The DeWitt equation in Quantum Field Theory and its applications

Im Fachbereich Physik der Freien Universität Berlin

eingereichte Dissertation

By:

Parikshit Dutta

November 2013

Selbstständigkeitserklärung

Hiermit versichere ich, die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt zu haben. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Berlin, 14.11.2013   *Parikshit Dutta*
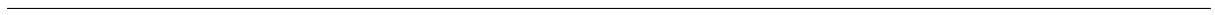
Zusammenfassung

Diese Arbeit behandelt eine Funktional-Differentialgleichung für die effektive Wirkung von Quantenfeldtheorien, erstmals hergeleitet von Bryce de Witt. Das Ziel war es, diese Gleichung, die einen alternativen Zugang zu Quantenfeldtheorien bietet, zu lösen. In dieser Arbeit ist es gelungen, eine formale Lösung dieser Gleichung in Form einer Schleifenentwicklung zu finden. Die Lösung ist insofern formal als sie die nackten, also nicht renormierten Parameter enthält. In diesem Zusammenhang werden weiters einige Probleme diskutiert, die bei der Definition der Gleichung selbst auftreten. Glücklicherweise ist Renormierung für gewisse Feldtheorien nicht erforderlich, die frei von Divergenzen sind, wie etwa einige spezielle supersymmetrische Theorien. Die eingangs erwähnten Techniken werden angewandt, um die effektive Wirkung des $N = 1$ Wess-Zumino-Modells in zwei Dimensionen zu ermitteln. Das Ergebnis wurde bis zur 2-Schleifen-Ordnung, also bis zur Ordnung $\hbar^2$ überprüft. Der Versuch, diese Lösung auch auf die supersymmetrische $N = 4$ Yang-Mills-Theorie anzuwenden, scheiterte in Ermangelung eines off-shell-Superraum-Formalismus für dieses Modell, und es wurden lediglich einige mögliche Lösungswege über den Lichtkegel-Superraum-Formalismus und seine Impulsraumdarstellung diskutiert. Weiters wurde die de Witt-Gleichung auf die Liouville-Feldtheorie angewandt, eine wichtige zweidimensionale konforme Feldtheorie, die in den letzten Jahren trotz offener Probleme häufig diskutiert wurde. Einige dieser offenen Punkte wurden mithilfe der de Witt-Gleichung analysiert. Mit der Schwinger-Dyson-Gleichung der nullten Mode, die äquivalent zur de Witt-Gleichung ist, war es möglich, die Polstruktur der Korrelationsfunktionen der primären Operatoren dieser Theorie zu untersuchen. Die Existenz einer dualen Gleichung gibt weitere Aufschlüsse über die Struktur der Korrelationsfunktionen dieser Theorie von einer sehr unterschiedlichen Perspektive. Im Zuge dieser Erörterungen werden einige relevante Konzepte eingeführt, die für die durchgeführten Berechnungen und die Diskussion der Ergebnisse von Nutzen sind. Die Dissertation endet mit einer Interpretation der Resultate und einem Ausblick auf offene Fragen, die in zukünftigen Arbeiten behandelt werden sollen.

Abstract

In this work we looked at a functional differential equation, written down by Bryce De-Witt, which is an equation for the Effective Action in Quantum Field Theory. The goal was to solve this equation, which gives a different perspective to approach Quantum Field Theory, than the usual path integral approach. In our work, we were able to write a formal solution for this equation, in terms of a loop expansion. It is formal in the sense, that all the parameters in the solution are bare parameters, and not renormalized ones. In this context we also discussed some problem with defining the equation itself. Luckily for some field theories, there is no need for renormalization, as they are completely free of divergences. These are some special supersymmetric theories. We employed this technique for writing down the Effective Action for the $N = 1$ Wess Zumino model in 2 dimensions, which is a finite model, and checked it to 2-Loops i.e. to order $\hbar^2$. Then we tried to employ this solution for $N = 4$ Super Yang Mills theory in 4 dimensions, but due to lack of an off-shell superspace formalism for this model, we were not able to do this, although we discussed some possible ways of doing this in Light Cone superspace formalism, and its momentum representation. Next we used this equation to study Liouville Field theory. This being a very important 2 dimensional conformal field theory, which has been much studied in the past few years, although there are some open points which are not yet understood. We employed this equation to analyze some of this questions. Firstly we wrote down the zero mode Schwinger Dyson equation (equivalent to the DeWitt equation). Then using this we were able to analyze the pole structure of the correlation functions of primary operators in this theory. Arguing the existence of another dual equation, we were able to shed some light on the structure of the correlation functions in this theory, from a very different perspective. During the course of this, we will introduce some relevant concepts, which will be useful in our discussion of the calculations and results. In the end we conclude with discussion of the results and some open questions, which we would like to address in the future.

Contents

Chapter 1

Introduction

Quantum Field Theory is a very interesting subject. It came about, when physicists tried to merge Quantum Mechanic with the theory of Special Relativity. Quantum Mechanics was the theory which came about to describe the wave-particle dual nature of matter (and light as a matter of fact). It probably began in the very beginning of the 19th century with the work of some great minds, like Max Planck (black body radiation), Albert Einstein (Photo electric effect), Niels Bohr (Hydrogen Atom) etc.. The mathematical frame work for the quantum theory was set up by Heisenberg and Schrodinger as we all know. Then Dirac wrote down his relativistic wave equation, which describes the electron and also the canonical quantization scheme. Then it really took into action, as Feynman, Schwinger and others constructed the Path Integral Formulation of quantum field theory, with the help of which they where able to construct Quantum Electro Dynamics. After that many people contributed to construct the Standard Model of particle physics which describes the known world around us. A key ingredient of all this was that their were divergences in quantum field theory, which made no sense. These divergences occurred while computing scattering amplitudes of particle. Then the marvelous idea of Renormalization arrived. Worked on by Gell Mann, Wilson, Gross, Politzer, Wilczek and others, they constructed the frame work that we now are accustomed with. Then came Supersymmetry and other ideas, and advances in the field kept on growing and growing. This short history of this vast subject is really not justified, but the history itself is so vast that describing it in proper way will take us very long. Instead we intend now to discuss the key concepts that we utilize for our study. In this chapter we will give a brief introduction to these key concepts, so that one can understand in words, what our goal is.

Quantizing a Classical System

As quantum mechanics is understood as a more fundamental theory of nature, the question arises, if one is given a classical system, how does one construct a quantum theory out of it. As we know a classical system is described by some variables, generally coordinates (q_i) and momenta (p_i)or velocity $(\dot{q}_i)$. From this variables one can construct the action S, a functional of q_i and $\dot{q}_i$, whose minimization gives the equations of motion, which provides the dynamics of the system. The action itself is the time integral over a functional L, called the Lagrangian of the system. In simple terms this is the kinetic energy minus the potential energy of the system, while another known quantity the Hamiltonian H, is the kinetic energy plus the potential energy of the system. The Hamiltonian itself is expressed as the Legendre transform of the Lagrangian as the variable describing

the two functions are the $(q_i, \dot{q}_i)$ and (q_i, p_i) respectively. Now mathematical frame work of classical mechanics tells us that the Hamiltonian can be expressed in terms of any canonical variables, (Q_i, P_i) which are related to the previous set (q_i, p_i) by a canonical transformation. A canonical transformation is one which keeps the form of Hamilton's equations invariant. Nevertheless once this mathematical structure is set up, one more or less has complete understanding of the classical system. The idea proposed by Dirac for quantizing a system, is that one promotes, the canonical variables to operators (denoted by ^ symbol), and then the commutation relation between these operators is given as follows:

$$[\hat{Q}_i, \hat{P}_i] = i\hbar \{Q_i, P_i\}_{P.B} \tag{1.1}$$

This is called the canonical quantization procedure. Now lets turn to the other perspective. We know that the classical system is described by the action $S(q_i)$. Feynman pointed out that one can quantize a system, and calculate the transition amplitude between different quantum states of the system, by summing over all possible trajectories with two fixed end points, weighted with the function $exp[-iS(q_i)/\hbar]$. More precisely for a given quantum system, the transition amplitude between two states is formally given by :

$$\int_{q_i^I}^{q_i^F} \mathcal{D}q_i \, exp[-\frac{i}{\hbar}S(q_i)] \tag{1.2}$$

Where I and F denote the initial and final positions. This is the well known path integral formulation. Generalizing to infinite number of coordinate variables, i.e. infinite degrees of freedom, the above version is generalized to path integrals in quantum field theory. There is another version of the path integral, which is the Hamiltonian formulation. In fact it is more fundamental than the above version (Lagrangian formulation), and is written as follows:

$$\int_{q_i^I}^{q_i^F} \mathcal{D}q_i \mathcal{D}p_i \, exp[-\frac{i}{\hbar}(\int dt H(q_i, p_i) - p_i \dot{q}_i)] \tag{1.3}$$

where after integration with the p_i's we obtain (1.2). Now the most fundamental property of the path integral is that it is invariant under the the change of integration variables variable $q_i \rightarrow q_i + \epsilon_i$ in functional space (ϵ_i being a constant in functional space). This means that the measure of the path integral is invariant under the redefinition, $\mathcal{D}q_i \rightarrow \mathcal{D}(q_i + \epsilon_i)$. This can also be understood from the fact that the path integral is analogous to the Fourier transform, and that is invariant under translation of the integral measure. We will discuss more about this later in Chapter 3. There is also one view point, which comes about from the Schwinger action principle. It states that the variation of the transition amplitude between two states is equal to the expectation value of the variation of the action between these states. From this principle and after modifying the Lagrangian by a source term, one obtains the Schwinger equation, which states the expectation value of the equation of motion in presence of a source term vanishes. If one finds the solution to this equation one can find all transition amplitudes in quantum field theory. The path integral is actually the solution of this equation, represented as a Fourier transform in functional space. This equation or a different version of it which we will call the DeWitt equation is a functional differential equation of no known type. It is in general plagued by singularities occurring from the expectation value of fields at the same point. Our work was in general to understand the structure of the solution of this equation. We were able to do that and in Chapter 4, we explicitly write down the formal solution. We must emphasize here that the solution, is unrenormalized one, but it is interesting to see

the structures in it nevertheless. More over the solution is an expansion in orders of $\hbar$, which is equivalent to an expansion in loops. Although there is a general understanding that the solution works, nevertheless it can only be checked order by order. We perform this check for a specific model later on. We also emphasize the analogy of the path integral in Euclidian signature to the Laplace transform and perform some examples in zero dimension case (which is space of ordinary integrals) to give a flavour of the DeWitt equation.

Using Supersymmetry

While there is problem of renormalization in common field theories, there are some special ones where this is not required. This is where Supersymmetry comes in. Supersymmetry is a special symmetry of a Lagrangian under consideration such that there exists a transformation of the bosonic degrees of freedom to the fermionic ones, and vice versa, such that the action is left invariant under it (more precisely the Lagrangian is left invariant upto a total derivative). This symmetry has a very far reaching consequence. Firstly in four dimensions, all the quadratic divergence in the theory is canceled. Secondly depending on the dimension of the theory, and "number of Supersymmetries" (there can be one such transformation or many for a given theory, giving rise to minimal or extended supersymmetry), the theory can be completely free of divergences and can be finite (i.e. even the logarithmic divergences cancel in 4 dimensions). For example in 4 dimension $N = 4$ supersymmetric Yang Mills Theory (a theory which has four independent supersymmetry transformations, although they transform into one another by action of $SU(4)$ group element) is finite. Another example is N=1 Wess Zumino model, which has only one supersymmetry and is finite in 2 dimensions. Thus in these theories, there is no need of renormalization, as the expectation value of the fields occurring in the DeWitt equation sum up to be finite in this case.

It is important to note that it is a bit subtle than what we pointed out. In fact for theories with extended Supersymmetries there in general is no known way to construct a covariant off-shell formalism with finite number of fields (on-shell means supersymmetry transformations leave the action invariant upto equations of motion of the fields). This means that the action is not supersymmetric off-shell. We will discuss this point in the Chapter on Supersymmetry and also later in the Chapter on $N = 4$ Super Yang Mills.

Thus for some specific models the solution does not need any renormalization. We discuss the solution in the context of $N = 1$ Wess Zumino model in 2 dimensions (using superspace construction) in Chapter 6. Also here we explicitly check the solution upto 3rd order in $\hbar$. Next we give a general overview of $N = 4$ Super Yang Mills theory. In particular the Light cone superfield formalism is discussed shortly. We intended to use this construction to write down the Effective Action for this model. Unfortunately this turned out to be very cumbersome and tedious. Formally speaking the solution holds for this model, and by construction is finite (as in Light cone superspace all the Green's functions are finite), but to implement the solution practically is difficult, because of non-local inverse derivatives appearing in the Light Cone formalism.

Here we want to point out that in superspace formalism, the measure of the path integral is linear. This means that the Jacobian of the transformation of the path integral measure under supersymmetry transformations, is identically one. This was shown in [40]. Although when there is no superspace representation of these theories, there could be a

functional determinant term i.e. an anomaly which arises from the non invariance of the path integral measure. We will not deal with these anomalies while using translation invariance of the measure to derive the DeWitt equation although this is a very important point for on-shell supersymmetric field theories.

Conformal Field Theory

Next we turn our attention to some very special field theories specifically in 2 dimensions. These are conformal field theories. They have the special property that under conformal transformation the action remains invariant. A conformal transformation in the strict sense is a transformation of the space time coordinates such that the angle of two arbitrary curves intersecting at a point is preserved. The action of such a transformation on the set of fields of the theory defines a conformal field theory. The Poincare' symmetry group is a sub group of the conformal symmetry group. More over just like there are definite transformation laws for the correlation functions for a given symmetry of the Lagrangian (like in non-Abelian gauge theories where the n-point functions have gauge indices and transform under action of the gauge group, and also in the general case where they transform under the action of the Poincare' group), the transformation laws of he correlation functions in a conformally invariant theory restricts their structure to a great deal. More over we will discuss the constituents of a given particular CFT, namely the primary fields, from which all other fields can be constructed, the energy momentum tensor, which is a very important object, along with some other important concepts in the Chapter on conformal field theory.

A very special Conformal Field Theory is Liouville Field Theory in 2 dimensions. It is important from the point of view of String theory, as it comes up in the quantization of strings in non-critical dimensions. Our interest in this theory is primarily from the point of view of quantum field theory. the Liouville potential is of exponential type. Hence for this case the DeWitt equation (or the equivalent Schwinger Dyson equation) is of simple form. This makes the study of this theory comparatively simpler in contrast to theories with other interaction terms. In our Chapter on this topic we will discuss how we can obtain the three point correlation function for this model. More over we will argue on the existence of another functional equation, which comes about due to regularizing the interaction term. Form this dual DeWitt equation, we will obtain the pole structure of the three point function, and shed some light on its uniqueness. This technique developed in this work is pretty simple and interesting, and also useful for other Liouville type theories. It would be worth while to investigate other theories in the same way.

In the end we will give short concluding remarks on the results obtained during this work and also discuss some open questions and future prospects.

Chapter 2

The Effective Action

2.1 Motivation

Let us look at the problem of determining the vacuum expectation value of a quantum field, which is to be determined as a function of the parameters of the Lagrangian. While dealing with the problem at a classical level, we can minimize the potential energy to find the expectation value of the field. But when dealing with the full quantum theory, the classical value can be altered by loop corrections. Thus the motivation is to find a function in the full quantum picture, whose minimum would give us the exact value of the vacuum expectation value of the field. There is a analogous situation in statistical systems. A quantum system can be viewed as a statistical one, with the quantum fluctuations replaced by thermal ones. At zero temperature the thermodynamic ground state is the state of lowest energy, but at non zero temperature it is that state which minimizes the Gibbs free energy. Let us take the example of a magnetic system. The Helmholtz free energy $A(H)$ is defined as follows:

$$Z(H) = e^{-\beta A(H)} = \int \mathcal{D}s \, exp\left[-\beta \int dx (\mathcal{H}[s] - Hs(x)) \right] \tag{2.1}$$

where H is the external magnetic field, $\mathcal{H}[s]$ is the spin energy density and $\beta = 1/kT$. Now we can find the magnetization of this system as follows:

$$
\begin{aligned}
-\frac{\partial A}{\partial H}\bigg|_{\beta=constant} &= -\frac{1}{\beta}\frac{\partial}{\partial H} log \, Z \\
&= \frac{1}{Z}\int dx \int \mathcal{D}s \, s(x) exp\left[-\beta \int dx(\mathcal{H}[s] - Hs(x)) \right] \\
&= \int dx \langle s(x) \rangle \equiv \mathcal{M}
\end{aligned}
\tag{2.2}
$$

Now the Gibbs free energy of the system is defined by the Legendre transform as follows:

$$G = A + \mathcal{M}H \tag{2.3}$$

From this definition one obtains the following:

$$\frac{\partial G}{\partial \mathcal{M}} = \frac{\partial A}{\partial \mathcal{M}} + \mathcal{M}\frac{\partial H}{\partial \mathcal{M}} + H$$

$$\begin{aligned} &= \frac{\partial H}{\partial \mathcal{M}}\frac{\partial A}{\partial H} + \mathcal{M}\frac{\partial H}{\partial \mathcal{M}} + H \\ &= H \end{aligned}$$

$$(2.4)$$

Where β has been kept fixed. Now if $H = 0$, One can find the extremum of the Gibbs free energy at the corresponding value of the magnetization $\mathcal{M}$. We know that the thermodynamically most stable state is the minimum of $G(\mathcal{M})$. Thus, this function gives a geometric picture of the preferred thermodynamic state, inclusive of all thermal fluctuations. By analogy a similar quantity can be constructed in quantum field theory, which we will discuss now.

2.2 Partition function and Effective action

The Euclidian path integral in D dimensional quantum field theory is defined as follows:

$$Z[J] \equiv exp\left[-\frac{1}{\hbar}W[J] \right] \equiv \int \mathcal{D}\phi\, exp\left[-\frac{1}{\hbar}\left(S[\phi] + \int d^D x J(x)\phi(x) \right) \right] \qquad (2.5)$$

Where the measure $\mathcal{D}\phi$ is formally normalized to unity, that is $Z[0] = 1$. The path integral is formally the sum over all field configurations and is the solution to the expectation value of the operator equation of motion of a given theory. $W[J]$ is the generator of connected Green's function of the theory, which are the Greens function such that in a diagramatic representation, cutting any line in the diagram does not make it fall into two separate ones. The connected Green's functions in the presence of a source J are then given by

$$W_n(x_1, ..., x_n; J) \equiv \langle \phi(x_1)\cdots\phi(x_2)\rangle_{connected} \equiv (-\hbar)^{n-1}\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)} \qquad (2.6)$$

Setting the current $J(x)$ to zero we obtain the full connected Greens function of the theory, which are the scattering amplitudes which defines the scattering processes.The classical field $\varphi(x)$ is the vacuum expectation value of the quantum field $\phi(x)$ and is defined as follows:

$$\varphi(x; J) = \frac{\delta W[J]}{\delta J(x)} \qquad (2.7)$$

Then effective action Γ is defined as the Legendre transform (in analogy to the Gibbs free energy) of $W[J]$ as follows:

$$\Gamma[\varphi] = W[J] - \int d^D x J(x)\varphi(x) \qquad (2.8)$$

Thus we can express $J(x)$ as a functional of $\varphi(x)$ through the inverse relationship:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x; \varphi) \qquad (2.9)$$

We will assume in the following that the relation between $J = J(x; \varphi)$ and $\varphi(x; J)$ can be freely inverted.Usually setting the right hand side to zero, and solving for a constant field configuration we get the vacuum expectation value of the field. On the other hand

if one chooses a constant field configuration φ_c, and evaluate Γ then one ends up with the effective potential for the theory defined as follows:

$$V_{eff}[\varphi_c] = \frac{1}{\mathcal{V}}\Gamma[\varphi_c] \tag{2.10}$$

Where Γ is divided by a infinite volume factor, $\mathcal{V} = \int d^D x$. Now in general from Γ we define the one particle irreducible n-point correlation functions of the model (the proper vertices of the theory) which are obtained by repeated functional derivative of Γ with respect to $\varphi(x)$:

$$\Gamma_n(x_1, ..., x_n; \varphi) \equiv \frac{(-1)^n}{\hbar} \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1) \cdots \delta\varphi(x_n)} \tag{2.11}$$

More over the W_ns and the Γ_ns are related with each other by relations of the following form:

$$\int d^D y W_2(x, y; \varphi)\Gamma_2(y, z; \varphi) = \delta^D(x - z) \tag{2.12}$$

and

$$W_3(x, y, z; \varphi) = \int d^D u\, d^D v\, d^D w\, W_2(x, u; \varphi)W_2(y, v; \varphi)W_2(z, w; \varphi)\Gamma_3(u, v, w; \varphi) \tag{2.13}$$

and so on for higher W_ns which can be seen by repeated functional derivatives of the first equation. It is to be noted that the J dependence of the W_ns are replaced by φ using the invertible relationships. The Effective action can also be expanded in powers of $\hbar$ as follows:

$$\Gamma[\varphi] = \Gamma^{(0)}[\varphi] + \hbar\Gamma^{(1)}[\varphi] + ... \tag{2.14}$$

where it is well known that $\Gamma^{(0)}[\varphi] = S[\varphi]$ is the classical action while higher order introduce quantum corrections, the first order being the trace log term. Our motivation is to find a more or less closed form for this expansion. To do this we will employ a functional differential equation. We will discuss this equation in the next chapter. There also exists different but analogous methods for calculating the effective action, which one can find in [61]. There the effective action is treated as a loop expansion and corresponds to every order in $\hbar$ expansion. There is also the background field formalism which is employed in [61] for this purpose. The reader is encouraged to read through the discussion on effective action in [61] for better understanding of the subject.[1].

---

[1] For more rigorous study of the subject please see [1],[3],[13],[14],[31],[39],[61]

Chapter 3

The DeWitt Equation

In 1965, B.DeWitt wrote down a functional equation for the Effective action in quantum field theory [1]. This equation relates the functional derivative of the full quantum effective action $[\varphi]$ to the functional derivative of the classical action. It is interesting in the sense that, while usual quantum field theory involves functional integration via the path integral formalism, here the information is encoded into a functional differential equation. In case the classical action is polynomial, this equation has only finite number of terms, which assumes a relatively simple form. More over this equation can serve as the generating functional for an infinite hierarchy of Schwinger-Dyson equations for the theory concerned.

The main problem with this equation is that it is hard to define properly. Although the path integral also faces similar problems, but there are established techniques such as renormalized perturbation theory which solves this problems, while in this case there are no such established techniques available. Most prominent of the problems is the fact that it involves functional derivative at same point which gives rise to short distance singularities which can not be renormalized using perturbative methods. So in some sense proper definition of the equation already requires some knowledge of the solution. So our approach will be to look for formal solution of this unrenormalized equation. It is also important to point out that the equation is of no known type which makes it even harder to solve.

3.1   Derivation of DeWitt Equation

Now we look at the formal definition of our main equation of interest, the DeWitt equation. The DeWitt equation is a functional differential equation for the effective action Γ of a given model in quantum field theory. Starting from the path integral, we see that the expectation value of a field functional $Q[\phi]$ in presence of a source J is defined in the following way (we employ Euclidian signature) :

$$\langle Q[\phi] \rangle_J := exp\left(\frac{1}{\hbar}W[J]\right) \int \mathcal{D}\phi Q[\phi] exp\left[-\frac{1}{\hbar}\left(S[\phi] + \int d^Dx J(x)\phi(x)\right)\right] \qquad (3.1)$$

This can be rewritten as follows:

$$\langle Q[\phi] \rangle_J := exp\left(\frac{1}{\hbar}W[J]\right) exp\left(-\frac{1}{\hbar}W\left[J - \hbar\frac{\delta}{\delta\phi}\right]\right) Q[\phi]\Big|_{\phi=0} \qquad (3.2)$$

Which is because of the following elementary identity:

$$f(x) = exp\left(x\frac{\partial}{\partial y}\right)f(y)\Big|_{y=0} \tag{3.3}$$

Which is just a Taylor series. Now upon expanding the expression for $W[J - \hbar\frac{\delta}{\delta\phi}]$ we obtain the following:

$$
\begin{aligned}
W\left[J - \hbar\frac{\delta}{\delta\phi}\right] =&\ W[J] - \hbar\int d^D x \frac{\delta W[J]}{\delta J(x)}\frac{\delta}{\delta\phi(x)} + \\
&-\hbar\sum_{n=2}^{\infty}\frac{1}{n!}\int d^D x_1 \cdots d^D x_n\, W_n(x_1,...,x_n;J)\frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}
\end{aligned} \tag{3.4}
$$

Inserting the above expression and expressing J as a functional of φ, using the inversion relation between them and once again using the elementary identity mentioned above we obtain the following:

$$
\begin{aligned}
\langle Q[\phi]\rangle_{J[\varphi]} =&\ \\
&{}^*_*\exp\left[\sum_{n=2}^{\infty}\frac{1}{n!}\int d^D x_1 \cdots d^D x_n\, W_n\left(x_1,...,x_n;J[\varphi]\right)\frac{\delta}{\delta\varphi(x_1)}\cdots\frac{\delta}{\delta\varphi(x_n)}\right]{}^*_* Q[\varphi]
\end{aligned} \tag{3.5}
$$

where the symbol ${}^*_*$ indicates that the functional differential operators act only on the external factor $Q[\varphi]$, but not on $J[\varphi]$ in W_n. It is important here that the sum in the exponent starts only at $n = 2$. Again there exists the identity, by DeWitt ,

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \left\langle\frac{\delta S[\phi]}{\delta\phi(x)}\right\rangle_{J=J[\varphi]} \tag{3.6}$$

This is because of the formal identity of the path integral being invariant under translation of the measure:

$$\int \mathcal{D}\phi\frac{\delta}{\delta\phi(x)}exp\left(-\frac{1}{\hbar}(S[\phi] + \int d^D x J(x)\phi(x))\right) = 0 \tag{3.7}$$

Leading to the following expression,

$$-J[\varphi](x) = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \left\langle\frac{\delta S[\phi]}{\delta\phi(x)}\right\rangle_{J=J[\varphi]} \tag{3.8}$$

Now inserting the functional $\delta S[\varphi]/\delta\varphi(x)$ for $Q[\varphi]$ we finally obtain the following equation for the effective action, which is the DeWitt equation,

$$
\begin{aligned}
\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} =&\ \\
&{}^*_*\exp\left[\sum_{n=2}^{\infty}\frac{1}{n!}\int d^D x_1 \cdots d^D x_n\, W_n\left(x_1,...,x_n;J[\varphi]\right)\frac{\delta}{\delta\varphi(x_1)}\cdots\frac{\delta}{\delta\varphi(x_n)}\right]{}^*_* \frac{\delta S[\varphi]}{\delta\varphi(x)}
\end{aligned} \tag{3.9}
$$

Observe that for polynomial actions $S[\phi]$ the functional differential operator reduces to a finite number of terms upon expansion of the exponential.

3.2   A simple example

We start by looking at a concrete example, consider the classically conformal ϕ^4 theory with the action

$$S[\phi] = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{4} \phi^4 \right) \tag{3.10}$$

Then the DeWitt equation can be written as follows:

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = \left\langle -\Box \phi(x) + \lambda \phi^3(x) \right\rangle_{J = J[\varphi]} \tag{3.11}$$

Which can be explicitly written as follows:

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = -\Box \varphi(x) + \lambda \varphi^3(x) + 3\lambda W_2(x, x; \varphi)\varphi(x) + \lambda W_3(x, x, x; \varphi) \tag{3.12}$$

Expressing W_2 and W_3 by means of (2.12) and (2.13) we see that all quantities in this equation can be expressed in terms of $\Gamma[\varphi]$ and its functional derivatives, so that the above indeed becomes a functional differential equation for $\Gamma[\varphi]$.

As they stand these equations, and in particular the basic functional equation (3.9), are formal. Nevertheless, there is already one useful application: the equation (3.9) can be used as a generating equation to derive the Schwinger-Dyson equations. With the standard formula for the one-particle irreducible n-point functions

$$\Gamma_n(x_1, \cdots, x_n) \equiv \Gamma_n(x_1, \cdots, x_n; \varphi)|_{\varphi = 0} \tag{3.13}$$

The first of which can be seen for the case of our example as follows:

$$\begin{aligned} \hbar \Gamma_2(x, y) &= (-\Box + 3\lambda W_2(x, x))\delta^D(x - y) \\ &- \lambda \int d^D u \, d^D v \, d^D w \, W_2(x, u) W_2(x, v) W_2(x, w) \Gamma_4(u, v, w, y) \end{aligned} \tag{3.14}$$

3.3   Examples in Zero dimensions

To get a flavour of the DeWitt equation we look at Zero dimensional field theory, where the path integral is an ordinary integral. Thus,

$$Z[j] = \int_c dx \, e^{xj - S[x]} \tag{3.15}$$

Where the contour c is some contour such that the integrand has the same value at the end points. In the path integral point of view this corresponds to choosing the end points to be lying on the classical path or trajectory (which is always the case). As a first step we can look at the Gamma function which is defined as follows for $Re[s] > 0$:

$$\Gamma[s] = \int_0^\infty dx \, x^{s-1} e^{-t} \tag{3.16}$$

19

The above is defined as a Mellin transform whose measure dt/t is invariant under dilation i.e. $x \to ax$. Under the following change of variables $x = e^t$, it can be defined as a Laplace transform (two sided) :

$$\Gamma[s] = \int_{-\infty}^{\infty} dt \, exp[ts - e^t] \tag{3.17}$$

Now we can see that the integrand is zero at both the limits i.e. ∞ and $-\infty$. It is easy to see the resemblance with the path integral now. Moreover the above integral has a translationaly invariant measure, i.e. $dx \to d(x + a)$. Hence we see the following identity which can be directly seen by integration byparts also:

$$\int_{-\infty}^{\infty} dt \, \frac{d}{dt} exp[ts - e^t] = 0$$
$$\Rightarrow s \int_{-\infty}^{\infty} dt \, exp[ts - e^t] - \int_{-\infty}^{\infty} dt \, e^t exp[ts - e^t] = 0$$
$$\Rightarrow s\Gamma[s] - \Gamma[s + 1] = 0$$
$$\Rightarrow \Gamma[s + 1] = s\Gamma[s] \tag{3.18}$$

Which is the functional equation for the Gamma function. This boils down to the property of bilateral (two sided) Laplace transform relating a function $f(t)$ with its derivative $f'(t)$,

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) \tag{3.19}$$

And thus,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} \tag{3.20}$$

It is straight forward to see the resemblance of the above equation to the DeWitt equation or equivalently the Schwing Dyson equation for the path integral in Quantum Field Theory. This is because the path integral itself is defined as a bilateral Laplace transform, as the end points of the path lie on a classical trajectory. Let us look at another example for this case. The Airy function has the following integral representation:

$$\mathcal{A}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, exp[i(zt + \frac{t^3}{3})] \tag{3.21}$$

Under the translation invariance of the measure, one obtains:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \frac{d}{dt} exp[i(zt + \frac{t^3}{3})] = 0$$
$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, (z + t^2) exp[i(zt + \frac{t^3}{3})] = 0$$
$$\Rightarrow z\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, exp[i(zt + \frac{t^3}{3})] - \frac{d^2}{dz^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, exp[i(zt + \frac{t^3}{3})]$$
$$\Rightarrow z\mathcal{A}(z) - \mathcal{A}''(z) = 0 \tag{3.22}$$

Which is the differential equation for the Airy function. Thus we understand the basic philosophy of this technique is to go from an integral representation of the function

(e.g. Gamma function) or functional (e.g. Z[J]) to its differential equation or functional differential equation respectively (of course one can run the argument the other way and state that $\Gamma[s]$ or Z[J] are the solution of the corresponding equations) . We will come back to more zero dimensional field theory examples when we discuss the solution of the DeWitt equation. It is important to note that while the equivalence is to a Laplace transform while dealing with the Euclidian path integral, analogous equivalence is to the Fourier transform while dealing with the Minkowski version of the path integral.

3.4 Equivalence of DeWitt equation and Schwinger Dyson equation

If one starts with the formal identity of the translation invariance of the path integral measure, one obtains (dropping $\hbar$ for simplicity) :

$$\int \mathcal{D}\phi \frac{\delta}{\delta\phi(x)} exp\bigg[- \big(S[\phi] + \int d^D x J(x)\phi(x)\big)\bigg] = 0$$
$$\int \mathcal{D}\phi \bigg(\frac{\delta S}{\delta\phi(x)} + J\bigg) exp\bigg[- \big(S[\phi] + \int d^D x J(x)\phi(x)\big)\bigg] = 0 \qquad (3.23)$$

Now as any pollynomial in $\phi(x)$ in the path integral can be replaced as follows:

$$\int \mathcal{D}\phi\, \phi^p(x) exp\bigg[-\big(S[\phi]+\int d^D x J(x)\phi(x)\big)\bigg] \equiv \int \mathcal{D}\phi\bigg(-\frac{\delta}{\delta J(x)}\bigg)^p exp\bigg[-\big(S[\phi]+\int d^D x J(x)\phi(x)\big)\bigg] \qquad (3.24)$$

Thus eq (3.23) can be rewritten as follows:

$$\int \mathcal{D}\phi\bigg(\frac{\delta S}{\delta\phi(x)}\bigg[-\frac{\delta}{\delta J(x)}\bigg] + J\bigg) exp\bigg[- \big(S[\phi] + \int d^D x J(x)\phi(x)\big)\bigg] = 0 \qquad (3.25)$$

Which is the Schwinger Dyson equation. The DeWitt equation is just the formulation of the above object in terms of connected Green's function of the theory. The point is just that whether one wants to solve the theory in terms of the external source J (where φ is a functional of J), or the classical field φ (where J is a functional of φ). In the first case the Schwinger Dyson equation is used, while in the later case the DeWitt equation is used. In either of the two cases we have a functional differential equation but with different variables.

Another important thing to point out here is the Schwinger action principle. It states that if there is a transition amplitude between two states ϕ_f at time t_f and ϕ_i at time t_i, then the variation of this amplitude is given by:

$$\delta\langle\phi_f t_f|\phi_i t_i\rangle = \int_{t_i}^{t_i} d^4 x\langle\phi_f t_f|\delta\mathcal{L}|\phi_i t_i\rangle \qquad (3.26)$$

Where $\mathcal{L}$ is the Lagrangian density. This is an equivalent quantization scheme to the path integral. Using this principle and then changing the Lagrangian in presence of a source term slightly, i.e. we vary the source $J(x)$ to $J(x)+\delta J(x)$, we obtain the following :

$$\frac{\delta}{\delta J(x)}\langle\phi_f t_f|\phi_i t_i\rangle = \langle\phi_f t_f|\hat{\phi}(x)|\phi_i t_i\rangle \qquad (3.27)$$

It is important to note that one has a c-number operation in the L.H.S of this relation, while one deals with an operator in the R.H.S. . Along with this there is the field equation of motion, in the operator formalism obtained by variation of the action with respect to ϕ :

$$\frac{\delta S}{\delta \phi(x)} + J(x) = 0 \tag{3.28}$$

Now the expectation value of this equation must vanish. Combining this with the action principle, we have the Schwinger Dyson equation (as polynomial in ϕ can be replaced by polynomial in $-\frac{\delta}{\delta J}$ by action principle):

$$\langle \phi_f t_f | \frac{\delta S}{\delta \phi(x)} \left[-\frac{\delta}{\delta J(x)} \right] + J(x) |\phi_i t_i \rangle = 0 \tag{3.29}$$

The path integral corresponds to solving this equation by performing a Laplace transform (or a Fourier transform in Minkowski space). Thus we see there exists two alternate view points, one starting with the path integral and the other starting with the Schwinger Dyson equation. We will use the later view point as a starting point in our later chapters.

Chapter 4

Formal solution of DeWitt equation

Here we will give the formal solution of the DeWitt equation which is the unrenormalized effective action functional and can be presented as follows (in 4 dimensions):

$$
\begin{aligned}
\Gamma[\varphi] \;=\; & S[\varphi] \;+\; \frac{\hbar}{2} \int d^4x \, \log\left[\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(x)}\right] \\[2mm]
& -\hbar \log\left[\exp\left(\frac{\hbar}{2}\int d^4u\, d^4v \, G_{cl}(u,v;\varphi)\frac{\delta^2}{\delta\eta(u)\delta\eta(v)}\right)\exp\left(-\hbar^{-1}S_{\mathrm{int}}[\varphi,\eta]\right)\Big|_{\eta=0}\right]_{\mathrm{1PI}}
\end{aligned} \tag{4.1}
$$

Where the subscript $1PI$ means that one-particle reducible diagrams are to be omitted in the expansion, and where the logarith removes disconnected diagrams from inside the brackets. The interaction part of the action is defined as follows:

$$
\begin{aligned}
S_{\mathrm{int}}[\varphi,\eta] \;:=\; & S[\varphi+\eta] - S[\varphi] - \int d^4u\, \eta(u)\frac{\delta S[\varphi+\eta]}{\delta\varphi(u)}\bigg|_{\eta=0} \\[2mm]
& -\frac{1}{2}\int d^4u\, d^4v\, \eta(u)\eta(v)\frac{\delta^2 S[\varphi+\eta]}{\delta\varphi(u)\delta\varphi(v)}\bigg|_{\eta=0} \\[2mm]
=\; & \frac{1}{3!}\frac{\delta^3 S}{\delta\varphi^3}\eta^3 + \frac{1}{4!}\frac{\delta^4 S}{\delta\varphi^4}\eta^4 + \cdots
\end{aligned} \tag{4.2}
$$

(where implicit integration is understood) And,

$$
\int d^4y\, G_{cl}(x,y;\varphi)\frac{\delta^2 S[\varphi]}{\delta\varphi(y)\delta\varphi(z)} = \delta^{(4)}(x-z) \tag{4.3}
$$

The S_{int} introduces φ dependent vertices, but note that the φ dependence arises only from 4-point vertices onwards. The functional Γ is to be calculated with the classical field dependent Green's function $G_{cl}(x,y,\varphi)$. This solution matches with the structure, discussed in [57] and also [57] (for ϕ^4 theory).

4.1   Sketch of the Derivation

To look at how the solution works we start with another Zero dimensional field theory example, namely $\lambda\phi^4$, a system with finitely many degrees of freedom, in terms of which

the results described in the above section can be explicitly illustrated, and where we do not have to worry about UV infinities. The action we look at is as follows:

$$S(x) = \frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j + \frac{\lambda}{4} \sum_{j=1}^{n} x_j^4 \tag{4.4}$$

where A_{ij} is a non-degenerate positive definite matrix. The generating function $W(J) \equiv W(J_1, \ldots, J_n)$ for the 'connected Green's functions' is then defined in analogy with partition function as:

$$e^{-W(J)} := \int_{\mathbb{R}^n} dx \, \exp\left[-S(x) - \sum_j x_j J_j\right] \tag{4.5}$$

where the integration measure dx is normalized in such a way that $W(0) = 0$. The generating function is easily seen to satisfy the differential equation

$$\sum_j A_{ij} \frac{\partial Z}{\partial J_j} + \lambda \frac{\partial^3 Z}{\partial J_i^3} - J_i Z(J) = 0 \tag{4.6}$$

or, in terms of $W(J)$,

$$\sum_j A_{ij} \frac{\partial W}{\partial J_j} + \lambda \left[\frac{\partial^3 W}{\partial J_i^3} - 3 \frac{\partial^2 W}{\partial J_i^2} \frac{\partial W}{\partial J_i} + \left(\frac{\partial W}{\partial J_i}\right)^3\right] = -J_i \tag{4.7}$$

When expressed in terms of the effective action, this is the finite-dimensional analog of the DeWitt equation (3.9), seen below. So in analogy with (2.7) let us define the 'classical field' by

$$\varphi_i(J) := \frac{\partial W(J)}{\partial J_i} \tag{4.8}$$

and introduce the 'effective action' $\Gamma(\varphi)$ in the usual way by Legendre transformation as described earlier. The DeWitt equation now reduces to a set of partial differential equations:

$$\frac{\partial \Gamma(\varphi)}{\partial \varphi_i} = \exp\left[\sum_{k \geq 2} \frac{1}{k!} \sum_{j_1, \ldots, j_k} W_{j_1 \cdots j_k}(J) \frac{\partial}{\partial \varphi_{j_1}} \cdots \frac{\partial}{\partial \varphi_{j_k}}\right] \frac{\partial S(\varphi)}{\partial \varphi_i} \tag{4.9}$$

where $W_{j_1 \cdots j_k} \equiv (-1)^{k-1} \partial_{j_1} \cdots \partial_{j_k} W$, and we have relations analogous to (2.12) and (2.13), that is, $\sum_j W_{ij}(J)\Gamma_{jk}(\varphi(J)) = \delta_{ik}$, and so on. We can now produce a formal solution of (4.9), re-deriving a result that was essentially obtained by [8]. From the general definition we directly obtain the following differential equation for $\Gamma(\varphi)$

$$\exp\left[-\Gamma(\varphi) + \sum_j \varphi_j \frac{\partial \Gamma(\varphi)}{\partial \varphi_j}\right] = \int_{\mathbb{R}^n} dx \, \exp\left[-S(x) + \sum_j x_j \frac{\partial \Gamma(\varphi)}{\partial \varphi_j}\right] \tag{4.10}$$

To evaluate the integral we split the 'effective action' into a 'classical' part $S(\varphi)$ and a 'quantum' part $\tilde{\Gamma}(\varphi)$ according to

$$\Gamma(\varphi) = \frac{1}{2} \sum_{i,j=1}^{N} \varphi_i A_{ij} \varphi_j + \frac{1}{4} \lambda \sum_{j=1}^{N} \varphi_j^4 + \tilde{\Gamma}(\varphi) \tag{4.11}$$

Shifting integration variables as $x_j \to x_j + \varphi_j$ in (4.10), a little algebra gives

$$\exp\left[-\tilde{\Gamma}(\varphi)\right] =$$

$$\int_{\mathbb{R}^n} dx \, \exp\left[-\frac{1}{2}\sum_{ij} x_i G_{ij}^{-1}(\varphi) x_j - \lambda \sum_j x_j^3 \varphi_j - \frac{\lambda}{4} \sum_j x_j^4 + \sum_j x_j \frac{\partial \tilde{\Gamma}(\varphi)}{\partial \varphi_j}\right] \tag{4.12}$$

with the classical 'field-dependent' Green's function $G_{ij}(\varphi)$

$$\sum_j \left(A_{ij} + 3\lambda \delta_{ij} \varphi_j^2\right) G_{jk}(\varphi) = \delta_{ik} \tag{4.13}$$

Performing the Gaussian integral, and using Wick's theorem in the form

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} d^n x \, f(x) \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} C_{ij} x_i x_j\right) =$$

$$(\det C)^{-1/2} \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}^{-1} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j}\right) f(y)\bigg|_{y=0} \tag{4.14}$$

the expression (4.12) can be re-written in the form

$$\exp\left[-\tilde{\Gamma}(\varphi)\right] = \left(\det G_{ij}(\varphi)\right)^{1/2} \exp\left(\frac{1}{2}\sum_{i,j} G_{ij}(\varphi) \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j}\right)$$

$$\exp\left[-\lambda \sum_j \varphi_j \eta_j^3 - \frac{\lambda}{4} \sum_j \eta_j^4 + \sum_j \eta_j \frac{\partial \tilde{\Gamma}(\varphi)}{\partial \varphi_j}\right]_{\eta=0} \tag{4.15}$$

Let us pause to explain this formula. The determinant prefactor just produces the well known semi-classical (one-loop) correction $\propto \log\left(\det G_{ij}(\varphi)\right)$ to the classical action. As for the remaining terms, and ignoring the last term $\propto \eta_j \partial\tilde{\Gamma}/\partial\varphi_j$, we would get the sum over all connected vacuum diagrams with the field-dependent propagator $G_{ij}(\varphi)$ (as the result of taking the logarithm on both sides). Although this last term would seem to make the equation completely untractable, a little bit of thought shows that this is not so. Because $\tilde{\Gamma}(\varphi)$ contains only one-particle irreducible contributions, the effect of this last term is precisely to remove the one-particle reducible diagrams from the expansion: because this term is linear in η, it can couple to the rest of any diagram only via a single line. In other words, *the quantum effective action is nothing but the sum of the one-loop correction and the sum over one-particle irreducible vacuum diagrams with at least two loops and with the field-dependent Green's function (4.3).* This is the result derived in [8] for the effective potential in quantum field theory.

Now one can analogously write the equation for the Effective Action in D dimensional field theory case. Starting from the partition function again and performing a background field split, one obtains the equation for the effective action:

$$
\Gamma[\varphi] \;=\; S[\varphi] \;+\; \frac{\hbar}{2}\int d^D x \,\log\left[\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(x)}\right]
$$

$$
-\hbar\log\left[\exp\left(\frac{\hbar}{2}\int d^D u\, d^D v\, G_{cl}(u,v;\varphi)\frac{\delta^2}{\delta\eta(u)\delta\eta(v)}\right)\exp\left(-\frac{1}{\hbar}\left(S_{\text{int}}[\varphi,\eta] - \int d^D x\,\eta(x)\frac{\delta\tilde\Gamma[\varphi]}{\delta\varphi(x)}\right)\Bigg|_{\eta=0}\right)\right]
$$

$$(4.16)$$

Where,

$$
\tilde\Gamma[\varphi] = \Gamma[\varphi] - S[\varphi]
$$

The last part is exactly what cancels all the *One Particle Reducible* diagrams. This completes our sketch of the derivation of the solution of the DeWitt equation. In a later chapter we will see an explicit example where the DeWitt equation is well defined and the solution (also well defined) can be checked order by order in $\hbar$. The most important thing here to note is that the solution is not perturbative in the coupling constant of the theory but in orders of $\hbar$ which is a natural small parameter in the quantum theory.

However, a direct verification of (4.1) to all orders is cumbersome. We will therefore postpone a discussion of this issue to the following chapters in terms of an example where the DeWitt equation is well defined. Let us just note that in conjunction with the explicit expression as a sum over $\varphi(x)$-dependent vacuum diagrams we can see directly from the DeWitt equation (3.9) that $\Gamma[\varphi]$ can only contain one-particle irreducible (1PI) diagrams: the action of the first functional derivative $\delta\Gamma[\varphi]/\delta\varphi(x)$ in particular leads to the cutting any one of the propagators in a diagram arising in the expansion (4.1). If we had a diagram which is not 1PI then there would be at least one propagator which joins two 1PI subdiagrams. The action of the functional derivative on this diagram would thus split the diagram in two parts at this propagator, leaving two disconnected diagrams. But on the r.h.s. of the DeWitt equation we have only connected Green's functions, $\delta^n W[J]/\delta J(x_1)\cdots\delta J(x_n)$. So there can be no disconnected diagrams on the r.h.s. of (3.9) and thus we can only have 1PI diagrams contributing to $\Gamma[\varphi]$, as expected.

The effective (Coleman-Weinberg) potential is obtained by specializing all formulas to x-independent fields $\varphi(x) = \varphi_0$ [4] and removing a formally infinite volume factor $\propto \int dx$. The main advantage of writing the effective potential as a sum over vacuum type diagrams is the following: rather than having to do all the combinatorics with 'antenna diagrams', one obtains the answer at each loop order 'in one stroke'. In particular the RG improved one-loop potential obtained by summing ladder bubble diagrams is directly obtained. This was, in fact, the first application of this formula in [8] where the effective potential was also determined to two loops for φ^4 theory. As shown there the formalism implies considerable simplifications in comparison with the textbook derivations of the Coleman-Weinberg potential.

At the end of this section we write the solution (4.1) for the finite dimensional integral with the action defined by (4.4), that is, the solutions to (4.9). In accordance with the explanation after (4.1) we include only 1PI and connected diagrams in the expansion

$$
\Gamma(\varphi_i) = S(\varphi_i) + \Gamma^{(1)}(\varphi_i) + \Gamma^{(2)}(\varphi_i) + \Gamma^{(3)}(\varphi_i) + \ldots \tag{4.17}
$$

where the indices denote the loop order. In this way we obtain

$$
\begin{aligned}
\Gamma^{(1)}(\varphi) &= -\frac{1}{2}\ln\det(G_{ij}) \\[2mm]
\Gamma^{(2)}(\varphi) &= -\left[-\frac{3\lambda}{4}\sum_i G_{ii}^2 + 3\lambda^2\sum_{i,j}\varphi_i\varphi_j G_{ij}^3\right] \\[2mm]
\Gamma^{(3)}(\varphi) &= -\left[\frac{3\lambda^2}{4}\sum_{i,j}G_{ij}^4 + \frac{9\lambda^2}{4}\sum_{ij}G_{ii}G_{ij}^2 G_{jj}\right. \\[2mm]
&\quad -27\lambda^3\sum_{i,j,k}\varphi_i\varphi_j G_{ij}G_{ik}^2 G_{jk}^2 - 27\lambda^3\sum_{i,j,k}\varphi_i\varphi_j G_{ij}^2 G_{ik}G_{jk}G_{kk} \\[2mm]
&\quad +54\lambda^4\sum_{i,j,k,l}\varphi_i\varphi_j\varphi_k\varphi_l G_{ij}G_{jk}G_{kl}G_{li}G_{ik}G_{jl} \\[2mm]
&\quad \left. +81\lambda^4\sum_{i,j,k,l}\varphi_i\varphi_j\varphi_k\varphi_l G_{ij}^2 G_{kl}^2 G_{ik}G_{jl}\right]
\end{aligned}
\tag{4.18}
$$

4.2 Convergence properties

Now we go back to the zero dimensional analysis to to look at some of the convergence properties of the solution. By construction, this series solution must satisfy the discrete DeWitt equation (4.9), and this claim can in principle be checked order by order. Equally important is the fact that the expansion, while being asymptotic, can have vastly better convergence properties for non-vanishing φ than the usual perturbation expansion in terms of the coupling constant λ. This is most easily seen by simplifying our zero-dimensional field theory even further to an integral over one variable. In this case the 'Green's function' (4.13) is simply $G(\varphi) \equiv (1 + 3\lambda\varphi^2)^{-1}$. For a given vacuum diagram with I internal lines we have

$$
I = \frac{3}{2}V_3 + 2V_4
\tag{4.19}
$$

where V_3 and V_4, respectively, denote the number of three- and four-point vertices in a given diagram ; note that in any vacuum diagram, the number of three-point vertices is *even*. The number of loops is equal to

$$
L = \frac{1}{2}V_3 + V_4 + 1
\tag{4.20}
$$

Therefore an arbitrary vacuum diagram with L loops will be proportional to

$$
\frac{\lambda^{V_4}(\lambda\varphi)^{V_3}}{(1 + 3\lambda\varphi^2)^I} \approx (\lambda\varphi^4)^{1-L}
\tag{4.21}
$$

(for $L = 1$, the relevant parameter is $\log(1 + 3\lambda\varphi^2)$). In other words, the loop expansion now coincides with an expansion in $(\lambda\varphi^4)^{-1}$: of course, this expansion should only be used in the appropriate region in field space and the space of couplings, where $\lambda\varphi^4$ is sufficiently large. So we see that the series can converge well even for large λ provided the value of the classical field φ is not too small (and different from zero)! We have checked this claim by numerical integration of a non-trivial example, which we give bellow. The important lesson, then, is that it is not simply the coupling constant λ (or its running analog $\lambda(\mu)$,

where μ is some renormalization scale) that governs the convergence properties of the effective action functional, but that one should also consider the question of convergence w.r.t. to the value of the field variables φ_j or $\varphi(x)$ as well.

4.3 Numerical results

To illustrate the efficiency of the expansion (4.1) we present some numerical results for the simple one-dimensional integral

$$\exp(-W(J)) = \int \frac{dx}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2 - \frac{\lambda}{4}x^4 - xJ\right] \tag{4.22}$$

in this section. To this aim, we go through the same steps as before, with the expansion (4.17) and $n = 1$ in (4.18). The loop expansion here becomes

$$
\begin{aligned}
\Sigma^{(0)}(\varphi) &\equiv S_{\text{cl}}(\varphi) = \frac{\varphi^2}{2} + \frac{\lambda\varphi^4}{4} \\
\Sigma^{(1)}(\varphi) &= S_{\text{cl}} - \frac{1}{2}\ln(G) \\
\Sigma^{(2)}(\varphi) &= \Sigma^{(1)} - \left(-\frac{3\lambda}{4}G^2 + 3\lambda^2\varphi^2 G^3\right) \\
\Sigma^{(3)}(\varphi) &= \Sigma^{(2)} - \left(\frac{3\lambda^2}{4}G^4 + \frac{9\lambda^2}{4}G^4 - 27\lambda^3\varphi^2 G^5 - 27\lambda^3\varphi^2 G^5 \right. \\
&\qquad\qquad \left. + 54\lambda^4\varphi^4 G^6 + 81\lambda^4\varphi^4 G^6\right)
\end{aligned}
\tag{4.23}
$$

where $G \equiv 1/(1 + 3\lambda\varphi^2)$, and where we have defined $\Sigma^{(i)} \equiv \sum_{0 \leq j \leq i} \Gamma^{(j)}$. The results for Γ_{exact} and $\Sigma^{(i)}$ for three exemplary values of λ and φ are given in the following table:

λ	1.0	1.0	100.0	200.0
φ	1.0	4.0	1.0	1.0
Γ_{exact}	1.4532	73.9458145683	28.353282939	53.6991599696
S_{cl}	0.75	72.0	25.5	50.5
$\Sigma^{(1)}$	1.4431	73.9459101483	28.353555132	53.6992974659
$\Sigma^{(2)}$	1.4431	73.9458145253	28.353282864	53.6915996015
$\Sigma^{(3)}$	1.4512	73.9458145667	28.353282912	53.6991599663

$$\tag{4.24}$$

Evidently the approximation converges rapidly even for large values of λ!

4.4 Iterative Procedure

In principle one could set up an iterative scheme to see that the above solution is the one for the effective action. One starts by defining the effective action with an iterative index:

$$\frac{\delta\Gamma^{(k)}[\varphi]}{\delta\varphi(x)} =$$

$$\underset{*}{*}\exp\left[\sum_{n=2}^{\infty}\frac{1}{n!}\int d^{D}x_{1}\cdots d^{D}x_{n}\,W_{n}^{(k-1)}\Big(x_{1},...,x_{n};J[\varphi]\Big)\frac{\delta}{\delta\varphi(x_{1})}\cdots\frac{\delta}{\delta\varphi(x_{n})}\right]\underset{*}{*}\frac{\delta S[\varphi]}{\delta\varphi(x)}$$

and also the following relation:

$$\int d^{D}y\,W_{2}^{(k-1)}(x,y;\varphi)\Gamma_{2}^{(k-1)}(y,z;\varphi) = \delta^{D}(x-z) \tag{4.25}$$

and other similar relations analogous to (2.13). Where W's and Γ's are defined as earlier with index k. If one starts with the definition:

$$\Gamma_{2}^{(0)}(y,z;\varphi) = \frac{1}{\hbar}\frac{\delta^{2}S[\varphi]}{\delta\varphi(y)\delta\varphi(z)} \tag{4.26}$$

then by performing the iteration, one exactly obtains the solution discussed as earlier in orders of $\hbar$. Thus one obtains,

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \lim_{k\to\infty}\frac{\delta\Gamma^{(k)}[\varphi]}{\delta\varphi(x)} \tag{4.27}$$

Although in principle one obtains the derivative of the Effective Action in this case, it is enough to define the correlation functions using this procedure. It is pointed out here that we first realized the solution of the DeWitt equation using this procedure. A similar construction of the effective action was done in [57], which also starts with a functional differential equation, and can be identified to be the DeWitt equation. In [57], one can also check the construction of the effective action for higher composite operators, which is also important for the study of bound states. We also mention here that the solution discussed here is for the bare and unrenormalized effective action. So the objects associated are still divergent. But there are some theories which are finite, and so without divergences. This is where we have to invoke supersymmetry which we will discuss in the next chapter.

Chapter 5

Brief Introduction to Supersymmetry

Supersymmetry arose from the observation of physicists that the normal space time symmetries in a quantum field theory can be extended to accommodate a symmetry between bosons and fermions, such that one transforms into another. These symmetry operations transform different members of the multiplet of a model into each other. These transformations are to be represented as linear operators acting on the vector space, which is the representation space of the multiplet. This structure is formulated by writing down a Lagrangian which transforms under these transformations upto a total derivative which leaves the action invariant. Although there has been lack of experimental evidence in this field, a lot of work has been devoted to construction of field theories which has these symmetries, and study of these structures. The important fact in these studies is that the normal concepts of quantum field theory allow for supersymmetry without any further more assumptions. It was found that space-time and internal symmetries can only be related by fermionic symmetry generators Q of spin 1/2 (called supersymmetry generators). So in the presence supersymmetry, multiplets contain particles of different spins. Supersymmetry transformations are thus generated by quantum operators Q, which will change fermions into bosons and vice versa:

$$Q\,|fermion\rangle = |boson\rangle \quad ; \quad Q\,|boson\rangle = |fermion\rangle \tag{5.1}$$

Now which particular bosons and fermions are related by this symmetry to each other by some Q, the total number of Q's, and the transformation of other properties, depend on the model under consideration. The Q's change the statistics and so the spin of the state. Spin is in some sense, a space time symmetry, as it is related to behaviour under spatial rotations. Q's also affect some of the internal quantum numbers of the states. It is this interesting property of combining internal symmetry with space time behaviour that makes supersymmetric theories a good candidate to study, in the attempt to unify all the fundamental interactions.

There are some crucial properties of the supersymmetry generators Q. We will not derive them but state them here. Firstly the Q's are invariant under translations, meaning the commutator of the Q's with the energy operator E or the momentum operator P is zero. Secondly the spectrum of the energy operator E (which is the Hamiltonian), contains no negative eigen values in a theory with supersymmetry. If one denotes the lowest energy state i.e. the vacuum as $|0\rangle$, this state will have zero energy, if Q and $Q^\dagger$ annihilate this state. Any state whose energy is not zero is not invariant under

supersymmetry transformations. This means that there are N superpartner states (from N supersymmetry generators Q), for every one particle state. The spin of these state and its partner will differ by $1/2$. So we have thirdly, each multiplet must contain at least one boson and one fermion, whose spins differ by $1/2$. A supermultiplet is defined as a set of quantum states which can be transformed into one another by one or more supersymmetry transformations. Now the translation invariance of Q implies that Q does not change the energy and the momentum and so all states in a multiplet of unbroken supersymmetry have the same mass. There has been no experimental evidence which show elementary particles to be accompanied by superpartners with different spin, but same masses. Thus if supersymmetry is to be present in nature, it can only be present as a spontaneously broken symmetry. If that is the case, then the ground state will not be invariant under all supersymmetric transformations, which is to say, $Q|0\rangle \neq 0$ for some Q. This means from earlier discussion that, supersymmetry is spontaneously broken if and only if the energy of the lowest lying state which is the vacuum is not exactly zero. Although spontaneous symmetry breaking may lift the mass degeneracy of the supermultiplet by giving different masses to different members, the multiplet structure i.e. the fermionic and bosonic states remain the same. Thus we still need super partners for the presence of supersymmetry. Theories with one spinorial generator is called $N = 1$ while theory with more number of spinorial charge Q, more specifically N number of them is called N extended supersymmetry. The general understanding about supersymmetry is that there exists extended supersymmetry at high energies, while at intermediate energies there exists $N = 1$ supersymmetry, while at low energies there is no supersymmetry. The last step is necessary to lift the mass degeneracy which is not seen in experiments.

One important point is that, any multiplet with extended supersymmetry will contain particles with spin $\geq N/4$. This means that spins $\geq 3/2$ must be present for $N > 4$ in 4 dimensions. Renormalizable flat-space theories cannot accommodate spins $\geq 3/2$. The presence of fields with spin $3/2$ requires the introduction of coupling constants with negative mass dimensions. This makes presence of such fields in the theory non-renormalizable. There is also a bound for gravity theories, since gravity cannot couple consistently with spins $\geq 5/2$, the limit in this case is $N = 8$.

5.1   Algebra

[1]It was found by Coleman and Mandula that any group of bosonic symmetries of the S-matrix in relativistic field theory is the direct product of the Poincare group with an internal symmetry group. The bosonic generators are thus the four momenta P_μ and the six Lorentz generators $M_{\mu\nu}$, plus a certain number of Hermitian internal symmetry generators, B_r. The algebra of the Poincaré group is as follows:

$$[P_\mu, P_\nu] = 0 \tag{5.2}$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\sigma}P_\sigma - \eta_{\mu\sigma}P_\rho) \tag{5.3}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}) \tag{5.4}$$

Now for the internal symmetry group we have,

$$[T_a, T_b] = if_{ab}^c T_c \tag{5.5}$$

---

[1]Here we discuss some basics definitions and constructions of superfields which will be used in our later chapters. If the reader is familiar with the topic then he or she can directly move on to the next chapter and for detailed study of the subject see [34],[35],[40],[41],[33].

While direct product of the two means that commutators of the Poincare generators and the internal symmetry groups vanish, i.e.

$$[T_a, P_\mu] = [T_a, M_{\mu\nu}] = 0 \tag{5.6}$$

Now all supersymmetry generators are fermionic, which is to say that they must change the spin of a state by a half-odd amount and change the statistics of the state. In presence of extra generators $Q_{\alpha i}$, ($\alpha = 1, 2$), the above algebra is extended to the following:

$$[Q_{\alpha,i}, P_\mu] = [\bar{Q}_{\dot\alpha}^i, P_\mu] = 0 \tag{5.7}$$

$$[Q_{\alpha i}, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_{\beta i} \tag{5.8}$$

$$[\bar{Q}_{\dot\alpha}^i, M_{\mu\nu}] = -\frac{1}{2}Q_{\dot\beta}^i(\bar\sigma_{\mu\nu})_{\dot\alpha}^{\dot\beta} \tag{5.9}$$

$$[Q_{\alpha i}, T_r] = (b_r)_i^j Q_{\alpha j} \tag{5.10}$$

$$[\bar{Q}_{\dot\alpha}^i, T_r] = -\bar{Q}_{\dot\alpha}^j(b_r)_j^i \tag{5.11}$$

$$\{Q_{\alpha i}, \bar{Q}_{\dot\beta}^j\} = 2\delta_i^j(\sigma^\mu)_{\alpha\dot\beta}P_\mu \tag{5.12}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\epsilon_{\alpha\beta}Z_{ij} \tag{5.13}$$

$$\{\bar{Q}_{\dot\alpha}^i, \bar{Q}_{\dot\beta}^i\} = -2\epsilon_{\dot\alpha\dot\beta}Z^{ij} \tag{5.14}$$

$$[Z_{ij}, Others] = 0$$

Where Z_{ij} is the Central Charge of the algebra, and is given by some linear combination of the generators T_rs as follows:

$$Z_{ij} = a_{ij}^r T_r \quad , \quad a_{ij}^r = -a_{ji}^r \quad , \quad Z^{ij} = (Z_{ij})^\dagger \quad , \quad Z_{ij} = Z_{ji} \tag{5.16}$$

and $\sigma^\mu = (\mathbb{I}, \sigma^i)$, where σ^i are the Pauli matrices, and $\mathbb{I}$ is the 2 dimensional identity matrix. Also analogously $\bar\sigma^\mu = (\mathbb{I}, -\sigma^i)$. Also we use the following definitions of the two indexed σ matrices,

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \quad , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{5.17}$$

where we use the following signature for $\eta^{\mu\nu} \equiv (+, -, -, -)$ and γ^μ are the Dirac gamma matrices. Now the supersymmetry transformations relate fermions to bosons. This is because they are spin $1/2$ operators and action of such an operator on a state changes its spin. Now one can divide the representation space on which these generators act into bosonic and fermionic sectors. The even elements of the algebra (i.e. the bosonic generators $P_\mu, M_{\mu\nu}, T_a$) map the subspaces into themselves, i.e. bosonic sector to bosonic sectors and fermionic sector to fermionic. The odd elements which are the supersymmetry generators, map the bosonic subspace to the fermionic subspace, and vice-versa.

Now the anticommutator relation $\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$ means that there must be a way of going from one subspace to the other and and back, such that the net result is action of the momentum operator P_μ on the original subspace. More over $Q\bar{Q} + \bar{Q}Q$ which is the double mapping, no dimensions are lost. From all these considerations, we get the following rule : *the fermionic and bosonic subspaces of the representation have the same dimension or more specifically the same degrees of freedom.*

5.2 States in a Supersymmetric theory

If there are N supersymmetry generator in a theory, then from the action of this generators on the states in the theory, one can find the spectrum of the theory. So for a

massless theory we can count the number of states knowing that the generators change the spin of the state by $1/2$:

$$
\begin{array}{lccccc}
\text{helicity:} & \lambda_0 & \lambda_0 + 1/2 & \lambda_0 + 1 & \dots & \lambda_0 + N/2 \\
\text{no. of states:} & C_0^N = 1 & C_1^N = N & C_2^N & \dots & C_N^N = 1
\end{array}
$$

Where C_0^N denote combinatorial coefficients. For example the counting for a massless $N = 4$ theory will be as follows: starting from a helicity -1 state, we have 4 helicity $-1/2$ state, 6 helicity 0 states, 4 helicity $1/2$ state and 1 helicity 1 state. For massive theories, the one particle states are described by the mass (m), the total spin (s) and the spin projection along the z-axis (s_3). The action of Q on a state with spin s will be a linear combination of states with spins $s + 1/2$ and $s - 1/2$. The counting is similar and can be checked in any text book on supersymmetry.

5.3 Superfield

The Supersymmetry algebra can be viewed as a Lie algebra with anticommuting parameters. This motivates one to write down a group element for the purpose:

$$
G(x, \theta, \bar{\theta}) = e^{i[-x_\mu P^\mu + \theta Q + \bar{\theta}\bar{Q}]} \tag{5.18}
$$

Where x^μ are the position coordinates, and $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ are anticommuting two component Weyl spinors, i.e. $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ and $\alpha, \dot{\alpha}$ take values $1, 2$ (when we write product of these objects without indices, a sum over the indices will be assumed). Product of two group elements is then given as follows:

$$
G(0, \eta, \bar{\eta})G(x^\mu, \theta, \bar{\theta}) = G(x^\mu + i\theta\sigma^\mu\bar{\eta} - i\eta\sigma^\mu\bar{\theta}, \theta + \eta, \bar{\theta} + \bar{\eta}) \tag{5.19}
$$

This can be understood as a motion in the parameter space of the group. This motion is generated by the differential operators Q and $\bar{Q}$:

$$
\eta Q + \bar{\eta}\bar{Q} = \eta^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \right) + \bar{\eta}_{\dot{\alpha}} \left(\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha \sigma^\mu_{\alpha\beta}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu \right) \tag{5.20}
$$

Now a superfield ϕ is a function of the complete superspace coordinate variables $(x, \theta, \bar{\theta})$, and can be understood as a power series expansion in the anticommuting variables θ and $\bar{\theta}$. For one pair of them we can write it as:

$$
\phi(x, \theta, \bar{\theta}) = A(x) + \theta\psi(x) + \bar{\theta}\chi(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\omega}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \tag{5.21}
$$

The higher powers of $\theta, \bar{\theta}$ vanish. The transformation of this superfield under supersymmetry is defined as follows:

$$
\begin{aligned}
\delta_\eta \phi(x, \theta, \bar{\theta}) &= (\eta Q + \bar{\eta}\bar{Q})\phi(x, \theta, \bar{\theta}) = \delta_\eta A(x) + \theta\delta_\eta\psi(x) + \bar{\theta}\delta_\eta\chi(x) + \theta\theta\delta_\eta m(x) \\
&\quad + \bar{\theta}\bar{\theta}\delta_\eta n(x) + \theta\sigma^\mu\bar{\theta}\delta_\eta v_\mu(x) + \theta\theta\bar{\theta}\delta_\eta\bar{\omega}(x) + \bar{\theta}\bar{\theta}\theta\delta_\eta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}\delta_\eta d(x)
\end{aligned} \tag{5.22}
$$

Where the transformation laws of the componenet fields can be obtained by matching powers of θ and $\bar{\theta}$ on both sides. One can verify that the linear combination of superfield is also a superfield and also product of superfields is also a superfield, which is because the

Supersymmetry generators are linear differential operators. In general the representation of the supersymmetry algebra in terms of the superfields are highly reducible. Thus constraints are imposed on the superfield to eliminate some of the extra fields. For example one can put the constraint, $\bar{D}\phi = 0$, where the D and $\bar{D}$ are super covariant derivatives defined as follows:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \tag{5.23}$$

$$\bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{5.24}$$

The condition $\bar{D}_{\dot\alpha}\phi = 0$ is called chiral superfield. Let us look at this example in more detail to get a flavour of superfields. The given constraint is easy to solve in terms of $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$ and θ because,

$$\bar{D}_{\dot\alpha}(x^\mu + i\theta\sigma^\mu\bar\theta) = 0 \quad , \quad \bar{D}_{\dot\alpha}\theta = 0 \tag{5.25}$$

Thus the superfield ϕ can be expressed in these variables as follows:

$$\begin{aligned}
\phi(x,\theta,\bar\theta) \quad &= A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\
&= A(x) + i\theta\sigma^\mu\bar\theta_\mu A(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box A(x) \\
&+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta\theta F(x)
\end{aligned} \tag{5.26}$$

Where, A is a complex scalar, ψ a two component Weyl spinor, and F a complex auxiliary field. Now the component fields here transform under supersymmetry as follows:

$$\begin{aligned}
\delta_\eta A(x) &= \sqrt{2}\eta\psi(x) \\
\delta_\eta\psi(x) &= i\sqrt{2}\sigma^\mu\bar\eta\partial_\mu A(x) + \sqrt{2}\eta F(x) \\
\delta_\eta F(x) &= i\sqrt{2}\bar\eta\bar\sigma^\mu\partial_\mu\psi(x)
\end{aligned} \tag{5.27}$$

These transformations close as follows under two supersymmetric transformations:

$$\begin{aligned}
(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)A(x) &= -2i(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)\partial_\mu A(x) \\
(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\psi(x) &= -2i(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)\partial_\mu\psi(x) \\
(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)F(x) &= -2i(\eta\sigma^\mu\bar\xi - \xi\sigma^\mu\bar\eta)\partial_\mu F(x)
\end{aligned} \tag{5.28}$$

Similarly one can construct the antichiral superfield ϕ^+ with the constraint $D_\alpha\phi^+ = 0$, which can be expressed in the variables $y^{+\mu} = x^\mu - i\theta\sigma^\mu\bar\theta$ and $\bar\theta$, with the following expansion:

$$\phi^+(x,\theta,\bar\theta) = A^*(y^+) + \sqrt{2}\bar\theta\bar\psi(y^+) + \bar\theta\bar\theta F^*(y^+) \tag{5.29}$$

Where $*$ means complex conjugation. Now using these two fields one can construct the action of the well known Wess Zumino Model in 4 dimensions as follows:

$$\int d^4x d^2\theta d^2\bar\theta\, \phi^+\phi + \int d^4x d^2\theta(\frac{1}{2}m\phi\phi + \frac{1}{3}g\phi\phi\phi + \lambda\phi) + \int d^4x d^2\bar\theta(\frac{1}{2}m\phi^+\phi^+ + \frac{1}{3}g\phi^+\phi^+\phi^+ + \lambda\phi^+) \tag{5.30}$$

Now in component form this can be written as follows after integration over θ variables, ($\int d\theta\,\theta = 1$):

$$\int d^4x \, A^*(x) \Box A(x) - i\bar{\psi}\partial_\mu \sigma^\mu \psi + F^* F + \left[m(AF - \frac{1}{2}\psi\psi) + g(A^2 F - \psi\psi A) + \lambda F + c.c \right] \quad (5.31)$$

Where $c.c$ denote complex conjugation.

5.4   On-shell realization of Supersymmetry

The above action it is clear that the equations of motion for F and F^* is simply algebraic and so they are called auxiliary fields. This means that they can be replaced in the Lagrangian in terms of the physical fields A, ψ and their complex counter parts. The on-shell action, after this replacement, is given as follows:

$$-i\bar{\psi}\partial_\mu \sigma^\mu \psi + A^* \Box A - \frac{1}{2}m\psi\psi - \frac{1}{2}m\bar{\psi}\bar{\psi} - g\psi\psi A - \bar{\psi}\bar{\psi}A^* - (\lambda + mA^* + gA^{*2})(\lambda + mA + gA^2)$$
$$(5.32)$$

Now there is a genral statement that one has for supersymmetric field theories, where auxiliary fields have been eliminated:

1. In these kind of theories, the supersymmetry transformations are non-linear in terms of the fields, and so checking supersymmetry invariance is not easy in general. also the number of bosonic fields and fermionic fields do not match.

2. The form of the supersymmetry transformations become model dependent. Although the action as a whole is supersymmetric, the kinetic part, the mass term and the interaction term are not invariant separately.

3. The supersymmetry algebra closes only on shell, which is to say they only close by using equations of motion.

5.5   Extended Supersymmetry

From the algebra described above it is understood that one generally requires infinite number of auxiliary fields to represent extended supersymmetry. This can be understood as follows: In presence of the Central Charge (which is zero for N=1 case), the superfield will be a function of $(x, \theta_i, \bar{\theta}_i, z)$ where i are internal symmetry indices and z denotes a set of bosonic variables that has to be added to realize the central charge and so a Taylor series expansion in z does not terminate. As a example let us look at N=2 supersymmetric theories. The group element in this case is the following:

$$G(x, \theta, \bar{\theta}, \zeta, \bar{\zeta}) = e^{i[-x_\mu P^\mu + \theta_i^\alpha Q_\alpha^i + \bar{\theta}_{\dot{\alpha}}^i \bar{Q}_i^{\dot{\alpha}}]} e^{i[\zeta T^{++} + \bar{\zeta} T^{--}]} \quad (5.33)$$

Here $T^{\pm\pm}$ are the coset generators for $SU(2)/U(1)$. These generators with the $U(1)$ generator T^0 form the $SU(2)$ algebra:

$$[T^{++}, T^{--}] = T^0 \quad , \quad [T^0, T^{\pm\pm}] = \pm 2T^{\pm\pm} \quad , \quad T^{\pm\pm} = T^1 \pm iT^2 \quad , \quad T^0 = 2T^3 \quad (5.34)$$

The new coordinates ζ do not transform under translations and supertranslations. It is interesting to note that the two new coordinates are even ones. Under supersymmetry ζ

stays inert while under $SU(2)$ $\theta_i, \bar{\theta}^i$ behave as isospinors and ζ transforms non-linearly. Now there can be another choice of representation. The $U(1)$ charge generator is represented on Q^i by the matrix τ^3:

$$[T^0, Q^i] = (\tau^3)^i_j Q^j \quad , \quad [T^{\pm\pm}, Q^i] = (\tau^{\pm\pm})^i_j Q^j \tag{5.35}$$

Where,

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \tau^{++} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \tau^{--} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.36}$$

Now the supersymmetry charges can be redefined for convenience:

$$Q^1_\alpha \equiv Q^+_\alpha \quad , \quad Q^2_\alpha \equiv Q^-_\alpha \quad , \quad \bar{Q}_{1\dot{\alpha}} \equiv \bar{Q}^-_{\dot{\alpha}} \quad , \quad \bar{Q}_{2\dot{\alpha}} \equiv -\bar{Q}^+_{\dot{\alpha}} \tag{5.37}$$

Then under action of the generator of the internal $SU(2)$ symmetry, we have:

$$\begin{aligned} [T^0, Q^+] = Q^+ \quad &, \quad [T^0, Q^-] = -Q^-, \\ [T^{++}, Q^+] = 0 \quad &, \quad [T^{++}, Q^-] = Q^+, \\ [T^{--}, Q^+] = Q^- \quad &, \quad [T^{--}, Q^-] = 0 \end{aligned} \tag{5.38}$$

and same for $\bar{Q}^\pm$, while the supersymmetry generators have the following algebra:

$$\begin{aligned} \{Q^+_\alpha, \bar{Q}^+_{\dot{\alpha}}\} = \{Q^-_\alpha, \bar{Q}^-_{\dot{\alpha}}\} = 0 \\ \{Q^+_\alpha, \bar{Q}^-_{\dot{\alpha}}\} = -\{Q^-_\alpha, \bar{Q}^+_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \end{aligned} \tag{5.39}$$

In this basis (called the *analytic basis*,[56]) one can analogously define the super covariant derivatives :

$$D^+_\alpha = \frac{\partial}{\partial\theta^{-\alpha}} \quad , \quad D^-_\alpha = -\frac{\partial}{\partial^{+\alpha}} + 2i\bar{\theta}^{-\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{5.40}$$

and similar definitions for $\bar{D}^\pm$. From these super covariant derivatives we can construct new superfields using constraints as follows:

$$D^+_\alpha \phi = \bar{D}^+_{\dot{\alpha}} \bar{\phi} = 0 \tag{5.41}$$

Notice that now the solution of this equation is in terms of the normal space time coordinates, the anticommuting theta variables and the parameters of the internal $SU(2)$ symmetry $(\zeta, \bar{\zeta})$. Thus an expansion of this field will have infinite number of terms in the variables $\zeta, \bar{\zeta}$, and this is precisely the reason why one has infinite number of auxiliary fields in extended supersymmetric theories. For higher supersymmetric theories, like $N = 3$ and $N = 4$ it is even more complicated to find such a superspace description (although for $N = 3$ such a space exists). Thus the problem of having an off-shell formulation of these theories is really difficult and yet a subject of ongoing research (check [56]).
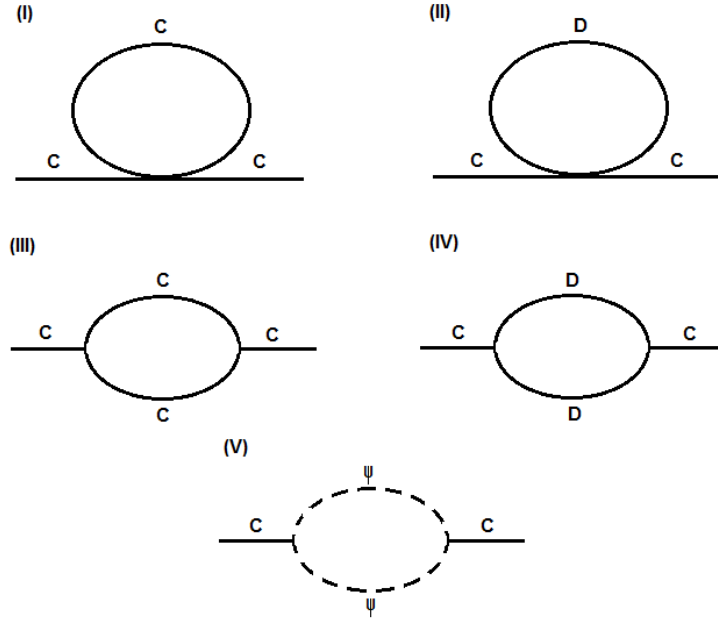
Figure 5.1: One loop contribution to self energy of scalar C

5.6 Cancellation of Divergences

It is interesting to point out one very fundamental property of supersymmetric theories which is the cancellation of all quadratic divergences (in 4 dimension). To see this lets go back to the Wess Zumino model in 4 dimensions with the Lagrangian:

$$\int d^4x d^2\theta d^2\bar{\theta}\, \phi^+\phi + \int d^4x d^2\theta(\frac{1}{2}m\phi\phi + \frac{1}{3}g\phi\phi\phi + \lambda\phi) + \int d^4x d^2\bar{\theta}(\frac{1}{2}m\phi^+\phi^+ + \frac{1}{3}g\phi^+\phi^+\phi^+ + \lambda\phi^+)$$

(5.42)

Writing this down in terms of component field and integrating out the auxiliary field, and also writing the complex scalar field (A) in terms of a scalar and pseudoscalar as $A = \frac{C+iD}{\sqrt{2}}$ and writing a 4 spinor $\Psi = (\psi, \bar{\psi})$, we have:

$$\int d^4x \left[\frac{1}{2}\partial_\mu C \partial^\mu C - \frac{1}{2}m^2 C^2 + \frac{1}{2}\partial^\mu D \partial_\mu D - \frac{1}{2}m^2 D^2 + \frac{1}{2}\bar{\Psi}(i\slashed{\partial} - m)\Psi \right.$$
$$\left. - \frac{mg}{\sqrt{2}}C(C^2 + D^2) - \frac{g^2}{4}(C^4 + D^4 + 2C^2 D^2) - \frac{g}{\sqrt{2}}\bar{\Psi}(C - iD\gamma^5)\Psi \right]$$

(5.43)

The one loop contribution to the self energy of the scalar C can be computed from the above diagrams (Shown in [43]). Let us write down the amplitude for the diagrams:

$$
\begin{aligned}
(I) &= -\frac{ig^2}{4}4.3 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2} = 3g^2 \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2} \\
(II) &= -\frac{ig^2}{2}2 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2} = g^2 \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2}
\end{aligned}
$$

$$
\begin{aligned}
(III) &= \left(-\frac{img}{\sqrt{2}}\right)^2 3.2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} \frac{i}{(p-q)^2 - m^2} \\
&= 3g^2 m^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m^2)((p-q)^2 - m^2)} \\
(IV) &= \left(-\frac{img}{\sqrt{2}}\right)^2 2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} \frac{i}{(p-q)^2 - m^2} \\
&= g^2 m^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m^2)((p-q)^2 - m^2)} \\
(V) &= -\left(-\frac{ig}{\sqrt{2}}\right)^2 2 \int \frac{d^4p}{(2\pi)^4} Tr\left\{\frac{i(\not{p}+m)}{p^2 - m^2} \frac{i(\not{p}-\not{q}+m)}{(p-q)^2 - m^2}\right\} \\
&= -2g^2 \left(\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} + \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p-q)^2 - m^2}\right. \\
&\quad \left. + \int \frac{d^4p}{(2\pi)^4} \frac{4m^2 - q^2}{(p^2 - m^2)((p-q)^2 - m^2)}\right)
\end{aligned}
$$

$$(5.44)$$

where q is the incoming momentum of the scalar field C. Adding the contributions, we obtain the following:

$$
2g^2 \left\{\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} - \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p-q)^2 - m^2} + \int \frac{d^4p}{(2\pi)^4} \frac{q^2 - 2m^2}{(p^2 - m^2)((p-q)^2 - m^2)}\right\}
$$

$$(5.45)$$

From the above it is clear that the quadratically divergent terms coming from the first two terms cancel, while the last term ony contributes to a logarithmic diverence. This is the most fascinating thing about supersymmetric theories, these theories are much well behaved than normal quantum field theories, and for some special cases, which we will discuss in our next chapter, the theories are completely finite i.e. even the logarithmic divergent terms cancel.

5.7 Vector Superfield

Now let us look at how to construct a supersymmetric gauge theory. For that we need to define a vector superfield. A vector superfield V satisfies the following reality condition:

$$V = V^\dagger \tag{5.46}$$

This superfield can be understood as an expansion in the anti-commuting parameters θ and $\bar{\theta}$ as follows:

$$
\begin{aligned}
V(x, \theta, \bar{\theta}) &= c(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{1}{2}\theta\theta[m(x) + in(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[m(x) - in(x)] - \theta\sigma^\mu\bar{\theta}v_\mu(x) \\
&\quad + i\theta\theta\bar{\theta}[\bar{\lambda} + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)] - i\bar{\theta}\bar{\theta}\theta[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_m\bar{\chi}(x)] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[d(x) + \frac{1}{2}\Box c(x)]
\end{aligned}
$$

$$(5.47)$$

Where the component fileds c, d, m, n and v_μ must be real. The supersymmetric generalization of a gauge transformation is given by:

$$V \to V + \phi + \phi^+ \tag{5.48}$$

Where $\phi + \phi^+$ is a Hermitian field and is a combination of a chiral and antichiral field. This is given in component form as follows:

$$
\begin{aligned}
\phi + \phi^+ \;=\; & A + A^* + \sqrt{2}(\theta\psi + \bar{\theta}\bar{\psi}) + \theta + \bar{\theta}\bar{\theta}F^* \\
& + i\theta\sigma^\mu\bar{\theta}\partial_\mu(A - A^*) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi \\
& + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\square(A + A^*)
\end{aligned}
\tag{5.49}
$$

The component fields transform under this transformations as follows:

$$
\begin{aligned}
& c \to c + A + A^* \\
& \chi \to \chi - i\sqrt{2}\psi \\
& m + in \to m + in - 2iF \\
& v_\mu \to v_\mu - i\partial_\mu(A - A^*) \\
& \lambda \to \lambda \\
& D \to D
\end{aligned}
$$

$$\tag{5.50}$$

There is a speciall gauge in which c, χ, m, n are zeores. Fixing this gauge breaks supersymmetry, but still allows the usual gauge transformation $v_\mu \to v_\mu + \partial_\mu a$. In this guage it is easy to calculate the powers of V ;

$$
\begin{aligned}
& V = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}d(x) \\
& V^2 = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}v_\mu v^\mu \\
& V^3 = 0
\end{aligned}
$$

$$\tag{5.51}$$

This choice is called Wess-Zumino gauge. Now to construct the supersymmetric field strength one observes:

$$
\begin{aligned}
& W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \\
& \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V
\end{aligned}
$$

$$\tag{5.52}$$

These superfields are chiral and gauge invariant. Chirality condition follows:

$$
\begin{aligned}
& \bar{D}_{\dot{\beta}}W_\alpha = 0 \\
& D_\beta\bar{W}_{\dot{\alpha}} = 0
\end{aligned}
$$

$$\tag{5.53}$$

And $\bar{D}\phi = D\phi^+ = 0$ is used to prove gauge invariance:

$$
W_\alpha \to -\frac{1}{4}\bar{D}\bar{D}D_\alpha(V + \phi + \phi^+) = W_\alpha - \frac{1}{4}\bar{D}\{\bar{D}, D_\alpha\}\phi = W_\alpha
\tag{5.54}
$$

In chiral variables, $y = x + i\theta\sigma\bar{\theta}$ and $y^+ = x - i\theta\sigma\bar{\theta}$:

$$
\begin{aligned}
V &= -\theta\sigma^\mu\bar{\theta}v_\mu(y) + i\theta\theta\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[d(y) - i\partial_\mu v^\mu(y)] \\
&= -\theta\sigma^\mu\bar{\theta}v_\mu(y^+) - i\bar{\theta}\bar{\theta}\theta\lambda(y^+) + i\theta\theta\bar{\theta}\bar{\lambda}(y^+) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[d(y^+) + i\partial_\mu v^\mu(y^+)]
\end{aligned}
\tag{5.55}
$$

While W_α, $W_{\dot\alpha}$ are given as follows:

$$
\begin{aligned}
W_\alpha &= -i\lambda_\alpha(y) + [\delta_\alpha^\beta d(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha^\beta(\partial_\mu v_\nu(y) - \partial_\nu v_\mu(y))]\theta_\beta + \theta\theta\sigma_{\alpha\dot\alpha}^\mu\partial_\mu\bar{\lambda}^{\dot\alpha}(y) \\
\bar{W}_{\dot\alpha} &= i\bar{\lambda}_{\dot\alpha}(y^+) + [\epsilon_{\dot\alpha\dot\beta}d(y^+) + \frac{i}{2}\epsilon_{\dot\alpha\dot\rho}(\bar{\sigma}^\mu\bar{\sigma}^\nu)_{\dot\beta}^{\dot\rho}(\partial_\mu v_\nu(y^+) - \partial_\nu v_m(y^+))]\bar{\theta}^{\dot\beta} - \epsilon_{\dot\alpha\dot\beta}\bar{\theta}\bar{\theta}\bar{\sigma}^{\mu\dot\beta\alpha}\partial_\mu\lambda_\alpha(y^+)
\end{aligned}
\tag{5.56}
$$

μ, ν are space time indices. It is clear that $W_\alpha, \bar{W}_{\dot\alpha}$ are composed of gauge invariant fields d, λ_α and $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. The fields W_α, $\bar{W}_{\dot\alpha}$, have the component fields λ_α and $\bar{\lambda}_{\dot\alpha}$, as the lowest dimensions. since W_α is chiral, then the $\theta\theta$ component of $W^\alpha W_\alpha$ is,

$$
W_\alpha W^\alpha|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{2}v^{\mu\nu}v_{\mu\nu} + d^2 + \frac{i}{4}v^{\mu\nu}v^{\rho\eta}\epsilon_{\mu\nu\rho\eta}
\tag{5.57}
$$

Where $\epsilon_{\mu\nu\rho\eta}$ are totally antisymmetric in all its indices. A similar form exists for the $\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$. Using these two one can construct the action for a supersymmetric field theory with vector super fields as follows:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{4}(W_\alpha W_{\dot\alpha}|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar{\theta}\bar{\theta}}) \\
\mathcal{S} &= \int d^4x\mathcal{L} = \int d^4x\left[\frac{1}{2}d^2 - \frac{1}{4}v^{\mu\nu}v_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda}\right]
\end{aligned}
\tag{5.58}
$$

Of course this is an action for Abelian supersymmetric gauge fields only without interaction. In general it is easy to generalize to non Abelian cases, simply by changing to the corresponding field strength and replacing the the partial derivatives ∂_μ with covariant ones D_μ, and taking a trace over all the gauge indices. This is precisely what we would get for $N = 1$ Super Yang Mills theory, with a non-Abelian gauge group.

Chapter 6

N=1 Wess-Zumino model in D=2

The $N = 1$ Wess-Zumino model in two space-time dimensions is UV finite order by order in perturbation theory (the generic non-supersymmetric theories having only logarithmic divergences in two dimensions, which are removed by imposing supersymmetry). The Euclidean version of the model can be written in terms of a single superfield $\Phi(z)$ with superspace coordinate $z \equiv (x, \theta)$, where θ is a two-component (anti-commuting) Majorana spinor with $\theta = \theta^*$. The superfield contains a real scalar A and a Majorana spinor ψ, as well as the auxiliary field F:

$$\Phi(x, \theta) = A(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x)$$

$$(6.1)$$

Also, the superfield is a real superfield, i.e. $\Phi = \Phi^\dagger$. For simplicity we restrict attention to the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}\Phi\bar{D}D\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 \tag{6.2}$$

We could replace the last two terms by an arbitrary polynomial $P(\Phi)$ here, but this would only make the formulas more cumbersome and not give any new insights. The supercovariant derivative is defined by

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + (\gamma^\mu\theta)_\alpha\partial_\mu \tag{6.3}$$

and,

$$\bar{D}^\alpha = -C^{\alpha\beta}D_\beta \tag{6.4}$$

where C is the charge conjugation matrix. And the matrices γ^μ are defined as follows:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.5}$$

The Lagrangian in component form is as follows:

$$\mathcal{L} = \frac{1}{2}A\Box A - \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^2 + \frac{1}{2}m(2AF - \bar{\psi}\psi) + g(A^2F - A\bar{\psi}\psi) \tag{6.6}$$

The above Lagrangian can be checked to be invariant under the following Supersymmetry transformation upto total derivatives:

$$\delta A = \epsilon_\alpha \delta_\alpha A, \quad \delta \psi_\beta = \epsilon_\alpha \delta_\alpha \psi_\beta, \quad \delta F = \epsilon_\alpha \delta_\alpha F \tag{6.7}$$

Where,

$$\delta_\alpha A = C_{\alpha\beta} \psi_\beta = -\bar{\psi}_\alpha, \quad \delta_\alpha \psi_\beta = \gamma^\mu_{\beta\alpha} \partial_\mu A + \delta_{\alpha\beta} F, \quad \delta_\alpha F = \gamma^\mu \partial_\mu \psi \tag{6.8}$$

Along with the closure of the supersymmetry algebra under two consecutive transformations on the fields:

$$[\delta_\xi, \delta_\eta] A = [(\bar{\eta}\gamma^\mu\xi) - (\bar{\xi}\gamma^\mu\eta)]\partial_\mu A \tag{6.9}$$
$$[\delta_\xi, \delta_\eta] \psi_\beta = [(\bar{\eta}\gamma^\mu\xi) - (\bar{\xi}\gamma^\mu\eta)]\partial_\mu \psi_\beta$$
$$[\delta_\xi, \delta_\eta] F = [(\bar{\eta}\gamma^\mu\xi) - (\bar{\xi}\gamma^\mu\eta)]\partial_\mu F$$

$$\tag{6.10}$$

Writing out the DeWitt equation for the three fields A, ψ and F we get

$$\frac{\delta\Gamma[A,F,\psi]}{\delta A(x)} = \Box A(x) + mF(x) + g\big[2A(x)F(x) - \psi(x)\psi(x)\big]$$
$$- g\hbar\left[2\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_F(x)} + \mathrm{Tr}\frac{\delta^2 W[J]}{\delta J_\psi(x)\delta J_{\bar{\psi}}(x)}\right]$$

$$\frac{\delta\Gamma[A,F,\psi]}{\delta\bar{\psi}(x)} = -\slashed{\partial}\psi(x) - m\psi(x) - 2gA(x)\psi(x) - \hbar g\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_\psi(x)}$$

$$\frac{\delta\Gamma[A,F,\psi]}{\delta F(x)} = F(x) + mA(x) + gA^2(x) - \hbar g\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_A(x)} \tag{6.11}$$

It can be noted that the DeWitt equation for the scalar A and the fermion ψ is finite, while that for the auxiliary field F is not, and we have to provide a normal ordering prescription for it to be well defined. Let us first look at the equation the scalar field A. If we just look at the supersymmetric Ward identity:

$$-\langle \bar{\psi}_\alpha(x)\psi_\beta(y)\rangle + \gamma^\mu_{\beta\alpha}\partial^y_\mu\langle A(x)A(y)\rangle + \delta_{\alpha\beta}\langle A(x)F(y)\rangle = 0 \tag{6.12}$$

Taking the trace of the above one obtains,

$$-Tr[\langle \bar{\psi}(x)\psi(y)\rangle - \slashed{\partial}\langle A(x)A(y)\rangle] + 2\langle A(x)F(y)\rangle = 0 \tag{6.13}$$

The second second term is odd under integration as it will contribute a factor of $\slashed{p}$ which will be zero after integration, which will arise due to coincident points of the Green's function, and hence we find:

$$2\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_F(x)} + \mathrm{Tr}\frac{\delta^2 W[J]}{\delta J_\psi(x)\delta J_{\bar{\psi}}(x)} = 0 \tag{6.14}$$

Which has to be true if supersymmetry is to be preserved and vacuum expectation value of the fields is to be zero. We can try to also check this to first order, where:

$$\left[2\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_F(x)} + \mathrm{Tr}\frac{\delta^2 W[J]}{\delta J_\psi(x)\delta J_{\bar{\psi}}(x)}\right]\bigg|_{1loop}$$

$$= 2 \int \frac{d^2p}{4\pi^2} \left(\frac{-m}{p^2 + m^2} \right) + Tr\left[\int \frac{d^2p'}{4\pi^2} \left(\frac{(i\not{p}' + m)}{p'^2 + m^2} \right) \right]$$

$$= 2 \int \frac{d^2p}{4\pi^2} \left(\frac{-m}{p^2 + m^2} \right) + 2 \int \frac{d^2p'}{4\pi^2} \left(\frac{m}{p'^2 + m^2} \right) \tag{6.15}$$

Which after proper regularization yields zero. Now we turn to the equation for the fermion ψ, which can be seen to be finite, simply because, the Greens function at the same point, between the scalar A and the fermion ψ is zero, i.e,

$$\frac{\delta^2 W[J]}{\delta J_A(x) \delta J_\psi(x)} = 0 \tag{6.16}$$

But when we look at the equation for the auxillary field F, we see that it is not finite simply because,

$$\left. \frac{\delta^2 W[J]}{\delta J_A(x) \delta J_A(x)} \right|_{1loop} \equiv \int \frac{d^2p}{4\pi^2} \left(\frac{1}{p^2 + m^2} \right) \tag{6.17}$$

which is logarithmically divergent. Therefore the equation for the auxiliary filed has to be made well defined. This can be removed by replacing the product $A^2(x)$ by the normal ordered product

$$:A(x)A(y): \equiv A(x)A(y) - \underline{A(x)A(y)} \tag{6.18}$$

and taking $x \to y$ afterwards. This singularity simply follows from the fact that if one expresses the auxiliary field F in terms of the physical field A, the non-linear terms in A must be rendered non-singular to make F itself well-defined as a quantum operator. [1] Consequently, the last component of the DeWitt equation must be replaced by

$$\frac{\delta \Gamma[A, F, \psi]}{\delta F(x)} = F(x) + mA(x) + g :A^2(x): -\hbar g \frac{\delta^2 W[J]}{\delta J_A(x) \delta J_A(x)} \tag{6.19}$$

and then *all* components of the DeWitt equation are free of singularities. In practice, the above replacement simply means that in the formal solution as a sum over vacuum diagrams there are no tadpole diagrams (these are anyway absent for a theory with only cubic vertices as they would lead to non-1PI diagrams in Γ which cannot be).

6.1 Transformation of the DeWitt equation under supersymmetry

It can be checked that the DeWitt equation transforms under supersymmetry as the supercurrent, which can be seen as follows:

$$\delta_\epsilon \frac{\delta \Gamma}{\delta A(x)} = -\bar{\epsilon}\gamma^\mu \partial_\mu \left[\frac{\delta \Gamma}{\delta \bar{\psi}(x)} \right] \tag{6.20}$$

$$\delta_\epsilon \frac{\delta \Gamma}{\delta \bar{\psi}(x)} = -\epsilon_\alpha \frac{\delta \Gamma}{\delta A(x)} - \gamma^\mu_{\alpha\beta}\epsilon_\beta \partial_\mu \left[\frac{\delta \Gamma}{\delta F(x)} \right] \tag{6.21}$$

---

[1] But note that, while $:A^2:$ is well-defined as an operator, it is singular as a c-number, while the converse is true for A^2!

$$\delta_\epsilon \frac{\delta\Gamma}{\delta F(x)} = -\bar\epsilon_\beta \frac{\delta\Gamma}{\delta\bar\psi^\beta(x)} \tag{6.22}$$

This is obvious as the L.H.S. of each of the equations form the components of a super-current source term, which couples with the super field. This is natural as they should be equal to the components of the supersymmetric source of the path integral.

6.2   On shell action and normal ordering

We can also integrate out the auxiliary field to write an on-shell action, which is as follows:

$$-\left[\frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(mA + gA^2)^2 + \frac{1}{2}\bar\psi\gamma^\mu\partial_\mu\psi + \frac{1}{2}(m + 2gA)\bar\psi\psi\right] \tag{6.23}$$

If we look at the the interaction term and calculate the tadpole diagrams contributing to the vacuum expectation value of the scalar field A, then we obtain the following:

$$-mgA^3(x)|_{1\text{loop}} \to -\frac{3g}{m}\int d^2p\frac{1}{p^2 + m^2}$$
$$-gA\bar\psi(x)\psi(x)|_{1\text{loop}} \to +\frac{2g}{m}\int d^2p\frac{1}{p^2 + m^2} \tag{6.24}$$

Which do not cancel obviously. After normal ordering the interaction part of the Lagrangian becomes the following:

$$-gA\bar\psi\psi - \frac{1}{2}g^2 : A^2 :: A^2 : -mgA : A^2 : \tag{6.25}$$

When one computes the expectation value of the scalar A analogous to case above, one finds that it is zero to one loop. Thus one sees that after normal ordering the equation for the auxiliary field F and then integrating it out, no divergence is introduced into the theory. This is to point out that this theory can be made finite even in the on-shell version. Another point to notice here is that the equation of motion for the auxiliary field is quadratic in the the scalar A, which means while integrating out the field F we unknowingly do add divergences. The procedure that we followed was we normal ordered the equation of motion for the auxiliary field first and then replaced it algebraically, thus introducing no divergence through the procedure. A similar check was performed for $N = 1$ Wess Zumino model in 4 dimensions, where we normal ordered the equation of motion of the auxiliary fields first before replacing it in the Lagrangian. This preserved the property that all quadratic divergences in loop diagrams are canceled.

6.3   DeWitt equation in Superspace

All the three DeWitt equations can be conveniently recast into a superspace equation. A similar normal ordering can be done in the superspace version of the Lagrangian and as it is much more convenient to work in it we would stick to the superspace description. So we have the functional derivative of the action as:

$$\frac{\delta S}{\delta\Phi} = -\frac{1}{2}\bar D D\Phi + m\Phi + g\Phi^2 \tag{6.26}$$

The arguments of the foregoing sections generalize directly to superspace. For the cubic Lagrangian above the DeWitt equation (3.9) takes an especially simple form, namely

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(z)} = \; :\frac{\delta S[\Phi]}{\delta\Phi(z)}: \; - \hbar g \frac{\delta^2 W[J]}{\delta J(z)\delta J(z)}\bigg|_{J=J[\Phi]} \tag{6.27}$$

or, more specifically

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(z)} = -\frac{1}{2}\bar{D}D\Phi(z) + m\Phi(z) + g :\Phi^2(z): -\hbar g \frac{\delta^2 W[J]}{\delta J(z)\delta J(z)}\bigg|_{J=J[\Phi]} \tag{6.28}$$

where $z \equiv (x^\mu, \theta)$ and $J(z)$ is the 'supersource field' $J(z) \equiv J_F + \bar{\theta}J_\psi + \bar{\theta}\theta J_A$. The normal ordering is understood to be in the sense of the component expressions given above. In the formal solution below this simply means that all tadpole diagrams are suppressed.

For the free superfield the superspace propagator is

$$G_2^{(0)}(z-z') = \tag{6.29}$$

$$= \big\langle 0\big|T[(A(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x))(A(x') + \bar{\theta}'\psi(x') + \frac{1}{2}\bar{\theta}'\theta'F(x'))]\big|0\big\rangle$$

$$= \exp\left[-\frac{1}{2}(\bar{\theta}-\bar{\theta}')(\gamma_\mu\partial^\mu + m)(\theta-\theta')\right]\triangle_F(x-y) \tag{6.30}$$

In analogy with (4.3) we define the classical Green's function in superspace

$$\int dz' \, G_{cl}(z,z';\Phi)\frac{\delta^2 S[\Phi]}{\delta\Phi(z')\delta\Phi(z'')} = \delta(z-z'') \tag{6.31}$$

(where the fermionic part of the δ-function is defined in the usual way as $\delta(\theta) = \theta$) so that $G_{cl}(z,z';\Phi) = G_2^{(0)}(z-z') + \cdots$.

By construction the supersymmetric DeWitt equation (6.27) is well defined, and we can therefore take over the formal solution given in the previous Chapter 4,

$$\begin{aligned}
\Gamma[\Phi] &= S[\Phi] + \frac{\hbar}{2}\int d^4z \ln\left[\frac{\delta^2 S}{\delta\Phi(z)\delta\Phi(z)}\right] \\
&\quad - \hbar \ln\left[\exp\left(\frac{\hbar}{2}G_{i,j}\frac{\delta^2}{\delta\tilde{\Phi}_i\delta\tilde{\Phi}_j}\right)\exp\left(-\frac{\tilde{S}_{int}}{\hbar}\right)\bigg|_{\tilde{\Phi}=0}\right]
\end{aligned} \tag{6.32}$$

Where G_{ij} is shorthand for $G_{cl}(z_i, z_j; \Phi)$ and $\tilde{S}_{int} = \frac{g}{3}\tilde{\Phi}^3$, and all the integrals are understood to be in superspace. Now if we expand the series we have the following:

$$\left[1 + \sum_{n=1}^\infty \frac{1}{n!}\left(\frac{\hbar}{2}G_{i,j}\frac{\delta^2}{\delta\tilde{\Phi}_i\delta\tilde{\Phi}_j}\right)^n\right]\left[1 + \sum_{m=1}^\infty \frac{1}{m!}\left(\frac{-\tilde{S}_{int}}{\hbar}\right)^m\right]\bigg|_{\tilde{\Phi}=0} \tag{6.33}$$

Because the dummy variable $\tilde{\Phi}$ is put to 0, and the interaction is cubic, only terms with the condition, $2n = 3m$ survive. Thus the first of this will be at two loops for $m = 2, n = 3$. Evaluating the corresponding term we get

$$\left[\frac{1}{3!}\left(\frac{\hbar}{2}G_{i,j}\frac{\delta^2}{\delta\tilde{\Phi}_i\delta\tilde{\Phi}_j}\right)^3\right]\left[\frac{1}{2!}\left(\frac{-\tilde{S}_{int}}{\hbar}\right)^2\right] = \frac{\hbar g^2}{3}\int_{z,w} G_{cl}^3(z,w;\Phi)$$

At the next order (three loops) we have $n = 6, m = 4$, and

$$\left[\frac{1}{6!}\left(\frac{\hbar}{2}G_{i,j}\frac{\delta^2}{\delta\tilde{\Phi}_i\delta\tilde{\Phi}_j}\right)^6\right]\left[\frac{1}{4!}\left(\frac{-\tilde{S}_{int}}{\hbar}\right)^4\right]$$

$$= \hbar^2\left[\frac{2}{3}g^4\int_{u,v,w,z}G_{cl}(u,v;\Phi)G_{cl}(u,w;\Phi)G_{cl}(u,z;\Phi)G_{cl}(v,w;\Phi)G_{cl}(v,z;\Phi)G_{cl}(w,z;\Phi)\right.$$

$$\left.+g^4\int_{u,v,w,z}G_{cl}^2(u,v;\Phi)G_{cl}^2(w,z;\Phi)G_{cl}(u,w;\Phi)G_{cl}(v,z;\Phi) + \frac{1}{2}\left(\frac{g^2}{3}\int_{z,w}G_{cl}^3(z,w;\Phi)\right)^2\right]$$

$$(6.34)$$

We recognize the last term as square of the term which we got for $n = 3, m = 2$ (two loops), which is removed by taking log of the entire expression as these diagrams are not connected. Hence summing up we get the following contribution to the effective action:

$$\Gamma = S + \frac{\hbar}{2}\int d^4z\ln\left[\frac{\delta^2S}{\delta\Phi(z)\delta\Phi(z)}\right] - \frac{\hbar^2g^2}{3}\int_{z,w}G_{cl}^3(z,w;\Phi)$$

$$-\frac{2}{3}\hbar^3g^4\int_{u,v,w,z}G_{cl}(u,v;\Phi)G_{cl}(u,w;\Phi)G_{cl}(u,z;\Phi)G_{cl}(v,w;\Phi)G_{cl}(v,z;\Phi)G_{cl}(w,z;\Phi)$$

$$-\hbar^3g^4\int_{u,v,w,z}G_{cl}^2(u,v;\Phi)G_{cl}^2(w,z;\Phi)G_{cl}(u,w;\Phi)G_{cl}(v,z;\Phi) + (\hbar^4) \qquad (6.35)$$

To check this we first calculate the second functional derivative of Γ, which is, up to order $\hbar^2$,

$$\frac{\delta^2\Gamma}{\delta\Phi(z_1)\delta\Phi(z_2)} = \frac{\delta^2S}{\delta\Phi(z_1)\delta\Phi(z_2)} - 2\hbar g^2 G_{cl}(z_1,z_2;\Phi)G_{cl}(z_1,z_2;\Phi)$$

$$-8\hbar^2g^4\int_{z,w}G_{cl}(z,z_2;\Phi)G_{cl}(z_2,z_1;\Phi)G_{cl}(z_1,w;\Phi)G_{cl}^2(z,w;\Phi)$$

$$-8\hbar^2g^4\int_{z,w}G_{cl}(z,z_1;\Phi)G_{cl}(z_1,w;\Phi)G_{cl}(z,z_2;\Phi)G_{cl}(z_2,w;\Phi)G_{cl}(z,w;\Phi)$$

$$(6.36)$$

Inverting the above we obtain the 2-point Green's function up to order $\hbar^2$,

$$-\frac{\delta^2W}{\delta J(z_1)\delta J(z_2)} = G_{cl}(z_1,z_2;\Phi) + 2\hbar g^2\int_{u,v}G_{cl}(z_1,u;\Phi)G_{cl}^2(u,v;\Phi)G_{cl}(v,z_2;\Phi)$$

$$+8h^2g^4\int_{u,v,z,w}\left[G_{cl}(z_1,u;\Phi)G_{cl}(z,v;\Phi)G_{cl}(v,u;\Phi)G_{cl}(u,w;\Phi)G_{cl}^2(z,w;\Phi)G_{cl}(v,z_2;\Phi)\right.$$

$$\left.+G_{cl}(z_1,u;\Phi)G_{cl}(z,u;\Phi)G_{cl}(u,w;\Phi)G_{cl}(z,v;\Phi)G_{cl}(v,w;\Phi)G_{cl}(z,w;\Phi)G_{cl}(v,z_2;\Phi)\right]$$

$$+4\hbar^2g^4\int_{u,v,w,z}G_{cl}(z_1,u;\Phi)G_{cl}^2(u,v;\Phi)G_{cl}(v,w;\Phi)G_{cl}^2(w,z;\Phi)G_{cl}(z,z_2;\Phi) \qquad (6.37)$$
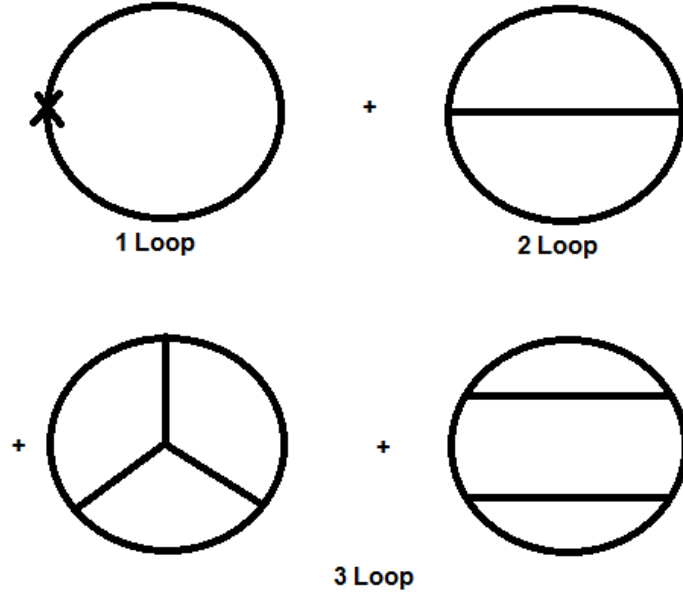
Figure 6.1: Diagrams contributing to Γ upto order $\hbar^3$

Now putting this in the DeWitt equation from the R.H.S, we obtain:

$$\frac{\delta S}{\delta \Phi(z)} - \hbar g \frac{\delta^2 W}{\delta J(z)\delta J(z)}$$

$$= \frac{\delta S}{\delta \Phi(z)} + \hbar g G_{cl}(z,z;\Phi) + 2\hbar^2 g^3 \int_{u,v} G_{cl}(z,u;\Phi)G_{cl}^2(u,v;\Phi)G_{cl}(v,z;\Phi)$$

$$+8h^3 g^5 \int_{u,v,z',w} \Bigg[G_{cl}(z,u;\Phi)G_{cl}(z',v;\Phi)G_{cl}(v,u;\Phi)G_{cl}(u,w;\Phi)G_{cl}^2(z',w;\Phi)G_{cl}(v,z;\Phi)$$

$$+G_{cl}(z,u;\Phi)G_{cl}(z',u;\Phi)G_{cl}(u,w;\Phi)G_{cl}(z',v;\Phi)G_{cl}(v,w;\Phi)G_{cl}(z',w;\Phi)G_{cl}(v,z;\Phi) \Bigg]$$

$$+4\hbar^3 g^5 \int_{u,v,w,z'} G_{cl}(z,u;\Phi)G_{cl}^2(u,v;\Phi)G_{cl}(v,w;\Phi)G_{cl}^2(w,z';\Phi)G_{cl}(z',z;\Phi) \tag{6.38}$$

This is exactly what we get from the l.h.s. by taking the first functional derivative of Γ. The solution upto 3rd order in $\hbar$ can be represented graphicaly which is shown in the above diagram.

Here the propagators are understood to be the Φ dependent classical propagator $G_{cl}(u,v;\Phi)$. In principle one can go on like this and write down the solution in orders of $\hbar$. The functional differential operator which acts on the interaction term to yield the complete set of diagrams, can be used to construct an algebraic procedure of computing higher loop oder corrections to the correlation functions of the theory. But after 4th order the computation becomes cumbersome, and one would require to write some computer codes to perform this. There are also many interesting aspects of this construction.

Observe that the action of the functional differential operator :

$$\int d^4u\, d^4v\, G_{cl}(u,v;\Phi) \frac{\delta}{\delta\Phi(u)} \frac{\delta}{\delta\Phi(v)} \tag{6.39}$$

The action of this on each of the Φ dependent diagrams adds one propagator to the diagram, i.e. the loop number of the diagram changes. Thus there seems to be some algebraic structure in this. Although we were not able to go into details of this structure we intend to do so in some future work.

Chapter 7

N=4 Super Yang Mills Theory

In 4 dimensions, there is a very interesting theory which is finite. This is $N = 4$ Supersymmetric Yang Mills theory. It was obtained first by compactifying $N = 1$ Supersymmetric Yang Mills theory from 10 dimensions to 4 dimensions. The theory has the maximal amount of supersymmetry allowed for a rigid supersymmetric theory in 4 dimensions. The symmetry generated by sixteen real supercharges (8 complex supercharges), has been shown to be finite [12], and possessing a vanishing β function. The field content of the theory, apart from the choice of the gauge group G, consists of 6 scalars, four Majorana spinors and one vector field. All these fields are in the adjoint representation of the gauge group G. Here it is important to point out that the propagators of the elementary fields are ultraviolet divergent because of the choice of the Wess Zumino gauge. These divergences are canceled once the contribution of the fields which are put to zero in the Wess Zumino gauge are taken into account. Thus in Wess Zumino gauge, because of these divergences one requires a wave function renormalization. But as is known, without choosing the Wess Zumino gauge, the action in the component field form becomes extremely cumbersome, and not very useful for practical computation. So we will start by describing the action in on-shell component field formalism.

The Lagrangian for N=4 Super Yang Mills Theory in component field form is given as follows:

$$
\begin{aligned}
\mathcal{L} = Tr \Bigg[&-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}(\mathcal{D}A_i)^2 - \frac{1}{2}(\mathcal{D}B_i)^2 \\
&+\frac{1}{4}g^2([A_i, A_j]^2 + [B_i, B_j]^2 + 2[A_i, B_j]^2) \\
&-\frac{1}{2}i\,\bar{\lambda}_K\gamma^\mu\mathcal{D}_\mu\lambda_K + \frac{1}{2}g\bar{\lambda}_K[(\alpha^j_{KL}A_j + \beta^j_{KL}\gamma^5 B_j), \lambda_l]\Bigg]
\end{aligned}
\tag{7.1}
$$

Where the indices K,L, run from 1 to 4, i and j indices from 1 to 3. Here A_i is a scalar, B_i a pseudoscalar λ_k a Majorana spinor and A_μ a vector boson. Now The field strength and the covariant derivative is defined as follows:

$$
G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]
\tag{7.2}
$$

$$
\mathcal{D}_\mu\lambda = \partial_\mu\lambda + ig[A_\mu, \lambda]
\tag{7.3}
$$

The matrices α and β are defined as using:

$$
\{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = -2\delta^{ij}
\tag{7.4}
$$

$$
[\alpha^i, \beta^j] = 0
\tag{7.5}
$$

These are antisymmetric and real matrices, and is given by:

$$\beta^1 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \qquad \beta^2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \beta^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \tag{7.6}$$

And

$$\alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \qquad \alpha^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \qquad \alpha^3 \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \tag{7.7}$$

Using this Lagrangian, we can try to write the DeWitt equation and implement the solution, but unfortunately this Lagrangian is not off-shell, and hence the DeWitt equation using this Lagrangian will not be finite. This can be intuitively understood as also from the point fact that although the action is invariant under supersymmetry transformation, the path integral measure is not, as the action is only invariant upto equations of motion, while the measure is off-shell, in principle. As this theory is a theory with extended supersymmetry, therefore we need infinite number of auxiliary fields to account for off-shell degrees of freedom.

7.1 Action from Dimensional compactification

The above Lagrangian was obtained by dimensional compactification from N=1 Super Yang Mills theory in 10 dimensions to 4 dimensions by [44]. Here we give a brief description of this procedure. Th Lagrangian for the N=1 Super Yang Mills theory is given as :

$$\mathcal{L} = Tr(-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}i\bar{\lambda}\mathcal{D}_\mu\lambda) \tag{7.8}$$

Where the field strength and the covariant derivative have similar definitions as earlier and the action is a 10 dimensional integral of the Lagrangian. More over we have λ, a 32 dimensional Majorana spinor and satisfies the following constraints:

$$\lambda^* = \lambda \tag{7.9}$$
$$\gamma^{11}\lambda = \lambda \tag{7.10}$$

Where γ^{11} is analogous to γ^5 in 4 dimensions. Now one can check that the action is invariant under the following transformations:

$$\delta A_\mu = -i\bar{\epsilon}\gamma_\mu\lambda \tag{7.11}$$
$$\delta\lambda = G_{\mu\nu}\sigma^{\mu\nu}\epsilon \tag{7.12}$$

where

$$\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \tag{7.13}$$

Where ϵ is a constant anti-commuting Majorana-Weyl spinor. We will not discuss this invariance here but it involves using the Fierz transformation in 10-dimensions. Also these transformations close on the fields using the equations of motion, as:

$$[\delta_1, \delta_2]\lambda = 2i(\epsilon_1\gamma^\nu\epsilon_2)\mathcal{D}_\nu\lambda \tag{7.14}$$
$$[\delta_1, \delta_2]A_\mu = 2i(\epsilon_1\gamma^\nu\epsilon_2)\mathcal{D}_\nu G_{\nu\mu} \tag{7.15}$$

Now when one does dimensional reduction by compactification, it is assumed that the extra 6 coordinates $x^{3+i}(i = 1, ..6)$ are compact. This is obtained by introducing a radii r^i for the coordinates, all fields being periodic in x^{3+i}. When we drop the dependence of the fields on the extra six coordinates and make the following identifications, we end up with the four dimensional action which we discussed above. The $A_{3+i}(i = 1, 2, 3)$ are identified with the pseudoscalars B_i and $A_{6+i}(i = 1, 2, 3)$ are identified with the scalar fields A_i. Then we have the vector field $A_\mu(\mu = 0, 1, 2, 3)$. Further, the Majorana-Weyl spinor λ describes four Majorana spinors λ_K in four dimensions. Moreover the supersymmetry transformation parameter ϵ describes also four Majorana spinor parameters ϵ_K. Thus we obtain the action which contains one vector, three scalars, three pseudoscalars and four Majorana fermions, which is invariant under supersymmetry transformation with an internal symmetry index.

7.2 Light Cone Superspace Formalism

The problem concerning the absence of an off-shell formalism, can be worked around in Light Cone Superspace formalism [12],[18] where only physical degrees of freedom are manifest. On the light cone we have $A^\mu B_\mu = A_i B_i + A_l A_l - A_0 B_0$, where we have denoted the longitudinal direction as l, time direction as 0, and the rest of transverse directions as i. Now from this one can define, $A_\pm = 2^{-1/2}(A_0 \pm A_l), A = 2^{-1/2}(A_1 + iA_2)$. Now in the light cone gauge one has the condition, $A^{+a} = 0$ where we have put back the gauge index a. Also A^{-a} can be algebraically replaced in the Lagrangian. And so the propagating degrees of freedom are : $A(x)$ a complex field describing the vector degree of freedom, 3 complex fields $C^{mn}(x)$(which is just linear combination of the scalar A_i and pseudoscalar B_i introduced in the previous section. Explicitly $2^{-1/2}(A_i + iB_i)$) describing 6 scalar particles, and complex Grassmann fields $\chi^m(x)$ describing 4 spin 1/2 particles. Now in this language the Lagrangian for the N=4 SUSY Yang Mills on the Light Cone gauge is given as follow, using the Grassmann parameter θ^m and its complex conjugate $\bar{\theta}_m$:

$$L = 72[-\bar{\phi}^a(\frac{\Box}{\partial^{+2}})\phi^a + \frac{4}{3}gf^{abc}[\frac{1}{\partial^+}\bar{\phi}^a\phi^b\bar{\partial}\phi^c + \frac{1}{\partial^+}\phi^a\bar{\phi}^b\partial\bar{\phi}^c]$$
$$-g^2 f^{abc}f^{ade}[\frac{1}{\partial^+}(\phi^b\partial^+\phi^c)\frac{1}{\partial^+}(\bar{\phi}^d\partial^+\bar{\phi}^e) + \frac{1}{2}\phi^b\bar{\phi}^c\phi^d\bar{\phi}^e]$$

$$(7.16)$$

Where the following super covariant derivative is introduced,

$$d^{\tilde{m}} = -\frac{\partial}{\partial\bar{\theta}_{\tilde{m}}} - \frac{i}{\sqrt{2}}\theta^{\tilde{m}}\partial^+ \quad \bar{d}_{\tilde{n}} = \frac{\partial}{\partial\theta^{\tilde{n}}} + \frac{i}{\sqrt{2}}\bar{\theta}_{\tilde{n}}\partial^+ \qquad (7.17)$$

The field ϕ and $\bar{\phi}$ are defined using the constraints:

$$d^{\tilde{m}}\phi = 0 \quad \bar{\phi} = \frac{1}{48}(\frac{\bar{d}^4}{\partial^{+2}})\phi \qquad (7.18)$$

$$d^{\tilde{m}}d^{\tilde{n}}\bar{\phi} = \frac{1}{2}\epsilon^{\tilde{m}\tilde{n}\tilde{p}\tilde{q}}\bar{d}_{\tilde{p}}\bar{d}_{\tilde{q}}\phi \qquad (7.19)$$

Where the "~" indices correspond to internal SU(4) indices and the rest, the gauge indices. For completeness purpose, we give the component field expansion of thesuperfield ϕ below:

$$\phi(x,\theta,\bar{\theta}) = \frac{1}{\partial^+}A(y) + i\frac{1}{\partial^+}\theta^{\tilde{m}}\bar{\chi}_{\tilde{m}}(y) + i\frac{1}{\sqrt{2}}\theta^{\tilde{m}}\theta^{\tilde{n}}\bar{C}_{\tilde{m}\tilde{n}}(y)$$

$$+\frac{1}{6}\sqrt{2}\theta^{\tilde{m}}\theta^{\tilde{n}}\theta^{\tilde{p}}\epsilon_{\tilde{m}\tilde{n}\tilde{p}\tilde{q}}\chi^{\tilde{q}}(y) + \frac{1}{12}\theta^{\tilde{m}}\theta^{\tilde{n}}\theta^{\tilde{p}}\theta^{\tilde{q}}\epsilon_{\tilde{m}\tilde{n}\tilde{p}\tilde{q}}\partial^+\bar{A}(y) \tag{7.20}$$

where we have supressed the gauge index in the above field expansion and $y = (x, \bar{x}, x^+, x^- - i2^{-1/2}\theta^{\tilde{m}}\bar{\theta}_{\tilde{m}})$. The supersymmetry transformations on the components are given as follows:

$$\delta A^a = i\epsilon^{\tilde{m}}\bar{\chi}_{\tilde{m}}^a \tag{7.21}$$

$$\delta C^{\tilde{m}\tilde{n}a} = -i(\epsilon^{\tilde{m}}\chi^{\tilde{n}a} - \epsilon^{\tilde{n}}\chi^{\tilde{m}a} + \epsilon^{\tilde{m}\tilde{n}\tilde{p}\tilde{q}}\bar{\epsilon}_p\bar{\chi}_q^a) \tag{7.22}$$

$$\delta\chi^{\tilde{m}a} = \sqrt{2}\epsilon^{\tilde{m}}\partial^+\bar{A}^a + \sqrt{2}\bar{\epsilon}_n\partial^+ C^{\tilde{m}\tilde{n}a} \tag{7.23}$$

The functional derivative in the light cone superspace are defined:

$$\frac{\delta\phi^a(x,\theta,\bar{\theta})}{\delta\phi^b(x',\theta',\bar{\theta}')} = \frac{d^4}{(4!)^2}\delta^4(x-x')\delta^4(\theta-\theta')\delta^4(\bar{\theta}-\bar{\theta}')\delta_b^a \tag{7.24}$$

and,

$$\frac{\delta\bar{\phi}^a(x,\theta,\bar{\theta})}{\delta\phi^b(x',\theta',\bar{\theta}')} = \frac{12\bar{d}^4 d^4}{(4!)^4}\delta^4(x-x')\delta^4(\theta-\theta')\delta^4(\bar{\theta}-\bar{\theta}')\delta_b^a \tag{7.25}$$

From this we can derive the DeWitt equation, which can be written in the following representative form considering that it takes up the same form as earlier.:

$$\frac{\delta\Gamma}{\delta\varphi^o(x)} = \frac{\delta S}{\delta\varphi^o(x)} - \frac{i\hbar}{2}\iint d^{12}y d^{12}z G_2^{mn}(y,z)\frac{\delta^3 S}{\delta\varphi^m(y)\delta\varphi^n(z)\delta\varphi^o(x)}$$

$$-\frac{\hbar^2}{6}\iiint d^{12}w d^{12}y d^{12}z G_3^{pmn}(w,y,z)\frac{\delta^4 S}{\delta\varphi^p(w)\delta\varphi^m(y)\delta\varphi^n(z)\delta\varphi^o(x)} \tag{7.26}$$

Where φ is the vacuum expectation value of ϕ and $z = (x,\theta,\bar{\theta})$, and understood to be the classical field with respect to which we solve for the effective action. Here the Greens functions are defined using a chiral source which satisfy the constraint equations same as ϕ, as follows:

$$G_n^{a_1..a_n}(z_1,..,z_n) = \frac{\delta^n W[J]}{\delta J^{a_1}(z_1)..\delta J^{a_n}(z_n)} \tag{7.27}$$

Where $W[J]$ is defined as follows:

$$exp\left[\frac{i}{\hbar}W[J]\right] = \int D\phi^a(z)exp\left(\frac{i}{\hbar}\int d^{12}z[L + \frac{1}{4}\phi^a\frac{\bar{d}^4}{\partial^{+4}}J^a]\right) \tag{7.28}$$

Using this it can be shown analogous to [12] that the DeWitt equation is finite. But the actual equation is pretty cumbersome. The finiteness is shown by shifting the supercovariant derivative d's to the external leg. and performing the integration over the theta functions, as any diagram can be written as a local function in the θ variables. By doing so, in the end one obtains product of delta functions of these anticommuting θ variables, which is zero.

7.3 Momentum space representation

It is worth while to point out that the momentum space representation of the action (discussed in [50]) could give a relatively easier technique at computing amplitudes in this model. In momentum light cone space representation, the action for N=4 Supersymmetric Yang Mills Theory can be written as follows:

$$
S = 72 \int d^4\theta d^4\bar{\theta} Tr\left[\int \frac{d^4p_1}{(2\pi)^4} 2\hat{\bar{\phi}}(-p_1)\frac{p_1^2}{p_{1-}^2}\hat{\phi}(p_1) \right.
$$

$$
-\frac{8ig}{3}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{1}{(p_{1-}+p_{2-})}\left(\bar{p}_2\hat{\bar{\phi}}(-p_1-p_2)[\hat{\phi}(p_1),\hat{\phi}(p_2)]+p_2\hat{\phi}(-p_1-p_2)[\hat{\bar{\phi}}(p_1),\hat{\bar{\phi}}(p_2)]\right)
$$

$$
-2g^2\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\left(\frac{(p_{1-}.p_{2-})^2}{(p_{2-}+p_{3-})^2}[\hat{\phi}(-p_1-p_2-p_3),\hat{\bar{\phi}}(p_1)][\hat{\bar{\phi}}(p_2)\hat{\bar{\phi}}(p_3)]\right.
$$

$$
\left.\left.-\frac{1}{2}[\hat{\bar{\phi}}(-p_1-p_2-p_3),\hat{\bar{\phi}}(p_1)][\hat{\phi}(p_2),\hat{\bar{\phi}}(p_3)]\right)\right]
$$

$$(7.29)$$

Where $\hat{\phi}$ is understood as the Fourier transform of the superfield ϕ (where $p_- \to i\partial_+$), only with respect to the space-time coordinates. Using this action we can write down the effective action for N=4 super Yang Mills as the following, as the effective action can be formally understood as the sum of all vacuum energy diagrams with modified classical propagators, it must be independent of representation, (whether in position space or momentum space):

$$
\Gamma[\hat{\varphi}] = S[\hat{\varphi}] + \frac{i\hbar}{2}\int d^4p \ln\left[\frac{\delta^2 S[\hat{\varphi}]}{\delta\hat{\varphi}(-p)\delta\hat{\varphi}(p)}\right] \tag{7.30}
$$

$$
-i\hbar\ln\left[\exp\left(\frac{i\hbar}{2}\int d^4p_1 d^4p_2\, G_{cl}^{ab}(p_1,p_2;\hat{\varphi})\frac{\delta^2}{\delta\hat{\eta}^a(p_1)\delta\hat{\eta}^b(p_2)}\right)\exp\left(i\hbar^{-1}S_{\text{int}}[\hat{\varphi},\hat{\eta}]\right)\bigg|_{\eta=0}\right]_{1PI}
$$

Where $\hat{\varphi}(p)$ is the Fourier transform of the classical field and the classical propagator, G_{cl} is defined as follows:

$$
\int d^4p \frac{\delta^2 S}{\delta\hat{\varphi}^a(p_1)\delta\hat{\varphi}^b(p)}G_{cl}^{bc}(p,p_2;\hat{\varphi}) = \delta^{ac}\delta^{(4)}(p_1-p_2)
$$

and S_{int} is defined as follows:

$$
\frac{8gf^{abc}}{3}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{1}{(p_{1-}+p_{2-})}\left(\bar{p}_2\hat{\bar{\eta}}^c(-p_1-p_2)\hat{\eta}^a(p_1)\hat{\eta}^b(p_2)+p_2\hat{\eta}^c(-p_1-p_2)\hat{\bar{\eta}}^a(p_1)\hat{\bar{\eta}}^b(p_2)\right)
$$

$$
+2g^2 f^{abc}f^{efc}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\left(\frac{(p_{1-}.p_{2-})^2}{(p_{2-}+p_{3-})^2}\left(\hat{\varphi}^a(-p_1-p_2-p_3)\hat{\eta}^b(p_1)\hat{\bar{\eta}}^e(p_2)\hat{\bar{\eta}}^f(p_3)\right.\right.
$$

$$
+\hat{\eta}^a(-p_1-p_2-p_3)\hat{\varphi}^b(p_1)\hat{\bar{\eta}}^e(p_2)\hat{\bar{\eta}}^f(p_3)+\hat{\eta}^a(-p_1-p_2-p_3)\hat{\eta}^b(p_1)\hat{\bar{\varphi}}^e(p_2)\hat{\bar{\eta}}^f(p_3)
$$

$$
\left.+\hat{\eta}^a(-p_1-p_2-p_3)\hat{\eta}^b(p_1)\hat{\bar{\eta}}^e(p_2)\hat{\bar{\varphi}}^f(p_3)\right)-\frac{1}{2}\left(\hat{\varphi}^a(-p_1-p_2-p_3)\hat{\bar{\eta}}^b(p_1)\hat{\eta}^e(p_2)\hat{\bar{\eta}}^f(p_3)\right.
$$

$$
+\hat{\eta}^a(-p_1-p_2-p_3)\hat{\bar{\varphi}}^b(p_1)\hat{\eta}^e(p_2)\hat{\bar{\eta}}^f(p_3)+\hat{\eta}^a(-p_1-p_2-p_3)\hat{\bar{\eta}}^b(p_1)\hat{\varphi}^e(p_2)\hat{\bar{\eta}}^f(p_3)
$$

$$
\left.\left.+\hat{\eta}^a(-p_1-p_2-p_3)\hat{\bar{\eta}}^b(p_1)\hat{\eta}^e(p_2)\hat{\bar{\varphi}}^f(p_3)\right)\right)
$$

$$
+2g^2 f^{abc}f^{efc}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\left(\frac{(p_{1-}.p_{2-})^2}{(p_{2-}+p_{3-})^2}\hat{\eta}^a(-p_1-p_2-p_3)\hat{\eta}^b(p_1)\hat{\bar{\eta}}^e(p_2)\hat{\bar{\eta}}^f(p_3)\right) \tag{7.31}
$$

55

It is clear from the form of S_{int} that the terms cubic in $\hat{\eta}$ produce the 3-point interaction vertex and is background field i.e. $\hat{\varphi}$ dependent, while the terms quartic in η form the 4-point interaction vertex. This interaction term is already very cumbersome to work with. Nevertheless its easier than the position representation, and directly yields the amplitudes.

We know that, the n-point amplitudes are just the n-point Green's functions of the theory, therefore we can calculate the amplitudes by the repeated functional derivative on the following equation:

$$
\frac{\delta^3 W[J]}{\delta J^a(p_1 \delta J^b(p_2)\delta J^c(p_3)} =
$$
$$
\int d^4 p_1' d^4 p_2' d^4 p_3' \frac{\delta^2 W[J]}{\delta J^a(p_1)\delta J^d(p_1')} \frac{\delta^2 W[J]}{\delta J^b(p_2)\delta J^e(p_2')} \frac{\delta^2 W[J]}{\delta J^c(p_3)\delta J^f(p_3')} \frac{\delta^3 \Gamma[\varphi]}{\delta\hat{\varphi}^d(p_1')\delta\hat{\varphi}^e(p_2')\delta\hat{\varphi}^f(p_3')}
$$
$$
(7.32)
$$

Thus the n-point amplitudes are obtained by differentiating the above equation equation (n-3)-times and setting both the source J and the background field φ to zero. The functional derivative of Γ with respect to J, is understood to be taken as follows:

$$
\frac{\delta}{\delta J^a(p)} \frac{\delta^3 \Gamma[\varphi]}{\delta\hat{\varphi}^d(p_1')\delta\hat{\varphi}^e(p_2')\delta\hat{\varphi}^f(p_3')} = \int d^4 p' \frac{\delta^2 W[J]}{\delta J^a(p)\delta J^c(p')} \frac{\delta^4 \Gamma[\varphi]}{\delta\hat{\varphi}^c(p')\delta\hat{\varphi}^d(p_1')\delta\hat{\varphi}^e(p_2')\delta\hat{\varphi}^f(p_3')}
$$
$$
(7.33)
$$

Thus the n-point amplitudes are then defined as follows:

$$
A(p_1, \sigma_1; ...; p_n, \sigma_n) = \left. \frac{\delta^n W[J]}{\delta J^{\sigma_1}(p_1)...\delta J^{\sigma_n}(p_n)} \right|_{J=0,\hat{\varphi}=0}
\qquad (7.34)
$$

7.4   Outlook

Although we were not able to shed more light on this problem, we still were able to understand many key concepts associated with it. Firstly absence of an off-shell formalism, hinders the use of this technique for $N = 4$ supersymmetric Yang Mills theory and extended on-shell supersymmetric field theories in general. This can be understood from the point of view, that while the supersymmetry transformations are on-shell, the path integral itself is off-shell. This means that starting with an action which is only on-shell supersymmetric, then while performing loop calculations, we will see that the divergences coming from the bosonic and fermionic terms do not cancel. In supersymmetric gauge theories, this on-shell choice of the action is due to the choice of the Wess Zumino gauge, with out which the action automatically becomes extremely cumbersome. Also another point in the discussion which we should mention is the that while using the formal translation invariance of the measure to derive the DeWitt equation, it was automatically assumed that the path integral measure is invariant under on-shell supersymmetric transformation. It was shown in [40] that when there exists a superspace representation of the full supersymmetry algebra, then the measure is invariant, as the super symmetry transformations act linearly on all the component fields. On-shell the supersymmetry transformations become non-linear, and hence the measure is not invariant under it, and thus it is expected that the translation invariance of the measure has

to account for an anomalous term. It would be interesting to look into this in a bit more detail, and see whether the anomalous term could give some insight into the on-shell version of this theory.[1]

---

[1]For more recent developements please see [19],[20],[21].

Chapter 8

Brief introduction to Conformal Field Theory in 2 dimensions

The interest in conformal invariance in two dimensions started after the work of Belavin, Polyakov, and Zamolodchikov [42]. These authors used the representation theory of Virasoro algebra, developed earlier, and showed how to construct completely solvable conformal field theories which are the so-called minimal models. A striking feature of the work put forward by them was that, in these conformal field theories, the Lagrangian or the Hamiltonian play a relatively minor role. The key idea is to try to solve for the correlation functions of the theory with out perturbative methods based on a local action. This is done only on criteria of self-consistency and symmetry. The interesting idea in this approach is that the product of local quantum operators can always be expressed as a linear combination of well-defined local operators. For example if we have two operators, we have:

$$\phi_i(x)\phi_j(y) = \sum_k C_{ij}^k(x-y)\phi_k(y) \tag{8.1}$$

Where $C_{ij}^k(x-y)$ are coefficients and not operators. This is called an operator product expansion which is a vital part in the study of conformal field theories. A conformal field theory, as the name suggests is a theory which is invariant under conformal transformations. These theories describe the critical behavior of systems at a second order phase transition. The simple example is the Ising model in two dimensions, with spins $\sigma_i = \pm 1$ on sites of a square lattice. The partition function is defined in this case as $Z = \sum_\sigma exp(-E/T)$, where the energy $E = -\epsilon \sum_{(ij)} \sigma_i \sigma_j$, where the sum is over nearest neighbors on the lattice sites and ϵ a constant. The model has a high temperature disorder phase (i.e. where the expectation value $\langle \sigma \rangle = 0$) and a low temperature ordered phase (i.e. where the expectation value $\langle \sigma \rangle \neq 0$). Thus there is a phase transition, at which the typical configuration have fluctuations on all length scales, so the theory which describes this model at this critical point should be scale independent (or more generally conformal invariant). Thus Conformal invariance (or scale invariance) is an important phenomenon close to phase transition in physical systems. For more detailed background study of this perspective the reader is urged to look at [63]. There one can look at calculation of critical exponents using variational perturbation theory introduced in [61], and also to calculate the same for the λ-transition of superfluid Helium [64], which matched the experimental results to a high degree of accuracy. One must note that there is deep relation between conformally invariant theories and nonconformally invariant physical systems close to their critical point. As pointed out in [63], there are many thermodynamic quantities,

which exhibit a power law with respect to the temperature close to criticality. For extensive study of this subject and systems close to criticality please refer to [63]. Also it is pointed out that the effective action is an effective tool for describing strongly coupled systems, and in the strong coupling limit the effective interaction changes the interaction to an anomalous power,[63],[59]. The description of such a strongly coupled theory is treated in [59], where the usual Gaussian random walk of the path integral is replaced by Levy walks (please see [59] for more details). In three or higher dimensions, conformal invariance does not give much more information than simple scale invariance. But in two dimensions, the conformal algebra becomes infinite dimensional, and hence gives more restrictions on the theory.[1]

Let us give a simple idea what we mean by a conformally invariant theory. Suppose we start with the action of the following form:

$$S = \int d^D x \, \mathcal{L}(\partial_x, g_{\mu\nu}(x), \phi(x)) \tag{8.2}$$

Where ϕ denote any field that might appear in the action, and we have specified the metric which we have specified separately. General coordinate invariance implies that:

$$S = S' \equiv \int d^D x' \, \mathcal{L}(\partial_{x'}, g'_{\mu\nu}(x'), \phi'(x')) \tag{8.3}$$

Under a general coordinate transformation, $x \to x'$, such that, $x^\mu = f^\mu(x'^\nu)$, a general tensor field of rank n will trnasform as :

$$\phi'_{\mu_1,\dots\mu_n}(x') = \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \dots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}} \phi_{\nu_1,\dots\nu_n}(f(x')) \tag{8.4}$$

Under this transformation a scalar function $\phi(x)$ will be left invariant, as $\phi'(x') = \phi(f(x'))$. Now if the transformation $x \to x'$ is conformal, which is to say the metric transforms by a Weyl factor $g^{\mu\nu} \to \Lambda(x)g^{\mu\nu}$, then using this property,

$$S = S'' \equiv \int d^D x' \, \mathcal{L}(\partial_{x'}, g_{\mu\nu}(f(x')), \phi'(x')) \tag{8.5}$$

This is the conformal symmetry of the action. We point out here that the metric remains unchanged, if we start with a flat space metric. It is then understood that the action of such theories is unchanged if we integrate the same Lagrangian, which is now expressed in terms of the new fields, over the new coordinates. Also this means that one can define the conformal transformation for theories in flat space, that are not coupled to gravity, and so one can start with an action in which no dynamical metric appears. This is the essence of the conformal invariant theories. Now we go into more detailed mathematical description of these theories, which will be required for our last chapter.

8.1 Transformation and Algebra

Now we discuss the transformations and the algebra of the generators of these transformations. A conformal transformation is a transformation of the coordinates such that it

---

[1]For more detailed reading please refer to [36],[42],[54],[55],[63].

is an invertible map and it leaves the metric tensor invariant upto a scale. This is defined as:

$$g'^{\mu\nu} = \Lambda(x)g^{\mu\nu}$$

The set of conformal transformations forms a group and the Poincare group forms a subgroup, since the latter corresponds to the case $\Lambda(x) = 1$. The word conformal means that the transformations do not change the angle between two arbitrary curves crossing each other. A field theory which is invariant under such transformations (much the same way as other symmetries) is called a conformal field theory. The globally well defined conformal transformations which are also called the special conformal group are the following:

$$
\begin{aligned}
&\text{(translation)} & \bar{x}^\mu &= x^\mu + a^\mu \\
&\text{(dilation)} & \bar{x}^\mu &= \alpha x^\mu \\
&\text{(rigid rotation)} & \bar{x}^\mu &= M^\mu_\nu x^\nu \\
&\text{(SCT)} & \bar{x}^\mu &= \frac{x^\mu - b^\mu(x^\nu x_\nu)}{1 - 2(b^\nu x_\nu) + (b^\nu b_\nu)(x^\sigma x_\sigma)}
\end{aligned}
\tag{8.6}
$$

where a^μ, b^μ are vectors, M^μ_ν an element of Lorentz Group and α is a real constant. Now the generators of the above transformations are the following:

$$
\begin{aligned}
&\text{(translation)} & P_\mu &= -i\partial_\mu \\
&\text{(dilation)} & D &= -ix^\mu\partial_\mu \\
&\text{(rigid rotation)} & L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&\text{(SCT)} & K_\mu &= -i[2x_\mu x^\nu\partial_\nu - (x^\nu x_\nu)\partial_\mu]
\end{aligned}
\tag{8.7}
$$

where SCT stands for Special Conformal Transformations. These generators follow the following algebra:

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})
\end{aligned}
\tag{8.8}
$$

Here we give some useful description and tools for the study of conformal field theory in 2 dimensions only. Firstly let us start by the looking at a conformal map in this case. Considering coordinates on the plane (x^0, x^1) and changing variables, $x^\mu \to w^\mu(x)$, the contravariant metric tensor transforms as,

$$g^{\mu\nu} \to \left(\frac{\partial w^\mu}{\partial x^\alpha}\right)\left(\frac{\partial w^\nu}{\partial x^\beta}\right)g^{\alpha\beta} \tag{8.9}$$

Because a conformal transformation is defined as when $g'^{\mu\nu}(w) \propto g^{\mu\nu}(z)$ this puts the following condition on the Jacobian of transformation,

$$\left(\frac{\partial w^0}{\partial x^0}\right)^2 + \left(\frac{\partial w^0}{\partial x^1}\right)^2 = \left(\frac{\partial w^1}{\partial x^0}\right)^2 + \left(\frac{\partial w^1}{\partial x^1}\right)^2 \tag{8.10}$$

$$\frac{\partial w^0}{\partial x^0}\frac{\partial w^1}{\partial x^0} + \frac{\partial w^0}{\partial x^1}\frac{\partial w^1}{\partial x^1} = 0 \tag{8.11}$$

The above conditions are equivalent to the following:

$$\frac{\partial w^1}{\partial x^0} = \frac{\partial w^0}{\partial x^1}, \quad \frac{\partial w^0}{\partial x^0} = -\frac{\partial w^1}{\partial x^1} \tag{8.12}$$

Which are the holomorphic Cauchy-Riemann equations. Also the conditions can be written as antiholomorphic Cauchy-Riemann equations as follows:

$$\frac{\partial w^1}{\partial x^0} = -\frac{\partial w^0}{\partial x^1}, \quad \frac{\partial w^0}{\partial x^0} = \frac{\partial w^1}{\partial x^1} \tag{8.13}$$

Thus we can use complex coordinates to represent these equations,

$$z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1, \quad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \tag{8.14}$$

The inverse transformation can be written as follows:

$$x^0 = \frac{1}{2}(z + \bar{z}), \quad x^1 = \frac{1}{2i}(z - \bar{z}), \quad \partial_0 = \partial_z + \partial_{\bar{z}}, \quad \partial_1 = i(\partial_z - \partial_{\bar{z}}) \tag{8.15}$$

In the holomorphic coordinates, the metric tensor for the flat case can be written as :

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{8.16}$$

Where the index μ takes value $z, \bar{z}$. Also in this language we can write the conformal transformation to be defined by the following condition:

$$\partial_{\bar{z}} w(z, \bar{z}) = 0 \tag{8.17}$$

Which is just to say that it is a holomorphic map,

$$z \to w(z) \tag{8.18}$$

8.2   Primary Field

A primary field is defined as a field which under conformal map transforms as follows:

$$\phi(z, \bar{z}) = \left(\frac{\partial w}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi(w, \bar{w}) \tag{8.19}$$

Where $\Delta, \bar{\Delta}$ are called the holomorphic and antiholomorphic conformal dimensions of the operator. Therefore, in a 2 dimensional conformal field theory, the correlation functions of primary operators transform as follows.

$$\langle \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle = \prod_{i=1}^{n} \left(\frac{\partial w_i}{\partial z_i}\right)^{\Delta_i} \left(\frac{\partial \bar{w}_i}{\partial \bar{z}_i}\right)^{\bar{\Delta}_i} \langle \phi_1(w_1, \bar{w}_1) \dots \phi_n(w_n, \bar{w}_n) \rangle \tag{8.20}$$

This puts enough constraint on the structure of these functions. More over the holomorphic and anti-holomorphic conformal dimensions Δ, and $\bar{\Delta}$ are in general different for fields with spins, but, here we will only consider spinless fields, so they turn out to be the same in our case.

8.3 2 and 3 point correlation functions

Global conformal symmetry (rotation, translation, dilation and special conformal transformation) restricts the structure of the two and three point function (expectation value with 2, 3 primary field) of spinless primary operators as follows:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{|z_1 - z_2|^{2(\Delta_1 + \Delta_2)}}, \quad \Delta_1 = \Delta_2 \tag{8.21}$$

and,

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = \frac{C_{123}}{|z_1 - z_2|^{2\gamma_3}|z_2 - z_3|^{2\gamma_1}|z_3 - z_1|^{2\gamma_2}}, \quad \gamma_1 = \Delta_2 + \Delta_3 - \Delta_1, etc. \tag{8.22}$$

And C_{12}, C_{123} are called structure constants, and are model dependent and also depend on the primary fields present in the correlator. It is these functions which we need to calculate to describe the theory. The structure of the higher point functions can not be fixed by global conformal invariance. In general they are functions of *cross ratios* which can for four points can be written as follows:

$$\frac{|z_1 - z_2||z_3 - z_4|}{|z_1 - z_3||z_2 - z_4|} \tag{8.23}$$

For n points there can be $n(n-3)/2$ anharmonic ratios. This makes finding higher point functions extremely difficult.

8.4 Energy Momentum Tensor

the energy momentum tensor is an important ingredient in conformal field theories. Classically it is defined as the variation of the action with respect to the metric tensor :

$$T^{\mu\nu} = -2\frac{\delta S}{\delta g_{\mu\nu}} \tag{8.24}$$

For the case at hand, it is interesting to note that the energy momentum tensor is traceless which can be seen as follows: Under scale transformation, the metric transforms as;

$$\delta g_{\mu\nu} = \epsilon g_{\mu\nu} \tag{8.25}$$

and the action would correspondingly transform as:

$$\delta S = \int d^2 x \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = -\frac{1}{2}\int d^2 x \sqrt{g}\epsilon T^{\mu}_{\mu} \tag{8.26}$$

Which must be zero in a conformal field theory as it is scale invariant. Therefore we have the following :

$$T^{\mu}_{\mu} = 0 \tag{8.27}$$

Also we have another condition. When there is a continuous symmetry in a field theory then we have a conserved current j_μ which is to say that $\partial^\mu j_\mu = 0$. For conformal field theory we have the conserved current $j_\mu = T_{\mu\nu}\epsilon^\nu$. For the special case of $\epsilon^\mu = constant$ we obtain the following:

$$\partial^\mu j_\mu = (\partial^\mu T_{\mu\nu})\epsilon^\nu = 0 \quad \Rightarrow \partial^\mu T_{\mu\nu} = 0 \tag{8.28}$$

In the case of a curved world sheet this becomes $\nabla_\mu T^{\mu\nu} = 0$. On the other hand if one has a general transformation $\epsilon^\mu(x)$ then we get again $T^\mu_\mu = 0$. When we write the energy momentum tensor in holomorphic coordinates :

$$T_{zz}(z,\bar{z}) = \frac{1}{2}(T_{00} - iT_{10}) \quad \text{and} \quad T_{\bar{z}\bar{z}}(z,\bar{z}) = \frac{1}{2}(T_{00} + iT_{10}) \tag{8.29}$$

While the condition (8.28) becomes :

$$\partial_{\bar{z}} T_{zz} = 0 \quad \text{and} \quad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{8.30}$$

While the tracelessness condition gives the following:

$$T_{z\bar{z}} = T_{\bar{z}z} = 0 \tag{8.31}$$

Thus the two non vanishing components of the energy momentum tensor are holomorphic or antiholomorphic.

8.5 The Conformal Algebra

Coming back to our complex coordinates, and looking at a holomorphic transformation, $z \mapsto w(z)$, we have:

$$ds^2 = dz d\bar{z} \mapsto \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} dz d\bar{z} \tag{8.32}$$

A Laurent expansion around $z = 0$ for a general infinitesimal conformal transformation can be written as,

$$z' = z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n \left(-z^{(n+1)}\right) \tag{8.33}$$

$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \left(-\bar{z}^{(n+1)}\right) \tag{8.34}$$

The generators corresponding to the transformations are:

$$l_n = -z^{n+1}\partial_z \quad , \quad \bar{l}_n = \bar{z}^{n+1}\partial_{\bar{z}} \tag{8.35}$$

As $n \in \mathbb{Z}$ there are infinite number of generators, and they obey the following commutation relations:

$$[l_m, l_n] = (m-n)l_{m+n} \quad , \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \quad , \quad [l_m, \bar{l}_n] = 0 \tag{8.36}$$

This is known as the Witt Algebra and is the classical counterpart of the well known Virasoro Algebra. We present here a general discussion. Performing a Laurent expansion of $T(z)$ we have:

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \quad \text{where} \quad L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \tag{8.37}$$

These modes satisfy the following commutation relation, which is the Virasoro Algebra,

$$[L_m, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{8.38}$$

Where c is called the central charge of the algebra which commutes with all the generators of the algebra. Given the central charge and the set of primary fields along with their conformal dimensions, and a technique for getting higher point function from the lower ones, one can define a conformal field theory completely. The method of obtaining the higher point functions from the lower ones is called conformal bootstrap. For example if one is given the 2- and 3-point function, then by using this method one can find the 4-point function. But in doing so one needs an extra structure called the conformal blocks which are functions of the cross ratios mentioned above. If one can find these conformal blocks then in principle one can write down any correlation function.

8.6 Conformal Ward Identities

The conformal Ward identities are obtained by variation of the correlation functions under a conformal transformation. Now when we look at a correlation function of primary fields and take an infinitesimal variation, we have the following:

$$z \to z' \tag{8.39}$$

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') \tag{8.40}$$

and the infinitesimal variation

$$\phi(z, \bar{z}) \to \phi(z, \bar{z}) + \delta\phi(z, \bar{z}) \tag{8.41}$$

Now for the expectation value of string of primary fields we have (assuming that the measure is invariant under conformal transformation):

$$
\frac{1}{Z} \int \mathcal{D}\phi \, \phi_1'(z_1, \bar{z}_1) \dots \phi_n'(z_n, \bar{z}_n) e^{-S[\phi']}
$$
$$
= \frac{1}{Z} \int \mathcal{D}\phi \, (\phi_1(z_1, \bar{z}_1) + \delta\phi_1(z_1, \bar{z}_1)) \dots (\phi_n(z_n, \bar{z}_n) + \delta\phi_n(z_n, \bar{z}_n)) e^{-S[\phi] - \delta S[\phi]}
$$
$$
= \frac{1}{Z} \int \mathcal{D}\phi \, \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) e^{-S[\phi]} + \frac{1}{Z} \int \mathcal{D}\phi \, \delta\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) e^{-S[\phi]} + \dots
$$
$$
+ \frac{1}{Z} \int \mathcal{D}\phi \, \phi_1(z_1, \bar{z}_1) \dots \delta\phi_n(z_n, \bar{z}_n) e^{-S[\phi]} - \frac{1}{Z} \int \mathcal{D}\phi \, \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \delta S[\phi] e^{-S[\phi]} \tag{8.42}
$$

Now as we have conformally invariant correlation functions, we obtain the following:

$$
\frac{1}{Z} \int \mathcal{D}\phi \, \delta\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) e^{-S[\phi]} + \dots + \frac{1}{Z} \int \mathcal{D}\phi \, \phi_1(z_1, \bar{z}_1) \dots \delta\phi_n(z_n, \bar{z}_n) e^{-S[\phi]}
$$
$$
- \frac{1}{Z} \int \mathcal{D}\phi \, \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \delta S[\phi] e^{-S[\phi]} = 0 \tag{8.43}
$$

Now from the definition of the energy momentum tensor we have in holomorphic coordinates:

$$\delta S = \oint ds^\mu \epsilon^\nu T_{\mu\nu} \tag{8.44}$$

μ, ν taking values $z, \bar{z}$. Also for the infinitesimal variation of the primary field, we have from 8.19 :

$$\delta\phi = \phi'(z,\bar{z}) - \phi(z,\bar{z}) = -(\Delta\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\bar{\Delta}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi) \tag{8.45}$$

Putting these together and in holomorphic coordinates:

$$\frac{1}{Z}\int \mathcal{D}\phi\, \delta\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)e^{-S[\phi]} + \cdots + \frac{1}{Z}\int \mathcal{D}\phi\, \phi_1(z_1,\bar{z}_1)\ldots\delta\phi_n(z_n,\bar{z}_n)e^{-S[\phi]}$$

$$= -\frac{1}{2\pi i}\oint_C dz[\epsilon(z)\langle T_{zz}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle + \bar{\epsilon}(\bar{z})\langle T_{z\bar{z}}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle]$$

$$+\frac{1}{2\pi i}\oint_C d\bar{z}[\bar{\epsilon}(\bar{z})\langle T_{\bar{z}\bar{z}}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle + \epsilon(z)\langle T_{\bar{z}z}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle]$$

$$\tag{8.46}$$

We have introduced a factor of $1/2\pi$ for convenience, which changes the normalization of the energy momentum tensor and the contour C encloses all the points $z_1,\ldots,z_n$. This is a general form of the Ward Identity which is valid also for expectation value of string of fields which are not primary. Restricting to primary fields and looking at the holomorphic part, we have the following:

$$-\frac{1}{Z}\int \mathcal{D}\phi\, \langle(\Delta_1\phi_1(z_1,\bar{z}_1)\partial_{z_1}\epsilon(z_1) + \epsilon(z_1)\partial_{z_1}\phi(z_1,\bar{z}_1))\ldots\phi_n(z_n,\bar{z}_n)e^{-S[\phi]} + \ldots$$

$$-\frac{1}{Z}\int \mathcal{D}\phi\, \phi_1(z_1,\bar{z}_1)\ldots(\Delta_n\phi_n(z_n,\bar{z}_n)\partial_{z_n}\epsilon(z_n) + \epsilon(z_n)\partial_{z_n}\phi(z_n,\bar{z}_n))e^{-S[\phi]}$$

$$= -\frac{1}{2\pi i}\oint_C dz\, \epsilon(z)\langle T_{zz}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle \tag{8.47}$$

where we have used the tracelessness of the energy momentum tensor ($T^\mu_\mu = \frac{1}{2}(T_{z\bar{z}} + T_{\bar{z}z}) = 0$). Thus (8.47) can be written as follows:

$$-\frac{1}{2\pi i}\oint_C dz\, \epsilon(z)\langle T_{zz}(z,\bar{z})\mathcal{X}\rangle = -\sum_{i=1}^{n}(\Delta_i\partial_{z_i}\epsilon(z_i) + \epsilon(z_i)\partial_{z_i})\langle\mathcal{X}\rangle$$

$$\Rightarrow -\frac{1}{2\pi i}\oint_C dz\, \epsilon(z)\langle T_{zz}(z,\bar{z})\mathcal{X}\rangle = -\frac{1}{2\pi i}\sum_{i=1}^{n}\oint_C dz\, \epsilon(z)\left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle\mathcal{X}\rangle$$

$$\tag{8.48}$$

where,

$$\langle\mathcal{X}\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, \phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)e^{-S[\phi]} \tag{8.49}$$

and we have used the residue theorem to rewrite the R.H.S. . This is the more common version of the Ward Identity. Here we will not do much more discussion on this topic, as we will not require this for our next chapters. Now we will discuss about the Coloumb Gas formalism, which we would require in the next chapter.

8.7   Coloumb Gas Formalism

It is interesting to note that the free bosonic field correlator $\langle\phi(z_i,\bar{z}_i)\phi(z_j,\bar{z}_j)\rangle = -\ln|z_i - z_j|^2$ resembles the electric potential energy of unit charges in 2 dimensions. So finding

the correlator of n-fields, is like the potential energy of n-unit charges. This analogy gives the name. The action for a free boson theory has the following action:

$$S = \frac{1}{4\pi} \int d^2 x \partial_\mu \phi \partial^\mu \phi \tag{8.50}$$

It is a conformal field theory with central charge $c = 1$. The holomorphic energy momentum tensor for this model is the follows:

$$T(z) = -\partial_z \phi \partial_z \phi \tag{8.51}$$

In the quantum case the above expression is to be normal ordered. $\partial\phi$ is a primary field with dimension 1. Now a vertex operator is defined as the exponential of the field ϕ:

$$V_\alpha(z, \bar{z}) = e^{i2\alpha\phi(z,\bar{z})} \tag{8.52}$$

The conformal dimension for this vertex operator is α^2. The vertex operators are understood to be normal ordered implicitly. Now the correlator of vertex operators is given by:

$$
\begin{aligned}
\langle V_{\alpha_1}(z_1, \bar{z}_1) \ldots V_{\alpha_n}(z_n, \bar{z}_n) \rangle \quad &= \int \mathcal{D}\phi \prod_{i=1}^{n} e^{i2\alpha_i\phi(z_i,\bar{z}_i)} \exp\left(-\frac{1}{4\pi} \int (\partial_\mu\phi)^2 d^2 x \right) \\
&= \prod_{i<j} |z_i - z_j|^{4\alpha_i\alpha_j}
\end{aligned}
\tag{8.53}
$$

Which is because, $\langle \phi(z_i, \bar{z}_i)\phi(z_j, \bar{z}_j) \rangle = -\ln|z_i - z_j|^2$. Moreover there is the neutrality condition, which is :

$$\sum_{i=1}^{n} \alpha_i = 0 \tag{8.54}$$

This can be seen as follows: if we split $\phi(z, \bar{z})$ into a constant mode ϕ_0 and a fluctuating part $\tilde{\phi}(z, \tilde{z})$, where they satisfy the following conditions:

$$\int dz\, d\bar{z}\phi_0 = \int dz\, d\bar{z}\, \phi(z, \bar{z}) \quad , \quad \int dz\, d\bar{z}\, \tilde{\phi}(z, \bar{z}) = 0 \tag{8.55}$$

Now if one splits the path integral and only look at the zero mode part (actually this is a bit subtle, in flat space the possible zero mode is not only a constant but there are eigen functions of the Laplacian with eigen value zero. On a sphere, there is no problem, as the only zero mode possible is a constant and the flat case can be understood as a limit of the result on the sphere), then we have:

$$\int_{-\infty}^{\infty} \mathcal{D}\phi_0\, e^{i2\sum_{i=1}^{n}\alpha_i} = \delta(2\sum_{i=1}^{n} \alpha_i) \tag{8.56}$$

Now the basic idea is to have a background charge in the system, this modifies the conformal dimensions of the vertex operators and the central charge. We will now drop the i in the definition of the vertex operators, just to make contact with the next chapter. Now If we have the action:

$$S = \frac{1}{4\pi} \int d^2 x \sqrt{g}[\partial_\mu \phi(x)\partial^\mu \phi(x) + QR(x)\phi(x)] \tag{8.57}$$

67

Where $R(x)$ is the scalar curvature of the surface which we consider. This can also be thought as charge $-Q$ at infinity (consider Q to be positive). Now the conformal dimension of the vertex operators become: $\alpha(Q - \alpha)$. More over in 2 dimensions we have the Gauss-Bonnet theorem, which says that:

$$\frac{1}{4\pi} \int d^2x \sqrt{g} R(x) = \chi \tag{8.58}$$

Where χ is the Euler character of the surface. For the sphere, $\chi = 2$. Moreover, the neutrality condition is transformed to the following:

$$\sum_{i=1}^{n} \alpha_i = Q \tag{8.59}$$

(This kind of Coulomb gas integrals are also important in Kosterlitz-Thouless Phase Transition in Condensed Matter Physics). On the sphere there is no problem in deriving the neutrality condition as the only zero mode on the sphere are constants. Now we discuss a very important concept, which we will need in the next chapter, which is a *Screening Charge Operator*. A screening charge operator, is conformal dimension zero operator, introduction of which does not change the conformal properties of a correlator. A local operator of conformal weight zero is an identity operator. In the above Coloumb representation there are two such vertex operators, $V_0(z)$ and $V_Q(z)$, but they are identity operators of the algebra (actually that is the case if are dealing with the minimal models, but not for Liouville like theories, any ways we exclude these two operators). The other possible choice are operators of the following type (writing in holomorphic coordinates and only the z part) :

$$\mathcal{Q} = \oint_C dz\, V_1(z) \tag{8.60}$$

where the 1 in the subscript represents conformal dimension 1. To see that this nonlocal operator, is conformally invariant, we note that under a conformal transformation ($z \to w$):

$$\mathcal{Q} \to \oint dz\, V_1(w) \frac{dw}{dz} = \mathcal{Q} = \oint dw\, V_1(w) \tag{8.61}$$

Of course the corresponding contour will also be changed. Now we will see how $\mathcal{Q}$ commutes with all the Virasoro generators. For this we have to compute:

$$
\begin{aligned}
[L_n, \mathcal{Q}] \quad &= \oint dz\, [L_n, V_1(z)] \\
&= \frac{1}{2\pi i} \oint dw \oint dz\, T(w) V_1(z) \\
&= \frac{1}{2\pi i} \oint dw \oint dz\, z^{n+1} \left[\frac{1 \times V_1(z)}{(w-z)^2} + \frac{\partial V_1(z)}{w-z} + reg \right] \\
&= \oint dz [(n+1) z^n V_1(z) + z^{n+1} \partial V_1(z)] \\
&= \oint dz\, \partial(z^{n+1} V_1(z)) = 0
\end{aligned}
$$

$$\tag{8.62}$$

Where we have written $1\times$ in the calculation just to formally show, that it is the conformal dimension times the expression.Now there are two conformal weight zero operators we

can use as we have a quadratic equation for the dimension $\alpha(Q - \alpha) = 1$. Let us denote the roots as α_+ and α_-. Now from the sum of roots and product of roots of a quadratic equation we have,

$$\alpha_+ + \alpha_- = Q \quad , \quad \alpha_+ \alpha_- = 1 \tag{8.63}$$

We represent the two screening charges depending on the α_+ and α_- as $\mathcal{Q}_+$ and $\mathcal{Q}_-$ respectively. In Minimal models (models with central charge less than 1), the screening charge operators are used to modify the charge of the correlator. One has to note that in Minimal models, there is only one operator for one conformal dimension. Any correlation function which does not satisfy the neutrality condition can be modified by adding these screening charges. Thus a correlator in a theory which can be described by the Coloumb gas system (as Minimal models are) exists if it satisfies the modified neutrality condition (as one can add screening charges to modify the naive neutrality condition, when it is not satisfied):

$$\sum_{i=1}^{n} \alpha_i + k\alpha_+ + l\alpha_- = Q \tag{8.64}$$

where k and l are positive integrs, corresponding to integer insertion of the screening charge operators, and the correlator is therefore:

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) \ldots V_{\alpha_n}(z_n \bar{z}_n) \mathcal{Q}_+^k \mathcal{Q}_-^l \rangle \tag{8.65}$$

We will not go any further in this topic, which in itself is a vast subject. For our later chapter we only need the definition of the screening charge operators, and so we end our discussion on general properties of conformal field theory here. In the next chapter we will only use the form of the three point function described earlier and formal properties of the screening charge operators for our analysis. It is important to note that as we will be dealing with the explicit action of Liouville Field Theory, not much knowledge is required of the operator formalism in conformal field theory, and only knowledge of functional methods in quantum field theory will be sufficient.

Chapter 9

Liouville Field Theory

Quantum Liouville field theory became a very important topic of study since it came up in the quantization of non-critical strings in the work of Polyakov [24]. Since then many people have tried to quantize this model. Liouville field theory being a conformal field theory in 2 dimensions, has the infinite dimensional symmetry generated by the Virasoro generators. Canonical quantization of this theory proved problematic and could not yield the full two and three point functions which are essential for describing the theory. An important break through happened when [16], [17], gave a proposal for the three point function in Liouville field theory. Al though it was not obtained by directly but by analytically continuing after obtaining it for a discrete set of points, it satisfied all the properties required and also passed many stringent tests. Then [23] derived a set of difference equations for the structure constants (which is the part left after fixing the position dependent part by conformal invariance). One intriguing point about the proposal is that it has dual set of poles, which can not be seen from the path integral. The predicted poles are at negative integer values of the Liouville coupling constant b, while the dual set are in the negative integer values of $1/b$. Some authors have claimed that it is due to presence of an extra exponential potential, that is introduced due to renormalization. This dual potential produces the pole structure and also fixes the three point function uniquely. Thus it is worth while to investigate this problem, as the complete understanding has not been obtained.

Liouville field theory is a good candidate as, the potential is of quite simple form to analyze with the technique discussed earlier. The DeWitt equation (or the Schwinger Dyson equation) is of simple form in this case and so the equation at least seems to be tractable. Now let us start by introducing the classical conformally invariant action in the next section.

9.1   Classical Conformal Invariance

The classical action for conformal invariant Liouville theory 2 dimensions (We stick to Euclidian metric for simplicity) is the following:

$$S = \frac{1}{4\pi} \int d^2x \sqrt{\hat{g}}[\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + Q_c\hat{R}\phi(x) + 4\pi\mu e^{2b\phi(x)}] \tag{9.1}$$

Where $Q_c = 1/b$ and $R(x)$ is the scalar curvature term. Under change of the metric as:

$$\hat{g}_{\mu\nu}(x) = e^{\sigma(x)}g_{\mu\nu}(x) \tag{9.2}$$

Where g is another arbitrary fixed metric. We will see that this Weyl factor σ, can be reabsorbed in a redefinition of ϕ. Now let us look at the term with scalar curvature term. this transforms as follows:

$$\sqrt{\hat{g}}\hat{R} = \sqrt{g}(R - \Delta\sigma) \tag{9.3}$$

Thus the kinetic term and the curvature term put together we:

$$\frac{1}{4\pi}\int \sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + Q_c R\phi + Q_c g^{\mu\nu}\partial_\mu\sigma\partial_\nu\phi]d^2x \tag{9.4}$$

where we have used,

$$-\int \sqrt{g}\phi(x)\Delta\sigma d^2x = \int \sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\sigma d^2x \tag{9.5}$$

(the Laplace-Beltrami operator Δ is w.r.t the metric $g^{\mu\nu}$). Thus the transformed kinetic term can be written as follows:

$$\frac{1}{4\pi}\int \sqrt{g}\left[g^{\mu}\nu\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} + Q_c R\hat{\phi} - \frac{Q_c^2}{2}\left(R\sigma + \frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right)\right]d^2x \tag{9.6}$$

where the Liouville field has been shifted as:

$$\hat{\phi}(x) = \phi(x) + \frac{Q_c}{2}\sigma(x) \tag{9.7}$$

Thus we see that the action remains form invariant upto a boundary term (last term in brackets). We will come back to this term shortly. Let us look ta the exponential term. Firstly the contribution from the determinant of the metric:

$$\sqrt{\hat{g}} = e^\sigma \sqrt{g} \tag{9.8}$$

And next the contribution from the shift of the field:

$$e^{2b\phi(x)} = e^{-bQ_c\sigma}e^{2b\hat{\phi}(x)} \tag{9.9}$$

This means that for this term to be invariant we have: $1 - bQ_c = 0$, which gives the classical value of the back ground charge $Q_c = 1/b$. Now we see that the action is invariant upto a boundary term. A confromal transformation corresponds to the case when (using holomorphic coordinates $z = x_0 + ix_1$)

$$\sigma = -log\left|\frac{\partial w}{\partial z}\right|^2 \tag{9.10}$$

Under such a transformation the action is defined on an annulus Γ, of outer radius $\tilde{R} \to \infty$, and inner radius $1/\tilde{R} \to 0$, around the point point $z = 0$, given as:

$$S[\phi] = \frac{1}{4\pi}\int_\Gamma d^2z[(\partial_a\phi)^2 + 4\pi\mu e^{2b\phi}] + \frac{Q_c}{\pi\tilde{R}}\int_{\partial\Gamma} dl\phi + 2Q^2 log\tilde{R} \tag{9.11}$$

This condition is also called the background charge $-Q$ at infinity. The Thus under conformal transformation (holomorphic coordinate transformations $z \to w(z)$), the Liouville field transforms as follows:

$$\phi(w,\bar{w}) = \phi(z,\bar{z}) - \frac{Q_c}{2}log\left|\frac{\partial w}{\partial z}\right|^2 \tag{9.12}$$

Quantum mechanically the background charge Q becomes $b + 1/b$. We will discuss this later when we will regularize the Schwinger Dyson equation.

9.2 Classical Solution

If we restrict to flat space, then the Lagrangian for this theory is of the following form (dropping the curvature term):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 2\pi\mu e^{2b\phi(x)} \tag{9.13}$$

This gives the following equation of motion:

$$\Box\phi(x) - 4\pi\mu b e^{2b\phi(x)} = 0 \tag{9.14}$$

This equation has the following solution, which was first given by Joseph Liouville himself, in terms of two arbitrary functions, F and G,

$$\phi(x) = \frac{1}{2b}ln\left[\frac{F'(z)G'(\bar{z})}{[1 - \pi\mu b^2 F(z)G(\bar{z})]^2}\right] \tag{9.15}$$

Another important point is that the solution of the Liouville equation of motion can be mapped to the solution of the Laplace equation i.e. the free field equation by a transformation called the Backlund transformations which are given as follows:

$$2\partial_z(\phi - \phi_f) = \sqrt{2\pi\mu}\alpha e^{2b(\phi+\phi_f)}$$
$$2\partial_{\bar{z}}(\phi + \phi_f) = \sqrt{2\pi\mu}\frac{1}{\alpha}e^{2b(\phi-\phi_f)}$$

$$\tag{9.16}$$

where $\Box\phi_f = 0$, and so is a free field and α is an arbitrary constant.

9.3 Energy Momentum Tensor, Primar Fields and Central Charge

The holomorphic and anti-holomorphic energy momentum tensors for Liouville field theory are given as :

$$T(z) = -(\partial_z\phi)^2 + Q\partial_z^2\phi$$
$$T(\bar{z}) = -(\partial_{\bar{z}}\phi)^2 + Q\partial_{\bar{z}}^2\phi$$

$$\tag{9.17}$$

The primary fields of Liouville field theory are the exponential vertex operators:

$$V_\alpha = e^{2\alpha\phi(x)} \tag{9.18}$$

The conformal dimension of this operators are $\Delta_\alpha = \alpha(Q - \alpha)$, where the $-\alpha^2$ comes from quantum correction. Under a conformal transformation ($z \to w(z)$) theses primary fields transform as:

$$e^{2\alpha\phi(z,\bar{z})} = \left(\frac{\partial w}{\partial z}\right)^{\Delta_\alpha}\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^{\Delta_\alpha}2\alpha\phi(w,\bar{w}) \tag{9.19}$$

73

From this definition one can find the scaling behaviour of correlator of any n-point function which is discussed in the earlier chapter. The central charge of the Liouville theory is given in terms of the Liouville coupling parameter b as:

$$C = 1 + 6Q^2 = 1 + 6\left(b + \frac{1}{b}\right)^2 \tag{9.20}$$

Using these one can write down the Ward identities as well, which are of the same form as discussed. We will not require them for our later discussion so we will not mention them here.

9.4 Correlation functions and Translation invariance of measure

One can define the partition function for Liouvile Field Theory as follows:

$$\int [D\phi] e^{-S[\phi] + \int d^2x \sqrt{g} J(x).\phi(x)} = Z[J] \tag{9.21}$$

And the correlation functions are defined as the partition function with delta function insertions of the following type: $J(x) = \sum 2\alpha_i (g(x_i))^{-1/2} \delta^{(2)}(x - x_i)$. And so

$$\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = Z[J] \tag{9.22}$$

On applying the translation invariance of the path integral measure we obtain the following:

$$\int [D\phi] \frac{\delta}{\delta\phi(x)} e^{-S[\phi] + J.\phi} = 0$$

$$\Rightarrow Z[J]J(x) - \left\langle \frac{\delta S}{\delta\phi(x)} \right\rangle = 0$$

$$\Rightarrow J(x)Z[J] = -\frac{1}{2\pi}\Delta \frac{\delta Z[J]}{\delta J(x)} + \frac{1}{4\pi}QR(x)Z[J] + 2\mu b Z[J_{x,b}] \tag{9.23}$$

Where Δ is the Laplace-Beltrami operator and $J(y)_{x,b}$ is defined as follows:

$$J(y)_{x,b} = J(y) + \frac{2b}{\sqrt{g(x)}} \delta^2(y - x) \tag{9.24}$$

This is because,

$$\langle e^{2b\phi(x)} \rangle_J = Z\left[J + \frac{\delta}{\delta\phi}\right] e^{2b\phi(x)} \Big|_{\phi(x)=0} \tag{9.25}$$

Now if we split up $\phi(x) = \phi_0 + \tilde{\phi}(x)$ where ϕ_0 is the zero mode of the Liouville field and $\tilde{\phi}(x)$ is the fluctuating part such that,

$$\int d^2x \sqrt{g} \, \phi_0 = \int d^2x \sqrt{g} \, \phi(x) \quad , \quad \int d^2x \sqrt{g} \, \tilde{\phi}(x) = 0 \tag{9.26}$$

and again perform the translation invariance of the path integral measure with respect to the zero mode only, we obtain the following:

$$\int [D\phi] \frac{\delta}{\delta\phi_0} e^{-S[\phi]+J.\phi} = 0$$

$$\Rightarrow \int d^2x \sqrt{g} \left[J(x) - \frac{1}{4\pi} Q R(x) \right] Z[J] = \int d^2x \sqrt{g} \left[2\mu b Z[J_{x,b}] \right]$$

$$\Rightarrow [2(\sum_i \alpha_i - Q)] Z[J] = 2\mu b \int d^2x \sqrt{g} Z[J_{x,b}] \tag{9.27}$$

Now for two insertions and after taking a flat space limit, (9.23) becomes :

$$(2\alpha_2 \delta^{(2)}(x - x_2) + 2\alpha_3 \delta^{(2)}(x - x_3) - \frac{1}{4\pi} Q R(x)) \langle e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle =$$

$$-\frac{1}{2\pi} \Box \langle \phi(x) e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle + 2\mu b \langle e^{2b\phi(x)} e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle \tag{9.28}$$

Now dropping the terms on the L.H.S (which are contact terms in principle), we can rewrite the equation as follows:

$$\frac{1}{2\pi} \Box \langle \phi(x) e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle = 2\mu b \langle e^{2b\phi(x)} e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle$$

Now making the following identification:

$$\phi(x) = \frac{\int [D\phi] \phi(x) e^{-S[\phi]+\int d^2x \sqrt{g(x)} J(x)\phi(x)}}{\int [D\phi] e^{-S[\phi]+\int d^2x \sqrt{g(x)} J(x)\phi(x)}}$$

$$= \frac{\int [D\phi] (\frac{1}{2} \partial_\alpha e^{2\alpha\phi(x)})|_{\alpha=0} e^{-S[\phi]+\int d^2x \sqrt{g(x)} J(x)\phi(x)}}{\int [D\phi] e^{-S[\phi]+\int d^2x \sqrt{g(x)} J(x)\phi(x)}}$$

$$\tag{9.30}$$

Which in our context we have:

$$\phi(x) = \frac{1}{2} \frac{\partial_\alpha \langle e^{2\alpha\phi(x)} e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle |_{\alpha=0}}{\langle e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle} \tag{9.31}$$

Now putting things together,

$$\frac{1}{4\pi} \Box \partial_\alpha \langle e^{2\alpha\phi(x)} e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle |_{\alpha=0} = 2\mu b \langle e^{2b\phi(x)} e^{2\alpha_2 \phi(x_2)} e^{2\alpha_3 \phi(x_3)} \rangle \tag{9.32}$$

Where the $\Box$ is the Laplacian and is with respect to x coordinates. This is precisely the equation which was used in [16] to check the proposal for three point function. This is basically the expectation value of the equation of motion.

9.5 1st functional equation

Let us look at a generic n-point function, then the above equation will be as follows:

$$[\sum_{i=1}^{n} \alpha_i - Q] \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = \mu b \int d^2x \sqrt{g(x)} \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{2b\phi(x)} \rangle \tag{9.33}$$

But the above is a recursive equation with a screening charge operator. Notice that the conformal dimension of the non-local screening operator on the right is zero and so the introduction of such an operator does not change the conformal structure. By setting one of the $\alpha = b$ one obtains again:

$$[\sum_{i=1}^{n} \alpha_i + b - Q]\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}e^{2b\phi(x)}\rangle = \mu b \int d^2y\sqrt{g(y)}\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}e^{2b\phi(x)}e^{2b\phi(y)}\rangle \quad (9.34)$$

Replacing the above in the first equation, one again obtains:

$$[\sum_{i=1}^{n} \alpha_i - Q][\sum_{i=1}^{n} \alpha_i + b - Q]\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\rangle$$

$$= (\mu b)^2 \int d^2x\sqrt{g(x)} \int d^2y\sqrt{g(y)}\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}e^{2b\phi(x)}e^{2b\phi(y)}\rangle \quad (9.35)$$

The above can be extended n-times, obtaining the following and denoting $\sum_{i=1}^{n} \alpha_i = \tilde{\alpha}$,

$$\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\rangle \prod_{j=0}^{m}(\tilde{\alpha} + jb - Q)$$

$$= (b\mu)^{m+1} \int \cdots \int \langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)} \prod_{j=0}^{m} e^{2b\phi(y_j)}\rangle d^2y_j\sqrt{g(y_j)} \quad (9.36)$$

It is clear form the above structure that, the n-point functions are singular at $\tilde{\alpha} = Q - mb$, which is also seen from the path integral. Moreover, the residues of the the n-point function are directly seen to be exactly what one obtains from the path integral. We also realize from here, that the correlator is related to arbitrary integer insertions of the screening charge operator.

9.6 Equation for the Structure Constants

Now when ever $\tilde{\alpha} = Q - mb$, then one can choose the right hand side to be evaluated with j insertions of the screening charge operator, such that it is a free field Coloumb gas integral. To elaborate on the computation later, we present here some insights. Firstly taking the ratio of eq 9.36, with m insertions and $m-1$ of the screening charge operator, gives the following:

$$(\tilde{\alpha} - Q + mb) \int \cdots \int \langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)} \prod_{j=0}^{m-1} e^{2b\phi(y_j)}\rangle d^2y_j\sqrt{g(y_j)}$$

$$= \mu b \int \cdots \int \langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)} \prod_{j=0}^{m} e^{2b\phi(y_j)}\rangle d^2y_j\sqrt{g(y_j)} \quad (9.37)$$

Using the above we have:

$$\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\rangle \prod_{j=0}^{m}(\tilde{\alpha}+jb-Q)$$

$$= (b\mu)^m(\tilde{\alpha}-Q+mb)\int\cdots\int\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\prod_{j=0}^{m-1} e^{2b\phi(y_j)}\rangle d^2y_j\sqrt{g(y_j)} \tag{9.38}$$

Let us see what the functional integral on the right hand side actually looks like. From the Liouville path integral after integrating over the zero mode, we have the following:

$$\int\cdots\int\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\prod_{j=0}^{m-1} e^{2b\phi(y_j)}\rangle = \Gamma\left(-\frac{(Q-\tilde{\alpha}-mb)}{b}\right)\int d\tilde{\phi}e^{-S_0(\tilde{\phi})}\times$$

$$\prod_{i=1}^{n} e^{2\alpha_i\tilde{\phi}(x_i)}\left(\int d^2y\sqrt{g(y)}e^{2b\tilde{\phi}(y)}\right)^m\left(\mu\int d^2x\sqrt{g(x)}e^{2b\tilde{\phi}(x)}\right)^{\frac{(Q-\tilde{\alpha}-mb)}{b}} \tag{9.39}$$

Where $S_0(\tilde{\phi})$ is the free field action. When $(\tilde{\alpha}-Q+mb)\to 0$, the R.H.S is what one gets when we expand the Liouville path integral perturbatively in powers of the cosmological constant. Also, when this constraint is satisfied, the R.H.S is a free field Coloumb Gas integral, which can be evaluated. Writing down this in usual notations in literature, and after some algebra we obtain the following relationship, when we restrict to three insertions:

$$(m!)(-1)^m b^m(\epsilon)C(\alpha_1,\alpha_2,\alpha_3)f(x_1,x_2,x_3)$$
$$= b^m(m!)(-1)^m f(x_1,x_2,x_3)I_m(\alpha_1,\alpha_2,\alpha_3) \tag{9.40}$$

Where, we have absorbed the powers of the μ in the definition of I_m in accordance with the literature which are just identified with the residues of the correlation function and explicitly written as [27]:

$$I_m = \left(\frac{-\pi\mu}{\gamma(-b^2)}\right)^m\frac{\prod_{j=1}^{m}\gamma(-jb^2)}{\prod_{k=0}^{m-1}[\gamma(2\alpha_1 b+kb^2)\gamma(2\alpha_2 b+kb^2)\gamma(2\alpha_3 b+kb^2)]} \tag{9.41}$$

and,

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \tag{9.42}$$

And $f(x_1,x_2,x_3)$ is the position dependent part which turns out to be the same on both sides (which is expected) and given by :

$$f(x_1,x_2,x_3) = |x_1-x_2|^{2(\Delta_3-\Delta_1-\Delta_2)}|x_2-x_3|^{2(\Delta_1-\Delta_2-\Delta_3)}|x_3-x_1|^{2(\Delta_2-\Delta_3-\Delta_1)} \tag{9.43}$$

Δ_i being the conformal dimensions, and $\epsilon = (\tilde{\alpha}+mb-Q)\to 0$. Again a similar equation holds for choice of parameters, $(\alpha_1+b,\alpha_2,\alpha_3)$, as:

$$((m-1)!)(-1)^{m-1}b^{m-1}(\epsilon)C(\alpha_1+b,\alpha_2,\alpha_3)f'(x_1,x_2,x_3)$$
$$= b^{m-1}((m-1)!)(-1)^{m-1}f'(x_1,x_2,x_3)I_{m-1}(\alpha_1+b,\alpha_2,\alpha_3) \tag{9.44}$$

Taking the ratio of the two equations, gives us the following functional equation for the structure constant, which is valid for for the condition $(\tilde{\alpha} + mb - Q) = 0$,

$$\frac{C(\alpha_1 + b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} = \frac{I_{m-1}(\alpha_1 + b, \alpha_2, \alpha_3)}{I_m(\alpha_1, \alpha_2, \alpha_3)} \tag{9.45}$$

Which explicitly can be written down as follows:

$$\frac{C(\alpha_1 + b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} =$$
$$-\frac{\gamma(-b^2)}{\pi\mu} \frac{\gamma(b(2\alpha_1 + b))\gamma(2b\alpha_1)\gamma(b(\alpha_2 + \alpha_3 - \alpha_1 - b))}{\gamma(b(\alpha_1 + \alpha_2 + \alpha_3 - Q))\gamma(b(\alpha_1 + \alpha_2 - \alpha_3))\gamma(b(\alpha_1 + \alpha_3 - \alpha_2))} \tag{9.46}$$

9.7   2nd functional equation

But there exists two screening operators in the theory as there are two solutions to the quadratic equation $\alpha(Q - \alpha) = 1$. A screening operator has conformal dimension zero and so adding it to the correlation function maintains the conformal symmetry, while modifying its net charge. Hence just like in Coloumb Gas Formalism, one has two screening operator, the other being the following:

$$\int d^2x \sqrt{g(x)} e^{\frac{2}{b}\phi(x)} \tag{9.47}$$

Let us start by looking at the 2nd functional equation for this screening charge. We will here first write down the functional equation and get the 2nd equation for the structure constant analogously. If one has an equation of the following form:

$$(\tilde{\alpha} - Q)\langle\prod_{i=1}^{n} e^{2\alpha_i(x_i)}\rangle = \tilde{\mu}\frac{1}{b}\int d^2x \sqrt{g(x)}\langle\prod_{i=1}^{n} e^{2\alpha_i(x_i)} e^{\frac{2}{b}\phi(x)}\rangle \tag{9.48}$$

Where $\tilde{\mu}$ is some undetermined constant, which we will discuss later. One can perform the same iterative scheme as earlier, and obtain the following:

$$\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\rangle \prod_{j=0}^{m}(\tilde{\alpha} + \frac{j}{b} - Q) = \int \cdots \int (\tilde{\mu}\frac{1}{b})^{m+1}\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)} \prod_{j=0}^{m} e^{\frac{2}{b}\phi(y_j)}\rangle d^2y_j \sqrt{g(y_j)} \tag{9.49}$$

Again when one of the products on the L.H.S is zero, then we have a pole in the correlator, and the right hand side is a Free Field Coloumb Gas Integral. Then we can again write down the functional equation for the structure constants, which yield a relation of the following type:

$$\frac{C(\alpha_1 + 1/b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} = \frac{\tilde{I}_{m-1}(\alpha_1 + 1/b, \alpha_2, \alpha_3)}{\tilde{I}_m(\alpha_1, \alpha_2, \alpha_3)} \tag{9.50}$$

Where the residues $\tilde{I}$ are the Free Field Coloumb Gas integrals computed with $1/b$ insertions. This can be written down explicitly as follows:

$$\frac{C(\alpha_1 + 1/b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} =$$

$$-\frac{\gamma(-b^{-2})}{\pi\tilde{\mu}} \frac{\gamma(\frac{(2\alpha_1+1/b)}{b})\gamma(2\frac{\alpha_1}{b})\gamma(\frac{(\alpha_2+\alpha_3-\alpha_1-1/b)}{b})}{\gamma(\frac{(\alpha_1+\alpha_2+\alpha_3-Q)}{b})\gamma(\frac{(\alpha_1+\alpha_2-\alpha_3)}{b})\gamma(\frac{(\alpha_1+\alpha_3-\alpha_2)}{b})} \tag{9.51}$$

9.8 DOZZ formula

Using the shift equations (9.46),(9.51), which is exactly the same obtained by [23] (actually only the first one was obtained, the second was assumed because of duality of the degenerate fields $-b/2$ and $-1/(2b)$) , one can obtain the DOZZ formula for the 3-point function (which for irrational values of b is unique) if one assumes that the shift equations are valid in the entire parameter space. For completeness we present the formula here:

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\tilde{\alpha})/b} \times$$

$$\frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_2 + \alpha_3 - \alpha_1)\Upsilon(\alpha_3 + \alpha_1 - \alpha_2)} \tag{9.52}$$

Also $\tilde{\mu}$ is identified as $(\pi\mu\gamma(b^2))^{b^{-2}}/(\pi\gamma(b^{-2}))$. Here it is to be pointed out that although from this analysis the value of the "dual cosmological constant" is not obtained, but it is enough to get the pole structure of the correlation function out, and then the dual cosmological constant can be fixed automatically as the R.H.S of the shift equations is for the same function i.e the structure constant once one analytically continues. We will later see that although the dual cosmological constant can not be fixed, its scaling with respect to μ can be obtained. We here also define the Υ functions as follows:

$$log\Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[(Q/2 - x)^2 e^{-t} - \frac{sinh^2((Q/2 - x)\frac{t}{2})}{sinh\frac{tb}{2}sinh\frac{t}{2b}}\right] \quad 0 < Re(x) < Re(Q) \tag{9.53}$$

Here $Q = b + 1/b$ and $\Upsilon_0 = \Upsilon'|_{x=0}$, prime denotes derivative with respect to x. From the above definition we can find the functional equation for the Υ function:

$$\Upsilon_b(Q - x) = \Upsilon_b(x) \tag{9.54}$$

Studying more properties, we find that it has simple zeros at $x = 0$ and $x = Q$. More over there are shift equations which this function satisfies as follows:

$$\begin{aligned}\Upsilon_b(x + b) &= \gamma(bx)b^{1-2bx}\Upsilon_b(x) \\ \Upsilon_b(x + 1/b) &= \gamma(x/b)b^{\frac{2x}{b}-1}\Upsilon_b(x)\end{aligned} \tag{9.55}$$

These shift relations are crucial for solving for the shift equations for the structure constant. These equations also show that there are more simple zeros at $x = -mb - n/b$

and $x = (m' + 1) + (n' + 1)/b$, where m, n, m', n' are non-negative integers. There is also the inverse shift relations:

$$\Upsilon_b(x - b) = \gamma(bx - b^2)^{-1} b^{2bx - 1 - 2b^2} \Upsilon_b(x)$$
$$\Upsilon_b(x - 1/b) = \gamma(x/b - 1/b^2)^{-1} b^{1 + \frac{2}{b^2} - \frac{2x}{b}} \Upsilon_b(x)$$

$$(9.56)$$

It is also important to point out that for irrational values of b the three point function is fixed uniquely if the shift equations are valid for the entire parameter space (actually it is not yet clear whether we need information for the entire parameter space or only these infinite set of planes, on which these equations principally hold is enough, and this a problem we still are looking at). For rational values of b, there in principle could be a function which is periodic in b and $1/b$. And hence the structure constant may be multiplied by a periodic function $g(\tilde{\alpha} - Q)$, such that $g(-mb - n/b) = 1$ for integer values of m and n to match with the values of the residues. We will later discuss some implications of this.

9.9   Source of the 2nd functional equation

We start by looking at the path integral which after evaluation of the zero mode integral is,

$$\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = \Gamma\left(-\frac{(Q - \tilde{\alpha})}{b}\right) \int d\tilde{\phi} e^{-S_0(\tilde{\phi})} \prod_{i=1}^{n} e^{2\alpha_i \tilde{\phi}(x_i)} \left(\mu \int d^2 x \sqrt{g(x)} e^{2b\tilde{\phi}(x)}\right)^{\frac{(Q - \tilde{\alpha})}{b}}$$

$$(9.57)$$

If, $(\tilde{\alpha} - Q) = -mb - 1/b$ then we can not evaluate the above integral because the exponent of the interacting term is not an integer. We know though from free Coloumb gas theory that there are screening operators which are of conformal dimension zero and commute with all the Virasoro generators, of the theory, although having non zero charge. Therefore adding a screening charge operator does not change the x-dependence of the correlators, but changes the net charge. In Liouville theory, similar to free field Coloumb Gas theory, we have two screening charges, as pointed out earlier. If we insert the second screening charge in the correlator, then we obtain the following:

$$\int d^2 x \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{\frac{2}{b}\phi(x)} \rangle = \Gamma\left(-\frac{(Q - \tilde{\alpha} - 1/b)}{b}\right) \int d\tilde{\phi} e^{-S_0(\tilde{\phi})} \times$$
$$\prod_{i=1}^{n} e^{2\alpha_i \tilde{\phi}(x_i)} \left(\int d^2 y \sqrt{g(y)} e^{\frac{2}{b}\tilde{\phi}(y)}\right) \left(\mu \int d^2 x \sqrt{g(x)} e^{2b\tilde{\phi}(x)}\right)^{\frac{(Q - \tilde{\alpha} - 1/b)}{b}} \qquad (9.58)$$

Now the exponent of the interaction term (μ dependent part) is an integer, and also the Coloumb gas constraint is satisfied so the integral is calculable. But now the argument of the Gamma function is a negative integer, and hence singular. This already gives us a hint that there are also poles for $1/b$ as we saw for b. In free field Coloumb Gas theory, one relates a correlator which does not satisfy the charge neutrality constraint to the one which does, by addition of suitable no. of screening charges. Similarly one ask the question how are the two correlators, with the screening charge and without the screening

charge related in this case. Is it really analogous to the first functional equation for b, i.e we look for the form of the functional equation, relating $\int d^2x \sqrt{g(x)} \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{\frac{2}{b}\phi(x)} \rangle$ to $\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle$. But we already have a form for the functional equation which we will now see. If we start with a bare action, and write down the Schwinger Dyson equation for the zero mode part, then we obtain the following:

$$[\sum_{i=1}^{n} \alpha_i - Q_B]\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = \mu_B b \int d^2x \sqrt{g(x)} \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{2b\phi(x)} \rangle \tag{9.59}$$

Where we have denoted the bare parameters with B. This is the expectation value of the operator equation of motion, which can be directly obtained form the Schwinger action principle. Without regularization and renormalization, there exists only the above equation. This is also because the classical dimension of the exponential operators are αQ_B, where $Q_B = 1/b$ and hence there is only one non-local conformal weight zero operator, and hence no background dependence coming from the $\int d^2x \sqrt{g} e^{2b\phi(x)}$ (if there is no need for regularizing, or Weyl invariant regularization exists) , while the L. H.S and R.H.S of the equation transform the same way under transformation of $J(x)$. Now when one regularizes the exponential operator in the integral on realizes, there is extra background dependence coming from the regularization, as we are unable to regularize in a Weyl invariant way. This is because of the extra Weyl factor coming from the point splitting of the Green's function at the same point, i.e.$G(x, x+dx) \equiv \frac{1}{2}ln(ds/L) + \frac{1}{4}ln(\sqrt{g(x)})$, where ds is the infinitesimal geodesic separation and L sets the overall scale. Thus we replace the bare Schwinger Dyson equation by a renormalized one.

$$[\sum_{i=1}^{n} \alpha_i - Q_R]\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = \mu_R b \beta \int d^2x \sqrt{g(x)} \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{2b\beta\phi(x)} \rangle \tag{9.60}$$

Where Q_R, μ_R and β are renormalization parameters. For the exponent to be background independent, we have the following condition:

$$1 - Q_R\beta b + \beta^2 b^2 = 0 \tag{9.61}$$

or,

$$Q_R = \frac{1}{\beta b} + \beta b \tag{9.62}$$

Now there are two choices of β for which we have same $Q_R = b + 1/b$, which are as follows: $\beta = 1, 1/b^2$ (we could have as well called $b\beta$ as b_R, or renormalized b, it is to show that for one choice, there is a classical screening equation counter part, while for the other there is none). These choices just reflect that there are two exponential operators for conformal weight 1. Hence we get two possible Schwinger Dyson equationns:

$$[\tilde{\alpha} - Q]\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = \mu b \int d^2x \sqrt{g(x)} \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{2b\phi(x)} \rangle \tag{9.63}$$

$$[\tilde{\alpha} - Q]\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} \rangle = \tilde{\mu}\frac{1}{b} \int d^2x \sqrt{g(x)} \langle \prod_{i=1}^{n} e^{2\alpha_i \phi(x_i)} e^{\frac{2}{b}\phi(x)} \rangle \tag{9.64}$$

81

Thus we already see that we have an equation for the 2nd screening charge operator which gives us the form of the relation we were looking for, which simply arises due regularizing the exponential operator in the quantum theory. The cosmological constants μ and $\tilde{\mu}$ are renormalized according to choice of β. We know that as adding a correlator α changes the scaling by $\frac{\alpha}{b}$, so the scaling for each choice will be μ, μ^{1/b^2}, upto some dimensionless constant. This can also be seen from the fact that after regularizing, the scaling term in front of the exponential operator, will be of dimension $[mass^2]^{1+(b\beta)^2}$. This means if we start with μ_B having dimension $[mass]^2$, then for each choice, after accounting for regularization (which is normal ordering in fact), the scaling will be $(\mu_B)^{1+b^2} \sim \mu$ and $(\mu_B)^{1+1/b^2} \sim \tilde{\mu}$, which means $\tilde{\mu}$ scales with exponent $1/b^2$ with respect to μ, which can be directly seen from the effective potential analysis [48]. Using these two functional equations, we get the two relationship for the structure constant as described earlier. It is also important to note that while the L.H.S. are the same for the equations the R.H.S. of the equation are different. This seems to make the duality between b and $1/b$ manifest.

9.10   Discussion

Having two functional equations, and adding the two, we obtain a third, namely:

$$[\tilde{\alpha} - Q]2\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}\rangle =$$

$$\mu b \int d^2x \sqrt{g(x)}\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}e^{2b\phi(x)}\rangle + \tilde{\mu}\frac{1}{b}\int d^2x \sqrt{g(x)}\langle\prod_{i=1}^{n} e^{2\alpha_i\phi(x_i)}e^{\frac{2}{b}\phi(x)}\rangle$$

$$(9.65)$$

Which was used by [29] to prove the uniqueness of the 3-point correlation function. The basic point of the above equation is that, the residues of the DOZZ formula can be recast in the following form using the recursion relations for the Υ functions:

$$I_m(\omega) = -m!(-1)^m(\mu_\omega\pi(\omega^2)^{2-Q\omega}\gamma(\omega^2))^m \frac{\Upsilon_0}{\Upsilon'(-m\omega)}\prod_{i=1}^{3}\frac{\Upsilon(2\alpha_i)}{\Upsilon(\tilde{\alpha}-\alpha_i)} \qquad (9.66)$$

Where $\omega = b, 1/b$ and $\mu_\omega = \mu, \tilde{\mu}$ respectively. Thus if one wants to analytically continue this expression then, it will be proportional to the DOZZ formula multiplied by an unknown function $g(\tilde{\alpha} - Q)$. But this function should satisfy the following condition, which is imposed by equation (9.65

$$g(\tilde{\alpha} - Q + b) + g(\tilde{\alpha} - Q + 1/b) = 2g(\tilde{\alpha} - Q) \qquad (9.67)$$

For irrational values of b this can only be true if $g(\tilde{\alpha} - Q) = constant$, which was also pointed out earlier. But then it was shown in [29], that this is the case for all values of b, if the function $g(\tilde{\alpha} - Q)$ is a continious function of b and $g(-nb - m/b) = 1$ for integer m, n(in fact only being continuous seems to be enough as there exists infinitely many irrational numbers within a distance ϵ of a rational number) , fixing the DOZZ formula uniquely.

It is important to point out that the Schwinger Dyson equation is obtained from the Schwinger action principle, and the path integral can be understood as the solution

of this functional equation. If on considers the bare Schwinger Dyson equation as the fundamental equation which defines Liouville Field Theory, then one automatically gets two solutions of the equation after regularization.

The other view is that as the residues calculated from the path integral is the one which we obtain from a Coulumb gas system with integer insertions of one of the screening charge operators, it is intuitive that when one analytically continues the three point function from these points to the entire complex plane, one obtains, an extra set of poles precisely at the points where $\tilde{\alpha} - Q = -n/b$ and the residue given by integrals of the same Coloumb gas systems, with integer insertions of the other screening charge operator. In both of this view point the second functional equation is fundamental in actually calculating the three point function. Actually both view points are intertwined as the screening charge operators are the ones which can be used as the regularized exponential interaction term in the Lagrangian.

Another important point we would like to make here is that the screening charge operators do not seem to be symmetric under the order of inserting them. Which is to say that the addition of the screening charge with b followed by $1/b$ is not the same as $1/b$ followed by b. This is also discussed in [45],[46],[47]. The two equations are non trivially intertwined. If one assumes that the order of addition of the the Screening Charges does not matter, then the equation after addition of the Screening Charge has to be completely symmetrized. We will not discuss much about this here any more, but just state that one can guess the totally symmetrized equation for addition of the Screening Charges after some short algebra.

Chapter 10

Conclusion

During this work, we were initially motivated at this alternative way of dealing with Quantum field theory. Here the primary object is not the path integral but a functional differential equation i.e. the DeWitt equation, (whose solution is obviously related to the path integral) using which one could aim to construct some algebraic way of computing correlation functions and Green's functions. In our attempt we were partially successful. We obtained a solution for our functional differential equation in a formal unrenormalized form. Although so, it is useful to understand the structure behind it. The solution was given in terms of "classical" objects. In doing so we had to define the classical propagator denoted $G_{cl}(u, v; \Phi)$ which is the inverse of the second derivative of the classical action and the interaction term which is also defined analogously, as higher derivatives of the classical action. These objects put together gives the solution as the sum over all One Particle Irreducible vacuum energy diagrams with this modified propagator and the modified vertices. The convergence properties of this solution is remarkable even for large values of the coupling constant, as was evident in the numerical results obtained. It shows that the φ_{cl} i.e. the classical field some how takes over the role of the renormalization scale. This is a very interesting observation and needs to be investigated in more detail.

The application of this solution can be to investigate finite supersymmetric field theories. As pointed out on the Chapter on the introduction on supersymmetry, there are some filed theories which are finite and do not need renormalization. For this very special field theories we can hope to impose these solution. As an example we constructed this for $N = 1$ Wess Zumino model in 2 dimensions in superspace. We showed that the Equation is finite i.e. free of any divergences. Using $N = 1$ superspace, which is a convenient way of looking at these theories, we wrote down the DeWitt equation in superspace and then its solution. The solution was then checked to satisfy the DeWitt equation order by order. Also a complete check is in principle difficult and cumbersome. This is because the solution is an expansion in number of loops of a diagram and thus is technically an infinite series. Nevertheless the procedure of how the solution works is clear once one goes through the calculations.

Next we tried to discuss the DeWitt equation in the context of $N = 4$ super Yang mills theory. As is well known that this is a finite theory in 4 dimensions and the subject of much research. Unfortunately the finiteness was shown by gauge fixing the theory, i.e. in Light cone gauge. In Light cone gauge only the physical degrees of freedom of the theory survive. Thus in this construction one is left with a theory which is free of auxiliary fields. The price that one has to pay is that the action becomes extremely complicated. There are many nonlocal inverse derivative terms appearing in the action, which makes

technical computation extremely difficult. This was our primary obstacle. Although we had the action from which we could construct the DeWitt equation, which would be manifestly finite, the resultant equation was extremely big and messy and almost useless for all computational purposes. thus we were not successful in our goal to use the solution for this specific model. Actually when we started out to work on this equation, the main goal was to use this to understand the structure of the Effective action for $N = 4$ Super Yang Mills theory. When we wrote down the DeWitt equation in covariant formalism, we realized that it was not finite, i.e. the expectation value of the fields appearing in the equation did not add upto a finite quantity. I was personally quite puzzled as I was expecting this to be finite. Soon we realized that the lack of an off-shell formalism for this model is the problem. Without the auxiliary fields, which in the loop diagrams are necessary for accounting for the degrees of freedom such that the fermionic ones exactly match the bosonic ones, the divergences which are due to loop contribution to the tree order do not cancel. Thus there was no obvious way of constructing a covariant finite DeWitt equation which would be easy to handle. So we moved on to some other approaches such as the Light Cone Superspace formalism. Unfortunately as discussed earlier, this was very cumbersome. Although one can implement the solution using this formalism, it is not easy for computational purposes. This was the point when we started looking at other possible uses of this equation.

This led us to Liouville Field Theory. Liouville Field Theory has been extensively studied for more than 30 years now. It comes up in the quantization of bosonic string theory models in non-critical dimensions. The Liouville field itself is the Weyl factor, which one obtains while taking the conformal gauge on the world sheet metric of the string [30],[24]. As this choice has to respect Weyl rescaling of the metric, to preserve Weyl invariance, the integration measure has to be Weyl invariant. This is because the measure itself is induced by the 2 dimensional metric [30]. This initially was a real problem as the Weyl invariant measure was non-linear, while standard path integral is formulated with a translation invariant measure. The way out was motivated by [49],[30],[38], where they showed that this problem can be over come by introducing a functional determinant while going from a Weyl invariant measure to a translationally invariant measure. This functional determinant or the Jacobian was shown to be proportional to the original bare action before renormalization. After this, the bare action was renormalized and effective coupling and other parameters were introduced in the action. It must also be pointed out that Weyl invariant quantization of a theory requires a Weyl invariant short distance regulator, which is not known. This leads to a Weyl factor while regularizing the exponential interaction, which directly comes from the contraction of the two point function, yielding a factor of $-\alpha^2$ to the conformal weight of every vertex operator.

Thus after this was understood. One could use regular Quantum field theory techniques to treat Liouville Theory. But this being a conformal field theory, the perturbative expansion of the correlators made little sense. Later the proposal by [16],[17], came out which gave an exact form for the three point function. As discussed earlier, although the proposal for the three point function passed many stringent tests, there were some open questions regarding the pole structure of the function. As pointed out, the three point function in Liouville Filed Theory, has additional poles at the dual set of points $\tilde{\alpha} - Q = -n/b$ in addition to the standard ones $\tilde{\alpha} - Q = -m/b$, which are evident from the path integral. This was intriguing and we wanted to know if we could use our technique to get some insights into this.

Using the translation invariance of the path integral measure, we were able to obtain a

functional equation for the correlation functions in Liouville theory. This equation relates the n-point function to the (n+1)-point function. Restricting to the zero mode equation, we were able to obtain a very simple relationship between these correlation functions. From this equations we were able to find the equation for the structure constant, which was obtained by [23]. Also the simple equation for the correlation functions have a dual, simply because of the fact that it relates the correlation functions with and without a Screening Charge operator insertion. A Screening Charge operator is a conformal weight zero operator whose insertion in the correlator changes the charge without changing the conformal properties of the correlator. It was observed that starting from the bare unrenormalized action, if one writes down the Schwinger Dyson equation for the zero mode part of the field ϕ and then regularizes the equation, there would be two distinct choices for regularizing the exponential potential. These are precisely the two choices for the screening charge operators corresponding to the each of the choices b and $1/b$. Thus having arrived at two distinct equations for the Screening Charges, we got the pole structure of the correlation functions as predicted. More over once the structure constant is analytically continued to all values for $(\tilde{\alpha} - Q)$, the dual value of the cosmological constant $\tilde{\mu}$ is automatically obtained in terms of the cosmological constant μ. It was also pointed out that if one starts from the Schwinger action principle as the fundamental input of the theory, then one obtains after regularizing, two equations for the correlation function, the DOZZ proposal being the unique solution to the two equations. It was also pointed out that for rational values of b there could be in principle a periodic function which can be present in the three point function. But using some results established in this context, which uses the continuous dependence of this function on b, the proposal can be fixed uniquely.

It is interesting to note that this procedure of relating correlation functions with and without insertions of screening charge operators could be generalized to other theories also. For this much more detailed work has to be done to understand the structure of screening charge operators in other theories.

Outlook

We intend to finish this report with some ideas and open questions for prospective future research. In that context let us list them down for completeness:

1. The relation of the expectation value of the field ϕ to the renormalization scale. As pointed out in solution for the DeWitt equation, Γ is expressed in terms of the φ_{cl} which is the vacuum expectation value of the quantum field. For some zero dimensional models the convergence of this expansion was extremely well, and it no longer required smallness of the coupling constant λ. In fact the convergence depended on an effective coupling which was a function of φ_{cl} This is interesting and could be studied in the context of quantum field theory.

2. From the structure of the solution $\Gamma[\varphi_{cl}]$ it seems that there could be some algebraic technique for computing higher loop corrections. This is because successive orders in the loop diagrams can be obtained by action of a functional differential operator as pointed out in the chapter on the formal solution to the DeWitt equation. We actually looked a bit into this, and we found that apart from the symmetry factors, the action of the functional differential operator on the set of graphs in n-loop

indeed produces the set of graphs in (n+1)-loop. It would be worth while if one is able to fix the symmetry factors for the graphs, then it could be an interesting way of computing amplitudes for a theory.

3. The problem of the off-shell Effective action for $N = 4$ was our main goal when we started this project. Unfortunately we were not able to solve this problem, although some understanding was obtained. Whether there is a way of formulating this theory in a more convenient way so that one could use the DeWitt equation to study it is an open problem. It will be very interesting to understand more about this, and also if the translation invariance of the measure which is used to derive the DeWitt equation is the reason for the covariant formalism not working in this case (existence of anomalies).

4. In the context of Conformal Field Theories, we examined Liouville theory with this tool, and by obtaining two separate equations for the correlation functions relating it to correlation functions with two distinct screening charge operators, we obtained the DOZZ formula. It would be interesting to see if this method can be generalized to other CFTs in general, where corresponding equations of the correlation functions will be related to insertions of screening charge operators. For example in Toda theories which are Liouville like theories with more number of exponential operators.

5. Other interesting avenues could be to see if the DeWitt equation can be turned into a renormalization flow equation by adding a regulator to the action, or to study other finite models with this technique.

Acknowledgement

First and foremost I would like to thank my supervisor Prof. Hermann Nicolai for suggesting me this problem to work on and also helping me with useful discussion during the three years of my stay at the Max Planck Institute for Gravitational Physics. I am in debt to him for all his support and guidance. I am also thankful to Prof. Krzysztof Meissner for useful discussions and collaboration during his stay at MPI. I am thankful to George Jorjadze for all his support and encouragement and enormous amount of helpful discussions, which has helped me learn many useful concepts, and also for his collaboration during his stay at MPI and Humboldt University Berlin. I am thankful to Stefan Fredenhagen for helping me out regarding many things at MPI and also for his discussions and kindness. I thank Prof. Hagen Kleinert, for many meetings and discussions we had and also for his help regarding many administrative things at Free University of Berlin. I am thankful to Prof. Herald Dorn for his insightful discussions and inputs regarding my work. I am grateful to my parents for their continued support during my studies without which I will be completely lost. I would thank my dear friends who have also been extremely encouraging. Then I would like to thank my colleagues at MPI and else where, in no particular order, Michael Koehn, Cossimo Restuccia, Philipp Fleig, Christine Gruber, Stefan Pfenninger, Pan Kessel, Johannes Thuerigen, Despoina Katsimpouri, Angelika Fertig and other members at MPI who have helped me with useful discussions and also served as delightful companions. I would also like to thank the Erasmus Mundus Joint Doctorate program, who funded by research work for the three years of my stay (specifically Grant Number 2010-1816 from EACEA). Also like to mention Prof. Remo Ruffini the director of the International Relativistic Astrophysics Program, and Prof. Pascal Chardonnet who has always being there to help us out and also arranged many useful schools during the course of my Ph.D.

List of Publications

- Parikshit Dutta, Krzysztof A. Meissner, Hermann Nicolai; *The DeWitt Equation in Quantum Field Theory*; Phys. Rev. D 87 (2013), 105019.

- *Schwinger Dyson approach to Liouvile Field Theory.* (In preparation and soon to appear. This is a work done in collaboration with George Jorjadze)

Bibliography

[1] B.S. DeWitt, *Dynamical Theory of Groups and Fields*, Lectures at Les Houches Summer school of Theoretical Physics, published in the proceedings volume, *Relativity,Groups and Topology*, 1964.

[2] B. A.Faizullaev, M. M. Musakhanov, *Two-Loop Effective Action for Theories with Fermions*, `hep-th/9412066`

[3] J. Glimm and A. Jaffe, *Quantum Physics: a Functional Integral Point of View*, Springer Verlag (1987).

[4] S. Coleman and E. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, Phys. Rev. **D7** (1973) 1888.

[5] K.A. Meissner and H. Nicolai, *Conformal Symmetry and the Standard Model*, Phys. Lett. **B648** (2007) 312, `hep-th/0612165`.

[6] W.A. Bardeen, *On Naturalness in the Standard Model*, preprint FERMILAB-CONF-95-391-T.

[7] K.A. Meissner and H. Nicolai, *Effective action, conformal anomaly and the issue of quadratic divergences*, Phys. Lett. **B660** (2008) 260, `arXiv:0710.2840`.

[8] R. Jackiw, *Functional evaluation of the effective potential*, Phys.Rev. **D9** (1974) 1686

[9] J.M. Cornwall, R. Jackiw and E. Tomboulis, *Effective action for composite operators*, Phys. Rev. **D10** (1974) 2428

[10] K.A. Meissner and H. Nicolai, *Renormalization group and effective potential in classically conformal theories*, Acta Phys. Polon. **B40** (2009) 2737, `arXiv:0809.1338`

[11] F. Gliozzi, J. Scherk and D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl. Phys. **B 122** (1977) 253-290.

[12] L. Brink, O. Lindgren, B.E.W. Nilsson, *The Ultra-violet finiteness of The N=4 Yang-Mills Theory*, Phys. Lett. **B123** (1983) 323

[13] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, McGraw Hill (1985).

[14] S. Weinberg, *The Quantum Theory of Fields*, Volumes 1-3, Cambridge Univ. Press, 2000

[15] R.V. Maluf and A.J. da Silva, *Two-loop effective potential for the Wess-Zumino model in 2+1 dimensions*, `arXiv:1207.1706 [hep-th]`

[16] H. Dorn and H.J. Otto, *Two and three point functions in Liouville theory*,Nucl. Phys B429 (1994) 375-388.

[17] A.B. Zamolodchikov and Al.B. Zomolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. **B477** (1996) 577-605.

[18] L. Brink, O. Lindgren, B.E.W. Nilsson, $N = 4$ *Yang-Mills theory on the light cone*, Nucl. Phys. **B212** (1983) 401

[19] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson, D.A. Kosower and R. Roiban, *On-shell methods in perturbative QCD*, Ann. Phys. **322** (2007) 1587, `arXiv:0704.2798`

[20] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson, D.A. Kosower and R. Roiban, *Simplifying multiloop integrals and ultraviolet divergences of gauge theory and gravity amplitudes*, Phys. Rev. **D85** (2012) 105014, `arXiv:1201.5366`

[21] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka and M. Staudacher, *Harmonic R-matrices for scattering amplitudes and spectral regularization*, `arXiv:1212.0850`

[22] J. Teschner,*Liouville theory revisited*, arXiv:hep-th/0104158.

[23] J. Teschner, *On the Liouville three point function*, Phys.Lett. **B363** (1995) 65-70.

[24] A. Polyakov. Phys.Lett., B103 (1981) 207

[25] T. Curtright and C. Thorn. Phys.Rev.Lett., 48 (1982) 1309; E.Braaten, T.Curtright and C.Thorn. Phys.Lett., B118 (1982) 115; Ann.Phys.,147 (1983) 365. .

[26] G. Jorjadze, G. Weigt, *Correlation functions and vertex operators of Liouville theory*, Phys.Lett., B 581 (1-2) (2004) 133-140.

[27] V. S. Dotsenko and V.A.Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, Nucl. Phys. B 240 (1), (1984) 312-348.

[28] V. S. Dotsenko and V.A.Fateev, *Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge C≤1*, Nucl. Phys. B 251(FS13) (5-6), 691-734 (1985).

[29] L. O'Raifeartaigh,J. M. Pawlowski and V. V. Sreedhar, *The two exponential Liouville Theory and the uniqueness of the three-point function*, arxiv.org/pdf/hep-th/0003247.

[30] E. D'Hoker and P. S. Kurzepa, *2-D Quantum Gravity and Liouville Theory*, Mod. Phys. Lett. (Vol.5, No.18) (1990), 1411-1421.

[31] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Publishing Company.

[32] D. Harlow, J. Maltz, E. Witten, *Analytic Continuation of Liouville Theory*, arXiv:1108.4417.

[33] P. O. Freund, *Intoduction to Supersymmetry*, Cambridge University Press.

[34] J. Wess,J. Bagger, *Supersymmetry and Supergravity*, 2nd edition, Princeton University Press.

[35] M. F. Sohnius, *Introducing Supersymmetry*, Physics Reports 128, Nos. 2-3 (1985) 39-204. North-Holland, Amsterdam.

[36] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Springer.

[37] E. D'Hoker, R. Jackiw, *Classical and Quantal Liouville field theory*, Physical Review D, Vol. 26, no. 12, (1982).

[38] E. D'Hoker, *Equivalence of Liouville Theory and 2-D Quantum Gravity*, Modern Physics Letters A, Vol. 6, No.9 (1991) 745-767.

[39] K. Fujikawa, H. Suzuki, *Path Integrals and Quantum Anomalies*, Oxford Science Publications.

[40] H. Nicolai, *A Transformation Formula For Integrals Over Superfields*, Physics Letters, Vol101 B, No. 6, (1981).

[41] H. Nicolai, *supersymmetric Functional Integration Measures*, Lectures at 1984 NATO Advanced Study Institutes on Supersymmetry, Bonn.

[42] A. A. Belavin, A. M. Polakov, A. .B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Physics B, Vol 241, Issue 2, (1984), 333-380.

[43] S. Krippendorf, F. Quevedo, O. Schlotterer, *Cambridge Lectures on Supersymmetry and Extra Dimensions*, arXiv:1011.1491 .

[44] F. Gliozzi, J. Scherk, D. Olive, *Supersymmetry, Supergravity theories and the Dual Spinor Model*, Nuclear Physics B 122 (1977), 253-290.

[45] J. Schnittger, *quantum Group Structure and Local Fields in the Algebraic Approach to 2D Gravity*, Theoretical and Mathematical Physics, Vol. 104, No.1, (1995).

[46] J. L. Geravais, J. Schnittger, *Continuous spins in 2D gravity: Chiral vertex operators and local fields*, Nuclear Physics B 431, (1994), 273-312.

[47] L. O'Raifeartaigh, J. M. Pawlowski, V. V. Sreedhar, *Duality in Quantum Liouville Theory*, arXiv:hep-th/9811090.

[48] C. B. Thorn, *The S-Matrix and the effective potential for the Liouville Quantum Field Theory*, Physics Letters Vol. 128B, No. 3,4, (1983), 207-212.

[49] J. Distler, H. Kawai, Nuclear Physics B321 (1989) 509.

[50] S. Ananth, S. Kovacs, S. Parikh, *A manifestly MHV Lagrangian for N=4 Yang Mills*, arxiv.org/pdf/1101.3540 .

[51] A. Zamolodchikov, A. Zamolodchikov, *Lectures on Liouville Theory and Matrix Models*.

[52] Yu Nakayama *Liouville Field Theory - A decade after the revolution*, arXiv:hep-th/0402009 .

[53] P. Ramon, *Field theory : A Modern Prime*, Addison-Wesley Publishing Company, 2nd Edition.

[54] P. Ginsparg, *Applied Conformal Field Theory*, arXiv:hep-th/9108028 .

[55] A. N. Schellekens, Introduction to Conformal Field theory, NIKHEF-H, Amsterdam, 96.

[56] A. S. Galperin, E. .A. Ivanov, V. .I. Ogievvetsky, E. S. Sokatchev, *Harmonic Superspaces*, Cambridge Monographs on Mathematical Physics.

[57] H. Kleinert, *Higher Effective Actions for Bose Systems*, Fortschritte der Physik 30, 187-232 (1982).

[58] H. Kleinert, *Effective action and field equation for BEC from weak to strong couplings*, J. Phys. B: Atomic, Molecular and Optical Physics. 46 (2013) 175401 (12pp).

[59] H. Kleinert, *Quantum Field Theory of Black-Swan Events*, Foundations of Physics, http://link.springer.com/article/10.1007%2Fs10701-013-9749-x/fulltext.html, http://www.klnrt.de/409.

[60] H. Kleinert, *Green function of the double fractional Fokker-Planck equation-Path integral and stochastic differential equations*, http://www.klnrt.de/406.

[61] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th edition, World Scientific, Singapore 2009.

[62] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation*, World Scientific, Singapore 2008.

[63] H. Kleinert, *Critical Properties of Φ^4-Theories*, World Scientific, Singapore 2001.

[64] H. Kleinert, *Theory and Satellite Experiment for Critical Exponent alpha of lambda-Transition in Superfluid Helium*, Phys. Lett. A 277, 205 (2000), (arXiv:cond-mat/9906107).