Chapter 4

Dirichlet Problems for Inhomogeneous PDEs

In the present chapter, we consider some Dirichlet problems for inhomogeneous higher order complex partial differential equations which have been correspondingly discussed in the last chapter. To solve them, we need the higher order Pompeiu operators which are higher order analogues of the classical Pompeiu operators. We begin with them.

4.1 Higher Order Pompeiu Operators

In [29], Vekua systematically studied the so-called Pompeiu operators. They are defined as

$$T_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{\zeta - z}\, \mathrm{d}\xi \mathrm{d}\eta, \tag{4.1}$$

$$\overline{T}_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{\overline{\zeta} - \overline{z}}\, \mathrm{d}\xi \mathrm{d}\eta, \tag{4.2}$$

and the so-called Π and $\overline{\Pi}$ operators defined as the Cauchy principle value integrals

$$\Pi_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\xi \mathrm{d}\eta, \tag{4.3}$$

$$\overline{\Pi}_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{(\overline{\zeta} - \overline{z})^2}\, \mathrm{d}\xi \mathrm{d}\eta, \tag{4.4}$$

where $\mathcal{D}$ is a domain in the complex plane, w is a suitable complex valued function defined in $\mathcal{D}$.

Now it is well-known that the operators T and Π play an important role to solve various linear or nonlinear boundary value problems for first and second order complex partial differential equations. So it happens that the operators T and Π have elegant properties such as continuity, differentiability, even unitarity in L^2 when w is in some certain function spaces. For example, one of the famous properties of T is its differentiability in the Sobolev sense as follows

$$\partial_{\overline{z}} T_{\mathcal{D}} w(z) = w(z), \ \partial_z T_{\mathcal{D}} w(z) = \Pi_{\mathcal{D}} w(z). \tag{4.5}$$

In [12], Begehr and Hile introduced kernel functions

$$K_{m,n}(z) = \begin{cases} \frac{(-m)!(-1)^m}{(n-1)!\pi} z^{m-1}\overline{z}^{n-1}, & m \leq 0; \\ \frac{(-n)!(-1)^n}{(m-1)!\pi} z^{m-1}\overline{z}^{n-1}, & n \leq 0; \\ \frac{1}{(m-1)!(n-1)!\pi} z^{m-1}\overline{z}^{n-1}\big[\log|z|^2 - \sum_{k=1}^{m-1}\frac{1}{k} - \sum_{l=1}^{n-1}\frac{1}{l}\big], & m,n \geq 1, \end{cases} \tag{4.6}$$

where m, n are integers with $m + n \geq 0$ but $(m, n) \neq (0, 0)$.

Using the above kernel functions, they defined a hierarchy of integral operators, more precisely,

$$T_{m,n,\mathcal{D}} w(z) = \int\int_{\mathcal{D}} K_{m,n}(z - \zeta) w(\zeta) \, \mathrm{d}\xi \mathrm{d}\eta. \tag{4.7}$$

Obviously,

$$T_{0,1,\mathcal{D}} = T_{\mathcal{D}}, \ T_{1,0,\mathcal{D}} = \overline{T}_{\mathcal{D}}, \tag{4.8}$$

and

$$T_{-1,1,\mathcal{D}} = \Pi_{\mathcal{D}}, \ T_{1,-1,\mathcal{D}} = \overline{\Pi}_{\mathcal{D}}. \tag{4.9}$$

Operators $T_{m,n,\mathcal{D}}$ are seen as the higher order analogues of the operator $T_{\mathcal{D}}$ by comparing their properties such as Lebegue integrability, continuity and differentiability and so on. They are called higher order Pompeiu operators. The following properties of $T_{m,n,\mathcal{D}}$ are needed in the sequel. They are partial results from [12].

Theorem 14 (Begehr and Hile [12])**.** *Let $\mathcal{D}$ be a bounded domain, suppose $m + n \geq 1$ and $w \in L^p(\mathcal{D})$, $p > 2$, then $T_{m,n,\mathcal{D}} w(z)$ exists as a Lebegue integral for all z in $\mathbb{C}$, $T_{m,n,\mathcal{D}}$ is continuous in $\mathbb{C}$. Especially, $T_{m,n,\mathcal{D}}$ is locally Hölder continuous in $\mathbb{C}$, more precisely, for $|z_1|, |z_2| \leq R$ with any $R > 0$,*

$$|T_{m,n,\mathcal{D}} w(z_1) - T_{m,n,\mathcal{D}} w(z_2)| \leq \begin{cases} M_1 |z_1 - z_2|, & m + n \geq 2, \\ M_2 |z_1 - z_2|^{(p-2)/p}, & m + n = 1, \end{cases} \tag{4.10}$$

where the constants M_1, M_2 only depend on $m, n, p, \mathcal{D}, R$. Moreover, in $\mathbb{C}$, there are the Sobolev derivatives

$$\partial_z T_{m,n,\mathcal{D}} w(z) = T_{m-1,n,\mathcal{D}} w(z), \ \ \partial_{\bar{z}} T_{m,n,\mathcal{D}} w(z) = T_{m,n-1,\mathcal{D}} w(z) \tag{4.11}$$

and

$$\partial_z T_{1,0,\mathcal{D}} w(z) = \partial_{\bar{z}} T_{0,1,\mathcal{D}} w(z) = w(z). \tag{4.12}$$

Proof. See Theorem 4.2, 4.3, 4.5, 5.3 and Corollary 5.5 in [12]. □

4.2 Dirichlet Problem for Inhomogeneous Polyharmonic Equations

In the present and next sections, we consider the corresponding Dirichlet problems discussed in the last chapter for inhomogeneous equations. According to the results of the last section for the homogeneous equations, the key is to find some special solutions for the inhomogeneous equations. By Theorem 14, this is no problem under suitably assumable conditions. In what follows, as the operators T and Π have been widely used to study various linear or nonlinear boundary value problems for the first or second order complex partial differential equations, we will find that higher order Pompeiu operators $T_{m,n}$ are useful in the study of some Dirichlet boundary value problems for higher order complex partial differential equations.

Let $f \in L^p(\mathbb{D}),\ p > 2$, by Theorem 14, we get

$$\partial_z^k \partial_{\bar{z}}^l T_{m,n,\mathbb{D}} f(z) = T_{m-k,n-l,\mathbb{D}} f(z), \quad 0 \leq k + l \leq m + n \tag{4.13}$$

in the Sobolev sense. Moreover,

$$T_{m-k,n-l,\mathbb{D}} f(z) \in H_{loc}(\mathbb{C}) \subset C(\mathbb{C}), \ \text{ as } \ 0 \leq k + l < m + n, \tag{4.14}$$

where $H_{loc}(\mathbb{C})$ denotes the set of all locally Hölder continuous functions in $\mathbb{C}$.

Noting (4.13), we know that $w(z) = T_{m,n,\mathbb{D}} f(z)$ is a weak solution of the inhomogeneous equations

$$(\partial_z^m \partial_{\bar{z}}^n) w(z) = f(z), \ \ z \in \mathbb{D}, \ f \in L^p(\mathbb{D}), \ p > 2. \tag{4.15}$$

Now we consider the so-called Dirichlet problem for the inhomogeneous polyharmonic equations [14]:

$$\begin{cases} (\partial_z\partial_{\overline{z}})^n w(z) = f(z), \quad z \in \mathbb{D}, f \in L^p(\mathbb{D}), \, p > 2, \\ (\partial_z\partial_{\overline{z}})^k w(\tau) = \gamma_k(\tau), \ \tau \in \partial\mathbb{D}, \gamma_k \in C(\partial\mathbb{D}), \, 0 \le k \le n-1. \end{cases} \tag{4.16}$$

By Theorem 10, (4.13) and (4.14), we have

Theorem 15. *The problem (4.16) is solvable and its unique solution is*

$$w(z) = T_{n,n,\mathbb{D}} f(z) + \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} [\gamma_{k-1}(\tau) - T_{n+1-k,n+1-k,\mathbb{D}} f(\tau)] g_k(z,\tau) \frac{d\tau}{\tau}, \tag{4.17}$$

where $z \in \mathbb{D}$, $T_{l,l,\mathbb{D}}$ $(1 \le l \le n)$ *are the higher order Pompeiu operators,* $g_k(z,\tau)$ $(1 \le k \le n)$ *are the former* n *higher order Poisson kernel functions.*

Proof. Note that by (4.13) and (4.14), the problem (4.16) is equivalent to the PHD problem of simplified form

$$\begin{cases} w - T_{n,n,\mathbb{D}} f \in Har_n^{\mathbb{C}}(\mathbb{D}), \quad f \in L^p(\mathbb{D}), \, p > 2, \\ (\partial_z\partial_{\overline{z}})^k [w - T_{n,n,\mathbb{D}} f] = \gamma_k - T_{n-k,n-k,\mathbb{D}}, \ \gamma_k \in C(\partial\mathbb{D}), \, 0 \le k \le n-1. \end{cases} \tag{4.18}$$

So it is obvious that Theorem 15 follows from Theorem 10. □

Noting (4.6) and (4.7), by Theorem 4, we can give the explicit expressions of the double integrals in (4.17). This is the following theorem.

Theorem 16. *Suppose that* $m, n \in \mathbb{Z}_+$, *for* $1 \le j \le n-5$, *let* $N_{m,n,j}(z,\zeta)$ *be a* *vertical sum of the following form*

$$
\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\cdots\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j}(k+1)\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 3!}\right]
\end{array}\right.\\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]
\end{array}\right.\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 3!}\right]\\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 4!}\right]
\end{array}\right.\\
\vdots\\
\frac{(-1)^{n-j-3}}{(n-j-2)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{3}(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-2)!}\right]
\end{array}\right.\\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\cdots\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 3!}\right]
\end{array}\right.\\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-3}(k+1)^4\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!\cdot 2!}\right]
\end{array}\right.\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 3!}\right]\\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 4!}\right]
\end{array}\right.\\
\vdots\\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{4}(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot(n-j-3)!}\right]
\end{array}\right.\\
\vdots
\end{array}\right.
$$

$$
\sum\begin{cases}
\vdots \\
\vdots \\
\frac{(-1)^{n-j-6}}{(n-j-5)!}\sum\begin{cases}
\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^6(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!}\right] \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 3!}\right]
\end{cases} \\
-\frac{1}{2!}\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^4\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!\cdot 2!}\right]
\end{cases} \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 3!}\right] \\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^3(k+2)^3(k+3)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 4!}\right]
\end{cases} \\
\frac{(-1)^{n-j-5}}{(n-j-4)!}\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!\cdot 2!}\right] \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!\cdot 3!}\right]
\end{cases} \\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-3)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-3)!\cdot 2!}\right]
\end{cases} \\
\frac{(-1)^{n-j-3}}{(n-j-2)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-2)!}\right] \\
\frac{(-1)^{n-j-2}}{(n-j-1)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-2)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-1)!}\right]
\end{cases}
\tag{4.19}
$$

and let

$$N_{m,n,n-4}(z,\zeta)=\sum\begin{cases}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)(k+2)\cdots(k+n-5)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-4)!}\right]\\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2(k+2)\cdots(k+n-5)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-4)!\cdot 2!}\right]\\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2(k+2)\cdots(k+n-5)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-4)!\cdot 2!}\right]\\ \frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)^2(k+2)^2\cdots(k+n-5)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-4)!\cdot 3!}\right]\end{cases}, \tag{4.20}$$

$$N_{m,n,n-3}(z,\zeta)=\sum\begin{cases}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)(k+2)\cdots(k+n-4)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-3)!}\right]\\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)^2(k+2)\cdots(k+n-4)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-3)!\cdot 2!}\right]\end{cases}, \tag{4.21}$$

$$N_{m,n,n-2}(z,\zeta)=\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)(k+2)\cdots(k+n-3)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-2)!}, \tag{4.22}$$

$$N_{m,n,n-1}(z,\zeta)=\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k(k+1)(k+2)\cdots(k+n-2)}+\frac{\Delta_{m,0}(z,\zeta)}{(n-1)!}, \tag{4.23}$$

where

$$\Delta_{m,\ell}(z,\zeta)=-\frac{1}{[(m-1)!]^2\pi}\Big\{\sum_{\substack{1\le s<\infty\\ 0\le p,q\le m-1\\ p=q+s+\ell}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\cdot[\overline{\zeta}^p\zeta^{q+s}z^\ell+\zeta^p\overline{\zeta}^{q+s}\overline{z}^\ell]+\sum_{\substack{1\le l\le m-1\\ 0\le p,q\le m-1\\ p=q+\ell}}\binom{m-1}{p}\binom{m-1}{q}l^{-1}\cdot 2[\overline{\zeta}^p\zeta^q z^\ell+\zeta^p\overline{\zeta}^q\overline{z}^\ell]\Big\}, \tag{4.24}$$

$\ell=0,1,2,\ldots.$ *Moreover,* $G_{m,n}(z,\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}K_{m,m}(\tau-\zeta)g_n(z,\tau)\,\frac{d\tau}{\tau}$, $g_n(z,\tau)$ *is the nth higher order Poisson kernel, then*

$$G_{m,n}(z,\zeta)=D_{m,1}(z,\zeta)+D_{m,2}(z,\zeta)+\cdots+D_{m,n-1}(z,\zeta), \tag{4.25}$$

where $D_{m,j}(z,\zeta)=(-1)^{n-j}\frac{1-|z|^{2j}}{j!}N_{m,n,j}(z,\zeta)$, $j=1,2,\ldots,n-1$. *In all above formulae, by convention,* $\prod_{\ell=\imath}^{\jmath}(k+\ell)=1$ *as* $\imath>\jmath$.

To prove Theorem 16, we need some lemmas as follows.

Lemma 17.

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\tau^k\,\frac{\mathrm{d}\tau}{\tau}=\delta_{k0},\ \ k\in\mathbb{Z},\tag{4.26}$$

where δ_{k0} is the Kronecker sign and $\mathbb{Z}$ is the set of all integers.

Proof. It is obvious since $\tau=e^{i\theta},\ \theta\in[0,2\pi)$. □

Lemma 18.

$$|\tau-\zeta|^{2n}=\sum_{p,q=0}^{n}\binom{n}{p}\binom{n}{q}\zeta^p\overline{\zeta}^q\overline{\tau}^p\tau^q,\ \ \tau\in\partial\mathbb{D},\zeta\in\mathbb{D},n\in\mathbb{Z}_+,\tag{4.27}$$

where $\mathbb{Z}_+$ is the set of all positive integers.

Proof. (4.27) follows from the fact $|\tau-\zeta|^2=|1-\overline{\tau}\zeta|^2=(1-\overline{\tau}\zeta)(1-\tau\overline{\zeta}),\ \tau\in\partial\mathbb{D}$. □

Lemma 19.

$$\log|\tau-\zeta|^2=-\sum_{s=1}^{\infty}s^{-1}[(\overline{\tau}\zeta)^s+(\tau\overline{\zeta})^s],\ \tau\in\partial\mathbb{D},\zeta\in\mathbb{D}.\tag{4.28}$$

Proof. Since $\tau\in\partial\mathbb{D}$, $\zeta\in\mathbb{D}$, therefore

$$\begin{aligned}\log|\tau-\zeta|^2&=\log|1-\overline{\tau}\zeta|^2\\&=\log(1-\overline{\tau}\zeta)(1-\tau\overline{\zeta})\\&=\log(1-\overline{\tau}\zeta)+\log(1-\tau\overline{\zeta})\\&=-\sum_{s=1}^{\infty}s^{-1}[(\overline{\tau}\zeta)^s+(\tau\overline{\zeta})^s].\end{aligned}$$

The last equality follows from the fact $\log(1-x)=-\sum_{s=1}^{\infty}\frac{x^s}{s},\ |x|<1$. □

Proof of Theorem 16. By (4.6),

$$K_{m,m}(\tau-\zeta)=\frac{1}{[(m-1)!]^2\pi}|\tau-\zeta|^{2(m-1)}\Big[\log|\tau-\zeta|^2-2\sum_{l=1}^{m-1}\frac{1}{l}\Big].\tag{4.29}$$

Noting (2.34)-(2.38), in order to get $G_{m,n}(z,\zeta)$, the key is to obtain

$$\Delta_{m,k-1}(z,\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}K_{m,m}(\tau-\zeta)d_{k-1}(z,\tau)\frac{\mathrm{d}\tau}{\tau},\ k\geq 2$$

in which $d_{k-1}(z,\tau)=(z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}$ and

$$\Delta_{m,0}(z,\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}K_{m,m}(\tau-\zeta)\frac{\mathrm{d}\tau}{\tau}.$$

By Lemma 18-19,

$$|\tau-\zeta|^{2(m-1)}\log|\tau-\zeta|^2=-\sum_{\substack{1\leq s<\infty\\0\leq p,q\leq m-1}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\cdot[\zeta^{p+s}\overline{\zeta}^q\overline{\tau}^{p+s}\tau^q+\zeta^p\overline{\zeta}^{q+s}\overline{\tau}^p\tau^{q+s}],\tag{4.30}$$

$$|\tau-\zeta|^{2(m-1)}d_{k-1}(z,\tau)=\sum_{0\leq p,q\leq m-1}\binom{m-1}{p}\binom{m-1}{q}\cdot[\zeta^p\overline{\zeta}^q z^{k-1}\overline{\tau}^{p+k-1}\tau^q+\zeta^p\overline{\zeta}^q\overline{z}^{k-1}\overline{\tau}^p\tau^{q+k-1}]\tag{4.31}$$

and

$$|\tau-\zeta|^{2(m-1)}\log|\tau-\zeta|^2 d_{k-1}(z,\tau)=-\sum_{\substack{1\leq s<\infty\\0\leq p,q\leq m-1}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\cdot[\zeta^{p+s}\overline{\zeta}^q z^{k-1}\overline{\tau}^{p+s+k-1}\tau^q+\zeta^p\overline{\zeta}^{q+s}\overline{z}^{k-1}\overline{\tau}^p\tau^{q+s+k-1}].\tag{4.32}$$

Applying (4.29), (4.31)-(4.32), by Lemma 17, we have

$$\Delta_{m,k-1}(z,\zeta)=-\frac{1}{[(m-1)!]^2\pi}\Big\{\sum_{\substack{1\leq s<\infty\\0\leq p,q\leq m-1\\p=q+s+k-1}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\cdot[\overline{\zeta}^p\zeta^{q+s}z^{k-1}+\zeta^p\overline{\zeta}^{q+s}\overline{z}^{k-1}]+\sum_{\substack{1\leq l\leq m-1\\0\leq p,q\leq m-1\\p=q+k-1}}\binom{m-1}{p}\binom{m-1}{q}l^{-1}\cdot 2[\overline{\zeta}^p\zeta^q z^{k-1}+\zeta^p\overline{\zeta}^q\overline{z}^{k-1}]\Big\}.$$

Applying (4.27), (4.29) and (4.30), by Lemma 17, we get

$$\Delta_{m,0}(z,\zeta) = -\frac{1}{[(m-1)!]^2\pi}\Big\{\sum_{\substack{1\le s<\infty\\ 0\le p,q\le m-1\\ p=q+s}} \binom{m-1}{p}\binom{m-1}{q} s^{-1} \cdot [\overline{\zeta}^p\zeta^{q+s} + \zeta^p\overline{\zeta}^{q+s}] + \sum_{\substack{1\le l\le m-1\\ 0\le p\le m-1}} 4\Big[\binom{m-1}{p}\Big]^2 l^{-1}|\zeta|^{2p}\Big\}.$$

Thus we complete the proof of Theorem 16. □

Remark 11. By Theorem 16, applying $G_{m,n}(z,\zeta)$, we can rewrite the unique solution of the problem (4.16) as

$$w(z) = \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_{k-1}(\tau) g_k(z,\tau)\frac{\mathrm{d}\tau}{\tau} + \int_{\mathbb{D}} f(\zeta)\Big\{K_{n,n}(z-\zeta) + \sum_{k=1}^{n} G_{n+1-k,k}(z,\zeta)\Big\}\mathrm{d}\xi\mathrm{d}\eta. \tag{4.33}$$

Similarly, the double integrals appearing in what follows can easily be given in terms of $G_{m,n}(z,\zeta)$. To avoid technical difficulty, we will not repeat them again in the sequel.

4.3 Dirichlet Problems for Inhomogeneous Poly-analytic-harmonic Equations

In this section, we discuss three kinds of Dirichlet problems for the higher order inhomogeneous complex mixed-partial differential equations of simplified form:

$$\begin{cases} (\partial_z^m \partial_{\overline{z}}^n) w = f, \ \ f \in L^p(\mathbb{D}), p > 2, m > n, \\ (\partial_z \partial_{\overline{z}})^j w = \gamma_j, \ \ \gamma_j \in C(\partial\mathbb{D}), 0 \le j < n, \\ (\partial_z^{n+k}\partial_{\overline{z}}^n) w = \sigma_k, \ \ \sigma_k \in C(\partial\mathbb{D}), 0 \le k < m-n \end{cases} \tag{4.34}$$

and

$$\begin{cases}(\partial_z^m\partial_{\overline{z}}^n)w=f,\ \ f\in L^p(\mathbb{D}),p>2,m<n,\\(\partial_z\partial_{\overline{z}})^jw=\rho_j,\ \ \rho_j\in C(\partial\mathbb{D}),0\leq j<m,\\(\partial_z^m\partial_{\overline{z}}^{m+k})w=\varrho_k,\ \ \varrho_k\in C(\partial\mathbb{D}),0\leq k<n-m\end{cases}\tag{4.35}$$

as well as

$$\begin{cases}(\partial_z^m\partial_{\overline{z}}^n)w=f,\ \ f\in L^p(\mathbb{D}),p>2,\\(\partial_z^m\partial_{\overline{z}}^j)w=\chi_j,\ \ \chi_j\in C(\partial\mathbb{D}),0\leq j<n,\\(\partial_z^k\partial_{\overline{z}}^n)w=\lambda_k,\ \ \lambda_k\in C(\partial\mathbb{D}),0\leq k<m.\end{cases}\tag{4.36}$$

By Theorem 11-13, (4.13) and (4.14), we have

Theorem 20. *Set*

$$A(t)=\begin{pmatrix}n! & (n+1)!t & \cdots & \frac{(m-2)!}{(m-n-2)!}t^{m-n-2} & \frac{(m-1)!}{(m-n-1)!}t^{m-n-1}\\0 & (n+1)! & \cdots & \frac{(m-2)!}{(m-n-3)!}t^{m-n-3} & \frac{(m-1)!}{(m-n-2)!}t^{m-n-2}\\\vdots & \vdots & \ddots & \vdots & \vdots\\0 & 0 & \cdots & (m-2)! & (m-1)!t\\0 & 0 & \cdots & 0 & (m-1)!\end{pmatrix},$$

$$a^*(t)=\begin{pmatrix}\sigma_0(t)-T_{m-n,0,\mathbb{D}}(t)\\\sigma_1(t)-T_{m-n-1,0,\mathbb{D}}(t)\\\vdots\\\sigma_{m-n-2}(t)-T_{2,0,\mathbb{D}}(t)\\\sigma_{m-n-1}(t)-T_{1,0,\mathbb{D}}(t)\end{pmatrix},\tag{4.37}$$

$$\Xi_l^*(z)=\frac{1}{n!(n+1)!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\overline{\det(A_l^*(\tau))}}{\tau-z}\mathrm{d}\tau,\tag{4.38}$$

and

$$\widetilde{\varphi}_l^*(z)=\int_0^z\int_0^{\zeta_{n-1}}\cdots\int_0^{\zeta_1}\Xi_l^*(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1}+\pi_l^*(z),\tag{4.39}$$

where $t\in\partial\mathbb{D}$, $\pi_l^*\in\Pi_{n-1}$, the matrix $A_l^*(t)$ is given by replacing the lth column *of $A(t)$ by $a^*(t)$, $0 \le l \le m-n-1$. Then*

$$w(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} g_k(z,\tau) \Big[\gamma^*_{k-1}(\tau) - \sum_{l=0}^{m-n-1} \frac{(n+l)!}{(n+l-k+1)!} \tau^{n+l-k+1} \overline{\partial_z^{k-1} \widetilde{\varphi}^*_l(\tau)} \Big] \frac{\mathrm{d}\tau}{\tau} + z^n \sum_{l=0}^{m-n-1} z^l \overline{\widetilde{\varphi}^*_l(z)} + T_{m,n,\mathbb{D}} f(z) \tag{4.40}$$

are all solutions of the problem (4.34) if and only if

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{z \det A^*_l(\tau)}{\tau - z} \frac{\mathrm{d}\tau}{\tau} = 0, \ z \in \mathbb{D}, \ 0 \le l \le m-n-1, \tag{4.41}$$

where $\gamma^*_{k-1}(\tau) = \gamma_{k-1}(\tau) - T_{m+1-k,n+1-k,\mathbb{D}} f(\tau)$, $g_k(z,\tau)\,(1 \le k \le n)$ are the former n higher order Poisson kernels.

Theorem 21. *Set*

$$A'(t) = \begin{pmatrix} m! & (m+1)!\bar{t} & \cdots & \frac{(n-2)!}{(n-m-2)!}\bar{t}^{n-m-2} & \frac{(n-1)!}{(n-m-1)!}\bar{t}^{n-m-1} \\ 0 & (m+1)! & \cdots & \frac{(n-2)!}{(n-m-3)!}\bar{t}^{n-m-3} & \frac{(n-1)!}{(n-m-2)!}\bar{t}^{n-m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-2)! & (n-1)!\bar{t} \\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix},$$

$$a^{\#}(t) = \begin{pmatrix} \varrho_0(t) - T_{n-m,0,\mathbb{D}}(t) \\ \varrho_1(t) - T_{n-m-1,0,\mathbb{D}}(t) \\ \vdots \\ \varrho_{n-m-2}(t) - T_{2,0,\mathbb{D}}(t) \\ \varrho_{n-m-1}(t) - T_{1,0,\mathbb{D}}(t) \end{pmatrix}, \tag{4.42}$$

$$\Xi^{\#}_l(z) = \frac{1}{m!(m+1)!\cdots(n-1)!} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\det(A^{\#}_l(\tau))}{\tau - z} \mathrm{d}\tau, \tag{4.43}$$

and

$$\widetilde{\psi}^{\#}_l(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Xi^{\#}_l(\zeta) \mathrm{d}\zeta \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_{m-1} + \pi^{\#}_l(z), \tag{4.44}$$

where $t \in \partial\mathbb{D}$, $\pi_l^{\#} \in \Pi_{m-1}$, *the matrix* $A_l^{\#}(t)$ *is given by replacing the* l*th column of* $A_l'(t)$ *by* $a^{\#}(t)$, $0 \le l \le n-m-1$. *Then*

$$\begin{aligned} w(z) =& \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\varrho_{k-1}^{\#}(\tau) \\ &- \sum_{l=0}^{n-m-1} \frac{(m+l)!}{(m+l-k+1)!} \overline{\tau}^{m+l-k+1} \partial_z^{k-1} \widetilde{\psi}_l^{\#}(\tau)\Big] \frac{\mathrm{d}\tau}{\tau} \\ &+ \overline{z}^m \sum_{l=0}^{n-m-1} \overline{z}^l \widetilde{\psi}_l^{\#}(z) + T_{m,n,\mathbb{D}} f(z) \end{aligned} \tag{4.45}$$

are all solutions of the problem (4.35) if and only if

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{z\, \overline{\det A_l^{\#}(\tau)}}{\tau - z} \frac{\mathrm{d}\tau}{\tau} = 0, \ z \in \mathbb{D}, \ 0 \le l \le n-m-1, \tag{4.46}$$

where $\rho_{k-1}^{\#}(\tau) = \rho_{k-1}(\tau) - T_{m+1-k,n+1-k,\mathbb{D}} f(\tau)$, $g_k(z,\tau)\,(1 \le k \le m)$ *are the former* n *higher order Poisson kernels.*

Theorem 22. *Set*

$$B(t) = \begin{pmatrix} 1 & \overline{t} & \overline{t}^2 & \cdots & \overline{t}^{n-1} \\ 0 & 1 & 2\overline{t} & \cdots & (n-1)\overline{t}^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-2)! & (n-1)!\overline{t} \\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix},$$

$$C(t) = \begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-1} \\ 0 & 1 & 2t & \cdots & (m-1)t^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix},$$

$$b^*(t) = \begin{pmatrix} \chi_0(t) - T_{0,n,\mathbb{D}} f(t) \\ \chi_1(t) - T_{0,n-1,\mathbb{D}} f(t) \\ \vdots \\ \chi_{n-1}(t) - T_{0,1,\mathbb{D}} f(t) \end{pmatrix}, \quad c^*(t) = \begin{pmatrix} \lambda_0(t) - T_{m,0,\mathbb{D}} f(t) \\ \lambda_1(t) - T_{m-1,0,\mathbb{D}} f(t) \\ \vdots \\ \lambda_{m-1}(t) - T_{1,0,\mathbb{D}} f(t) \end{pmatrix} \tag{4.47}$$

and

$$\Theta_p^*(z) = \frac{1}{1!2!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det B_p^*(\tau)}{\tau - z}\,\mathrm{d}\tau, \tag{4.48}$$

$$\Lambda_q^*(z) = \frac{1}{1!2!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det C_q^*(\tau)}}{\tau - z}\,\mathrm{d}\tau, \tag{4.49}$$

as well as

$$\mu_p^*(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Theta_p^*(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \kappa_p^*(z), \tag{4.50}$$

$$\nu_q^*(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Lambda_q^*(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1} + \xi_q^*(z), \tag{4.51}$$

where $t \in \partial\mathbb{D}$, $\kappa_p^* \in \Pi_{m-1}$, $\xi_q^* \in \Pi_{n-1}$, *the matrices* $B_p^*(t), C_q^*(t)$ *are respectively given by replacing the pth, qth column of* $B(t), C(t)$ *by* $b^*(t)$, $c^*(t)$, $0 \le p \le n-1$, $0 \le q \le m-1$. *Then*

$$w(z) = T_{m,n,\mathbb{D}}f(z) + \sum_{p=0}^{n-1} \overline{z}^p \mu_p^*(z) + \sum_{q=0}^{m-1} z^q \overline{\nu_q^*(z)} \tag{4.52}$$

are all solutions of the problem (4.36) if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\overline{\det B_p^*(\tau)}}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0,\ \ 0 \le p \le n-1 \tag{4.53}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\det C_q^*(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0,\ \ 0 \le q \le m-1 \tag{4.54}$$

for $z \in \mathbb{D}$.

Proofs of Theorems 20-22. From (4.13) and (4.14), the problems (4.34-(4.36) are respectively equivalent to the following ones:

$$\begin{cases} w - T_{m,n,\mathbb{D}}f \in M_{m,n}(\mathbb{D}),\ \ f \in L^p(\mathbb{D}), p > 2, m > n, \\ (\partial_z\partial_{\overline{z}})^j[w - T_{m,n,\mathbb{D}}f] = \gamma_j - T_{m-j,n-j,\mathbb{D}}f,\ \gamma_j \in C(\partial\mathbb{D}), 0 \le j < n, \\ (\partial_z^{n+k}\partial_{\overline{z}}^n)[w - T_{m,n,\mathbb{D}}f] = \sigma_k - T_{m-n-k,0,\mathbb{D}}f,\ \sigma_k \in C(\partial\mathbb{D}), 0 \le k < m-n \end{cases} \tag{4.55}$$

and

$$\begin{cases} w - T_{m,n,\mathbb{D}} f \in M_{m,n}(\mathbb{D}), \quad f \in L^p(\mathbb{D}), p > 2, m < n, \\ (\partial_z \partial_{\overline{z}})^j [w - T_{m,n,\mathbb{D}} f] = \rho_j - T_{m-j,n-j,\mathbb{D}} f, \ \rho_j \in C(\partial \mathbb{D}), 0 \leq j < m, \\ (\partial_z^m \partial_{\overline{z}}^{m+k})[w - T_{m,n,\mathbb{D}} f] = \varrho_k - T_{n-m-k,0,\mathbb{D}} f, \ \varrho_k \in C(\partial \mathbb{D}), 0 \leq k < n - m \end{cases} \tag{4.56}$$

as well as

$$\begin{cases} w - T_{m,n,\mathbb{D}} f \in M_{m,n}(\mathbb{D}), \quad f \in L^p(\mathbb{D}), p > 2, \\ (\partial_z^m \partial_{\overline{z}}^j)[w - T_{m,n,\mathbb{D}} f] = \chi_j - T_{0,n-j,\mathbb{D}} f, \ \chi_j \in C(\partial \mathbb{D}), 0 \leq j < n, \\ (\partial_z^k \partial_{\overline{z}}^n)[w - T_{m,n,\mathbb{D}} f] = \lambda_k - T_{m-k,0,\mathbb{D}} f, \ \lambda_k \in C(\partial \mathbb{D}), 0 \leq k < m. \end{cases} \tag{4.57}$$

So, by Theorems 11-13, we complete the proofs of Theorems 20-22. □