Chapter 2

Decompositions of Functions

In this chapter, we introduce the theory of polyanalytic, polyharmonic as well as poly-analytic-harmonic functions and are mainly concerned with the decompositions for them. The excellent book of Balk [3] contains the detailed theory of polyanalytic and of polyharmonic functions of one complex variable. Another book [1] due to Aronszajn, Cresse and Lipkin is an eminent one for polyharmonic functions of several complex variables.

2.1 Polyanalytic Functions

It is well known that analytic functions are the main object of classical complex analysis. They have three kinds of definitions respectively due to Cauchy, Riemann and Weierstrasse which are derivative definition, differential equation definition and series definition. All of them are equivalent. To extend analytic functions of one variable, many generalized analogues are yielded such as generalized analytic functions, polyanalytic functions and metaanalytic functions etc. [3, 29]. In another direction, the generalization for several variables yielded the analytic functions of several complex variables which are not related to the research object of this dissertation although several complex variables is an important branch of modern complex analysis.

2.1.1 Definition

One definition of analytic functions is in terms of Cauchy-Riemann operator $\partial_{\overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. That is, if a continuously differentiable function f satisfies $\partial_{\overline{z}} f = 0$ in some domain of the complex plane, then f is analytic in the domain. This is the differential equation definition due to Riemann. A natural extension of this definition is to iterate the Cauchy-Riemann operator. In this way, the generalized analogues are the so-called polyanalytic functions.

For simplicity, in what follows, we always suppose that Ω is a simply connected (bounded or unbounded) domain in the complex plane with smooth boundary $\partial\Omega$.

Definition 1. If $f \in C^n(\Omega)$ satisfies the equation $\partial_{\overline{z}}^n f = 0$ in Ω, then f is called an n-analytic function in Ω, concisely, a polyanalytic function.

The set of polyanalytic functions of order n in Ω is simply denoted by $H_n(\Omega)$. Especially, $H_1(\Omega)$ is the set of all analytic functions in Ω. However, $H(\Omega)$ denotes the set of all Hölder continuous functions in Ω.

2.1.2 Decomposition

Polyanalytic functions are closely related to usual analytic functions. Any polyanalytic function can be decomposed to a direct sum of some analytic functions with certain coefficients.

Theorem A. *Let Ω be a simply connected (bounded or unbounded) domain in the complex plane with smooth boundary $\partial\Omega$. If $f \in H_n(\Omega)$, then for any $z_0 \in \Omega$, there uniquely exist functions $f_j \in H_1(\Omega)$, $j = 0, 1, \ldots, n-1$ such that*

$$f(z) = \sum_{j=0}^{n-1} (\overline{z} - \overline{z}_0)^j f_j(z), \quad z \in \Omega. \tag{2.1}$$

Proof. First, we verify the existence by induction. For $n = 1$, let $f_0(z) = f(z)$, $z \in \Omega$, then (2.1) is obvious. Suppose that (2.1) holds as $n = m$, if $f \in H_{m+1}(\Omega)$, then $\partial_{\overline{z}} f \in H_m(\Omega)$. So there exist $g_j \in H_1(\Omega)$, $j = 0, 1, \ldots, m-1$ such that

$$\partial_{\overline{z}} f(z) = \sum_{j=0}^{m-1} (\overline{z} - \overline{z}_0)^j g_j(z), \quad z \in \Omega.$$

Therefore,

$$\partial_{\overline{z}} \big[f(z) - \sum_{j=0}^{m-1} (\overline{z} - \overline{z}_0)^{j+1} g_j(z) \big] = 0,$$

that is to say $f - \sum_{j=0}^{m-1}(\overline{z} - \overline{z}_0)^{j+1} g_j \in H_1(\Omega)$. Let $f_0 = f - \sum_{j=0}^{m-1}(\overline{z} - \overline{z}_0)^{j+1} g_j$ and $f_{j+1} = g_j$, then (2.1) also holds for $n = m+1$. By induction, (2.1) also holds for any $n \in \mathbb{N}$.

Next, we go to the uniqueness. Let

$$\sum_{j=0}^{n-1}(\overline{z} - \overline{z}_0)^j f_j(z) = 0,$$

using operators $\partial_{\overline{z}}^j$ $(j = 1, 2, \ldots, n-1)$ to act on both sides of the last equality, it is easy to get that $f_j(z) = 0$, $z \in \Omega$, $j = 0, 1, \ldots, n-1$. □

Remark 1. In [17], Du and Wang established the above theorem with a different form whereas it is implicitly included in the book [3] of Balk. Let $H^n(\Omega) = \{(f_0(z), f_1(z), \ldots, f_{n-1}(z)) : f_j \in H_1(\Omega), z \in \Omega, j = 0, 1, \ldots, n-1\}$, from the above theorem, we know that $H_n(\Omega)$ and $H^n(\Omega)$ are isomorphic as complex vector spaces. So we call f_j in (2.1) the analytic jth decomposition component of the polyanalytic function f. We also call $H^n(\Omega)$ n-analytic space or the decomposition space for polyanalytic functions.

Let $\overline{H}_n(\Omega)$ denotes the set of all functions satisfying $\partial_z^n f(z) = 0$, $z \in \Omega$. Since $\overline{\partial_z f} = \partial_{\overline{z}} \overline{f}$, similarly or directly following from the above theorem, we also get

Corollary 1. *Let $f \in \overline{H}_n(\Omega)$, then for any $z_0 \in \Omega$,*

$$f(z) = \sum_{j=0}^{n-1}(z - z_0)^j \overline{f_j(z)}, \ z \in \Omega, \tag{2.2}$$

where $f_j \in H_1(\Omega)$, $j = 0, 1, \ldots, n-1$. The decomposition (2.2) *is unique.*

2.2 Polyharmonic Functions

It is well known that harmonic functions are intimately related to analytic functions. Any real harmonic function can be decomposed as a sum of an analytic function and its conjugate function which is antianalytic. That is, any real harmonic function can be the real part of some analytic function. In this section, this idea of decomposition is also valid for the generalized analogues of harmonic functions which are called polyharmonic functions. Any polyharmonic function can be decomposed into a sum of some polyanalytic function and its conjugate.

2.2.1 Definition

The definition given here is different from the usual manner (see [1, 3]). In what follows, we always use polyharmonic operators $(\partial_z \partial_{\overline{z}})^n$ $(n \geq 1)$ to define polyharmonic functions, in particular, $\partial_z \partial_{\overline{z}}$ is the harmonic operator.

Definition 2. If a real valued function $f \in C^{2n}(\Omega)$ satisfies the equation $(\partial_z \partial_{\overline{z}})^n f = 0$ in Ω, then f is called an n-harmonic function in Ω, concisely, a polyharmonic function.

The set of polyharmonic functions of order n in Ω is simply denoted by $Har_n(\Omega)$. Especially, $Har_1(\Omega)$ is the set of all harmonic functions in Ω. Sometimes we need consider $Har_n^{\mathbb{C}}(\Omega) = \{f + ig : f, g \in Har_n(\Omega)\}$ consisting of all complex polyharmonic functions of order n in Ω.

In addition, we introduce the function spaces $H^j_{1,z_0}(\Omega) = \{\, \varphi \in H_1(\Omega) : \varphi^{(k)}(z_0) = 0,\ z_0 \in \Omega,\ 0 \leq k < j \,\}$ and $\Pi^j_{1,z_0}(\Omega) = \{\, ic(z - z_0)^j : c \in \mathbb{R},\ z, z_0 \in \Omega \,\}$, where $\mathbb{R}$ denotes the set of all real numbers and $j = 0, 1, 2, \ldots$. Obviously, for $j > 1$, $H^j_{1,z_0}(\Omega)$ is the set of all analytic functions which have at least jth order zero at $z_0 \in \Omega$ whereas $H^0_{1,z_0}(\Omega) = H_1(\Omega)$. Of course, $\Pi^j_{1,z_0}(\Omega) \subset H^j_{1,z_0}(\Omega) \subset H_1(\Omega)$. If $\varphi, \widetilde{\varphi} \in H^j_{1,z_0}(\Omega)$ and $\varphi - \widetilde{\varphi} \in \Pi^j_{1,z_0}(\Omega)$, then we say that φ and $\widetilde{\varphi}$ are equivalent and write that $\varphi \sim_j \widetilde{\varphi}$. Moreover, define $\sim = \cup_j \sim_j$, that is, $f \sim g$ if $f \sim_j g$ for some $j \in \mathbb{N}$. Especially, for example, $0 \sim_j ic(z - z_0)^j$ for any nonzero $c \in \mathbb{R}$.

2.2.2 Decomposition

With the above preliminaries, the following decomposition fact for polyharmonic functions holds.

Theorem B. *Let Ω be a simply connected (bounded or unbounded) domain in the complex plane with smooth boundary $\partial\Omega$. If $f \in Har_n(\Omega)$, then for any $z_0 \in \Omega$, there exist functions $f_j \in H^j_{1,z_0}(\Omega)$, $j = 0, 1, \ldots, n-1$ such that*

$$f(z) = 2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z}-\overline{z}_0)^j f_j(z)\Big\}, \quad z\in\Omega, \tag{2.3}$$

where $\Re$ *denotes the real part. The above decomposition expression of* f *is unique in the sense of the equivalence relation* $\sim$*, more precisely,* $\sim_j$ *for* f_j*. That is, if* (2.3) *also holds for* $\hat{f}_j \in H^j_{1,z_0}(\Omega)$*,* $j = 0, 1, \ldots, n-1$*, then* $\hat{f}_j \sim_j f_j$*,* $j = 0, 1, \ldots, n-1$.

Proof. Write $f = f_n$, where n denotes the order of the polyharmonic function $f \in Har_n(\Omega)$. We will prove (2.3) by induction.

As a basic fact of real harmonic function, (2.3) is obvious for $n = 1$.

Suppose that (2.3) holds for $n-1$ $(n > 2)$, i.e., for any $f_{n-1} \in Har_{n-1}(\Omega)$ and $z_0 \in \Omega$,

$$f_{n-1}(z) = 2\Re\Big\{\sum_{j=0}^{n-2}(\overline{z}-\overline{z}_0)^j f_{n-1,j}(z)\Big\}, \quad z\in\Omega,$$

where $f_{n-1,j} \in H^j_{1,z_0}(\Omega)$, $j = 0, 1, \ldots, n-2$. Therefore, for any $f_n \in Har_n(\Omega)$ and $z_0 \in \Omega$, since $(\partial_z\partial_{\overline{z}})f_n \in Har_{n-1}(\Omega)$,

$$(\partial_z\partial_{\overline{z}})f_n(z) = 2\Re\Big\{\sum_{j=0}^{n-2}(\overline{z}-\overline{z}_0)^j g_{n-1,j}(z)\Big\}, \quad z\in\Omega$$

holds for some $g_{n-1,j} \in H^j_{1,z_0}(\Omega)$, $j = 0, 1, \ldots, n-2$.

For $1 \le j \le n-1$, define

$$f_{n,j}(z) = j^{-1}\int_{z_0}^{z} g_{n-1,j-1}(z)\mathrm{d}z, \quad z\in\Omega.$$

One shall find that the above definition is reasonable since $g_{n-1,j-1} \in H_1(\Omega)$ and Ω is simply connected. Thus $\partial_z f_{n,j} = j^{-1}g_{n-1,j-1}$. Further, set

$$\widetilde{f}_n(z) = 2\Re\Big\{\sum_{j=1}^{n-1}(\overline{z}-\overline{z}_0)^j f_{n,j}(z)\Big\}, \quad z\in\Omega.$$

By straight calculation, we have $(\partial_z\partial_{\overline{z}})(f_n - \widetilde{f}_n) = 0$, that is, $f_n - \widetilde{f}_n$ is a usual harmonic function. So there exists an analytic function $f_{n,0}$ such that $f_n - \widetilde{f}_n = 2\Re\{f_{n,0}\}$. Hence, (2.3) also holds for n and then for any $n \in \mathbb{N}$ by induction.

For the uniqueness, set

$$2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z}-\overline{z}_0)^j f_j(z)\Big\}=0,$$

taking operators $(\partial_z\partial_{\overline{z}})^j$ $(j=1,2,\ldots,n-1)$ acting on its both sides, we get

$$\Re\big[\partial_z^j f_j\big]=0.$$

So $\partial_z^j f_j \equiv ic_j$, c_j is some real constant. Therefore, $f_j \sim_j 0$, $j=0,1,\ldots,n-1$. $\square$

Remark 2. If we set $F(z)=\sum_{j=0}^{n-1}(\overline{z}-\overline{z}_0)^j f_j(z)$, then $F\in H_n(\Omega)$ and (2.3) can be rewritten as follows

$$f(z)=F(z)+\overline{F(z)},\ \ z\in\Omega. \tag{2.4}$$

So (2.4) is certainly a natural extension of decomposition from harmonic functions to polyharmonic functions at least in this form, and is also the exact version of the weak decomposition theorem appeared in [9]. However, (2.3) is the exact version of the decomposition theorem there.

Remark 3. For any $z_0\in\Omega$, let $Har^n(\Omega,z_0)$ denote the n dimensional real vector space $\{\,(f_0(z),f_1(z),\ldots,f_{n-1}(z)) : f_j\in H^j_{1,z_0}(\Omega),\ z\in\Omega,\ 0\le j\le n-1\,\}$ and $Har^n(\Omega)=\{\,Har^n(\Omega,z_0) : z_0\in\Omega\,\}$. Thus Theorem B shows that $Har_n(\Omega)$ and $Har^n(\Omega,z_0)$ are isomorphic as real vector spaces. In this sense, we call $f_j\ (\in H^j_{1,z_0}(\Omega)\,)$ the analytic jth decomposition component of the polyharmonic function f at z_0. And we also call $Har^n(\Omega,z_0)$ the decomposition space for polyharmonic functions at z_0 or (n,z_0)-harmonic space and $Har^n(\Omega)$ n-harmonic space cluster, respectively.

Remark 4. If we write $f_j(z)=(z-z_0)^j h_j(z),\ h_j\in H_1(\Omega)$, then

$$f(z)=2\Re\Big\{\sum_{j=0}^{n-1}|z-z_0|^{2j}h_j(z)\Big\},\quad z\in\Omega \tag{2.5}$$

is just a result of Balk [3] although its analogue may have appeared earlier [15, 28]. So we call (2.3) Goursat decomposition form whereas (2.5) Balk decomposition form. One will find that the Goursat decomposition form plays an important role in the calculation of higher order Poisson kernel functions, see Corollary 2 below.

Corollary 2. *Let the sequence of functions* $\{f_n\}$ *defined in* Ω *satisfy*

1. f_1 *is a harmonic function in* Ω*, i.e.,* $f_1 \in Har_1(\Omega)$*;*

2. $(\partial_z \partial_{\overline{z}}) f_n = f_{n-1}$ *in* Ω *for* $n > 1$*.*

Then $f_n \in Har_n(\Omega)$ *for* $n > 1$*, and*

$$\partial_z f_{n,j} = j^{-1} f_{n-1,j-1}, \quad 1 \leq j \leq n-1, \tag{2.6}$$

where $f_{n,j}$ *is the analytic* j*th decomposition component of the* n*-harmonic function* f_n*. It must be noted that* (2.6) *holds in the sense of the equivalence relation* $\sim$*. More precisely,* $\sim_j$ *for* $f_{n,j}$ *and* $\sim_{j-1}$ *for* $f_{n-1,j-1}$*,* $j = 1, 2, \ldots, n-1$*.*

Remark 5. Corollary 2 provides a fundament to our calculation in what follows about kernel functions (i.e., higher order Poisson kernels) appeared in [9].

2.3 Poly-analytic-harmonic Functions

In this section, we define a new class of functions which are called poly-analytic-harmonic functions. All and the same, we also consider their decompositions in terms of analytic functions.

2.3.1 Definition

As in the last two sections, polyanalytic functions and polyharmonic functions are respectively defined by the partial differential operators $\partial_{\overline{z}}^n$ and $(\partial_z \partial_{\overline{z}})^n$, we use the mixed partial differential operators $\partial_z^m \partial_{\overline{z}}^n \, (m \neq n)$ to define a class of functions which will play an important role in solving some Dirichlet boundary value problems in Chapter 3 and 4.

Definition 3. If $f \in C^{m+n}(\Omega)$ satisfies the equation $(\partial_z^m \partial_{\overline{z}}^n) f = 0 \, (m \neq n)$ in Ω, then f is called a poly-analytic-harmonic function in Ω.

Let $M_{m,n}(\Omega) = \{ f \in C^{m+n}(\Omega) : (\partial_z^m \partial_{\overline{z}}^n) f(z) = 0, \, z \in \Omega \}$, especially, $M_{0,n}(\Omega) = H_n(\Omega)$ and $M_{n,0}(\Omega) = \overline{H}_n(\Omega)$ as well as $M_{n,n}(\Omega) = Har_n^{\mathbb{C}}(\Omega)$.

2.3.2 Decomposition

By the decompositions of polyanalytic functions and polyharmonic functions, we get two kinds of decompositions for poly-analytic-harmonic functions as follows.

To do so, let Π_n denote the set of all complex polynomials of degree at most n. We define another equivalence relation $\backsim_n$ as follows:

$$\text{If } f - g \in \Pi_n \text{ for } f, g \in H_1(\Omega), \text{ then } f \backsim_n g.$$

In addition, we set $\backsim = \cup_n \backsim_n$, that is, $f \backsim g$ if $f \backsim_n g$ for some $n \in \mathbb{N}$.

Theorem C (Harmonic Decomposition). *If $f \in M_{m,n}(\Omega)$, where $m, n > 1$ and $m \neq n$, then for any $z_0 \in \Omega$,*
1. as $m > n$,

$$\begin{aligned} f(z) =& 2\Re\Big\{\sum_{k=0}^{n-1} (\overline{z} - \overline{z}_0)^k \varphi_k(z)\Big\} + 2i\Re\Big\{\sum_{k=0}^{n-1} (\overline{z} - \overline{z}_0)^k \widehat{\varphi}_k(z)\Big\} \\ &+ (z - z_0)^n \sum_{l=0}^{m-n-1} \frac{l!}{(n+l)!} (z - z_0)^l \overline{\widetilde{\varphi}_l(z)}, \quad z \in \Omega, \end{aligned} \tag{2.7}$$

where $\varphi_k, \widehat{\varphi}_k \in H^k_{1,z_0}(\Omega)$ and $\widetilde{\varphi}_l \in H_1(\Omega)$;
2. as $m < n$,

$$\begin{aligned} f(z) =& 2\Re\Big\{\sum_{s=0}^{m-1} (\overline{z} - \overline{z}_0)^s \psi_s(z)\Big\} + 2i\Re\Big\{\sum_{s=0}^{m-1} (\overline{z} - \overline{z}_0)^s \widehat{\psi}_s(z)\Big\} \\ &+ (\overline{z} - \overline{z}_0)^m \sum_{t=0}^{n-m-1} \frac{t!}{(m+t)!} (\overline{z} - \overline{z}_0)^t \widetilde{\psi}_t(z), \quad z \in \Omega, \end{aligned} \tag{2.8}$$

where $\psi_s, \widehat{\psi}_s \in H^s_{1,z_0}(\Omega)$ and $\widetilde{\psi}_t \in H_1(\Omega)$. (2.7) *and* (2.8) *are unique in the sense of equivalence relations $\sim$ and $\backsim$, more precisely, $\sim_k$ for $\varphi_k, \widehat{\varphi}_k$ and $\sim_s$ for $\psi_s, \widehat{\psi}_s$ whereas $\backsim_{n-1}$ for all $\widetilde{\varphi}_l$ and $\backsim_{m-1}$ for all $\widetilde{\psi}_t$.*

Proof. We only prove (2.7). Similarly (2.8) follows. As $m > n$, from $(\partial_z^m \partial_{\overline{z}}^n) f(z) = 0$, $z \in \Omega$, we know that $(\partial_z \partial_{\overline{z}})^n f \in \overline{H}_{m-n}(\Omega)$. So by Corollary 1,

$$(\partial_z \partial_{\overline{z}})^n f(z) = \sum_{l=0}^{m-n-1} (z - z_0)^l \overline{\phi_l(z)}, \ z \in \Omega,$$

where $\phi_l \in H_1(\Omega)$. Let

$$\widetilde{\varphi}_l(z) = \int_{z_0}^{z} \int_{z_0}^{\zeta_{n-1}} \cdots \int_{z_0}^{\zeta_1} \phi_l(\zeta)\, \mathrm{d}\zeta \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_{n-1},\ z \in \Omega$$

and

$$f_2(z) = \sum_{l=0}^{m-n-1} \frac{l!}{(n+l)!}(z-z_0)^l \overline{\widetilde{\varphi}_l(z)},\ \ z \in \Omega, \tag{2.9}$$

obviously, $\partial_z^n \widetilde{\varphi}_l(z) = \phi_l(z),\ z \in \Omega$. Furthermore,

$$(\partial_z \partial_{\overline{z}})^n [(z-z_0)^n f_2(z)] = \sum_{l=0}^{m-n-1} (z-z_0)^l \overline{\phi_l(z)},\ z \in \Omega.$$

Therefore, $f - (z-z_0)^n f_2 \in Har_n^{\mathbb{C}}(\Omega)$. Thus

$$\begin{aligned} f(z) - (z-z_0)^n f_2(z) &= 2\Re\Big\{ \sum_{k=0}^{n-1} (\overline{z} - \overline{z}_0)^k \varphi_k(z) \Big\} + 2i\Re\Big\{ \sum_{k=0}^{n-1} (\overline{z} - \overline{z}_0)^k \widehat{\varphi}_k(z) \Big\} \\ &\triangleq f_1(z),\ z \in \Omega, \end{aligned} \tag{2.10}$$

where $\varphi_k, \widehat{\varphi}_k \in H_{1,z_0}^k(\Omega)$. So (2.7) follows from (2.9) and (2.10).

Now we turn to the uniqueness. If

$$f_1(z) + (z-z_0)^n f_2(z) = 0,\ z \in \Omega,$$

where f_1 is given by (2.10) and f_2 is given by (2.9), then, applying the operators $\partial_z^{n+l} \partial_{\overline{z}}^n\ (l = 0, 1, \dots, m-n-1)$ to the last equality, we get

$$\partial_z^n \widetilde{\varphi}_l(z) = \phi_l(z) \equiv 0,\ z \in \Omega$$

for $0 \le l \le m-n-1$. So $\widetilde{\varphi}_l \in \Pi_{n-1} \restriction_\Omega$ which denotes the set of all complex polynomials of degree at most n restricted to Ω, that is, $\widetilde{\varphi}_l \backsim_{n-1} 0$. This is just the uniqueness of $\widetilde{\varphi}_l$ in the sense of equivalence relation $\backsim_{n-1}$. Thus $f_2(z) = 0$ and then $f_1(z) = 0$ follows, in which $z \in \Omega$. So the equivalence uniqueness of $\varphi_k, \widehat{\varphi}_k$ are given by Theorem B. $\square$

Only using Theorem A and Corollary 1, we also get

Theorem D (Canonical Decomposition)**.** *If $f \in M_{m,n}(\Omega)$, where $m, n > 1$ and $m \neq n$, then for any $z_0 \in \Omega$,*

$$f(z) = \sum_{p=0}^{n-1} (\overline{z} - \overline{z}_0)^p \mu_p(z) + \sum_{q=0}^{m-1} (z - z_0)^q \overline{\nu_q(z)} \tag{2.11}$$

where $\mu_p, \nu_q \in H_1(\Omega)$. (2.11) is unique in the sense of equivalence relation $\backsim$ for μ_p and ν_q. More precisely, $\backsim_{m-1}(\backsim_{n-1})$ for μ_p while $\nu_q(\mu_p)$ is unique, $p = 0, 1, \ldots, n-1$, $q = 0, 1, \ldots, m-1$.

Proof. Using Theorem A and Corollary 1, the proof is similar to the one of Theorem C. □

Remark 6. If we set

$$f_3(z) = 2\Re\Big\{ \sum_{s=0}^{m-1} (\overline{z} - \overline{z}_0)^s \psi_s(z) \Big\} + 2i\Re\Big\{ \sum_{s=0}^{m-1} (\overline{z} - \overline{z}_0)^s \widehat{\psi}_s(z) \Big\},$$

$$f_4(z) = \sum_{t=0}^{n-m-1} \frac{t!}{(m+t)!} (\overline{z} - \overline{z}_0)^t \widetilde{\psi}_t(z),$$

$$f_5(z) = \sum_{p=0}^{n-1} (\overline{z} - \overline{z}_0)^p \mu_p(z), \quad f_6(z) = \sum_{q=0}^{m-1} (z - z_0)^q \overline{\nu_q(z)},$$

where $z \in \Omega$, then

$$f(z) = f_1(z) + (z - z_0)^n f_2(z),\ m > n,$$
$$f(z) = f_3(z) + (\overline{z} - \overline{z}_0)^m f_4(z),\ m < n,$$
$$f(z) = f_5(z) + f_6(z),$$

where $f_1(z), f_2(z)$ are given by (2.10) and (2.9). Obviously, $f_1(z), f_3(z)$ are complex polyharmonic functions, $f_2(z), f_6(z)$ are anti-polyanalytic functions and $f_4(z), f_5(z)$ are polyanalytic functions. So we call the decompositions (2.7) and (2.8) harmonic decompositions whereas the decomposition (2.11) is canonical decomposition.

Remark 7. As in [19], all above theorems can be simplified as follows

$$Har_n(\Omega) = 2\Re\Big\{ \sum_{j=0}^{n-1} \oplus (\overline{z} - \overline{z}_0)^j \, (H/\Pi)^j_{1,z_0}(\Omega) \Big\}, \tag{2.12}$$

where $(H/\Pi)^j_{1,z_0}(\Omega)$ denotes the set of all equivalence classes about $\sim_j$, $j = 0, 1, \ldots, n-1$ and $\sum_{j=0}^{n-1} \oplus a_j := a_0 \oplus a_1 \oplus \cdots \oplus a_{n-1}$ which denotes the direct sum of $a_0, a_1, \ldots, a_{n-1}$.

$$H_n(\Omega) = \sum_{j=0}^{n-1} \oplus (\overline{z} - \overline{z}_0)^j \, H_1(\Omega). \tag{2.13}$$

$$\overline{H}_n(\Omega) = \sum_{j=0}^{n-1} \oplus (z - z_0)^j \, \overline{H}_1(\Omega). \tag{2.14}$$

$$M_{m,n}(\Omega) = Har^{\mathbb{C}}_n(\Omega) \oplus (z - z_0)^n \overline{H}_{m-n}(\Omega) \ (m > n). \tag{2.15}$$

$$M_{m,n}(\Omega) = Har^{\mathbb{C}}_m(\Omega) \oplus (\overline{z} - \overline{z}_0)^m H_{n-m}(\Omega) \ (m < n). \tag{2.16}$$

$$M_{m,n}(\Omega) = H_n(\Omega) \oplus \overline{H}_m(\Omega). \tag{2.17}$$

All the decompositions (2.12)-(2.17) are understood in the sense of the equivalence relations $\sim$ and $\backsim$.

2.4 Higher Order Poisson Kernels

From now on, let $\Omega = \mathbb{D}$ which is the unit disc in the complex plane, $\partial\mathbb{D}$ is its boundary, i.e., the unit circle in the complex plane.

In [9], Begehr, Du and Wang considered the Dirichlet problem for polyharmonic functions (PHD problem) in Chapter 3. They have found that the PHD problem is uniquely solvable and its unique solution is connected with a sequence $\{ g_n(z, \tau) \}_{n=1}^{\infty}$ of real valued functions of two variables defined on $\mathbb{D} \times \partial\mathbb{D}$ which are called kernel functions of the solution (simply, kernel functions).

By induction, they guessed and stated that the kernel functions have the following properties:

1. $(\partial_z \partial_{\overline{z}}) g_1(z, \tau) = 0$ and $(\partial_z \partial_{\overline{z}}) g_n(z, \tau) = g_{n-1}(z, \tau)$ for $n > 1$;

2. $\lim_{z\to t,\,|t|=1,\,|z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau} = \gamma(t)$ for any $\gamma \in C(\partial\mathbb{D})$;

3. $\lim_{z\to t,\,|t|=1,\,|z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau) g_2(z,\tau)\frac{\mathrm{d}\tau}{\tau} = 0$ for any $\gamma \in C(\partial\mathbb{D})$;

4. $\lim_{z\to t,\,|t|=1,\,|z|<1} g_n(z,\tau) = 0$ uniformly holds for $\tau \in \partial\mathbb{D}$, $n > 2$.

So their final work is looking for some method to calculate the kernel functions. Though the kernel functions satisfy some certain induction relation (see property 1), the method used in [9] is complicated. All calculations are done up to $g_6(z,\tau)$. In the present section, we will develop a method to calculate all the kernel functions on the basis of Corollary 2.

Unfortunately, only from the above properties 1-4, the kernel functions are not uniquely defined. There is another property of the kernel functions, namely,

5. $g_n(z,\tau) \in C^{2n}(\mathbb{D})$ as a function of z with fixed $\tau \in \partial\mathbb{D}$ and $g_n(z,\tau)$, $\partial_z g_n(z,\tau)$, $\partial_{\overline{z}} g_n(z,\tau) \in C(\mathbb{D}\times\partial\mathbb{D})$, $n = 1, 2, \dots$.

In one moment, we will find that the kernel functions with the above properties are related to the classical Poisson kernel. So we give the following definition.

Definition 4. If a sequence $\{\, g_n(z,\tau)\,\}_{n=1}^{\infty}$ of real valued functions of two variables defined on $\mathbb{D}\times\partial\mathbb{D}$ satisfies the above properties 1-5, then $\{\, g_n(z,\tau)\,\}_{n=1}^{\infty}$ is called a sequence of higher order Poisson kernels, more precisely, $g_n(z,\tau)$ is called the nth order Poisson kernel.

By Theorem B and Corollary 2, we have

Theorem 3. *If $\{\, g_n(z,\tau)\,\}_{n=1}^{\infty}$ is a sequence of higher order Poisson kernels defined on $\mathbb{D}\times\partial\mathbb{D}$, i.e., $\{\, g_n(z,\tau)\,\}_{n=1}^{\infty}$ fulfills the above properties 1-5, then, for $n > 1$, there exist functions $g_{n,0}(z,\tau), g_{n,1}(z,\tau), \dots, g_{n,n-1}(z,\tau)$ defined on $\mathbb{D}\times\partial\mathbb{D}$ such that*

$$g_n(z,\tau) = 2\Re\Big\{\sum_{j=0}^{n-1} \overline{z}^j g_{n,j}(z,\tau)\Big\},\ z \in \mathbb{D},\ \tau \in \partial\mathbb{D} \tag{2.18}$$

with

$$\partial_z g_{n,j}(z,\tau) = j^{-1} g_{n-1,j-1}(z,\tau) \tag{2.19}$$

for $1 \le j \le n-1$ and

$$\partial_z^k g_{n,j}(0,\tau) = 0 \tag{2.20}$$

for $0 \le k \le j-1$ *with respect to* $\tau \in \partial\mathbb{D}$ *as well as*

$$g_{n,0}(z,\tau) = -\sum_{j=1}^{n-1} z^{-j} g_{n,j}(z,\tau). \tag{2.21}$$

However,

$$g_1(z,\tau) = \frac{1}{1-z\overline{\tau}} + \frac{1}{1-\overline{z}\tau} - 1 \tag{2.22}$$

is the Poisson kernel. Such $\{\, g_n(z,\tau) \,\}_{n=1}^{\infty}$ *is unique. Moreover, the decomposition components* $g_{n,j}(z,\tau) \in C(\mathbb{D} \times \partial\mathbb{D})$ *satisfy* $g_{n,j}(\cdot,\tau) \in H^j_{1,0}(\mathbb{D})$ *for fixed* $\tau \in \partial\mathbb{D}$ *and* $\partial_z g_{n,j}(z,\tau) \in C(\mathbb{D} \times \partial\mathbb{D})$, $n = 1, 2, \ldots,$ $j = 0, 1, \ldots, n-1$.

Proof. At first, we consider the existence of the sequence. By the classical theory of harmonic functions [27], the Poisson kernel satisfies the properties which $g_1(z,\tau)$ fulfills. So we can set

$$g_1(z,\tau) = \frac{1}{1-z\overline{\tau}} + \frac{1}{1-\overline{z}\tau} - 1.$$

From the properties 1 and 5, $g_n(z,\tau) \in Har_n(\mathbb{D})$ as a function of z with fixed τ. Noting the properties 3-4, by Theorem B and Corollary 2, we get $g_n(z,\tau)$ in view of (2.18)-(2.21) by induction.

Next, we consider the uniqueness of the sequence. To do so, we need a fact: if some sequence $\{\, \widetilde{g}_n(z,\tau) \,\}_{n=1}^{\infty}$ of real functions defined on $\mathbb{D} \times \partial\mathbb{D}$ satisfies the properties 1, 3-5 and $2'$. $\lim_{z\to t,\, |t|=1,\, |z|<1} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\tau)\widetilde{g}_1(z,\tau)\frac{d\tau}{\tau} = 0$ for any $\gamma \in C(\partial\mathbb{D})$, then $\widetilde{g}_n(z,\tau) \equiv 0$, $n = 1, 2 \ldots$.

In fact, since $\widetilde{g}_1(z,\tau)$ is harmonic as a function of $z \in \mathbb{D}$ with fixed $\tau \in \partial\mathbb{D}$, for any $\gamma \in C(\partial\mathbb{D})$,

$$h(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\tau)\widetilde{g}_1(z,\tau)\frac{d\tau}{\tau}$$

is also a harmonic function in $\mathbb{D}$ by the property 5 and a simple consequence of Lemma 7 in Chapter 3 and can be continuously extended to $\overline{\mathbb{D}}$ with vanishing boundary values by the above property $2'$. By the property 1 and the maximal module principle of harmonic functions, $h(z) \equiv 0,\ z \in \overline{\mathbb{D}}$.

For any fixed $z \in \mathbb{D}$, set

$$(\widetilde{g}_1)_z(\tau) = \widetilde{g}_1(z,\tau),$$

then $(\widetilde{g}_1)_z$ is a function of τ on $\partial\mathbb{D}$. From the property 5, we know that $(\widetilde{g}_1)_z \in C(\partial\mathbb{D})$. Thus

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \widetilde{g}_1^2(z,\tau)\frac{\mathrm{d}\tau}{\tau} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\widetilde{g}_1)_z(\tau)\widetilde{g}_1(z,\tau)\frac{\mathrm{d}\tau}{\tau} \equiv 0,\ z\in\mathbb{D}.$$

Note that $\widetilde{g}_1(z,\tau)$ is real, so $\widetilde{g}_1(z,\tau) \equiv 0$, $z \in \mathbb{D}$, $\tau \in \partial\mathbb{D}$. Then, from the properties 1, 3-4, for any $n \geq 2$, $\widetilde{g}_n(z,\tau)$ is harmonic as a function of $z \in \mathbb{D}$ with fixed $\tau \in \partial\mathbb{D}$ and

$$\lim_{z\to t,\,|t|=1,\,|z|<1} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\tau)\widetilde{g}_n(z,\tau)\frac{\mathrm{d}\tau}{\tau} = 0$$

for any $\gamma \in C(\partial\mathbb{D})$, i.e., the property $2'$ is valid for $\widetilde{g}_n(z,\tau)$ while $n \geq 2$. Since $\widetilde{g}_n(z,\tau)$ are real, repeating the proof for $\widetilde{g}_1(z,\tau)$, we get $\widetilde{g}_n(z,\tau) \equiv 0$ for $n \geq 2$. Hence, the uniqueness of the sequence $\{\, g_n(z,\tau) \,\}_{n=1}^{\infty}$ follows from the above fact.

Finally, it is easy to know that the properties of $g_{n,j}(z,\tau)$ follow from Corollary 2 and the same properties of $g_{1,0}(z,\tau)$ as well as (2.19)-(2.21). □

By Theorem 3, now we will calculate the higher order Poisson kernels by induction. To do so, set

$$g_{1,0}(z,\tau) = \frac{1}{1-z\overline{\tau}} - \frac{1}{2} = \sum_{k=1}^{\infty}(z\overline{\tau})^k + \frac{1}{2}, \tag{2.23}$$

therefore

$$\begin{aligned} g_1(z,\tau) = 2\Re\{g_{1,0}(z,\tau)\} &= \sum_{k=1}^{\infty}\left((z\overline{\tau})^k + (\overline{z}\tau)^k\right) + 1 \\ &= \sum_{k=2}^{\infty}\left((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\right) + 1. \end{aligned} \tag{2.24}$$

By (2.19)-(2.20),

$$\begin{aligned} g_{2,1}(z,\tau) &= \int_0^z g_{1,0}(\zeta,\tau)\mathrm{d}\zeta \\ &= \sum_{k=1}^{\infty}\frac{1}{k+1}z^{k+1}\overline{\tau}^k + \frac{1}{2}z \\ &= \sum_{k=2}^{\infty}\frac{1}{k}z^k\overline{\tau}^{k-1} + \frac{1}{2}z. \end{aligned} \tag{2.25}$$

By (2.21),

$$
\begin{aligned}
g_{2,0}(z,\tau) &= -\Big[\sum_{k=2}^{\infty}\frac{1}{k}(z\overline{\tau})^{k-1}+\frac{1}{2}\Big] \\
&= \frac{1}{z\overline{\tau}}\log(1-z\overline{\tau})+\frac{1}{2}.
\end{aligned} \tag{2.26}
$$

Substituting (2.25)-(2.26) into (2.18) , we get

$$
\begin{aligned}
g_2(z,\tau) &= -(1-|z|^2)\Big[\sum_{k=2}^{\infty}\frac{1}{k}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+1\Big] \\
&= -(1-|z|^2)\Big[\frac{1}{z\overline{\tau}}\log(1-z\overline{\tau})+\frac{1}{\overline{z}\tau}\log(1-\overline{z}\tau)+1\Big].
\end{aligned} \tag{2.27}
$$

Similarly,

$$
\begin{aligned}
g_{3,2}(z,\tau) =&\frac{1}{2}\int_0^z g_{2,1}(\zeta,\tau)\mathrm{d}\zeta \\
=&\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z^2\Big], \\
g_{3,1}(z,\tau) =& \int_0^z g_{2,0}(\zeta,\tau)\mathrm{d}\zeta \\
=& -\Big[\sum_{k=2}^{\infty}\frac{1}{k^2}z^k\overline{\tau}^{k-1}+\frac{1}{2}z\Big], \\
g_{3,0}(z,\tau) =&\Big[\sum_{k=2}^{\infty}\frac{1}{k^2}(z\overline{\tau})^{k-1}+\frac{1}{2}\Big] \\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big],
\end{aligned}
$$

so

$$
\begin{aligned}
g_3(z,\tau) =&(1-|z|^2)\Big[\sum_{k=2}^{\infty}\frac{1}{k^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+1\Big] \\
&-\frac{1-|z|^4}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big].
\end{aligned} \tag{2.28}
$$

Again,

$$
g_{4,3}(z,\tau) =\frac{1}{3}\int_0^z g_{3,2}(\zeta,\tau)\mathrm{d}\zeta
$$

$$
\begin{aligned}
&= \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!}z^3\Big],\\
g_{4,2}(z,\tau) &= \frac{1}{2}\int_0^z g_{3,1}(\zeta,\tau)\mathrm{d}\zeta\\
&= -\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z^2\Big],\\
g_{4,1}(z,\tau) &= \int_0^z g_{3,0}(\zeta,\tau)\mathrm{d}\zeta\\
&= \Big[\sum_{k=2}^{\infty}\frac{1}{k^3}z^{k}\overline{\tau}^{k-1}+\frac{1}{2}z\Big]\\
&\quad -\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big],\\
g_{4,0}(z,\tau) &= \Big\{-\Big[\sum_{k=2}^{\infty}\frac{1}{k^3}(z\overline{\tau})^{k-1}+\frac{1}{2}\Big]\\
&\quad +\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]\Big\}\\
&\quad +\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]\\
&\quad -\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big],
\end{aligned}
$$

thus

$$
\begin{aligned}
g_4(z,\tau) =& -(1-|z|^2)\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3}\big((z\overline{\tau})^{k-1}+(z\overline{\tau})^{k-1}\big)+1\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}\big((z\overline{\tau})^{k-1}+(z\overline{\tau})^{k-1}\big)+\frac{1}{2!}\Big]\Big\}\\
&+\frac{1-|z|^4}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)}\big((z\overline{\tau})^{k-1}+(z\overline{\tau})^{k-1}\big)+\frac{1}{2!}\Big]\\
&-\frac{1-|z|^6}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(z\overline{\tau})^{k-1}\big)+\frac{1}{3!}\Big]. \qquad (2.29)
\end{aligned}
$$

Furthermore,

$$
g_{5,4}(z,\tau) = \frac{1}{4}\int_0^z g_{4,3}(\zeta,\tau)\mathrm{d}\zeta
$$

$$
\begin{aligned}
&= \frac{1}{4!}\Big[\sum_{k=2}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} z^{k+3}\overline{\tau}^{k-1} + \frac{1}{2\cdot 4!} z^4\Big], \\
g_{5,3}(z,\tau) &= \frac{1}{3}\int_0^z g_{4,2}(\zeta,\tau)\mathrm{d}\zeta \\
&= -\frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)} z^{k+2}\overline{\tau}^{k-1} + \frac{1}{2\cdot 3!} z^3\Big], \\
g_{5,2}(z,\tau) &= \frac{1}{2}\int_0^z g_{4,1}(\zeta,\tau)\mathrm{d}\zeta \\
&= \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} z^{k+1}\overline{\tau}^{k-1} + \frac{1}{2\cdot 2!} z^2\Big] \\
&\quad - \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)^2} z^{k+1}\overline{\tau}^{k-1} + \frac{1}{2\cdot 2!\cdot 2!} z^2\Big]\Big\}, \\
g_{5,1}(z,\tau) &= \int_0^z g_{4,0}(\zeta,\tau)\mathrm{d}\zeta \\
&= -\Big[\sum_{k=2}^{\infty} \frac{1}{k^4} z^{k}\overline{\tau}^{k-1} + \frac{1}{2} z\Big] \\
&\quad + \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} z^{k}\overline{\tau}^{k-1} + \frac{1}{2\cdot 2!} z\Big] \\
&\quad + \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} z^{k}\overline{\tau}^{k-1} + \frac{1}{2\cdot 2!} z\Big] \\
&\quad - \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)} z^{k}\overline{\tau}^{k-1} + \frac{1}{2\cdot 3!} z\Big], \\
g_{5,0}(z,\tau) &= \Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^4} (z\overline{\tau})^{k-1} + \frac{1}{2}\Big] \\
&\quad - \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 2!}\Big] \\
&\quad - \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 2!}\Big] \\
&\quad + \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 3!}\Big]\Big\} \\
&\quad - \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 2!}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]\Big\}\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]\\
&-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 4!}\Big],
\end{aligned}
$$

therefore

$$
\begin{aligned}
g_5(z,\tau)=&(1-|z|^2)\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+1\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\Big\}\\
&-\frac{1-|z|^4}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\Big\}\\
&+\frac{1-|z|^6}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\\
&-\frac{1-|z|^8}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{4!}\Big].
\end{aligned}
\tag{2.30}
$$

For $g_6(z,\tau)$:

$$
\begin{aligned}
g_{6,5}(z,\tau)=&\frac{1}{5}\int_0^z g_{5,4}(\zeta,\tau)\mathrm{d}\zeta\\
=&\frac{1}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)}z^{k+4}\overline{\tau}^{k-1}+\frac{1}{2\cdot 5!}z^5\Big],\\
g_{6,4}(z,\tau)=&\frac{1}{4}\int_0^z g_{5,3}(\zeta,\tau)\mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
=&-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)}z^{k+3}\overline{\tau}^{k-1}+\frac{1}{2\cdot 4!}z^4\Big],\\
g_{6,3}(z,\tau)=&\frac{1}{3}\int_0^z g_{5,2}(\zeta,\tau)\mathrm{d}\zeta\\
=&\frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!}z^3\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!\cdot 2!}z^3\Big]\Big\},\\
g_{6,2}(z,\tau)=&\frac{1}{2}\int_0^z g_{5,1}(\zeta,\tau)\mathrm{d}\zeta\\
=&-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z^2\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}z^2\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}z^2\Big]\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}z^2\Big]\Big\},\\
g_{6,1}(z,\tau)=&\int_0^z g_{5,0}(\zeta,\tau)\mathrm{d}\zeta\\
=&\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5}z^{k}\overline{\tau}^{k-1}+\frac{1}{2}z\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big]\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!}z\Big]\Big\}\\
&-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}z\Big]\Big\}
\end{aligned}
$$

$$+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}z^k\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!}z\Big]$$

$$-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)}z^k\overline{\tau}^{k-1}+\frac{1}{2\cdot 4!}z\Big],$$

$$g_{6,0}(z,\tau)=-\Big\{\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5}(z\overline{\tau})^{k-1}+\frac{1}{2}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]\Big\}$$

$$-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]\Big\}$$

$$+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]$$

$$-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 4!}\Big]\Big\}$$

$$+\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]$$

$$+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}\Big]\Big\}$$

$$-\frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]$$

$$- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)^2(k+2)}(z\overline{\tau})^{k-1} + \frac{1}{2\cdot 3!\cdot 2!}\Big]\Big\}$$

$$+ \frac{1}{4!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)(k+3)}(z\overline{\tau})^{k-1} + \frac{1}{2\cdot 4!}\Big]$$

$$- \frac{1}{5!}\Big[\sum_{k=2}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)}(z\overline{\tau})^{k-1} + \frac{1}{2\cdot 5!}\Big],$$

so

$$g_6(z,\tau) = -(1-|z|^2)\Big\{\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^5}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + 1\Big]$$

$$- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!}\Big]$$

$$- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!}\Big]$$

$$+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)(k+2)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{3!}\Big]\Big\}$$

$$- \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!}\Big]$$

$$- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)^2}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!\cdot 2!}\Big]\Big\}$$

$$+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)(k+2)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{3!}\Big]$$

$$- \frac{1}{4!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)(k+3)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{4!}\Big]\Big\}$$

$$+ \frac{1-|z|^4}{2!}\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!}\Big]$$

$$- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)^2}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!\cdot 2!}\Big]$$

$$- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)^2}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!\cdot 2!}\Big]$$

$$+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}\big) + \frac{1}{2!\cdot 3!}\Big]\Big\}$$

$$
\begin{aligned}
&- \frac{1-|z|^6}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!\cdot 2!}\Big]\Big\}\\
&+ \frac{1-|z|^8}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{4!}\Big]\\
&- \frac{1-|z|^{10}}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{5!}\Big].
\end{aligned} \tag{2.31}
$$

For $g_7(z,\tau)$:

$$
\begin{aligned}
g_{7,6}(z,\tau) =& \frac{1}{6}\int_0^z g_{6,5}(\zeta,\tau)\mathrm{d}\zeta\\
=& \frac{1}{6!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)}z^{k+5}\overline{\tau}^{k-1}+\frac{1}{2\cdot 6!}z^6\Big],\\
g_{7,5}(z,\tau) =& \frac{1}{5}\int_0^z g_{6,4}(\zeta,\tau)\mathrm{d}\zeta\\
=& -\frac{1}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)(k+4)}z^{k+4}\overline{\tau}^{k-1}+\frac{1}{2\cdot 5!}z^5\Big],\\
g_{7,4}(z,\tau) =& \frac{1}{4}\int_0^z g_{6,3}(\zeta,\tau)\mathrm{d}\zeta\\
=& \frac{1}{4!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}z^{k+3}\overline{\tau}^{k-1}+\frac{1}{2\cdot 4!}z^4\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)(k+3)}z^{k+3}\overline{\tau}^{k-1}+\frac{1}{2\cdot 4!\cdot 2!}z^4\Big]\Big\},\\
g_{7,3}(z,\tau) =& \frac{1}{3}\int_0^z g_{6,2}(\zeta,\tau)\mathrm{d}\zeta\\
=& -\frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!}z^3\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!\cdot 2!}z^3\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!\cdot 2!}z^3\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)^2}z^{k+2}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!\cdot 3!}z^3\Big]\Big\},\\
g_{7,2}(z,\tau)=&\frac{1}{2}\int_0^z g_{6,1}(\zeta,\tau)\mathrm{d}\zeta\\
=&\frac{1}{2!}\Big\{\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z^2\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}z^2\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}z^2\Big]\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}z^2\Big]\Big\}\\
&-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}z^2\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^3}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 2!\cdot 2!}z^2\Big]\Big\}\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}z^2\Big]\\
&-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)(k+3)}z^{k+1}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!\cdot 4!}z^2\Big]\Big\},\\
g_{7,1}(z,\tau)=&\int_0^z g_{6,0}(\zeta,\tau)\mathrm{d}\zeta\\
=&-\Big\{\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^6}z^{k}\overline{\tau}^{k-1}+\frac{1}{2}z\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big]\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 3!}z\Big]\Big\}\\
&-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}z^{k}\overline{\tau}^{k-1}+\frac{1}{2\cdot 2!}z\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)^2} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 2!\cdot 2!} z\Big]\Big\} \\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)(k+2)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 3!} z\Big] \\
&- \frac{1}{4!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)(k+2)(k+3)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 4!} z\Big]\Big\} \\
&+ \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^5(k+1)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 2!} z\Big] \\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)^2} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 2!\cdot 2!} z\Big] \\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)^2} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 2!\cdot 2!} z\Big] \\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)^2(k+2)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 2!\cdot 3!} z\Big]\Big\} \\
&- \frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)(k+2)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 3!} z\Big] \\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)^2(k+2)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 3!\cdot 2!} z\Big]\Big\} \\
&+ \frac{1}{4!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)(k+2)(k+3)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 4!} z\Big] \\
&- \frac{1}{5!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)(k+3)(k+4)} z^k \overline{\tau}^{k-1} + \frac{1}{2\cdot 5!} z\Big],
\end{aligned}
$$

$$
\begin{aligned}
g_{7,0}(z,\tau) =& \Big\{\Big\{\Big\{\Big[\sum_{k=2}^{\infty} \frac{1}{k^6} (z\overline{\tau})^{k-1} + \frac{1}{2}\Big] \\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^5(k+1)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 2!}\Big] \\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^5(k+1)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 2!}\Big] \\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{1}{k^4(k+1)(k+2)} (z\overline{\tau})^{k-1} + \frac{1}{2\cdot 3!}\Big]\Big\}
\end{aligned}
$$

$$-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]\Big\}$$

$$+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]$$

$$-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 4!}\Big]\Big\}$$

$$-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]$$

$$+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}\Big]\Big\}$$

$$+\frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!\cdot 2!}\Big]\Big\}$$

$$-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 4!}\Big]$$

$$+\frac{1}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)(k+4)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 5!}\Big]\Big\}$$

$$-\frac{1}{2!}\Big\{\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]$$

$$-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]$$

$$
\begin{aligned}
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}\Big]\Big\}\\
&-\frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!}\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^3}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 2!\cdot 2!}\Big]\Big\}\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 3!}\Big]\\
&-\frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 2!\cdot 4!}\Big]\Big\}\\
&+\frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!}\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!\cdot 2!}\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!\cdot 2!}\Big]\\
&+\frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)^2}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 3!\cdot 3!}\Big]\Big\}\\
&-\frac{1}{4!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 4!}\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)(k+3)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 4!\cdot 2!}\Big]\Big\}\\
&+\frac{1}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)(k+4)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 5!}\Big]\\
&-\frac{1}{6!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)}(z\overline{\tau})^{k-1}+\frac{1}{2\cdot 6!}\Big],
\end{aligned}
$$

so

$$
\begin{aligned}
g_7(z,\tau)=&(1-|z|^2)\Big\{\Big\{\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^6}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+1\Big]\\
&-\frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]\\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\Big\}\\
&- \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\Big\}\\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\\
&- \frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{4!}\Big]\Big\}\\
&- \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 3!}\Big]\Big\}\\
&+ \frac{1}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!\cdot 2!}\Big]\Big\}\\
&- \frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{4!}\Big]\\
&+ \frac{1}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)(k+4)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{5!}\Big]\Big\}\\
&- \frac{1-|z|^4}{2!}\Big\{\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^5(k+1)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 3!}\Big]\Big\}\\
&- \frac{1}{2!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^3}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 2!\cdot 2!}\Big]\Big\}\\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 3!}\Big]\\
&- \frac{1}{4!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{2!\cdot 4!}\Big]\Big\}\\
&+ \frac{1-|z|^6}{3!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^4(k+1)(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!\cdot 2!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)^2(k+2)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!\cdot 2!}\Big]\\
&+ \frac{1}{3!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)^2}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{3!\cdot 3!}\Big]\Big\}\\
&- \frac{1-|z|^8}{4!}\Big\{\Big[\sum_{k=2}^{\infty}\frac{1}{k^3(k+1)(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{4!}\Big]\\
&- \frac{1}{2!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)^2(k+2)(k+3)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{4!\cdot 2!}\Big]\Big\}\\
&+ \frac{1-|z|^{10}}{5!}\Big[\sum_{k=2}^{\infty}\frac{1}{k^2(k+1)(k+2)(k+3)(k+4)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{5!}\Big]\\
&- \frac{1-|z|^{12}}{6!}\Big[\sum_{k=2}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)}\big((z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}\big)+\frac{1}{6!}\Big].
\end{aligned}
\tag{2.32}
$$

From the above calculations of $g_1(z,\tau), g_2(z,\tau), \ldots, g_7(z,\tau)$, we find that all of them are sums with certain summands which take on some nice orderliness precisely stated in the following Remark 8.

In order to get $g_n(z,\tau)$, we introduce a vertical sum

$$\sum \left\{ \begin{array}{l} a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \right. =: a_1 + a_2 + \cdots + a_n. \tag{2.33}$$

So we can also write $g_1(z,\tau), g_2(z,\tau), \ldots, g_7(z,\tau)$ in terms of some vertical sums. For example,

$$g_5(z,\tau) = \sum \left\{ \begin{array}{l} (1-|z|^2) \sum \left\{ \begin{array}{c} \Big[\sum_{k=2}^{\infty} \frac{1}{k^4} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + 1 \Big] \\ -\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + \frac{1}{2!} \Big] \\ -\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + \frac{1}{2!} \Big] \\ \frac{1}{3!} \Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + \frac{1}{3!} \Big] \end{array} \right. \\ -\dfrac{1-|z|^4}{2!} \sum \left\{ \begin{array}{c} \Big[\sum_{k=2}^{\infty} \frac{1}{k^3(k+1)} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + \frac{1}{2!} \Big] \\ -\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)^2} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + \frac{1}{2!\cdot 2!} \Big] \end{array} \right. \\ \dfrac{1-|z|^6}{3!} \Big[\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2(k+1)(k+2)} \big((z\overline{\tau})^{k-1} + (z\overline{\tau})^{k-1} \big) + \frac{1}{3!} \Big] \\ -\dfrac{1-|z|^8}{4!} \Big[\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} \big((z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1} \big) + \frac{1}{4!} \Big]. \end{array} \right.$$

Obviously, in the above form of vertical sum, $g_5(z,\tau)$ is an excellent example which is possessed of the nice circulatory structure and sequential properties in detail stated in Remark 8 and Appendix A.

In general, applying vertical sums, we have

Theorem 4. *For $1 \leq j \leq n-5$, let $W_{n,j}(z,\tau)$ be a vertical sum of the form*

$$
\sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l}
\cdots \sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l}
\Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j}(k+1)\cdots(k+j-1)} + \frac{1}{j!} \Big] \\
-\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2!} \Big] \\
-\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2!} \Big] \\
\frac{1}{3!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 3!} \Big]
\end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l}
\Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2!} \Big] \\
-\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 2!} \Big]
\end{array} \right. \\
\frac{1}{3!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 3!} \Big] \\
-\frac{1}{4!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 4!} \Big]
\end{array} \right. \\
\qquad \vdots \\
\frac{(-1)^{n-j-3}}{(n-j-2)!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)} + \frac{1}{j! \cdot (n-j-2)!} \Big]
\end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l}
\cdots \sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l}
\Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2!} \Big] \\
-\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 2!} \Big] \\
-\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 2!} \Big] \\
\frac{1}{3!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 3!} \Big]
\end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l}
\Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 2!} \Big] \\
-\frac{1}{2!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-3}(k+1)^4\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 2! \cdot 2!} \Big]
\end{array} \right. \\
\frac{1}{3!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 3!} \Big] \\
-\frac{1}{4!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)} + \frac{1}{j! \cdot 2! \cdot 4!} \Big]
\end{array} \right. \\
\qquad \vdots \\
\frac{(-1)^{n-j-4}}{(n-j-3)!} \Big[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)} + \frac{1}{j! \cdot 2! \cdot (n-j-3)!} \Big]
\end{array} \right. \\
\qquad \vdots
\end{array} \right.
$$

$$
\sum \left\{ \begin{array}{l}
\vdots \\
\vdots \\
\frac{(-1)^{n-j-6}}{(n-j-5)!} \sum \left\{ \begin{array}{l}
\sum \left\{ \begin{array}{l}
\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^6(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 2!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 2!} \right] \\
\frac{1}{3!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 3!} \right]
\end{array} \right. \\
-\frac{1}{2!} \sum \left\{ \begin{array}{l}
\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 2!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^4\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 2!\cdot 2!} \right]
\end{array} \right. \\
\frac{1}{3!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 3!} \right] \\
-\frac{1}{4!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^3(k+2)^3(k+3)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{1}{j!\cdot(n-j-5)!\cdot 4!} \right]
\end{array} \right. \\
\frac{(-1)^{n-j-5}}{(n-j-4)!} \sum \left\{ \begin{array}{l}
\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^5(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{1}{j!\cdot(n-j-4)!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{1}{j!\cdot(n-j-4)!\cdot 2!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{1}{j!\cdot(n-j-4)!\cdot 2!} \right] \\
\frac{1}{3!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{1}{j!\cdot(n-j-4)!\cdot 3!} \right]
\end{array} \right. \\
\frac{(-1)^{n-j-4}}{(n-j-3)!} \sum \left\{ \begin{array}{l}
\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)} + \frac{1}{j!\cdot(n-j-3)!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)} + \frac{1}{j!\cdot(n-j-3)!\cdot 2!} \right]
\end{array} \right. \\
\frac{(-1)^{n-j-3}}{(n-j-2)!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)} + \frac{1}{j!\cdot(n-j-2)!} \right] \\
\frac{(-1)^{n-j-2}}{(n-j-1)!} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-2)} + \frac{1}{j!\cdot(n-j-1)!} \right]
\end{array} \right. \tag{2.34}
$$

with $d_{k-1}(z,\tau) = (z\overline{\tau})^{k-1} + (\overline{z}\tau)^{k-1}$*, and let*

$$W_{n,n-4}(z,\tau) = \sum \begin{cases} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^4(k+1)(k+2)\cdots(k+n-5)} + \frac{1}{(n-4)!}\right] \\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^2(k+2)\cdots(k+n-5)} + \frac{1}{(n-4)!\cdot 2!}\right] \\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)^2(k+2)\cdots(k+n-5)} + \frac{1}{(n-4)!\cdot 2!}\right] \\ \frac{1}{3!}\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)^2(k+2)^2\cdots(k+n-5)} + \frac{1}{(n-4)!\cdot 3!}\right] \end{cases}, \tag{2.35}$$

$$W_{n,n-3}(z,\tau) = \sum \begin{cases} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)(k+2)\cdots(k+n-4)} + \frac{1}{(n-3)!}\right] \\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)^2(k+2)\cdots(k+n-4)} + \frac{1}{(n-3)!\cdot 2!}\right] \end{cases}, \tag{2.36}$$

$$W_{n,n-2}(z,\tau) = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)(k+2)\cdots(k+n-3)} + \frac{1}{(n-2)!}, \tag{2.37}$$

$$W_{n,n-1}(z,\tau) = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k(k+1)(k+2)\cdots(k+n-2)} + \frac{1}{(n-1)!}. \tag{2.38}$$

If $\{ g_n(z,\tau) \}_{n=1}^{\infty}$ *is a sequence of higher order Poisson kernel functions defined on* $\mathbb{D} \times \partial\mathbb{D}$ *, then*

$$g_n(z,\tau) = D_1(z,\tau) + D_2(z,\tau) + \cdots + D_{n-1}(z,\tau), \tag{2.39}$$

where $D_j(z,\tau) = (-1)^{n-j}\frac{1-|z|^{2j}}{j!}W_{n,j}(z,\tau)$, $j = 1, 2, \ldots, n-1$. *In all above formulae, by convention,* $\prod_{\ell=\imath}^{\jmath}(k+\ell) = 1$ *as* $\imath > \jmath$.

Remark 8. Among all above formulae, $g_1(z,\tau), \ldots, g_6(z,\tau)$ are the same ones obtained by a different method in [9]. $g_7(z,\tau)$ and $g_n(z,\tau)$ are new. Carefully observing all above vertical sums $W_{n,j}(z,\tau)$, $j = 1, 2, \ldots, n-1$, one may find that the vertical sums take on some structural orderliness. More precisely, there is a distinct circulatory structure of the following vertical sum

$$\frac{(-1)^{p-1}}{p!}\sum\left\{\begin{array}{l}\frac{(-1)^{q-1}}{q!}\alpha\sum\left\{\begin{array}{c}\varepsilon\\ -\frac{1}{2!}\zeta\\ -\frac{1}{2!}\zeta\\ \frac{1}{3!}\varsigma\end{array}\right.\\ \frac{(-1)^{q}}{(q+1)!}\beta\sum\left\{\begin{array}{c}\varpi\\ -\frac{1}{2!}\omega\end{array}\right.\\ \frac{(-1)^{q+1}}{(q+2)!}\mu\gamma\\ \frac{(-1)^{q+2}}{(q+3)!}\nu\delta\end{array}\right., \tag{2.40}$$

where α, β, μ, ν are 1 or 0, all of which are nonzero or only one of which is nonzero, the latter only happens when $j = n-4, n-3, n-2, n-1$, $1 \leq p \leq n-4$ and $0 \leq q \leq n-4$. However, $\varepsilon, \zeta, \varsigma, \varpi, \omega, \gamma, \delta$ are sums of the form

$$\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{m_1}(k+1)^{m_2}(k+2)^{m_3}\cdots(k+n-2)^{m_{n-1}}}+\frac{1}{\vartheta}, \tag{2.41}$$

where $m_1, m_2, \ldots, m_{n-1}$ are nonnegative integers satisfying

$$m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0 \text{ and } m_1 + m_2 + \cdots + m_{n-1} = n-1, \tag{2.42}$$

whereas ϑ is a product of some factorials which takes on some evident regularity, i.e., ϑ is the product of $j!$ and all denominators of the coefficients appearing before the vertical sum symbols and the sum which it belongs to. Moreover, when $\alpha = \beta = \gamma = \delta = 1$, the multiplicities have the following sequential properties:

(1) From ε to ζ and ϖ to ω, m_1 decreases by 1 whereas m_2 simultaneously increase by 1;

(2) From ζ to ς, m_1 decreases by 1 whereas m_3 simultaneously increase by 1;

(3) From ε to ϖ, ϖ to γ and γ to δ, m_1 decreases by 1 for each step whereas m_{q+1}, m_{q+2} and m_{q+3} sequentially increases by 1.

It must be noted that the new multiplicities also satisfy (2.42) all the same. In addition, for $W_{n,j}(z, \tau)$, there are $n-j-1$ vertical sums as its summands in the outmost vertical sum. From the top down, these vertical sums respectively have 2^{n-j-3}, 2^{n-j-4}, …, 2, 1, 1 summands of the form as (2.41). The above property (3) holds for the variance of the multiplicities about the first summand of the form as (2.41) between two adjacent vertical sums and the coefficients appearing before the vertical sum symbols are in turn 1, $-\frac{1}{2!}$, …, $\frac{(-1)^{n-j-3}}{(n-j-2)!}$, $\frac{(-1)^{n-j-2}}{(n-j-1)!}$. Interestingly, any one of these vertical sums has similar structure and properties as the outmost vertical sum.

Just because of the above sequential properties of the multiplicities and the nice circulatory structure, we can sequentially define $W_{n,j}(z,\tau)$ as the vertical sum (2.34) only from the first summand $\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j}(k+1)\cdots(k+j-1)}+\frac{1}{j!}$.

To prove Theorem 4, we need the following lemma.

Lemma 5. *If* $\{\,g_n(z,\tau)\,\}_{n=1}^{\infty}$ *is a sequence of higher order Poisson kernels defined on* $\mathbb{D}\times\partial\mathbb{D}$, *then*

$$g_{n,j}(z,\tau)=(-1)^{n-j+1}\frac{1}{j!}\widetilde{W}_{n,j}(z,\tau),\ j=\begin{cases}1,2,\ldots,n-1,\ n>1,\\ 0,\ n=1,\end{cases}\tag{2.43}$$

where $g_{n,j}(z,\tau)$ *are the same ones in Theorem 3 and* $\widetilde{W}_{n,j}(z,\tau)$ *are given by replacing all numerators* $d_{k-1}(z,\tau)$ *by* $z^{k+j-1}\overline{\tau}^{k-1}$ *and all numerators* 1 *by* $\frac{z^j}{2}$ *in all summands as* (2.41) *of* $W_{n,j}(z,\tau)$. *Thus*

$$g_{n,0}(z,\tau)=\begin{cases}\sum_{j=1}^{n-1}z^{-j}(-1)^{n-j}\frac{1}{j!}\widetilde{W}_{n,j}(z,\tau),\ n>1,\\ \sum_{k=2}^{\infty}(z\overline{\tau})^{k-1}+\frac{1}{2},\ n=1.\end{cases}\tag{2.44}$$

Proof. By (2.23) and (2.38) as well the definition of $\widetilde{W}_{n,j}(z,\tau)$, (2.43)-(2.44) holds for $m=1$. That is, the claim of Lemma 5 is real for $m=1$. Suppose that the claim is also real for $m=n-1$ ($n>2$), by (2.19), (2.20) and the definition of $\widetilde{W}_{n,j}(z,\tau)$, for $1\le j\le n-1$,

$$\begin{aligned}g_{n,j}(z,\tau)=&j^{-1}\int_0^z g_{n-1,j-1}(\zeta,\tau)\mathrm{d}\zeta\\ =&j^{-1}\int_0^z(-1)^{n-j+1}\frac{1}{(j-1)!}\widetilde{W}_{n-1,j-1}(\zeta,\tau)\mathrm{d}\zeta\\ =&(-1)^{n-j+1}\frac{1}{j!}\widetilde{W}_{n,j}(z,\tau).\end{aligned}\tag{2.45}$$

By (2.21), we have

$$g_{n,0}(z,\tau)=\sum_{j=1}^{n-1} z^{-j}(-1)^{n-j}\frac{1}{j!}\widetilde{W}_{n,j}(z,\tau).$$

By induction method, the claim holds for all $n\in\mathbb{N}$. □

Proof of Theorem 4. Noting the definitions of $W_{n,j}(z,\tau)$ and $\widetilde{W}_{n,j}(z,\tau)$, by (2.18), (2.43) and (2.44), we have

$$\begin{aligned} g_n(z,\tau) =&2\Re\Big\{\sum_{j=0}^{n-1}\overline{z}^j g_{n,j}(z,\tau)\Big\}\\ =&2\Re\Big\{\sum_{j=1}^{n-1}(-1)^{n-j}\frac{1}{j!}(z^{-j}-\overline{z}^j)\widetilde{W}_{n,j}(z,\tau)\Big\}\\ =&\sum_{j=1}^{n-1}(-1)^{n-j}\frac{1}{j!}(1-|z|^{2j})W_n(z,\tau). \end{aligned} \tag{2.46}$$

Thus we complete the proof of Theorem 4. □