Appendix B: Higher Order Schwarz Kernels

Let $\Omega = \mathbb{H}$ be the upper half plane, its boundary $\partial\mathbb{H}$ is the real axis. As another application of the decomposition for polyharmonic functions, in this appendix, we consider higher order analogues of the classical Schwarz kernel in $\mathbb{H}$ which are called higher order Schwarz kernels.

Let $\breve{H}(\partial\mathbb{H})$ denote the set of all functions f defined on $\partial\mathbb{H}$ satisfying the following properties (see [25]):

(i) $f \in H(I)$, I is a sufficiently large closed real interval containing the origin in its interior;

(ii) $|f(x_1) - f(x_2)| \leq A|x_1^{-1} - x_2^{-1}|$, $x_1, x_2 \in \partial\mathbb{H} \setminus I$, where A is a positive constant.

Definition. If a sequence $\{ G_n(z,x) \}_{n=1}^{\infty}$ of real valued functions of two variables defined on $\mathbb{H} \times \partial\mathbb{H}$ satisfies

1′. $G_n(z,x) \in C^{2n}(\mathbb{H})$ as a function of z with fixed $x \in \partial\mathbb{H}$, $G_n(z,x)$, $\partial_z G_n(z,x)$, $\partial_{\overline{z}} G_n(z,x) \in C(\mathbb{H} \times \partial\mathbb{H})$ and $G_n(z,x)$, $\partial_z G_n(z,x)$, $\partial_{\overline{z}} G_n(z,x) \in \breve{H}(\partial\mathbb{H})$ as a function of x with fixed $z \in \mathbb{H}$;

2′. $(\partial_z \partial_{\overline{z}}) G_1(z,x) = 0$ and $(\partial_z \partial_{\overline{z}}) G_n(z,x) = G_{n-1}(z,x)$ for $n > 1$;

3′. $\lim_{z \to x,\, x \in \partial\mathbb{H},\, z \in \mathbb{H}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Upsilon(t) G_1(z,t) \mathrm{d}t = \Upsilon(x)$ for any $\Upsilon \in \breve{H}(\partial\mathbb{H})$;

4′. $\lim_{z \to x,\, x \in \partial\mathbb{H},\, z \in \mathbb{H}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Upsilon(t) G_2(z,t) \mathrm{d}t = 0$ for any $\Upsilon \in \breve{H}(\partial\mathbb{H})$;

5′. $\lim_{z \to x,\, x \in \partial\mathbb{H},\, z \in \mathbb{H}} G_n(z,x') = 0$ uniformly holds for $x' \in \partial\mathbb{H}$, $n > 2$.

Then $\{ G_n(z,x) \}_{n=1}^{\infty}$ is called a sequence of higher order Schwarz kernels, more precisely, $G_n(z,x)$ is the nth order Schwarz kernel.

It must be noted that all integrals in the above and what follows are understood as principle value integrals whether they are singular or not.

Note that (see [25])

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Big[\frac{1}{t - z} - \frac{1}{t - \overline{z}} \Big] \mathrm{d}t = 1, \ z \in \mathbb{H}$$

and by Sokhotzki-Plemelj formula

$$\lim_{z\to x,\, x\in\partial\mathbb{H},\, z\in\mathbb{H}} \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\Upsilon(t)\Big[\frac{1}{t-z}-\frac{1}{t-\overline{z}}\Big]\mathrm{d}t=\Upsilon(x)$$

for any $\Upsilon\in\breve{H}(\partial\mathbb{H})$. By Theorem B and Corollary 2, then we have the following existence and uniqueness theorem for a sequence of higher order Schwarz kernels.

Theorem. *If* $\{\,G_n(z,x)\,\}_{n=1}^{\infty}$ *is a sequence of higher order Schwarz kernels defined on* $\mathbb{H}\times\partial\mathbb{H}$ *, i.e.,* $\{\,G_n(z,x)\,\}_{n=1}^{\infty}$ *fulfills the above properties* $1'$-$5'$*, then, for* $n>1$*, there exist functions* $G_{n,0}(z,x),G_{n,1}(z,x),\dots,G_{n,n-1}(z,x)$ *defined on* $\mathbb{H}\times\partial\mathbb{H}$ *such that*

$$G_n(z,x)=2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z}+i)^jG_{n,j}(z,x)\Big\},\ z\in\mathbb{H},\,x\in\partial\mathbb{H}$$

with

$$\partial_z G_{n,j}(z,x)=j^{-1}G_{n-1,j-1}(z,x)$$

for $1\le j\le n-1$ *and*

$$\partial_z^k G_{n,j}(i,x)=0$$

for $0\le k\le j-1$ *with respect to* $x\in\partial\mathbb{H}$ *as well as*

$$G_{n,0}(z,x)=-\sum_{j=1}^{n-1}(z+i)^jG_{n,j}(z,x).$$

However,

$$G_1(z,x)=\frac{1}{i}\Big[\frac{1}{x-z}-\frac{1}{x-\overline{z}}\Big].$$

is the classical Schwarz kernel of complex form. Such $\{\,G_n(z,x)\,\}_{n=1}^{\infty}$ *is unique.*

Proof. By Theorem B and Corollary 2, the proof is similar as the one of Theorem 3 in Chapter 2. □

Open Problem 1. As the higher order Poisson kernels, it is interesting to get the explicit expressions of higher order Schwarz kernels. Different from Chapter 2, here we only give the expressions of $G_2(z,x)$ and $G_3(z,x)$. In view of the above theorem, by straight calculations, we get

$$G_2(z,x)=\frac{z-\overline{z}}{i}\log\Big|\frac{x-z}{x-i}\Big|^2$$

and

$$G_3(z,x) = \frac{z-\overline{z}}{2i}\left[x^2 \log\left|\frac{x-z}{x-i}\right|^2 - 2\Re\left\{z^2 \log\left(\frac{x-z}{x-i}\right)\right\} + \Re\{(2x+z+i)(z-i)\}\right].$$

Although the others can be given by induction, are there a unified expression for all higher order Schwarz kernels as the one for the higher order Poisson kernels?

Open Problem 2. As the higher order Poisson kernels are used to solve the Dirichlet problem for polyharmonic functions in the unit disc, whether can we use the higher order Schwarz kernels to solve the following Dirichlet problem for polyharmonic functions in the upper half plane?

Polyharmonic Dirichlet Problem in the Upper Half Plane Find a function $U \in Har_n(\mathbb{H})$ satisfying the Dirichlet type boundary conditions

$$[(\partial_z \partial_{\overline{z}})^j U]^+(t) = \Upsilon_j(t),\ t \in \partial\mathbb{H},\ 0 \le j < n,$$

where $\Upsilon_j \in \breve{H}(\partial\mathbb{H})$.