I} & 0 \\ \mathbf{V} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.5}$$

where $\mathbf{V}$ is given by[29]

$$\mathbf{V}\nu_1 = \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu_1(s)ds, \quad \theta \in (-T, 0). \tag{5.6}$$

Proof. By equation (4.15)

$$\mathbf{L}[\nu, D\mathbf{R}z]$$
$$= \begin{cases} -\frac{\varphi^{\alpha\prime}(\theta+T)}{\mathbf{M}\left[\varphi^{\alpha\prime}(-T-)\right]}\mathbf{M}\left[\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu(s-T)ds + e^{-\mathbf{B}T}D\mathbf{R}z\right] + \nu(\theta+T), & \theta \in [-2T, -T), \\ -\frac{\varphi^{\alpha\prime}(\theta-T)}{\mathbf{M}\left[\varphi^{\alpha\prime}(-T-)\right]}\mathbf{M}\left[\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu(s-T)ds + e^{-\mathbf{B}T}D\mathbf{R}z\right] + & \\ \quad + \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu(s-T)ds + e^{-\mathbf{B}(\theta+T)}D\mathbf{R}z, & \theta \in [-T, 0], \end{cases}$$

[29]$\mathbf{V}$ is called a Volterra operator. It is not needed to know what is a Volterra operator for our cause. We prove all the properties of it that are needed for this dissertation. For a general definition and analysis of Volterra operators, see Väth [52].

where $(\nu, z) \in \mathbb{B}_p^s \times \mathbb{R}^{N_1}$.

Using this and the definition of $\widetilde{\mathbf{L}}$ (Definition 5.3), we have

$$\widetilde{\mathbf{L}}(\nu_1, \nu_2, D\mathbf{R}z) \\ = \begin{pmatrix} -\frac{\varphi^{\alpha\prime}(\theta)}{\mathbf{M}\left[\varphi^{\alpha\prime}(-T-)\right]}\mathbf{M}\left[\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu_1(s)ds + e^{-\mathbf{B}T}D\mathbf{R}z\right] + \nu_2(\theta) \\ -\frac{\varphi^{\alpha\prime}(\theta-T)}{\mathbf{M}\left[\varphi^{\alpha\prime}(-T-)\right]}\mathbf{M}\left[\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu_1(s)ds + e^{-\mathbf{B}T}D\mathbf{R}z\right] + \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu_1(s)ds + e^{-\mathbf{B}(\theta+T)}D\mathbf{R}z \end{pmatrix},$$

where $(\nu_1, \nu_2, z) \in \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T, 0) \times \mathbb{R}^{N_1}$ and $\theta \in [-T, 0]$ in each row.

Set

$$\mathbf{c}_1(\nu_1) := -\frac{\mathbf{M}\int_{-T}^{0} e^{\mathbf{B}s}\mathbf{A}\nu_1(s)ds}{\mathbf{M}\left[\varphi^{\alpha\prime}(-T-)\right]}, \\ \mathbf{c}_2(z) := -\frac{\mathbf{M}e^{-\mathbf{B}T}D\mathbf{R}z}{\mathbf{M}\left[\varphi^{\alpha\prime}(-T-)\right]}.$$

Then $\widetilde{\mathbf{L}}$ can be written in a more elegant way (remember that $\mathbf{M}$ is a linear operator (Section 1)):

$$\widetilde{\mathbf{L}}[\nu_1, \nu_2, D\mathbf{R}z] = \begin{pmatrix} (\mathbf{c}_1(\nu_1) + \mathbf{c}_2(z)) \cdot \varphi^{\alpha\prime}(\theta) + \nu_2(\theta) \\ (\mathbf{c}_1(\nu_1) + \mathbf{c}_2(z)) \cdot \varphi^{\alpha\prime}(\theta - T) + \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\nu_1(s)ds + e^{-\mathbf{B}(\theta+T)}D\mathbf{R}z \end{pmatrix}.$$

The linear operator $\widetilde{\mathbf{L}_\Pi}$ (formula (5.3)) can be written as

$$\widetilde{\mathbf{L}_\Pi}\begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} = \mathcal{F} + \mathcal{V} = \left[\underbrace{\begin{pmatrix} \mathbf{F}^1 & 0 & \mathbf{F}^3 \\ \mathbf{F}^4 & 0 & \mathbf{F}_1^6 + \mathbf{F}_2^6 \\ \mathbf{F}^7 & 0 & \mathbf{F}^9 \end{pmatrix}}_{=:\mathcal{F}} + \underbrace{\begin{pmatrix} 0 & \mathbf{I} & 0 \\ \mathbf{V} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:\mathcal{V}}\right]\begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix}, \qquad (5.7)$$

where

1. $\mathbf{F}^1\nu_1 = \mathbf{c}_1(\nu_1) \cdot \varphi^{\alpha\prime}(\theta)$,
2. $\mathbf{F}^3 z = \mathbf{c}_2(z) \cdot \varphi^{\alpha\prime}(\theta)$,
3. $\mathbf{F}^4\nu_1 = \mathbf{c}_1(\nu_1) \cdot \varphi^{\alpha\prime}(\theta - T)$,
4. $\mathbf{F}_1^6 z = \mathbf{c}_2(z) \cdot \varphi^{\alpha\prime}(\theta - T)$, $\mathbf{F}_2^6 z = e^{-\mathbf{B}(\theta+T)}D\mathbf{R}z$,
5. $\mathbf{F}^7\nu_1 = \mathbf{E}^{\mathbb{R}}\left[((\mathbf{V} + \mathbf{F}^4)\nu_1)(0)\right]$,
6. $\mathbf{F}^9 z = \mathbf{E}^{\mathbb{R}}\left[(\mathbf{F}_1^6 z + \mathbf{F}_2^6 z)(0)\right]$,

and $\mathbf{V}$ is given in (5.6). □

In the next lemma we show that $\mathcal{F}$ is a finite-rank operator and find its basis. For this we denote the standard base of $\mathbb{R}^{N_1}$ by $r_1, \dots, r_{N-1}$.

Lemma 5.7. *The operator $\mathcal{F}$ is a finite-rank operator with dim Range$(\mathcal{F}) < 2N-1$. Moreover, if we define the basis functions, $e_1, \dots, e_{2N-1} \in \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1}$, as*

$$e_1 := \begin{pmatrix} \varphi^{\alpha\prime}(\theta) \\ \varphi^{\alpha\prime}(\theta-T) \\ 0 \end{pmatrix},$$

$$e_i := \begin{pmatrix} 0 \\ e^{-\mathbf{B}(\theta+T)} D\mathbf{R} r_{i-1} \\ 0 \end{pmatrix}, \quad i = 2, \dots, N,$$

$$e_i := \begin{pmatrix} 0 \\ 0 \\ r_{i-N}, \end{pmatrix}, \quad i = N+1, \dots, 2N-1,$$

then

$$\text{Range}(\mathcal{F}) \subset \text{span}\{e_i\}_{i=1}^{2N-1},$$

and

$$\mathcal{F}\begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} = \sum_{i=1}^{2N-1} f_i e_i, \tag{5.8}$$

where

$$\begin{aligned} f_1 &= [\mathbf{c}_1(\nu_1) + \mathbf{c}_2(z)], \\ f_i &= z_{i-1}, \quad i = 2, \dots, N, \\ f_i &= \langle \mathbf{F}^7\nu_1 + \mathbf{F}^9 z), r_{i-N}\rangle_{\mathbb{R}^{N_1}}, \quad i = N+1, \dots, 2N-1. \end{aligned}$$

Proof. By relation (5.7) for $\mathcal{F}$

$$\begin{aligned} \mathcal{F}\begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} &= \begin{pmatrix} \mathbf{F}^1 & 0 & \mathbf{F}^3 \\ \mathbf{F}^4 & 0 & \mathbf{F}_1^6 + \mathbf{F}_2^6 \\ \mathbf{F}^7 & 0 & \mathbf{F}^9 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{F}^1\nu_1 + \mathbf{F}^3 z \\ \mathbf{F}^4\nu_1 + \mathbf{F}_1^6 z + \mathbf{F}_2^6 z \\ \mathbf{F}^7\nu_1 + \mathbf{F}^9 z \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \mathbf{F}^1\nu_1 + \mathbf{F}^3 z \\ \mathbf{F}^4\nu_1 + \mathbf{F}_1^6 z \\ 0 \end{pmatrix}}_{(A)} + \underbrace{\begin{pmatrix} 0 \\ \mathbf{F}_2^6 z \\ 0 \end{pmatrix}}_{(B)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \mathbf{F}^7\nu_1 + \mathbf{F}^9 z \end{pmatrix}}_{(C)}. \end{aligned}$$

We write (A),(B),(C) in the basis $\{e_i\}$.

<u>(A).</u> By the expressions for $\mathbf{F}$'s in the proof of Lemma 5.6, (A) equals

$$(A) = \begin{pmatrix} [\mathbf{c}_1(\nu_1) + \mathbf{c}_2(z)] \cdot \varphi^{\alpha\prime}(\theta) \\ [\mathbf{c}_1(\nu_1) + \mathbf{c}_2(z)] \cdot \varphi^{\alpha\prime}(\theta - T) \\ 0 \end{pmatrix} = [\mathbf{c}_1(\nu_1) + \mathbf{c}_2(z)] e_1(\theta) = f_1 e_1.$$

(B). By the expressions for $\mathbf{F}$'s in the proof of Lemma 5.6, (B) equals

$$(B) = \begin{pmatrix} 0 \\ e^{-\mathbf{B}(\theta+T)} D\mathbf{R}z \\ 0 \end{pmatrix} = \sum_{i=2}^{N} z_{i-1} \begin{pmatrix} 0 \\ e^{-\mathbf{B}(\theta+T)} D\mathbf{R}r_i \\ 0 \end{pmatrix} = \sum_{i=2}^{N} f_i e_i.$$

(C). Note that $\mathbf{F}^7\nu_1 + \mathbf{F}^9 z \in \mathbb{R}^{N_1}$. Hence (C) equals

$$(C) = \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^{N-1} \langle \mathbf{F}^7\nu_1 + \mathbf{F}^9 z, r_i \rangle_{\mathbb{R}^{N_1}} r_i \end{pmatrix} = \sum_{i=N+1}^{2N-1} f_i e_i.$$

□

5.2 Finite-dimension reduction

In this subsection we first show that it is sufficient to study only the eigenvalues of $\mathbf{L}_\Pi$. Then we create a linear problem that checks if a nonzero λ is an eigenvalue of $\mathbf{L}_\Pi$.

Lemma 5.8. *A complex number $\lambda \neq 0$ satisfies $\lambda \in \sigma(\mathbf{L}_\Pi)$ (spectrum of $\mathbf{L}_\Pi$) if and only if λ is an eigenvalue of $\mathbf{L}_\Pi$.*

Proof. By Lemma 5.4, $\sigma(\widetilde{\mathbf{L}_\Pi}) = \sigma(\mathbf{L}_\Pi)$. Hence we show that $\lambda \neq 0 \in \sigma(\widetilde{\mathbf{L}_\Pi})$ if and only if λ is an eigenvalue of $\widetilde{\mathbf{L}_\Pi}$. We show next that the operator $\widetilde{\mathbf{L}_\Pi}^2$ is compact, and by [15, chapter VII4.5, Theorems 5 and 6] this shows that the spectrum of $\widetilde{\mathbf{L}_\Pi}$ is composed only of zero and possibly eigenvalues.

By Lemma 5.6, $\widetilde{\mathbf{L}_\Pi} = \mathcal{F} + \mathcal{V}$, where $\mathcal{F}$ is a finite-rank operator. Then

$$\widetilde{\mathbf{L}_\Pi}^2 = \mathcal{F}^2 + \mathcal{F}\mathcal{V} + \mathcal{V}\mathcal{F} + \mathcal{V}^2.$$

All the terms in the previous equation besides $\mathcal{V}^2$ are finite-rank operators. Every finite-rank operator is obviously compact. It remains to show that $\mathcal{V}^2$ is compact.

By equation (5.5), $\mathcal{V}^2$ from $\mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1}$ to itself is given by

$$\mathcal{V}^2 = \begin{pmatrix} \mathbf{V} & 0 & 0 \\ 0 & \mathbf{V} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence to show that $\mathcal{V}^2$ is compact, we need to show that $\mathbf{V}$ is compact as an operator on the space $\mathbb{B}_p^{sw}$, and on the space $\mathbb{W}_p^s(-T,0)$.

The operator $\mathbf{V}$ is continuous from the space $\mathbb{W}_p^s(-T,0)$ (and $\mathbb{B}_p^{sw}$) to the space $\mathbb{W}_p^1(-T,0)$. The latter is compactly embedded into $\mathbb{B}_p^{sw}$ and $\mathbb{W}_p^s(-T,0)$. The proof is complete by noticing that $\mathbb{W}_p^s(-T,0)$ (and $\mathbb{B}_p^{sw}$) are embedded in $\mathbb{L}_p(-T,0)$. Hence $\mathbf{V} : \mathbb{W}_p^s(-T,0)$ or $\mathbb{B}_p^{sw} \to \mathbb{W}_p^s(-T,0)$ is a compact operator. □

If $\mathbf{A} = 0$ in problem (1.1)–(1.3), then the delay term disappears. The next lemma determines the eigenvalues of $\widetilde{\mathbf{L}_\Pi}$ in this case.

Lemma 5.9. *Let* $\mathbf{A} = 0$. *Then* $\sigma(\widetilde{\mathbf{L}_\Pi}) = \{$*eigenvalues of* $\mathbf{F}^9\} \cup \{0\}$, *where* 0 *is an eigenvalue of* $\widetilde{\mathbf{L}_\Pi}$ *with infinite geometrical multiplicity.*

Proof. By Lemma 5.8 it is enough to look for the eigenvalues of $\widetilde{\mathbf{L}_\Pi}$.

If $\mathbf{A} = 0$, then $c_1(\nu_1)$=0 and $\mathbf{V} = 0$. Hence the operators $\mathbf{F}^1$, $\mathbf{F}^4$ and $\mathbf{F}^7$ equal zero as well. By relation (5.7), λ is an eigenvalue of $\widetilde{\mathbf{L}_\Pi}$ if and only if

$$\widetilde{\mathbf{L}_\Pi}\begin{pmatrix}\nu_1\\ \nu_2\\ z\end{pmatrix} = \begin{pmatrix}0 & \mathbf{I} & \mathbf{F}^3\\ 0 & 0 & \mathbf{F}^6_1+\mathbf{F}^6_2\\ 0 & 0 & \mathbf{F}^9\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ z\end{pmatrix} = \begin{pmatrix}\nu_2+\mathbf{F}^3 z\\ (\mathbf{F}^6_1+\mathbf{F}^6_2)z\\ \mathbf{F}^9 z\end{pmatrix} = \lambda\begin{pmatrix}\nu_1\\ \nu_2\\ z\end{pmatrix}. \tag{5.9}$$

Consider $\lambda \neq 0$. From the previous relation it follows that if λ is an eigenvalue $\widetilde{\mathbf{L}_\Pi}$, then it is an eigenvalue of $\mathbf{F}^9$. On the other hand, if λ is an an eigenvalue of $\mathbf{F}^9$ with eigenvector z, then λ is an eigenvalue of $\widetilde{\mathbf{L}_\Pi}$ with the eigenfunction:

$$\begin{pmatrix}\frac{1}{\lambda}((\mathbf{F}^6_1+\mathbf{F}^6_2)z+\mathbf{F}^3 z\\ \frac{1}{\lambda}(\mathbf{F}^6_1+\mathbf{F}^6_2)z\\ z\end{pmatrix}.$$

Consider now $\lambda = 0$. By (5.9 $\lambda = 0$ is an eigenvalue of $\widetilde{\mathbf{L}_\Pi}$ if and only if

$$\begin{pmatrix}\nu_2+\mathbf{F}^3 z\\ (\mathbf{F}^6_1+\mathbf{F}^6_2)z\\ \mathbf{F}^9 z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

which shows that for every $\nu_1 \in \mathbb{B}^{sw}_p$, every vector of the following form is an eigenfunction corresponding to eigenvalue $\lambda = 0$.

$$\begin{pmatrix}\nu_1\\0\\0\end{pmatrix}$$

□

Remark 5.10. If $b = 0$ (as in Lemma 5.9), then the delay has no effect on problem (1.1)–(1.3), and it is equivalent to a problem without delay. It is possible to build a Poincaré map to this problem (without delay), and calculate its linearization explicitly (in a similar way as in [25]). The linearization is then the same as the matrix $\mathbf{F}^9$.

The operator $\widetilde{\mathbf{L}_\Pi}$ is a sum of two operators, one of them is a finite-rank operator. The next lemma shows that the problem of finding the eigenvalues of a finite-rank operator is a finite-dimensional problem.

Lemma 5.11. *Let* $\mathbf{Q}$ *be a linear finite-rank operator in a Banach space* $\mathbb{B}$ *(with range* $\mathcal{R}(\mathbf{Q}) \subset \mathbb{Y}$, *where* $\mathbb{Y}$ *is finite-dimensional). Then* $\lambda \neq 0$ *and* ρ^λ *are an eigenvalue and an eigenfunction of* $\mathbf{Q}$ *respectively, if and only if they are an eigenvalue and an eigenfunction of the finite-dimensional operator*

$$\mathbf{Q}|_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}.$$

Proof. The direction $\Leftarrow$ is straightforward.

For $\Rightarrow$: If ρ^λ is an eigenfunction of $\mathbf{Q}$, then $\mathbf{Q}\rho^\lambda = \lambda\rho^\lambda$. Hence $\rho^\lambda = \frac{1}{\lambda}\mathbf{Q}\rho^\lambda \in \mathcal{R}(\mathbf{Q}) \subset \mathbb{Y}$. $\square$

For the dimension reduction we would have to examine the resolvent set of $\mathcal{V}$, i.e., study the inverse of $(\lambda\mathcal{I} - \mathcal{V})$ for $\lambda \neq 0$.

Before stating the theorem we remark that the inverse of $(\lambda\mathbf{I} - \mathbf{V})$ is well known for Volterra operators when ρ is a $\mathbb{R}$-valued function. Then it is calculated via the Neumann series [30]. However, this is not true for $\mathbb{R}^N$-valued functions in general and particularly not in our case. The reason is that the matrices $\mathbf{A}, \mathbf{B}$ in equation (5.6) are not necessarily commutative (which is needed for the Neumann series).

Lemma 5.12. *The operator*

$$(\lambda\mathcal{I} - \mathcal{V}) : \mathbb{L}_p(-T,0) \times \mathbb{L}_p(-T,0) \times \mathbb{R}^{N_1} \to \mathbb{L}_p(-T,0) \times \mathbb{L}_p(-T,0) \times \mathbb{R}^{N_1}$$

is invertible for every $\lambda \neq 0$. Its inverse is given by

$$(\lambda\mathcal{I} - \mathcal{V})^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \\ q \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix},$$

where

$$\begin{aligned} \nu_1 &= (\lambda^2\mathbf{I} - \mathbf{V})^{-1}[\rho_2 + \lambda\rho_1], \\ \nu_2 &= \lambda\nu_1 - \rho_1, \\ z &= \frac{1}{\lambda}q, \end{aligned} \tag{5.10}$$

and $(\lambda^2\mathbf{I} - \mathbf{V})^{-1} : \mathbb{L}_p(-T,0) \to \mathbb{L}_p(-T,0)$ is given by

$$(\lambda^2\mathbf{I} - \mathbf{V})^{-1}\rho = \frac{1}{\lambda^2}\rho - \frac{1}{\lambda^2}(\mathbf{B} - \frac{1}{\lambda^2}\mathbf{A}) \int_{-T}^{\theta} e^{(\mathbf{B} - \frac{1}{\lambda^2}\mathbf{A})(s-\theta)}\rho ds + \frac{1}{\lambda^2}\mathbf{B} \int_{-T}^{\theta} e^{(\mathbf{B} - \frac{1}{\lambda^2}\mathbf{A})(s-\theta)}\rho ds, \tag{5.11}$$

for $\rho \in \mathbb{L}_p(-T,0)$.

Proof. **Step I.** Take $\lambda \neq 0$. By relation (5.5) (for the operator $\mathcal{V}$)

$$(\lambda\mathbf{I} - \mathcal{V}) \begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{I} & -\mathbf{I} & 0 \\ -\mathbf{V} & \lambda\mathbf{I} & 0 \\ 0 & 0 & \lambda\mathbf{I} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ z \end{pmatrix} = \begin{pmatrix} \lambda\nu_1 - \nu_2 \\ -\mathbf{V}\nu_1 + \lambda\nu_2 \\ \lambda z \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ q \end{pmatrix}.$$

It is straightforward that

$$z = \frac{1}{\lambda}q$$

[30] To calculate via the Neumann series, note that $(\lambda\mathbf{I} - \mathbf{V})^{-1} = \frac{1}{\lambda}(\mathbf{I} - \frac{1}{\lambda}\mathbf{V})^{-1}$. Then use the Neumann series $(\mathbf{I} - \frac{1}{\lambda}\mathbf{V})^{-1} = \sum_{n=0}^{\infty}(\frac{1}{\lambda}\mathbf{V})^n$.

and that

$$\nu_2 = \lambda\nu_1 - \rho_1,$$
$$-\mathbf{V}\nu_1 + \lambda\nu_2 = \rho_2.$$

Plug the expression for ν_2 from the first line in the previous equation into the second line:

$$-\mathbf{V}\nu_1 + \lambda^2\nu_1 - \lambda\rho_1 = \rho_2.$$

Isolate ν_1:

$$\nu_1 = (\lambda^2\mathbf{I} - \mathbf{V})^{-1}[\rho_2 + \lambda\rho_1].$$

This proves equation (5.10), provided that $\lambda^2\mathbf{I}-\mathbf{V}$ is invertible in $\mathbb{L}_p(-T,0)$, which we show in Step II.

Step II. Set $\mu = \lambda^2$. We are looking for the inverse of $\mu I - \mathbf{V}$.
Step II.I. Existence of an inverse. The operator $\mathbf{V}$ is compact [12, Chapter II, Proposition 4.7], while $\mu\mathbf{I}-\mathbf{V}$ is a Fredholm operator with index zero [12, Chapter XI, Proposition 3.3]. According to a corollary of Fredholm alternative ([12, Chapter VII, Corollary 7.10]), $\mu\mathbf{I} - \mathbf{V}$ has an inverse if and only if the only solution to the problem

$$(\mu I - \mathbf{V})\varrho = 0 \tag{5.12}$$

is $\varrho = 0$.

If $(\mu I - \mathbf{V})\varrho = 0$, then $\varrho = \frac{1}{\mu}\mathbf{V}\varrho$. Since $\mathbf{V} : \mathbb{L}_p(-T,0) \to \mathbb{W}^1_p(-T,0)$, $\varrho \in \mathbb{W}^1_p(-T,0)$. Differential relation (5.12) using the expression for $\mathbf{V}$ (relation (5.6))

$$\begin{cases} \mu\varrho' - \mathbf{A}\varrho + \mathbf{B}\underbrace{\int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\varrho(s)ds}_{=\mu\varrho} = \mu\varrho' + (\mu\mathbf{B} - \mathbf{A})\varrho = 0, \\ \varrho(-T) = 0, \end{cases}$$

where the initial condition follows relation (5.12). By the previous equation $\varrho \in \mathbb{W}^2_p(-T,0) \subset C^1(-T,0)$ (the space of continuously differentiable function). Thus the theory of ordinary differential equations implies that the last equation has a unique solution $\varrho = 0$.

Step II.II. Inverse on a dense subspace. Consider $(\mu I - \mathbf{V})$ as an operator from $C^\infty[-T,0]$ to $C^\infty[-T,0]$. Choose $\rho \in C^\infty[-T,0]$. We look for $\varrho \in C^\infty(-T,0)$ such that

$$\mu\varrho - \int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\varrho(s)ds = \rho. \tag{5.13}$$

It is possible to differentiate both sides since ϱ, $\rho \in C^{\infty}[0,T]$:

$$\begin{cases} \mu\varrho' - \mathbf{A}\varrho(\theta) + \mathbf{B}\int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\varrho(s)ds = \rho', \\ \varrho(-T) = \rho(-T)0. \end{cases}$$

By equation (5.13), $\mathbf{B}\int_{-T}^{\theta} e^{\mathbf{B}(s-\theta)}\mathbf{A}\varrho(s)ds = \mu\mathbf{B}\varrho - \mathbf{B}\rho$. Hence the previous equation becomes

$$\begin{cases} \mu\varrho' + (\mu\mathbf{B} - \mathbf{A})\varrho = \rho' + \mathbf{B}\rho, \\ \varrho(-T) = 0. \end{cases}$$

By the semigroup theory [39] the solution is

$$(\mu\mathbf{I} - \mathbf{V})^{-1}\rho = \varrho = \frac{1}{\mu}\rho - \frac{1}{\mu}(\mathbf{B} - \frac{1}{\mu}\mathbf{A})\int_{-T}^{\theta} e^{(\mathbf{B}-\frac{1}{\mu}\mathbf{A})(s-\theta)}\rho ds + \frac{1}{\mu}\mathbf{B}\int_{-T}^{\theta} e^{(\mathbf{B}-\frac{1}{\mu}\mathbf{A})(s-\theta)}\rho ds. \tag{5.14}$$

By Step II.I this ϱ is a unique solution from $\mathbb{L}_p(-T,0)$ for equation 5.13.
Step II.III. Extension to $\mathbb{L}_p$. The operator $(\mu I - \mathbf{V})^{-1}$ from equation (5.14) is a bounded linear operator both from $\mathbb{L}_p(-T,0)$ into itself.

By [1, Corollary 2.30], $C^{\infty}(-T,0)$ is dense in $\mathbb{L}_p(-T,0)$, hence $(\mu I - \mathbf{V})^{-1}$ has a unique extension to $\mathbb{L}_p(-T,0)$ given by (5.14).

Step II.IV. Replacing μ by λ^2 yields equation (5.11). □

The next lemma is the main lemma of this section. It shows that checking if a complex number $\lambda \neq 0$ is an eigenvalue of $\mathbf{L}_{\Pi}$ is equivalent to finding the eigenvalues of a finite-rank operator. By the previous lemma it is a finite-dimensional problem.

Lemma 5.13. *A complex $\lambda \neq 0$ is an eigenvalue of $\widetilde{\mathbf{L}_{\Pi}}$ with eigenfunction $\zeta := (\nu_1, \nu_2, z)$ if and only if*

$$\rho := (\lambda I - \mathcal{V})\zeta \tag{5.15}$$

is an eigenfunction with eigenvalue 1 of the finite-rank operator $\mathcal{F}(\lambda I - \mathcal{V})^{-1} : \mathbb{L}_p \times \mathbb{L}_p \times \mathbb{R}^{N_1} \to \mathbb{L}_p \times \mathbb{L}_p \times \mathbb{R}^{N_1}$, i.e,

$$\mathcal{F}(\lambda I - \mathcal{V})^{-1}\rho = \rho. \tag{5.16}$$

Proof. $\underline{\Rightarrow:}$ A complex number $\lambda \neq 0$ is an eigenvalue of $\widetilde{\mathbf{L}_{\Pi}}$ if there exists $\zeta := (\nu_1, \nu_2, z) \in \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1}$ such that (by relation (5.4))

$$(\lambda\mathcal{I} - \mathcal{V} - \mathcal{F})\zeta = 0. \tag{5.17}$$

Let $\rho := (\lambda\mathcal{I} - \mathcal{V})\zeta \in \mathbb{L}_p \times \mathbb{L}_p \times \mathbb{R}^{N_1}$. The operator $(\lambda\mathcal{I} - \mathcal{V})$ is invertible for every $\lambda \neq 0$ by Lemma 5.12, hence $\zeta = (\lambda\mathcal{I} - \mathcal{V})^{-1}\rho$. Then (5.17) implies that

$$0 = (\lambda\mathcal{I} - \mathcal{V} - \mathcal{F})(\lambda\mathcal{I} - \mathcal{V})^{-1}\rho = (\mathcal{I} - \mathcal{F}(\lambda\mathcal{I} - \mathcal{V})^{-1})\rho.$$

$\underline{\Leftarrow:}$ Let ρ satisfy (5.16). Then $\rho \in Range(\mathcal{F})$. From the expression of range $\mathcal{F}$ from Lemma 5.7, it follows that $\rho \in \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1}$, and hence from the expression for $(\lambda\mathcal{I} - \mathcal{V})^{-1}$ from Lemma 5.12, it follows that $(\lambda\mathcal{I} - \mathcal{V})^{-1}\rho \in \mathbb{B}_p^{sw} \times \mathbb{W}_p^s(-T,0) \times \mathbb{R}^{N_1}$. Set $\zeta = (\lambda\mathcal{I} - \mathcal{V})^{-1}\rho$ to complete the proof. □

5.3 The matrix of the linear operator $\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}$

Definition 5.14. By Lemma 5.7, for every e_i, $i=1,\dots,2N-1$, we can write

$$\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}(e_i)=f_{1,i}e_1+f_{2,i}e_2+\cdots+f_{n,i}e_n.$$

The matrix $[\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}]=\{f_{j,i}\}$ is called the matrix of the operator $\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}$. The eigenvalues of $\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}:\mathbb{L}_p\to\mathbb{L}_p$ are the same as those of $[\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}]$.

In this section we calculate the matrix $[\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}]$. Recall the linearization $D\mathbf{R}$ of the lift operators given by (4.17).

Define $Q(e_i,\lambda)$, $i=1,\dots,n$, $\lambda\neq 0\in\mathbb{C}$, as in the following formulas using Lemma 5.12 and e_i from Lemma 5.7.

For $i=1$:

$$\begin{aligned}\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}e_1&=\mathcal{F}\begin{pmatrix}\underbrace{(\lambda^2\mathbf{I}-\mathbf{V})^{-1}\left[\varphi^{\alpha\prime}(\theta-T)+\lambda\varphi^{\alpha\prime}(\theta)\right]}_{=:Q(e_1,\lambda)}\\ \lambda Q(e_1,\lambda)-\varphi^{\alpha\prime}(\theta)\\ 0\end{pmatrix}\\&=\mathcal{F}\begin{pmatrix}Q(e_1,\lambda)\\ \lambda Q(e_1,\lambda)-\varphi^{\alpha\prime}(\theta)\\ 0\end{pmatrix}.\end{aligned}$$

For $i=2,\dots,N$:

$$\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}e_i=\mathcal{F}\begin{pmatrix}\underbrace{(\lambda^2\mathbf{I}-\mathbf{V})^{-1}e^{-\mathbf{B}(\theta+T)}D\mathbf{R}r_{i-1}}_{=:Q(e_i,\lambda)}\\ \lambda Q(e_i,\lambda)\\ 0\end{pmatrix}=\mathcal{F}\begin{pmatrix}Q(e_i,\lambda)\\ \lambda Q(e_i,\lambda)\\ 0\end{pmatrix}.$$

For $i=N+1,\dots,2N-1$:

$$\begin{aligned}\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}e_i&=\frac{1}{\lambda}\mathcal{F}\begin{pmatrix}0\\0\\r_{i-N}\end{pmatrix}\\&=\frac{1}{\lambda}\Bigg(\mathbf{c}_2(r_{i-N})e_1+e_{i-N+1}\\&\quad+\sum_{j=N+1}^{2N-1}\left[\mathbf{E}^{\mathbb{R}}\left\langle(\mathbf{F}_1^6+\mathbf{F}_2^6)r_{i-n}\right](0),r_{j-N}\right\rangle_{\mathbb{R}^{N_1}}e_j\Bigg).\end{aligned}$$

Then, by the expression for $\mathcal{F}$ in (5.7) for $i = 1$:

$$[\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}e_1] = \begin{array}{l} \left(\begin{array}{c} \mathbf{c}_1\big(Q(e_1,\lambda)\big) \\ 0 \\ \dots \\ 0 \\ \langle \mathbf{E}^{\mathbb{R}}\big[(\mathbf{V}+\mathbf{F}^4)Q(e_1,\lambda)\big](0), r_1\rangle_{\mathbb{R}^{N_1}} \\ \dots \\ \langle \mathbf{E}^{\mathbb{R}}\big[(\mathbf{V}+\mathbf{F}^4)Q(e_1,\lambda)\big](0), r_{N-1}\rangle_{\mathbb{R}^{N_1}} \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \dots\\ 0\end{array}}\right\} N-1 \\ \\ \\ \\ \end{array} .$$

For $i = 2, \dots, N$:

$$[\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}e_i] = \begin{array}{l} \left(\begin{array}{c} \mathbf{c}_1\big(Q(e_i,\lambda)\big) \\ 0 \\ \dots \\ 0 \\ \langle \mathbf{E}^{\mathbb{R}}\big[(\mathbf{V}+\mathbf{F}^4)Q(e_i,\lambda)\big](0), r_1\rangle_{\mathbb{R}^{N_1}} \\ \dots \\ \langle \mathbf{E}^{\mathbb{R}}\big[(\mathbf{V}+\mathbf{F}^4)Q(e_i,\lambda)\big](0), r_{n-1}\rangle_{\mathbb{R}^{N_1}} \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \dots\\ 0\end{array}}\right\} N-1 \\ \\ \\ \\ \end{array} .$$

For $i = N+1, \dots, 2N-1$:

$$[\mathcal{F}(\lambda\mathcal{I}-\mathcal{V})^{-1}e_i] = \left(\begin{array}{c} \frac{1}{\lambda}\mathbf{c}_2(r_{i-N}) \\ 0 \\ \dots \\ 0 \\ \frac{1}{\lambda} \\ 0 \\ \dots \\ 0 \\ \frac{1}{\lambda}\langle \mathbf{E}^{\mathbb{R}}[(\mathbf{F}_1^6+\mathbf{F}_2^6)r_{i-n}](0), r_1\rangle_{\mathbb{R}^{N_1}} \\ \dots \\ \frac{1}{\lambda}\langle \mathbf{E}^{\mathbb{R}}[(\mathbf{F}_1^6+\mathbf{F}_2^6)r_{i-n}](0), r_{N-1}\rangle_{\mathbb{R}^{N_1}} \end{array}\right) \begin{array}{c} \\ \\ \\ \\ \longleftarrow \text{ i-N+1} \\ \\ \\ \\ \\ \\ \\ \end{array} \quad \left.\vphantom{\begin{array}{c}0\\ \dots\\ 0\\ \frac{1}{\lambda}\\ 0\\ \dots\\ 0\end{array}}\right\} N-1 \quad .$$

Chapter II

Controlling the Controller: Partial Hysteresis Differential Equations and Pyragas Control

6. Setting of the problem. Existence and uniqueness

In this section we establish the setting for Chapter II. The main equation is presented. In the bulk of the section we prove fundamental properties of the problem, specifically, existence and uniqueness of solutions. In order to show these, we reduce the problem to an equivalent infinite-dimensional ordinary differential equation.

6.1 Setting of the problem

An equation is called a **hysteresis-delay partial differential equation (HDPDE)** if it is a differential equation that has both hysteresis and delay terms (either in the equation itself or in its boundary conditions). Let $Q \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the following HDPDE:

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \quad t > 0, \quad x \in Q, \tag{6.1}$$

with the boundary condition

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} = \mathcal{H}(\mathbf{M}u)(t)k(x) + b(x)(\mathbf{M}u(t-2T) - \mathbf{M}u(t)), \quad t > 0, \quad x \in \partial Q, \tag{6.2}$$

and the initial conditions

$$u(x,t) = \varphi(x,t), \quad t \in (-2T, 0), \quad x \in Q, \tag{6.3}$$
$$u(x,0+) = \psi(x), \quad x \in Q, \tag{6.4}$$

where $u(x,0+)$ is in the sense of traces from the right. Here

- $b(x)$ and $k(x)$ are given functions from $C^\infty(\partial Q)$,
- $\mathbf{M}$ is a linear average function defined as

$$\mathbf{M}\varphi = \int_Q m(x)\varphi(x)dx,$$

  where φ is a given function from the space $L_2(Q)$. For functions $u(x,t)$ we use the notation

$$\mathbf{M}u(t) = \mathbf{M}[u(\cdot,t)],$$

- the hysteresis operator $\mathcal{H}$ is defined in Chapter I (Definition 1.2),
- $2T$ is the period of the specific periodic solution whose stability we want to change. We remind the reader that the delay term is added as a Pyragas control once a periodic solution with period $2T$ is already known. We choose $2T$ instead of T to be compatible with Chapter I. See the explanation in the introduction.

Definition 6.1. We call the term $\mathbf{M}u(t-2T)-\mathbf{M}u(t)$ (without $b(x)$) in the boundary condition (6.2) a **Pyragas term**, since it appears due to applying Pyragas control to the problem (see Section 2 in introduction). We use the name "**Pyragas term**" along this chapter for similar terms of this form.

Discussion 6.2. The initial condition at $t = 0$ may seem redundant at first, since we can use the first initial condition to imposes a condition on $u(x, 0+)$ in the sense of traces, i.e., $u(x, 0+) = \varphi(x, 0-)$. However, if we define $u(x, 0+)$ in that way, it will belong to the space $L_2(Q)$. This turns out to be not regular enough for showing existence and uniqueness of solutions (see Lemma 6.10).

The following condition corresponds to [25, Condition 2.1] (there it was defined for problem (6.1)–(6.4) without delay). We assume that this condition holds for the remainder of the chapter.

Condition 6.3. The functions m and k satisfy the following conditions

$$\int_{\partial Q} k(x)d\Gamma > 0, \quad \int_Q m(x)dx > 0.$$

6.2 Definitions: spaces and solutions

Denote by $L_2 := L_2(Q)$ the space of square-integrable Lebesgue functions on Q, and by $W_2^1 := W_2^1(Q)$ the Sobolev space of weakly differentiable functions, with the norms

$$\|\varphi\|_{L_2} = \left(\int_Q |\varphi(x)|^2 dx\right)^{\frac{1}{2}},$$
$$\|\varphi\|_{W_2^1} = \left(\int_Q |\varphi(x)|^2 + |\nabla\varphi(x)|^2 dx\right)^{\frac{1}{2}}.$$

Define similarly for $k \in \mathbb{N}$ the spaces $W_2^k := W_2^k(Q)$ of k times weakly differentiable functions, with the norm

$$\left(\int_Q \sum_{|\alpha|\le k} |D^\alpha\varphi(x)|^2 dx\right)^{\frac{1}{2}},$$

where α is a multi-index. Define the spaces $W_2^{1/2}(\partial Q)$ and $W_2^{3/2}(\partial Q)$ as all functions $\varphi : \partial Q \to \mathbb{R}$ such that the following norm is finite:

$$\|\varphi\|_{W_2^{1/2}(\partial Q)} := \inf\left\{\|u\|_{W_2^1(Q)} : u \in W_2^1(Q), u\big|_{\partial Q} = \varphi\right\},$$
$$\|\varphi\|_{W_2^{3/2}(\partial Q)} := \inf\left\{\|u\|_{W_2^2(Q)} : u \in W_2^2(Q), u\big|_{\partial Q} = \varphi\right\}.$$

Let $\mathbb{B}$ be a Banach space. Denote by $L_2(a, b; \mathbb{B})$ $(a < b)$ the space of square-integrable $\mathbb{B}$-valued functions, and by $W_2^1(a, b; \mathbb{B})$ the Sobolev space of weakly differentiable $\mathbb{B}$-valued functions with the norms

$$\|\varphi\|_{L_2(a,b;\mathbb{B})} = \left(\int_a^b \|\varphi(s)\|_{\mathbb{B}}^2 ds\right)^{\frac{1}{2}},$$
$$\|\varphi\|_{W_2^1(a,b;\mathbb{B})} = \left(\int_a^b \|\varphi(s)\|_{\mathbb{B}}^2 + \|\varphi'(s)\|_{\mathbb{B}}^2 ds\right)^{\frac{1}{2}}.$$

Using this notation, the space of initial data for problem (6.1)–(6.4) is denoted by

$$\mathcal{W} := W_2^1\big(-2T, 0; L_2(Q)\big) \times W_2^1(Q). \tag{6.5}$$

Finally, solutions will belong to the anisotropic Sobolev space $W^{2,1}$ defined as

$$W^{2,1}(Q \times (a,b)) := \left\{u \in L_2(a,b; W_2^2(Q)) : u_t \in L_2(a,b; L_2(Q))\right\},$$

where u_t is the weak t-derivative of u. Note that $W^{2,1}(Q\times(a,b)) \subset C\big[a,b; W_2^1(Q)\big]$ [35].

Definition 6.4. Let $T_1 > 0$. A function $u(x,t)$ is called a **(strong) solution to problem (6.1)–(6.4) on** $[-\mathbf{2T}, \mathbf{T_1}]$ with initial data $(\varphi,\psi) \in \mathcal{W}$ if $u \in W_2^1(-2T, 0; L_2(Q)) \cap W^{2,1}(Q\times(0,T_1))$ $\big($this implies also that, $u \in L_2(-2T, T_1; L_2(Q))\big)$, and u satisfies equation (6.1) and relation (6.3) for a.e. t, and relations (6.2) and (6.4) in the sense of traces.

A function u is called a **(strong) solution to problem (6.1)–(6.4) on** $[-\mathbf{2T}, \infty)$ if it is a solution to this problem on $[-2T, T_1]$ for every $T_1 > 0$.

A **switching time** for a function $u(t)$ is defined similarly to Definition 1.3.

We use the following basis for the spaces L_2 and W_2^1.

Notation 6.5. Let $\{\lambda_j\}_{j=0}^{\infty}$ and $\{e_j(x)\}_{j=0}^{\infty}$ be the eigenvalues and eigenfunctions of the spectral problem for the Laplacian

$$\begin{aligned} -\Delta e_j &= \lambda_j e_j, \\ \left.\frac{\partial e_j}{\partial \nu}\right|_{\partial Q} &= 0. \end{aligned} \tag{6.6}$$

The following is true by [38, Chapter IV, 1.3].

1. The eigenvalues satisfy $0 = \lambda_0 < \lambda_1 < \dots$,
2. The set $\{e_j\}$ forms an orthonormal basis for L_2, $e_0(x) = \sqrt{mes(Q)}$,
3. The set $\{e_j\}$ forms a basis for W_2^1 (while $\{e_j/\sqrt{\lambda_j + 1}\}$ forms an orthonormal basis),
4. If $f \in W_2^1(Q)$, define f^L to be coefficients of f in $L_2(Q)$, and f^H to be the coefficients of f in $W_2^1(Q)$ with respect to the basis $\{e_j/\sqrt{\lambda_j+1}\}$. Then the following connection exists:

$$f^L = \frac{1}{\sqrt{1+\lambda_j}} f^H.$$

5. If $v = \sum_{j=0}^{\infty} v_j e_j \in L_2(Q)$, then the norm
$$\|v\|^2_{L_2(Q)}$$
is equivalent to
$$\sum_{j=0}^{\infty} |v_j|^2.$$

6. If $v = \sum_{j=0}^{\infty} v_j e_j \in W_2^1(Q)$, then the norm
$$\|v\|^2_{W_2^1(Q)}$$
is equivalent to the norm
$$\sum_{j=0}^{\infty} (1+\lambda_j)|v_j|^2.$$

6.3 Existence and uniqueness of solutions for a fixed hysteresis value

The value of the hysteresis $\mathcal{H}(\mathbf{M}u)(t)$ in equation (6.1) can be ± 1. Hence we define two versions of problem (6.1)–(6.4). In the one where $\mathcal{H}(\mathbf{M}u)(t) = +1$ we denote the unknown function as u_+, in the other, where $\mathcal{H}(\mathbf{M}u)(t) = -1$, as u_- (cf. Section 1.4):

$$\begin{aligned}
&\frac{\partial u_+(x,t)}{\partial t} = \Delta u_+(x,t), \quad t > 0, \quad x \in Q,\\
&\left.\frac{\partial u_+}{\partial \nu}\right|_{\partial Q} = k(x) + b(x)(\mathbf{M}u_+(t-2T) - \mathbf{M}u_+(t)), \quad t > 0, \quad x \in \partial Q,\\
&u_+(x,t) = \varphi_+(x,t), \quad t \in (-2T, 0), \quad x \in Q,\\
&u_+(x,0+) = \psi_+(x) \quad x \in Q,
\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}
&\frac{\partial u_-(x,t)}{\partial t} = \Delta u_+(x,t), \quad t > 0, \quad x \in Q,\\
&\left.\frac{\partial u_-}{\partial \nu}\right|_{\partial Q} = -k(x) + b(x)(\mathbf{M}u_-(t-2T) - \mathbf{M}u_-(t)), \quad t > 0, \quad x \in \partial Q,\\
&u_-(x,t) = \varphi_-(x,t), \quad t \in (-2T, 0), \quad x \in Q,\\
&u_-(x,0+) = \psi_-(x), \quad x \in Q.
\end{aligned} \tag{6.8}$$

Solutions to problems (6.7) and (6.8) are defined as in Definition 6.4.

Remark 6.6. The results in this subsection are proved only for problem (6.7), as the proofs for problem (6.8) are similar with obvious changes.

Lemma 6.7. *There exists a unique solution to problem (6.7).*

The proof of Lemma 6.7 is given at the end of this subsection.

We use the method of steps (see Footnote 4). In the first step $t \in [0, 2T]$, and hence $u_+(x, t-2T) = \varphi(x, t-2T)$ in the boundary condition of equation (6.7). This leads to the problem (where we replaced u_+ by u for readability)

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \quad t \in (0, 2T), x \in Q, \tag{6.9}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} = k(x) + b(x)(\mathbf{M}\varphi(t-2T) - \mathbf{M}u(t)), \quad x \in \partial Q, \tag{6.10}$$

$$u(x, 0+) = \psi, x \in Q. \tag{6.11}$$

We call u a **solution to problem (6.9)–(6.11) on $[\mathbf{0}, \mathbf{T_1}]$** if $u \in L_2(-2T, T_1; L_2(Q)) \cap W^{2,1}(Q \times (0, T_1))$ and u satisfies equation (6.9) for almost every t in $[0, T_1]$, and relations (6.10) and (6.11) in the sense of traces.

It is straightforward that if u is a solution to problem (6.9)–(6.11) on $[0, 2T]$, then u (extended to $[-2T, 0]$ as φ) is a solution to problems (6.7) on $[-2T, 2T]$.

Define the spatially nonlocal linear operator $\mathbf{B} : \mathbb{L}_2(Q) \to W_2^1(\partial Q)$ as

$$\mathbf{B}[\varphi](x) := b(x)\mathbf{M}\varphi. \tag{6.12}$$

It is easy to see that if we set

$$g(t) = \mathbf{M}\varphi(t-2T),$$

then $g \in W_2^1(0, 2T)$ and problem (6.9)–(6.11) is written with the aid of $\mathbf{B}$ and g as

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \quad t \in (0, 2T), \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial Q} + \mathbf{B}u = b(x)g(t) + k(x), \\ u(x, 0) = \psi(x). \end{cases} \tag{6.13}$$

We prove in the next two lemmas a general result about existence and uniqueness of solutions to parabolic problems.

Notation 6.8. The unbounded operator $\mathbf{A}$ and bounded operator $\bar{\mathbf{A}}$ are defined as

$$\mathbf{A} : L_2(Q) \to L_2(Q),$$
$$\mathbf{A}v = \Delta v,$$
$$\mathcal{D}om(\mathbf{A}) = \left\{ v \in W_2^2(Q) : \left.\frac{\partial v}{\partial \nu}\right|_{\partial Q} + \mathbf{B}v = 0 \right\},$$

and

$$
\begin{aligned}
&\bar{\mathbf{A}} : W_2^2(Q) \to L_2(Q) \times W_2^{1/2}(\partial Q),\\
&\bar{\mathbf{A}} = (\Delta v, \frac{\partial v}{\partial \nu}\Big|_{\partial Q} + \mathbf{B}v).
\end{aligned}
$$

Note that $\mathbf{A}$ and $\bar{\mathbf{A}}$ corresponds to the same problem.

Lemma 6.9. *The operator* $\mathbf{A}$ *generates an analytic semigroup.*

Proof. A closed operator $\mathbf{A}$ with dense domain generates an analytic semigroup if and only if [37, Definition 1.3.1 and Theorem 1.3.3] there exists $\omega \in \mathbb{R}$ such that

(1) there exist $0 < \delta < \frac{\pi}{2}$ such that $\Theta := \{\lambda \neq \omega : |\arg(\lambda - \omega)| < \frac{\pi}{2} + \delta\} \subset \rho(\mathbf{A})$, and

(2) there exists $C > 0$ such that if $\lambda \in \Theta$, $\lambda \neq \omega$, then

$$
\|(\mathbf{A} - \lambda \mathbf{I})^{-1}\| \leq \frac{C}{|\lambda - \omega|}.
$$

We prove that $\mathbf{A}$ satisfies those two conditions via [49, Theorem 21.1, Lemma 21.1]. We use the following notation[31]:

$$
\begin{aligned}
&\mathbf{A}_q = \Delta - q^2 \mathbf{I},\\
&\mathbf{A}_j^1 = 0, \quad j = 0, 1,\\
&\mathbf{B}_1^0 : W_2^2(Q) \to L_2(\partial Q), \quad \mathbf{B}_1^0 v = \frac{\partial v}{\partial \nu}\big|_{\partial Q},\\
&\mathbf{B}_{1l}^1 = 0, \quad l = 0, 1,\\
&\mathbf{B}_{10}^2 : W_2^1(Q) \to L_2(\partial Q) \text{ and } W_2^{\frac{3}{2}}(Q) \to W_2^{\frac{1}{2}}(\partial Q), \quad \mathbf{B}_{10}^2 v = b(x) \int_Q v(x) dx,\\
&\mathbf{B}_{11}^2 = 0,
\end{aligned}
$$

where $\arg q$ is in some sector Θ with angle less than π. There are three conditions to be satisfies in the above-mentioned lemma and theorem from [49]. Condition 21.1 is satisfied by [47, Chapter 3], while condition 21.2 is satisfied by [32, Section 5.1], and condition 21.3 is satisfied by [37, Chapter 3].

Denote, as in [49],

$$
\mathcal{L}(q) : \mathbb{W}_2^2(Q) \to L_2(Q) \times W_2^{1/2}(\partial Q), \quad \mathcal{L}(q) = (\mathbf{A}_q, \mathbf{B}_1^0 + \mathbf{B}_{10}^2).
$$

By [49, Theorem 21.1, Lemma 21.1] the following holds.

[31] This system is in the form of equations (21.1)–(21.2) in [49], with the parameters: $m = 1$, $m_\mu = 1$, $k = 0$, $l = 1$ and $p_l = 1/2$.

1. There exists $q_1 > 0$ such that for $q \in \{q \in \Theta : |q| \geq q_1\}$ the operator $\mathcal{L}(q)$ has a bounded inverse
$$\mathcal{L}^{-1}(q) : L_2 \times \mathbb{W}_2^{1/2}(\partial Q) \to \mathbb{W}_2^2(Q).$$
This shows that condition (1) for a generator with $\omega = q_1^2$ holds when $\lambda = q^2$.

2. The following estimate holds for every $u \in \mathcal{D}om(\mathbf{A})$ and f such that $(\mathbf{A} - q^2\mathbf{I})u = f$:
$$q^2\|u\|_{L_2(Q)} = q^2\|(\mathbf{A} - q^2\mathbf{I})^{-1}f\|_{L_2(Q)} \leq Const\|f\|_{L_2(Q)},$$
which shows condition (2) for the generator with $\omega = q_1^2$ holds when $\lambda = q^2$.

Finally $\mathcal{D}om(\mathbf{A})$ is dense in $L_2(Q)$ by [40, Lemma 1]. $\square$

Lemma 6.10. *For any $F \in L_2\big(0, 2T; L_2(Q)\big)$, $h \in W_2^1(Q)$ and $T_1 > 0$, there exists a unique solution $u \in W^{2,1}\big(Q \times (0, T_1)\big)$ to the following problem:*
$$\begin{cases} u_t = \Delta u + F(t)(x), \quad t \in (0, 2T), \quad x \in Q, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial Q} + \mathbf{B}u = 0, \quad x \in \partial Q, \\ u(x, 0) = h(x), \quad x \in Q. \end{cases} \tag{6.14}$$

Proof. Equation (6.14) can be written as
$$\begin{aligned} &v'(t) - \mathbf{A}v(t) = F(t), \\ &v(0) = h. \end{aligned} \tag{6.15}$$
This is the same form as equation (1.1) in [3, Chapter I, section 1.1][32]. By [3, Chapter I, section 1.3, Theorem 3.7 and the conditions in the beginning of 3.3] if

1. $\mathbf{A}$ generates an analytic semigroup, and
2. $h \in W_2^1(Q)$,

then there exists a unique solution. The result follows from the fact that the operator $\mathbf{A}$ generates an analytic semigroup by Lemma 6.9. $\square$

Lemma 6.11. *There exists no more than one solution to problem (6.9)–(6.11) on $[0, T_1]$.*

Proof. Let u_1, u_2 be two solutions of problem (6.9)–(6.11) with the same initial data (φ, ψ). If we set $v = u_1 - u_2$, then v satisfies the problem
$$\begin{aligned} &\frac{\partial v(x,t)}{\partial t} = \Delta v(x,t), \quad t \in (0, 2T), x \in Q, \\ &\left.\frac{\partial v}{\partial \nu}\right|_{\partial Q} + b(x)\mathbf{M}v(t) = 0 \quad x \in \partial Q, \\ &v(x, 0) = 0. \end{aligned}$$
The preceding problem satisfies the conditions of Lemma 6.14 and hence it has a unique solution $v \equiv 0$. $\square$

[32] Setting the space E in their notation to be $L_2(Q)$.

To be able to use Lemma 6.10, we convert problem (6.13) into a nonlocal problem with homogeneous Neumann boundary conditions. For this we use auxiliary functions v_b, v_k, which are defined as solutions to the following problems:

$$\begin{cases} \Delta v_b = f_b(x), \quad x \in Q, \\ \left.\dfrac{\partial v_b}{\partial \nu}\right|_{\partial Q} + \mathbf{B} v_b = b(x), \quad x \in \partial Q, \end{cases} \tag{6.16}$$

$$\begin{cases} \Delta v_k = f_k(x), \quad x \in Q, \\ \left.\dfrac{\partial v_k}{\partial \nu}\right|_{\partial Q} + \mathbf{B} v_k = k(x), \quad x \in \partial Q. \end{cases} \tag{6.17}$$

where f_b, $f_k \in L_2(Q)$ are chosen such that there exist solutions $v_b, v_k \in W_2^2(Q)$ for problems (6.16) and (6.17) respectively. The existence of such f_b, f_k is shown in Lemma 6.12.

Define $w(x,t)$ as

$$w(x,t) := u(x,t) - g(t)v_b(x) - v_k(x).$$

Following this definition, w satisfies the problem

$$\begin{cases} w_t = u_t - g'(t)v_b(x) = \Delta u - g'(t)v_b(x) = \Delta w + \Delta v_k + g(t)\Delta v_b - g'(t)v_b(x), \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\partial Q} + \mathbf{B} w = b(x)g(t) + k(x) - b(x)g(t) - k(x) = 0, \\ w(x,0) = \psi(x) - g(0)v_b(x) - v_k(x). \end{cases}$$

If we set

$$\begin{aligned} F(t)(x) &:= \Delta v_k + g(t)\Delta v_b - g'(t)v_b(x), \\ h(x) &:= \varphi(x,0) - g(0)v_b(x) - v_k(x), \end{aligned}$$

then $F \in L_2\big(0, 2T; L_2(Q)\big)$ and $h \in W_2^1(Q)$. This leads to the problem

$$\begin{cases} w_t = \Delta w + F(t)(x), \quad t \in (0, 2T), \quad x \in Q, \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\partial Q} + \mathbf{B} w = 0, \quad x \in \partial Q, \\ w(x,0) = h(x), \quad x \in Q. \end{cases} \tag{6.18}$$

The next lemma shows that problems (6.16) and (6.17) have solutions.

Lemma 6.12. *There exist $f_b, f_k \in L_2(Q)$ such that there exist solutions to problems (6.16) and (6.17).*

Proof. We prove the lemma for problem (6.16), since the proof for problem (6.17) is similar.

Consider the operator

$$\bar{\mathbf{A}}_0 : W_2^2(Q) \to L_2(Q) \times W_2^{1/2}(Q), \tag{6.19}$$

$$\bar{\mathbf{A}}_0 v = \left(\Delta v, \left.\frac{\partial v}{\partial \nu}\right|_{\partial Q} \right) = 0. \tag{6.20}$$

It is well known [38] that the Fredholm index of $\bar{\mathbf{A}}_0$ is zero.

We would like to show that $\bar{\mathbf{A}}$ is a compact perturbation of $\bar{\bar{\mathbf{A}}}$. This will follow if the operator $\mathbf{B} : W_2^2(Q) \to W_2^{1/2}(Q)$ is a compact operator. Indeed, the fact that $b \in C^\infty(\bar{Q})$ shows that

$$\mathbf{B} : W_2^2(Q) \to W_2^1(\partial Q)$$

is bounded, while $W_2^1(Q)$ is compactly embedded into $W_2^{1/2}(\partial Q)$ (see e.g. [35]).

The compactness of $\mathbf{B}$ shows [12, Chapter XI, Theorem 3.11], that $\bar{\mathbf{A}}$ is a Fredholm operator and $ind(\bar{\mathbf{A}}) = 0$. The fact that the Fredholm index of $\bar{\mathbf{A}}$ is zero, implies that $dimN(\bar{\mathbf{A}}) = codimR(\bar{\mathbf{A}}) < \infty$ (where N denotes the kernel, and R denotes the range).

It is clear that problem (6.16) has a solution if and only if $(f_b, b) \in R(\mathbf{A})$, which by the previous paragraph means that (f_b, b) is orthogonal in $L_2(Q) \times W_2^{1/2}(\partial Q)$ to finitely many elements (c_j, d_j), $j = 1, \ldots, \bar{M}$:

$$\langle (f_b, b), (c_j, d_j) \rangle = 0, \quad j = 1, \ldots, \bar{M}.$$

Since $\langle (f_b, b), (c_j, d_j) \rangle = \langle f_b, c_j \rangle + \langle b, d_j \rangle$, the previous equations imply that

$$\langle f_b, c_j \rangle = -\langle b, d_j \rangle, \quad j = 1, \ldots, \bar{M}.$$

Choose $f_b(x) = -\sum_{j=1}^{\bar{M}} \frac{\langle b, d_j \rangle c_j(x)}{\|c_j\|^2_{L_2(Q)}}$ to obtain the desired result. □

Lemma 6.7. The proof is given in the method of steps (see Footnote 4).

In each step Lemma 6.11, the way that we constructed problem (6.18) and the fact that $h \in W_2^1(Q)$ (since φ, g, v_b, $v_k \in W_2^1(Q)$), imply that problem (6.9)–(6.11) has a unique solution if problem (6.18) has a solution. The latter follows from Lemma 6.10 from the fact that problem (6.18) has the structure of problem (6.14). □

6.4 Reduction to an infinite-dimensional system of ordinary differential equations for a fixed hysteresis value

In this section we present an equivalent infinite-dimensional system of ordinary differential equations for problem (6.1)–(6.4). To do so we represent the coefficients in this problem in the basis $\{e_j\}$ from Notation 6.5.

Recall the basis e_j for $L_2(Q)$ and $W_2^1(Q)$ from Notation 6.5. It follows from Definition 6.4 of a solution that if u is a solution to problem (6.7), then $u(\cdot,t)$ is in the space $W_2^2(Q)$ for a.e. fixed $t \geq 0$. Hence each solution u can be written in the $\{e_j\}$ basis as

$$u(x,t) = \sum_{j=0}^{\infty} u_j(t)e_j(x), \tag{6.21}$$

where the converges is in the L_2 norm and

$$u_j(t) := \langle u(\cdot,t), e_j\rangle_{L_2} = \int_Q u(x,t)e_j(x)dx.$$

Similarly the weak t-derivative of u is in the space $L_2(Q)$ for a.e. t, and is written as a series in $L_2(Q)$ by

$$u_t(x,t) = \sum_{j=0}^{\infty} u_j'(t)e_j(x).$$

Define similar coefficients for the other terms in problem (6.1)–(6.4) and in problems (6.16) and (6.17).

$$\begin{aligned}
&m_j = \int_Q m(x)e_j(x)dx,\ k_j = \int_{\partial Q} k(x)e_j(x)d\Gamma,\ b_j = \int_{\partial Q} b(x)e_j(x)d\Gamma,\\
&\varphi_j(t) = \int_Q \varphi(x,t)e_j(x)dx,\ \psi_j = \int_Q \psi e_j(x)dx,\\
&v_{b,j} = \int_Q v_b(x)e_j(x)dx,\ v_{k,j} = \int_Q v_k(x)e_j(x)dx.
\end{aligned}$$

Note that the $\sum_j^\infty \sqrt{1+\lambda_j}\psi_j$ converges in W_2^1 since $\psi \in W_2^1(Q)$ by assumption. We also note that b_j, k_j are not the Fourier coefficients of $b(x), k(x)$. However, by [22, Lemma 2.1] the following holds (this is used in the next subsection in the proof of Lemma 7.6).

Lemma 6.13. *The following series converge:*

$$\sum_{j=1}^{\infty}\left(\frac{|b_j|^2}{\lambda_j}\right) < \infty,$$
$$\sum_{j=1}^{\infty}\left(\frac{|k_j|^2}{\lambda_j}\right) < \infty.$$

Proof. The proof follows from [22, Lemma 2.1]. There it is shown for $k, b \in H^{1/2}(\partial Q)$. $\square$

The following is compatible with Condition 4.2 in [25].

Condition 6.14. The function $m(x)$ has only $M < \infty$ nonzero $m_j(x)$ in its series representation, i.e,

$$m(x) = \sum_{j=0}^{M-1} m_j e_j(x).$$

Here we assumed, for simplicity, that the first M coefficients are the nonzero ones.

Lemma 6.15. *By Condition 6.14 the average of $u(x,t)$ satisfies*

$$\mathbf{M}u(t) = \sum_{j=0}^{M-1} m_j u_j(t).$$

Proof. Plug the series representation of $m(x)$ into the definition of $\mathbf{M}u(t)$:

$$\begin{aligned}\mathbf{M}u(t) &= \int_Q m(x)u(x,t)dx = \int_Q \sum_{j=0}^{\infty} m_j(x)e_j(x)u(x,t)dx = \sum_{j=0}^{\infty} \int_Q m_j(x)u(x,t)e_j(x)dx \\ &= \sum_{j=0}^{\infty} m_j u_j(t) = \sum_{j=0}^{M-1} m_j u_j(t),\end{aligned}$$

where the last equality follows from Condition 6.14. $\square$

The next two problems are infinite-dimensional ordinary differential equations versions of problems (6.7) and (6.8).

$$\begin{aligned}&u'_{j,+}(t) = k_j - \lambda_j u_{j+} + b_j\big(\mathbf{M}u_+(t-2T) - \mathbf{M}u_+(t)\big), \quad t > 0, \\ &u_{j,+}(t) = \varphi_j(t), \quad t \in (-2T, 0), \\ &u_{j,+}(0+) = \psi_j.\end{aligned} \tag{6.22}$$

$$\begin{aligned}&u'_{j,-}(t) = -k_j - \lambda_j u_{j-} + b_j\big(\mathbf{M}u_-(t-2T) - \mathbf{M}u_-(t)\big), \quad t > 0, \\ &u_{j,-}(t) = \varphi_j(t), \quad t \in (-2T, 0), \\ &u_{j,-}(0+) = \psi_j.\end{aligned} \tag{6.23}$$

Here $j \in \mathbb{N} \cup \{0\}$, $u_\pm(x,t) = \sum_{j=0}^{\infty} u_{j,\pm}(t)e_j(x)$, $\mathbf{M}u_\pm$ is given by Lemma 6.15 and the initial data $\varphi_{j,\pm} \in W_2^1(-2T, 0)$, $\psi_{j,\pm} \in \mathbb{R}$ are such that

$$\begin{aligned}\varphi_\pm(x,t) &= \sum_{j=0}^{\infty} \varphi_{j,\pm}(t)e_j(x) \in W_2^1(-2T, 0, L_2(Q)), \\ \psi_\pm(x) &= \sum_{j=0}^{\infty} \psi_{j,\pm}e_j(x) \in W_2^1(Q),\end{aligned} \tag{6.24}$$

with the norms

$$\|\varphi_\pm\|_{W_2^1(-2T,0;L_2(Q))} = \left(\sum_{j=0}^{\infty} \|\varphi_{j,\pm}\|^2_{L_2(-2T,0)} + \sum_{j=0}^{\infty} \|\varphi'_{j,\pm}\|^2_{L_2(-2T,0)}\right)^{\frac{1}{2}},$$

$$\|\psi_\pm\|_{W_2^1(Q)} = \left(\sum_{j=0}^{\infty}(1+\lambda_j)|\psi_{j,\pm}|^2\right)^{\frac{1}{2}}.$$

These norms are equivalent to the norms of $W_2^1(-2T,0;L_2(Q))$ and $W_2^1(Q)$ respectively defined in Section 6.2 (see Notation 6.5).

Definition 6.16. A function $u_+(x,t) := \sum_{j=0}^{\infty} u_{j,+}(t)e_j(x)$ is called a solution to problem (6.22) with initial data $(\varphi_+,\psi_+) \in W_2^1(-2T,0,L_2(Q)) \times W_2^1(Q)$ if $u_+ \in W_2^1(-2T,T_1,L_2(Q) \cap W^{2,1}(0,T_1)$ satisfies problem (6.22) for almost every $t \in [-2T,T_1]$ for every $T_1 > 0$, and the initial condition $u_j(0+) = \psi$ in the sense of traces from the left.

In the same manner we define a solution to problem (6.23).

The following result shows a connection between the PDE problem (6.7) (problem (6.8)) and problem (6.22) (problem (6.23)).

Theorem 6.17. *A function $u_+(x,t)$ is a strong solution to the PDE problem (6.7) (problem (6.8)) if and only if $\sum_{j=0}^{\infty} u_{j,+}(t)e_j(x)$ $\left(\sum_{j=0}^{\infty} u_{j,-}(t)e_j(x)\right)$ satisfy the ODE problem (6.22) (problem (6.23)).*

Proof. We prove the result only for problems (6.7) and (6.22). The proof for problems (6.8) and (6.23) follows in a similar manner. We write $u := u_+$ and $u_j := u_{j,+}$ for readability.

<u>$\Rightarrow$:</u> Let u be a solution to the PDE problem (6.7). We show now that u_j is the solution to the j-th equation in the ODE problem (6.22) for every j. Since $u \in W_2^1(-2T,T_1,L_2(Q) \cap W^{2,1}(0,T_1)$ for every $T_1 > 0$ by definition of a solution for problem (6.7), then u also solves problem (6.22).

Multiply the first equation in problem (6.7) by $e_j(x)$ and integrate over Q. Integration by parts yields for a.e. $t > 0$

$$\int_Q u_t e_j dx = \int_Q \Delta u e_j dx = \int_{\partial Q} \left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} e_j d\Gamma - \int_Q \nabla u \nabla e_j dx. \tag{6.25}$$

The first term on the left hand side of the previous equality equals ${u_j}'(t)$. The last term in the previous equality equals by integration by parts

$$-\int_Q \nabla u \nabla e_j dx = \underbrace{-\int_{\partial Q} u \left.\frac{\partial e_j}{\partial \nu}\right|_{\partial Q} d\Gamma}_{=0} + \int_Q u \Delta e_j dx.$$

Using the boundary conditions of u from problem (6.7) and the fact that λ_j are the eigenvalues of (6.6), the right hand side of equation (6.25) equals

$$\int_{\partial Q} k(x)e_j(x)d\Gamma + \int_{\partial Q} b(x)e_j(x)\mathbf{M}u(t-2T)d\Gamma \\ - \int_{\partial Q} b(x)e_j(x)\mathbf{M}u(t)d\Gamma - \int_{Q} \lambda_j u(x,t)e_j(x)dx,$$

which implies that

$$u_j{}'(t) = k_j - \lambda_j u_j(t) + b_j(\mathbf{M}u(t-2T) - \mathbf{M}u(t)).$$

$\underline{\Leftarrow}$: We show that the ODE problem (6.22) has no more than one solution. Since by Lemmas 6.11 and 6.10 there is a unique solution to the PDE problem (6.7), and since the coefficients of every solution to problem (6.7) are a solution to problem (6.22), by the above argument this will show that a solution to problem (6.22) is a solution to problem (6.7).

The proof uses the method of steps (see Footnote 4).

Consider the j-th equation ($j = 0, \dots, M-1$) of problem (6.22) and recall Lemma 6.15:

$$u_j'(t) = k_j - \lambda_j u_j(t) + b_j \left(\underbrace{\sum_{j=0}^{M-1} m_j u_j(t-2T) - \sum_{j=0}^{M-1} m_j u_j(t)}_{:=U^M(t)} \right). \tag{6.26}$$

In the first step, the system is of the form

$$\tilde{u}'(t) = \mathbf{A}u(t) + \mathbf{B}(t),$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ and $\mathbf{B}(t) \in L_2(\mathbb{R}^N)$ and $\tilde{u} = (u_1, .., u_{M-1})$. It is well known that such a system has a unique solution in the space $W_2^1(0, T_1)$ for each $T_1 > 0$.

For $j = M+1, \dots, \infty$, the equation for u_j is

$$u_j' = k_j - \lambda_j u_j(t) + b_j U^M(t),$$

where U^M is defined in equation (6.26). The unique solution to the preceding equation is

$$u_j(t) = \psi_j e^{-\lambda_j t} + \int_0^t e^{\lambda_j(s-t)}(k_j + b_j U^M(s))ds,$$

where each u_j is in the space $W_2^1(0, T_1)$ for each $T_1 > 0$.

Continuing in the method of steps (see Footnote 4) shows that for every j, $(\varphi_j, \psi_j) \in W_2^1(-2T, 0) \times \mathbb{R}$ and every $T_1 > 0$, problem (6.22) has no more than one solution in the space $W_2^1(-2T, 0) \cap W_2^1(0, T_1)$. In particular, this shows that there is no more than one solution to problem (6.22). $\square$

6.5 Existence and uniqueness of solution to the hysteresis-delay partial differential equation

We begin by introducing an infinite-dimensional ODE version of problem (6.1)–(6.4). Consider the following infinite-dimensional system of ordinary differential equations:

$$
\begin{aligned}
&u_j'(t) = \mathcal{H}(\mathbf{M}u)(t)k_j - \lambda_j u_j + b_j\big(\mathbf{M}u(t-2T) - \mathbf{M}u(t)\big), \quad t > 0,\\
&u_j(t) = \varphi_j(t), \quad t \in (-2T, 0),\\
&u_j(0+) = \psi_j,
\end{aligned}
\tag{6.27}
$$

where $j \in \mathbb{N} \cup \{0\}$, $u(x,t) = \sum_{j=0}^{\infty} u_j(t)e_j(x)$, $\mathbf{M}u$ is given by Lemma 6.15 and the initial data is defined similarly as for problems (6.22) and (6.23). Solutions to problem (6.27) are defined similarly as in Definition 6.4.

The next lemma shows that switchings do not accumulate for problem (6.27) (cf. Lemma 1.10).

Lemma 6.18. *For every $(\varphi, \psi) \in \mathcal{W}$ and $T_1 > 0$, there exists a positive integer*

$$\bar{N} := \bar{N}(\varphi, x, T_1) > 0$$

such that for a time sequence $0 < t_1, t_2, \ldots, t_{\bar{N}}$, if $u(x,t)$ is defined on $Q \times [0, t_{\bar{N}}]$, $t_1, t_2, \ldots, t_{\bar{N}}$ are switching times of u, and u is a solution to problem (6.27) on $[-2T, t_{\bar{N}}]$, then

$$t_{\bar{N}} > T_1$$

and u is a solution to problem (6.27) on $[-2T, T_1]$.

Proof. Switching times of u are times at which $\mathcal{H}(\mathbf{M}u)(t)$ changes its value. By Corollary 6.15

$$\mathcal{H}(\mathbf{M}u)(t) = \mathcal{H}\left(\sum_{j=0}^{M-1} m_j u_j\right)(t).$$

The switchings are decided by the dynamics of problem (6.26) from the previous proof. Then the result follows from Lemma 1.10 in Chapter I since $\varphi_j \in L_2(-2T, 0)$. $\square$

By the previous lemma the following definition is an alternative definition for solutions of problem (6.27).

Definition 6.19 (solution to ODE problem (6.27))**.** Given $T_1 > 0$, a function $u(x,t) := \sum_{j=0}^{\infty} u_j(t)e_j(x)$ is a solution to problem (6.27) with initial data $(\varphi, \psi) \in W_2^1(-2T, 0, L_2(Q)) \times W_2^1(Q)$ if

1. u has finitely many switching times $t_1 < t_2 < \cdots < t_j$ in the interval $[0, T_1]$ (or possibly no switching times at all).

2. u equals the solution $u_+^{(1)}(t)$ of problem (6.22) with initial data $(\varphi, \psi) \in \mathcal{W}$, for $t \in [-2T, t_1]$ (or $t \in [-2T, T_1]$ if there are no switching times).

3. If there is at least one switching time, define $t_{j+1} := T_1$ (if $t_j < T_1$). Then for every $2 \le i \le j+1$ (or every $2 \le i \le j$ if $t_j = T_1$) the following hold, for $t \in [t_{i-1}, t_i]$,

   3.1. For even i: the solution satisfies

   $$u(x,t) = u_-^{(i)}(x, t - t_{i-1}),$$

   where $u_-^{(i)}$ is the solution to problem (6.23) with initial data

   $$(u(x, s + t_{i-1})_{s\in(-2T,0)}, u(x, t_{i-1})) \in \mathcal{W}.$$

   3.2. For odd $i > 1$: the solution satisfies

   $$u(x,t) = u_+^{(i)}(x, t - t_{i-1}),$$

   where $u_+^{(i)}$ is the solution to problem (6.22) with initial data

   $$(u(x, s + t_{i-1})_{s\in(-2T,0)}, u(x, t_{i-1})) \in \mathcal{W}.$$

Combining Definition 6.19 and Lemmas 6.17 and 6.18 yields the following lemma.

Lemma 6.20. The following takes place.

1. For every $(\varphi, \psi) \in \mathcal{W}$ and $T_1 > 0$, there exists a positive integer

   $$\bar{N} := \bar{N}(\varphi, x, T_1) > 0$$

   such that for a time sequence $0 < t_1, t_2, \dots, t_{\bar{N}}$, if $u(x,t)$ is defined on $Q \times [0, t_{\bar{N}}]$, $t_1, t_2, \dots, t_{\bar{N}}$ are switching times of u, and u is a solution to problem (6.1)–(6.4) on $[-2T, t_{\bar{N}}]$, then

   $$t_{\bar{N}} > T_1$$

   and u is a solution to problem (6.1)–(6.4) on $[-2T, T_1]$.

2. A function $u(x,t)$ is a solution to problem (6.1)–(6.4) on $[-2T, T_1]$ if and only if $u(x,t) = \sum_{j=0}^{\infty} u_j(t) e_j(x)$ is a solution to problem (6.27) on $[-2T, T_1]$.

By the previous lemma, the following definition is equivalent Definition 6.4.

Definition 6.21 (solution to problem (6.1)–(6.4))**.** Given $T_1 > 0$, a function $u \in L_2(-2T, T_1; L_2(Q)) \cap W^{2,1}(Q \times [0, T_1])$ is a solution to problem (6.1)–(6.4) with initial data $(\varphi, \psi) \in \mathcal{W}$ if

1. u has finitely many switching times $t_1 < t_2 < \dots < t_j$ in the interval $[0, T_1]$ (or possibly no switching times at all).

2. $u(x,t)$ equals the solution $u_+^{(1)}(x,t)$ of problem (6.7) with initial data $(\varphi,\psi) \in \mathcal{W}$, for $t \in [-2T, t_1]$ (or $t \in [-2T, T_1]$ if there are no switching times).

3. If there is at least one switching time, define $t_{j+1} := T_1$ (if $t_j < T_1$). Then for every $2 \leq i \leq j+1$ (or every $2 \leq i \leq j$ if $t_j = T_1$) the following hold, if $t \in [t_{i-1}, t_i]$,

   3.1. For even i: the solution satisfies

   $$u(x,t) = u_-^{(i)}(x, t - t_{i-1}),$$

   where $u_-^{(i)}$ is the solution to problem (6.23) with initial data

   $$(u(x, s + t_{i-1})_{s \in (-2T, 0)}, u(x, t_{i-1})) \in \mathcal{W}.$$

   3.2. For odd $i > 1$: the solution satisfies

   $$u(x,t) = u_+^{(i)}(x, t - t_{i-1}),$$

   where $u_+^{(i)}$ is the solution to problem (6.22) with initial data

   $$(u(x, s + t_{i-1})_{s \in (-2T, 0)}, u(x, t_{i-1})) \in \mathcal{W}.$$

Finally we state the main theorem in this section.

Theorem 6.22. *For every $(\varphi,\psi) \in \mathcal{W}$ there exists a unique solution to problem (6.1)–(6.4) on $[-2T, \infty)$.*

Proof. The proof is similar to the proof of Theorem 1.12 in Chapter I, using Lemmas 6.7 and 6.20. □

7. Conditional stability and reduction to a system of finitely-many ODEs

In this subsection we define the stability of a solution and prove that stability is determined by finitely many hysteresis-delay ordinary differential equations.

We being by dividing the system of ordinary differential equations into two complementary subsystems.

Definition 7.1 (Guiding-guided decomposition)**.** Recall that in Condition 6.14 we assumed, for simplicity that the first M coefficients of $m(x)$ in its series representation are non-zero. Recall also that by Corollary 6.15,

$$\mathbf{M}u(t) = \sum_{j=0}^{M-1} m_j u_j(t).$$

Consider for $j = 0, \dots, M-1$ the independent system

$$\begin{aligned} &u_j'(t) = k_j - \lambda_j u(t) + b_j \left(\sum_{j=0}^{M-1} m_j u_j(t-2T) - \sum_{j=0}^{M-1} m_j u_j(t) \right), \quad t > 0 \\ &u_j(t) = \varphi_j(t), \quad t \in (-2T, 0), \\ &u_j(0+) = \psi_j, \end{aligned} \tag{7.1}$$

and for $j = M, \dots, \infty$ the (non-independent) system

$$\begin{aligned} &u_j'(t) = k_j - \lambda_j u(t) + b_j \left(\sum_{j=0}^{M-1} m_j u_j(t-2T) - \sum_{j=0}^{M-1} m_j u_j(t) \right), \quad t > 0 \\ &u_j(t) = \varphi_j(t), \quad t \in (-2T, 0), \\ &u_j(0+) = \psi_j, \end{aligned} \tag{7.2}$$

where the initial data is defined similarly as for problems (6.22) and (6.23).

We call problem (7.1) the **guiding system** and problem (7.2) the **guided system**. This terminology is fitting since Theorem 7.6 shows that the stability of a periodic solution to the guided system depends on that of a periodic solution to the guiding system.

For any vector $u = \{u_j\}_{j\geq 0}$ we use the following notation

$$\bar{u} = \{u_j\}_{j=0,\dots,M-1}, \quad u^0 = \{u_j\}_{j\geq M}.$$

The decomposition of problem (6.27) implies a corresponding decomposition of the space $W_2^1(Q)$:

$$W_2^1 = \bar{W} \times W^0,$$

where the norm of $\bar{W}$ and W^0 are given by

$$\|\bar{u}\|_{\bar{W}} = \left(\sum_{j=0,\dots,M-1} (1+\lambda_j)|u_j|^2 \right)^{\frac{1}{2}}, \quad \|u^0\|_{W^0} = \left(\sum_{j=M,\dots,\infty} (1+\lambda_j)|u_j|^2 \right)^{\frac{1}{2}}.$$

Throughout this section, we make the following assumption. It is a natural assumption in light of [22, 25] (where existence of periodic solutions of the heat equation with hysteresis on the boundary were proved).

Assumption 7.1. Assume that

1. Problem (6.1)–(6.4) has a periodic solution u_p with period $2T$ generated by initial data $(\varphi^\alpha, \psi^\alpha) \in \mathcal{W}$.
2. The function $\bar{u}^p$ satisfies Assumption 2.12.

Since the switching time of u_p depends only on $\bar{u}_p$, then due to Assumption 2.12 u_p has exactly two switching points along its period, at $t = T$ and $t = 2T$. Hence we make the additional notation :

$$(\varphi^\beta, \psi^\beta) = \big(u_p(s+T)|_{s\in(-2T,0)}, u_p(T)\big),$$

which is the value of u_p in the phase space $\mathcal{W}$ at its switching time T.

Definition 7.2. The orbits $\Gamma_1 = (\Gamma_1^t, \Gamma_1^x)$, $\Gamma_2 = (\Gamma_2^t, \Gamma_2^x) \subset \mathcal{W}$ are defined as

$$\Gamma_1 := \{(u_p(s+t))\big|_{s\in(-2T,0)}, u_p(t))\big|t \in [0,T]\},$$
$$\Gamma_2 := \{(u_p(s+t))\big|_{s\in(-2T,0)}, u_p(t))\big|t \in [T,2T]\}.$$

The **orbit of the periodic solution** in the space $\mathcal{W}$ equals then

$$\Gamma = (\Gamma^t, \Gamma^x) = \Gamma_1 \cup \Gamma_2.$$

Definition 7.3. A $2T$-periodic solution u_p of problem (6.1)–(6.4) is called **stable**[33] (or **orbitally stable**) if for any neighbourhood Λ^x of Γ^x in $W_2^1(Q)$ there exist neighbourhoods Ω_1 of Γ_1 and Ω_2 of Γ_2 in $\mathcal{W}$ such that if

$$(\varphi,\psi) \in \Omega_1,\, \mathbf{M}\psi < \beta \text{ or } (\varphi,\psi) \in \Omega_2,\, \mathbf{M}\psi \geq \beta,$$

then the solution to problem (6.1)–(6.4) with initial data (φ, ψ), belongs to Λ^x for $t \geq 0$.

A $2T$-periodic solution u_p is called **asymptotically stable** (or **orbitally asymptotically stable**), if in addition to the previous requirements, there exist neighbourhoods Θ_1 of Γ_1 and Θ_2 of Γ_2 in $\mathcal{W}$ such that if

$$(\varphi,\psi) \in \Theta_1,\, \mathbf{M}\psi < \beta \text{ or } (\varphi,\psi) \in \Theta_2,\, \mathbf{M}\psi \geq \beta,$$

[33] The main goal in this chapter is to stabilize an unstable periodic solution from [25]. The instability in [25] was in the space $W_2^1(Q)$. Hence here, while we have to take perturbations in the space $\mathcal{W}$ (since it is the phase space), we show only that solutions stay in the neighbourhood of the periodic solution in the space $W_2^1(Q)$.

then the solution u of problem (6.1)–(6.4) with the initial data (φ, ψ) satisfies

$$dist(u(t), \Gamma^x) \to 0 \text{ as } t \to \infty,$$

where the distance is taken in the $W_2^1(Q)$ space.

Remark 7.4. By equivalence of norms (see Notation 6.5), stability can be defined in the same manner for u_p as a solution to the ODE problem (6.27). Moreover, a periodic solution to the PDE problem (6.1)–(6.4) is stable if and only if it is stable for problem (6.27). In the rest of the section we discuss stability of u_p as a solution to the infinite-dimensional system of ordinary differential equations problem (6.27).

Remark 7.5. Recall the definition of the Poincaré map $\mathbf{P}$ from Chapter I (Definition 3.7). Since $W_2^1(a,b) \subset W_p^s(a,b)$ (for $a < b$ and p, s satisfying Conditions 2.1 and 4.16), we can create a Poincare map $\mathbf{P}$ for $(\bar{\varphi}^\alpha, \bar{\psi}^\alpha)$ (the initial data that generates the periodic solution to the guiding system (7.1)) such that $(\bar{\varphi}^\alpha, \bar{\psi}^\alpha)$ is a fixed point of $\mathbf{P}$. Recall also Definition 3.9 of an asymptotically stable fixed point for $\mathbf{P}$. It is used in the statement of the next theorem.

Theorem 7.6. *Let u_p be a periodic solution to PDE problem (6.1)–(6.4) satisfying Assumption 7.1. If $(\bar{\varphi}^\alpha, \bar{\psi}^\alpha)$ is an asymptotically stable fixed point of the corresponding Poincaré map $\mathbf{P}$, then u_p is asymptotically stable periodic solution to problem (6.1)–(6.4).*

Proof. In light of Definition 6.21, and Theorem 6.17, we treat u_p as a solution to problem (6.27).

Assume that $(\bar{\varphi}^\alpha, \bar{\psi}^\alpha)$ is an asymptotically stable fixed point of $\mathbf{P}$.

By Lemma 3.18 $\bar{u}_p$ is an asymptotically stable solution (in the sense of Definition 2.9). Hence it is sufficient to show that the solution to the guided system is asymptotically stable. Choose $\varepsilon > 0$.

Let u be a solution to problem (6.27) with initial data (φ, ψ). Denote the Pyragas term of problem (6.27) as

$$G(t) := \sum_{j=0,\dots,M-1} m_j u_j(t - 2T) - \sum_{j=0,\dots,M-1} m_j u_j(t). \tag{7.3}$$

By Lemma 3.19 for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\|(\bar{\varphi}, \bar{\psi}) - (\bar{\varphi}^\alpha, \bar{\psi}^\alpha)\|_{W_2^1 \times \mathbb{R}^M} \le \delta,$$

then $|G(t)| \le \varepsilon$ for all $t \ge 0$ and

$$G(t) \to 0 \text{ as } t \to \infty.$$

We use the following notation: Let $t_0, t_1, \dots$ be the switching times of u, and assume, without loss of generality, that $\mathbf{M}u(t_0) = \alpha$. We leave the proof of the theorem for $t \in [0, t_0]$ to the reader, as its method is similar to the rest of the proof.

Denote

$$
\begin{aligned}
&(\varphi^{(1)}, \psi^{(1)}) := (\varphi, \psi) \text{ if } i = 1,\\
&(\varphi^{(i)}, \psi^{(i)}) := (u(s+t_i)\big|_{s\in(-2T,0)}, u(t_i)) \in \mathcal{W} \text{ if } i \neq 1,
\end{aligned}
$$

and denote by $u^{(i)}$ the solution between the switchings t_i and t_{i+1} (i.e., $u^{(i)}$ is a restriction of u on $[t_i, t_{i+1}]$). In addition, use the notation

$$
c_i := \begin{cases} \|\psi^{(i)} - \psi^{\alpha}\|_{W_2^1(Q)}, & \text{if } i \text{ is even},\\ \|\psi^{(i)} - \psi^{\beta}\|_{W_2^1(Q)}, & \text{if } i \text{ is odd}. \end{cases}
$$

Following problem (7.2) and the Definition 6.19 of a solution, the equation for the guided system $u_j^{(i)}$, $j = 0, \dots, M-1$ when $i \in \mathbb{N}$ is even is

$$
\begin{aligned}
&u_j^{(i)\prime}(t) = k_j - \lambda_j u_j^{(i)} + b_j G(t), \quad t \in (t_i, t_{i+1}),\\
&u_j^{(i)}(t_i) = \psi_j^{(i)},\\
&u_j^{(i)}(t) = \varphi_j^{(i)}(t),\ t \in (t_i - 2T, t_i).
\end{aligned}
$$

The history in the previous equation is “contained” inside $G(t)$, hence it is equivalent to

$$
\begin{aligned}
&u_j^{(i)\prime} = k_j - \lambda_j u_j^{(i)} + b_j G(t), \quad t \in (t_i, t_{i+1}),\\
&u_j^{(i)}(t_i) = \psi_j^{(i)}.
\end{aligned}
$$

Solving the previous equation yields

$$
u_j^{(i)}(t) = e^{-\lambda_j(t-t_i)}(\psi_j^{(i)} - \frac{k_j}{\lambda_j}) + \frac{k_j}{\lambda_j} + e^{-\lambda_j t}\int_{t_i}^{t} e^{\lambda_j r} b_j G(r)dr.
$$

Step I. In this step we show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|(\varphi, \psi) - (\varphi^{\alpha}, \psi^{\alpha})\|_{\mathcal{W}} \le \delta$, then $c_i \le \varepsilon$ for all $i \ge 1$ and $c_i \to 0$ as $i \to \infty$.

Let i be even and $j \ge M$. Recall that $u_j^{(i)}(t_{i+1}) = \psi_j^{(i+1)}$, and that T and $2T$ are the switching times of u_p in its period (Assumption 7.1). The term c_{i+1} then equals

$$
\begin{aligned}
&u^{(i)}(t_{i+1}) - \psi^{\beta}\\
&= e^{-\lambda_j(t_{i+1}-t_i)}\left(\psi_j^{(i)} - \frac{k_j}{\lambda_j}\right) - e^{-\lambda_j T}\left(\psi_j^{\alpha} - \frac{k_j}{\lambda_j}\right) + e^{-\lambda_j t_{i+1}}\int_{t_i}^{t_{i+1}} e^{\lambda_j r} b_j G(r)dr,
\end{aligned} \tag{7.4}
$$

where we use the fact that $\psi^{\beta} = u_p(T)$. The square of the W^0 norm of $u^{(i)}(t_{i+1}) - \psi^{\beta}$

is

$$
\begin{aligned}
c_{i+1}^2 &= \sum_{j\geq M}(1+\lambda_j)\left|e^{-\lambda_j(t_{i+1}-t_i)}\left(\psi_j^{(i)}-\frac{k_j}{\lambda_j}\right)-e^{-\lambda_j T}\left(\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right)+e^{-\lambda_j t_{i+1}}\int_{t_i}^{t_{i+1}}e^{\lambda_j r}G(r)dr\right|^2\\
&\leq 2\Bigg(\underbrace{\sum_{j\geq M}(1+\lambda_j)\left|e^{-\lambda_j(t_{i+1}-t_i)}\left(\psi_j^{(i)}-\frac{k_j}{\lambda_j}\right)-e^{-\lambda_j T}\left(\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right)\right|^2}_{=:(A)}\\
&\quad+\underbrace{\sum_{j\geq M}(1+\lambda_j)\left|e^{-\lambda_j t_{i+1}}\int_{t_i}^{t_{i+1}}e^{\lambda_j r}b_jG(r)dr\right|^2}_{=:(B)}\Bigg).
\end{aligned}
\tag{7.5}
$$

Part (A) is estimated in Step I.I, and part (B) in Step I.II.

Step I.I. By relation (7.5) and Lemma 9.10(1).

$$
\begin{aligned}
(A) &= \sum_{j\geq M}(1+\lambda_j)\Bigg|e^{-\lambda_j(t_{i+1}-t_i)}\left(\psi_j^{(i)}-\frac{k_j}{\lambda_j}\right)-e^{-\lambda_j(t_{i+1}-t_i)}\left(\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right)\\
&\quad+e^{-\lambda_j(t_{i+1}-t_i)}\left(\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right)-e^{-\lambda_j T}\left(\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right)\Bigg|^2\\
&= \sum_{j\geq M}(1+\lambda_j)\left|e^{-\lambda_j(t_{i+1}-t_i)}(\psi_j^{(i)}-\psi_j^{\alpha})+\left(e^{-\lambda_j(t_{i+1}-t_i)}-e^{-\lambda_j T}\right)\left(\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right)\right|^2\\
&\leq (1+\chi)\sum_{j\geq M}(1+\lambda_j)\left|\psi_j^{(i)}-x_j^{\alpha}\right|^2 e^{-2\lambda_j(t_{i+1}-t_i)}\\
&\quad+\bar{\chi}\sum_{j\geq M}(1+\lambda_j)\left|\psi_j^{\alpha}-\frac{k_j}{\lambda_j}\right|^2\left|e^{-\lambda_j(t_{i+1}-t_i)}-e^{-\lambda_j T}\right|^2.
\end{aligned}
\tag{7.6}
$$

Let $\lambda^* = \min_{j\geq M}\lambda_j > 0$. Fix $\chi > 0$ such that

$$
\gamma := (1+2\chi)e^{-2\lambda^* T} < 1. \tag{7.7}
$$

By Lemma 3.12 ($\mathbf{t}_\beta$ is locally Lipschitz continuous), if the perturbation of the guiding system is small enough, then there exists a constant C_1 such that (see notation of spaces in Chapter I Section 1.3).

$$
\begin{aligned}
|t_{i+1}-t_i-T| &\leq C_1\|(\bar{\varphi}^{(i)},\bar{\psi}^{(i)})-(\varphi^{\alpha},x^{\alpha})\|_{\left(L_p(-2T,0)\right)^M\times\mathbb{R}^M}\\
&\leq C_1\|(\bar{\varphi}^{(i)},\bar{\psi}^{(i)})-(\varphi^{\alpha},x^{\alpha})\|_{\left(W_2^1(-2T,0)\right)^M\times\mathbb{R}^M}.
\end{aligned}
$$

Now choose i large enough such that $\|(\bar{\varphi}^{(i)},\bar{\psi}^{(i)})-(\varphi^{\alpha},x^{\alpha})\|_{(W_2^1(-2T,0))^M\times\bar{W}}$ is small enough and relation (7.7) will yield that

$$
(1+\chi)e^{-2\lambda_j(t_{i+1}-t_i)} \leq (1+2\chi)e^{-2\lambda^* T} < 1.
$$

Combining the preceding inequality with relation (7.6) shows that

$$(A) \le \gamma \|\psi^{(i)} - \psi^\alpha\|^2_{W^0} + \bar{\chi} \sum_{j \ge M} (1+\lambda_j) \left| \psi_j^\alpha - \frac{k_j}{\lambda_j} \right|^2 \left| e^{-\lambda_j (t_{i+1}-t_i)} - e^{-\lambda_j T} \right|^2. \quad (7.8)$$

Choose (by taking large i) the initial data $(\bar{\varphi}^{(i)}, \bar{\psi}^{(i)})$ such that $\|(\bar{\varphi}^{(i)}, \bar{\psi}^{(i)}) - (\varphi^\alpha, x^\alpha)\|_{\left(W_2^1(-2T,0)\right)^M \times \mathbb{R}^M}$ is small enough such that $t_{i+1} - t_i > \frac{T}{2}$. Then

$$\begin{aligned} \left| e^{-\lambda_j (t_{i+1}-t_i)} - e^{-\lambda_j T} \right| &\le e^{-\lambda_j \frac{T}{2}} \left| t_{i+1} - t_i - T \right| \\ &\le C_2 \left\| (\bar{\varphi}^{(i)}, \bar{\psi}^{(i)}) - (\bar{\varphi}^\alpha, \bar{\psi}^\alpha) \right\|_{\left(W_2^1(-2T,0)\right)^M \times \mathbb{R}^M}. \end{aligned}$$

The sum $\sum_{j \ge M} (1+\lambda_j) \left| \psi_j^\alpha - \frac{k_j}{\lambda_j} \right|^2$ in relation (7.8) converges due to Lemma 6.13. Relation (7.8) implies then that

$$(A) \le \gamma \underbrace{\|\psi^{(i)} - \psi^\alpha\|^2_{W_2^1(\mathbb{R}^\infty)}}_{c_i^2} + C_3 \left\| (\bar{\varphi}^{(i)}, \bar{\psi}^{(i)}) - (\bar{\varphi}^\alpha, \bar{\psi}^\alpha) \right\|^2_{\left(W_2^1(-2T,0)\right)^M \times \mathbb{R}^M}.$$

Note that $\left\| (\bar{\varphi}^{(i)}, \bar{\psi}^{(i)}) - (\bar{\varphi}^\alpha, \bar{\psi}^\alpha) \right\|_{\left(W_2^1(-2T,0)\right)^M \times \mathbb{R}^M} \to 0$ for even $i \to \infty$ since we assumed that $(\varphi^\alpha, \psi^\alpha)$ is an asymptotically stable fixed point of $\mathbf{P}$.

Repeat the same process for odd i with replacing α by β (and vice versa) to show this estimate for every i.

Step I.II. Now we estimate (B) in relation (7.5). We have

$$\left| e^{-\lambda_j t_{i+1}} \int_{t_i}^{t_{i+1}} e^{\lambda_j r} b_j G(r) ds \right|^2 \le \left(e^{-\lambda_j t_{i+1}} \int_{t_i}^{t_{i+1}} |e^{\lambda_j r} b_j G(r)| ds \right)^2.$$

Choose an arbitrary $\bar{\gamma} > 0$. As we mentioned after relation (7.3), if $\|(\bar{\varphi}, \bar{\psi}) - (\bar{\varphi}^\alpha, \bar{\psi}^\alpha)\|_{\left(W_2^1(-2T,0)\right)^M \times \mathbb{R}^M}$ is small enough, then $|G(r)| \le \bar{\gamma}$, and hence

$$\left| e^{-\lambda_j t_{i+1}} \int_{t_i}^{t_{i+1}} e^{\lambda_j r} b_j G(r) ds \right|^2 \le \frac{b_j^2 \bar{\gamma}^2}{\lambda_j^2} \left(1 - e^{-\lambda_j (t_{i+1}-t_i)}\right)^2 \le \frac{b_j^2 \bar{\gamma}^2}{\lambda_j^2}.$$

Use this in (B) from relation (7.5):

$$(B) \le \sum_{j \ge M} \bar{\gamma}^2 \left(\frac{b_j^2}{\lambda_j^2} + \frac{b_j^2}{\lambda_j} \right) = C_4 \bar{\gamma}^2,$$

where the last inequality follows Lemma 6.13. To prove that $(B) \to 0$ as $i \to \infty$ we note that $\sup_{t \in [t_i, t_{i+1}]} |G(t)| \to 0$ as $i \to \infty$ by Lemma 3.19. Hence the choice of $\bar{\gamma}$ goes to 0 as $i \to \infty$.

This process can be repeated for odd i. Hence Steps I.I and I.II in addition to Lemma 9.10 show what was claimed in the beginning of Step I.

Step II. Choose $\varepsilon > 0$. We show that if $c_i \leq \varepsilon$, then $dist(u_i(t), \Gamma) \leq C\varepsilon$ for every $t \in [t_i, t_{i+1}]$, where $C > 0$ is some constant. We show this only for even i, as odd i follows similarly.

Step II.I. In the same manner that we achieved relation (7.4), the following relation holds for every $t_i \leq t \leq t_i + T$, $t < t_{i+1}$.

$$\begin{aligned} u_j^{(i)}(t) - \varphi_j^\alpha(t - t_i) &= e^{-\lambda_j(t-t_i)} \left(\psi_j^{(i)} - \frac{k_j}{\lambda_j} \right) - e^{-\lambda_j(t-t_i)} \left(\psi_j^\alpha - \frac{k_j}{\lambda_j} \right) + e^{-\lambda_j t} \int_{t_i}^t e^{\lambda_j r} b_j G(r) dr \\ &= e^{-\lambda_j(t-t_i)} \left(\psi_j^{(i)} - \psi_j^\alpha \right) + e^{-\lambda_j t} \int_{t_i}^t e^{\lambda_j r} b_j G(r) dr. \end{aligned}$$

Applying the W^0 norm yields

$$\begin{aligned} \|u^{(i)}(t) - \varphi^\alpha(t - t_i)\|_{W^0}^2 &= \sum_{j \geq M} (1 + \lambda_j) \left| e^{-\lambda_j(t-t_i)} \left(\psi_j^{(i)} - \psi_j^\alpha \right) + e^{-\lambda_j t} \int_{t_i}^t e^{\lambda_j r} G(r) dr \right|^2 \\ &\leq C_5 \sum_{j \geq M} (1 + \lambda_j) \left| e^{-\lambda_j(t-t_i)} \left(\psi_j^{(i)} - \psi_j^\alpha \right) \right|^2 \\ &+ C_5 \sum_{j \geq M} (1 + \lambda_j) \left| e^{-\lambda_j t} \int_{t_i}^t e^{\lambda_j r} b_j G(r) dr \right|^2 \\ &\leq \underbrace{C_5 \sum_{j \geq M} (1 + \lambda_j) \left| \psi_j^{(i)} - \psi_j^\alpha \right|^2}_{\leq \|\psi^{(i)} - \psi^\alpha\|_{W_2^1(Q)}^2} + C_5 \sum_{j \geq M} (1 + \lambda_j) \left| e^{-\lambda_j t} \int_{t_i}^t e^{\lambda_j r} b_j G(r) dr \right|^2. \end{aligned}$$

The first term in the last line is bounded by ε and goes to 0 using Step I, and we use the same method that we used in Step I.II for the second term.

Step II.II. If $t_{i+1} - t_i > T$ and $t_i + T \leq t \leq t_{i+1}$, then

$$u_j^{(i)}(t) - \varphi_j^\alpha(T) = e^{-\lambda_j(t-t_i)} \left(\psi_j^{(i)} - \frac{k_j}{\lambda_j} \right) - e^{-\lambda_j T} \left(\psi_j^\alpha - \frac{k_j}{\lambda_j} \right) + e^{-\lambda_j t} \int_{t_i}^t e^{\lambda_j r} b_j G(r) dr,$$

which is bounded similarly to Step I, with t replacing t_{i+1} and using the fact that $t - t_i - T \leq t_{i+1} - t_i - T$ is small. □

8. Application: Stabilization of a periodic solution

In this section we stabilize unstable periodic solutions. We consider a system taken from [25], where it is known that an unstable periodic solution exists. We give sufficient conditions under which stabilization of the periodic solution from [25] is possible. Finally, we give a family of examples in which we stabilize an unstable slow-oscillating[34] periodic solution given in [25].

In this section we consider PDE problems having the form of problem (6.1)–(6.4) with $b(x) \equiv 0$. It was first studied in [25], where they considered a thermal control problem. It is shown in [25] that this problem has, for some choice of parameters, an unstable periodic solution with a period $2T$ (we remark that the definition of instability in [25] is different from the one in this dissertation, though see Lemma 8.1 for the connection between the problems). Recall that this periodic solution is also a solution for nonzero $b(x)$, since the delay was chosen as Pyragas control (see Section 2 in introduction).

The goal in this section is to **stabilize** this unstable periodic solution. By **stabilizing** we mean that we find a function $b(x) \not\equiv 0$ such that the periodic solution to problem (6.1)–(6.4) with this $b(x)$ is stable, while it is unstable for $b(x) \equiv 0$. The main result of the section is Theorem 8.5, which provides a family of systems in which stabilization is achieved.

This section uses in the notation from Chapter I, as we use the theory from there to study the stability of the guiding system. Writing the guiding system (7.1) in the notation of problem (1.1)–(1.3) from Chapter I yields

$$\begin{aligned} &u'(t) = k\mathcal{H}(\mathbf{M}u)(t) - \mathbf{\Lambda}u + \mathbf{A}[u(t-2T) - u(t)], \quad \text{for } t > 0, \\ &u(t) = \varphi(t), \quad \text{for } t \in (-2T, 0), \\ &u(0+) = x, \end{aligned} \tag{8.1}$$

[34] By slow-oscillating periodic solution, we mean a system with an unstable periodic solution u_p of period p_1 such that the system has another periodic solution with period $p_2 < p_1$. Hence, solution u_p oscillates slower than the solution with period p_2.

where

$$\mathbf{A} = \begin{pmatrix} m_0 b_0 & m_1 b_0 & \dots & m_{M-1} b_0 \\ m_0 b_1 & m_1 b_1 & \dots & m_{M-1} b_1 \\ \dots & \dots & \dots & \dots \\ m_0 b_{M-1} & m_1 b_{M-1} & \dots & m_{M-1} b_{M-1} \end{pmatrix} = \begin{pmatrix} b_0 \\ \dots \\ \dots \\ \dots \\ b_{M-1} \end{pmatrix} \begin{pmatrix} m_0 & \dots & m_{M-1} \end{pmatrix},$$

$$k = \begin{pmatrix} k_0 \\ \dots \\ \dots \\ \dots \\ k_{M-1} \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_{M-1} \end{pmatrix},$$

$$\varphi = (\varphi_1, \dots, \varphi_{M-1})^T, \quad x = (x_1, \dots, x_{M-1})^T. \tag{8.2}$$

Note especially that the first row of Λ is zero.

If $b_0 = b_1 = \dots = b_{M-1} = 0$ this system is equivalent to the guiding system in [25], for which there were given conditions under which a system of this kind has an unstable periodic solution. The next lemma shows the connection between stability in this chapter and the stability in [25].

Lemma 8.1. *Consider the linear operator $\mathbf{L}_\Pi$ (from Section 5) that corresponds to problem (8.1). If $b_0 = b_1 = \dots = b_{M-1} = 0$, then the eigenvalues of $\mathbf{L}_\Pi$ equal a union of the eigenvalues of the matrix from [25, formula (4.17)] and zero.*

Moreover, the periodic solution u_p is stable if and only if the corresponding periodic solution in [25] is stable.

Proof. By Lemma 5.9 if $b_0 = b_1 = \dots = b_{M-1} = 0$, then the eigenvalues of $\widetilde{\mathbf{L}_\Pi}$ (which are equal to those of $\mathbf{L}_\Pi$ by Lemma 5.4) are $\{$eigenvalues of materix $\mathbf{F}^9\} \cup \{0\}$ (where $\mathbf{F}^9$ is given in the proof of Lemma 5.6). A direct calculation shows that $\mathbf{F}^9$ equals the matrix [25, formula (4.17)].

As for stability, we note that when $b = 0$, then the history φ plays no rule in the stability in Definition 7.3, and the definition is equivalent to Definition [Definition 3.3] [25] of stability. □

We consider the case $M = 3$, which is the simplest case in which an unstable periodic solution exists in [25]. For simplicity, choose

$$b := b_0 = b_1 = b_2 \in \mathbb{R}.$$

Problem (8.1) takes the form

$$\begin{aligned} u_0'(t) &= k_0 \mathcal{H}(\mathbf{M}u)(t) + b(\mathbf{M}u(t-2T) - \mathbf{M}u(t)), \\ u_1'(t) &= k_1 \mathcal{H}(\mathbf{M}u)(t) - \lambda_1 u_1(t) + b(\mathbf{M}u(t-2T) - \mathbf{M}u(t)), \\ u_2'(t) &= k_2 \mathcal{H}(\mathbf{M}u)(t) - \lambda_2 u_2(t) + b(\mathbf{M}u(t-2T) - \mathbf{M}u(t)), \end{aligned} \tag{8.3}$$

with initial conditions defined as in (8.1).

For simplicity again, fix m_1, m_2, and treat b, m_0 as parameters. By [22, 25] if $b \equiv 0$, then there exists a unique periodic solution that satisfies Assumption 7.1 for each m_0 and $b = 0$ for certain choices of $\beta - \alpha$. Due to the way that we chose the delay, it is also a periodic solution when $b \neq 0$ (but not necessarily unique). Following Theorem 7.6, it is sufficient to study the stability (in the sense of Definition 2.9) of the guiding system (7.1) (which equals (8.3) for $M = 3$).

Assume that $\beta - \alpha$ is such that a periodic solution that satisfies Assumption 7.1 exists. By abuse of notation we denote $u_p(b, m_0) := u_p(0, m_0)$ to be a periodic solution to the guiding system (8.3) (where the parameters are used to denote that there may be a different periodic solution for each choice of b and m_0).

For each b and m_0, the linear operator $\mathbf{L}_\Pi(b, m_0)$ from Section 4 is defined for $u_p(b, m_0)$. By Theorem 4.18, the stability of $u_p(b, m_0)$ depends on the spectral radius of $\mathbf{L}_\Pi(b, m_0)$. In Section 5 we showed that the spectral radius is decided by the eigenvalues of $\mathbf{L}_\Pi$. In Lemma 5.13 we showed that there exists a polynomial (namely, $det\big(\mathbf{I} - [\mathcal{F}(\mu\mathbf{I} - \mathcal{V})^{-1}]\big)$ such that $\mu \neq 0$ is an eigenvalue of $\mathbf{L}_\Pi$ if and only if it is a root of this polynomial. Denote this polynomial by $\mathbf{J}(b, m_0, \mu)$ (again, it depends on the parameters b, m_0 of the system, and now also on $\mu \neq 0$).

Treat μ as $\mu = \mu_1 + i\mu_2$ and $\mathbf{J}$ as

$$\mathbf{J}(b, m_0, \mu_1, \mu_2) = \mathbf{J}_1(b, m_0, \mu_1, \mu_2) + i\mathbf{J}_2(b, m_0, \mu_1, \mu_2),$$

where $\mathbf{J}_1$ and $\mathbf{J}_2$ are real-valued. Identifying the complex plane with $\mathbb{R}^2$, $\mathbf{J}$ becomes an operator from $\mathbb{R}^4 \to \mathbb{R}^2$. Note that $\mathbf{J}_1$, $\mathbf{J}_2$ and their partial derivatives by b, m_0, μ_1, μ_2 are continuously dependent on b, m_0, μ_1, μ_2. This is straightforward from Section 5.3.

Lemma 8.2. *If μ be a root of $\mathbf{J}(b, m_0, \mu)$, then its complex conjugate $\bar{\mu}$ is also a root of $\mathbf{J}(b, m_0, \mu)$.*

Proof. The proof follows once noting that the coefficients of the matrix $[\mathcal{F}(\mu\mathbf{I} - \mathcal{V})^{-1}]$ are real, see Section 5.3. □

By the definition of $\mathbf{J}$ and Lemma 8.2 for $b = 0$ and a fixed m_0, the polynomial $\mathbf{J}$ has

- two simple complex conjugated non-real roots or one simple non-zero real root,
- one root at zero.

The following assumption gives conditions on the nonzero roots. It is valid without further mention throughout the rest of the section.

Assumption 8.3. For $b = 0$, we assume that there exist m_1, m_2 and $\tilde{m}$ such that

1. There are two complex conjugated roots (or one real root) of $\mathbf{J}$ on the unit sphere for $m_0 = \tilde{m}$,

2. There are two complex conjugated roots (or one real root) of $\mathbf{J}$ outside unit sphere for $m_0 < \tilde{m}$,

3. All the roots of $\mathbf{J}$ are inside unit sphere for $m_0 > \tilde{m}$,

By Lemma 8.2 when $m_0 = \tilde{m}$, $\mathbf{J}$ has one root not below the imaginary axis. Denote it by $\tilde{\mu}_1 + i\tilde{\mu}_2$.

Remark 8.4. Periodic solution that satisfy Assumption 8.3 can be build from [25, Example 4.1]. The periodic solutions there are such that the stable ones have a smaller period than the unstable ones. Hence, we call the stable periodic solutions "fast-oscillating periodic solutions", and the unstable periodic solutions "slow oscillating periodic solution". We will stabilize now such slow-oscillating periodic solutions, which are often the desirable ones in application.

The stabilization Theorem 8.5 will contain two conditions. We develop them in two steps before stating the theorem.

Step I. The spectrum is upper semicontinuous [30, Chapter IV.3.1]. Hence for every $\varepsilon > 0$ there is a $\delta > 0$ such that if

$$\|(b,m) - (0,\tilde{m})\|_{\mathbb{R}^2} \leq \delta,$$

then the eigenvalues of $\mathbf{L}_\Pi(b, m_0)$ are contained in the set

$$S := B_\varepsilon(\tilde{\mu}_1 + i\tilde{\mu}_2) \cup B_\varepsilon(\tilde{\mu}_1 - i\tilde{\mu}_2) \cup B_\varepsilon(0). \tag{8.4}$$

Let us prove that if

$$\det \begin{pmatrix} \dfrac{\partial \mathbf{J}_1}{\partial \mu_1} & \dfrac{\partial \mathbf{J}_1}{\partial \mu_2} \\ \dfrac{\partial \mathbf{J}_2}{\partial \mu_1} & \dfrac{\partial \mathbf{J}_2}{\partial \mu_2} \end{pmatrix} \neq 0 \text{ at the point } (0, \tilde{m}, \tilde{\mu}_1, \tilde{\mu}_2), \tag{8.5}$$

then there is one eigenvalue in $B_\varepsilon(\tilde{\mu}_1 + i\tilde{\mu}_2)$, one eigenvalue in $B_\varepsilon(\tilde{\mu}_1 - i\tilde{\mu}_2)$, and all the rest are in $B_\varepsilon(0)$. The balls are taken in the complex plane, and ε should be small enough such that the balls do not intersect.

If (8.5) holds, then the implicit function theorem yields that there is a neighbourhood of $(0, \tilde{m})$ and functions $\mu_1(b, m_0)$, $\mu_2(b, m_0)$ such that

$$\mathbf{J}(b, m_0, \mu_1, \mu_2) = 0$$

in this neighbourhood if and only if

$$\mu_1 = \mu_1(b, m_0),\, \mu_2 = \mu_2(b, m_0).$$

In particular, $\mathbf{J}(b, m_0, \lambda_1.\lambda_2)$ has one root in $B_\varepsilon(\tilde{\mu}_1 + i\tilde{\mu}_2)$. This completes the proof, since by the way that we defined $\mathbf{J}$, $\mu \neq 0$ is an eigenvalue of $\mathbf{L}_\Pi(b, m_0)$ if and only if $\mathbf{J}(b, m_0, \mu_1, \mu_2) = 0$.

Step II. We want to show that a change of b pushes the eigenvalues inside the unit sphere. Change to polar coordinates and define the operator $\widetilde{\mathbf{J}} : \mathbb{R}^4 \to \mathbb{R}^2$ as

$$\widetilde{\mathbf{J}}(b, m_0, \theta, r) = \mathbf{J}(b, m_0, r\cos\theta, r\sin\theta).$$

The operators $\widetilde{\mathbf{J}}_1$ and $\widetilde{\mathbf{J}}_2$ are defined as they were for $\mathbf{J}$. The operator $\widetilde{\mathbf{J}}$ vanishes at $(0, \tilde{m}, \tilde{\theta}, 1)$, where $\tilde{\theta}$ is the angle of $(\tilde{\mu}_1, \tilde{\mu}_2)$ in the complex plane. If the following condition holds

$$det \begin{pmatrix} \dfrac{\partial \widetilde{\mathbf{J}}_1}{\partial b} & \dfrac{\partial \widetilde{\mathbf{J}}_1}{\partial \theta} \\ \dfrac{\partial f_2}{\partial b} & \dfrac{\partial f_1}{\partial \theta} \end{pmatrix} \neq 0 \text{ at the point } (0, \tilde{m}, \tilde{\theta}, 1), \tag{8.6}$$

then by the implicit function theorem there is a neighbourhood of $(\tilde{m}, 1)$ and function $b(m_0, r)$, $\theta(m_0, r)$ such that

$$f(r, m_0, b, \theta) = 0,$$

in this neighbourhood if and only if

$$b = b(m_0, r), \, \theta = \theta(m_0, r).$$

The following statement is the main theorem in this section. See also Figure 8.

Theorem 8.5. *If Assumption 8.3 and relations (8.5) and (8.6) take place, then there exists $m^* < \tilde{m}$ and $b^* \neq 0$ such that $u_p(0, m^*)$ is an unstable periodic solution to problem (8.3), and $u_p(b^*, m^*)$ is a stable periodic solution to problem (8.3).*

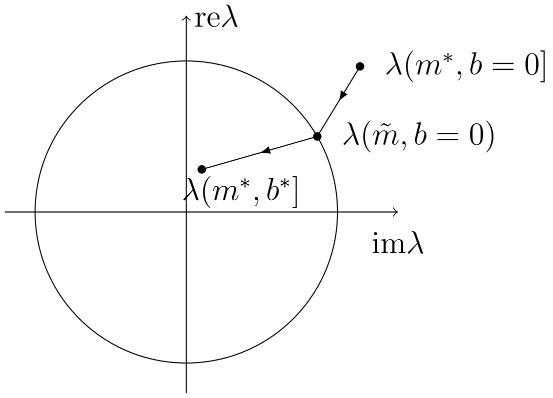

Figure 4: Theorem 8.5. The eigenvalue of $\mathbf{J}$ above the imaginary axis is depicted on the unit circle.

Proof. Choose $0 < \varepsilon$ small enough, such that if relation (8.5) holds, then there is a neighbourhood $\mathcal{N}$ of $(0, \tilde{m})$ such that if $(b, m_0) \in \mathcal{N}$, then $\mathbf{J}(b, m_0, \mu_1, \mu_2)$ has only one eigenvalue in $B_\varepsilon(\tilde{\mu}_1 + i\tilde{\mu}_2)$. By Lemma 8.2 it is sufficient to determine this eigenvalue to study stability.

Since relation (8.6) holds, there is a neighbourhood $\widetilde{\mathcal{N}}$ of $(\tilde{m}, 1)$ such that if $(m_0, r) \in \widetilde{\mathcal{N}}$ then the result in Step II holds. Choose $m^* < \tilde{m}$ and $r^* < 1$ such that $(m^*, r^*) \in \widetilde{\mathcal{N}}$.

Since $m^* < \tilde{m}$, the (μ_1, μ_2) root of $\mathbf{J}(0, m^*, \mu_1, \mu_2)$ above the imaginary axis is outside the unit sphere (i.e. $u_p(0, m^*)$ is an unstable periodic solution). Since $r^* < 1$, then $u_p(m^*, r^*)$ is a stable periodic solution. $\square$

Chapter III

Appendix

9. Appendix: Generalities

This appendix collects all general notation and lemmas which are used in this dissertation.

9.1 List of spaces

Table 1 summarizes the spaces used in this dissertation.

Chapter I		
Symbol	**Description**	**Definition**
$\mathbb{R}^{N_1}$		$\mathbb{R}^{N-1}$
$L_p(a,b)$	Lebesgue space	p-integrable functions on the real line
$\mathbb{L}_p(a,b)$	N copies of $L_p(a,b)$	$(L_p(a,b))^N$
		L_p with weak derivative which is L_p
$\mathbb{W}_p^1(a,b)$	N copies of $W_p^1(a,b)$	$(W_p^1(a,b))^N$
$W_p^s(a,b)$, $0<s<1$	Fractional Sobolev space	$\Vert\varphi\Vert_{L_p(a,b)} + \left(\int_a^b\int_a^b \frac{\vert\varphi(t)-\varphi(s)\vert^p}{\vert t-s\vert^{1+sp}} dsdt\right)^{\frac{1}{p}}$
	Seminorm of $W_p^s(a,b)$	$\left(\int_a^b\int_a^b \frac{\vert\varphi(t)-\varphi(s)\vert^p}{\vert t-s\vert^{1+sp}} dsdt\right)^{\frac{1}{p}}$
$\mathbb{W}_p^s(a,b)$	N copies of W_p^s	$(W_p^s(a,b))^N$
$\mathbb{B}_p^s(a,b)$, $-2T<a<b$,		$\mathbb{L}_p(-2T,b)\cap\mathbb{W}_p^s(a,b)$
$\mathbb{B}_p^{sw}$	Intermediate working space	$\mathbb{L}_p(-T,0)\cap\mathbb{W}_p^s(-\sigma,0)$
Chapter II		
$L_2 := L_2(Q)$	Lebesgue space	Square integrable functions on Q
$W_2^k := W_2^k(Q)$	Sobolev space	k times weakly differentiable functions whose derivatives belong L_2
$W_2^{1/2}(\partial Q)$, $W_2^{3/2}(\partial Q)$	Fractional Sobolev spaces	
$L_2(a,b;\mathbb{B})$		square-integrable $\mathbb{B}$-valued function
$W_2^1(a,b;\mathbb{B})$		Sobolev space of weakly differentiable $\mathbb{B}$-valued functions
$\mathcal{W}$		$W_2^1\big(-2T,0;L_2(Q)\big)\times W_2^1(Q)$
$W^{2,1}(Q\times(a,b))$		anisotropic Sobolev space

Table 1: Spaces

The following embedding is used often in the dissertation.

Lemma 9.1. *The space $\mathbb{W}_p^1$ is embedded in the space $\mathbb{W}_p^s$ (where the bound depends on the size of the domain) [13, Proposition 2.2].*

9.2 Finite difference lemma: Fréchet derivatives of $\mathbb{W}_p^1$ functions

A version of the lemma in this section can be found in Mikhailov [38][Chapter III, 3.4]. The version that appears here is a modification of the theorem there, with three main differences:

1. While in Mikhailov it is required for the function to be compactly supported, we require it to be defined on a larger region.

2. While Mikhailov handles the L_2 norm, we handle the L_p, $p \geq 1$ norm (which is the case in this dissertation, see Condition 4.16).

3. We give an estimate of the nonlinear term (under extra conditions on the function). This is needed to estimate the nonlinearity in the proofs of Lemma 4.28 and Lemma 4.30.

Lemma 9.2. *Let $p \geq 1$, $Q' = (a, b)$ and $Q \Subset Q'$ be a bounded interval.*

1. *If $f \in W_p^1(Q')$, then for δ small enough*

$$\|f(\cdot + \delta) - f(\cdot) - \delta f'(\cdot)\|_{L_p(Q)} = o(|\delta|), \tag{9.1}$$

where o depends on f.

2. *If f additionally in $W_p^2(a, c)$ and $W_p^2(c, b)$ for some $a < c < b$, $c \in Q$, then*

$$\|f(\cdot + \delta) - f(\cdot) - \delta f'(\cdot)\|_{L_p(Q)} = O(|\delta|^{1+\frac{1}{p}}), \tag{9.2}$$

where O depends on f.

Proof. **Step I.** We first prove the following claim: if $f \in W_p^1(Q')$ is compactly supported in Q', then relation (9.1) holds. If in addition f belongs to the spaces $W_p^2(a, c)$ and $W_p^2(c, b)$, then relation (9.2) holds.

Consider positive δ for simplicity. Assume for the time being that $f \in \mathring{C}^1(Q')$ (continuously differentiable functions with support in Q'). We extend f to be zero out of Q'. Due to the compact support of f, the extension makes it $W_p^1(\mathbb{R})$. Equation (9.1) is equivalent to

$$\left\| \frac{f(\cdot + \delta) - f(\cdot)}{\delta} - f'(\cdot) \right\|_{L_p(\mathbb{R})} \to 0 \text{ as } \delta \to 0.$$

Since we assume for the moment that f also has a classical derivative (which equals the weak one), then

$$\frac{f(\theta + \delta) - f(\theta)}{\delta} - f'(\theta) = \frac{1}{\delta} \left[\int_\theta^{\theta+\delta} f'(s) - f'(\theta) ds \right].$$

Taking it to the p-th power, integrate over the whole interval on both sides, and use Hölder's inequality:

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \frac{f(\theta + \delta) - f(\theta)}{\delta} - f'(\theta) \right|^p d\theta &\leq \frac{1}{\delta} \int_{-\infty}^{\infty} \left| \int_\theta^{\theta+\delta} f'(s) - f'(\theta)) ds \right|^p d\theta \\ &\leq \frac{1}{\delta} \int_0^\delta \int_{-\infty}^{\infty} |f'(\theta + r) - f'(\theta)|^p \, d\theta dr. \end{aligned}$$

Restricting both sides to $Q' = (a, b)$ yields

$$\left\| \frac{f(\cdot + \delta) - f(\cdot)}{\delta} - f'(\cdot) \right\|_{L_p(Q')}^p \leq \frac{1}{\delta} \int_0^\delta \underbrace{\int_a^b |f'(\theta + r) - f'(\theta)|^p \, d\theta \, dr}_{(*)}. \tag{9.3}$$

Now we explain why inequality (9.3) is true for general $W_p^1(Q')$ functions with compact support (which is extended by zero out of Q'). For this we refer the reader to the proof of Mikahilov [38, Chapter III, Section 3.4, Theorem 3]. The general idea is to approximate f by its averaging function with, with a sufficiently small averaging radius, ρ. The averaging functions are in $\mathring{C}^1(Q')$ and hence satisfy inequality (9.3). The space $\mathring{C}^1(Q')$ is dense in $W_p^1(Q')$ with zero trace, and hence taking the limit as $\rho \to 0$ yields the appropriate result.

To obtain equation (9.1) we need to use the theorem for continuity in the mean of functions in the space $L_p(Q')$, $p \geq 1$. The proof of this lemma is exactly the same as the proof for $L_1(Q')$ and $L_2(Q')$ in [38, III.2,Theorem 4]. This continuity implies that for every $\varepsilon > 0$, there is a $\gamma = \gamma(f) > 0$ such that if $|r| \leq |\delta| \leq \gamma$, then the last term in inequality (9.3) (denoted as (*)) satisfies

$$(*) \leq \varepsilon^p.$$

Using this and inequality (9.3) we get that

$$\left\|\frac{f(\theta+\delta)-f(\theta)}{\delta} - f'\right\|_{L_p(Q)}^p \leq \varepsilon^p,$$

for $|\delta| \leq \gamma$. This shows equation (9.1) for functions which are compactly supported in Q'.

To show Equation (9.2) we assume that f is $W_p^2(a,c)$ and $W_p^2(c,b)$, and develop

(*):

$$
\begin{aligned}
(*) = & \int_a^{c-\delta} |f'(\theta+r) - f'(\theta)|^p \, d\theta + \int_{c-\delta}^{c} |f'(\theta+r) - f'(\theta)|^p \, d\theta \\
& + \int_c^b |f'(\theta+r) - f'(\theta)|^p \, d\theta \\
\leq & \int_a^{c-\delta} \underbrace{\left| \int_0^r f''(\theta+s) ds \right|^p}_{\leq \left(\int_0^r |f''(\theta+s)| ds \right)^p} d\theta + \int_{c-\delta}^{c} |f'(\theta+r)|^p \, d\theta \\
& + \int_{c-\delta}^{c} |f'(\theta)|^p \, d\theta + \int_c^b \underbrace{\left| \int_0^r f''(\theta+s) ds \right|^p}_{\leq \left(\int_0^r |f''(\theta+s)| ds \right)^p} d\theta \\
\leq & \, r^{p-1} \underbrace{\int_a^{c-\delta} \int_0^r |f''(\theta+s)|^p \, ds d\theta}_{= \int_0^r \int_a^{c-\delta+s} |f''(\theta)|^p d\theta ds} + \int_{c-\delta}^{c-r} |f'(\theta+r)|^p \, d\theta + \int_{c-r}^{c} |f'(\theta+r)|^p \, d\theta \\
& + \delta \, |f'(\theta)|^p_{L_\infty(a,c)} + r^{p-1} \underbrace{\int_c^b \int_0^r |f''(\theta+s)|^p \, ds d\theta}_{\int_0^r \int_{c+s}^{b+s} |f''(\theta)|^p d\theta ds} \\
\leq & \, r^p \, \|f''\|^p_{L_p(a,c)} + (\delta - r) \, |f'|^p_{L_\infty(a,c)} \\
& + r \, \|f'\|^p_{L_\infty(c,b)} + \delta \, |f'(\theta)|^p_{L_\infty(a,c)} + r^p \, \|f''\|^p_{L_p(c,b)} \, .
\end{aligned}
$$

Plugging this back into equation (9.3) instead of (*) yields

$$
\begin{aligned}
& \left\| \frac{f(\theta+\delta) - f(\theta)}{\delta} - f' \right\|^p_{L_p(Q')} \\
& \leq \frac{1}{\delta} \int_0^\delta r^p \, \|f''\|^p_{L_p(a,c)} + \delta \, |f'|^p_{L_\infty(a,c)} \\
& \quad + r \, \|f'\|^p_{L_\infty(c,b)} + \delta \, |f'(\theta)|^p_{L_\infty(a,c)} + r^p \, \|f''\|^p_{L_p(c,b)} \, dr \\
& \leq \frac{1}{p+1} \, \|f''\|^p_{L_p(a,c)} \, \delta^p + \delta \, |f'\|^p_{L_\infty(a,b)} + \frac{1}{p+1} \, \|f''\|^p_{L_p(c,b)} \, \delta^p .
\end{aligned}
$$

This shows equation (9.2) for f which is compactly supported in Q'.

Step II. Since Q is compactly contained in the interior of Q', there exist subsets $Q_1, Q_2 \Subset Q'$ such that $Q \subset Q_1 \subset Q_2 \subset Q'$. Take a function η such that

$$
\eta(\theta) = \begin{cases} 1 & \theta \in Q_1, \\ 0 & \theta \in Q'/Q_2 \end{cases}, \quad \eta \in C^\infty(\bar{Q}')
$$

Observe that the product of f and η, is such that $f\eta(\theta) \in W^1_p(Q')$ is compactly supported in Q'. Step I implies that

$$
\|f\eta(\theta+\delta) - f\eta(\theta) - \delta (f\eta)'(\theta)\|_{L_p(Q')} \to 0 \text{ as } \delta \to 0.
$$

Choose $|\delta| < dist\{\partial Q, \partial Q_1\}$. Then

$$\int_{Q'} |f\eta(\theta+\delta) - f\eta(\theta) - \delta(f\eta)'(\theta)|^p d\theta$$
$$= \int_{Q'/Q} |f\eta(\theta+\delta) - f\eta(\theta) - \delta(f\eta)'(\theta)|^p d\theta + \int_{Q} |f\eta(\theta+\delta) - f\eta(\theta) - \delta(f\eta)'(\theta)|^p d\theta \to 0$$

as $\delta \to 0$. Since both terms on the right hand side in the previous equations are non-negative, then

$$\int_{Q} |f\eta(\theta+\delta) - f\eta(\theta) - \delta(f\eta)'(\theta)|^p d\theta \to 0 \text{ as } \delta \to 0.$$

By the definition of η, $f\eta(\theta) = f(\theta)$ if $\theta \in Q$. Then, the way in which we selected δ, shows that the previous equation equals

$$\int_{Q} |f(\theta+\delta) - f(\theta) - \delta f'|^p d\theta \to 0 \text{ as } \delta \to 0.$$

Equation (9.2) follows in a similar manner. $\square$

9.3 Besov's inequality

Lemma 9.3. *Let $p > 1$ and $s \in [0,1]$. Let $a < b$ and $0 < \sigma < b-a$. If $f \in W^s_p(a, b+\sigma)$ and $\delta \in (0, \sigma/2)$, then*

$$\|f(\theta+\delta) - f(\theta)\|_{\mathbb{L}_p(a,b)} \le Const\, \delta^s \|f\|_{W^s_p(a,b+\sigma)}.$$

If $f \in W^s_p(a-\sigma, b)$ and $\delta \in (-\sigma/2, 0)$, then

$$\|f(\theta-\delta) - f(\theta)\|_{\mathbb{L}_p(a,b)} \le Const\, \delta^s \|f\|_{W^s_p(a-\sigma,b)}.$$

In both cases, $Const > 0$ does not depend on f.

Proof. For $s < 1$ the proof follows from [5, Chap. IV, Eq. (14)]. For $s = 1$, the result is obtained by representing $f(\theta+\delta) - f(\theta) = \int_0^\delta f'(\theta+s)\, ds$. $\square$

9.4 Properties of fractional Sobolev spaces

Lemma 9.4 (Fractional Sobolev functions are continuous in the norm)**.** *Let $Q, Q' \subset \mathbb{R}$ be bounded domains such that $Q \Subset Q'$. If $ps < 1$, then any function f in $\mathbb{W}^s_p(Q')$ is continuous in the following sense. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$|y| \le \delta,$$

then

$$\|f(\cdot + y) - f(\cdot)\|_{\mathbb{W}^s_p(Q)} \le \varepsilon,$$

where $\delta < dist(\partial Q, \partial Q')$ (for the statement to be well-defined).

Proof. Choose $\varepsilon > 0$.

Since $ps < 1$, the set $C_0^\infty(Q')$ (compactly supported smooth function in Q') is dense in $\mathbb{W}_p^s(Q')$ [51, 4.3.2, Theorem 1]. Then there exists $\tilde{f} \in C_0^\infty(Q')$ such that

$$\|f - \tilde{f}\|_{\mathbb{W}_p^s(Q')} \leq \frac{\varepsilon}{3}.$$

Note that if $f \in \mathbb{W}_p^s(Q')$, then $\|f\|_{\mathbb{W}_p^s(Q)} \leq \|f\|_{\mathbb{W}_p^s(Q')}$. Hence the previous inequality implies that

$$\|f - \tilde{f}\|_{\mathbb{W}_p^s(Q)} \leq \|f - \tilde{f}\|_{\mathbb{W}_p^s(Q')} \leq \frac{\varepsilon}{3},$$
$$\|f(\cdot + y) - \tilde{f}(\cdot + y)\|_{\mathbb{W}_p^s(Q)} \leq \|f - \tilde{f}\|_{\mathbb{W}_p^s(Q')} \leq \frac{\varepsilon}{3},$$

where $|y| < dist(\partial Q, \partial Q')$.

Since $\tilde{f} \in C_0^\infty(Q')$, there is a $0 < \delta < dist(\partial Q, \partial Q')$ such that if $|y| \leq \delta$, then

$$\|\tilde{f}(\cdot + y) - \tilde{f}(\cdot)\|_{\mathbb{W}_p^s(Q)} \leq Const \|\tilde{f}(\cdot + y) - \tilde{f}(\cdot)\|_{C^1(\bar{Q})} \leq \frac{\varepsilon}{3}.$$

Hence, if $|y| < \delta$, then

$$\begin{aligned}
&\|f(\cdot + y) - f(\cdot)\|_{\mathbb{W}_p^s(Q)} \\
&\leq \|f(\cdot + y) - \tilde{f}(\cdot + y)\|_{\mathbb{W}_p^s(Q)} + \|\tilde{f}(\cdot + y) - \tilde{f}(\cdot)\|_{\mathbb{W}_p^s(Q)} + \|\tilde{f}(\cdot) - f(\cdot)\|_{\mathbb{W}_p^s(Q)} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

□

Lemma 9.5. *If $f_1 \in \mathbb{W}_p^s(a,b)$ and $f_2 \in \mathbb{W}_p^s(b,c)$, where $s < \frac{1}{p}$, $p > 1$, then the function*

$$f(\theta) = \begin{cases} f_1(\theta), & \theta \in [a,b), \\ f_2(\theta), & \theta \in [b,c], \end{cases}$$

belongs to $\mathbb{W}_p^s(a,c)$.

Proof. Define the space

$$\widetilde{\mathbb{W}}_p^s(a,b) = \{\varphi | \varphi \in \mathbb{W}_p^s(\mathbb{R}), supp\, \varphi \in [a,b]\}$$

by Triebel [51, Chapter 4.3.2, Theorem 1], if $s \leq \frac{1}{p}$ and $s - \frac{1}{p} \neq integer$ then

$$\widetilde{\mathbb{W}}_p^s(a,b) = \mathbb{W}_p^s(a,b).$$

This means that $supp\, f_1 \in [a,b]$ and hence the function

$$\tilde{f}_1(\theta) = \begin{cases} f_1(\theta), & \theta \in [a,b), \\ 0, & \theta \in [b,c], \end{cases}$$

is in the space $\mathbb{W}_p^s(a,c)$. In the same way, the function

$$\tilde{f}_2(\theta) = \begin{cases} 0, & \theta \in [a,b), \\ f_2(\theta), & \theta \in [b,c], \end{cases}$$

is in the space $\mathbb{W}_p^s(a,c)$. Since $f = \tilde{f}_1 + \tilde{f}_2$, then $f \in \mathbb{W}_p^s(a,c)$. □

The following lemma is proved in [8].

Lemma 9.6. *Let $f \in \mathbb{W}_p^s(a,c)$, $s < \frac{1}{p}$, and b such that $a < b < c$. Then*

$$\|f\|_{\mathbb{W}_p^s(a,c)} \leq C\big(\|f\|_{\mathbb{W}_p^s(a,b)} + \|f\|_{\mathbb{W}_p^s(b,c)}\big),$$

where $C \to \infty$ as $\min\{b-a, c-b\} \to 0$.

9.5 Spectral radius and equivalent norms

Definition 9.7 (spectral radius)**.** Let $\mathbf{L} : X \to X$ be a bounded linear operator. The spectral radius of $\mathbf{L}$ is defined as

$$r(\mathbf{L}) = \sup\{|\lambda|; \lambda \in \sigma(\mathbf{L})\}.$$

By [12, Chapter VII.3] the limit $\|\mathbf{L}^n\|^{\frac{1}{n}}$ exists and

$$r(\mathbf{L}) = \lim_{n\to\infty} \|\mathbf{L}^n\|^{\frac{1}{n}}.$$

Lemma 9.8. *Let X be a Banach space with a norm $\|\cdot\|$. Let $\mathbf{L} : X \to X$ be a bounded linear operator. If the spectral radius satisfies*

$$r(\mathbf{L}) < 1,$$

then there exists an equivalent norm on X, $\|\cdot\|^*$ such that

$$\|\mathbf{L}\|^* < 1$$

and if $\varphi \in X$, then

$$\|\varphi\| \leq \|\varphi\|^*.$$

Proof. Let $c \in \mathbb{R}$ be such that

$$r(\mathbf{L}) < c < 1.$$

Let $\varphi \in X$. Define the norm

$$\|\varphi\|^* := \sum_{n=0}^{\infty} \frac{\|\mathbf{L}^n\varphi\|}{c^n}.$$

Note that the series converge:

$$\|\varphi\|^* = \sum_{n=0}^{\infty} \frac{\|\mathbf{L}^n\varphi\|}{c^n} \leq \sum_{n=0}^{\infty} \frac{\|\mathbf{L}^n\|\|\varphi\|}{c^n},$$

where the convergence of the last series is by the root test for the convergence of a series.

$$\lim_{n\to\infty} \left(\frac{\|\mathbf{L}^n\|}{c^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\|\mathbf{L}^n\|^{\frac{1}{n}}}{c} = \frac{r(\mathbf{L})}{c} < 1. \tag{9.4}$$

We leave it to the reader to check the norm properties of $\|\cdot\|^*$.

Next we show that the norms are equivalent. On the one hand, $\|\varphi\| \leq \|\varphi\|^*$, since

$$\|\varphi\| \leq \|\varphi\| + \underbrace{\sum_{n=1}^{\infty} \frac{\|\mathbf{L}^n \varphi\|}{c^n}}_{\geq 0} = \|\varphi\|^*.$$

On the other hand, $\|\varphi\|^* \leq Const\|\varphi\|$ by (9.4). Finally, we show that $\|\mathbf{L}\|^* < 1$.

$$\|\mathbf{L}\varphi\|^* = \sum_{n=0}^{\infty} \frac{\|\mathbf{L}^{n+1}\varphi\|}{c^n} = c\sum_{n=0}^{\infty} \frac{\|\mathbf{L}^{n+1}\varphi\|}{c^{n+1}} \leq c\|\varphi\|^*.$$

□

9.6 $\mathbb{R}^N$ norm of an integral of an $\mathbb{L}_p$ function

Lemma 9.9. *Let $f \in \mathbb{L}_p(a,b)$, $p > 1$. Then*

$$\left\| \int_a^b f(s)ds \right\|_{\mathbb{R}^N} \leq (b-a)^{\frac{p-1}{p}} \|f\|_{\mathbb{L}_p(a,b)}.$$

Proof. The proof follows from the calculation.

$$\left\| \int_a^b \varphi(s)ds \right\|_{\mathbb{R}^N} \leq \int_a^b \|\varphi(s)\|_{\mathbb{R}^N} ds = \|\varphi\|_{\mathbb{L}_1(a,b)} \leq (b-a)^{\frac{p-1}{p}} \|\varphi\|_{\mathbb{L}_p(a,b)},$$

where the last inequality follows from Hölder's inequality. □

9.7 Technical Lemmas

The following lemma is used in the proof of Theorem 7.6.

Lemma 9.10. *The following takes place.*

1. *For every $\chi > 0$, there exists $\bar{\chi} > 0$ such that*

$$(a+b)^2 \leq (1+\chi)a^2 + \bar{\chi}b^2, \tag{9.5}$$

   for every $a, b \in \mathbb{R}$.

2. *Let a sequence $c_1, c_2, \ldots$ of nonnegative numbers satisfy the inequalities*

$$c_{i+1} \leq \gamma c_i + f_i, \quad i \in \mathbb{N},$$

   where f_i are positives, $f_i \to 0$ as $i \to \infty$ and $\gamma < 1$ is independent of i. For every $\varepsilon > 0$ there is $\delta > 0$, such that if

$$c_1 \leq \delta,$$
$$f_i \leq \delta \text{ for all } i \in \mathbb{N},$$

   then $c_i \leq \varepsilon$ for all $i \in \mathbb{N}$ and $c_i \to 0$ as $i \to \infty$.

Proof. 1. We use "Young's inequality with epsilon"

$$2ab \le \chi a^2 + \frac{b^2}{\chi}.$$

Hence

$$(a+b)^2 = a^2 + b^2 + 2ab \le a^2 + b^2 + \chi a^2 + \frac{b^2}{\chi} = (1+\chi)a^2 + (1+\frac{1}{\chi})b^2.$$

Set $\bar{\chi} = 1 + \frac{1}{\chi}$ to get relation (9.5).

2. Fix an arbitrary $\varepsilon > 0$. Choose $\bar{\gamma} > 0$ such that $\gamma + \bar{\gamma} < 1$, and $\delta > 0$ such that $\delta \le (1-\gamma-\bar{\gamma})\varepsilon$. Assume that $f_i \le \delta$ for all $i \in \mathbb{N}$ and that $c_1 \le \delta$. Then $c_1 \le \varepsilon$ and

$$c_2 \le \gamma c_1 + f_1 \le \gamma\varepsilon + (1-\gamma-\bar{\gamma})\varepsilon \le (1-\bar{\gamma})\varepsilon \le \varepsilon.$$

Continue by induction. If $c_i \le \varepsilon$, then

$$c_{i+1} \le \gamma c_i + f_i \le (1-\bar{\gamma})\varepsilon \le \varepsilon,$$

which shows that $c_i \le \varepsilon$ for all $i \in \mathbb{N}$.

Now take an arbitrary $\varepsilon_1 > 0$. By the lemma's assumption, there exists $N_1 > 0$ such that $f_i \le (1-\gamma-\bar{\gamma})\varepsilon_1$ if $i \ge N_1$. Thus, for every $i \ge N_1$, if $c_i \ge \varepsilon_1$ then

$$c_{i+1} \le \gamma c_i + (1-\gamma-\bar{\gamma})\varepsilon_1 \le (1-\bar{\gamma})c_i,$$

and if $c_i \le \varepsilon_1$ then

$$c_{i+1} \le \gamma c_i + (1-\gamma-\bar{\gamma})\varepsilon_1 \le (1-\bar{\gamma})\varepsilon_1 \le \varepsilon_1,$$

which implies that there exists $N_2 \ge N_1$ such that $c_i \le \varepsilon_1$ for all $i \ge N_2$. □

9.8 Nonlinear auxiliary function

Lemma 9.11. *Let a function f be defined as*

$$f(\theta) := \begin{cases} a\theta & \theta \in [-\delta, 0] \\ 0 & \theta \in (0, \infty), \end{cases} \tag{9.6}$$

where $a \in \mathbb{R}$ and $\delta \in (0,1) \cap (0,T)$. Then for $s \in (0,1)$

$$\|f\|_{\mathbb{W}_p^s(-\delta, T-\delta)} \le Const\, \delta^{1-s+\frac{1}{p}}, \tag{9.7}$$

where $Const > 0$ does not depends on δ and T.

Proof. We have

$$
\begin{aligned}
&\|f\|_{L_p(-\delta,T-\delta)} \\
&= \left(\int_{-\delta}^{T-\delta} |f(\theta)|^p d\theta\right)^{\frac{1}{p}} = \left(\int_{-\delta}^{0} |a\theta|^p d\theta\right)^{\frac{1}{p}} = \frac{a}{p+1}\delta^{1+\frac{1}{p}} \le Const\,\delta^{1+\frac{1}{p}-s}.
\end{aligned}
$$

Now we estimate the seminorm

$$
\begin{aligned}
&\left(\int_{-\delta}^{T-\delta}\int_{-\delta}^{T-\delta}\left(\frac{|f(t)-f(r)|^p}{|t-r|^{1+sp}}\right)drdt\right)^{\frac{1}{p}} \\
&= \left(\int_{-\delta}^{T-\delta}\int_{-\delta}^{0}\left(\frac{|f(t)-f(r)|^p}{|t-r|^{1+sp}}\right)drdt + \int_{-\delta}^{T-\delta}\int_{0}^{T-\delta}\left(\frac{|f(t)|^p}{|t-r|^{1+sp}}\right)drdt\right)^{\frac{1}{p}} \\
&= |a|\Bigg(\underbrace{\int_{-\delta}^{0}\int_{-\delta}^{0}\left(|t-r|^{p-1-sp}\right)drdt}_{(A)} + \underbrace{\int_{0}^{T-\delta}\int_{-\delta}^{0}\left(\frac{|r|^p}{|t-r|^{1+sp}}\right)drdt}_{(B)} \\
&\quad + \underbrace{\int_{-\delta}^{0}\int_{0}^{T-\delta}\left(\frac{|t|^p}{|t-r|^{1+sp}}\right)drdt}_{(C)}\Bigg)^{\frac{1}{p}}.
\end{aligned}
$$

Estimate each part separately. We have

$$
(A) = \frac{2}{(p-sp)(p-sp+1)}\delta^{p-sp+1}.
$$

After changing the order of integration, $(B) = (C)$. We must calculate only (C) then:

$$
\begin{aligned}
(C) \underbrace{=}_{0\le t\le r} &\int_{-\delta}^{0}\int_{0}^{T-\delta}\left((-t)^p(r-t)^{-1-sp}\right)drdt = -\int_{-\delta}^{0}\frac{1}{sp}(-t)^p\left[(r-t)^{-sp}\right]\Bigg|_{r=0}^{r=T-\delta} dt \\
&= \int_{-\delta}^{0}\frac{1}{sp}(-t)^p((-t)^{-sp}-(T-\delta-t)^{-sp})dt \le \int_{-\delta}^{0}\frac{1}{sp}(-t)^{p-sp}dt \\
&= -\frac{1}{(sp)(p-sp+1)}\left[(-t)^{p-sp+1}\right]\Bigg|_{-\delta}^{0} = \frac{\delta^{p-sp+1}}{(sp)(p-sp+1)}.
\end{aligned}
$$

Taking it to the power of $\frac{1}{p}$ both on (A) and (C) (=(B)), we obtain estimate (9.7). $\square$

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