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**Multilevel Structural Equation Modeling of
Multitrait-Multimethod-Multioccasion Data**

Dissertation

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Summary

Many psychologists agree with the statement that the multitrait-multimethod (MTMM) analysis developed by Campbell and Fiske (1959) is one of the most important methodological developments in the social and behavioral sciences (see e.g., Kenny, 1995). MTMM measurement designs allow researchers to scrutinize the *convergent* and *discriminant* validity of their measures. The numerous advantages of multimethod research (Eid, 2006) as well as the increasing interest in longitudinal research have led many statisticians to develop new models for analyzing multitrait-multimethod-multioccasion (MTMM-MO) data (e.g., Burns, Walsh, & Gomez, 2003; Burns & Haynes, 2006; Cole & Maxwell, 2003; Courvoisier, 2006; Courvoisier, Nussbeck, Eid, Geiser, & Cole, 2008; Crayen, Geiser, Scheithauer, & Eid, 2011; Geiser, 2008, 2009; Geiser, Eid, Nussbeck, Courvoisier, & Cole, 2010; Grimm, Pianta, & Konold, 2009). Currently, the most common way to analyze MTMM data is via structural equation models (SEMs; Eid, 2000). Using structural equation models for analyzing longitudinal MTMM data bears many advantages such as (a) separating different sources of variance (e.g., due to trait, occasion-specific, method, and measurement error influences), (b) testing theoretical assumptions via model test indices, (c) relating latent method variables to external variables. However, researchers often struggle with choosing the appropriate structural equation model for their particular MTMM-MO measurement design. According to Eid et al. (2008) the model selection process should be guided by the types of methods used in the MTMM measurement design. For example, measurement designs with *interchangeable* methods imply that methods are randomly chosen from a common set of equivalent methods (e.g., multiple student ratings for teaching quality). As a consequence, measurement designs using interchangeable methods result out of a multistage sampling procedure and thus imply a hierarchical (multilevel) data structure (e.g., raters nested in targets). In contrast, measurement designs with *structurally different* methods result whenever methods are fixed. Structurally different methods are methods which cannot be easily replaced by one another (e.g., physiological measures, self-ratings, teacher ratings). In this thesis four different multilevel structural equation models (ML-SEMs) are proposed for analyzing longitudinal MTMM data combining structurally different and interchangeable methods. Specifically, a latent state (LS-COM) model (see chapter 2), a latent change (LC-COM) model (see chapter 3), a latent state-trait (LST-COM) model (see chapter 4) and a latent growth curve (LGC-COM) model (see chapter 5) is formally defined. The abbreviation *COM* stands for the combination of structurally different and interchangeable methods. In addition, the statistical performance of each model is investigated via four simulation studies (see Part III). According to the results of the simulation studies, the models perform well in general. Across all simula-

tion studies the amount of improper solutions (Heywood cases) as well as parameter estimate bias (peb) was below 5%. No convergence problems with respect to the H0 model were found. The average standard error bias (seb) was also below the critical cutoff value of .1 for most parameters. However, with increasing model complexity (number of parameters) larger sample sizes on both levels are needed. The results of the simulation studies are discussed and practical guidelines for empirical applications are given (see Section 11.1). Finally, the advantages and limitations of the models are discussed and an outlook on future research topics is provided.

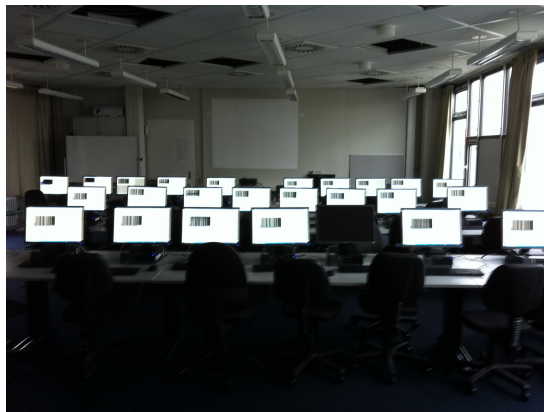
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The picture below shows 25 PCs working for me. Note the low working motivation of the PC in the first row!



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Part I

Theoretical Background

Chapter 1

Introduction

“The MTMM matrix represents one of the most important discoveries in the social and behavioral sciences.”

(Kenny, 1995, p. 123)

1.1 Multitrait-multimethod analysis

Considering the potential devastating impacts of invalid and inaccurate measurement of important aspects of human life, the need for valid measures in social and behavioral sciences can hardly be overestimated. According to Courvoisier et al. (2008) “invalid measurements bear risks like over- or underestimation of treatment effects, they may lead to the wrong diagnosis, they may indicate a suboptimal treatment, or, in the worst case, they might even not detect a relevant symptom at all” (p. 270). Therefore, many psychologists agree with the statement that any decision or diagnostic judgment should be based on the best information available, which implies that the information is valid, reliable, objective, and specific to a given problem (Courvoisier et al., 2008; Nussbeck, 2008).

Currently, one of the most common strategies to scrutinize the validity of a given measure is via multitrait-multimethod (MTMM) analysis (Eid, 2000; Eid & Nussbeck, 2009). Since its invention by Campbell and Fiske in 1959, MTMM has had an undeniable impact in psychology. As Kenny (1995) notices: “The MTMM matrix represents one of the most important discoveries in the social and behavioral sciences. It is as important an invention in the behavioral science field as the microscope is in biology and the telescope is in astronomy” (p. 123). Moreover, Sternberg (1992) states that the article by Campbell and Fiske (1959) entitled “Convergent and discriminant validation by the multitrait matrix” is one of the most influential articles in psychology (see also Eid & Nussbeck, 2009). Today, the article has been cited over 5,000 times¹ and MTMM analysis is known as a gold standard for scrutinizing the validity of a measure (Carretero-Dios, Eid, & Ruch, 2011). Before discussing the numerous advantages of MTMM analysis in greater detail, the meaning of the terms *validity* as well as *validation* shall be clarified. According to Borsboom, Mellenbergh, and van Heerden (2004) the term *validation* refers to specific activities that researchers undertake

¹Information retrieved from <http://apps.webofknowledge.com> [retrieved July, 2012]

in order to verify or disprove a test to be valid. On the other hand, the term *validity* has been conceptualized either in sense of a *property* of a given test (Borsboom et al., 2004) or in sense of the adequacy of *interpretations* of test results (Messick, 1980, 1989, 1995):

Validity is an integrated evaluative judgment of the degree to which empirical evidence and theoretical rationales support the adequacy and appropriateness of inferences and actions based on test scores or other modes of assessment. [...] Broadly speaking, then, validity is an inductive summary of both the existing evidence and the potential consequences of score interpretation and use.

(Messick, 1995, p. 13)

In contrast to that, Borsboom et al. (2004) argue that validity is rather a matter of the adequacy of the measurement, not of the interpretation of test scores:

It is our intent to convince the reader that most of the validity literature either fails to articulate the validity problem clearly or misses the point entirely. Validity is not complex, faceted, or dependent on nomological networks and social consequences of testing. It is a very basic concept and was correctly formulated, for instance, by Kelley (1927, p. 14) when he stated that a test is valid if it measures what it purports to measure.

(Borsboom et al., 2004, p. 1061)

Despite this on-going philosophical dispute, most authors of the standards of educational and psychological testing (e.g., APA, AERA & NCME, 1999) tend to agree with Messick's definition of validity (see Eid & Schmidt, in press). Different facets of validity have been proposed over the years, for example *convergent* and *discriminant* validity, *content* validity, *criterion-related* validity and *face* validity (see Campbell & Fiske, 1959; Messick, 1995; Shadish, Cook, & Campbell, 2002). The authors of the standards of educational and psychological testing consider these different facets of validity as subordinate to the main concept of construct validity (Eid & Schmidt, in press). Construct validity, in the sense of Messick (1989), refers to "the *adequacy* and *appropriateness* of *inferences* and *actions* based on test scores or other modes of assessments" (p. 13). In order to verify that the interpretations of test scores are adequate, researchers usually try to provide evidence that the hypothesized relationships between theoretical entities are linked consistently to relationships between observed entities. For example, suppose that a test X measures empathy. As a consequence, the test scores of this empathy test should also be empirically related to scores of a test measuring aggressiveness (substantial negative intercorrelations). Besides that, the test scores of test X should be empirically unrelated to test scores of another test Y which is assumed to be theoretically unrelated to empathy (e.g., food preference). As the above quote of Borsboom et al. (2004) indicates, researchers usually try to scrutinize these hypothesized (inter)relationships among theoretical and observed entities with a nomological net (see Figure 1.1). Cronbach and Meehl (1955) were the first to propose this validation method. However, both researchers also emphasized that test validation is an *on-going process* which involves numerous kinds of studies,

for instance (a) studies of group differences, (b) studies of interrelationships between other tests, (c) studies of the internal structure (e.g., factor analysis), (d) studies of stability and change of test scores, and (e) process analysis (for more details see Eid & Schmidt, in press). Nevertheless, the approach by Cronbach and Meehl (1955) implies some shortcomings. First, Cronbach and Meehl (1955) did not explicitly stress the advantages of multimethod measurement and how to properly study method bias. For instance, a self-report measure of empathy may be more positively biased than warranted. It would be impossible to investigate the degree of method biases (e.g., bias due to self-reports) in measurement designs when using only one method (Geiser, 2008). Another methodological shortcoming refers to the link between the theoretical and observation entities (i.e., adequacy of measurement) depicted in Figure 1.1. In particular, it is unclear how to statistically test whether the link between the theoretical and empirical entities is correct. Moreover, it is unclear how to separate different sources of variances (e.g., measurement-error from true-score variance) from one another in the classical approach. In contrast to Cronbach and Meehl (1955), Campbell and Fiske (1959) highlighted the importance of *multimethod* investigations in the social and behavioral sciences. They suggested using multitrait-multimethod correlation matrices to investigate the convergent and discriminant validity of the given measures. Convergent validity is indicated by high positive correlations of test scores of two different scales that are theoretically related (e.g., two different empathy scales). Discriminant validity is indicated if two theoretically unrelated attributes are also empirically unrelated. For instance, only low or no associations between test scores of intelligence and empathy scales are assumed, given that both constructs are considered to be distinct.

One of the main reasons for using multiple methods is to disentangle different sources of influence such as effects of construct score influences, rater influences, measurement-error influences, and/or temporal influences (Courvoisier et al., 2008; Kenny, 1995). According to Campbell and Fiske (1959) at least two constructs and two methods are needed to separate trait from method effects. It is important to note that the term *method* is not clearly defined in psychology (see Geiser, 2008; Podsakoff, MacKenzie, Lee, & Podsakoff, 2003). For instance, different tests (e.g. speeded test vs. non-speeded test), different types of assessment (e.g., questionnaire vs. physiological measures), different items (e.g., positive vs. negative coded items), different persons (e.g. multiple raters vs. expert ratings), or different measurement occasions can be conceived as method (Eid & Nussbeck, 2009; Geiser, 2008).

Another advantage of multimethod measurement designs is that they are more informative than single-method designs (Geiser, 2008). Moreover, multimethod measurement designs allow examining the degree of *method specificity* (e.g., method or rater biases) as well as the *generalizability* of these method effects across constructs (Eid, 2006; Geiser, 2008). Method specificity refers to the amount of observed or true score variance that is due to method influences (see Eid, 1995). For example, method specificity may be reflected by the amount of observed variance of self-report measures that cannot be predicted by other reports (e.g., parent report) (Eid, 2000; Geiser, 2008). MTMM measurement designs allow separating different variance components from one another

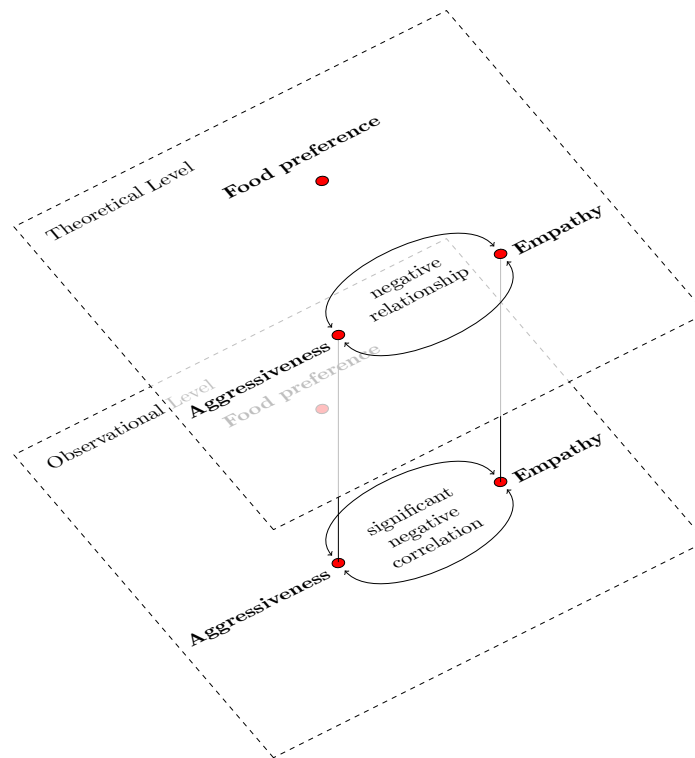


Figure 1.1: Nomological network for validating a new empathy scale following Cronbach and Meehl (1955). Nodes represent theoretical or empirical quantities in the nomological network. Double arrows represent probabilistic or deterministic relationships among the theoretical or empirical quantities. Lines without any arrows reflect the operationalization or the measurement of the theoretical quantities.

and investigating construct² and method influences (e.g., method bias). In the classical MTMM approach by Campbell and Fiske (1959) four criteria for the evaluation of multitrait-multimethod correlation matrix are proposed (see also Eid, 2010, pp. 851-852):

1. The entries in the validity diagonal referring to correlations between the same constructs measured by different methods (i.e., monotrait-heteromethod block) should be significantly different from zero and sufficiently large. This desideratum concerns the degree of convergent validity.
2. Correlations between the same constructs measured by different methods (i.e., monotrait-heteromethod block) should be higher than the correlations between different constructs measured by different methods (i.e., heterotrait-heteromethod block). This desideratum concerns the degree of discriminant validity.
3. Similarly, correlations between different constructs measured by the same method (i.e., heterotrait-monomethod block) should be smaller than correlations between the same constructs measured by different methods (i.e., monotrait-heteromethod block). This desideratum also concerns the degree of discriminant validity.
4. Finally, “the same or a similar pattern of constructs should be shown in all of the heterotrait

²Following Geiser (2008) a distinction between traits and constructs is made. Throughout the entire thesis, the term trait is used to refer to stable person-specific influences that can be separated from occasion-specific influences (Steyer, Ferring, & Schmitt, 1992; Steyer, Schmitt, & Eid, 1999; Eid, 1995). The term construct is used to refer to the attributes (e.g., teaching quality, life satisfaction etc.) that were measured.

triangles of both the monomethod and the heteromethod blocks” (Eid, 2010, pp. 851-852). This desideratum also concerns the degree of discriminant validity, given that it implies that the associations between different constructs are similar across different methods as well as method combinations.

Despite the numerous advantages of the classical approach by Campbell and Fiske (1959) there are also some limitations. According to Eid (2010) “the application of these criteria is difficult if the measures differ in their reliabilities” (p. 852). That is, because the correlations between the different measures can be distorted in different ways due to measurement error influences. Furthermore, there is no statistical test whether or not the criteria are fulfilled in empirical applications or whether or not the psychometric model as such fits the data (Eid, 2010). Moreover, Campbell and Fiske (1959) did not explicate how to account for temporal effects which are present in almost any measurement. That is, measurement almost never takes place in a situational vacuum (see Steyer et al., 1999). Hence, many important questions cannot be answered with the classical MTMM approach:

- How strong is the influence of the measurement error?
- How adequate is the measurement model (the link between theoretical and observed entities)?
- Do the psychometric properties of the instrument change over time?
- Does the construct change over time?
- Does the method bias change over time?

With the development of more sophisticated statistical methods such as confirmatory factor analysis (CFA) and structural equation models [SEMs, see e.g., Bollen (1989), Jöreskog (1979)] many of these problems could be resolved. Over the years, CFA and SEM modeling approaches for modeling MTMM data have been increasingly applied to social and behavioral data (e.g., Dumenci, 2000; Eid, 2000; Eid & Diener, 2006). The main advantages of MTMM-SEMs are (Dumenci, 2000; Eid, Lischetzke, & Nussbeck, 2006; Eid et al., 2008): (a) they allow separating measurement influences from individual difference with respect to construct or method effects, (b) they allow relating different construct or method variables to other external variables, and (c) they allow scrutinizing the fit of the statistical model (e.g., with χ^2 fit statistics). Several SEM-based MTMM models have been proposed over the years (Eid, 2000; Eid, Lischetzke, Nussbeck, & Trierweiler, 2003; Eid et al., 2008; Kenny & Kashy, 1992; Marsh & Hocevar, 1988; Marsh, 1993; Marsh & Grayson, 1994; Pohl & Steyer, 2010; Widaman, 1985; Wothke, 1995). For an overview and detailed discussion of existing MTMM-SEMs see Eid et al. (2006) as well as Geiser (2008). In the next section, important developments of MTMM-SEMs for longitudinal measurement designs are discussed.

1.2 Multitrait-multimethod-multioccasion analysis

Change is an inevitable feature of human life. People think, feel, and/or behave differently over time. Therefore it is not surprising that social and behavioral scientists share great interest in

studying the change or stability of attributes (e.g., empathy), method effects (e.g., rater bias), and psychometric properties (e.g., reliability, convergent and discriminant validity). The importance of longitudinal measurement designs is also reflected in the increasing number of publications devoted to this research area (Geiser, 2008). For instance, Khoo, West, Wu, and Kwok (2006) note that 32% of studies published in *Developmental Psychology* in 2002 included longitudinal measurement designs. Biesanz, West, and Kwok (2003) found that 24% of studies published in the *Journal of Personality: Personality Process and Individual Differences* from 2000 to 2001 included longitudinal measurement designs. Moreover, longitudinal MTMM studies are more informative than cross-sectional MTMM studies (Geiser, 2008). Only with respect to longitudinal MTMM analysis is it possible to explicitly model temporal effects. Longitudinal MTMM analyses allow testing crucial assumptions referring to measurement invariance or the existence of indicator-specific effects. These assumptions cannot be tested with respect to cross-sectional MTMM study designs. Despite the numerous advantages of longitudinal modeling, only “few attempts have been made to develop and use appropriate models for longitudinal MTMM data so far” (Geiser, 2008, p. 19). Example of researchers who have contributed to this research field are Burns et al. (2003), Burns and Haynes (2006), Cole and Maxwell (2003), Courvoisier (2006), Courvoisier et al. (2008), Geiser (2008), Geiser et al. (2010), Grimm et al. (2009), Scherpenzeel and Saris (2007). The work by Courvoisier (2006) and Geiser (2008) is essential for the understanding of the models presented in this thesis. Therefore, the models by these authors are discussed in greater detail here.

Geiser (2008) proposed a latent state and a latent change model for longitudinal MTMM designs (see Figure 1.2 and Figure 1.3). Geiser’s model³ is an extension of the multiple indicator CTC($M-1$) model for cross-sectional data proposed by Eid et al. (2003). The starting point of the latent state model is the decomposition of the observed scores into latent state S_{ijkl} as well as error variables E_{ijkl} , where i = indicator, j = construct, k = method, and l = occasion of measurement:

$$Y_{ijkl} = S_{ijkl} + E_{ijkl}.$$

Next, a reference (standard) method is chosen in order to contrast different methods from another (see Eid, 2000; Eid et al., 2003; Geiser, 2008). Without loss of generality, the reference method is denoted by $k = 1$. The remaining methods ($k \neq 1$) serve as non-reference methods. Geiser, Eid, and Nussbeck (2008) provide detailed guidelines for choosing an appropriate reference method. In order to define method variables that reflect the amount of observed variance of a non-reference indicator (e.g., teacher report) that is not due to the reference method (e.g., student self-report), it is necessary to regress the latent state variables pertaining to the non-reference methods S_{ijkl} on the latent state variables pertaining to the reference method S_{ij1l} . In other words, the latent state variables of the reference method are used as predictors of the latent state

³For the sake of clarity, the models in Figure 1.2 and Figure 1.3 are depicted with common latent state (change) factors S_{j1l} (S_{j1l}^{BC}) and with common latent method (change) factors M_{jkl} (M_{jkl}^{BC}), and not with indicator-specific latent factors S_{ij1l} , S_{ij1l}^{BC} , M_{ijkl} , and M_{ijkl}^{BC} . That means that the index i denoting the indicator (item) is not needed in these figures, and is therefore dropped. However, it is worth noting that Geiser (2008) introduced the model in his original work with indicator-specific latent variables (i.e., S_{ij1l} , S_{ij1l}^{BC} , M_{ijkl} , and M_{ijkl}^{BC}), and then introduced homogeneous assumptions with respect to these latent variables.

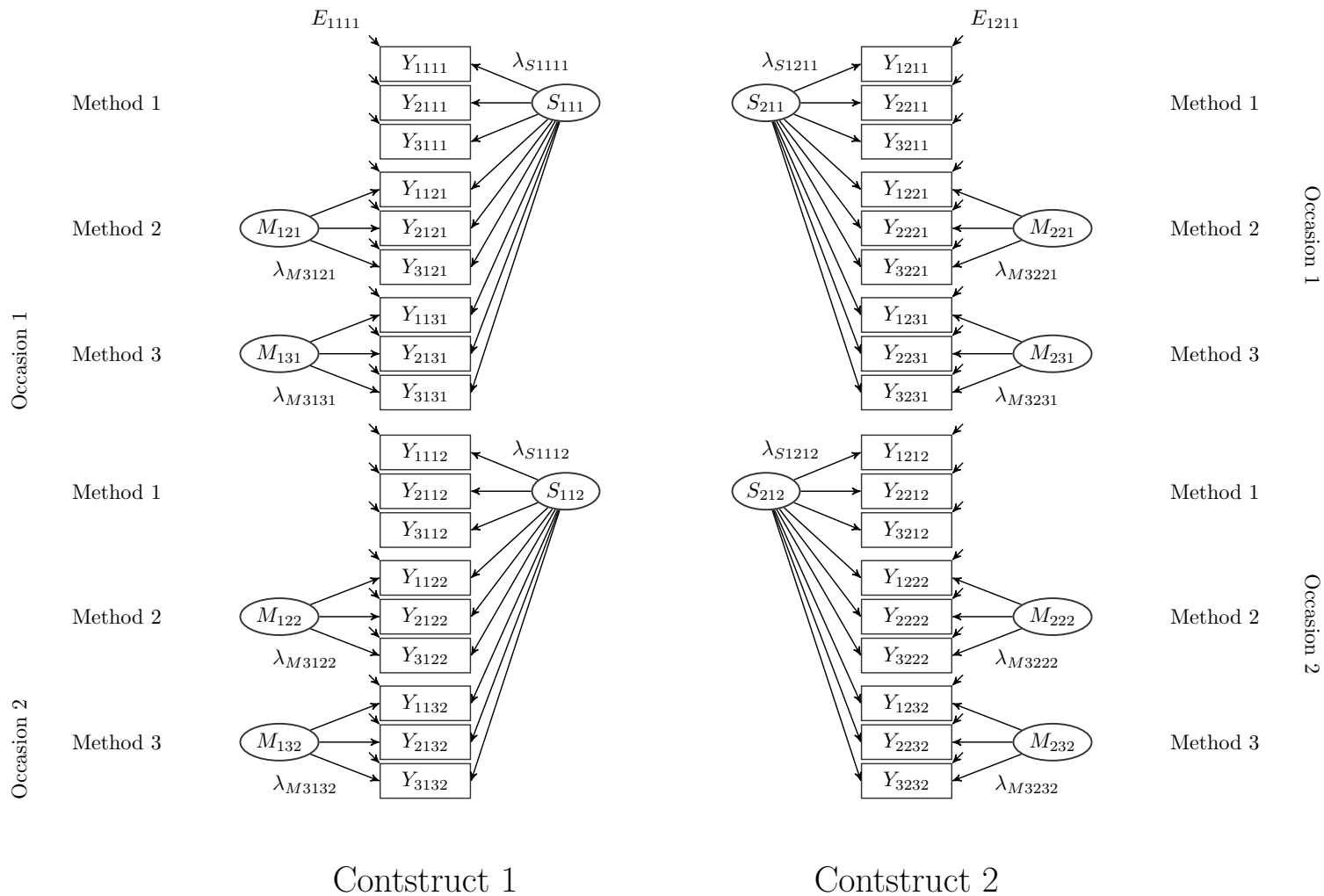


Figure 1.2: Path diagram of the CS-C($M - 1$) model by Geiser (2008) for three indicators, two constructs, three methods, and two occasions of measurement. Y_{ijkl} = observed variable (i = indicator, j = construct, k = method, l = occasion of measurement). S_{j1l} = common latent state factor. M_{jkl} = common latent method factor. E_{ijkl} = error variable. All latent correlations are omitted for clarity.

variables of the non-reference methods. Generally, this latent regression can be expressed by the following equation:

$$E(S_{ijk}|S_{ij1}) = \alpha_{ijk} + \lambda_{S_{ij1}}S_{ij1}, \quad \forall k \neq 1.$$

The residuals of this regression are the latent method variables M_{ijkl} . These latent method variables reflect the over- or underestimation of the non-reference method (e.g., teacher report) with respect to the reference method (e.g., self-report) at a given occasion of measurement. Given that these latent method variables are defined as residuals, the general properties of residuals hold as well (Geiser, 2008; Steyer & Eid, 2001), for example:

$$\begin{aligned} E(M_{ijkl}) &= 0, \\ Cov(S_{ij1}, M_{ijkl}) &= 0. \end{aligned}$$

In addition to that, the latent method variables M_{ijkl} are assumed to be homogeneous across items for the same construct, method, and measurement occasion. More specifically, all M_{ijkl} are linear functions of each other:

$$M_{ijkl} = \lambda_{M_{ijkl}}M_{jk1}.$$

The complete measurement equation of an observed indicator in the latent state MTMM-SEM by Geiser (2008) can be expressed as follows:

$$Y_{ijkl} = \begin{cases} \alpha_{ij1} + \lambda_{S_{ij1}}S_{ij1} + E_{ij1}, & \forall k = 1, \\ \alpha_{ijk} + \lambda_{S_{ij1}}S_{ij1} + \lambda_{M_{ijkl}}M_{jk1} + E_{ijk}, & \forall k \neq 1. \end{cases}$$

With respect to an alternative parametrization of the model, it is possible to study true interindividual differences in intraindividual change of constructs as well as method effects (see Geiser, 2008; Geiser et al., 2010). The measurement equation above may be rewritten as follows:

$$Y_{ijkl} = \begin{cases} \alpha_{ij1} + \lambda_{S_{ij1}}[S_{ij1} + (S_{ij1} - S_{ij1})] + E_{ij1}, & \forall k = 1, \\ \alpha_{ijk} + \lambda_{S_{ij1}}[S_{ij1} + (S_{ij1} - S_{ij1})] + \lambda_{M_{ijkl}}[M_{jk1} + (M_{jk1} - M_{jk1})] + E_{ijk}, & \forall k \neq 1. \end{cases}$$

The latent difference variables ($S_{ij1} - S_{ij1}$) represent interindividual differences in intraindividual change with respect to the reference method. In contrast, the latent difference variables ($M_{jk1} - M_{jk1}$) reflect the interindividual differences in intraindividual change with respect to method-specific deviations from the reference method. These latent difference variables may serve as explanatory or dependent variables in further latent regression analysis. For example, a researcher might incorporate potential covariates (e.g., gender, age) in order to explain the deviation of change scores from the reference method. Besides that, Geiser (2008) proposed a model in order to study the true change of subsequent latent variables (so called neighbor change models) as well as models for investigating time-invariant item-specific effects. The major advantage of these models lies in the combination of longitudinal measurement theories such as the latent state/change theory (Steyer, 1988; Steyer et al., 1992, 1999) and the Correlated-Trait-Correlated-(Method-1) modeling

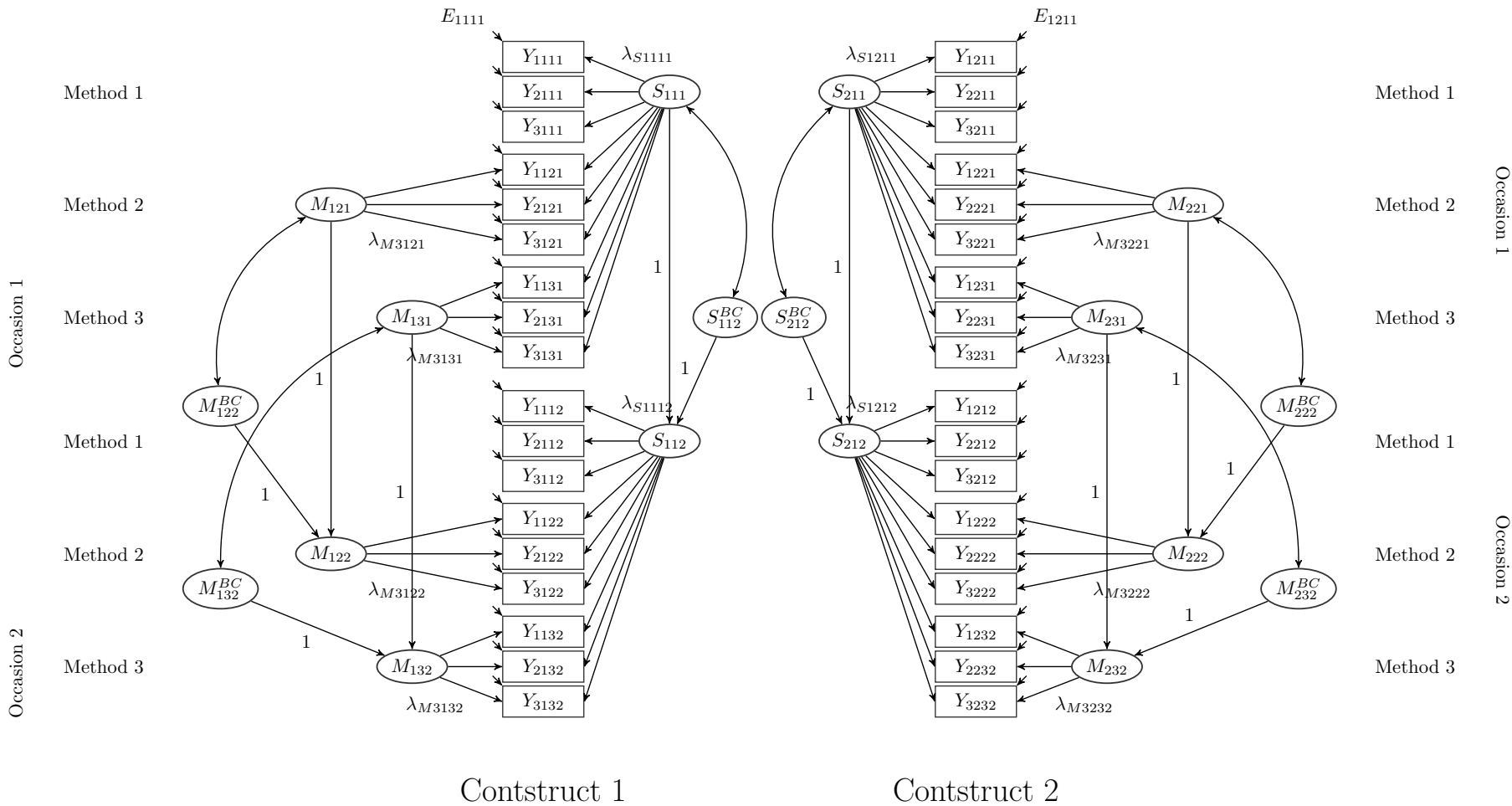


Figure 1.3: Path diagram of the change version of the CS-C(M-1) model by Geiser (2008) for three indicators, two constructs, three methods, and two occasions of measurement. Y_{ijkl} = observed variable (i = indicator, j = construct, k = method, l = occasion of measurement). S_{j1l} = common latent state factor. S_{j1l}^{BC} = common change factor for the reference method from measurement occasion 1 to l (i.e., superscript BC = baseline change). M_{jkl} = common method factor. M_{jkl}^{BC} = common change method factor for the non-reference method from measurement occasion 1 to l (i.e., superscript BC = baseline change). E_{ijkl} = error variable. All latent correlations are omitted for clarity.

approach (e.g., Eid, 2000; Eid et al., 2003). First, all of these models enable researchers to analyze the entire MTMM-MO (multitrait-multimethod-multioccasion) matrix, whereas previous attempts generally focused on modeling the occasion-specific MTMM covariance matrices. Temporal effects such as the stability and change of constructs as well as method effects can be fully investigated. Second, all of the models account for measurement error influences and thus allow studying *true* convergent and discriminant validity of the given measures. Third, crucial assumptions (e.g. the degree of measurement invariance assumptions or the existence of indicator-specific effects) can be tested via standard model fit statistics. Fourth, in the CTC(M-1) modeling approach (see Eid, 2000), different components of variances can be separated. For example, the models allow calculating coefficients of *occasion-specific consistency*, *method specificity*, *indicator-specificity*, and *reliability* (for more details see Geiser, 2008). Fifth, essential psychometric properties regarding the existence, uniqueness, admissible transformations, and meaningfulness of the latent variables have been demonstrated. Moreover, the limits of the applicability of the models have been scrutinized by extensive simulation studies (e.g., Crayen, 2008; Geiser, 2008) as well as by empirical applications (Crayen et al., 2011; Geiser et al., 2010).

The multitrait-multimethod latent state-trait (MM-LST) model by Courvoisier (2006) combines the advantages of latent state-trait theory (Steyer et al., 1999) and the CTC(M-1) modeling approach (Eid, 2000; Eid et al., 2003). This model is especially useful when researchers seek to analyze true discriminant and convergent validity on the level of *occasion-specific* variables (i.e., measures depending on momentary or situational effects) as well as on the level of *trait* variables (i.e., free of situational and measurement error effects). According to the LST theory, the latent state variables S_{ijkl} can be further decomposed into latent trait T_{ijkl} and latent state-residual variables O_{ijkl} . The observed variables are therefore given by:

$$Y_{ijkl} = \begin{cases} S_{ijkl} + E_{ijkl}, \\ T_{ijkl} + O_{ijkl} + E_{ijkl}. \end{cases}$$

T_{ijkl} is the latent trait variable and reflects person-specific influences (i.e., consistent person-specific effects across time). O_{ijkl} represents effects of the situations and/or person-situation-interactions. E_{ijkl} is the measurement error. In order to derive trait and occasion-specific method variables, one has to choose a reference (standard) method and regress the trait/occasion-specific variables of the non-reference method on the trait/occasion-specific variables of the reference method. The latent residuals (TM_{ijkl} , OM_{ijkl}) of these latent regressions can be interpreted as trait or occasion-specific method variables.

$$\begin{aligned} T_{ijkl} &= E(T_{ijkl}|T_{ij1l}) + TM_{ijkl}, \\ O_{ijkl} &= E(O_{ijkl}|O_{ij1l}) + OM_{ijkl}. \end{aligned}$$

Given that TM_{ijkl} and OM_{ijkl} are defined as residuals with respect to T_{ij1l} and O_{ij1l} , the general properties of residuals apply again (for more details see Courvoisier, 2006). The complete equation

for the observed variables is given by:

$$Y_{ijkl} = \begin{cases} = \alpha_{ij1l} + \lambda_{T_{ij1l}} T_{ij} + \lambda_{O_{ij1l}} O_{ijl} + E_{ij1l}, & \forall k = 1, \\ = \alpha_{ijkl} + \lambda_{T_{ij1l}} T_{ij} + \lambda_{TM_{ijkl}} TM_{ijk} + \lambda_{O_{ij1l}} O_{ijl} + \lambda_{OM_{ijkl}} OM_{jkl} + E_{ijkl}, & \forall k \neq 1. \end{cases}$$

The latent trait variable T_{ij} reflects person-specific influences measured by the reference method, with the intercept α_{ijkl} and factor loading $\lambda_{T_{ij1l}}$. The residual O_{ijl} represents effects of the situations and/or person-situation-interactions measured by the reference method. The parameter $\lambda_{O_{ij1l}}$ denotes the factor loadings for this latent factor. By definition the residual variable O_{ijl} is uncorrelated with the latent trait variable T_{ij} . The latent trait method variable TM_{ijk} represents the method-specific influence of method k on the trait level weighted by its factor loading $\lambda_{TM_{ijkl}}$. The variance of this variables reflects the amount of person-specific variance of an observed variable pertaining to the non-reference method which is not explained by the latent trait variable measured by the reference method. Similarly, OM_{jkl} is the latent residual and represents the method-specific influences of method k on the occasion level, weighted by its factor loading $\lambda_{OM_{ijkl}}$. With regard to the definition of the model, different variance components can be studied (see Courvoisier et al., 2008, pp. 274-275):

1. The trait consistency coefficient $TCon(Y_{ijkl})$: This coefficient represents the proportion of observed variance due to stable interindividual differences (i.e., general trait level) as measured by the reference method.
2. The occasion-specificity coefficient $OSpe(Y_{ijkl})$: This coefficient represents the proportion of observed variance due to occasion-specific interindividual differences (i.e., the momentaneous oscillation around the stable trait) as measured by the reference method.
3. The trait-specific method coefficient $TMSpe(Y_{ijkl})$: This coefficient represents the part of the observed variable that is uniquely due to the method deviation from the reference trait.
4. The occasion-specific method coefficient $OMSpe(Y_{ijkl})$: This coefficient represents the part of the observed variable that is uniquely due to the method deviation from the occasion-specific reference state residual variable.

Researchers who are interested in studying the degree of true convergent validity on the level of trait variables need to compare the consistency coefficient $Con(Y_{ijkl})$ and the trait-specific method coefficient $TMSpe(Y_{ijkl})$. Conversely, researchers who are interested in analyzing the degree of convergent validity on the level of occasion-specific residual variables need to compare the occasion-specificity coefficient $OSpe(Y_{ijkl})$ and the occasion-specific method coefficient $OMSpe(Y_{ijkl})$.

The modeling approach by Courvoisier (2006) entails many advantages for studying longitudinal MTMM data. For example, the MM-LST model is especially useful for studying variability processes (i.e., occasion-specific oscillation around a time-invariant trait or method effects). Besides, the latent variables of the model are clearly defined and have a clear psychometric meaning.

Despite the numerous advantages of the models by Geiser (2008) as well as Courvoisier (2006), the models are only adequate for analyzing MTMM measurement designs incorporating structurally

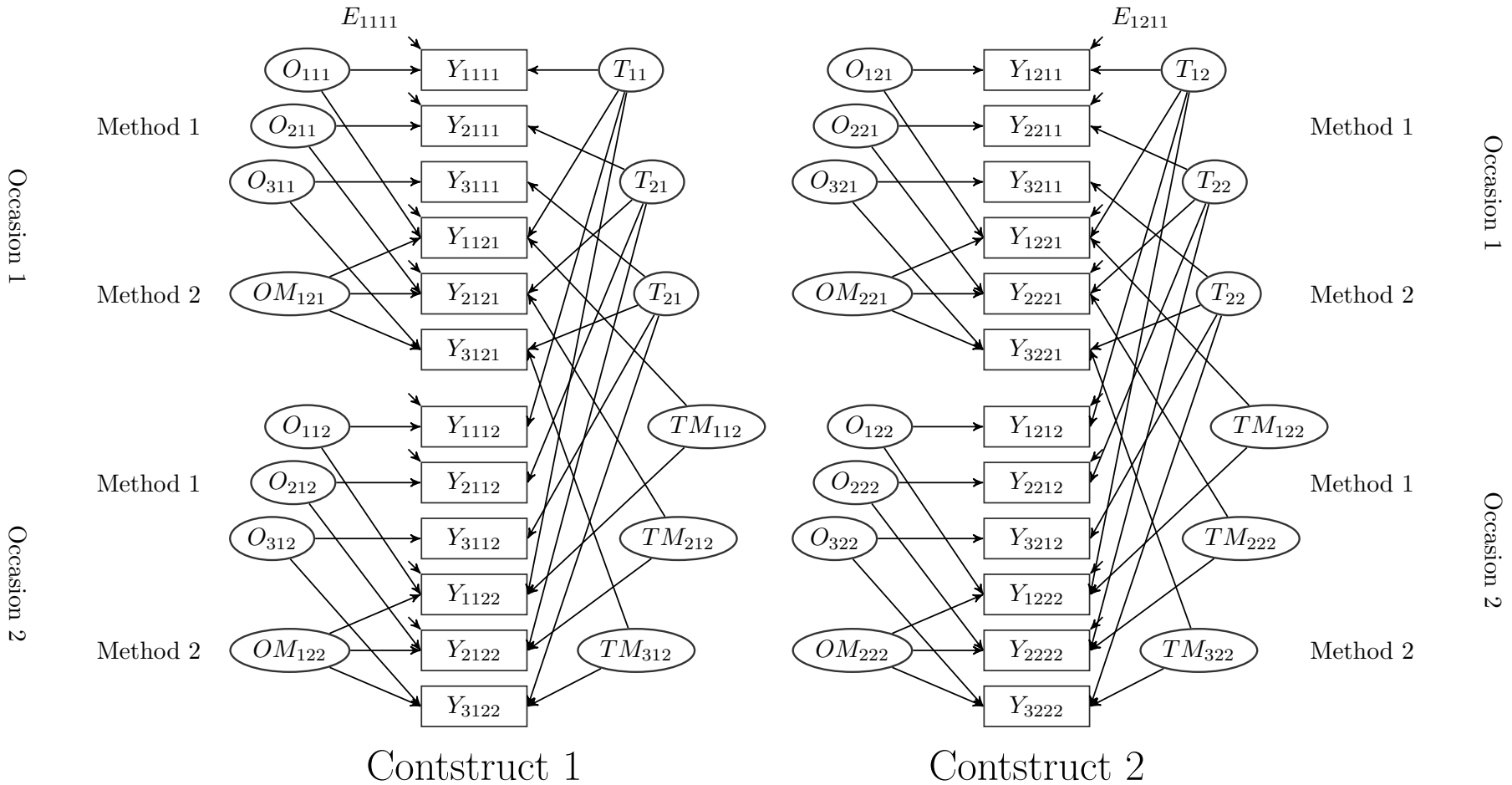


Figure 1.4: Path diagram of the MTMM-LST model by Courvoisier (2006) for two indicators, two constructs, two methods, and two occasions of measurement. Y_{ijkl} = observed variable (i = indicator, j = construct, k = method, l = occasion of measurement). T_{ij} = latent trait factor. TM_{ijk} = trait method factor. O_{ijl} = occasion-specific factor, OM_{jkl} = occasion and method-specific factor, E_{ijkl} = error variable. All latent correlations and all factor loadings are omitted for clarity.

different methods (e.g., student self-report, parent report, physiological measures). However, many researchers conduct longitudinal MTMM studies using a combination of structurally different and interchangeable methods (see e.g., Ciarrochi & Heaven, 2009; Dai, De Meuse, & Peterson, 2010; Denissen, Schönbrodt, van Zalk, Meeus, & van Aken, 2011; Ho, 2010; Violato, Lockyer, & Fidler, 2008). Especially in organizational and industrial psychology studies, it is common to use a combination of structurally different (e.g., employee's self-report, supervisor report) and interchangeable methods (e.g., colleagues reports, customers reports), for example, in so-called 360 degree feedback designs (Mahlke et al., 2012). A detailed explanation of the meaning of the terms "structurally different" and "interchangeable" methods will be given in the next section. However, a very simplistic interpretation of the terms shall be given now in order to understand why the models by Geiser (2008) as well as Courvoisier (2006) are not appropriate in general for these types of measurement designs: Interchangeable methods such as multiple colleagues (or customers) ratings for employees' social competencies are more or less exchangeable, given that these ratings stem out of a uniform rater population (see also Eid et al., 2008; Mahlke et al., 2012). In contrast, structurally different methods (e.g., employee's self-report and supervisor report) cannot easily be replaced by one another, given that these methods stem out of different method (e.g., rater) populations. In addition, whereas interchangeable methods can be selected randomly for a particular target, structurally different methods are fixed beforehand for a particular target (see Eid et al., 2008). As a consequence of this sampling process, MTMM measurement designs with a combination of structurally different and interchangeable methods imply a multilevel data structure (Eid et al., 2008). Note that the models by Geiser (2008) as well as Courvoisier (2006) are defined as single level structural equation models. Thus, these models are not in general appropriate for measurement designs with interchangeable methods (Eid et al., 2008; Geiser, 2008; Nussbeck, Eid, Geiser, Courvoisier, & Lischetzke, 2009). Only under certain circumstances, for example, when a small and equal number of raters per target is used, the models by Geiser (2008) as well as Courvoisier (2006) can also be used for analyzing measurement designs with interchangeable methods (see Nussbeck et al., 2009). Nevertheless, models that are more general and flexible than the existing models are needed for analyzing longitudinal MTMM measurement designs with a combination of structurally different and interchangeable methods. In the next section, the differences between measurement designs with structurally different and/or interchangeable methods are discussed in more detail.

1.3 Different models for different types of methods

Some models are more appropriate for particular data structures than others. The simplest explanation may be that a given model A fits the data better than an alternative model B. Comparing different models (e.g., A and B) according to the standard fit criteria is presumably the most popular model selection strategy nowadays. However, such data driven model selection strategies (e.g. testing all available models) may not always be the best solution. As Eid et al. (2008) noted, data driven model selection strategies are highly arbitrary and may even "increase the likelihood of improper solutions, convergence problems, and poor model fit" (p. 231). Even if a particular model fits the data well, the model results may not be readily interpretable (see Eid et al., 2008).

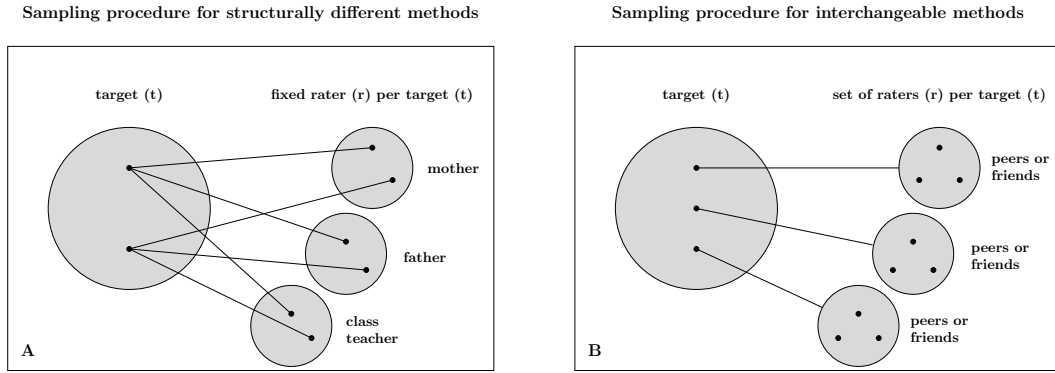


Figure 1.5: Sampling procedure for different kinds of methods. Figure A refers to the sampling procedure of measurement designs with structurally different (fixed) methods. Figure B refers to the sampling procedure of measurement designs with interchangeable (random) methods.

Moreover, in cases of equivalent models (e.g., latent state vs. latent change model), the model fit criteria cannot be used for model comparisons. For that reason, Eid et al. (2008) strongly recommended using theory-driven model selection strategies in MTMM studies. According to Eid et al. (2008) different types of methods require different types of MTMM models. In total, three types of measurement designs were distinguished by Eid et al. (2008): (1) measurement designs with structurally different methods, (2) measurement designs with interchangeable methods, and (3) measurement designs with a combination of structurally different and interchangeable methods. Measurement designs with structurally different methods use methods that are fixed beforehand for a given target. For example, the rating of a mother is fixed for a given child. In other words, structurally different methods cannot be randomly selected out of a common set of methods. The sampling procedure of measurement designs with structurally different methods requires selecting a target t out of a set of targets T and then observe the ratings of the different raters ($r = 1, 2, 3$) on indicators i for constructs j (see Figure A of 1.5). The simplest case of the random experiment for measurement designs with three structurally different methods can be represented as follows:

$$\Omega = \Omega_T \times \Omega_{1ij} \times \dots \times \Omega_{2ij} \times \dots \times \Omega_{3ij}.$$

Ω_T refers to the possible set of targets, Ω_{1ij} represents the first possible set of ratings (e.g., self ratings) on indicator i and construct j , Ω_{2ij} refers to the second possible set of ratings (e.g., mother ratings for each child) on indicator i and construct j and Ω_{3ij} refers to the third possible set of structurally different methods (e.g., objective test scores for each child) with respect to indicator i and construct j . Finally, \times is the Cartesian product set operator. Note that all methods (self-rating, mother rating, objective test) may reflect different perspectives on the target (child), given that all methods stem out of different non-interchangeable method populations. Researchers who are interested in analyzing “pure” method effects (i.e., method effects that are unrelated with the trait and only due to the influences of a particular method) should *not* aggregate both ratings by taking the mean of both ratings. Instead, researchers should rather contrast different methods against each other which is automatically done in the Correlated-Tait-Correlated-(Method)-1 modeling approach (Eid, 2000; Eid et al., 2003).

A number of MTMM-SEMs have been proposed for analyzing longitudinal as well as cross-sectional measurement designs with structurally different methods (Courvoisier, 2006; Courvoisier et al., 2008; Crayen et al., 2011; Geiser, 2008; Geiser et al., 2010; Eid, 2000; Eid et al., 2003). These models offer many advantages. However, they are not appropriate for analyzing measurement designs with interchangeable methods. Measurement designs with interchangeable methods imply a different data structure than measurement designs with structurally different methods. Measurement designs with interchangeable methods use methods that can be randomly sampled out of a uniform distribution of methods. For example, a study with interchangeable methods may use multiple peer ratings for the evaluation of students social competency or multiple colleague ratings for leadership quality. Given that peers per student (or colleagues per employee) stem out of the same rater population, these methods (ratings) can be conceived as interchangeable. It is important to note that interchangeability does not mean that the values of these ratings are the same across different raters per target, but rather that the raters pertain to the same rater pool. In addition, measurement designs with interchangeable methods imply a multistage sampling procedure (see Figure B of 1.5). In the first step the target t is selected out of a set of possible targets T and in the second step the different raters r are selected out of a set of possible target-specific raters R . Finally, the rating of each rater r of a particular target t on indicator i and construct j is observed. The simplest case of the random experiment for interchangeable methods can be represented as follows:

$$\Omega = \Omega_T \times \Omega_R \times \Omega_{ijk}.$$

Ω_T refers to the set of possible targets and Ω_R is the set of possible raters for each target. Ω_{ijk} is the set of possible outcomes that contains the values of ratings for each indicator i and each construct j . That means that Ω_{ijk} maps the ratings of each raters for each target on indicator i and for construct j into the set of real numbers \mathbb{R} . The conceptual distinction between structurally different methods and interchangeable methods relates to the distinction between fixed and random factors in analysis of variance (Hays, 1994) and is well summarized by Shrout (1995) saying: “The nature of the description of interrater consensus often varies according to the research perspective. In formal methodological terms, the different perspectives may vary in terms of whether targets and/or judges are considered to be fixed or random” (p. 82). Eid et al. (2008) proposed a multilevel MTMM-SEM for measurement designs with interchangeable methods. The major advantage of this multilevel modeling strategy is its flexibility. For example, the model allows to study “true” unique rater bias for a *varying* number of raters per target. In contrast to the CTC($M-1$) approach for structurally different methods (see above), the latent trait variables are defined as latent means in this framework. The “true” (measurement error free) average mean of the interchangeable methods (peer ratings) is defined as trait (see Eid et al., 2008). As a consequence, the trait values are free of measurement error influences and rater-specific influences (see Eid et al., 2008). The unique rater bias is then given by the deviation of a particular rater’s true-score from this true average mean (see Eid et al., 2008). Finally, “true” convergent and discriminant validity can be studied as well. Given that many empirical studies use a combination of both structurally different and interchangeable methods, Eid et al. (2008) formulated a model that combines both

modeling approaches for cross-sectional data. With respect to the extended or multilevel CTC($M-1$) model (Eid et al., 2008; Carretero-Dios et al., 2011) it is possible investigating trait effects as well as “unique” and “common” method effects. The main advantage of this model is that method influences can be studied on both levels (rater- and target-level). In addition, researchers may also study different variance components (Eid et al., 2008, p. 245):

- The consistency coefficient indicates the degree of convergent validity. The consistency coefficient reflects the amount of true variance of a non-reference indicator that is explained by the reference method (e.g., the amount of true variance of the peer ratings that is shared with the self-rating).
- The common method specificity coefficient represents the amount of true variance of a non-reference indicator that is not explained by the reference method (e.g., self-rating) but that is common to all interchangeable methods (e.g., peer-ratings)
- The unique method specificity coefficient reflects the amount of true variance of a non-reference indicator that is neither shared with the reference method (e.g., self-rating) nor shared with other interchangeable methods (e.g., peers) but that is specific to a particular rater. Hence, this coefficient reflects the variability that is only due to the unique view of interchangeable raters.

Even though Eid et al. (2008) presented a basic framework for modeling both structurally different and interchangeable methods, they did not consider an extension to longitudinal data. The aim of this thesis is to formally define different multilevel CTC($M-1$) models for longitudinal MTMM measurement designs with structurally different and interchangeable methods. These new MTMM-SEMs combine the advantages and flexibilities of the CTC($M-1$) modeling approach, of longitudinal modeling strategies (i.e., latent state, latent change, latent state-trait, and latent growth curve modeling), and of multilevel (i.e., multirater) modeling strategies. Finally, all of the presented models will be tested empirically with respect to extensive Monte Carlo simulation studies.

1.4 Aims and structure of the present work

The aim of this present work is to develop appropriate structural equation models for longitudinal MTMM measurement designs that imply a combination of structurally different and interchangeable methods. Given that many longitudinal MTMM measurement designs do not only incorporate one type of method (e.g., either structurally different or interchangeable methods), but a combination of different types of methods, there is a great need for such longitudinal MTMM-SEMs. For example, in educational psychology many researchers are interested in studying teaching quality by using multiple student ratings, teacher self ratings and/or the ratings of school principals. In development psychology, researchers analyze the development of social competency by using students’ self-ratings, peer ratings, and parent ratings. Furthermore, many organizational psychologists use multiple source (360 degree) feedback designs in order to investigate the stability and change of leadership quality. All of these measurement designs imply a combination of structurally different and interchangeable methods. To my knowledge, no appropriate models have yet

been proposed for analyzing such measurement designs. The aim of this present work is to fill this gap and to develop models that are appropriate for such complex measurement designs. Additionally, the models will be formulated with respect to the four longitudinal modeling frameworks (i.e., latent state, latent change, latent state-trait, and latent growth curve framework). The latent state (LS-COM) model is a good starting point for modeling change over time. This model often serves as a baseline model for testing and establishing crucial assumptions such as the degree of measurement invariance (Geiser, 2008, 2012). Moreover, the stability and change of trait and method effects can be studied by analyzing the correlations between latent variables. In order to explicitly model “true” change of trait and/or method effects, two latent change versions are formulated. The first latent baseline change model (baseline LC-COM model) allows modeling “true” interindividual differences in intraindividual change with respect to the reference method. Hence, this model is useful whenever researchers are solely interested in modeling “true” change of the construct over time. In addition, an extended latent baseline change model (extended baseline LC-COM model) is proposed that enables researchers studying “true” change of (common/unique) methods effects over time. The third model combines the advantages of latent state-trait theory and the MTMM-MO modeling framework for different types of methods. This model is called LST-COM model and is useful for studying “true” convergent and discriminant validity on the level of occasion-specific variables and on the level of trait (free of occasion-specific influence and measurement error) variables. This model is especially appropriate for investigating variability processes (i.e., occasion-specific oscillation around a time-invariant trait). The fourth model (LGC-COM model) can be seen as a more “general” variant of the LST-COM model that entails the LST-COM model as a special case. The LGC-COM model is suitable for modeling different forms of growth with respect to the trait (linear, quadratic, cubic) separately from occasion-specific, method-specific and measurement error influences. All of the models will be formulated based on the stochastic measurement theory (c.f. Steyer, 1989; Steyer & Eid, 2001; Suppes & Zinnes, 1963; Zimmermann, 1975). The advantage of this approach is that (1) all latent variables are clearly defined, (2) the psychometric properties of each model with regard to existence, uniqueness, admissible transformations and meaningfulness are explicitly demonstrated and (3) all additional assumptions and implications for deriving testable consequences with respect to the latent covariances and mean structure of the latent variables are unfold. In addition to that, the applicability and limitations of each model are investigated in extensive Monte Carlo simulation studies.

Part II

Formal Definitions

Chapter 2

Formal definition of the latent state (LS-COM) model

2.1 A gentle introduction

In this chapter, a latent state model for longitudinal MTMM data incorporating a combination of structurally different and interchangeable methods is formally defined. The model is called LS-COM model. The abbreviation “LS-COM” was chosen for simplicity. The first part of the abbreviation “LS” indicates which modeling approach for longitudinal data analysis is used. In this case a latent state (LS) model. In the subsequent chapters, two latent change (LC), one latent state-trait (LST) as well as one latent growth curve (LGC) model will be also defined. The second part of the abbreviation “COM” stands for the combination of structurally different as well as interchangeable methods. The latent state (LS-COM) model represents a good starting point for modeling complex MTMM-MO data structures, given that it implies no restrictions with respect to the latent variance-covariance matrix of the model (Geiser, 2008, 2012). With the LS-COM model, it is possible studying the change and stability of constructs as well as method effects across time. Moreover, the LS-COM model allows explicitly modeling the measurement error as well as the hierarchical data structure of complex longitudinal MTMM measurement designs. Thus, the LS-COM model enables researchers to study method effects on different levels (e.g., common and unique method bias) and on different occasions of measurement. Hence, the LS-COM model makes it possible studying the stability and change of common (rater-unspecific) as well as the unique (rater-specific) method bias. In addition, the LS-COM model allows studying the degree of “true” convergent and discriminant validity of the given measures and testing important assumptions such as the degree of measurement invariance and/or the existence of indicator-specific effects. Before the LS-COM model is formally defined in this chapter, a gentle introduction is provided. In the gentle introduction, the main steps of the formal model definition are summarized and explained.

Step 1: Random experiment

In order to define an appropriate model for longitudinal MTMM measurement designs with a combination of structurally different and interchangeable methods, it is important to consider the sampling procedure for such complex data structures. The sampling procedure may be best characterized in terms of the random experiment. Any single experiment, trial, or observation

which can be repeated numerous times can be conceived as a random experiment (Behrends, 2013; Eid, 1995; Steyer, 1988). A random experiment also constitutes the probability space $(\Omega, \mathfrak{A}, P)$ upon which random (observed or unobserved) variables can be defined. For a detailed explanation of the components of the probability space, see Eid (1995), Steyer (1988, 1989), Steyer and Eid (2001), as well as Steyer, Nagel, Partchev, and Mayer (in press). The simplest case of the random experiment that characterizes the sampling procedure for longitudinal measurement designs with structurally different and interchangeable methods is the Cartesian products of the following sets:

$$\Omega = \Omega_T \times \Omega_{TS_1} \times \dots \times \Omega_{TS_l} \times \Omega_R \times \Omega_{RS_1} \times \dots \times \Omega_{RS_l} \times \Omega_{ijk_1} \times \dots \times \Omega_{ijk_l}.$$

The above equation states that target t (e.g., Sophia) has been chosen from a set of targets Ω_T and is considered in a situation Ω_{TS_l} . Then, a rater r (e.g., Elias) is selected from a set of raters Ω_R in a situation Ω_{RS_l} . The rating v (e.g., 4) is an element of Ω_{ijk_l} , where i = item/indicator, j = construct, k = method, and l = occasion of measurement. In this case, the possible outcome $\omega = (t, ts, r, rs, v) = (\text{Sophia in a situation } ts \text{ on occasion of measurement } l, \text{ Elias in a situation } rs \text{ on occasion of measurement } l, 4)$. The phrase “in a situation on occasion of measurement l ” is used to express that targets and raters are assessed at the same occasion of measurement l , but may still be affected differently by target- or rater-specific inner as well as outer situational influences (cf. Geiser, 2008; Steyer, 1988). In total, the random experiment for longitudinal measurement designs with structurally different and interchangeable methods implies five different types of mappings. The mapping of the possible outcomes to the set of targets $p_T : \Omega \rightarrow \Omega_T$, the mapping of the possible outcomes to the set of target-situations $p_{TS_l} : \Omega \rightarrow \Omega_{TS_l}$, the mapping of the possible outcomes to the set of raters $p_R : \Omega \rightarrow \Omega_R$, the mapping of the possible outcomes to the set of rater-situations $p_{RS_l} : \Omega \rightarrow \Omega_{RS_l}$, and the mapping of the possible outcomes to the set of real numbers $Y_{rtijkl} : \Omega \rightarrow \mathbb{R}$. The values of the variable Y_{rtijkl} are the observed values of an indicator i of construct j , assessed by the non-reference interchangeable method k , on the l^{th} occasion of measurement for target t rated by rater r . Thus, the variable Y_{rtijkl} may also be conceived as a level-1 observed variable. In contrast to that, the values of the variable Y_{tijkl} are the observed values of an indicator i of construct j , assessed by a structurally different method k , on the l^{th} occasion of measurement of the target t . For example, the self-rating of the target t (e.g., Sophia) may additionally be considered on occasion of measurement l . Given that these ratings are not rater- but only target-specific, the values of these (self-)ratings are measured by the observed variables Y_{tijkl} . In other words, the observed variables Y_{tijkl} are measured on level-2 (i.e., the target-level). Note that it is also possible to assess the ratings of another structurally different method (e.g., parent or teacher) for target t on occasion of measurement l .

Throughout this work, one and only one set of interchangeable methods/raters per target (e.g., peer-ratings) will be considered. On the other hand, the models presented here are not restricted to any specific number of structurally different methods/raters (e.g., self-ratings, parent ratings, teacher ratings, etc.). In order to differentiate between the different types of methods, the index k is chosen to avoid an additional index or superscript (c.f. Eid et al., 2008). The index k will be used for the distinction between reference and non-reference method. For the sake of simplicity,

the first method (e.g., self-rating) is chosen as reference method or gold standard. In order to refer to the reference method, the index $k = 1$ is used. Throughout this work, the second method $k = 2$ will refer to the set of interchangeable methods (e.g., peer-ratings). Every additional structurally different method (e.g., parent or teacher rating) is indicated by $k > 2$. In summary, $k = 1$ refers to the reference method, whereas $k \neq 1$ refers to the non-reference methods, which can either be the interchangeable method ($k = 2$) or another structurally different method ($k > 2$).

Step 2: Latent state variables as true-score variables

According to latent state theory (see Geiser, 2012; Steyer et al., 1992), each observed variable can be decomposed into a latent state and an error variable. The latent state variables in latent state theory correspond to the true-score variables in classical test theory (CTT, see Geiser, 2008).

$$\begin{aligned} Y_{tij1l} &= S_{tij1l} + E_{tij1l}, & k = 1, & \text{(reference method),} \\ Y_{rtij2l} &= S_{rtij2l} + E_{rtij2l}, & k = 2, & \text{(interchangeable non-reference method),} \\ Y_{tijkkl} &= S_{tijkkl} + E_{tijkkl}, & k > 2, & \text{(structurally different non-reference method).} \end{aligned}$$

The latent state and error variables are defined in terms of conditional expectations $E(\cdot|\cdot)$:

$$S_{tij1l} \equiv E(Y_{tij1l}|p_T, p_{TS_i}), \quad (2.1)$$

$$E_{tij1l} \equiv Y_{tij1l} - E(Y_{tij1l}|p_T, p_{TS_i}), \quad (2.2)$$

$$S_{rtij2l} \equiv E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}), \quad (2.3)$$

$$E_{rtij2l} \equiv Y_{rtij2l} - E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}), \quad (2.4)$$

$$S_{tijkkl} \equiv E(Y_{tijkkl}|p_T, p_{TS_i}), \quad k > 2, \quad (2.5)$$

$$E_{tijkkl} \equiv Y_{tijkkl} - E(Y_{tijkkl}|p_T, p_{TS_i}), \quad k > 2. \quad (2.6)$$

According to the above Definitions 2.1 to 2.6, it is clear that some latent state variables are target specific, whereas other latent state variables are rater-target specific. For instance, the values of $E(Y_{tij1l}|p_T, p_{TS_i})$ represent the true-scores of the reference method (e.g., student self-ratings) on indicator i , for construct j on measurement occasion l . In contrast, the values of $E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})$ are the true-scores of the i^{th} indicator for construct j on measurement occasion l measured by a non-reference method that we expect for a rater in a rater-situation for a target in a target-situation. In other words, the latent state variables are measured on different levels (rater- or target-level). In order to define level-2 state variables on the basis of the level-1 state variables the target- and occasion-specific expectations of the level-1 state variables are considered. Moreover, if the latent state variables are measured on the same level (e.g., the target-level), the latent state variables may also be contrasted against (regressed on) each other. This latent regression approach (so called CTC($M-1$) modeling framework), in which different perspectives are contrasted against each other was, first proposed by Eid (2000) as well as by Eid et al. (2003) for multiple indicator MTMM-SEMs.

Step 3: Conditional Expectations of the latent state variables

By definition, interchangeable methods are randomly drawn from of a set of similar or uniform methods (i.e., the same rater population). With respect to this definition, the expected value of the “true” interchangeable ratings per target at occasion of measurement l defines the target’s state on occasion of measurement l measured by the interchangeable methods. This is expressed by the following equation:

$$S_{tij2l} \equiv E(S_{rtij2l}|p_T, p_{TS_i}) \quad (2.7)$$

$$= E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_{TS_i}]. \quad (2.8)$$

According to Equation 2.7, the conditional expectations of the latent state variables S_{rtij2l} given the target (p_T) in a target-situation (p_{TS_i}) can be defined as target (rater-unspecific) latent state variables. Consequently, the latent state variables S_{tij2l} may be considered as “true means” of the interchangeable ratings per target on occasion l for indicator i measuring construct j .

Step 4: Definition of latent residual method variables

The residuals of the latent regression analysis in Equation 2.7 are referred to as unique method variables and are defined as:

$$\begin{aligned} UM_{rtij2l} &\equiv S_{rtij2l} - E(S_{rtij2l}|p_T, p_{TS_i}) \\ &= S_{rtij2l} - S_{tij2l}. \end{aligned}$$

A value of the unique method variable UM_{rtij2l} reflects the unique deviations of a particular rater r from the expected value of the interchangeable raters at occasion of measurement l . Moreover, given that UM_{rtij2l} is defined as residual with respect to the S_{tij2l} , it follows that $E(UM_{rtij2l}) = 0$ and $Cov(S_{tij2l}, UM_{rtij2l}) = 0$. In order to define latent residual method variables on level-2 (target-level), the latent state variables of the non-reference methods S_{tijkl} (for $k \neq 1$) are regressed on the latent state variable S_{tij1l} of the reference method. In general, this latent regression analysis can be expressed by the following equation:

$$E(S_{tijkl}|S_{tij1l}) = \alpha_{tijkl} + \lambda_{S_{ijkl}} S_{tij1l}, \quad k \neq 1.$$

The main advantage of this latent regression analysis is that different method variables can be defined that are uncorrelated with the reference latent state variables S_{tij1l} . Consider, for example, that the level-2 latent state variables S_{tij2l} of the interchangeable method are regressed on the latent state variables of the reference method S_{tij1l} . Then the residual variables of this latent regression can be defined as common method variables:

$$CM_{tij2l} \equiv S_{tij2l} - E(S_{tij2l}|S_{tij1l}).$$

The common method variables CM_{tij2l} capture the amount of “true” variance of the non-reference interchangeable methods that is common to the interchangeable methods (e.g., general view of the peers), but is not shared with the reference method (e.g., self-report) on occasion of measurement l . The common method variables CM_{tij2l} reflect the “true” occasion-specific common method bias of the interchangeable methods that is unrelated with the reference method. This unrelatedness follows by definition, given that common method variables are defined as residuals with respect to the reference method and therefore the general properties of residuals hold (see Steyer & Eid, 2001). Specifically, the following properties hold:

$$\begin{aligned} E(CM_{tij2l}) &= 0, \\ Cov(S_{tij1l}, CM_{tij2l}) &= 0, \\ Cov(UM_{rtij2l}, CM_{tij2l}) &= 0. \end{aligned}$$

Similarly, “true” occasion-specific method variables M_{tijkl} that pertain to the other structurally different non-reference method (e.g., parent or teacher report, $k > 2$) can be defined as:

$$M_{tijkl} \equiv S_{tijkl} - E(S_{tijkl} | S_{tij1l}).$$

By definition, these latent method variables reflect the true amount of method influences specific to the remaining non-reference structurally different methods on occasion of measurement l . The latent method variables M_{tijkl} reflect the true over- or underestimation of the target self-report (reference method) by the true rating of the supervisor (non-reference structurally different method) on occasion of measurement l . Moreover, the variables M_{tijkl} may be correlated with CM_{tij2l} . These correlations indicate, for example, whether or not method bias generalizes across different constructs $j \neq j'$ and/or different occasion of measurement $l \neq l'$.

Step 5: Definition of latent method factors

In order to obtain an identified model, it is assumed that all latent method variables UM_{rtij2l} , CM_{tij2l} , and M_{tijkl} are homogeneous across items. Specifically, it is assumed that the method variables pertaining to the same kind of method UM_{rtij2l} , CM_{tij2l} , M_{tijkl} are positive linear functions of each, respectively (see also Courvoisier, 2006; Eid, 1995; Geiser, 2008; Steyer, 1988; Steyer & Eid, 2001). Thus, these latent variables only differ by a multiplicative constant such that

$$\begin{aligned} UM_{rtij2l} &= \lambda_{UM_{ij2l}} UM_{rtj2l}, \\ CM_{tij2l} &= \lambda_{CM_{ij2l}} CM_{tj2l}, \\ M_{tijkl} &= \lambda_{M_{ijkl}} M_{tjkl}, \end{aligned} \quad k > 2.$$

Based on this assumption, it is possible to define latent method factors, namely: CM_{tj2l} , UM_{rtj2l} , and M_{tjkl} . The existence of these latent occasion-specific method factors is demonstrated in Section

2.3 of the subsequent chapter. Note that the index i for the indicator has been dropped, because these variables are common to all indicators with the same indices r , t , j , k , and l . In summary, the measurement equations for any observed variable of the LS-COM model are given by:

$$Y_{tij1l} = S_{tij1l} + E_{tij1l}, \quad (2.9)$$

$$Y_{tijkl} = \alpha_{tijkl} + \lambda_{Sijkl}S_{tij1l} + \lambda_{Mijkl}M_{tjkl} + E_{tijkl}, \quad k > 2, \quad (2.10)$$

$$Y_{rtij2l} = \alpha_{rtij2l} + \lambda_{Sij2l}S_{tij1l} + \lambda_{CMij2l}CM_{tj2l} + \lambda_{UMij2l}UM_{rtj2l} + E_{rtij2l}. \quad (2.11)$$

According to the above Equations 2.9 to 2.11, the observed variables pertaining to the reference method Y_{tij1l} measure a latent state variable S_{tij1l} as well as a measurement error variable E_{tij1l} . Note that there are no additional method variables present in Equation 2.9. The expressions

$$\begin{aligned} \alpha_{tijkl} + \lambda_{Sijkl}S_{tij1l}, & \quad k > 2, \\ \alpha_{tj2l} + \lambda_{Sij2l}S_{tij1l}, \end{aligned}$$

in Equations 2.10 and 2.11 refer to the latent regression analyses of the latent state variables pertaining to the non-reference method on the latent reference state variables. The residuals of the latent regression analyses are defined as method factors (M_{tjkl} , CM_{tj2l}) as described above. As a consequence, each of method factor (M_{tjkl} , CM_{tj2l}) is weighted by a factor loading parameter (λ_{Mijkl} , λ_{CMij2l}). In addition to that, the observed variables of non-reference interchangeable method Y_{rtij2l} also measure a unique method factor UM_{rtj2l} weighted by a factor loading parameter λ_{UMij2l} . It is important to note that the values of the latent method factors have different meanings. For example, a value of UM_{rtj2l} reflects the difference between the true rating of a particular rater r on occasion of measurement l from the expected value of all raters for target t on occasion of measurement l . Given that this method bias is specific to the true over- or under-estimation of a particular rater r on occasion of measurement l , it may be called occasion-specific unique method bias. A value of CM_{tj2l} reflects the “true” common view of the interchangeable raters on occasion of measurement l that is not shared with the reference method (e.g., target’s self-report) on the same occasion of measurement. The term “common” is used, given that this method bias is shared with other interchangeable raters for target t , but not with the reference method. Thus, values of CM_{tj2l} represent “true” occasion-specific common method bias. Finally, a value of M_{tjkl} reflects the “true” method bias of another structurally different method (e.g., parent or teacher rating) that is not shared with the reference method. For simplicity, it is assumed that the measurement error variables (E_{tij1l} , E_{tijkl} , E_{rtij2l}) are uncorrelated with each other (see e.g. Definition 2.2). Figure 2.4 illustrates a LS-COM model for three indicators, two constructs, three methods (1 = reference method, 2 = interchangeable method 3 = another structurally different reference method), and two occasions of measurement. For the sake of clarity, all correlations among the latent variables were omitted in the figure. Note that the model illustrated in Figure 2.4 refers to a LS-COM model with indicator-specific latent state factors.

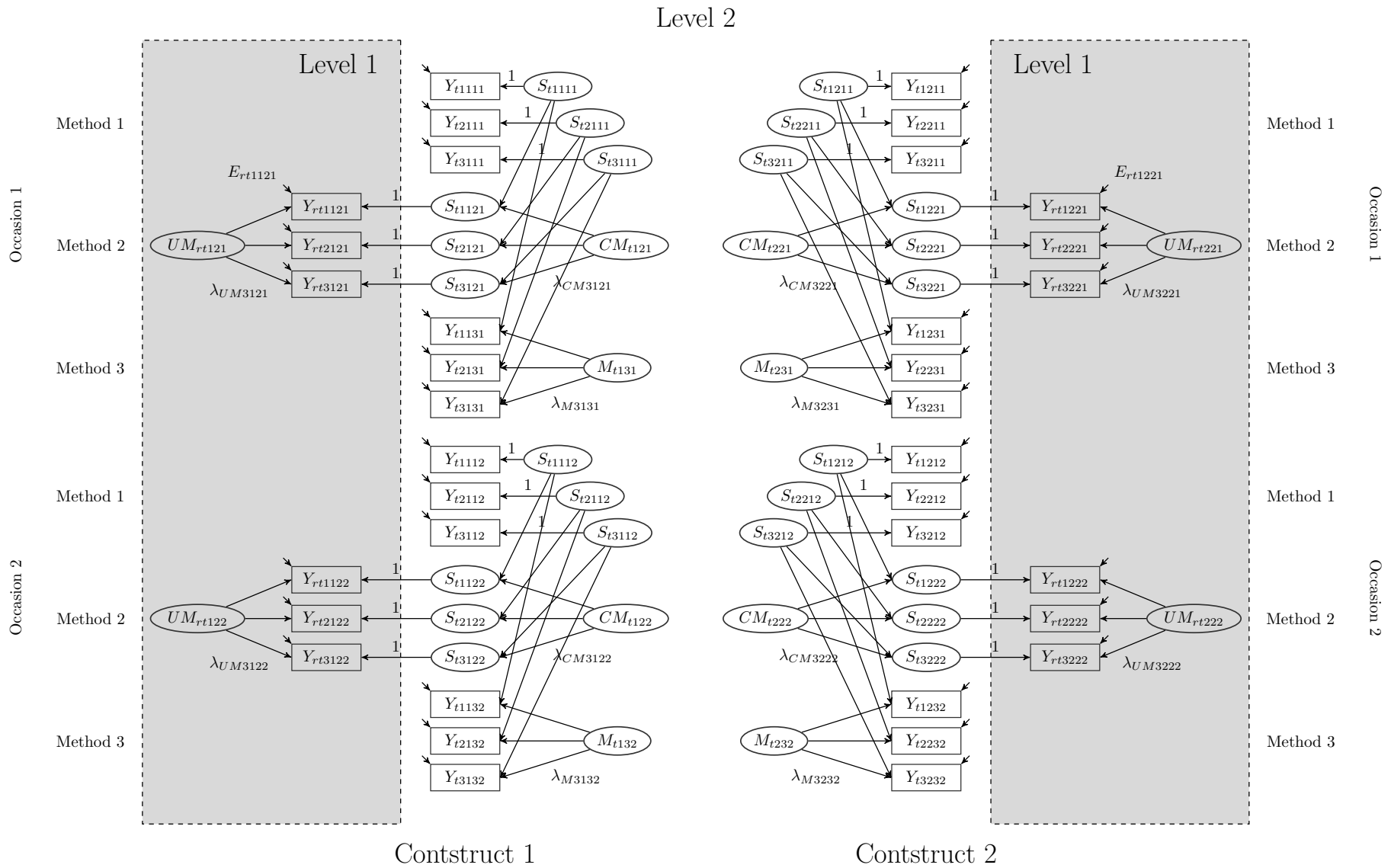


Figure 2.1: Path diagram of the LS-COM model with indicator-specific state factors.

Path diagram of the LS-COM model with indicator-specific state factors incorporating three methods at two measurement occasions for two constructs. All correlations between latent variables were omitted for clarity.

2.2 Formal definition of the LS-COM model

In the following sections, the LS-COM model is formally defined based on the stochastic measurement theory following the approach by Steyer and Eid (2001). For simplicity, the model is defined for just three methods. The first method ($k = 1$) refers to the reference method which is assumed to be structurally different method (e.g., students' self-report) relative to the other methods. The second method ($k = 2$) refers to the set of interchangeable methods (e.g., peer reports for a particular student). It is important to note that the set of interchangeable methods could also be chosen as reference method, however, this is not done in the present work. Guidelines for choosing an appropriate reference method are given in Section 11.1 as well as in Geiser et al. (2008). The third method ($k = 3$) refers to another structurally different method (e.g., parent rating).

Definition 2.1 (LS-COM model)

The random variables $\{Y_{111111}, \dots, Y_{rtijkl}, \dots, Y_{abcdef}\}$ and $\{Y_{11111}, \dots, Y_{tijk}, \dots, Y_{bcdef}\}$ on a probability space $(\Omega, \mathfrak{A}, P)$ are variables of a LS-COM model if the following conditions hold:

- (a) $(\Omega, \mathfrak{A}, P)$ is a probability space such that $\Omega = \Omega_T \times \Omega_{TS_1} \times \dots \times \Omega_{TS_l} \times \Omega_R \times \Omega_{RS_1} \times \dots \times \Omega_{RS_l} \times \Omega_{ijk1} \times \dots \times \Omega_{ijkl}$.
- (b) The projections $p_T : \Omega \rightarrow \Omega_T$, $p_{TS_l} : \Omega \rightarrow \Omega_{TS_l}$, $p_R : \Omega \rightarrow \Omega_R$, and $p_{RS_l} : \Omega \rightarrow \Omega_{RS_l}$ are random variables on $(\Omega, \mathfrak{A}, P)$.
- (c) The variables $Y_{tijk} : \Omega_T \times \Omega_{TS_1} \times \dots \times \Omega_{TS_l} \rightarrow \mathbb{R}$ and $Y_{rtijkl} : \Omega_T \times \Omega_{TS_1} \times \dots \times \Omega_{TS_l} \times \Omega_R \times \Omega_{RS_1} \times \dots \times \Omega_{RS_l} \rightarrow \mathbb{R}$, for which $r \in R \equiv \{1, \dots, a\}$, $t \in T \equiv \{1, \dots, b\}$, $i \in I \equiv \{1, \dots, c\}$, $j \in J \equiv \{1, \dots, d\}$, $k \in K \equiv \{1, \dots, e\}$, $l \in L \equiv \{1, \dots, f\}$ are random variables on $(\Omega, \mathfrak{A}, P)$ with finite first- and second-order moments.
- (d) Without loss of generality, the first method ($k = 1$) is selected as reference method. The second method ($k = 2$) refers to the set of interchangeable methods which serve as non-reference methods. All other methods ($k > 2$) refer to structurally different methods which serve as non-reference methods. Then, the following variables are random variables on $(\Omega, \mathfrak{A}, P)$ with finite first- and second-order moments:

Rater-level (level-1):

$$S_{rtij2l} \equiv E(Y_{rtij2l} | p_T, p_{TS_l}, p_R, p_{RS_l}), \quad (2.12)$$

$$UM_{rtij2l} \equiv S_{rtij2l} - E(S_{rtij2l} | p_T, p_{TS_l}), \quad (2.13)$$

$$E_{rtij2l} \equiv Y_{rtij2l} - E(Y_{rtij2l} | p_T, p_{TS_l}, p_R, p_{RS_l}). \quad (2.14)$$

Target-level (level-2):

$$S_{tij1l} \equiv E(Y_{tij1l} | p_T, p_{TS_l}), \quad (2.15)$$

$$S_{tij2l} \equiv E(S_{rtij2l} | p_T, p_{TS_l}), \quad (2.16)$$

$$S_{tijk} \equiv E(Y_{tijk} | p_T, p_{TS_l}), \quad \forall k > 2, \quad (2.17)$$

$$CM_{tij2l} \equiv S_{tij2l} - E(S_{tij2l} | S_{tij1l}), \quad (2.18)$$

$$M_{tijk} \equiv S_{tijk} - E(S_{tijk} | S_{tij1l}), \quad \forall k > 2, \quad (2.19)$$

$$E_{tij1l} \equiv Y_{tij1l} - E(Y_{tij1l} | p_T, p_{TS_l}), \quad (2.20)$$

$$E_{tijk} \equiv Y_{tijk} - E(Y_{tijk} | p_T, p_{TS_l}), \quad \forall k > 2. \quad (2.21)$$

- (e) For each construct j , measured by a non-reference method ($k \neq 1$) on occasion of measurement l with item i , there is a constant $\alpha_{tijk} \in \mathbb{R}$ as well as a constant $\lambda_{S_{ijkl}} \in \mathbb{R}_+$ such that

$$E(S_{tijk}|S_{tij1l}) = \alpha_{tijk} + \lambda_{S_{ijkl}} S_{tij1l}. \quad (2.22)$$

- (f) For each construct j , measured by a non-reference method k (in this case $k = 2$) on occasion of measurement l and for each pair $(i, i') \in I \times I'$, ($i \neq i'$) there is a constant $\lambda_{CM_{ii'j2l}} \in \mathbb{R}_+$ such that

$$CM_{tij2l} = \lambda_{CM_{ii'j2l}} CM_{ti'j2l}. \quad (2.23)$$

- (g) For each construct j , measured by a non-reference method $k > 2$ on occasion of measurement l and for each pair $(i, i') \in I \times I'$, ($i \neq i'$) there is a constant $\lambda_{M_{ii'jkl}} \in \mathbb{R}_+$ such that

$$M_{tijk} = \lambda_{M_{ii'jkl}} M_{ti'jkl}, \quad \forall k > 2. \quad (2.24)$$

- (h) For each construct j , measured by a non-reference method k (in this case $k = 2$) on occasion of measurement l and for each pair $(i, i') \in I \times I'$, ($i \neq i'$) there is a constant $\lambda_{UM_{ii'j2l}} \in \mathbb{R}_+$ such that

$$UM_{rtij2l} = \lambda_{UM_{ii'j2l}} UM_{rti'j2l}. \quad (2.25)$$

Remarks. In order to define the model six indices are needed: r for rater, t for target, i for indicator, j for construct, k for method, and l for the occasion of measurement. The index r denotes whether a variable is measured on the rater-level or not, whereas the index k represents the type of method (reference or non-reference method). Note that the model is defined for one and only one set of interchangeable methods which is always represented by $k = 2$. According to Equation 2.16, the latent state variables S_{tij2l} represent the conditional expectation of S_{rtij2l} given a value of (p_T) and a value of (p_{TS_i}) . This latent regression may also be written as $E[Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}]|p_T, p_{TS_i}$. The latent variables S_{tij2l} thus represent the expected values of the distribution of the interchangeable ratings given a target in a specific situation on an occasion of measurement l . The residual of this latent regression is defined as unique method variable (see Equation 2.13). A value of this method variable is the deviation of the true-scores of one rater (for target t) from the expected value of all possible raters (for target t) for indicator i , construct j , and occasion of measurement l . It can be also considered as “true” over- or underestimation of a trait value by a particular rater on a particular measurement occasion l . This bias might be due to the effect of a rater and/or the interaction between rater and target. Each of the observed variables has its own associated latent true-score variable S_{rtij2l} , S_{tij1l} , or S_{tijk} as well as measurement error variable E_{rtij2l} , E_{tij1l} , or E_{tijk} . According to Equation 2.22 the latent state variables of the non-reference methods S_{tij2l} and S_{tijk} are regressed on the latent state variables of the reference method variables S_{tij1l} pertaining to the same indicator i , construct j , and measurement occasion l . The residuals of these latent regression analyses define the two remaining method variables M_{tijk} and CM_{tij2l} . In Equation 2.23 to 2.25 below, common (i.e., to all indicators) method factors are defined. Specifically, it is expected that all latent method variable CM_{tij2l} , M_{tijk} , UM_{rtij2l} belonging to the same construct, method, and occasion are positive linear functions of each other, respectively. In other words, it is assumed that these latent residual variables are perfectly correlated and only differ by a multiplicative constant.

2.3 Existence

According to Definition 2.1 the latent method variables, belonging to the same construct j , method k , and occasion of measurement l , but different indicators i and i' are positive linear transformations of each other. As a consequence, these latent residual variables can be represented by

common (i.e., to all indicators) method factors. The following theorem demonstrates the existence of the latent method factors (CM_{tj2l} , M_{tijk} , and UM_{rtj2l}).

Theorem 2.1 (Existence)

The random variables $\{Y_{111111}, \dots, Y_{rtijkl}, \dots, Y_{abcdef}\}$ and $\{Y_{11111}, \dots, Y_{tijk}, \dots, Y_{bcdef}\}$ are (CM_{tj2l} , M_{tijk} , UM_{rtj2l})-congeneric variables of a LS-COM model if and only if the Conditions a to e of Definition 2.1 hold and for each $r \in R$, $t \in T$, $i, \in I$, $j \in J$, $k \in K$, $l \in L$, there are real-valued random variables CM_{tj2l} , M_{tijk} , and UM_{rtj2l} on a probability space $(\Omega, \mathfrak{A}, P)$ and $(\lambda_{CM_{tj2l}}, \lambda_{M_{tijk}}, \lambda_{UM_{rtj2l}}) \in \mathbb{R}_+$ such that:

$$CM_{tj2l} = \lambda_{CM_{tj2l}} CM_{tj2l}, \quad (2.26)$$

$$M_{tijk} = \lambda_{M_{tijk}} M_{tijk}, \quad \forall k > 2, \quad (2.27)$$

$$UM_{rtj2l} = \lambda_{UM_{rtj2l}} UM_{rtj2l}. \quad (2.28)$$

Proofs. 1 Existence of latent variables

2.26 (1) For all i, j, k, l , assume that $CM_{tj2l} \equiv CM_{t1j2l}$ as well as $\lambda_{CM_{tj2l}} \equiv \lambda_{CM_{t1j2l}}$. Inserting these parameters in Equation 2.23 of the above definition, yields Equation 2.26:

$$CM_{tj2l} = \lambda_{CM_{tj2l}} CM_{tj2l} \text{ (repeated).}$$

(2) Similarly, according to Equation 2.26, CM_{tj2l} can be expressed as

$$CM_{tj2l} = \frac{CM_{tj2l}}{\lambda_{CM_{tj2l}}} \text{ as well as } CM_{tj2l} = \frac{CM_{ti'j2l}}{\lambda_{CM_{ti'j2l}}}.$$

If both equations are set equal, it follows from that: $CM_{tj2l} = \frac{\lambda_{CM_{tj2l}}}{\lambda_{CM_{ti'j2l}}} CM_{ti'j2l}$. Let $\lambda_{CM_{ii'j2l}} \equiv \frac{\lambda_{CM_{tj2l}}}{\lambda_{CM_{ti'j2l}}}$, then the Equation 2.23 is obtained:

$$CM_{tj2l} = \lambda_{CM_{ii'j2l}} CM_{ti'j2l} \text{ (repeated).}$$

2.27 (1) For all i, j, k, l , assume that $M_{tijk} \equiv M_{t1ijk}$ as well as $\lambda_{M_{tijk}} \equiv \lambda_{M_{t1ijk}}$. Inserting these parameters in Equation 2.24 of the above definition, yields Equation 2.27:

$$M_{tijk} = \lambda_{M_{tijk}} M_{tijk} \text{ (repeated).}$$

(2) Furthermore, according to Equation 2.27, M_{tijk} can be expressed as

$$M_{tijk} = \frac{M_{tijk}}{\lambda_{M_{tijk}}} \text{ as well as } M_{tijk} = \frac{M_{ti'jkl}}{\lambda_{M_{ti'jkl}}}.$$

By setting both equations equal, it follows from that: $M_{tijk} = \frac{\lambda_{M_{tijk}}}{\lambda_{M_{ti'jkl}}} M_{ti'jkl}$. Let $\lambda_{M_{ii'jkl}} \equiv \frac{\lambda_{M_{tijk}}}{\lambda_{M_{ti'jkl}}}$, then the Equation 2.24 is obtained:

$$M_{tijk} = \lambda_{M_{ii'jkl}} M_{ti'jkl} \text{ (repeated).}$$

2.28 (1) For all i, j, k, l , assume that $UM_{rtj2l} \equiv UM_{rt1j2l}$ as well as $\lambda_{UM_{rtj2l}} \equiv \lambda_{UM_{rt1j2l}}$. Inserting these parameters in Equation 2.25 of the above definition, yields Equation 2.28:

$$UM_{rtj2l} = \lambda_{UM_{rtj2l}} UM_{rtj2l} \text{ (repeated).}$$

(2) Further, according to Equation 2.28, UM_{rtj2l} can be expressed as

$$UM_{rtj2l} = \frac{UM_{rtj2l}}{\lambda_{UM_{rtj2l}}} \text{ as well as } UM_{rtj2l} = \frac{UM_{rti'j2l}}{\lambda_{UM_{rti'j2l}}}.$$

By setting both equations equal, it follows from that:

$$UM_{rtij2l} = \frac{\lambda_{UMijkl}}{\lambda_{UMi'jkl}} UM_{rti'jkl}.$$

Let $\lambda_{UMii'jkl} \equiv \frac{\lambda_{UMijkl}}{\lambda_{UMi'jkl}}$, then the Equation 2.25 is obtained:

$$UM_{rtijkl} = \lambda_{UMii'jk2l} UM_{rti'jkl} \text{ (repeated).}$$

□

Remarks. The above theorem clarifies that the assumptions made in Conditions 2.23 to 2.25 of the above definition imply the existence of common factors CM_{tj2l} , M_{tjkl} , UM_{rtj2l} . It is important to note that the term *common* refers to the fact that each factor is assumed to be common to all indicators, belonging to the same construct, same (non reference) method, and the same occasion of measurement. Put another way, it is assumed that each of the method variables belonging to the same construct, the same method, and the same occasion of measurement [CM_{tj2l} , M_{tjkl} , UM_{rtij2l}] are positive linear functions of each other and only differ by a multiplicative constant. The proof of the theorem shows that the method variables CM_{tj2l} , M_{tjkl} , and UM_{rtij2l} are not *uniquely* defined. In fact, there is a whole family of residual variables which could serve as common latent method factors (CM_{tj2l} , M_{tjkl} , and UM_{rtj2l}). The uniqueness of the latent method factors is discussed in the uniqueness theorem (see Section 2.4).

2.4 Uniqueness

The latent factors (CM_{tj2l} , M_{tjkl} , UM_{rtj2l}) are not completely uniquely defined in LS-COM models. If such models are defined with (CM_{tj2l} , M_{tjkl} , UM_{rtij2l})-congeneric variables, all of these variables and corresponding coefficients are uniquely defined only up to similarity transformation. That means that the latent variables in the LS-COM model as well as their corresponding coefficients are only uniquely defined up to a multiplication with a positive real number. The next theorem clarifies these statements in greater detail.

Theorem 2.2 (Admissible transformations and uniqueness)

1. Admissible Transformations

$\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijkl}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl} \rangle$ is a LS-COM model with:

$$\mathbf{S}_{rt} \equiv (S_{111111} \cdots S_{rtij2l} \cdots S_{abcd2f})^T, \quad (2.29)$$

$$\mathbf{S}_t \equiv (S_{111111} \cdots S_{tijk} \cdots S_{bcdef})^T, \quad (2.30)$$

$$\mathbf{UM}_{rt} \equiv (UM_{111111} \cdots UM_{rtj2l} \cdots UM_{abd2f})^T, \quad (2.31)$$

$$\mathbf{CM}_t \equiv (CM_{11111} \cdots CM_{tj2l} \cdots CM_{bd2f})^T, \quad (2.32)$$

$$\mathbf{M}_t \equiv (M_{11111} \cdots M_{tjkl} \cdots M_{bcdef})^T, \quad (2.33)$$

$$\mathbf{E}_{rt} \equiv (E_{111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^T, \quad (2.34)$$

$$\mathbf{E}_t \equiv (E_{111111} \cdots E_{tijk} \cdots E_{bcdef})^T, \quad (2.35)$$

$$\alpha_{tijk} \equiv (\alpha_{11111} \cdots \alpha_{tijk} \cdots \alpha_{bcdef})^T, \quad (2.36)$$

$$\lambda_S \equiv (\lambda_{11111} \cdots \lambda_{Sijkl} \cdots \lambda_{cdef})^T, \quad (2.37)$$

$$\lambda_{UM} \equiv (\lambda_{11111} \cdots \lambda_{UMij2l} \cdots \lambda_{cd2f})^T, \quad (2.38)$$

$$\lambda_{CM} \equiv (\lambda_{11111} \cdots \lambda_{CMij2l} \cdots \lambda_{cd2f})^T, \quad (2.39)$$

$$\lambda_M \equiv (\lambda_{11111} \cdots \lambda_{Mijkl} \cdots \lambda_{cdef})^T, \quad (2.40)$$

and if for all $r \in R, t \in T, i \in I, j \in J, k \in K, l \in L$:

$$UM'_{rtj2l} \equiv \beta_{UMj2l} UM_{rtj2l}, \quad (2.41)$$

$$CM'_{tj2l} \equiv \beta_{CMj2l} CM_{tj2l}, \quad (2.42)$$

$$M'_{tjkl} \equiv \beta_{Mjkl} M_{tjkl}, \quad k > 2, \quad (2.43)$$

$$\lambda'_{UMij2l} \equiv \lambda_{UMij2l} / \beta_{UMj2l}, \quad (2.44)$$

$$\lambda'_{CMij2l} \equiv \lambda_{CMij2l} / \beta_{CMj2l}, \quad (2.45)$$

$$\lambda'_{Mijkl} \equiv \lambda_{Mijkl} / \beta_{Mjkl}, \quad k > 2, \quad (2.46)$$

where $\beta_{CMj2l}, \beta_{UMj2l}$, and $\beta_{Mjkl} \in \mathbb{R}$, as well as $\beta_{CMj2l}, \beta_{UMj2l}$, and $\beta_{Mjkl} > 0$, then $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}'_{rt}, \mathbf{CM}'_t, \mathbf{M}'_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijkl}, \lambda'_{UMij2l}, \lambda'_{CMij2l}, \lambda'_{Mijkl} \rangle$ is a LS-COM model, too, with:

$$\mathbf{S}_{rt} \equiv (S_{111111} \cdots S_{rtij2l} \cdots S_{abcd2f})^T, \quad (2.47)$$

$$\mathbf{S}_t \equiv (S_{111111} \cdots S_{tijk} \cdots S_{bcdef})^T, \quad (2.48)$$

$$\mathbf{UM}'_{rt} \equiv (UM'_{111111} \cdots UM'_{rtj2l} \cdots UM'_{abd2f})^T, \quad (2.49)$$

$$\mathbf{CM}'_t \equiv (CM'_{111111} \cdots CM'_{tj2l} \cdots CM'_{bd2f})^T, \quad (2.50)$$

$$\mathbf{M}'_t \equiv (M'_{111111} \cdots M'_{tjkl} \cdots M'_{bcdef})^T, \quad (2.51)$$

$$\mathbf{E}_{rt} \equiv (E_{111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^T, \quad (2.52)$$

$$\mathbf{E}_t \equiv (E_{111111} \cdots E_{tijk} \cdots E_{bcdef})^T, \quad (2.53)$$

$$\alpha_{tijk} \equiv (\alpha_{111111} \cdots \alpha_{tijk} \cdots \alpha_{bcdef})^T, \quad (2.54)$$

$$\lambda_{\mathbf{S}} \equiv (\lambda_{111111} \cdots \lambda_{Sijkl} \cdots \lambda_{cdef})^T, \quad (2.55)$$

$$\lambda'_{\mathbf{UM}} \equiv (\lambda'_{111111} \cdots \lambda'_{UMij2l} \cdots \lambda'_{cd2f})^T, \quad (2.56)$$

$$\lambda'_{\mathbf{CM}} \equiv (\lambda'_{111111} \cdots \lambda'_{CMij2l} \cdots \lambda'_{cd2f})^T, \quad (2.57)$$

$$\lambda'_{\mathbf{M}} \equiv (\lambda'_{111111} \cdots \lambda'_{Mijkl} \cdots \lambda'_{cdef})^T. \quad (2.58)$$

2. Uniqueness

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijkl}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl} \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}'_{rt}, \mathbf{CM}'_t, \mathbf{M}'_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijkl}, \lambda'_{UMij2l}, \lambda'_{CMij2l}, \lambda'_{Mijkl} \rangle$ are LS-COM models, then for each $i \in I, j \in J, k \in K, l \in L$, there are $\beta_{CMj2l}, \beta_{UMj2l}$, and $\beta_{Mjkl} \in \mathbb{R}_+$ such that Equations 2.41 to 2.58 hold.

Proofs. 2 Admissible transformations and uniqueness

1. Admissible transformations

If UM_{rtij2l}, CM_{tj2l} , and M_{tjkl} are replaced by $UM'_{rtj2l}, CM'_{tj2l}, M'_{tjkl}$ as well as $\lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl}$ by the corresponding $\lambda'_{UMij2l}, \lambda'_{CMij2l}, \lambda'_{Mijkl}$, then:

$$\begin{aligned} UM_{rtij2l} &= \lambda'_{UMij2l} UM'_{rtj2l} \\ &= \left(\frac{1}{\beta_{UMj2l}} \right) \lambda_{UMij2l} \beta_{UMj2l} UM_{rtj2l} \\ &= \lambda_{UMij2l} UM_{rtj2l}, \end{aligned}$$

$$\begin{aligned} CM_{tj2l} &= \lambda'_{CMij2l} CM'_{tj2l} \\ &= \left(\frac{1}{\beta_{CMj2l}} \right) \lambda_{CMij2l} \beta_{CMj2l} CM_{tj2l} \\ &= \lambda_{CMij2l} CM_{tj2l}, \end{aligned}$$

$$\begin{aligned}
M_{tijkkl} &= \lambda'_{M_{ijkl}} M'_{tijkkl} \\
&= \left(\frac{1}{\beta_{M_{jkl}}} \right) \lambda_{M_{ijkl}} \beta_{M_{jkl}} M_{tijkkl} \\
&= \lambda_{M_{ijkl}} M_{tijkkl}.
\end{aligned}$$

In a similar way, if UM_{rtj2l} , CM_{tj2l} , and M_{tijkkl} are replaced by $\frac{UM'_{rtj2l}}{\beta_{UM_{j2l}}}$, $\frac{CM'_{tj2l}}{\beta_{CM_{j2l}}}$, and $\frac{M'_{tijkkl}}{\beta_{M_{jkl}}}$ as well as $\lambda_{UM_{ij2l}}$, $\lambda_{CM_{ij2l}}$, $\lambda_{M_{ijkl}}$ by $\beta_{UM_{j2l}} \lambda'_{UM_{ij2l}}$, $\beta_{CM_{j2l}} \lambda'_{CM_{ij2l}}$, $\beta_{M_{jkl}} \lambda'_{M_{ijkl}}$, then:

$$\begin{aligned}
UM_{rtij2l} &= \lambda_{UM_{ij2l}} UM_{rtj2l} \\
&= \beta_{UM_{j2l}} \lambda'_{UM_{ij2l}} \cdot \frac{UM'_{rtj2l}}{\beta_{UM_{j2l}}} \\
&= \lambda'_{UM_{ij2l}} UM'_{rtj2l},
\end{aligned}$$

$$\begin{aligned}
CM_{tj2l} &= \lambda_{CM_{ij2l}} CM_{tj2l} \\
&= \beta_{CM_{j2l}} \lambda'_{CM_{ij2l}} \cdot \frac{CM'_{tj2l}}{\beta_{CM_{j2l}}} \\
&= \lambda'_{CM_{ij2l}} CM'_{tj2l},
\end{aligned}$$

$$\begin{aligned}
M_{tijkkl} &= \lambda_{M_{ijkl}} M_{tijkkl} \\
&= \beta_{M_{jkl}} \lambda'_{M_{ijkl}} \cdot \frac{M'_{tijkkl}}{\beta_{M_{jkl}}} \\
&= \lambda'_{M_{ijkl}} M'_{tijkkl}.
\end{aligned}$$

2. Uniqueness

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijkkl}, \lambda_{tijkkl}, \lambda_{UM_{ij2l}}, \lambda_{CM_{ij2l}}, \lambda_{M_{ijkl}} \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}'_{rt}, \mathbf{CM}'_t, \mathbf{M}'_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijkkl}, \lambda_{tijkkl}, \lambda'_{UM_{ij2l}}, \lambda'_{CM_{ij2l}}, \lambda'_{M_{ijkl}} \rangle$ are LS-COM models, then $\lambda_{UM_{ij2l}} UM_{rtj2l} = \lambda'_{UM_{ij2l}} UM'_{rtj2l}$. Consequently, for all $j \in J$, $k \in K$, and $l \in L$:

$$UM'_{rtj2l} = \frac{\lambda_{UM_{ij2l}}}{\lambda'_{UM_{ij2l}}} UM_{rtj2l}.$$

Given that the ratio of $\lambda_{UM_{ij2l}}$ and $\lambda'_{UM_{ij2l}}$ has to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, a real constant can be defined for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\beta_{UM_{j2l}} \equiv \frac{\lambda_{UM_{ij2l}}}{\lambda'_{UM_{ij2l}}}.$$

Again, assume that both \mathcal{M} and \mathcal{M}' are LS-COM models, then $\lambda_{CM_{ij2l}} CM_{tj2l} = \lambda'_{CM_{ij2l}} CM'_{tj2l}$. Consequently, for all $j \in J$, $k \in K$, and $l \in L$:

$$CM'_{tj2l} = \frac{\lambda_{CM_{ij2l}}}{\lambda'_{CM_{ij2l}}} CM_{tj2l}.$$

Given that the ratio of $\lambda_{CM_{ij2l}}$ and $\lambda'_{CM_{ij2l}}$ has to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, a real constant can be defined for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\beta_{CM_{j2l}} \equiv \frac{\lambda_{CM_{ij2l}}}{\lambda'_{CM_{ij2l}}}.$$

If both \mathcal{M} and \mathcal{M}' are LS-COM models, then $\lambda_{M_{ijkl}}M_{tjkl} = \lambda'_{M_{ijkl}}M'_{tjkl}$. Consequently, for all $j \in J$, $k \in K$, and $l \in L$:

$$M'_{tjkl} = \frac{\lambda_{M_{ijkl}}}{\lambda'_{M_{ijkl}}} M_{tjkl}.$$

Given that the ratio of $\lambda_{M_{ijkl}}$ and $\lambda'_{M_{ijkl}}$ has to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, a real constant can be defined for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\beta_{M_{ijkl}} \equiv \frac{\lambda_{M_{ijkl}}}{\lambda'_{M_{ijkl}}}.$$

□

Remarks. The above theorem implies that the latent method factors UM_{rtj2l} , CM_{tj2l} , and M_{tjkl} as well as their corresponding factor loadings $\lambda_{UM_{ij2l}}$, $\lambda_{CM_{ij2l}}$, and $\lambda_{M_{ijkl}}$ are uniquely defined up to similarity transformations, that is, up to a multiplication with a positive real number. Hence, the latent method factors as well as their corresponding factor loadings are measured on a ratio scale.

2.5 Meaningfulness

In the following section, meaningful statements regarding the latent variables in the LS-COM models as well as their corresponding coefficients are addressed. Meaningful statements are statements that remain invariant across the admissible transformations (Geiser, 2008; Steyer & Eid, 2001). The next corollary lists a selection of meaningful statements regarding the latent method factors and their corresponding factor loadings.

Theorem 2.3 (Meaningfulness)

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{S_{ijkl}}, \lambda_{UM_{ij2l}}, \lambda_{CM_{ij2l}}, \lambda_{M_{ijkl}} \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}'_{rt}, \mathbf{CM}'_t, \mathbf{M}'_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{S_{ijkl}}, \lambda'_{UM_{ij2l}}, \lambda'_{CM_{ij2l}}, \lambda'_{M_{ijkl}} \rangle$ are LS-COM models, then for $\omega_1, \omega_2 \in \Omega$; $r, r' \in R$, $t, t' \in T$, $i, i' \in I$, $j, j' \in J$, $k, k' \in K$, and $l, l' \in L$:

$$\frac{\lambda_{UM_{ij2l}}}{\lambda_{UM_{i'j'2l}}} = \frac{\lambda'_{UM_{ij2l}}}{\lambda'_{UM_{i'j'2l}}}, \quad (2.59)$$

$$\frac{\lambda_{CM_{ij2l}}}{\lambda_{CM_{i'j'2l}}} = \frac{\lambda'_{CM_{ij2l}}}{\lambda'_{CM_{i'j'2l}}}, \quad (2.60)$$

$$\frac{\lambda_{M_{ijkl}}}{\lambda_{M_{i'j'kl}}} = \frac{\lambda'_{M_{ijkl}}}{\lambda'_{M_{i'j'kl}}}, \quad (2.61)$$

$$\frac{\lambda_{UM_{ij2l}}}{\lambda_{UM_{i'j'2l}}} - \frac{\lambda_{UM_{ij'2l'}}}{\lambda_{UM_{i'j'2l'}}} = \frac{\lambda'_{UM_{ij2l}}}{\lambda'_{UM_{i'j'2l}}} - \frac{\lambda'_{UM_{ij'2l'}}}{\lambda'_{UM_{i'j'2l'}}}, \quad (2.62)$$

$$\frac{\lambda_{CM_{ij2l}}}{\lambda_{CM_{i'j'2l}}} - \frac{\lambda_{CM_{ij'2l'}}}{\lambda_{CM_{i'j'2l'}}} = \frac{\lambda'_{CM_{ij2l}}}{\lambda'_{CM_{i'j'2l}}} - \frac{\lambda'_{CM_{ij'2l'}}}{\lambda'_{CM_{i'j'2l'}}}, \quad (2.63)$$

$$\frac{\lambda_{M_{ijkl}}}{\lambda_{M_{i'j'kl}}} - \frac{\lambda_{M_{ij'k'l'}}}{\lambda_{M_{i'j'k'l'}}} = \frac{\lambda'_{M_{ijkl}}}{\lambda'_{M_{i'j'kl}}} - \frac{\lambda'_{M_{ij'k'l'}}}{\lambda'_{M_{i'j'k'l'}}}, \quad (2.64)$$

$$\frac{UM_{rtj2l}(\omega_1)}{UM_{rtj2l}(\omega_2)} = \frac{UM'_{rtj2l}(\omega_1)}{UM'_{rtj2l}(\omega_2)}, \quad (2.65)$$

for $UM_{rtj2l}(\omega_2)$ and $UM'_{rtj2l}(\omega_2) \neq 0$,

$$\frac{CM_{tj2l}(\omega_1)}{CM_{tj2l}(\omega_2)} = \frac{CM'_{tj2l}(\omega_1)}{CM'_{tj2l}(\omega_2)}, \quad (2.66)$$

for $CM_{tj2l}(\omega_2)$ and $CM'_{tj2l}(\omega_2) \neq 0$,

$$\frac{M_{tjkl}(\omega_1)}{M_{tjkl}(\omega_2)} = \frac{M'_{tjkl}(\omega_1)}{M'_{tjkl}(\omega_2)}, \quad (2.67)$$

for $M_{tjkl}(\omega_2)$ and $M'_{tjkl}(\omega_2) \neq 0$,

$$\frac{UM_{rtj2l}(\omega_1)}{UM_{rtj2l}(\omega_2)} - \frac{UM_{r't'j'2l'}(\omega_1)}{UM_{r't'j'2l'}(\omega_2)} = \frac{UM'_{rtj2l}(\omega_1)}{UM'_{rtj2l}(\omega_2)} - \frac{UM'_{r't'j'2l'}(\omega_1)}{UM'_{r't'j'2l'}(\omega_2)}, \quad (2.68)$$

for $UM_{rtj2l}(\omega_2)$, $UM_{r't'j'2l'}(\omega_2)$, $UM'_{rtj2l}(\omega_2)$, $UM'_{r't'j'2l'}(\omega_2) \neq 0$,

$$\frac{CM_{tj2l}(\omega_1)}{CM_{tj2l}(\omega_2)} - \frac{CM_{t'j'2l'}(\omega_1)}{CM_{t'j'2l'}(\omega_2)} = \frac{CM'_{tj2l}(\omega_1)}{CM'_{tj2l}(\omega_2)} - \frac{CM'_{t'j'2l'}(\omega_1)}{CM'_{t'j'2l'}(\omega_2)}, \quad (2.69)$$

for $CM_{tj2l}(\omega_2)$, $CM_{t'j'2l'}(\omega_2)$, $CM'_{tj2l}(\omega_2)$, $CM'_{t'j'2l'}(\omega_2) \neq 0$,

$$\frac{M_{tjkl}(\omega_1)}{M_{tjkl}(\omega_2)} - \frac{M_{t'j'k'l'}(\omega_1)}{M_{t'j'k'l'}(\omega_2)} = \frac{M'_{tjkl}(\omega_1)}{M'_{tjkl}(\omega_2)} - \frac{M'_{t'j'k'l'}(\omega_1)}{M'_{t'j'k'l'}(\omega_2)}, \quad (2.70)$$

for $M_{tjkl}(\omega_2)$, $M_{t'j'k'l'}(\omega_2)$, $M'_{tjkl}(\omega_2)$, $M'_{t'j'k'l'}(\omega_2) \neq 0$,

$$\lambda_{UMij2l}^2 \text{Var}(UM_{rtj2l}) = \lambda_{UMij2l}'^2 \text{Var}(UM'_{rtj2l}), \quad (2.71)$$

$$\lambda_{CMij2l}^2 \text{Var}(CM_{tj2l}) = \lambda_{CMij2l}'^2 \text{Var}(CM'_{tj2l}), \quad (2.72)$$

$$\lambda_{Mijkl}^2 \text{Var}(M_{tjkl}) = \lambda_{Mijkl}'^2 \text{Var}(M'_{tjkl}), \quad (2.73)$$

$$\text{Corr}(UM_{rtj2l}, UM_{r't'j'2l'}) = \text{Corr}(UM'_{rtj2l}, UM'_{r't'j'2l'}), \quad (2.74)$$

$$\text{Corr}(CM_{tj2l}, CM_{t'j'2l'}) = \text{Corr}(CM'_{tj2l}, CM'_{t'j'2l'}), \quad (2.75)$$

$$\text{Corr}(M_{tjkl}, M_{t'j'k'l'}) = \text{Corr}(M'_{tjkl}, M'_{t'j'k'l'}), \quad (2.76)$$

where $\text{Var}(\cdot)$ denotes variance and $\text{Corr}(\cdot)$ denotes correlation.

Proofs. 3 Meaningfulness

For simplicity, the proofs for the Equations 2.59, 2.65, 2.68, 2.71, and 2.74 are presented as examples. The proofs for the remaining statements follow the same principles and are straightforward. Thus, these proofs are not reported here.

2.59 Replacing λ'_{UMij2l} , $\lambda'_{UMi'j'2l}$ in Equation 2.59 by $\left(\frac{\lambda_{UMij2l}}{\beta_{UMj2l}}\right)$ and $\left(\frac{\lambda_{UMi'j'2l}}{\beta_{UMj2l}}\right)$, respectively, verifies the equality

$$\frac{\lambda'_{UMij2l}}{\lambda'_{UMi'j'2l}} = \frac{\frac{\lambda_{UMij2l}}{\beta_{UMj2l}}}{\frac{\lambda_{UMi'j'2l}}{\beta_{UMj2l}}} = \frac{\lambda_{UMij2l}}{\lambda_{UMi'j'2l}}.$$

2.65 Replacing UM'_{rtj2l} by $\beta_{UMj2l}UM_{rtj2l}$ verifies the equality

$$\frac{UM'_{rtj2l}(\omega_1)}{UM'_{rtj2l}(\omega_2)} = \frac{\beta_{UMj2l}UM_{rtj2l}(\omega_1)}{\beta_{UMj2l}UM_{rtj2l}(\omega_2)} = \frac{UM_{rtj2l}(\omega_1)}{UM_{rtj2l}(\omega_2)}.$$

2.68 Replacing UM'_{rtj2l} by $\beta_{UMj2l}UM_{rtj2l}$ and $UM'_{r't'j'2l'}$ by $\beta_{UMj'2l'}UM_{r't'j'2l'}$ verifies the equality

$$\begin{aligned} \frac{UM_{rtj2l}(\omega_1)}{UM_{rtj2l}(\omega_2)} - \frac{UM_{r't'j'2l'}(\omega_1)}{UM_{r't'j'2l'}(\omega_2)} &= \frac{UM'_{rtj2l}(\omega_1)}{UM'_{rtj2l}(\omega_2)} - \frac{UM'_{r't'j'2l'}(\omega_1)}{UM'_{r't'j'2l'}(\omega_2)} \\ &= \frac{\beta_{UMj2l}UM_{rtj2l}(\omega_1)}{\beta_{UMj2l}UM_{rtj2l}(\omega_2)} - \frac{\beta_{UMj'2l'}UM_{r't'j'2l'}(\omega_1)}{\beta_{UMj'2l'}UM_{r't'j'2l'}(\omega_2)} \\ &= \frac{UM_{rtj2l}(\omega_1)}{UM_{rtj2l}(\omega_2)} - \frac{UM_{r't'j'2l'}(\omega_1)}{UM_{r't'j'2l'}(\omega_2)}. \end{aligned}$$

2.71 Replacing λ_{UMij2l}^2 by $\lambda_{UMij2l}^2\beta_{UMj2l}^2$ as well as $Var(UM_{rtj2l})$ by $Var\left(\frac{UM'_{rtj2l}}{\beta_{UMj2l}}\right)$ verifies the equality

$$\begin{aligned} \lambda_{UMij2l}^2 Var(UM_{rtj2l}) &= \lambda_{UMij2l}^2 \beta_{UMj2l}^2 Var\left(\frac{UM'_{rtj2l}}{\beta_{UMj2l}}\right) \\ &= \lambda_{UMij2l}^2 \beta_{UMj2l}^2 \frac{1}{\beta_{UMj2l}^2} Var(UM'_{rtj2l}) \\ &= \lambda_{UMij2l}^2 Var(UM'_{rtj2l}). \end{aligned}$$

2.74 Replacing UM_{rtj2l} and $UM_{r't'j'2l'}$ in Equation 2.74 by $\frac{UM'_{rtj2l}}{\beta_{UMj2l}}$ and $\frac{UM'_{r't'j'2l'}}{\beta_{UMj'2l'}}$

$$\begin{aligned} Corr(UM_{rtj2l}, UM_{r't'j'2l'}) &= Corr\left(\frac{UM'_{rtj2l}}{\beta_{UMj2l}}, \frac{UM'_{r't'j'2l'}}{\beta_{UMj'2l'}}\right) \\ &= Cor(UM'_{rtj2l}, UM'_{r't'j'2l'}). \end{aligned}$$

□

Remarks. With respect to the factor loadings λ_{UMij2l} , λ_{CMij2l} , λ_{Mijkl} , as well as their corresponding latent method factors UM_{rtj2l} , CM_{rtj2l} , and M_{rtjkl} , statements regarding the absolute values of the parameters are *not* meaningful (see also Geiser, 2008). The reason is that admissible transformations (e.g., multiplication with a positive real number) would result in different values of the parameters. Nevertheless, statements regarding the ratio of specific values of the factor loadings or the ratio of values of associated latent method factors are meaningful. For example, it is meaningful to say, that a value on the unique method factor for target A is x-times larger than the value on the unique method factor for target B (see also Geiser, 2008). Given that the products $\lambda_{UMij2l}^2 Var(UM_{rtj2l})$ are also invariant to similarity transformations [see Equation 2.71], it follows that statements regarding unique method specificity are meaningful. In fact, any statement with respect to the ratio of variances (i.e., consistency, method specificity, reliability) are meaningful. Finally, statements with respect to latent correlations between method factors is meaningful [see Equations 2.74 to 2.76]. Hence, latent correlations between latent state and/or method variables can be interpreted.

2.6 Testability

In the following section the covariance structure of the LS-COM model is discussed. In order to derive testable consequences for the covariance structure of the LS-COM model, it is necessary to introduce additional assumptions. These assumptions define a more restrictive variant of the LS-COM model. These assumptions are called conditional regressive independence (CRI) assumptions. Models that fulfill this assumption will be called LS-COM model with CRI. This section is structured as follows. First, the conditional regressive independence assumptions are introduced.

Second, the covariances that are equal to zero as a consequence of the model definition are presented. These covariances must be fixed to zero in empirical applications. Third, covariances that are assumed to be zero by further definitions are presented. The latter type of covariances may be set to zero in empirical applications for parsimony. Finally, permissible covariances that are estimable and substantively meaningful are discussed.

Definition 2.2 (LS-COM model with conditional regressive independence (RCI))
 $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl} \rangle$ is called a LS-COM model of $(UM_{rtj2l}, CM_{tij2l}, M_{tijkl})$ -congeneric variables with conditional regressive independence if and only if Definition 2.1 and Theorem 2.1 apply and

$$E(Y_{tijk}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'})) = E(Y_{tijk}|p_T, p_{TS_i}), \quad (2.77)$$

$$\begin{aligned} E(Y_{rtij2l}|p_T, p_{TS_1}, \dots, p_{TS_S}, p_R, p_{RS_1}, \dots, p_{RS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'})) \\ = E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}), \end{aligned} \quad (2.78)$$

$$\begin{aligned} E(Y_{rtij2l}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'})) \\ = E(Y_{rtij2l}|p_T, p_{TS_i}), \end{aligned} \quad (2.79)$$

where $(i, j, k, l) \neq (i, j, k, l)'$.

Remarks. The assumption stated in Equation 2.77 means that given a target (p_T) and a situation (p_{TS_i}) on a measurement occasion l , an observed (level-2) variable Y_{tijk} (belonging to a structurally different method) does neither depend on other target-situations, nor on the values of other Y_{tijk} or Y_{rtij2l} variables. Similarly, Equation 2.78 means that given a target (p_T), a target-situation (p_{TS_i}), a rater (p_R), and a rater-situation (p_{RS_i}) on a measurement occasion l , the observed (level-1) variable Y_{rtij2l} (pertaining to an interchangeable method) is conditionally regressively independent of other rater or target-situations as well as of other observed variables. According to Equation 2.79, an observed (level-1) variable Y_{rtij2l} (pertaining to an interchangeable method) does also not depend on other target-situations, or on the values of other Y_{tijk} or Y_{rtij2l} variables, given a target (p_T) and a situation (p_{TS_i}) on a measurement occasion l . As a consequence of these additional definitions, error variables belonging to different occasions of measurement as well as measurement levels (level-1 and level-2) are uncorrelated with each other.

2.6.1 Zero covariances based on model definition

The definition of the observed and latent variables has consequences for the covariance structure of the observed and latent variables. The next theorem summarizes the covariances that are zero as a consequence of the model definition of the LS-COM model with CRI.

Theorem 2.4 (Testability)

If $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), S_{rtij2l}, S_{tijk}, UM_{rtj2l}, CM_{tj2l}, M_{tjkl}, E_{rtij2l}, E_{tijk}, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl} \rangle$ is called a LS-COM model with conditional regressive independence (RCI), then for $r \in R, t \in T, i, i' \in I, j, j' \in J, k, k' \in K, l, l' \in L$ where i can be equal to i' , j to j' , k to k' and l to l' but $(ijkl) \neq (ijkl)'$:

Uncorrelateness of latent residual variables:

$$\text{Cov}(E_{rt(ij2l)}, E_{rt(ij2l)'}) = 0, \quad (2.80)$$

$$\text{Cov}(E_{t(ijkl)}, E_{t(ijkl)'}) = 0, \quad (2.81)$$

$$\text{Cov}(E_{rt(ij2l)}, E_{t(ijkl)'}) = 0. \quad (2.82)$$

Uncorrelateness of latent residual and latent state variables:

$$\text{Cov}(S_{tijkkl}, E_{ti'j'k'l'}) = 0, \quad (2.83)$$

$$\text{Cov}(S_{rtij2l}, E_{ti'j'k'l'}) = 0, \quad (2.84)$$

$$\text{Cov}(S_{ti'j'k'l'}, E_{rtij2l}) = 0, \quad (2.85)$$

$$\text{Cov}(S_{rtij2l}, E_{rti'j'2l'}) = 0. \quad (2.86)$$

Uncorrelateness of latent residual and latent method variables:

$$\text{Cov}(UM_{rtj2l}, E_{ti'j'k'l'}) = 0, \quad (2.87)$$

$$\text{Cov}(CM_{tj2l}, E_{ti'j'k'l'}) = 0, \quad (2.88)$$

$$\text{Cov}(M_{tjkl}, E_{ti'j'k'l'}) = 0, \quad (2.89)$$

$$\text{Cov}(UM_{rtj2l}, E_{rti'j'2l'}) = 0, \quad (2.90)$$

$$\text{Cov}(CM_{tj2l}, E_{rti'j'2l'}) = 0, \quad (2.91)$$

$$\text{Cov}(M_{tjkl}, E_{rti'j'2l'}) = 0. \quad (2.92)$$

Uncorrelateness of latent state and latent method variables:

$$\text{Cov}(S_{tij1l}, CM_{tj2l}) = 0, \quad (2.93)$$

$$\text{Cov}(S_{tij1l}, UM_{rtj'2l'}) = 0, \quad (2.94)$$

$$\text{Cov}(S_{tij1l}, M_{tjkl}) = 0. \quad (2.95)$$

Uncorrelateness of latent method variables:

$$\text{Cov}(CM_{tj2l}, UM_{rtj'2l'}) = 0, \quad (2.96)$$

$$\text{Cov}(M_{tjkl}, UM_{rtj'2l'}) = 0. \quad (2.97)$$

Proofs. 4 Testability

The subsequent proofs are based on Definition 2.2 as well as important properties of residual variables, namely that residual variables are always uncorrelated with their regressors as well as with numerically measurable functions of their regressors (see Steyer, 1988, 1989; Steyer & Eid, 2001; Steyer et al., in press). Therefore, any expression of the form $\text{Cov}[f(X), f(Y - E(Y|X))]$ equals zero as a consequence of the definition of residual variables. For a detail description and proofs of the properties of residual variables see Steyer et al. (in press, Chapter 9.4). In the following, the proofs for a selection of the above mentioned zero-correlations are provided.

2.80 Consider the covariance $\text{Cov}(E_{rt(ij2l)}, E_{rt(ij2l)'})$ with $E_{rt(ij2l)}$ and $E_{rt(ij2l)'}$ being defined as follows:

$$\begin{aligned} E_{rt(ij2l)} &\equiv Y_{rt(ij2l)} - E(Y_{rt(ij2l)} | p_T, p_{TS_i}, p_R, p_{RS_i}), \\ E_{rt(ij2l)'} &\equiv Y_{rt(ij2l)'} - E(Y_{rt(ij2l)'} | p_T, p_{TS_{i'}}, p_R, p_{RS_{i'}}). \end{aligned}$$

According to Bauer (1978, p. 54, Satz 9.4), $E_{rt(ij2l)'}$ is a $(p_T, p_{TS_{i'}}, p_R, p_{RS_{i'}}, Y_{rt(ij2l)'})$ -measurable function and with respect to the supposition made in Definition 2.2 it is admissible to replace $E(Y_{rt(ij2l)} | p_T, p_{TS_i}, p_R, p_{RS_i})$ by

$$E(Y_{rt(ij2l)} | p_T, p_{TS_1}, \dots, p_{TS_S}, p_R, p_{RS_1}, \dots, p_{RS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'}).$$

Hence, for all $(i, j, k, l) \neq (i, j, k, l)'$, E_{rtij2l} is also a residual with respect to the regressors $p_T, p_{TS_i'}, p_R, p_{RS_i'}$, and $Y_{rt(ij2l)'}$. Given that residuals are always uncorrelated with their regressors as well as with functions of their regressors (see Steyer, 1988, 1989; Steyer & Eid, 2001; Steyer et al., in press), it follows that $Cov(E_{rt(ij2l)}, E_{rt(ij2l)'}) = 0$.

2.81 In a similar way, the covariance $Cov(E_{t(ijkl)}, E_{t(ijkl)'})$ can be expressed as:

$$Cov \{ [Y_{tijkl} - E(Y_{tijkl}|p_T, p_{TS_i})], [Y_{t(ijkl)'} - E(Y_{t(ijkl)'}|p_T, p_{TS_i'})] \}.$$

Again, $E_{t(ijkl)'} \equiv Y_{t(ijkl)'} - E(Y_{t(ijkl)'}|p_T, p_{TS_i'})$ and therefore $E_{t(ijkl)'}$ is a $(p_T, p_{TS_i'}, Y_{t(ijkl)'})$ -measurable function. According to Definition 2.2, one can replace $E(Y_{tijkl}|p_T, p_{TS_i})$ by

$$E(Y_{tijkl}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)}), (Y_{rt(ij2l)})).$$

Consequently, for $(i, j, k, l) \neq (i, j, k, l)'$, E_{tijkl} is also a residual with respect to the regressors $p_T, p_{TS_i'}$, and $Y_{t(ijkl)'}$. Thus, $E_{t(ijkl)}$ and $E_{t(ijkl)'}$ are uncorrelated, for all $(i, j, k, l) \neq (i, j, k, l)'$.

2.82 Consider the covariance $Cov(E_{rt(ij2l)}, E_{t(ijkl)'})$ with $E_{rt(ij2l)}$ and $E_{t(ijkl)'}$ being defined as follows:

$$\begin{aligned} E_{rt(ij2l)} &\equiv Y_{rt(ij2l)} - E(Y_{rt(ij2l)}|p_T, p_{TS_i}, p_R, p_{RS_i}), \\ E_{t(ijkl)'} &\equiv Y_{t(ijkl)'} - E(Y_{t(ijkl)'}|p_T, p_{TS_i'}). \end{aligned}$$

According to Definition 2.2, one can replace $E(Y_{rt(ij2l)}|p_T, p_{TS_i})$ by

$$E(Y_{rt(ij2l)}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)}), (Y_{rt(ij2l)})).$$

Therefore, for all $(i, j, k, l) \neq (i, j, k, l)'$, $E_{rt(ij2l)}$ is also a residual with respect to the regressors $p_T, p_{TS_i'}$, and $Y_{t(ijkl)'}$. Given that residuals are always uncorrelated with their regressors as well as functions of their regressors (see Steyer, 1988, 1989; Steyer & Eid, 2001; Steyer et al., in press), it follows that $Cov(E_{rt(ij2l)}, E_{t(ijkl)'}) = 0$.

2.83 The covariance $Cov(S_{tijkl}, E_{ti'j'k'l'})$ can be expressed as

$$Cov \{ [E(Y_{tijkl}|p_T, p_{TS_i})], [Y_{ti'j'k'l'} - E(Y_{ti'j'k'l'}|p_T, p_{TS_i'})] \}.$$

Again, according to Definition 2.1 the latent state variables S_{tijkl} are defined as conditional expectation of Y_{tijkl} given the target (p_T) and the rater (p_{TS_i}). Thus, the latent state variables are (p_T, p_{TS_i}) -measurable functions. Furthermore, the latent residual variables $E_{ti'j'k'l'}$ are defined as residuals with respect to any $(p_T, p_{TS_i'})$ -measurable function, given that the supposition made in Definition 2.2 allows replacing $E(Y_{ti'j'k'l'}|p_T, p_{TS_i'})$ by

$$E(Y_{ti'j'k'l'}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)}), (Y_{rt(ij2l)})).$$

As a consequence, for all $(i, j, k, l) \neq (i, j, k, l)'$ the latent residual variables $E_{ti'j'k'l'}$ are residuals with respect to the regressors p_T, p_{TS_i} , and Y_{tijkl} . It follows that $Cov(S_{tijkl}, E_{ti'j'k'l'}) = 0$.

2.84 The covariance $Cov(S_{rtij2l}, E_{ti'j'k'l'})$ can be expressed as follows:

$$Cov \{ [E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})], [Y_{ti'j'k'l'} - E(Y_{ti'j'k'l'}|p_T, p_{TS_i'})] \}.$$

According to Definition 2.2 one can replace $E(Y_{ti'j'k'l'}|p_T, p_{TS_i'})$ by

$$E(Y_{ti'j'k'l'}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)}), (Y_{rt(ij2l)})).$$

Therefore, the latent variable $E_{ti'j'k'l'}$ is also a residual with respect to the regressors p_T and p_{TS_i} , and $Y_{rt(ij2l)}$. Given that S_{rtij2l} is defined as

$$S_{rtij2l} \equiv E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}),$$

the correlation between S_{rtij2l} and $E_{ti'j'k'l'}$ must be zero.

2.85 The covariances $Cov(S_{ti'j'k'l'}, E_{rtij2l})$ can be written as follows:

$$Cov \left\{ [E(Y_{ti'j'k'l'} | p_T, p_{TS_i'})], [Y_{rtij2l} - E(Y_{rtij2l} | p_T, p_{TS_i'}, p_R, p_{RS_i'})] \right\}.$$

According to Definition 2.2, one can replace $E(Y_{rt(ij2l)} | p_T, p_{TS_i})$ by

$$E \left(Y_{rt(ij2l)} | p_T, p_{TS_1}, \dots, p_{TS_s}(Y_{t(ijkl)'}, (Y_{rt(ij2l)'}) \right).$$

Therefore, for all $(i, j, k, l) \neq (i, j, k, l)'$, $E_{rt(ij2l)}$ is also a residual with respect to the regressors p_T , $p_{TS_i'}$, and $Y_{t(ijkl)'}$. Given that residuals are always uncorrelated with their regressors as well as functions of their regressors (see Steyer, 1988, 1989; Steyer & Eid, 2001; Steyer et al., in press), it follows that $Cov(S_{ti'j'k'l'}, E_{rtij2l}) = 0$.

2.87 The covariance $Cov(UM_{rtj2l}, E_{ti'j'k'l'})$ is equivalent to

$$\begin{aligned} \frac{1}{\lambda_{UMij2l}} Cov(UM_{rtij2l}, E_{ti'j'k'l'}) &= \frac{1}{\lambda_{UMij2l}} Cov(S_{rtij2l} - E(Y_{rtij2l} | p_T, p_{TS_i}), E_{ti'j'k'l'}) \\ &= \frac{1}{\lambda_{UMij2l}} Cov(S_{rtij2l}, E_{ti'j'k'l'}) - Cov(E(Y_{rtij2l} | p_T, p_{TS_i}), E_{ti'j'k'l'}) \\ &= \frac{1}{\lambda_{UMij2l}} Cov(S_{rtij2l}, E_{ti'j'k'l'}) - Cov(S_{tijkl}, E_{ti'j'k'l'}). \end{aligned}$$

It follows that $Cov(UM_{rtj2l}, E_{ti'j'k'l'}) = 0$, if $Cov(S_{rtij2l}, E_{ti'j'k'l'}) = 0$ and $Cov(S_{tijkl}, E_{ti'j'k'l'}) = 0$. It has already been shown that $Cov(S_{rtij2l}, E_{ti'j'k'l'}) = 0$ and $Cov(S_{tijkl}, E_{ti'j'k'l'}) = 0$ hold.

2.88 Similarly, the $Cov(CM_{tj2l}, E_{ti'j'k'l'})$ is equivalent to

$$\begin{aligned} \frac{1}{\lambda_{CMij2l}} Cov(CM_{tij2l}, E_{ti'j'k'l'}) &= \frac{1}{\lambda_{CMij2l}} Cov(S_{tij2l} - E(S_{tij2l} | S_{tij1l}), E_{ti'j'k'l'}) \\ &= \frac{1}{\lambda_{CMij2l}} Cov(S_{tij2l}, E_{ti'j'k'l'}) - Cov((\alpha_{tij2l} + \lambda_{Sij2l} S_{tij1l}), E_{ti'j'k'l'}) \\ &= \frac{1}{\lambda_{CMij2l}} Cov(S_{tij2l}, E_{ti'j'k'l'}) - \lambda_{Sij2l} Cov(S_{tij1l}, E_{ti'j'k'l'}). \end{aligned}$$

It follows that $Cov(CM_{tj2l}, E_{ti'j'k'l'}) = 0$ if $Cov(S_{tij2l}, E_{ti'j'k'l'}) = 0$ and $Cov(S_{tij1l}, E_{ti'j'k'l'}) = 0$. Again, it has already been shown that $Cov(S_{tij2l}, E_{ti'j'k'l'}) = 0$ and $Cov(S_{tij1l}, E_{ti'j'k'l'}) = 0$ hold.

2.90 The covariance $Cov(UM_{rtj2l}, E_{rti'j'2l'})$ can be rewritten as

$$\begin{aligned} \frac{1}{\lambda_{UMij2l}} Cov(UM_{rtij2l}, E_{rti'j'k'l'}) &= \frac{1}{\lambda_{UMij2l}} Cov(S_{rtij2l} - E(Y_{rtij2l} | p_T, p_{TS_i}), E_{rti'j'k'l'}) \\ &= \frac{1}{\lambda_{UMij2l}} Cov(S_{rtij2l}, E_{rti'j'k'l'}) - Cov(E(Y_{rtij2l} | p_T, p_{TS_i}), E_{rti'j'k'l'}) \\ &= \frac{1}{\lambda_{UMij2l}} Cov(S_{rtij2l}, E_{rti'j'k'l'}) - Cov(S_{tijkl}, E_{rti'j'k'l'}). \end{aligned}$$

It follows that $Cov(UM_{rtj2l}, E_{rti'j'k'l'}) = 0$, if $Cov(S_{rtij2l}, E_{rti'j'k'l'}) = 0$ and $Cov(S_{tijkl}, E_{rti'j'k'l'}) = 0$. Given that $Cov(S_{rtij2l}, E_{rti'j'k'l'}) = 0$ and $Cov(S_{tijkl}, E_{rti'j'k'l'}) = 0$ the equation $Cov(UM_{rtj2l}, E_{rti'j'2l'}) = 0$ holds.

2.93 The covariance $Cov(S_{tij1l}, CM_{tj2l})$ equals zero, if $\frac{1}{\lambda_{CMij2l}} Cov(S_{tij1l}, CM_{tij2l})$ is zero. By definition, the variable CM_{tj2l} is a residual with respect to S_{tij1l} . Thus, for the same construct and occasion of measurement both variables are uncorrelated with each other.

2.94 $UM_{rtj'2l'}$ is a function of $UM_{rti'j'2l'}$

$$UM_{rtj'2l'} = \frac{UM_{rti'j'2l'}}{\lambda_{UMi'j'2l'}},$$

it follows that $Cov(S_{tij1l}, UM_{rtj'2l'}) = 0$, if $Cov(S_{tij1l}, UM_{rti'j'2l'}) = 0$. The latent variables S_{tij1l} and $UM_{rti'j'2l'}$ are defined as follows:

$$\begin{aligned} S_{tij1l} &\equiv E(Y_{tij1l} | p_T, p_{TS_i}) \\ UM_{rti'j'2l'} &\equiv E(Y_{rti'j'2l'} | p_T, p_{TS_i'}, p_R, p_{RS_i'}) - E(Y_{rti'j'2l'} | p_T, p_{TS_i'}). \end{aligned}$$

As consequence, $UM_{rti'j'2l'}$ is a residual with respect to the regressors $p_T, p_{TS_{i'}}$, and $Y_{rti'j'2l'}$. In other words, $UM_{rti'j'2l'}$ is a $(p_T, p_{TS_{i'}}, Y_{rti'j'2l'})$ -measurable function. According to Definition 2 one can replace $E(Y_{rti'j'2l'}|p_T, p_{TS_{i'}})$ by

$$E(Y_{rti'j'2l'}|p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)}), (Y_{rt(ij2l)})).$$

Hence, $UM_{rti'j'2l'}$ is also a residual with respect to the regressors to $p_T, p_{TS_{i'}}$, and Y_{tijkl} . In other words, $UM_{rti'j'2l'}$ is also a function of S_{tij1l} and therefore both variables are uncorrelated with each other.

2.96 Again, $UM_{rtj'2l'}$ is a function of $UM_{rti'j'2l'}$, and CM_{tj2l} is a function of CM_{tij2l} .

$$UM_{rtj'2l'} = \frac{UM_{rti'j'2l'}}{\lambda_{UMi'j'2l'}}, \quad CM_{tj2l} = \frac{CM_{tij2l}}{\lambda_{CMij2l}}.$$

Consequently, $Cov(CM_{tj2l}, UM_{rtj'2l'})$ is zero if $Cov(CM_{tij2l}, UM_{rti'j'2l'})$ is zero. According to the following equation

$$CM_{tij2l} \equiv S_{tij2l} - E(S_{tij2l}|S_{tij1l}),$$

CM_{tij2l} is defined as residual with respect to the latent regression $E(S_{tij2l}|S_{tij1l})$. As a consequence, it follows that $Cov(CM_{tij2l}, UM_{rti'j'2l'})$ is zero, because $UM_{rti'j'2l'}$ is also a function of S_{tij1l} .

2.96 Similarly, equation $Cov(M_{tjkl}, UM_{rtj'2l'}) = 0$ can be shown. First, $UM_{rtj'2l'}$ is a function of $UM_{rti'j'2l'}$ as well as M_{tjkl} is a function of M_{tijkl} .

$$UM_{rtj'2l'} = \frac{UM_{rti'j'2l'}}{\lambda_{UMi'j'2l'}}, \quad M_{tjkl} = \frac{M_{tijkl}}{\lambda_{Mijkl}}.$$

Consequently, $Cov(M_{tjkl}, UM_{rtj'2l'})$ is zero, if $Cov(M_{tijkl}, UM_{rti'j'2l'})$ is zero. Given that M_{tijkl} is a function of S_{tijkl}

$$M_{tijkl} \equiv S_{tijkl} - E(S_{tijkl}|S_{tij1l}), \quad \forall k > 2.$$

It follows that $Cov(M_{tjkl}, UM_{rtj'2l'})$ is zero, because $UM_{rti'j'2l'}$ is also a function of S_{tij1l} . □

2.6.2 Covariance structure: LS-COM model with CRI

According to Theorem 2.4 not all covariances between latent variables are permitted in the LS-COM model with CRI. In the next section, the covariance structure for LS-COM models is illustrated for three indicators \times two traits \times three methods \times two occasions of measurements in matrix form. Note that this $3 \times 2 \times 3 \times 2$ measurement design does not represent the simplest case of the model. For example, it would be possible to specify an LS-COM model with just two indicators i , two constructs j , two methods k (one structurally different and one set of interchangeable methods), and two occasions of measurement l . Therefore, the model presented here (for a $3 \times 2 \times 3 \times 2$ measurement design) is more general. The complete covariance matrix of observed variables is 36×36 (i.e., $ijkl \times ijkl$) dimensional. The total covariance matrix $\sum_{\mathbf{T}}$ of a LS-COM model with CRI can be partitioned into a within $\sum_{\mathbf{W}}$ and a between $\sum_{\mathbf{B}}$ matrix of the same size (36×36):

$$\sum_{\mathbf{T}} = \sum_{\mathbf{W}} + \sum_{\mathbf{B}}.$$

Then, the within matrix is given by:

$$\sum_{\mathbf{W}} = \Lambda_{\mathbf{W}} \Phi_{\mathbf{W}} \Lambda_{\mathbf{W}}^{\mathbf{T}} + \Theta_{\mathbf{W}},$$

where $\Lambda_{\mathbf{W}}$ refers to the factor loading matrix of the unique method factors of size 36×20 (i.e., $ijkl \times (i+k-1)jl$). The elements of this matrix are denoted by λ_{UMij2l} , where i =indicator, j =construct, $k=2$ (set of interchangeable methods), l =measurement occasion. $\Lambda_{\mathbf{W}}^T$ refers to the transposed within factor loading matrix of size 36×20 (i.e., $ijkl \times (i+k-1)jl$). $\Phi_{\mathbf{W}}$ refers to the within variance and covariance matrix of the unique method factors with the dimension of 20×20 [i.e., $(i+k-1)jl \times (i+k-1)jl$], and $\Theta_{\mathbf{W}}$ is the diagonal residual covariance matrix of size 36×36 . In a similar way, the between matrix $\sum_{\mathbf{B}}$ of size 36×36 (i.e., $ijkl \times ijkl$) is given by:

$$\sum_{\mathbf{B}} = \Lambda_{\mathbf{B}} \Phi_{\mathbf{B}} \Lambda_{\mathbf{B}}^T + \Theta_{\mathbf{B}}.$$

Again, $\Lambda_{\mathbf{B}}$ of size 36×20 (i.e., $ijkl \times (i+k-1)jl$) refers to the between factor loadings matrix of the latent factors on the target-level. Thus, the elements of this matrix are λ_{Stijkl} , $\lambda_{CMtij2l}$, λ_{Mtijk} . $\Lambda_{\mathbf{B}}^T$ refers to the transposed matrix of the between factor loadings. $\Phi_{\mathbf{B}}$ refers to the between variance and covariance matrix of the between latent variables with the dimension of 20×20 [i.e., $(i+k-1)jl \times (i+k-1)jl$]. Finally, $\Theta_{\mathbf{B}}$ refers to the between residual variance and covariance matrix of size 36×36 (i.e., $ijkl \times ijkl$). In order to illustrate the complete covariance matrix of the LS-COM model for 3 indicators, 2 constructs, 3 methods, and 2 occasions of measurement, the index (j, l) which can take the following values (in the given ordering), is defined: (1, 1), (1, 2), (2, 1), (2, 2). The index (1, 1) indicates that a given parameter (e.g., factor loading, latent variable) refers to the first construct $j = 1$ measured on the first occasion of measurement $l = 1$. In addition, the function $Pos((j, l))$ is defined. The function maps the index (j, l) on its position \mathbf{p} with respect to the ordering above. The function therefore takes the values given in Table 2.1. The matrix $\mathbf{I}_{\mathbf{p}}$,

Function	Values			
(j, l)	(1,1)	(1,2)	(2,1)	(2,2)
$\mathbf{p}=Pos((j, l))$	1	2	3	4

Table 2.1: Function for the mapping of the index (j, l) to \mathbf{p} .

where $\mathbf{p} \in \mathbb{N} = \{1, \dots, 4\}$ is defined as 4×4 (i.e., $jl \times jl$) matrix with a one on the \mathbf{p}^{th} diagonal element and zeros elsewhere. For example, the matrix $\mathbf{I}_{\mathbf{p}}$ for $\mathbf{p} = 2$ is given by:

$$\mathbf{I}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The function of the matrix $\mathbf{I}_{\mathbf{p}}$ is to define the structure of $\Lambda_{\mathbf{W}}$ and to map the indices (j, l) to their correct position in $\Lambda_{\mathbf{W}}$. Therefore, $\Lambda_{\mathbf{W}}$ of size 36×20 (i.e., $ijkl \times (i+k-1)jl$) is written as the sum over the Kronecker products $\mathbf{I}_{\mathbf{p}}$ and $\Lambda_{\mathbf{W}_{\mathbf{p}}}$:

$$\Lambda_{\mathbf{W}} = \sum_{\mathbf{p}=1}^4 \mathbf{I}_{\mathbf{p}} \otimes \Lambda_{\mathbf{W}_{\mathbf{p}}}.$$

$\mathbf{I}_{\mathbf{p}}$ refers to a matrix of size 4×4 (i.e., $jl \times jl$) for the mapping function of \mathbf{p} , \otimes is the Kronecker product, and $\mathbf{\Lambda}_{\mathbf{w}_{\mathbf{p}}}$ refers to the matrix of size 9×5 (i.e., $ik \times i+k-1$) including the within factor loadings of the unique method factors. Hence, $\mathbf{\Lambda}_{\mathbf{w}_{\mathbf{p}}}$ is given by:

$$\mathbf{\Lambda}_{\mathbf{w}_{\mathbf{p}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{UM1j2l} & 0 \\ 0 & 0 & 0 & \lambda_{UM2j2l} & 0 \\ 0 & 0 & 0 & \lambda_{UM3j2l} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The elements λ_{UM1j2l} , λ_{UM2j2l} , λ_{UM3j2l} are greater than 0, whereas all other elements are necessarily zero. The complete within variance and covariance matrix $\mathbf{\Phi}_{\mathbf{w}}$ of size 20×20 [i.e., $(i+k-1)jl \times (i+k-1)jl$] can be expressed as follows:

$$\mathbf{\Phi}_{\mathbf{w}} = \mathbf{E} [(\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}} - \mathbf{E}[\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}}])(\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}} - \mathbf{E}[\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}}]^{\mathbf{T}})],$$

where $\mathbf{E}(\cdot)$ is the expected value. Note that $\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}}$ refers to the vector of size 20×1 (i.e., $(i+k-1)jl \times 1$) including all latent factors, except for the common method factor CM_{tjkl} :

$$\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}} = \begin{pmatrix} S_{t1111}, S_{t2111}, S_{t3111}, UM_{rt121}, M_{t131}, S_{t1112}, S_{t2112}, S_{t3112}, UM_{rt122}, M_{t132}, \\ S_{t1211}, S_{t2211}, S_{t3211}, UM_{rt221}, M_{t231}, S_{t1212}, S_{t2212}, S_{t3212}, UM_{rt222}, M_{t232} \end{pmatrix}^{\mathbf{T}}.$$

Note that for any $j, j' \in J$ and $l, l' \in L$ the unique method variables UM_{rtj2l} are uncorrelated with any latent state variable $S_{tj'1l'}$ or any latent method variable $M_{tj'3l'}$ on the target-level. Furthermore, note that the latent method variables $CM_{tj'2l'}$ are not represented in the vector $\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}}$. However, the remaining latent variables (S_{tijkl} , M_{tjkl}) are included in the vector $\mathbf{V}_{\mathbf{\Phi}_{\mathbf{w}}}$, given that the covariance matrices $\sum_{\mathbf{w}}$ and $\sum_{\mathbf{B}}$ have to be equally sized for matrix addition. The structure of the covariance matrix $\sum_{\mathbf{w}}$ is illustrated in Figure 2.2. According to this figure, permissible variances and covariances of latent unique method variables are represented as gray colored cells. White colored cells refer to correlations restricted to zero.

Finally, the within error matrix $\mathbf{\Theta}_{\mathbf{w}}$ of size 36×36 (i.e., $ijkl \times ijkl$) is given by

$$\mathbf{\Theta}_{\mathbf{w}} = \sum_{\mathbf{p}=1}^4 \mathbf{I}_{\mathbf{p}} \otimes \mathbf{\Theta}_{\mathbf{w}_{\mathbf{p}}},$$

where $\sum_{\mathbf{p}=1}^4$ is the matrix of size 4×4 (i.e., $jl \times jl$) for the mapping function of the index \mathbf{p} , \otimes is the Kronecker product, and $\mathbf{\Theta}_{\mathbf{w}_{\mathbf{p}}}$ is the variance-covariance matrix of the level-1 residual variables.

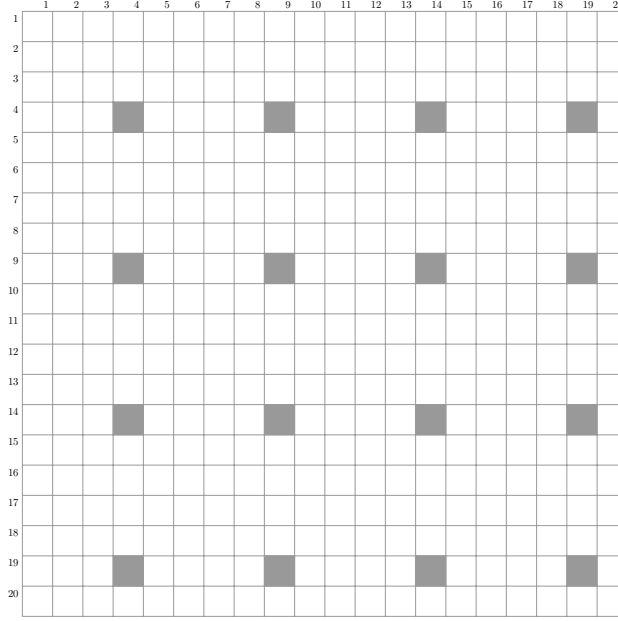


Figure 2.2: Within variance-covariance matrix $\Phi_{\mathbf{W}}$ of the LS-COM with $1=S_{t1111}$, $2=S_{t2111}$, $3=S_{t3111}$, $4=UM_{rt121}$, $5=M_{t131}$, $6=S_{t1112}$, $7=S_{t2112}$, $8=S_{t3112}$, $9=UM_{rt122}$, $10=M_{t132}$, $11=S_{t211}$, $12=S_{t211}$, $13=S_{t211}$, $14=UM_{rt221}$, $15=M_{t231}$, $16=S_{t1212}$, $17=S_{t2212}$, $18=S_{t3212}$, $19=UM_{rt222}$, $20=M_{t232}$. White colored cells indicate zero correlations, gray colored cells indicate permissible correlations.

Thus, $\Theta_{\mathbf{W}_p}$ is given by:

$$\Theta_{\mathbf{W}_p} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Var(E_{rt1j2l}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Var(E_{rt2j2l}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Var(E_{rt3j2l}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for which $r \in R$, $t \in T$, $i \in I$, $j \in J$, $k \in K$, $l \in L$, and for which only the elements $Var(E_{rtij2l}) > 0$. The between factor loading matrix $\Lambda_{\mathbf{B}}$ of size 36×20 [i.e., $ijkl \times (i+k-1)jl$] is given by:

$$\Lambda_{\mathbf{B}} = \sum_{\mathbf{p}=1}^4 \mathbf{I}_{\mathbf{p}} \otimes \Lambda_{\mathbf{B}_p}.$$

Again, $\mathbf{I}_{\mathbf{p}}$ refers to the matrix of size 4×4 described above, \otimes is the Kronecker product, and $\Lambda_{\mathbf{B}_p}$ refers to the between factor loading matrix of the latent variables on the level-2. Hence, $\Lambda_{\mathbf{B}_p}$ is

given by:

$$\Lambda_{\mathbf{B}j\mathbf{l}} = \begin{pmatrix} \lambda_{S1j1l} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{S2j1l} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{S3j1l} & 0 & 0 \\ \lambda_{S1j2l} & 0 & 0 & \lambda_{CM1j2l} & 0 \\ 0 & \lambda_{S2j2l} & 0 & \lambda_{CM2j2l} & 0 \\ 0 & 0 & \lambda_{S3j2l} & \lambda_{CM3j2l} & 0 \\ \lambda_{S1j3l} & 0 & 0 & 0 & \lambda_{M1j3l} \\ 0 & \lambda_{S2j3l} & 0 & 0 & \lambda_{M2j3l} \\ 0 & 0 & \lambda_{S3j3l} & 0 & \lambda_{M3j3l} \end{pmatrix}.$$

Then, the between variance and covariance matrix $\Phi_{\mathbf{B}}$ of size 20×20 [i.e., $(i+k-1)jl \times (i+k-1)jl$] is given by

$$\Phi_{\mathbf{B}} = \mathbf{E} [(\mathbf{V}_{\Phi_{\mathbf{B}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\mathbf{B}}}])(\mathbf{V}_{\Phi_{\mathbf{B}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\mathbf{B}}}]^{\mathbf{T}}),$$

where $\mathbf{V}_{\Phi_{\mathbf{B}}}$ refers to the vector of size 20×1 [i.e., $(i+k-1)jl \times 1$] including all latent factors on the target-level, namely:

$$\mathbf{V}_{\Phi_{\mathbf{B}}} = \begin{pmatrix} S_{t1111}, S_{t2111}, S_{t3111}, CM_{t121}, M_{t131}, S_{t1112}, S_{t2112}, S_{t3112}, CM_{t122}, M_{t132}, \\ S_{t1211}, S_{t2211}, S_{t3211}, CM_{t221}, M_{t231}, S_{t1212}, S_{t2212}, S_{t3212}, CM_{t222}, M_{t232} \end{pmatrix}^{\mathbf{T}}.$$

Note that the expected values of the latent method factors (i.e., CM_{tj2l} and M_{tjkl}) equal zero, given that these latent variables are defined as latent residuals. In contrast to that, the expected values of S_{tj1l} can be freely estimated. The mean structure of the model is discussed in detail in Section 2.5. Another consequence of the model definition is that all elements corresponding to latent correlations between latent state variables S_{tj1l} and the latent method variable (CM_{tj2l} and M_{tj3l}) pertaining to the *same* construct j and *same* occasion of measurement l equal to zero. The structure of the covariance matrix $\Sigma_{\mathbf{B}}$ is illustrated in Figure 2.3. Again, permissible variances and covariances of latent variables are represented as gray colored cells. White colored cells refer to zero correlations. Cells in lighter gray correspond to correlations that may be fixed to zero for parsimony in empirical applications.

The matrix $\Theta_{\mathbf{B}}$ of size 36×36 of the level-2 latent error variables is finally given by:

$$\Theta_{\mathbf{B}} = \sum_{\mathbf{p}=1}^4 \mathbf{I}_{\mathbf{p}} \otimes \Theta_{\mathbf{B}_{\mathbf{p}}},$$

where $\sum_{\mathbf{p}=1}^4$ is again the matrix of size 4×4 for the mapping function of the index \mathbf{p} , \otimes is the Kronecker product, and $\Theta_{\mathbf{B}_{\mathbf{p}}}$ is the variance-covariance matrix of the level-2 residual variables.

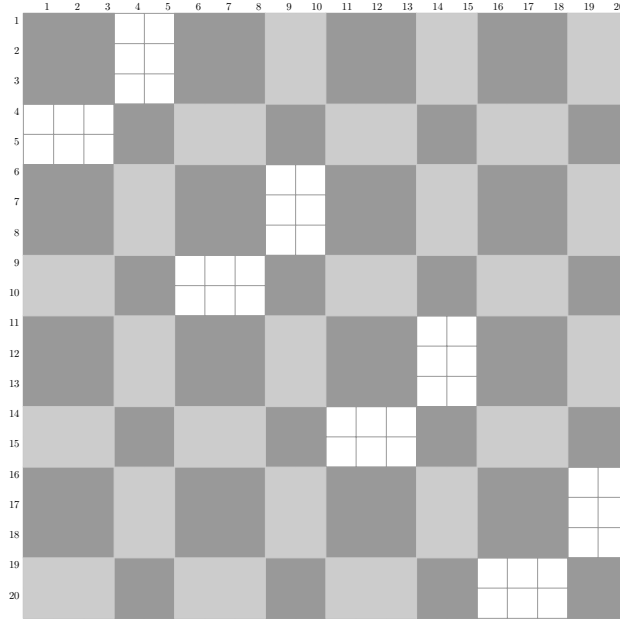


Figure 2.3: Between variance-covariance matrix $\Phi_{\mathbf{B}}$ of the LS-COM with $1=S_{t1111}$, $2=S_{t2111}$, $3=S_{t3111}$, $4=CM_{t121}$, $5=M_{t131}$, $6=S_{t1112}$, $7=S_{t2112}$, $8=S_{t3112}$, $9=CM_{t122}$, $10=M_{t132}$, $11=S_{t211}$, $12=S_{t211}$, $13=S_{t211}$, $14=CM_{t221}$, $15=M_{t231}$, $16=S_{t212}$, $17=S_{t212}$, $18=S_{t212}$, $19=CM_{t222}$, $20=M_{t232}$. White colored cells indicate zero correlations, dark gray colored cells indicate permissible correlations. Light gray colored cells indicate correlations that may be fixed to zero for parsimony.

Thus, $\Theta_{\mathbf{B}_p}$ is given by:

$$\Theta_{\mathbf{B}_p} = \begin{pmatrix} \text{Var}(E_{t1j1l}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{Var}(E_{t1j1l}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{Var}(E_{t1j1l}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Var}(E_{t1j1l}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{Var}(E_{t1j1l}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{Var}(E_{t1j3l}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{Var}(E_{t1j3l}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{Var}(E_{t1j3l}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{Var}(E_{t1j3l}) \end{pmatrix},$$

for which $t \in T$, $i \in I$, $j \in J$, $k \in K$, and $l \in L$ only the element $\text{Var}(E_{tijkl}) > 0$, and all other elements necessarily fixed to zero. Ultimately, the total variance-covariance matrix of the observed variables of size 36×36 is given by:

$$\Sigma_{\mathbf{T}} = \Lambda_{\mathbf{B}} \Phi_{\mathbf{B}} \Lambda_{\mathbf{B}}^{\mathbf{T}} + \Theta_{\mathbf{B}} + \Lambda_{\mathbf{W}} \Phi_{\mathbf{W}} \Lambda_{\mathbf{W}}^{\mathbf{T}} + \Theta_{\mathbf{W}}.$$

The variance-covariance matrices presented in this section were used in the simulation of the LS-COM model in Chapter 7 except for one additional restriction. In the Monte Carlo simulation study common latent state factors S_{tj1l} were assumed. Specifically, it was assumed that the indicator-specific latent state variables S_{tij1l} pertaining to the same construct j , same method k and same occasion of measurement l are perfectly correlated and can be therefore represented by common latent factors. Figure 2.4 shows a path diagram of an LS-COM model with common latent state factors.

2.6.3 Interpretation of non-zero covariances and correlations

The following correlations are permissible with respect to the definition of the LS-COM model. Consequently, these correlations can be estimated and interpreted.

1. The correlations $Cor(S_{tij1l}, S_{ti'j'1l})$ between indicator-specific latent state variables of the reference method belonging to the same construct j and the same occasion of measurement l , but different indicator i and i' can be interpreted as degree of homogeneity of the indicators of the reference method (see Geiser, 2008). High positive correlations indicate that the construct measured by these indicators is unidimensional and may be also represented by a common latent state factor.
2. The correlations between latent state factors of the reference method belonging to the same indicator i and the same occasion of measurement l , but different constructs j and j' can be interpreted as discriminant validity with respect to the reference method (see Geiser, 2008). Two different correlations can be distinguished: (A) The latent correlations $Cor(S_{tij1l}, S_{ti'j'1l})$ between state factors of the reference method belonging to the same indicator i across different constructs j and j' . High correlations indicate low discriminant validity of the constructs on occasion of measurement l with respect to the reference method. (B) The correlations $Cor(S_{tij1l}, S_{ti'j'1l})$ between latent state factors of the reference method belonging the same occasion of measurement l , but different indicators i and i' as well as different constructs j and j' . These correlations can be interpreted as discriminant validity coefficients with respect to the reference method that are corrected for indicator-specific effects.
3. The correlations $Cor(CM_{tj2l}, CM_{tj'2l})$ between latent common method factors belonging to the same occasion of measurement l , but different constructs j and j' indicate that the common method effects measured on the same occasion of measurement l generalize across different constructs j and j' . Correlations close to zero indicate that the common method effects are construct-specific. Substantial correlations result, for example, if the “true” peer effects that are not with the self-report generalize over different constructs. A negative correlation would be given, if peers underestimate the self-reported empathy of a particular child on occasion of measurement l , but overestimate the self-reported aggressiveness on the same occasion of measurement l .
4. The correlations $Cor(CM_{tj2l}, M_{tjkl})$ between common method factors pertaining to the same construct j and the same occasion of measurement l , but different non-reference method k and kt reflect the partial correlations of two different non-reference methods corrected for the self-report (see Geiser, 2008). For example, teachers (i.e., non-reference structurally different method) as well as peers (i.e., non-reference interchangeable method) both have similar view concerning the aggressiveness of the child that is not shared with the self-report of the child.
5. The correlations $Cor(UM_{rtj2l}, UM_{rtj'2l})$ between latent unique method factors belonging to the same occasion of measurement l , but different constructs j and j' , can be interpreted

in a similar way, namely as generalizability of the unique method effects across constructs on the same occasion of measurement. Note that the unique method factor represents the “true” peer rating, that are neither shared with other peers nor shared with the self-report of the target. In other words, these correlations reflect whether or not “true” specific peer effects (that are not shared with other peers) generalize across different constructs (e.g., aggressiveness and empathy). Correlations close to zero indicate that the unique method effect is construct-specific.

6. The correlations $Cor(S_{tij1l}, CM_{tj'2l})$ and $Cor(S_{tij1l}, M_{tj'2l})$ between method factors belonging to the non-reference method and latent state variables pertaining to the reference method, for the same occasion of measurement l , but for different constructs j and j' can be interpreted as discriminant validity coefficient that is corrected for method influences of the reference method (see Geiser, 2008). In many empirical applications these coefficients will be close to zero. It is therefore recommended to fix these correlations to zero for parsimony.
7. The correlations $Cor(CM_{tj2l}, M_{tj'kl})$ between method factors pertaining to the same occasion of measurement l , but different constructs j and j' and different non reference methods k and k' indicate discriminant validity between method factors corrected for the discriminant validity with respect to the reference method. Significant correlations indicate that the association between methods cannot be completely explained by the reference method. For example, the over- or underestimation of students' self-reported aggressiveness by peers is associated with the over- or underestimation of students' self-reported empathy by teachers. Hence, peers and teachers share something in common that is not reflected by the self-report of the students.
8. The correlations between latent state factors for the reference method belonging to the same indicator i , the same construct j , but different occasions of measurement l and l' can be interpreted as construct stability coefficients (see Geiser, 2008). Two different types of correlations can be distinguished: First, the correlations between the same latent indicator-specific state factors over time. (A) These correlations $Cor(S_{tij1l}, S_{tij1l'})$ can be interpreted as stability coefficients not corrected for indicator-specific effects. (B) Second, the correlations $Cor(S_{tij1l}, S_{t'i'j1l'})$ between latent state factors belonging to the same construct j , but different indicators i and i' as well as different occasions of measurement l and l' represent construct stability corrected for indicator-specific effects.
9. (A) The correlations $Cor(S_{tij1l}, S_{tij'1l'})$ between latent state factors belonging to the reference method of the same indicator i , but different constructs j and j' and different occasion of measurement l and l' can be interpreted as discriminant validity coefficients with respect to the reference method that are corrected for common occasion-specific influences (see Geiser, 2008). (B) The correlations $Cor(S_{tij1l}, S_{t'i'j'1l'})$ between latent state factors belonging to the reference method of different indicators i and i' , different constructs j and j' , and different occasions of measurement l and l' can be interpreted as discriminant validity coefficients that are corrected for indicator-specific and common occasion-specific influences.

10. The correlations between method factors belonging to the same construct j , the same non-reference method k , but different occasions of measurement l and l' can be interpreted as degree of stability of construct-specific method effects (see Geiser, 2008). For instance, the teachers consistently over- or underestimate students' empathy skills with respect to self-reports over time. Given that there are three different method factor, there are also three different correlations coefficients. (A) The correlations $Cor(CM_{tj2l}, CM_{tj2l'})$ reflect the degree of stability of the construct-specific common method effects. (B) The correlations $Cor(UM_{tj2l}, UM_{tj2l'})$ represent the stability of construct-specific unique-method effects. (C) The correlations $Cor(M_{tjkl}, M_{tjkl'})$ capture the degree of stability of the construct-specific method (e.g., teacher ratings) effects.
11. The correlations $Cor(CM_{tj2l}, CM_{tj2l'})$, $Cor(UM_{tj2l}, UM_{tj2l'})$, and $Cor(M_{tjkl}, M_{tjkl'})$ between method factors of pertaining to the same non-reference method k , but different constructs j and j' and different occasions of measurement l and l' can be interpreted as generalizability of method effects corrected for common occasion-specific effects (see Geiser, 2008). For example, high correlations $Cor(M_{tjkl}, M_{tjkl'})$ indicate that teacher consistently over- or underestimate students' self-reports over time, regardless which construct (e.g., aggressiveness or empathy) is considered. High correlations $Cor(UM_{tj2l}, UM_{tj2l'})$ indicate that specific peers consistently deviate from the general view of all peers for a particular target across different measurement occasions.
12. The correlations between method factors pertaining to level-2 non-reference methods and latent state variables belonging to the reference method of the same construct j , but different occasions of measurement l and l' are not easy to interpret. In most empirical applications these correlations will not be significant. Nevertheless these correlations are permissible and estimable. (A) Significant correlations of $Cor(S_{tij1l}, CM_{tj2l'})$ would indicate that the part of "true" peer ratings that are shared with other peers, but not shared with students' self-reports at time l can predict children's self-reported empathy scores at time l' . (B) Significant correlations of $Cor(S_{tij1l}, M_{tj2l'})$ may indicate that the part of "true" teacher ratings that is not shared with children's self-reports at time l can predict children's self-reported empathy scores at time l' .
13. The correlations $Cor(S_{tij1l}, CM_{tj2l'})$ and $Cor(S_{tij1l}, M_{tj2l'})$ between the method factors belonging to the level-2 non-reference methods and the latent state variable belonging to the reference method for different constructs j and j' as well as for different occasions of measurement l and l' are most likely to be close to zero in empirical applications. Significant correlations would reflect coefficients of discriminant validity corrected for common method effects and common occasion-specific influences (see Geiser, 2008). In most application it is recommended to fix these correlations to zero for parsimony.
14. The correlations $Cor(CM_{tj2l}, M_{tjkl'})$ between method factors pertaining to the same construct, but different level-2 non-reference methods and different occasions of measurement l and l' indicate the partial correlations between two different non-reference method corrected

for the reference method and common occasion-specific influences (see Geiser, 2008).

15. The correlations $Cor(CM_{tj2l}, M_{tj'kl'})$ between method factors pertaining to different level-2 non-reference methods, different constructs j and j' and different occasion of measurement l and l' indicate discriminant validity of methods effects corrected for construct-specific and common occasion-specific influences (see Geiser, 2008).

2.7 General measurement equations and variance decompositions

In the following section the general measurement equations of LS-COM models are discussed. Based on the definition of the LS-COM model different variance coefficients can be defined. These coefficients can be meaningfully interpreted as shown in Theorem 2.3. Note that the independence among latent variables derived in Theorem 2.4 are important prerequisites for separating different variance components from one another. Hence, a LS-COM model that fulfills these requirements is restated first. In a second step the general measurement equations of the LS-COM model are derived. In a third step, the additive variance decomposition of the observed variables is introduced and the different variance coefficients are discussed.

Definition 2.3

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl} \rangle$ be a LS-COM model according to the Definition 2.1 and Theorem 2.1 and:

$$\begin{aligned} \mathbf{S}_{rt} &\equiv (S_{111111} \cdots S_{rtij2l} \cdots S_{abcd2f})^T, \\ \mathbf{S}_t &\equiv (S_{111111} \cdots S_{tijk} \cdots S_{bcdef})^T, \\ \mathbf{UM}_{rt} &\equiv (UM_{111111} \cdots UM_{rtj2l} \cdots UM_{abd2f})^T, \\ \mathbf{CM}_t &\equiv (CM_{1111} \cdots CM_{tj2l} \cdots CM_{bd2f})^T, \\ \mathbf{M}_t &\equiv (M_{1111} \cdots M_{tijk} \cdots M_{bdef})^T, \\ \mathbf{E}_{rt} &\equiv (E_{111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^T, \\ \mathbf{E}_t &\equiv (E_{111111} \cdots E_{tijk} \cdots E_{bcdef})^T, \\ \alpha_{tijk} &\equiv (\alpha_{1111} \cdots \alpha_{tijk} \cdots \alpha_{bcdef})^T, \\ \lambda_S &\equiv (\lambda_{1111} \cdots \lambda_{Sijk} \cdots \lambda_{cdef})^T, \\ \lambda_{UM} &\equiv (\lambda_{1111} \cdots \lambda_{UMij2l} \cdots \lambda_{cd2f})^T, \\ \lambda_{CM} &\equiv (\lambda_{1111} \cdots \lambda_{CMij2l} \cdots \lambda_{cd2f})^T, \\ \lambda_M &\equiv (\lambda_{1111} \cdots \lambda_{Mijkl} \cdots \lambda_{cdef})^T. \end{aligned}$$

Remarks. According to the Definition 2.3 a LS-COM model with common method factors (i.e., CM_{tj2l} , M_{tijk} , UM_{rtj2l}) is defined. All indicators Y_{rtij2l} belonging to the same construct, same method, and same measurement occasion measure a latent state factor S_{tijk} and two construct- and occasion specific method factors, namely CM_{tj2l} and UM_{rtj2l} . Moreover, all indicators Y_{tijk} belonging to the same construct, same method, and same measurement occasion measure a latent state factor S_{tijk} and a construct- and occasion specific method factor, called M_{tijk} . The complete

measurement equations for the observed variables are given by:

$$Y_{tij1l} = S_{tij1l} + E_{tij1l}, \quad (2.98)$$

$$Y_{tijkl} = \alpha_{tijkl} + \lambda_{Sijkl}S_{tij1l} + \lambda_{Mijkl}M_{tjkl} + E_{tijkl}, \quad k > 2, \quad (2.99)$$

$$Y_{rtij2l} = \alpha_{rtij2l} + \lambda_{Sij2l}S_{tij1l} + \lambda_{CMij2l}CM_{tj2l} + \lambda_{UMij2l}UM_{rtj2l} + E_{rtij2l}. \quad (2.100)$$

2.7.1 Variance decomposition

According to the Equations 2.98 to 2.100, the variance of the observed variables can be additively decomposed as follows:

$$Var(Y_{tij1l}) = Var(S_{tij1l}) + Var(E_{tij1l}), \quad (2.101)$$

$$Var(Y_{tijkl}) = \lambda_{Sijkl}^2 Var(S_{tij1l}) + \lambda_{Mijkl}^2 Var(M_{tjkl}) + Var(E_{tijkl}), \quad k > 2, \quad (2.102)$$

$$Var(Y_{rtij2l}) = \lambda_{Sij2l}^2 Var(S_{tij1l}) + \lambda_{CMij2l}^2 Var(CM_{tj2l}) + \lambda_{UMij2l}^2 Var(UM_{rtj2l}) + Var(E_{rtij2l}). \quad (2.103)$$

Due to the additive variance decomposition, it is possible to define different variance components. The true intraclass correlation (ICC), as well as the coefficients of true consistency, true (common and unique) method specificity, and reliability can be defined. The true ICC reflects the amount of true-score variance that is explained by true interindividual differences between targets. The true ICC coefficients can also be interpreted as true rater-consistency on the target-level, given that this coefficient reflects the amount true-score variance that is shared by the methods (e.g., raters) on the target-level. Note that this coefficient is corrected for measurement error influences as well as specific (unique) rater influences. The true ICC is defined on the basis of the true-score variables pertaining to the interchangeable methods, τ_{rtij2l} :

$$ICC(\tau_{rtij2l}) = \frac{\lambda_{Sij2l}^2 Var(S_{tij1l}) + \lambda_{CMij2l}^2 Var(CM_{tj2l})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

The true consistency coefficient represents the amount of true-score variance that is explained by the latent state variable of the reference method at time l . The square root of the consistency coefficient can be interpreted in terms of true convergent validity with respect to the reference method:

$$CON(\tau_{tijkl}) = \frac{\lambda_{Sijkl}^2 Var(S_{tij1l})}{Var(Y_{tijkl}) - Var(E_{tijkl})}, \quad k > 2,$$

$$CON(\tau_{rtij2l}) = \frac{\lambda_{Sij2l}^2 Var(S_{tij1l})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

Furthermore, different coefficients of true method specificity can be defined. Method specificity coefficients represent the proportion of true-score variance that is due to method specific influences.

In total, three method specificity coefficients can be defined:

$$\begin{aligned}
 MS(\tau_{tijkkl}) &= \frac{\lambda_{Mijkl}^2 \text{Var}(M_{tjkl})}{\text{Var}(Y_{tijkkl}) - \text{Var}(E_{tijkkl})}, & k > 2, \\
 CMS(\tau_{rtij2l}) &= \frac{\lambda_{CMij2l}^2 \text{Var}(CM_{tj2l})}{\text{Var}(Y_{rtij2l}) - \text{Var}(E_{rtij2l})}, \\
 UMS(\tau_{rtij2l}) &= \frac{\lambda_{UMij2l}^2 \text{Var}(UM_{rtj2l})}{\text{Var}(Y_{rtij2l}) - \text{Var}(E_{rtij2l})}.
 \end{aligned}$$

The $MS(Y_{tijkkl})$ coefficient represents the proportion of true-score variance that is due to method specific influences of the non-reference structurally different methods. For example, this coefficient reflect the amount of true variance that is due to the true over- or underestimation of the employee's self-report (target) by the supervisor (structurally different rater). The $CMS(Y_{rtij2l})$ coefficient represents the proportion of true-score variance that is due to method specific influences of the common view of the interchangeable methods. This coefficient reflects the amount of true variance that is due to the true over- or underestimation of the employee's self-report (target) with respect to the general view of the colleagues (interchangeable methods). In contrast to that, the $UMS(Y_{rtij2l})$ coefficient represents the proportion of true variance that is due to method specific influences of the unique view of a interchangeable method (e.g., a particular rater) that is neither shared with the self-report (e.g., reference method) nor with other raters (e.g., the general view of the colleagues). In addition, total method specificity with respect to the true-score variables τ_{rtij2l} of the interchangeable methods can be calculated:

$$\begin{aligned}
 TMS(\tau_{rtij2l}) &= CMS(\tau_{rtij2l}) + UMS(\tau_{rtij2l}) \\
 &= 1 - CON(\tau_{rtij2l}).
 \end{aligned}$$

The reliability $Rel(\cdot)$ as well as unreliability $Unrel(\cdot)$ coefficients for the observed variables are given by:

$$\begin{aligned}
 Rel(Y_{tij1l}) &= 1 - \frac{\text{Var}(E_{tij1l})}{\text{Var}(Y_{tij1l})} = 1 - Unrel(Y_{tij1l}), \\
 Rel(Y_{tijkkl}) &= 1 - \frac{\text{Var}(E_{tijkkl})}{\text{Var}(Y_{tijkkl})} = 1 - Unrel(Y_{tijkkl}), & \forall k > 2, \\
 Rel(Y_{rtij2l}) &= 1 - \frac{\text{Var}(E_{rtij2l})}{\text{Var}(Y_{rtij2l})} = 1 - Unrel(Y_{rtij2l}).
 \end{aligned}$$

2.8 Mean structure

With respect to longitudinal studies many researcher seek to investigate mean changes over time. In this section, the latent variable mean structure of the LS-COM model is discussed. The following theorem shows the consequence of the model definition for the observed and latent variables.

Theorem 2.5 (Mean structure)

If $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), S_{rtij2l}, S_{tijkl}, UM_{rtj2l}, CM_{tj2l}, M_{tjkl}, E_{rtij2l}, E_{tijkl}, \alpha_{tijkl}, \lambda_{S_{ijkl}}, \lambda_{UM_{ij2l}}, \lambda_{CM_{ij2l}}, \lambda_{M_{ijkl}} \rangle$ is called a LS-COM model with CRI. Without loss of generality, $k=1$ method is chosen as reference method, then the following mean structure holds for all $r \in R \equiv \{1, \dots, a\}$, $t \in T \equiv \{1, \dots, b\}$, $i \in I \equiv \{1, \dots, c\}$, $j \in J \equiv \{1, \dots, d\}$, $k \in K \equiv \{1, \dots, e\}$, $l \in L \equiv \{1, \dots, f\}$:

$$E(Y_{rtij2l}) = E(\alpha_{tij2l}) + \lambda_{tij2l}E(S_{tij1l}), \quad (2.104)$$

$$E(Y_{tijkl}) = E(\alpha_{tijkl}) + \lambda_{tijkl}E(S_{tij1l}), \quad \forall k > 2. \quad (2.105)$$

$$E(S_{tij1l}) = E(Y_{tij1l}), \quad (2.106)$$

$$E(CM_{tj2l}) = 0, \quad (2.107)$$

$$E(UM_{rtj2l}) = 0, \quad (2.108)$$

$$E(M_{tjkl}) = 0, \quad \forall k > 2, \quad (2.109)$$

$$E(E_{tijkl}) = 0, \quad \forall k \neq 2, \quad (2.110)$$

$$E(E_{rtij2l}) = 0, \quad (2.111)$$

where $E(\cdot)$ denotes expected value.

Proofs. 5 Mean structure

According to Equation 2.9, $Y_{tij1l} = S_{tij1l} + E_{tij1l}$. Thus, $E(Y_{tij1l}) = E(S_{tij1l}) + E(E_{tij1l})$. Given that E_{tij1l} are defined as residuals, it follows that $E(E_{tij1l}) = 0$. Therefore, $E(Y_{tij1l}) = E(S_{tij1l})$ (see Equation 2.106). Furthermore, the latent variables CM_{tj2l} , UM_{rtj2l} , M_{tjkl} , E_{tijkl} , and E_{rtij2l} are also defined as latent residuals variables. By definition, residuals have an expected value of zero (Steyer, 1989; Steyer & Eid, 2001). Thus, Equations 2.107 to 2.111 follow directly by definition of the latent variables. Next, according to Equation 2.10, $Y_{tijkl} = \alpha_{tijkl} + \lambda_{S_{ijkl}}S_{tij1l} + \lambda_{M_{ijkl}}M_{tjkl} + E_{tijkl}$. Thus, $E(Y_{tijkl}) = E(\alpha_{tijkl}) + E(\lambda_{S_{ijkl}}S_{tij1l}) + E(\lambda_{M_{ijkl}}M_{tjkl}) + E(E_{tijkl})$. Given that, $E(\lambda_{M_{ijkl}}M_{tjkl})$ and $E(E_{tijkl})$ equals zero, the equation simplifies to Equation 2.105. Similarly, Equation 2.11 is given by $Y_{rtij2l} = \alpha_{tij2l} + \lambda_{S_{ij2l}}S_{tij1l} + \lambda_{CM_{ij2l}}CM_{tj2l} + \lambda_{UM_{ij2l}}UM_{rtj2l} + E_{rtij2l}$. Hence, $E(Y_{rtij2l}) = E(\alpha_{tij2l}) + E(\lambda_{S_{ij2l}}S_{tij1l}) + E(\lambda_{CM_{ij2l}}CM_{tj2l}) + E(\lambda_{UM_{ij2l}}UM_{rtj2l}) + E(E_{rtij2l})$. Again, $E(\lambda_{CM_{ij2l}}CM_{tj2l})$, $E(\lambda_{UM_{ij2l}}UM_{rtj2l})$, and $E(E_{rtij2l})$ are zero according to Equation 2.107, 2.108, and 2.111 and therefore the above equation simplifies to Equation 2.104. \square

Remarks. Equations 2.104 and 2.105 clarify that the expected value of an observed variable is equal to the expected value of the corresponding state factor if and only if $\alpha_{tijkl} = 0$ and $\lambda_{tijkl} = 1$. According to Equation 2.106, the expected values of the reference state factors are identical to the expected values of the indicators pertaining to the reference method. Equations 2.107 to 2.109 show very important implications of the model definition, namely that the method factors (CM_{tj2l} , UM_{rtj2l} and M_{tjkl}) are defined as residuals and therefore have expected values of zero. The same holds for the measurement error variables (see Equation 2.110 and 2.111).

2.9 Identifiability

An important prerequisite for parameter estimation refers to the problem of model identification. A model is said to be identified, if and only if each parameter of the model (e.g., means, variances, and covariances of the latent variables) can be uniquely determined with respect to the information in the data (e.g., means, variances, and covariances of the observed variables). A parameter is uniquely determined, if there is one and only one mathematical solution for each parameter in the model. In order to demonstrate the identification of a model, it is necessary to

assign a scale to each latent factor (Bollen, 1989). A general rule in structural equation modeling is to constrain the variance of the latent variable to a non-zero value or to fix one factor loading per factor to 1 (Bollen, 1989; Bollen & Curran, 2006). With respect to longitudinal SEMs usually the first factor loading per factor is fixed to 1, given that these restrictions still allow to investigate the change or stability of factor variance over time (Geiser, 2008). The next theorem implies that each parameter of the LS-COM model is identified, if at least 1 construct is measured by 2 methods on 2 occasions of measurement, with 2 indicators per method and if the state as well as method factors on the rater- and target-level are substantially correlated.

Theorem 2.6 (Identification of the LS-COM covariance structure)

Let $\mathcal{M} \equiv ((\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{sijkl}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl})$ be a LS-COM model of $(UM_{rtj2l}, CM_{tij2l}, M_{tijk})$ -congeneric variables with conditional regressive independence, then the parameter of the matrices $\Lambda_{\mathbf{B}}$, $\Lambda_{\mathbf{W}}$, $\Phi_{\mathbf{B}}$, $\Phi_{\mathbf{W}}$, $\Theta_{\mathbf{B}}$, and $\Theta_{\mathbf{W}}$ are identified, if either one factor loading λ_{sijkl} , λ_{CMij2l} , λ_{UMij2l} , λ_{Mijkl} for each factor S_{tij1l} , CM_{tij2l} , UM_{rtj2l} and M_{tijk} or the variance of the factors are set to any real value larger than 0, and

- (a) if $i = 2, j \geq 1, k \geq 2, l \geq 2$ and $\Phi_{\mathbf{B}}$ as well as $\Phi_{\mathbf{W}}$ contain permissible intercorrelations among the latent variables (i.e., nonzero elements in the off-diagonal), otherwise
- (b) if $i \geq 3, j \geq 1, k \geq 2, l \geq 2$.

Remarks. Assuming that the first factor loading parameters per latent factor (i.e., λ_{S1j1l} , λ_{M1jkl} , λ_{CM1j2l} , and λ_{UM1j2l}) are fixed to one and assuming that the latent method factors on level-1 and level-2 are substantially correlated with each other. Then each parameter of the model has a unique mathematical solution for a $2 \times 1 \times 2 \times 2$ MTMM-MO design. It is worth noting that the total covariance matrix $\Sigma_{\mathbf{T}}$ of the LS-COM model can be partitioned into a between covariance matrix $\Sigma_{\mathbf{B}}$ and a within covariance matrix $\Sigma_{\mathbf{W}}$ (see Section 2.6.2). The between covariance matrix $\Sigma_{\mathbf{B}}$ of any LS-COM model is a special case of the covariance matrix of a CS-C($M-1$) model for structurally different methods proposed by Geiser (2008) for the same dimension. The covariance matrix $\Sigma_{\mathbf{B}}$ of the LS-COM model is a special case, given that the residual variances of Y_{rtij2l} are set to zero on level-2. Therefore, the between covariance matrix $\Sigma_{\mathbf{B}}$ is a restrictive variant of the the variance-covariance matrix of a CS-C($M-1$) model for same number of indicators, constructs, methods, and occasions. Hence, the identification of the model on the target-level is proven by Geiser (2008). The within covariance matrix $\Sigma_{\mathbf{W}}$ is equivalent to the covariance matrix of a CFA-model. Thus, the “Three-Measurement-Rule” and “Two-Measurement-Rule” apply (see Bollen, 1989). In other words, the model on the within (rater) level is identified for two indicator per unique method factor if both factor are substantially correlated. If both factors are uncorrelated, at least three indicator per unique method factor are required for model identification.

2.10 Measurement invariance

Whenever researchers wish to compare test scores of different occasions of measurement (or of different groups), they have to ensure that the given measures assess the *same* constructs. That relates to the question whether or not the psychometric properties of a measure have changed over time or groups. In case of measurement non-invariance, it is not guaranteed that the differences in test scores can be directly be interpreted as differences or change in the level of an attribute (Geiser, 2008; Meredith, 1993; Tisak & Tisak, 2000). Therefore, establishing measurement invariance (MI) is an essential prerequisite for the analysis of test scores changes in longitudinal or in multigroup

studies. According to Widaman and Reise (1997, see also Meredith, 1993; Meredith & Horn, 2001) four different levels of measurement invariance can be distinguished:

1. factorial measurement invariance,
2. weak (or metric) measurement invariance,
3. strong measurement invariance, and
4. strict measurement invariance.

Factorial measurement invariance is the less restrictive form of MI and solely requires that the number of factors as well as the factor pattern of the latent factor loading parameters are similar across time points or groups. Weak factorial measurement invariance holds, if and only if the factor loading parameters per latent factor (state or method factors) are time-invariant. In addition to weak measurement invariance, strong measurement invariance requires that the intercepts of the observed variables are the same over time. In addition to strong MI, strict MI is established if and only if the residual variances of the observed variables are the same over time. The main advantage of MTMM-SEMs is to directly test the degree of measurement invariance via χ^2 fit statistics. Note that it is also possible to impose further restrictions on the factor structure. For example, researcher may impose additional constraints on the latent means, latent variances and/or covariances structure of the latent variables. In the next chapter, a latent change version of the LS-COM model is formally defined. With respect to this model latent difference variables are introduced. One important prerequisite of this (latent change) model is strong measurement invariance. In empirical applications, researchers should therefore test these restrictions before specifying a latent change model.

Definition 2.4 (LS-COM model with CRI and strong MI)

$\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{UM}_{rt}, \mathbf{CM}_t, \mathbf{M}_t, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2l}, \lambda_{CMij2l}, \lambda_{Mijkl} \rangle$ is called a LS-COM model of $(UM_{rtj2l}, CM_{tj2l}, M_{tijk})$ -congeneric variables with conditional regressive independence and with strong measurement invariance iff Definition 2.1, Theorem 2.1, Definition 2.2 hold and for each indicator i , construct j , method k and for each pair $(l, l') \in L \times L'$, $(l \neq l')$ there is a constant $\alpha_{tijk} \in \mathbb{R}$, a constant $\lambda_{Sijk} \in \mathbb{R}_+$, a constant $\lambda_{CMij2} \in \mathbb{R}_+$, a constant $\lambda_{UMij2} \in \mathbb{R}_+$, as well as a constant $\lambda_{Mijk} \in \mathbb{R}_+$, such that

$$\alpha_{tijk} \equiv \alpha_{tijk} = \alpha_{tijk'}, \quad (2.112)$$

$$\lambda_{Sijk} \equiv \lambda_{Sijk} = \lambda_{Sijk'}, \quad (2.113)$$

$$\lambda_{CMij2} \equiv \lambda_{CMij2l} = \lambda_{CMij2l'}, \quad (2.114)$$

$$\lambda_{UMij2} \equiv \lambda_{UMij2l} = \lambda_{UMij2l'}, \quad (2.115)$$

$$\lambda_{Mijk} \equiv \lambda_{Mijk} = \lambda_{Mijk'}, \quad \forall k > 2. \quad (2.116)$$

Remarks. With respect to the Definition 2.4 LS-COM model with CRI and MI is established by imposing restrictions on the level-2 intercepts as well as the factor loading parameters for each factor pertaining to the same indicator, construct and method. In particular, the level-2 intercepts α_{tijk} pertaining to the same indicator, construct and method are set equal across occasions of measurement. Furthermore, the latent factor loading parameters of the factors $(S_{tj1l}, M_{tijk}, CM_{tj2l},$ and $UM_{rtj2l})$ pertaining to the same indicator, construct, method but different

occasions of measurement are set equal. With respect to these restrictions, it is possible to define time-invariant intercepts (α_{tijk}) as well as time-invariant latent factor loading parameters (λ_{Sijk} , λ_{CMij2} , λ_{UMij2} , λ_{Mijk}).

Glossary

Box 2.1 (General)

Probability space \mathcal{E} projections

$(\Omega, \mathfrak{A}, P)$	<i>Probability space</i>
$p_T : \Omega \rightarrow \Omega_T$	<i>Mapping into a set of targets</i>
$p_{TS_i} : \Omega \rightarrow \Omega_{TS_i}$	<i>Mapping into a set of target-specific situations</i>
$p_R : \Omega \rightarrow \Omega_R$	<i>Mapping into a set of raters</i>
$p_{RS_i} : \Omega \rightarrow \Omega_{RS_i}$	<i>Mapping into a set of rater-specific situations</i>

Observed \mathcal{E} measurement error variables

Y_{tij1l}	<i>Observed variables pertaining to the reference (here: structurally different, $k = 1$) method (e.g., the self-ratings of the targets t of construct j on measurement occasion l with indicator i).</i>
Y_{rtij2l}	<i>Observed variables pertaining to the non-reference (here: interchangeable, $k = 2$) method (e.g., the ratings of the interchangeable raters r for particular targets t of construct j on measurement occasion l with indicator i).</i>
Y_{tijkl}	<i>Observed variables pertaining to the non-reference (here: another structurally different, $k \neq 1, 2$) method (e.g., the ratings of the boss for target t of construct j on measurement occasion l with indicator i).</i>
E_{tij1l}	<i>Measurement error variables pertaining to indicator i, construct j, reference method $k = 1$, and measurement occasion l.</i>
E_{rtij2l}	<i>Measurement error variables pertaining to indicator i, construct j, non-reference method $k = 2$, and measurement occasion l.</i>
E_{tijkl}	<i>Measurement error variables pertaining to indicator i, construct j, non-reference method $k \neq 1, 2$, and measurement occasion l.</i>

Box 2.2 (Latent State Model)*Latent variables of the LS-COM model*

S_{tij1l}	<i>target-specific latent state variables of the reference (here: structurally different, $k = 1$) method of construct j on measurement occasion l assessed by indicator i</i>
S_{rtij2l}	<i>rater-target-specific latent state variables of the non-reference (here: interchangeable, $k = 2$) method of construct j on measurement occasion l assessed by indicator i</i>
S_{tij2l}	<i>target-specific latent state variables of the non-reference (interchangeable, $k = 2$) method of construct j on measurement occasion l assessed by indicator i (i.e., latent group mean of the interchangeable ratings for a particular target)</i>
S_{tijk}	<i>target-specific latent state variables of other non-reference (here: structurally different, $k \neq 1, 2$) methods of construct j on measurement occasion l assessed by indicator i</i>
UM_{rtij2l}	<i>rater-target-specific latent unique method variables of the non-reference method $k = 2$ of construct j on measurement occasion l assessed by indicator i (e.g., unique view of a particular rater which is neither shared with the self-rating (here: reference method) nor with the common view of all raters for indicator i assessing construct j on measurement occasion l)</i>
CM_{tij2l}	<i>target-specific latent common method variables of the non-reference method $k = 2$ of construct j on measurement occasion l assessed by indicator i (e.g., common view of the raters which is not shared with the self-rating of a particular target for indicator i assessing construct j on measurement occasion l)</i>
M_{tijk}	<i>target-specific latent method variables of other non-reference methods $k \neq 1, 2$ of construct j on measurement occasion l assessed by indicator i (e.g., the unique view of the boss that is not shared with the self-rating of the target for indicator i assessing construct j on measurement occasion l)</i>

Box 2.3**Definition of the latent variables of the LS-COM**

$$\begin{aligned}
S_{tij1l} &\equiv E(Y_{tij1l}|p_T, p_{TS_i}), \\
S_{rtij2l} &\equiv E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}), \\
S_{tijkkl} &\equiv E(Y_{tijkkl}|p_T, p_{TS_i}), \quad \forall k > 2, \\
E_{tij1l} &\equiv Y_{tij1l} - E(Y_{tij1l}|p_T, p_{TS_i}), \\
E_{rtij2l} &\equiv Y_{rtij2l} - E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}), \\
E_{tijkkl} &\equiv Y_{tijkkl} - E(Y_{tijkkl}|p_T, p_{TS_i}), \quad \forall k > 2, \\
S_{tij2l} &\equiv E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_{TS_i}], \\
UM_{rtij2l} &\equiv E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i}) - \\
&\quad E(E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_{TS_i}), \\
CM_{tij2l} &\equiv E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_{TS_i}] - \\
&\quad E(E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_{TS_i}] | E(Y_{tij1l}|p_T, p_{TS_i})), \\
M_{tijkkl} &\equiv E(Y_{tijkkl}|p_T, p_{TS_i}) - \\
&\quad E(E(Y_{tijkkl}|p_T, p_{TS_i}) | E(Y_{tij1l}|p_T, p_{TS_i})).
\end{aligned}$$

Chapter 3

Formal definition of the baseline latent change (LC-COM) model

3.1 A gentle introduction

One of the key interests in psychology is to study interindividual differences in intraindividual change. The main advantage of longitudinal MTMM-SEMs is that “true” (i.e., measurement error free) change of constructs as well as method effects can be studied over time. In the following chapter, two latent change models are defined according to the stochastic measurement theory (Steyer, 1989; Steyer & Eid, 2001; Zimmermann, 1975). Latent change (LC) or latent difference (LD) models (McArdle & Hamagami, 2001; Steyer, Eid, & Schwenkmezger, 1997; Steyer, Partchev, & Shanahan, 2000) can be seen as an alternative parametrization of latent state (LS) models with strong measurement invariance. With respect to latent change (difference) models it is possible to explicitly model “true” interindividual differences in intraindividual change with regard to latent difference variables (see Steyer et al., 1997, 2000). In addition, it is possible to relate latent difference variables to manifest or latent background variables (e.g., age, gender, treatment groups etc.) in order to explain interindividual differences in “true” change scores. The latent difference (change) variables (denoted by S_{tij1l}^{BC}) are obtained by a simple tautological decomposition of the latent state variables pertaining to occasions of measurement l , where $l > 1$:

$$S_{tij1l} = S_{tij11} + (S_{tij1l} - S_{tij11}) = S_{tij11} + S_{tij1l}^{BC}, \quad \forall l > 1. \quad (3.1)$$

According to Equation 3.1, “true” change is modeled with respect to the initial status on the first measurement occasion $l = 1$ (i.e., baseline). Specifically, the latent change variables $(S_{tij1l} - S_{tij11}) = S_{tij1l}^{BC}$ represent the true intraindividual change of target t , measured by the reference method $k = 1$, indicator i , construct j from measurement occasion $l = 1$ (the initial status or baseline) to measurement occasion l . Due to this tautological reformulation of the latent state variables (see Equation 3.2), the model is called *baseline* latent change (baseline LC-COM) model. The superscript (BC) of the latent difference variables indicate that change is modeled with respect to the initial status (the baseline). In contrast to that, researchers may also investigate true change between each pair of subsequent measurement occasions which can be easily done by the following

parametrization:

$$S_{tij1l} = S_{tij1(l-1)} + (S_{tij1l} - S_{tij1(l-1)}) = S_{tij1(l-1)} + S_{tij1l}^{NC}, \quad \forall l > 1. \quad (3.2)$$

According to Equation 3.2, true change is not modeled with regard to the initial status, but rather with regard to the preceding measurement occasion [i.e., $(S_{tij1l} - S_{tij1(l-1)})$]. Hence, this model is called *neighbor* latent change (neighbor LC-COM) model. The superscript (NC) indicates that true change is studied with respect to the values of preceding (neighbor) latent state variables. Again, neighbor latent change models do not impose any additional restrictions and represent a simple restatement of latent state models with strong measurement invariance (Geiser, 2008). Moreover, reformulating a baseline change model into a neighbor change model and vice versa is straightforward and would even yield identical solutions for two occasions of measurement (see Equations 3.1 and 3.2). For the sake of simplicity, this thesis covers only baseline LC-COM change models. Precisely, *two* baseline LC-COM models are formally defined. With regard to the first model the main focus is on analyzing “true” change with respect to the latent *state* variables pertaining to the reference methods. This model will only differ slightly from the latent state model (LS-COM) with strong measurement invariance. The second model (extended baseline LC-COM) model enables researchers to study true change with respect to the reference as well as non-reference methods. Put differently, the extended LC-COM model allows studying “true” change of constructs as well as method effects. Moreover, it is possible to investigate “true” change of method bias on different levels (change of common or unique method bias). This modeling approach is particular beneficial with respect to intervention studies, given that “true” change of pure method effects (e.g. “true” change with respect to the over- or underestimation of the reference method that is specific to method influence of the non-reference methods) is studied and explained by manifest or latent background variables. The tautological decomposition of the latent method variables follows in a similar way as presented above:

$$\begin{aligned} UM_{rtij2l} &= UM_{rtij21} + (UM_{rtij2l} - UM_{rtij21}) = UM_{rtij21} + UM_{rtij2l}^{BC}, & \forall l > 1, \\ CM_{tij2l} &= CM_{tij21} + (CM_{tij2l} - CM_{tij21}) = CM_{tij21} + CM_{tij2l}^{BC}, & \forall l > 1, \\ M_{tijk1} &= M_{tijk1} + (M_{tijk1} - M_{tijk1}) = M_{tijk1} + M_{tijk1}^{BC}, & \forall k > 2 \wedge \forall l > 1. \end{aligned}$$

In order to define latent change (LC) models strong measurement invariance is necessary (see Proof 10 below). If strong measurement invariance does not hold, the meaning of the latent difference variables is ambiguous and thus the variables should not be interpreted (Geiser, 2008). Furthermore, with respect to strong measurement invariance latent state (LS) and latent change (LC) models are algebraically equivalent. Figure 3 shows the extended baseline LC-COM model for three indicators ($i = 1,2,3$), two constructs ($j = 1,2$), three methods ($k = 1,2,3$) and two occasions of measurement ($l = 1,2$).

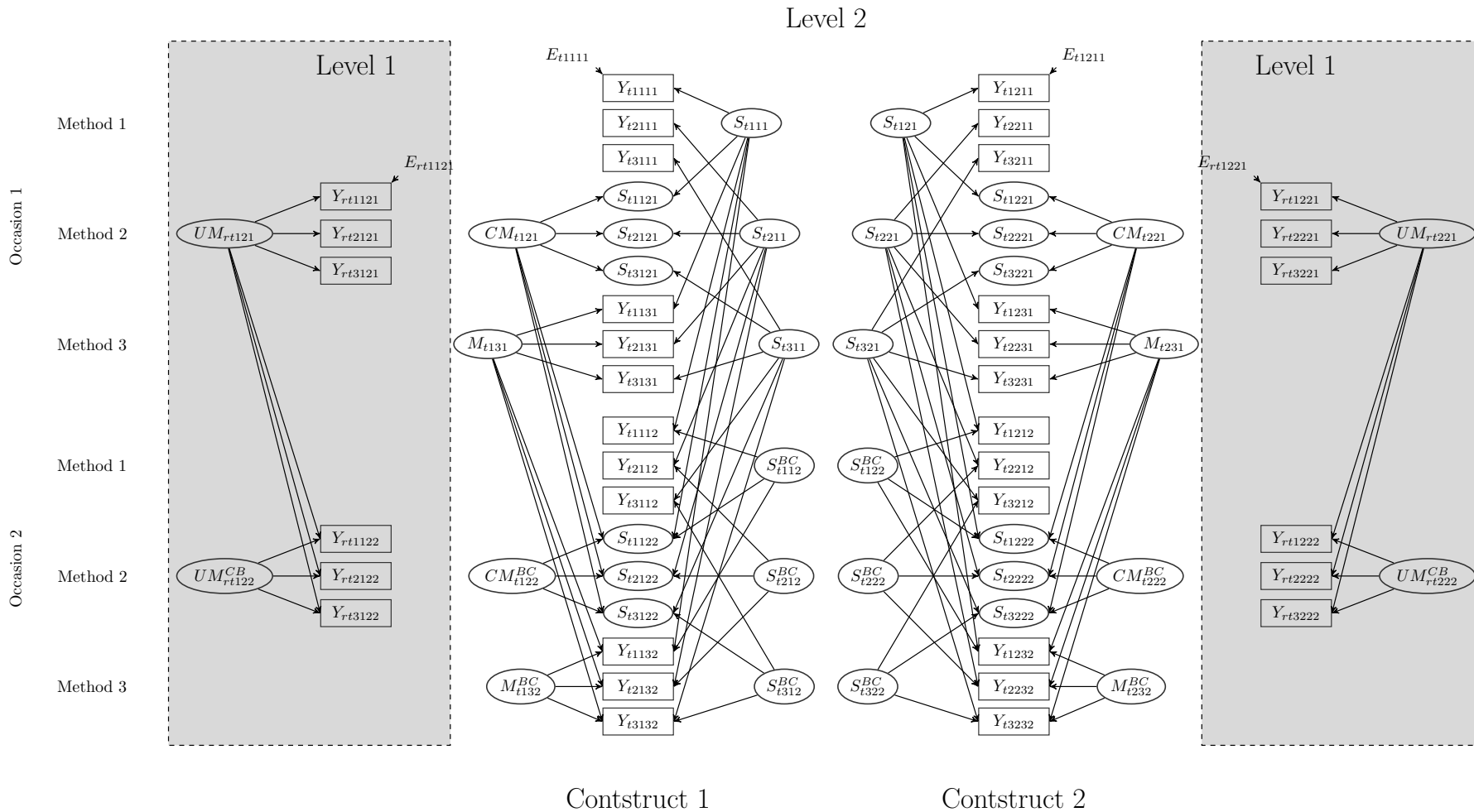


Figure 3.1: Path diagram of the extended baseline LC-COM model with indicator-specific latent state and change factors . An extended baseline LC-COM model with indicator-specific latent state and change factors incorporating for three indicators ($i=1,2,3$), two constructs ($j=1,2$), three methods ($k=1,2,3$) and two occasions of measurement ($l=1,2$). All factor loadings as well as correlations between latent variables were omitted for clarity. Measurement error variables E_{rtijkl} and E_{tijkl} are only depicted for the first indicator pertaining to method 1 and 2.

3.2 Definition of the simple baseline LC-COM model

Definition 3.1 (Simple baseline LC-COM model)

The random variables $\{Y_{111111}, \dots, Y_{abcdef}\}$ and $\{Y_{111111}, \dots, Y_{bcdef}\}$ on a probability space $(\Omega, \mathfrak{A}, P)$ are variables of a baseline LC-COM model if the conditions made in Definition 2.4 hold.

(a) For all $i \in I, j \in J, k \in K, l \in L$, and $\forall l > 1$, let

$$S_{tij1l}^{BC} \equiv (S_{tij1l} - S_{tij11}),$$

be also real-valued random variables on $(\Omega, \mathfrak{A}, P)$ with finite first- and second-order moments.

(b) Then the measurement equations of any observed variable $Y_{(r)tijkl}$ pertaining to the same indicator i , construct j , and measurement occasion l (where $\forall l > 1$), can be rewritten as follows:

$$Y_{tij1l} = S_{tij11} + S_{tij1l}^{BC} + E_{tij1l}, \quad \forall l > 1, \quad (3.3)$$

$$Y_{tijkl} = \alpha_{tijk} + \lambda_{S_{ijk}} S_{tij11} + \lambda_{S_{ijk}} S_{tij1l}^{BC} + \lambda_{M_{ijk}} M_{tjkl} + E_{tijkl}, \quad \forall k > 2, l > 1, \quad (3.4)$$

$$Y_{rtij2l} = \alpha_{tij2} + \lambda_{S_{ij2}} S_{tij11} + \lambda_{S_{ij2}} S_{tij1l}^{BC} + \lambda_{CM_{ij2}} CM_{tj2l} + \lambda_{UM_{ij2}} UM_{rtj2l} + E_{rtij2l}, \quad \forall l > 1. \quad (3.5)$$

Remarks. According to the definition 3.1, it is clear that the latent baseline change (LC-COM) model is an alternative parametrization of the LS-COM model with CRI and strong measurement invariance. Consequently, the measurement equations of the observed variables in the LS-COM model can be rewritten as stated in the equations of Condition (b). With regard to these equations, it is assumed that any latent state variable S_{tij1l} of measurement occasion l , where $l > 1$ can be fully decomposed into a initial state S_{tij11} and the difference of both states $(S_{tij1l} - S_{tij11})$. The difference of both latent state variables is defined as a latent change variable representing true interindividual differences in intraindividual change with respect to the reference method. Given that the LC-COM model is algebraically equivalent to the LS-COM model, the psychometric properties of the LC-COM model regarding existence, uniqueness, admissible transformation and meaningfulness of the latent variables remain unaltered as shown for the LS-COM model with CRI and strong MI. It is important to note that the occasion index l has been dropped from the intercepts and factor loadings to express that these parameters are time-invariant. Note that the same correlations that are zero as a consequence of the LS-COM model definition have to be also constrained in the LC-COM model. Furthermore, it is recommended to fix any correlations between latent state variables S_{tij1l} and latent method variables $UM_{rtj2l}, CM_{tj2l}, M_{tjkl}$ to zero. As a direct consequence of these restrictions, all latent difference variables S_{tij1l}^{BC} are also uncorrelated with all other latent method variables $UM_{rtj2l}, CM_{tj2l}, M_{tjkl}$ in the LC-COM model.

3.3 Definition of the extended baseline LC-COM model

In the following section, an extended version of the LC-COM model is presented. With respect to the extended LC-COM model it is possible to study “true” intraindividual change with regard to trait and method effects. Moreover, it is possible to investigate the true change of common and unique rater bias. These latent change method effects can be explained by other covariates (e.g., gender, age, treatment group).

Definition 3.2 (Extended Baseline LC-COM model)

The random variables $\{Y_{111111}, \dots, Y_{abcdef}\}$ and $\{Y_{11111}, \dots, Y_{bcdef}\}$ on a probability space $(\Omega, \mathfrak{A}, P)$ are variables of an extended baseline LC-COM model if the conditions in Definition 3.1 hold.

(a) For all $i \in I, j \in J, k \in K, l \in L$ and $\forall l > 1$, let

$$S_{tij1l}^{BC} \equiv (S_{tij1l} - S_{tij11}), \quad (3.6)$$

$$UM_{rtij2l}^{BC} \equiv (UM_{rtij2l} - UM_{rtij21}), \quad (3.7)$$

$$CM_{tij2l}^{BC} \equiv (CM_{tij2l} - CM_{tij21}), \quad (3.8)$$

$$M_{tijk1l}^{BC} \equiv (M_{tijk1l} - M_{tijk1}), \quad (3.9)$$

be also real-valued random variables on $(\Omega, \mathfrak{A}, P)$ with finite first- and second-order moments.

(b) Then the measurement equations of any observed variable Y_{tij1l}, Y_{tijk1l} , or Y_{rtij2l} pertaining to the same indicator i and construct j and measurement occasion l (where $l > 1$), can be rewritten as follows:

$$Y_{tij1l} = S_{tij11} + S_{tij1l}^{BC} + E_{tij1l}, \quad \forall l > 1, \quad (3.10)$$

$$Y_{tijk1l} = \alpha_{tijk} + \lambda_{Sijk} S_{tij11} + \lambda_{Sijk} S_{tij1l}^{BC} + \lambda_{Mijk} M_{tijk1} + \lambda_{Mijk} M_{tijk1l}^{BC} + E_{tijk1l}, \quad \forall k > 2, l > 1, \quad (3.11)$$

$$Y_{rtij2l} = \alpha_{tij2} + \lambda_{Sij2} S_{tij11} + \lambda_{Sij2} S_{tij1l}^{BC} + \lambda_{CMij2} CM_{tij21} + \lambda_{CMij2} CM_{tij2l}^{BC} + \lambda_{UMij2} UM_{rtij21} + \lambda_{UMij2} UM_{rtij2l}^{BC} + E_{rtij2l}, \quad \forall l > 1. \quad (3.12)$$

(c) Definition of common latent common method difference variables. For each construct j , measured by a non-reference method k (in this case, $k = 2$) on occasion of measurement $l, l > 1$ and for each pair $(i, i') \in I \times I', (i \neq i')$ there is a constant $\lambda_{CMii'j2} \in \mathbb{R}_+$, such that

$$CM_{tij2l}^{BC} = \lambda_{CMii'j2} CM_{ti'j2l}^{BC}. \quad (3.13)$$

(d) Definition of common latent method difference variables. For each construct j , measured by a non-reference method k (in this case $k > 2$) on occasion of measurement $l, l > 1$ and for each pair $(i, i') \in I \times I', (i \neq i')$ there is a constant $\lambda_{Mii'jk} \in \mathbb{R}_+$, such that

$$M_{tijk1l}^{BC} = \lambda_{Mii'jk} M_{ti'jkl}^{BC}, \quad \forall k > 2. \quad (3.14)$$

(e) Definition of common latent unique method difference variables. For each construct j , measured by a non-reference method k (in this case $k = 2$) on occasion of measurement $l, l > 1$ and for each pair $(i, i') \in I \times I', (i \neq i')$ there is a constant $\lambda_{UMii'j2} \in \mathbb{R}_+$, such that

$$UM_{rtij2l}^{BC} = \lambda_{UMii'j2} UM_{rti'j2l}^{BC}. \quad (3.15)$$

Remarks. The above Definition 3.2 implies that the latent change version of the LC-COM model is simply a reformulation of the latent state version of the LC-COM model (described in Chapter 2). In addition to that, strong measurement invariance is a necessary condition for defining and interpreting latent difference variables. According to Equations 3.6 to 3.9, latent difference variables can be construed. For example, the latent state difference variables are construed by this tautological expression: $S_{tij1l} = S_{tij11} + (S_{tij1l} - S_{tij11})$, where $l > 1$. Put differently, a

later reference state variable is perfectly determined by the initial latent reference state and the latent difference between the initial and the later reference state. In the same logic, the latent difference method variables are defined. According to Equations 3.13 to 3.15, it is assumed that latent difference method variables are perfectly correlated with each other. In other words, latent difference method variables belonging to the same construct j , same method k , and same occasion of measurement, but different indicators i and i' only differ by a multiplicative constant (i.e., $\lambda_{CMii'j2}$, $\lambda_{UMii'j2}$, $\lambda_{Mii'jk}$, where $k > 2$). Thus, these latent difference method variables are positive linear functions of each other, respectively.

3.4 Existence

The conditions made in Definition 1 logically imply that the method variables, belonging to the same construct j , method k , and occasion of measurement l , where $l > 1$, but different indicators i and i' are linear functions of each other. The following theorem entails the existence of the latent method factors CM_{ij2l}^{BC} , M_{ijkl}^{BC} , and UM_{rtij2l}^{BC} .

Theorem 3.1 (Existence)

The random variables $\{Y_{111111}, \dots, Y_{abcdef}\}$ and $\{Y_{11111}, \dots, Y_{bcdef}\}$ are CM_{ij2l}^{BC} , M_{ijkl}^{BC} , UM_{rtij2l}^{BC} -congeneric variables of a LC-COM model if the conditions a of Definition 3.1 hold and for each $r \in R$, $t \in T$, $i \in I$, $j \in J$, $k \in K$, $l \in L$, there are real-valued random variables M_{ij2l}^{BC} , M_{ijkl}^{BC} , and UM_{rtij2l}^{BC} on a probability space $(\Omega, \mathfrak{A}, P)$ and $(\lambda_{CMij2}, \lambda_{Mijk}, \lambda_{UMij2}) \in \mathbb{R}^+$, such that:

$$CM_{ij2l}^{BC} = \lambda_{CMij2} CM_{ij2}^{BC}, \quad \forall l > 1, \quad (3.16)$$

$$M_{ijkl}^{BC} = \lambda_{Mijk} M_{ijkl}^{BC}, \quad \forall k > 2, l > 1, \quad (3.17)$$

$$UM_{rtij2l}^{BC} = \lambda_{UMij2} UM_{rtij2l}^{BC}, \quad \forall l > 1. \quad (3.18)$$

Proofs. 6 Existence.

3.16 For all i, j, k, l , assume that $CM_{ij2l}^{BC} \equiv CM_{t1j2l}^{BC}$ as well as $\lambda_{CMij2} \equiv \lambda_{CMi1j2}$. Inserting these parameters in Equation 3.13 of the above definition, yields Equation 3.16:

$$CM_{ij2l}^{BC} = \lambda_{CMij2} CM_{ij2}^{BC} \text{ (repeated).}$$

Similarly, according to Equation 3.16, CM_{ij2l}^{BC} can be expressed as

$$CM_{ij2l}^{BC} = \frac{CM_{ij2l}^{BC}}{\lambda_{CMij2}} \text{ as well as } CM_{ij2l}^{BC} = \frac{CM_{ij2l}^{BC}}{\lambda_{CMi'j2}}.$$

If both Equations are set equal, it follows $CM_{ij2l}^{BC} = \frac{\lambda_{CMij2}}{\lambda_{CMi'j2}} CM_{ij2l}^{BC}$. Let $\lambda_{CMii'j2} \equiv \frac{\lambda_{CMij2}}{\lambda_{CMi'j2}}$, than Equation 3.13 is obtained:

$$CM_{ij2l}^{BC} = \lambda_{CMii'j2} CM_{ij2l}^{BC} \text{ (repeated).}$$

The proofs for Equation 3.17 and 3.18 follow the same logic and therefore will be left to the reader. □

Remarks. The above theorem clarifies that the assumptions made in Definitions 3.13 to 3.15 are equivalent to assuming common latent difference variables CM_{ij2l}^{BC} , M_{ijkl}^{BC} , UM_{rtij2l}^{BC} . Again, the term ‘‘common’’ refers to the fact that each latent method difference factor is assumed to be common for all indicators, belonging to the same construct, same (non-reference) method, and the same occasion of measurement. The proof of this theorem also shows that the latent method difference variables CM_{ij2l}^{BC} , M_{ijkl}^{BC} , UM_{rtij2l}^{BC} are not *uniquely* defined. The uniqueness of the latent method factors is discussed in the next Section 3.5.

3.5 Uniqueness

The latent factors $(CM_{tj2l}^{BC}, M_{tjkl}^{BC}, UM_{rtj2l}^{BC})$ are not uniquely defined in the extended baseline LC-COM model. If such models are defined with $(CM_{tj2l}^{BC}, M_{tjkl}^{BC}, UM_{rtj2l}^{BC})$ -congeneric variables, all of these parameters are defined up to similarity transformations. That is to say that these latent variables are only uniquely defined up to multiplications with a positive real number. In other words, the latent change method variables are measured on a ratio scale. The next theorem concerns the uniqueness and admissible transformations of parameters in the extended latent baseline LC-COM models.

Theorem 3.2 (Uniqueness)

1. Admissible transformations

$\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk}, \rangle$ is a baseline LC-COM model with:

$$\mathbf{S}_{rt} \equiv (S_{111111} \cdots S_{rtij2l} \cdots S_{abcd2f})^T, \quad (3.19)$$

$$\mathbf{S}_t \equiv (S_{111111} \cdots S_{tijkl} \cdots S_{bcdef})^T, \quad (3.20)$$

$$\mathbf{S}_t^{BC} \equiv (S_{111112}^{BC} \cdots S_{tijkl}^{BC} \cdots S_{bcdef}^{BC})^T, \quad (3.21)$$

$$\mathbf{UM}_{rt} \equiv (UM_{111111} \cdots UM_{rtj2l} \cdots UM_{abd2f})^T, \quad (3.22)$$

$$\mathbf{UM}_{rt}^{BC} \equiv (UM_{111112}^{BC} \cdots UM_{rtj2l}^{BC} \cdots UM_{abd2f}^{BC})^T, \quad (3.23)$$

$$\mathbf{CM}_t \equiv (CM_{111111} \cdots CM_{tj2l} \cdots CM_{bd2f})^T, \quad (3.24)$$

$$\mathbf{CM}_t^{BC} \equiv (CM_{111112}^{BC} \cdots CM_{tj2l}^{BC} \cdots CM_{bd2f}^{BC})^T, \quad (3.25)$$

$$\mathbf{M}_t \equiv (M_{111111} \cdots M_{tijkl} \cdots M_{bdef})^T, \quad (3.26)$$

$$\mathbf{M}_t^{BC} \equiv (M_{111112}^{BC} \cdots (M_{tijkl}^{BC} \cdots (M_{bdef}^{BC}))^T, \quad (3.27)$$

$$\mathbf{E}_{rt} \equiv (E_{111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^T, \quad (3.28)$$

$$\mathbf{E}_t \equiv (E_{111111} \cdots E_{tijkl} \cdots E_{bcdef})^T, \quad (3.29)$$

$$\alpha_{tijk} \equiv (\alpha_{111111} \cdots \alpha_{tijk} \cdots \alpha_{bcde})^T, \quad (3.30)$$

$$\lambda_{\mathbf{S}} \equiv (\lambda_{111} \cdots \lambda_{Sijk} \cdots \lambda_{cde})^T, \quad (3.31)$$

$$\lambda_{\mathbf{UM}} \equiv (\lambda_{111} \cdots \lambda_{UMij2} \cdots \lambda_{cd2})^T, \quad (3.32)$$

$$\lambda_{\mathbf{CM}} \equiv (\lambda_{111} \cdots \lambda_{CMij2} \cdots \lambda_{cd2})^T, \quad (3.33)$$

$$\lambda_{\mathbf{M}} \equiv (\lambda_{111} \cdots \lambda_{Mijk} \cdots \lambda_{cde})^T. \quad (3.34)$$

and if for all $r \in R, t \in T, i \in I, j \in J, k \in K, l \in L$:

$$UM_{rtj2l}'^{BC} \equiv \beta_{UMj2} UM_{rtj2l}^{BC}, \quad (3.35)$$

$$CM_{tj2l}'^{BC} \equiv \beta_{CMj2} CM_{tj2l}^{BC}, \quad (3.36)$$

$$M_{tjkl}'^{BC} \equiv \beta_{Mjk} M_{tjkl}^{BC}, \quad \forall k > 2, \quad (3.37)$$

$$\lambda_{UMij2}' \equiv \lambda_{UMij2} / \beta_{UMj2}, \quad (3.38)$$

$$\lambda_{CMij2}' \equiv \lambda_{CMij2} / \beta_{CMj2}, \quad (3.39)$$

$$\lambda_{Mijk}' \equiv \lambda_{Mijk} / \beta_{Mjk}, \quad \forall k > 2, \quad (3.40)$$

$$(3.41)$$

where $\beta_{CMj2}, \beta_{UMj2}$, and $\beta_{Mjk} \in \mathbb{R}$, as well as $\beta_{CMj2}, \beta_{UMj2}$, and $\beta_{Mjk} > 0$, then $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk},$

$\lambda_{\mathbf{Sijk}}, \lambda'_{\mathbf{UMij2}}, \lambda'_{\mathbf{CMij2}}, \lambda'_{\mathbf{Mijk}}, \rangle$ is a baseline LC-COM change model, too, with:

$$\mathbf{S}_{\mathbf{rt}} \equiv (S_{111111} \cdots S_{rtij2l} \cdots S_{abcd2f})^{\mathbf{T}}, \quad (3.42)$$

$$\mathbf{S}_{\mathbf{t}} \equiv (S_{111111} \cdots S_{tijk} \cdots S_{bcdef})^{\mathbf{T}}, \quad (3.43)$$

$$\mathbf{S}_{\mathbf{t}}^{\mathbf{BC}} \equiv (S_{11112}^{\mathbf{BC}} \cdots S_{tijk}^{\mathbf{BC}} \cdots S_{bcdef}^{\mathbf{BC}})^{\mathbf{T}}, \quad (3.44)$$

$$\mathbf{UM}_{\mathbf{rt}} \equiv (UM_{111111} \cdots UM_{rtij2l} \cdots UM_{abd2f})^{\mathbf{T}}, \quad (3.45)$$

$$\mathbf{UM}_{\mathbf{rt}}^{\mathbf{BC}} \equiv (UM_{11112}^{\mathbf{BC}} \cdots UM_{rtij2l}^{\mathbf{BC}} \cdots UM_{abd2f}^{\mathbf{BC}})^{\mathbf{T}}, \quad (3.46)$$

$$\mathbf{CM}_{\mathbf{t}} \equiv (CM_{111111} \cdots CM_{tj2l} \cdots CM_{bd2f})^{\mathbf{T}}, \quad (3.47)$$

$$\mathbf{CM}_{\mathbf{t}}^{\mathbf{BC}} \equiv (CM_{11112}^{\mathbf{BC}} \cdots CM_{tj2l}^{\mathbf{BC}} \cdots CM_{bd2f}^{\mathbf{BC}})^{\mathbf{T}}, \quad (3.48)$$

$$\mathbf{M}_{\mathbf{t}} \equiv (M_{111111} \cdots M_{tijk} \cdots M_{bcdef})^{\mathbf{T}}, \quad (3.49)$$

$$\mathbf{M}_{\mathbf{t}}^{\mathbf{BC}} \equiv (M_{11112}^{\mathbf{BC}} \cdots M_{tijk}^{\mathbf{BC}} \cdots M_{bcdef}^{\mathbf{BC}})^{\mathbf{T}}, \quad (3.50)$$

$$\mathbf{E}_{\mathbf{rt}} \equiv (E_{111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^{\mathbf{T}}, \quad (3.51)$$

$$\mathbf{E}_{\mathbf{t}} \equiv (E_{111111} \cdots E_{tijk} \cdots E_{bcdef})^{\mathbf{T}}, \quad (3.52)$$

$$\alpha_{\mathbf{tijk}} \equiv (\alpha_{111111} \cdots \alpha_{tijk} \cdots \alpha_{bcde})^{\mathbf{T}}, \quad (3.53)$$

$$\lambda_{\mathbf{S}} \equiv (\lambda_{111} \cdots \lambda_{Sijk} \cdots \lambda_{cde})^{\mathbf{T}}, \quad (3.54)$$

$$\lambda'_{\mathbf{UM}} \equiv (\lambda'_{111} \cdots \lambda'_{UMij2} \cdots \lambda'_{cd2})^{\mathbf{T}}, \quad (3.55)$$

$$\lambda'_{\mathbf{CM}} \equiv (\lambda'_{111} \cdots \lambda'_{CMij2} \cdots \lambda'_{cd2})^{\mathbf{T}}, \quad (3.56)$$

$$\lambda'_{\mathbf{M}} \equiv (\lambda'_{111} \cdots \lambda'_{Mijk} \cdots \lambda'_{cde})^{\mathbf{T}}. \quad (3.57)$$

2. Uniqueness

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{\mathbf{rt}}, \mathbf{S}_{\mathbf{t}}, \mathbf{S}_{\mathbf{t}}^{\mathbf{BC}}, \mathbf{UM}_{\mathbf{rt}}, \mathbf{UM}_{\mathbf{rt}}^{\mathbf{BC}}, \mathbf{CM}_{\mathbf{t}}, \mathbf{CM}_{\mathbf{t}}^{\mathbf{BC}}, \mathbf{M}_{\mathbf{t}}, \mathbf{M}_{\mathbf{t}}^{\mathbf{BC}}, \mathbf{E}_{\mathbf{rt}}, \mathbf{E}_{\mathbf{t}}, \alpha_{\mathbf{tijk}}, \lambda_{\mathbf{Sijk}}, \lambda_{\mathbf{UMij2}}, \lambda_{\mathbf{CMij2}}, \lambda_{\mathbf{Mijk}}, \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{\mathbf{rt}}, \mathbf{S}_{\mathbf{t}}, \mathbf{S}_{\mathbf{t}}^{\mathbf{BC}}, \mathbf{UM}_{\mathbf{rt}}, \mathbf{UM}_{\mathbf{rt}}^{\mathbf{BC}}, \mathbf{CM}_{\mathbf{t}}, \mathbf{CM}_{\mathbf{t}}^{\mathbf{BC}}, \mathbf{M}_{\mathbf{t}}, \mathbf{M}_{\mathbf{t}}^{\mathbf{BC}}, \mathbf{E}_{\mathbf{rt}}, \mathbf{E}_{\mathbf{t}}, \alpha_{\mathbf{tijk}}, \lambda_{\mathbf{Sijk}}, \lambda'_{\mathbf{UMij2}}, \lambda'_{\mathbf{CMij2}}, \lambda'_{\mathbf{Mijk}}, \rangle$ are baseline LC-COM change models, then there are for each $i \in I, j \in J, k \in K, l \in L, \beta_{CMj2}, \beta_{UMj2},$ and $\beta_{Mjk} \in \mathbb{R}^+$ such that Equations 3.35 to 3.57 hold.

Proofs. 7 Uniqueness & admissible transformations

1. Admissible transformations

If $UM_{rtij2l}^{\mathbf{BC}}, CM_{tj2l}^{\mathbf{BC}},$ and $M_{tijk}^{\mathbf{BC}}$ are replaced by $UM_{rtij2l}'^{\mathbf{BC}}, CM_{tj2l}'^{\mathbf{BC}}, M_{tijk}'^{\mathbf{BC}}$ as well as $\lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk}$ by the corresponding $\lambda'_{UMij2}, \lambda'_{CMij2}, \lambda'_{Mijk},$ then:

$$\begin{aligned} UM_{rtij2l}^{\mathbf{BC}} &= \lambda'_{UMij2} UM_{rtij2l}'^{\mathbf{BC}} = \lambda_{UMij2} UM_{rtij2l}^{\mathbf{BC}} \\ &= \left(\frac{1}{\beta_{UMj2}} \right) \lambda_{UMij2} \cdot \beta_{UMj2} UM_{rtij2l}^{\mathbf{BC}} \\ &= \lambda_{UMij2} UM_{rtij2l}^{\mathbf{BC}}, \end{aligned}$$

$$\begin{aligned} CM_{tj2l}^{\mathbf{BC}} &= \lambda'_{CMij2} CM_{tj2l}'^{\mathbf{BC}} = \lambda_{CMij2} CM_{tj2l}^{\mathbf{BC}} \\ &= \left(\frac{1}{\beta_{CMj2}} \right) \lambda_{CMij2} \cdot \beta_{CMj2} CM_{tj2l}^{\mathbf{BC}} \\ &= \lambda_{CMij2} CM_{tj2l}^{\mathbf{BC}}, \end{aligned}$$

$$\begin{aligned} M_{tijk}^{\mathbf{BC}} &= \lambda'_{Mijk} M_{tijk}'^{\mathbf{BC}} = \lambda_{Mijk} M_{tijk}^{\mathbf{BC}} \\ &= \left(\frac{1}{\beta_{Mjk}} \right) \lambda_{Mijk} \cdot \beta_{Mjk} M_{tijk}^{\mathbf{BC}} \\ &= \lambda_{Mijk} M_{tijk}^{\mathbf{BC}}. \end{aligned}$$

In a similar way, if UM_{rtj2l}^{BC} , CM_{tj2l}^{BC} , M_{tjkl}^{BC} are replaced by $\frac{UM'_{rtj2l}{}^{BC}}{\beta_{UMj2}}$, $\frac{CM'_{tj2l}{}^{BC}}{\beta_{CMj2}}$, and $\frac{M'_{tjkl}{}^{BC}}{\beta_{Mjk}}$ as well as λ_{UMij2} , λ_{CMij2} , λ_{Mijk} by $\beta_{UMj2}\lambda'_{UMij2}$, $\beta_{CMj2}\lambda'_{CMij2}$, $\beta_{Mjk}\lambda'_{Mijk}$, then:

$$\begin{aligned} UM_{rtj2l}^{BC} &= \lambda_{UMij2}UM_{rtj2l}^{BC} \\ &= \beta_{UMj2}\lambda'_{UMij2} \cdot \frac{UM'_{rtj2l}{}^{BC}}{\beta_{UMj2}} \\ &= \lambda'_{UMij2}UM'_{rtj2l}{}^{BC}, \end{aligned}$$

$$\begin{aligned} CM_{tj2l}^{BC} &= \lambda_{CMij2}CM_{tj2l}^{BC} \\ &= \beta_{CMj2}\lambda'_{CMij2} \cdot \frac{CM'_{tj2l}{}^{BC}}{\beta_{CMj2}} \\ &= \lambda'_{CMij2}CM'_{tj2l}{}^{BC}, \end{aligned}$$

$$\begin{aligned} M_{tjkl}^{BC} &= \lambda_{Mijk}M_{tjkl}^{BC} \\ &= \beta_{Mjk}\lambda'_{Mijk} \cdot \frac{M'_{tjkl}{}^{BC}}{\beta_{Mjk}} \\ &= \lambda'_{Mijk}M'_{tjkl}{}^{BC}. \end{aligned}$$

2. Uniqueness

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk} \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda'_{UMij2}, \lambda'_{CMij2}, \lambda'_{Mijk} \rangle$ are baseline LC-COM change models, then $\lambda_{UMij2}UM_{rtj2l}^{BC} = \lambda'_{UMij2}UM'_{rtj2l}{}^{BC}$. Consequently, for all $j \in J$, $k \in K$, and $l \in L$:

$$UM'_{rtj2l}{}^{BC} = \frac{\lambda_{UMij2}}{\lambda'_{UMij2}}UM_{rtj2l}^{BC}.$$

Given that the ratio of λ_{UMij2} and λ'_{UMij2} has to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, a real constant can be defined for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\beta_{UMj2} \equiv \frac{\lambda_{UMij2}}{\lambda'_{UMij2}}.$$

Again, assume that both \mathcal{M} and \mathcal{M}' are LC-COM models, then

$\lambda_{CMij2}CM_{tj2l}^{BC} = \lambda'_{CMij2}CM'_{tj2l}{}^{BC}$. Consequently, for all $j \in J$, $k \in K$, and $l \in L$:

$$CM'_{tj2l}{}^{BC} = \frac{\lambda_{CMij2}}{\lambda'_{CMij2}}CM_{tj2l}^{BC}.$$

Given that the ratio of λ_{CMij2} and λ'_{CMij2} have to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, a real constant can be defined for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\beta_{CMj2} \equiv \frac{\lambda_{CMij2}}{\lambda'_{CMij2}}.$$

Again, if both \mathcal{M} and \mathcal{M}' are LC-COM models, then $\lambda_{Mijk}M_{tjkl}^{BC} = \lambda'_{Mijk}M'_{tjkl}{}^{BC}$. Consequently, for all $j \in J$, $k \in K$, and $l \in L$:

$$M'_{tjkl}{}^{BC} = \frac{\lambda_{Mijk}}{\lambda'_{Mijk}}M_{tjkl}^{BC}.$$

Given that the ratio of λ_{Mijk} and λ'_{Mijk} have to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, a real constant can be defined for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\beta_{Mjk} \equiv \frac{\lambda_{Mijk}}{\lambda'_{Mijk}}.$$

□

Remarks. The above theorem implies that the latent method factors UM_{rtj2l}^{BC} , CM_{tj2l}^{BC} , and M_{tjkl}^{BC} as well as their corresponding factor loadings λ_{UMij2} , λ_{CMij2} , and λ_{Mijk} are uniquely defined up to similarity transformations, that is, up to a multiplication with a positive real number. Hence, the latent method factors as well as their corresponding factor loadings are measured at the ratio level.

3.6 Meaningfulness

In the following section, meaningful statements regarding parameters of the LC-COM model are addressed. The next theorem lists a selection of meaningful statements regarding the latent method factors and their corresponding factor loadings.

Theorem 3.3 (Meaningfulness)

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{sijk}, \lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk} \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{sijk}, \lambda'_{UMij2}, \lambda'_{CMij2}, \lambda'_{Mijk} \rangle$ are baseline LC-COM change models, then for $\omega_1, \omega_2 \in \Omega$; $r, r' \in R$, $t, t' \in T$, $i, i' \in I$, $j, j' \in J$, $k, k' \in K$, and $l, l' \in L$:

$$\frac{\lambda_{UMij2}}{\lambda_{UMi'j2}} = \frac{\lambda'_{UMij2}}{\lambda'_{UMi'j2}}, \quad (3.58)$$

$$\frac{\lambda_{CMij2}}{\lambda_{CMi'j2}} = \frac{\lambda'_{CMij2}}{\lambda'_{CMi'j2}}, \quad (3.59)$$

$$\frac{\lambda_{Mijk}}{\lambda_{Mi'jk}} = \frac{\lambda'_{Mijk}}{\lambda'_{Mi'jk}}, \quad (3.60)$$

$$\frac{\lambda_{UMij2}}{\lambda_{UMi'j2}} - \frac{\lambda_{UMij'2}}{\lambda_{UMi'j'2}} = \frac{\lambda'_{UMij2}}{\lambda'_{UMi'j2}} - \frac{\lambda'_{UMij'2}}{\lambda'_{UMi'j'2}}, \quad (3.61)$$

$$\frac{\lambda_{CMij2}}{\lambda_{CMi'j2}} - \frac{\lambda_{CMij'2}}{\lambda_{CMi'j'2}} = \frac{\lambda'_{CMij2}}{\lambda'_{CMi'j2}} - \frac{\lambda'_{CMij'2}}{\lambda'_{CMi'j'2}}, \quad (3.62)$$

$$\frac{\lambda_{Mijk}}{\lambda_{Mi'jk}} - \frac{\lambda_{Mij'k'}}{\lambda_{Mi'j'k'}} = \frac{\lambda'_{Mijk}}{\lambda'_{Mi'jk}} - \frac{\lambda'_{Mij'k'}}{\lambda'_{Mi'j'k'}}, \quad (3.63)$$

$$\frac{UM_{rtj2l}^{BC}(\omega_1)}{UM_{rtj2l}^{BC}(\omega_2)} = \frac{UM'_{rtj2l}{}^{BC}(\omega_1)}{UM'_{rtj2l}{}^{BC}(\omega_2)}, \quad (3.64)$$

for $UM_{rtj2l}^{BC}(\omega_2)$ and $UM'_{rtj2l}{}^{BC}(\omega_2) \neq 0$,

$$\frac{CM_{tj2l}^{BC}(\omega_1)}{CM_{tj2l}^{BC}(\omega_2)} = \frac{CM'_{tj2l}{}^{BC}(\omega_1)}{CM'_{tj2l}{}^{BC}(\omega_2)}, \quad (3.65)$$

for $CM_{tj2l}^{BC}(\omega_2)$ and $CM'_{tj2l}{}^{BC}(\omega_2) \neq 0$,

$$\frac{M_{tjkl}^{BC}(\omega_1)}{M_{tjkl}^{BC}(\omega_2)} = \frac{M'_{tjkl}{}^{BC}(\omega_1)}{M'_{tjkl}{}^{BC}(\omega_2)}, \quad (3.66)$$

for $M_{tjkl}^{BC}(\omega_2)$ and $M'_{tjkl}{}^{BC}(\omega_2) \neq 0$,

$$\frac{UM_{rtj2l}^{BC}(\omega_1)}{UM_{rtj2l}^{BC}(\omega_2)} - \frac{UM_{r't'j'2l'}^{BC}(\omega_1)}{UM_{r't'j'2l'}^{BC}(\omega_2)} = \frac{UM'_{rtj2l}{}^{BC}(\omega_1)}{UM'_{rtj2l}{}^{BC}(\omega_2)} - \frac{UM'_{r't'j'2l'}{}^{BC}(\omega_1)}{UM'_{r't'j'2l'}{}^{BC}(\omega_2)}, \quad (3.67)$$

for $UM_{rtj2l}^{BC}(\omega_2)$, $UM_{r't'j'2l'}^{BC}(\omega_2)$, $UM'_{rtj2l}{}^{BC}(\omega_2)$, $UM'_{r't'j'2l'}{}^{BC}(\omega_2) \neq 0$,

$$\frac{CM_{tj2l}^{BC}(\omega_1)}{CM_{tj2l}^{BC}(\omega_2)} - \frac{CM_{t'j'2l'}^{BC}(\omega_1)}{CM_{t'j'2l'}^{BC}(\omega_2)} = \frac{CM'_{tj2l}{}^{BC}(\omega_1)}{CM'_{tj2l}{}^{BC}(\omega_2)} - \frac{CM'_{t'j'2l'}{}^{BC}(\omega_1)}{CM'_{t'j'2l'}{}^{BC}(\omega_2)}, \quad (3.68)$$

for $CM_{tj2l}^{BC}(\omega_2)$, $CM_{t'j'2l'}^{BC}(\omega_2)$, $CM'_{tj2l}{}^{BC}(\omega_2)$, $CM'_{t'j'2l'}{}^{BC}(\omega_2) \neq 0$,

$$\frac{M_{tjkl}^{BC}(\omega_1)}{M_{tjkl}^{BC}(\omega_2)} - \frac{M_{t'j'k'l'}^{BC}(\omega_1)}{M_{t'j'k'l'}^{BC}(\omega_2)} = \frac{M'_{tjkl}{}^{BC}(\omega_1)}{M'_{tjkl}{}^{BC}(\omega_2)} - \frac{M'_{t'j'k'l'}{}^{BC}(\omega_1)}{M'_{t'j'k'l'}{}^{BC}(\omega_2)}, \quad (3.69)$$

for $M_{tjkl}^{BC}(\omega_2)$, $M_{t'j'k'l'}^{BC}(\omega_2)$, $M'_{tjkl}{}^{BC}(\omega_2)$, $M'_{t'j'k'l'}{}^{BC}(\omega_2) \neq 0$,

$$\lambda_{UMij2}^2 \text{Var}(UM_{rtj2l}^{BC}) = \lambda'_{UMij2}{}^2 \text{Var}(UM'_{rtj2l}{}^{BC}), \quad (3.70)$$

$$\lambda_{CMij2}^2 \text{Var}(CM_{tj2l}^{BC}) = \lambda'_{CMij2}{}^2 \text{Var}(CM'_{tj2l}{}^{BC}), \quad (3.71)$$

$$\lambda_{Mijk}^2 \text{Var}(M_{tjkl}^{BC}) = \lambda'_{Mijk}{}^2 \text{Var}(M'_{tjkl}{}^{BC}), \quad (3.72)$$

$$\text{Corr}(UM_{rtj2l}^{BC}, UM_{r't'j'2l'}^{BC}) = \text{Corr}(UM'_{rtj2l}{}^{BC}, UM'_{r't'j'2l'}{}^{BC}), \quad (3.73)$$

$$\text{Corr}(CM_{tj2l}^{BC}, CM_{t'j'2l'}^{BC}) = \text{Corr}(CM'_{tj2l}{}^{BC}, CM'_{t'j'2l'}{}^{BC}), \quad (3.74)$$

$$\text{Corr}(M_{tjkl}^{BC}, M_{t'j'k'l'}^{BC}) = \text{Corr}(M'_{tjkl}{}^{BC}, M'_{t'j'k'l'}{}^{BC}), \quad (3.75)$$

where $\text{Var}(\cdot)$ denotes variance and $\text{Corr}(\cdot)$ denotes correlation.

Proofs. 8 Meaningfulness.

The proofs for Equations 3.58, 3.64, 3.67, 3.70, and 3.73 are provided for illustration purposes. The remaining statements follow straightforward.

3.58 Replacing λ'_{UMij2} , $\lambda_{UMi'j2}$ in Equation 3.58 by $\left(\frac{\lambda_{UMij2}}{\beta_{UMj2}}\right)$ and $\left(\frac{\lambda_{UMi'j2}}{\beta_{UMj2}}\right)$, respectively, verifies the equality

$$\frac{\lambda'_{UMij2}}{\lambda_{UMi'j2}} = \frac{\lambda_{UMij2}}{\beta_{UMj2}} = \frac{\lambda_{UMij2}}{\lambda_{UMi'j2}}.$$

3.64 Replacing $UM'_{rtj2l}{}^{BC}$ by $\beta_{UMj2}UM_{rtj2l}^{BC}$ verifies the equality

$$\frac{UM'_{rtj2l}{}^{BC}(\omega_1)}{UM'_{rtj2l}{}^{BC}(\omega_2)} = \frac{\beta_{UMj2}UM_{rtj2l}^{BC}(\omega_1)}{\beta_{UMj2}UM_{rtj2l}^{BC}(\omega_2)} = \frac{UM_{rtj2l}^{BC}(\omega_1)}{UM_{rtj2l}^{BC}(\omega_2)}.$$

3.67 Again, replacing UM_{rtj2l}^{BC} by $\beta_{UMj2}UM_{rtj2l}^{BC}$ and $UM'_{r't'j'2l'}{}^{BC}$ by $\beta_{UMj2}UM_{r't'j'2l'}^{BC}$ verifies the equality

$$\begin{aligned}
\frac{UM_{rtj2l}^{BC}(\omega_1)}{UM_{rtj2l}^{BC}(\omega_2)} - \frac{UM_{r'tj'2l'}^{BC}(\omega_1)}{UM_{r'tj'2l'}^{BC}(\omega_2)} &= \frac{UM'_{rtj2l}{}^{BC}(\omega_1)}{UM'_{rtj2l}{}^{BC}(\omega_2)} - \frac{UM'_{r'tj'2l'}{}^{BC}(\omega_1)}{UM'_{r'tj'2l'}{}^{BC}(\omega_2)} \\
&= \frac{\beta_{UMj2}UM_{rtj2l}^{BC}(\omega_1)}{\beta_{UMj2}UM_{rtj2l}^{BC}(\omega_2)} - \frac{\beta_{UMj'2}UM_{r'tj'2l'}^{BC}(\omega_1)}{\beta_{UMj'2}UM_{r'tj'2l'}^{BC}(\omega_2)} \\
&= \frac{UM_{rtj2l}^{BC}(\omega_1)}{UM_{rtj2l}^{BC}(\omega_2)} - \frac{UM_{r'tj'2l'}^{BC}(\omega_1)}{UM_{r'tj'2l'}^{BC}(\omega_2)}.
\end{aligned}$$

3.70 Replacing λ_{UMij2}^2 by $\lambda_{UMij2}'^2\beta_{UMj2}^2$ as well as $\text{Var}(UM_{rtj2l}^{BC})$ by $\text{Var}\left(\frac{UM'_{rtj2l}{}^{BC}}{\beta_{UMj2}}\right)$ verifies the equality

$$\begin{aligned}
\lambda_{UMij2}^2 \text{Var}(UM_{rtj2l}^{BC}) &= \lambda_{UMij2}'^2 \beta_{UMj2}^2 \cdot \text{Var}\left(\frac{UM'_{rtj2l}{}^{BC}}{\beta_{UMj2}}\right) \\
&= \lambda_{UMij2}'^2 \beta_{UMj2}^2 \cdot \frac{1}{\beta_{UMj2}^2} \text{Var}(UM'_{rtj2l}{}^{BC}) \\
&= \lambda_{UMij2}'^2 \text{Var}(UM'_{rtj2l}{}^{BC}).
\end{aligned}$$

3.73 Replacing UM_{rtj2l}^{BC} and $UM_{r'tj'2l'}^{BC}$ in Equation 3.73 by $\frac{UM'_{rtj2l}{}^{BC}}{\beta_{UMj2}}$ and $\frac{UM'_{r'tj'2l'}{}^{BC}}{\beta_{UMj'2}}$

$$\begin{aligned}
\text{Corr}(UM_{rtj2l}^{BC}, UM_{r'tj'2l'}^{BC}) &= \text{Corr}\left(\frac{UM'_{rtj2l}{}^{BC}}{\beta_{UMj2}}, \frac{UM'_{r'tj'2l'}{}^{BC}}{\beta_{UMj'2}}\right) \\
&= \text{Corr}(UM'_{rtj2l}{}^{BC}, UM'_{r'tj'2l'}{}^{BC}).
\end{aligned}$$

□

Remarks. With respect to the factor loadings λ_{UMij2} , λ_{CMij2} , λ_{Mijk} , as well as their corresponding latent method factors UM_{rtj2l}^{BC} , CM_{rtj2l}^{BC} , and M_{ijkl}^{BC} statements regarding the absolute values of the parameters are *not* meaningful as already explained in the previous chapter. Nevertheless, statements regarding the ratio of specific values of the factor loadings or the associated latent method factors are meaningful. Hence, it is meaningful to say that the unique method change factor for target A is x-times larger or smaller than the unique method change factor for target B (see also Geiser, 2008). Similarly, statements regarding variance components of the latent factors as well as latent correlations among the latent method change factors also meaningful.

3.7 Testability

In this section the covariance structure of the baseline LC-COM change models is addressed. First, zero covariances are discussed. In empirical applications, it is important to restrict these covariances to zero. Second, admissible (freely estimated) covariances and their interpretations are regarded.

3.7.1 Zero covariances based on the model definition

The definition of the observed and latent variables has consequences for the covariance structure of the observed and latent variables. The next theorem summarizes the covariances which are zero by definition of the LC-COM model with conditional regressive independence.

Theorem 3.4 (Testability: consequences of model definition)

If $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk} \rangle$ is called an extended baseline LC-COM change model according to Definition 3.2. Then for $r \in R$, $t \in T$, $i, i' \in I$, $j, j' \in J$, $k, k' \in K$, $l, l' \in L$ where i can be equal to i' , j to j' , k to k' , and l to l' , and $l > 1$, but $(ijkl), (ij2l) \neq (ijkl)'$:

$$\text{Cov}(S_{tijkl}^{BC}, E_{ti'j'k'l'}) = 0, \quad (3.76)$$

$$\text{Cov}(S_{rtij2l}^{BC}, E_{ti'j'k'l'}) = 0, \quad (3.77)$$

$$\text{Cov}(S_{ti'j'k'l}^{BC}, E_{rtij2l}) = 0, \quad (3.78)$$

$$\text{Cov}(S_{rtij2l}^{BC}, E_{rti'j'2l'}) = 0, \quad (3.79)$$

$$\text{Cov}(UM_{rtj2l}^{BC}, E_{ti'j'k'l'}) = 0, \quad (3.80)$$

$$\text{Cov}(CM_{tj2l}^{BC}, E_{ti'j'k'l'}) = 0, \quad (3.81)$$

$$\text{Cov}(M_{tjkl}^{BC}, E_{ti'j'k'l'}) = 0, \quad (3.82)$$

$$\text{Cov}(UM_{rtj2l}^{BC}, E_{rti'j'2l'}) = 0, \quad (3.83)$$

$$\text{Cov}(CM_{tj2l}^{BC}, E_{rti'j'2l'}) = 0, \quad (3.84)$$

$$\text{Cov}(M_{tjkl}^{BC}, E_{rti'j'2l'}) = 0, \quad (3.85)$$

$$\text{Cov}(S_{tij1l}^{BC}, UM_{rtj'2l'}^{BC}) = 0, \quad (3.86)$$

$$\text{Cov}(CM_{tj2l}^{BC}, UM_{rtj'2l'}^{BC}) = 0, \quad (3.87)$$

$$\text{Cov}(M_{tjkl}^{BC}, UM_{rtj'2l'}^{BC}) = 0. \quad (3.88)$$

Moreover, all zero-correlations stated in Theorem 2.4 hold as well.

Remarks. In addition to the zero covariances stated in Theorem 3.4, it is strongly recommended to fix the following covariances between latent state different variables S_{tij1l}^{BC} and latent method difference variables $(CM_{tj'2l'}^{BC}, M_{tj'kl'}^{BC})$ belonging to different constructs $j \neq j'$ as well as different occasions of measurement $l \neq l'$, where $l, l' > 1$ to zero:

$$\text{Cov}(S_{tij1l}^{BC}, CM_{tj'2l'}^{BC}) = 0,$$

$$\text{Cov}(S_{tij1l}^{BC}, M_{tj'kl'}^{BC}) = 0.$$

Proofs. 9 *Testability*

3.76 The covariance $Cov(S_{tijkl}^{BC}, E_{ti'j'k'l'})$ can be expressed as follows

$$\begin{aligned} Cov(S_{tijkl}^{BC}, E_{ti'j'k'l'}) &= Cov[(S_{tijkl} - S_{tijk1}), E_{ti'j'k'l'}] \\ &= Cov(S_{tijkl}, E_{ti'j'k'l'}) - Cov(S_{tijk1}, E_{ti'j'k'l'}), \end{aligned} \quad \forall l > 1.$$

Given that $Cov(S_{tijkl}, E_{ti'j'k'l'})$ as well as $Cov(S_{tijk1}, E_{ti'j'k'l'})$ are zero as a consequence of the model definition, the covariance $Cov(S_{tijkl}^{BC}, E_{ti'j'k'l'})$ must be zero as well.

3.77-3.79 The proofs for Equations 3.77 to 3.79 follow straightforward and will be left to the reader.

3.80 The covariance $Cov(UM_{rtj2l}^{BC}, E_{ti'j'k'l'})$ is equivalent to

$$\begin{aligned} Cov(UM_{rtj2l}^{BC}, E_{ti'j'k'l'}) &= Cov[(UM_{rtj2l} - UM_{rtj21}), E_{ti'j'k'l'}] \\ &= Cov(UM_{rtj2l}, E_{ti'j'k'l'}) - Cov(UM_{rtj21}, E_{ti'j'k'l'}), \end{aligned} \quad \forall l > 1.$$

Given that $Cov(UM_{rtj2l}, E_{ti'j'k'l'})$ as well as $Cov(UM_{rtj21}, E_{ti'j'k'l'})$ are zero as a consequence of the model definition, the covariance $Cov(UM_{rtj2l}^{BC}, E_{ti'j'k'l'})$ must be zero as well. Hence, Equation 3.80 holds, too.

3.81-3.85 The proofs for Equations 3.81 and 3.85 follow straightforward and will be left to the reader.

3.86 The covariance $Cov(S_{tij1l}^{BC}, UM_{rtj'2l'})$ equals zero, given that

$$\begin{aligned} Cov(S_{tij1l}^{BC}, UM_{rtj'2l'}) &= Cov[(S_{tij1l} - S_{tij11}), (UM_{rtj'2l'} - UM_{rtj'21})] \\ &= Cov(S_{tij1l}, UM_{rtj'2l'}) - Cov(S_{tij1l}, UM_{rtj'21}) \\ &\quad - Cov(S_{tij11}, UM_{rtj'2l'}) + Cov(S_{tij11}, UM_{rtj'21}), \end{aligned} \quad \forall l \neq l' \wedge \forall l' > 1.$$

It has been already shown that $Cov(S_{tij1l}, UM_{rtj'2l'})$, $Cov(S_{tij1l}, UM_{rtj'21})$, $Cov(S_{tij11}, UM_{rtj'2l'})$, as well as $Cov(S_{tij11}, UM_{rtj'21})$ are zero by definition. Therefore Equation 3.86 holds.

□

3.7.2 Covariance structure: LC-COM model with CRI

In this section the covariance structure of an extended LC-COM model for three indicators \times two traits \times three methods \times two occasions of measurement is presented. This model is algebraic equivalent to a LS-COM model with conditional regressive independence (CRI) and strong measurement invariance (MI) presented in Chapter 2. Therefore, the total covariance matrices $\sum_{\mathbf{T}}$ of the observed variables of both models (LS-COM and LC-COM model) are identical. Nevertheless the reformulation of a LS-COM model into a LC-COM model may be reasonable for answering particular substantive research questions concerning true intraindividual change. Again, the total covariance matrix $\sum_{\mathbf{T}}$ is partitioned into a within $\sum_{\mathbf{W}}$ and a between $\sum_{\mathbf{B}}$ matrix:

$$\sum_{\mathbf{T}} = \sum_{\mathbf{W}} + \sum_{\mathbf{B}}$$

The within matrix of size 36×36 is given by:

$$\sum_{\mathbf{W}} = \Lambda_{\mathbf{W}} \Phi_{\mathbf{W}}^{BC} \Lambda_{\mathbf{W}}^{\mathbf{T}} + \Theta_{\mathbf{W}}$$

$\Lambda_{\mathbf{W}}$ refers to the matrix of the time-invariant factor loadings pertaining to the unique method (change) factors. The elements of this matrix are denoted by λ_{UMij2} , where i =indicator, j =construct,

$k=2$ (i.e., interchangeable method). Note that the index l for the measurement occasion has been dropped, given that these parameters are assumed to be time-invariant. $\Lambda_{\mathbf{W}}^T$ refers to the transposed within factor loading matrix. $\Phi_{\mathbf{W}}^{BC}$ refers to the within variance and covariance matrix of the unique method (baseline change) variables, and $\Theta_{\mathbf{W}}$ is the diagonal residual covariance matrix, which is identical to the within residual covariance matrix of the LS-COM model with CRI and strong MI. The between matrix $\Sigma_{\mathbf{B}}$ of size 36×36 is given by:

$$\Sigma_{\mathbf{B}} = \Lambda_{\mathbf{B}} \Phi_{\mathbf{B}}^{BC} \Lambda_{\mathbf{B}}^T + \Theta_{\mathbf{B}}$$

$\Lambda_{\mathbf{B}}$ refers to the matrix of the time-invariant between-level factor loadings. The elements of this matrix are λ_{Stijk} , λ_{CMtij2} , λ_{Mtijk} . $\Lambda_{\mathbf{B}}^T$ refers to the transposed vector. $\Phi_{\mathbf{B}}^{BC}$ refers to the between variance and covariance matrix of the between latent (baseline change) variables. $\Theta_{\mathbf{B}}$ refers to the diagonal between residual covariance matrix and is again identical to the between residual covariance matrix of the LS-COM model with CRI and strong MI.

In order to define the factor loading matrices $\Lambda_{\mathbf{W}}$ and $\Lambda_{\mathbf{B}}$ properly, the function $Pos((j, l))$ is needed. This function is clearly defined in Section 2.6.2. However, in contrast to the LS-COM model, the matrices $\mathbf{I}_{\mathbf{p}}$, where $\mathbf{p} \in \mathbb{N} = \{1, \dots, 4\}$ are defined as follows:

$$\mathbf{I}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{I}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{I}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{I}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $\Lambda_{\mathbf{W}}$ of size 36×20 can now be written as the sum over the Kronecker products of $\mathbf{I}_{\mathbf{p}}$ and $\Lambda_{\mathbf{W}_{\mathbf{p}}}$:

$$\Lambda_{\mathbf{W}} = \sum_{\mathbf{p}=1}^4 \mathbf{I}_{\mathbf{p}} \otimes \Lambda_{\mathbf{W}_{\mathbf{p}}}.$$

$\Lambda_{\mathbf{W}_{\mathbf{p}}}$ of size 9×5 contains the time invariant within-level factor loadings and is given by:

$$\Lambda_{\mathbf{W}_{\mathbf{p}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{UM1j2} & 0 \\ 0 & 0 & 0 & \lambda_{UM2j2} & 0 \\ 0 & 0 & 0 & \lambda_{UM3j2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The elements λ_{UM1j2} , λ_{UM2j2} , λ_{UM3j2} are greater than 0 and denote the latent factor loadings of the within latent factors UM_{rtjkl} , respectively UM_{rtjkl}^{BC} . The remaining elements of this matrix are necessarily zero. The within variance and covariance matrix $\Phi_{\mathbf{W}}^{BC}$ of size 20×20 is structurally

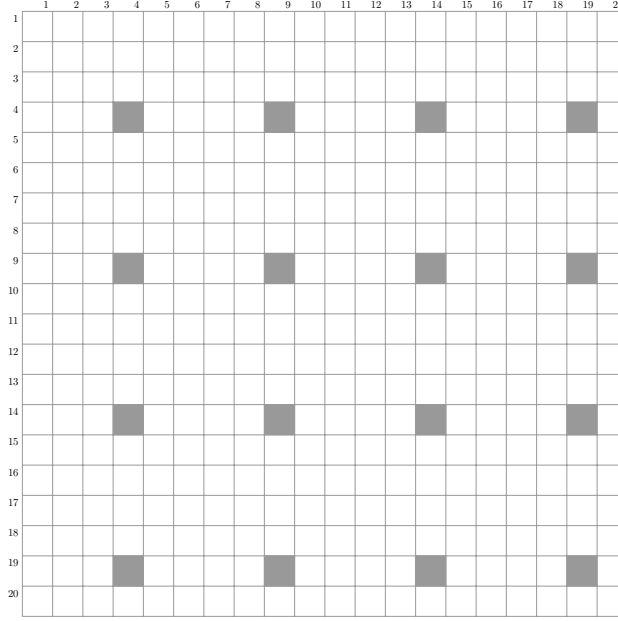


Figure 3.2: Within variance-covariance matrix $\Phi_{\mathbf{W}}^{\text{BC}}$ of the LC-COM with $1=S_{t_{1111}}$, $2=S_{t_{2111}}$, $3=S_{t_{3111}}$, $4=UM_{rt121}$, $5=M_{t_{131}}$, $6=S_{t_{1112}}^{\text{BC}}$, $7=S_{t_{2112}}^{\text{BC}}$, $8=S_{t_{3112}}^{\text{BC}}$, $9=UM_{rt122}^{\text{BC}}$, $10=M_{t_{132}}^{\text{BC}}$, $11=S_{t_{211}}$, $12=S_{t_{211}}$, $13=S_{t_{211}}$, $14=UM_{rt221}$, $15=M_{t_{231}}$, $16=S_{t_{1212}}^{\text{BC}}$, $17=S_{t_{2212}}^{\text{BC}}$, $18=S_{t_{3212}}^{\text{BC}}$, $19=UM_{rt222}^{\text{BC}}$, $20=M_{t_{232}}^{\text{BC}}$. White colored cells indicate zero correlations, gray colored cells indicate permissible correlations.

equivalent to the latent covariance matrix of the LS-COM model and is given by:

$$\Phi_{\mathbf{W}}^{\text{BC}} = \mathbf{E} \left[(\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}}])(\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}}]^{\text{T}} \right].$$

$\mathbf{E}(\cdot)$ is the expected value and $\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}}$ refers to the vector of size 20×1 including all latent factors, except for the common method (difference) factor $(CM_{t_{jkl}}, CM_{t_{jkl}}^{\text{BC}})$, namely

$$\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}} = \begin{pmatrix} S_{t_{1111}}, S_{t_{2111}}, S_{t_{3111}}, UM_{rt121}, M_{t_{131}}, S_{t_{1112}}^{\text{BC}}, S_{t_{2112}}^{\text{BC}}, S_{t_{3112}}^{\text{BC}}, UM_{rt122}^{\text{BC}}, M_{t_{132}}^{\text{BC}}, \\ S_{t_{1211}}, S_{t_{2211}}, S_{t_{3211}}, UM_{rt221}, M_{t_{231}}, S_{t_{1212}}^{\text{BC}}, S_{t_{2212}}^{\text{BC}}, S_{t_{3212}}^{\text{BC}}, UM_{rt222}^{\text{BC}}, M_{t_{232}}^{\text{BC}} \end{pmatrix}^{\text{T}}.$$

Given that $\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}}$ refers to the vector of the within (rater-specific) latent variables, any UM_{rtj2l} or UM_{rtj2l}^{BC} is uncorrelated with any other variables on the target-level (i. e., $S_{t_{i'j'1l'}}$, $S_{t_{i'j'1l'}}^{\text{BC}}$, $M_{t_{j'3l'}}$, $M_{t_{j'3l'}}^{\text{BC}}$, $CM_{t_{j'2l'}}$, $CM_{t_{j'2l'}}^{\text{BC}}$). Therefore, $\mathbf{V}_{\Phi_{\mathbf{W}}^{\text{BC}}}$ contains only the latent variance and covariances of the unique method (change) factors. The structure of the covariance matrix $\Sigma_{\mathbf{W}}$ is illustrated in Figure 3.2. Given that the LC-COM model is just a reparametrization of the LS-COM model, both covariance matrices of the models are identical.

The matrix of the between factor loading $\Lambda_{\mathbf{B}}$ of size 36×20 is given by:

$$\Lambda_{\mathbf{B}} = \sum_{\mathbf{p}=1}^4 \mathbf{I}_{\mathbf{p}} \otimes \Lambda_{\mathbf{B}_{\mathbf{p}}}.$$

$\Lambda_{\mathbf{B}_p}$ refers to the between factor loadings matrix of size 9×5 :

$$\Lambda_{\mathbf{9} \times \mathbf{3}} = \begin{pmatrix} \lambda_{S1j1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{S2j1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{S3j1} & 0 & 0 \\ \lambda_{S1j2} & 0 & 0 & \lambda_{CM1j2} & 0 \\ 0 & \lambda_{S2j2} & 0 & \lambda_{CM2j2} & 0 \\ 0 & 0 & \lambda_{S3j2} & \lambda_{CM3j2} & 0 \\ \lambda_{S1j3} & 0 & 0 & 0 & \lambda_{M1j3} \\ 0 & \lambda_{S2j3} & 0 & 0 & \lambda_{M2j3} \\ 0 & 0 & \lambda_{S3j3} & 0 & \lambda_{M3j3} \end{pmatrix}.$$

Again, note that the factor loading matrices $\Lambda_{\mathbf{W}_p}$ and $\Lambda_{\mathbf{B}_p}$ are structurally equivalent to the matrices presented in Section 2.6.2. The only differences between the LS-COM and LC-COM model is that different contrast matrices \mathbf{I}_p are defined. The between variance and covariance matrix $\Phi_{\mathbf{B}}^{\text{BC}}$ of size 20×20 is also equivalent with the covariance matrix $\Phi_{\mathbf{B}}$ of the LS-COM model, namely:

$$\Phi_{\mathbf{B}}^{\text{BC}} = \mathbf{E} \left[(\mathbf{V}_{\Phi_{\mathbf{B}}^{\text{BC}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\mathbf{B}}^{\text{BC}}}])(\mathbf{V}_{\Phi_{\mathbf{B}}^{\text{BC}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\mathbf{B}}^{\text{BC}}}]^{\text{T}} \right].$$

$\mathbf{E}(\cdot)$ is the expected value and $\mathbf{V}_{\Phi_{\mathbf{B}}^{\text{BC}}}$ refers to the vector of size 20×1 including all latent factors on the target-level, namely:

$$\mathbf{V}_{\Phi_{\mathbf{B}}^{\text{BC}}} = \begin{pmatrix} S_{t1111}, S_{t2111}, S_{t3111}, CM_{t121}, M_{t131}, S_{t1112}^{BC}, S_{t2112}^{BC}, S_{t3112}^{BC}, CM_{t122}^{BC}, M_{t132}^{BC}, \\ S_{t1211}, S_{t2211}, S_{t3211}, CM_{t221}, M_{t231}, S_{t1212}^{BC}, S_{t2212}^{BC}, S_{t3212}^{BC}, CM_{t222}^{BC}, M_{t232}^{BC} \end{pmatrix}^{\text{T}}.$$

As stated above, all covariances between latent state variables S_{tj1l} (respectively S_{tj1l}^{BC}) and any latent method factor CM_{tj2l} (respectively CM_{tj2l}^{BC}) or M_{tj3l} (respectively M_{tj3l}^{BC}) pertaining to the *same* construct j and *same* occasion of measurement l necessarily equal zero. In Figure 3.3 the between covariance matrix $\Phi_{\mathbf{B}}^{\text{BC}}$ of the extended LC-COM model is given for the sake of completeness.

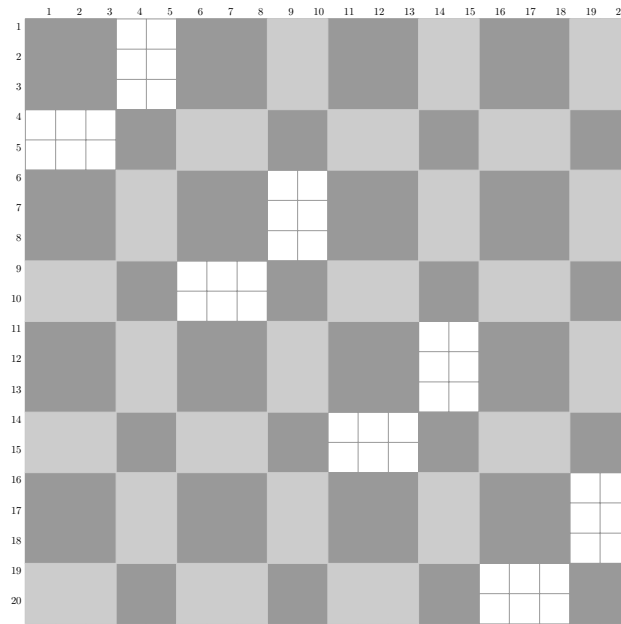


Figure 3.3: Between variance-covariance matrix $\Phi_{\mathbf{B}}^{BC}$ of the LC-COM with $1=S_{t_{1111}}$, $2=S_{t_{2111}}$, $3=S_{t_{3111}}$, $4=CM_{t_{121}}$, $5=M_{t_{131}}$, $6=S_{t_{1112}}^{BC}$, $7=S_{t_{2112}}^{BC}$, $8=S_{t_{3112}}^{BC}$, $9=CM_{t_{122}}^{BC}$, $10=M_{t_{132}}^{BC}$, $11=S_{t_{211}}$, $12=S_{t_{211}}$, $13=S_{t_{211}}$, $14=CM_{t_{221}}$, $15=M_{t_{231}}$, $16=S_{t_{212}}^{BC}$, $17=S_{t_{212}}^{BC}$, $18=S_{t_{212}}^{BC}$, $19=CM_{t_{222}}^{BC}$, $20=M_{t_{232}}^{BC}$. White colored cells indicate zero correlations, dark gray colored cells indicate permissible correlations. Light gray colored cells indicate correlations that may be fixed to zero for parsimony.

The complexity of variance-covariance matrices of the extended LC-COM model with indicator-specific latent factors can be reduced by assuming common latent factors for the reference state/change variables as well as for the latent method variables. A path diagram of an extended baseline LC-COM model with common latent state and change factors is depicted in Figure 3.7.2.

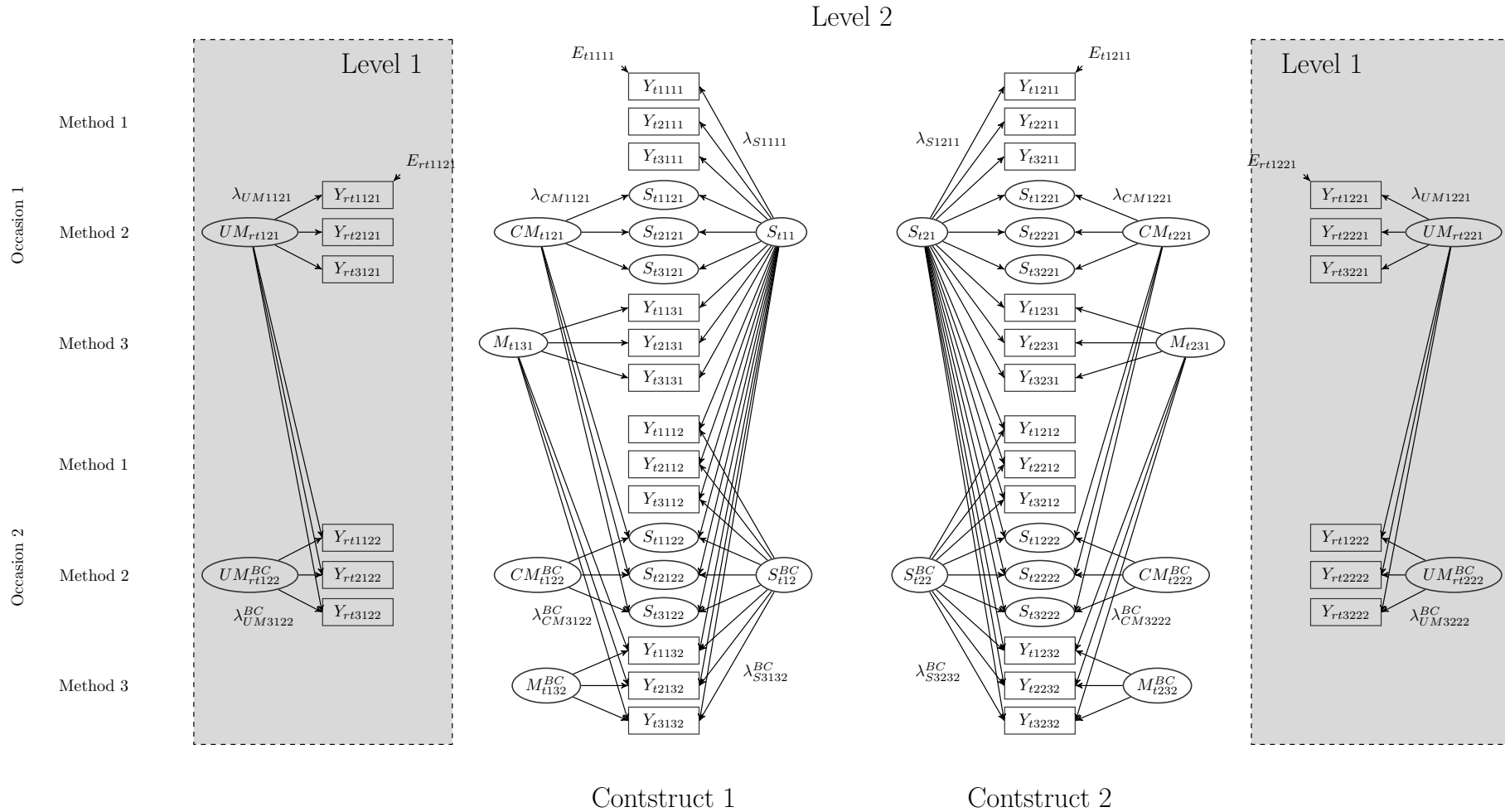


Figure 3.4: Path diagram of the extended LC-COM model with common latent state and change factors.

An extended LC-COM model with common latent state and change factors incorporating three indicators ($i=1,2,3$), two constructs ($j=1,2$), three methods ($k=1,2,3$) and two occasions of measurement ($l=1,2$). All correlations between latent variables were omitted for clarity. Measurement error variables E_{rtijkl} and E_{tijk} are only depicted for the first indicator pertaining to method 1 and 2.

3.7.3 Interpretation of non-zero covariances and correlations

The following correlations are permissible with respect to the definition of the LC-COM model. Consequently, these correlations can be estimated and interpreted.

1. The correlations $Cor(S_{tij11}, S_{tij'1l}^{BC})$ between the initial reference state factors (T1) and the latent change variables pertaining to the same indicator i , and the same construct j , reflect the associations between the initial status and change of the targets (see Geiser, 2008). High positive correlations indicate that targets with high latent state scores (e.g., high teaching motivation at T1) tend to also have higher latent change scores from measurement occasion l to l' compared to those targets with lower latent state scores at T1.
2. The correlations $Cor(S_{tij11}, S_{tij'1l'}^{BC})$ between the initial reference state factors (T1) and the latent change variables pertaining to different construct j and j' can in some cases be interpreted as predictive validity (see Geiser, 2008). For example, teachers with higher teaching motivation at T1 might benefit more from an intervention than teachers with lower teaching motivation at T1. Therefore, the high initial status on teaching motivation might also predict greater increase in teaching quality from time 1 to l .
3. The correlations $Cor(S_{tij1l}, S_{tij'1l'}^{BC})$ between latent change variables pertaining to the same constructs j reflect the relationship between difference scores pertaining to different time points (see Geiser, 2008). For instance, researcher might be interested in whether or not the teaching quality increases or decreases persistently after two or more interventions. High positive correlations indicate that individuals with higher state change scores between different time points (1 and l) tend to also have higher latent state change scores between time points (1 and l').
4. The correlations $Cor(S_{tij1l}, S_{tij'1l'}^{BC})$ between latent change scores pertaining to different constructs j and j' indicate that true change with regard to construct j (e.g., teaching ability) is associated with true change with regard to another construct j' (e.g., teaching quality). Positive correlations indicate that individuals with higher change scores with respect to construct j also tend to have higher change scores with respect to construct j' . These correlations can also be interpreted as discriminant validity of change (see Geiser, 2008). Low correlations mirror high discriminant validity of change.
5. The correlations between initial method factors and method difference factors belonging to the same construct j and same method k represent the association of method-specific deviation from the reference method at T1 with the method-specific deviation in change (see Geiser, 2008). Two types of correlations can be distinguished: (i) The correlations $Cor(M_{tjk1}, M_{tjkl}^{BC})$ between initial method factors and method difference factors pertaining to the same construct and the same structurally different method; (ii) the correlations $Cor(CM_{tj21}, CM_{tj2l}^{BC})$ between initial common method factors and common method difference factor pertaining to the same construct, as well as (iii) the correlations $Cor(UM_{rtj21}, UM_{rtj2l}^{BC})$ between initial unique method factors and unique method difference factors pertaining to the same construct.

6. The correlations $Cor(M_{t_j k 1}, M_{t_j' k l}^{BC})$ between the initial method factors and method difference factors belonging to the same method, but different constructs j and j' are difficult to interpret (see also Geiser, 2008). In most empirical applications these correlations will not substantially differ from zero. However, significant correlations would indicate that the method bias at T1 (e.g., over- or underestimation of the teaching motivation by the school principal with respect to the teacher self-rating) is associated in some way with the method bias change between time points 1 and l with respect to the teaching quality. Again, correlations [$Cor(CM_{t_j 21}, CM_{t_j' 2l}^{BC})$ and $Cor(UM_{rt_j 21}, UM_{rt_j' 2l}^{BC})$] between the initial common or unique method factors and common or unique method difference factors belonging to different constructs j and j' can be estimated, too.
7. The correlations [$Cor(M_{t_j k 1}, CM_{t_j 2l}^{BC})$, and $Cor(CM_{t_j 21}, M_{t_j k l}^{BC})$] between initial method factors and method difference factors pertaining to the same construct j , but different methods k and k' reflect the association between the method specific deviation of method k from the reference method at T1 and the method specific deviation in change for method k' (see Geiser, 2008). The correlations are also relatively difficult to interpret. An example of this correlation would be the correlation between the over- or underestimation of the teaching motivation by the school principal with respect to the teacher's self-report (reference method) at T1 and the change in the over- or underestimation by the student ratings with respect to the teacher's self-report.
8. The correlations [$Cor(M_{t_j k 1}, CM_{t_j' 2l}^{BC})$, and $Cor(CM_{t_j 21}, M_{t_j' k l}^{BC})$] can be estimated for method factors at T1 and method difference factors pertaining to different constructs j and j' and different methods k and k' . For most applications these correlations will not differ significantly from zero.
9. The correlations [$Cor(M_{t_j k l}^{BC}, M_{t_j' k l'}^{BC})$, $Cor(CM_{t_j 2l}^{BC}, CM_{t_j' 2l'}^{BC})$, and $Cor(UM_{rt_j 2l}^{BC}, UM_{rt_j' 2l'}^{BC})$] between method difference factors pertaining to the same construct and method represent the association between the method specific deviation in change scores pertaining to different measurement occasions (see Geiser, 2008). High positive values would indicate, for example, that the change scores of the method specific deviation of the student ratings from the teacher's self-rating at time l correlate with the change scores of the method specific deviation of the student ratings from the teacher's self-rating at time l' . In other words, the true change of method bias for a particular method k follows the same direction (i.e., increase or decrease in method bias) across different time points l and l' .
10. The correlations [$Cor(M_{t_j k l}^{BC}, M_{t_j' k l'}^{BC})$, $Cor(CM_{t_j 2l}^{BC}, CM_{t_j' 2l'}^{BC})$, and $Cor(UM_{rt_j 2l}^{BC}, UM_{rt_j' 2l'}^{BC})$] can be estimated for method difference factors pertaining to the same method k , but different constructs j and j' . These correlations can be interpreted as discriminant validity of change corrected for the influence of the reference method (see Geiser, 2008).
11. The correlations $Cor(M_{t_j k l}^{BC}, CM_{t_j 2l}^{BC})$ between method difference factors belonging to the same construct j , but different methods k and k' can be interpreted as the convergent validity of change. For instance, students and the school principal may agree in their ratings of the

teacher's development over the course of time, above what can already be predicted by the teacher's self-report. High positive correlations indicate that the non-reference methods share a "common view in change" that is not shared with the reference method (see Geiser, 2008).

12. The correlations between method difference factors pertaining to different constructs j and j' , different methods k and k' , and different time points l and l' characterize the "common view in change" shared by different method difference factors but not shared with the reference method with respect to different constructs j and j' . In most applications these correlations will not significantly differ from zero (see Geiser, 2008).

3.7.4 Correlations that should be set to zero for parsimony

The following correlations are permissible by definition, but shall be set to zero for parsimony in empirical applications. It is most likely that these correlations will not substantially differ from zero, and thus will be negligible:

1. The correlations between initial state factors of the reference method and any latent method difference factor.

$$\begin{aligned} \text{Cor}(S_{ti j 11}, M_{tj' k' l}^{BC}), & \quad \forall k > 2 \wedge \forall j \neq j' \wedge \forall l > 1 \\ \text{Cor}(S_{ti j 11}, CM_{tj' 2l}^{BC}), & \quad \forall j \neq j' \wedge \forall l > 1 \end{aligned}$$

2. The correlations between any initial latent method factor and latent reference state difference factor.

$$\begin{aligned} \text{Cor}(M_{tj k 1}, S_{ti' j' 1l}^{BC}), & \quad \forall i \neq i' \wedge \forall j \neq j' \wedge \forall k > 2 \wedge \forall l > 1 \\ \text{Cor}(CM_{tj 2l}, S_{ti' j' 1l}^{BC}), & \quad \forall i \neq i' \wedge \forall j \neq j' \wedge \forall l > 1 \end{aligned}$$

3. The correlation between latent reference state difference factors and any latent method difference factors.

$$\begin{aligned} \text{Cor}(M_{tj k l}^{BC}, S_{ti' j' 1l'}^{BC}), & \quad \forall i \neq i' \wedge \forall j \neq j' \wedge \forall k > 2 \wedge \forall l \neq l' \wedge \forall l > 1 \\ \text{Cor}(CM_{tj 2l}, S_{ti' j' 1l'}^{BC}), & \quad \forall i \neq i' \wedge \forall j \neq j' \wedge \forall l \neq l' \wedge \forall l > 1 \end{aligned}$$

3.8 General measurement equations and variance decompositions

In the following section the general measurement equations of extended latent baseline LC-COM models are discussed. Again, based on the definition of the extended latent baseline LC-COM model different variance coefficients can be defined. In Theorem 3.3 it has already been shown that these variance coefficients can be meaningfully interpreted. Again, the independence among latent variables discussed in Theorem 3.4 as well as Theorem 2.4 are important requirements for defining different variance coefficients. The next Definition 3.3 defines an extended latent baseline LC-COM model with common latent change method factors. On the basis of this definition the general measurement equations of extended latent baseline LC-COM models are introduced. In the next step, different variance coefficients are discussed.

Definition 3.3

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk} \rangle$ be an extended baseline LC-COM model according to Definition 3.2 and Theorem 3.1, and:

$$\begin{aligned} \mathbf{S}_t^{BC} &\equiv (S_{11112}^{BC} \cdots S_{tijk}^{BC} \cdots S_{bcdef}^{BC})^T, \\ \mathbf{UM}_{rt}^{BC} &\equiv (UM_{11112}^{BC} \cdots UM_{rtj2l}^{BC} \cdots UM_{abd2f}^{BC})^T, \\ \mathbf{CM}_t^{BC} &\equiv (CM_{1112}^{BC} \cdots CM_{tj2l}^{BC} \cdots CM_{bd2f}^{BC})^T, \\ \mathbf{M}_t^{BC} &\equiv (M_{1112}^{BC} \cdots M_{tijk}^{BC} \cdots M_{bdef}^{BC})^T. \end{aligned}$$

All other latent variables of the LS-COM model (see Definition 2.3) remain unaltered.

Remarks. In the above Definition 3.3 an extended latent baseline LC-COM model with common latent difference method factors is defined. Note that the latent difference variables were construed by the following tautological equations:

$$\begin{aligned} S_{tij1l} &= S_{tij11} + (S_{tij1l} - S_{tij11}), \\ CM_{tij2l} &= CM_{tij21} + (CM_{tij2l} - CM_{tij21}), \\ UM_{tij2l} &= UM_{tij21} + (UM_{tij2l} - UM_{tij21}), \\ M_{tijk} &= M_{tijk1} + (M_{tijk} - M_{tijk1}), \quad \forall k > 2, \end{aligned}$$

where $(S_{tij1l} - S_{tij11}) \equiv S_{tijk}^{BC}$, $(CM_{tij2l} - CM_{tij21}) \equiv CM_{tj2l}^{BC}$, $(UM_{rtij2l} - UM_{rtij21}) \equiv UM_{rtij2l}^{BC}$, and $(M_{tijk} - M_{tijk1}) \equiv M_{tijk}^{BC}$. Furthermore, according to the statements in Definition 3.2 it was shown (see Theorem 3.1) that latent difference method variables $(CM_{tj2l}^{BC}, UM_{rtij2l}^{BC}, M_{tijk}^{BC})$ pertaining to the same construct j , same non-reference method k , and same occasion of measurement l are positive linear transformations of each other, respectively. Hence, it was assumed that these latent difference variables only differ by a multiplicative constant. Consequently, latent difference method factors $(CM_{tj2l}^{BC}, UM_{rtij2l}^{BC}, M_{tijk}^{BC})$ were construed. According to Theorem 3.4 as well as 2.4 the measurement equations for the observed variables are given by:

$$Y_{tij1l} = S_{tij11} + S_{tijk}^{BC} + E_{tij1l}, \quad \forall l > 1, \quad (3.89)$$

$$\begin{aligned} Y_{tijk} &= \alpha_{tijk} + \lambda_{Sijk} S_{tij11} + \lambda_{Sijk} S_{tijk}^{BC} + \\ &\quad \lambda_{Mijk} M_{tijk1} + \lambda_{Mijk} M_{tijk}^{BC} + E_{tijk}, \quad \forall k > 2, l > 1, \end{aligned} \quad (3.90)$$

$$\begin{aligned} Y_{rtij2l} &= \alpha_{tij2} + \lambda_{Sij2} S_{tij11} + \lambda_{Sij2} S_{tijk}^{BC} + \\ &\quad \lambda_{CMij2} CM_{tj21} + \lambda_{CMij2} CM_{tj2l}^{BC} + \\ &\quad \lambda_{UMij2} UM_{rtj21} + \lambda_{UMij2} UM_{rtj2l}^{BC} + E_{rtij2l}, \quad \forall l > 1. \end{aligned} \quad (3.91)$$

3.8.1 Variance decomposition

Based on the above Equations 3.89 to 3.91, the variance of the observed variables can be decomposed as follows:

$$\begin{aligned} Var(Y_{tij1l}) = & Var(S_{tij11}) + Var(S_{tij1l}^{BS}) + \\ & 2Cov(S_{tij11}, S_{tij1l}^{BC}) + Var(E_{tij1l}), \end{aligned} \quad \forall l > 1, \quad (3.92)$$

$$\begin{aligned} Var(Y_{tijk1}) = & \lambda_{S_{ijk}}^2 Var(S_{tij11}) + \lambda_{S_{ijk}}^2 Var(S_{tij1l}^{BC}) + \\ & 2(\lambda_{S_{ijk}}^2)Cov(S_{tij11}, S_{tij1l}^{BC}) + \lambda_{M_{ijk}}^2 Var(M_{tjk1}) + \\ & \lambda_{M_{ijk}}^2 Var(M_{tjk1}^{BC}) + 2(\lambda_{M_{ijk}}^2)Cov(M_{tjk1}, M_{tjk1}^{BC}) + \\ & Var(E_{tijk1}), \end{aligned} \quad \forall k > 2, \forall l > 1, \quad (3.93)$$

$$\begin{aligned} Var(Y_{rtij2l}) = & \lambda_{S_{ij2}}^2 Var(S_{tij11}) + \lambda_{S_{ij2}}^2 Var(S_{tij1l}^{BC}) + \\ & 2(\lambda_{S_{ij2}}^2)Cov(S_{tij11}, S_{tij1l}^{BC}) + \\ & \lambda_{CM_{ij2}}^2 Var(CM_{tj21}) + \lambda_{CM_{ij2}}^2 Var(CM_{tj2l}^{BC}) + \\ & 2(\lambda_{CM_{ij2}}^2)Cov(CM_{tj21}, CM_{tj2l}^{BC}) + \\ & \lambda_{UM_{ij2}}^2 Var(UM_{rtj21}) + \lambda_{UM_{ij2}}^2 Var(UM_{rtj2l}^{BC}) + \\ & 2(\lambda_{UM_{ij2}}^2)Cov(UM_{rtj21}, UM_{rtj2l}^{BC}) + Var(E_{rtij2l}), \end{aligned} \quad \forall l > 1. \quad (3.94)$$

According to the above Equations 3.92 to 3.94 the variance decomposition of the observed variables implies not only additive variance components, but also the latent covariances between the initial state S_{tjk1} and the latent difference variables S_{tij1l}^{BC} . Given that researchers might find it difficult to interpret different variance components with regard to this variance decomposition, an alternative variance decomposition is proposed. The following variance decomposition is based on the observed difference scores, given that the observed difference scores are of particular interest for researchers studying true interindividual differences in intraindividual change. A convenient side effect of this variance decomposition is that no covariance structure between the initial state and latent difference (change) variables has to be considered (see also Geiser, 2008). In order to decompose the total variance of the observed difference (change) scores, strong measurement invariance has to be assumed. Only if strong measurement invariance holds, the total variance of any observed difference score [$Y_{tijk1}^{BC} \equiv (Y_{tijk1} - Y_{tijk1})$] as well as [$Y_{rtij2l}^{BC} \equiv (Y_{rtij2l} - Y_{rtij21})$, $\forall l > 1$] can be

decomposed as follows (see Proof 10):

$$\begin{aligned} \text{Var}(Y_{tij1l}^{BC}) = & \text{Var}(S_{tij1l}^{BC}) + \text{Var}(E_{tij1l}) + \\ & \text{Var}(E_{tij1l}), \end{aligned} \quad \forall l > 1, \quad (3.95)$$

$$\begin{aligned} \text{Var}(Y_{tijkl}^{BC}) = & \lambda_{Sijk}^2 \text{Var}(S_{tij1l}^{BC}) + \lambda_{Mijk}^2 \text{Var}(M_{tjkl}^{BC}) + \\ & \text{Var}(E_{tijk1}) + \text{Var}(E_{tijkl}), \end{aligned} \quad \forall k > 2, l > 1, \quad (3.96)$$

$$\begin{aligned} \text{Var}(Y_{rtij2l}^{BC}) = & \lambda_{Sij2}^2 \text{Var}(S_{tij1l}^{BC}) + \\ & \lambda_{CMij2}^2 \text{Var}(CM_{tj2l}^{BC}) + \\ & \lambda_{UMij2}^2 \text{Var}(UM_{rtj2l}^{BC}) + \\ & \text{Var}(E_{rtij21}) + \text{Var}(E_{rtij2l}), \end{aligned} \quad \forall l > 1. \quad (3.97)$$

With respect to the Equations 3.95 to 3.97, it is clear that the amount of “true” change with respect to reference state or method variables can be investigated. However, it is important to note that the error variances at both time points are part of the equation (see Geiser, 2008). This is a direct consequence of the rules of variances and covariances (see Steyer & Eid, 2001, Box F.1., p. 343). On the basis of this additive variance decomposition, it is possible to define the consistency coefficient of true change:

$$\begin{aligned} \text{CON}(\tau_{tijkl}^{BC}) &= \frac{\lambda_{Sijk}^2 \text{Var}(S_{tij1l}^{BC})}{\text{Var}(Y_{tijkl}^{BC}) - \text{Var}(E_{tijk1}) - \text{Var}(E_{tijkl})}, \quad \forall k > 2, \\ \text{CON}(\tau_{rtij2l}^{BC}) &= \frac{\lambda_{Sij2}^2 \text{Var}(S_{tij1l}^{BC})}{\text{Var}(Y_{rtij2l}^{BC}) - \text{Var}(E_{rtij21}) - \text{Var}(E_{rtij2l})}. \end{aligned}$$

The consistency coefficient of true change represents the proportion of true variance of the observed change scores that is determined by the change of the reference method. The consistency coefficient of true change may also be interpreted as index of true convergent validity of change (Geiser, 2008). In addition, different coefficients of method specificity indicating true method change can be calculated. These coefficients represent the proportion of true variance of an observed change score that is determined by “pure” method change. The term “pure” refers to the fact that the proportion of true change of an observed change score is investigated that is not explained by the true change of the reference method. This amount of true variance is free of change of the reference method, but solely due to the change of a non-reference method. In total, three method specificity coefficients of “true” method change can be defined: the method specificity coefficient of the non-reference structurally different method $MS(\tau_{tijkl}^{BC})$, the common method specificity coefficient of the non-reference interchangeable methods $CMS(\tau_{rtij2l}^{BC})$ and the unique method coefficient of the

non-reference interchangeable methods $UMS(\tau_{rtij2l}^{BC})$:

$$\begin{aligned} MS(\tau_{tijkl}^{BC}) &= \frac{\lambda_{Mijk}^2 \text{Var}(M_{tijkl}^{BC})}{\text{Var}(Y_{tijkl}^{BC}) - \text{Var}(E_{tijk1}) - \text{Var}(E_{tijkl})}, & \forall k > 2, \\ CMS(\tau_{rtij2l}^{BC}) &= \frac{\lambda_{CMij2}^2 \text{Var}(CM_{tj2l}^{BC})}{\text{Var}(Y_{rtij2l}^{BC}) - \text{Var}(E_{rtij21}) - \text{Var}(E_{rtij2l})}, \\ UMS(\tau_{rtij2l}^{BC}) &= \frac{\lambda_{UMij2}^2 \text{Var}(UM_{rtj2l}^{BC})}{\text{Var}(Y_{rtij2l}^{BC}) - \text{Var}(E_{rtij21}) - \text{Var}(E_{rtij2l})}. \end{aligned}$$

Moreover, the total method change coefficient $TMS(\tau_{rtij2l}^{BC})$ with respect to the true change scores of the interchangeable methods is given by:

$$TMS(\tau_{rtij2l}^{BC}) = \frac{\lambda_{UMij2}^2 \text{Var}(UM_{rtj2l}^{BC}) + \lambda_{CMij2}^2 \text{Var}(CM_{tj2l}^{BC})}{\text{Var}(Y_{rtij2l}^{BC}) - \text{Var}(E_{rtij21}) - \text{Var}(E_{rtij2l})}$$

Finally, the reliability coefficient for the observed change scores can be defined as follows:

$$\begin{aligned} Rel(Y_{tijkl}^{BC}) &= 1 - \frac{\text{Var}(E_{tj11}) + \text{Var}(E_{tj1l})}{\text{Var}(Y_{tj1l}^{BC})}, & \forall k > 2, \\ Rel(Y_{rtij2l}^{BC}) &= 1 - \frac{\text{Var}(E_{tjkl}) + \text{Var}(E_{tjkl})}{\text{Var}(Y_{tjkl}^{BC})}. \end{aligned}$$

Subsequently, the unreliability coefficients are defined as follows:

$$\begin{aligned} Unrel(Y_{tijkl}^{BC}) &= 1 - Rel(Y_{tijkl}^{BC}) = \frac{\text{Var}(E_{tjkl}) + \text{Var}(E_{tjkl})}{\text{Var}(Y_{tjkl}^{BC})}, & \forall k > 2, \\ Unrel(Y_{rtij2l}^{BC}) &= 1 - Rel(Y_{rtij2l}^{BC}) = \frac{\text{Var}(E_{rtij21}) + \text{Var}(E_{rtij2l})}{\text{Var}(Y_{rtij2l}^{BC})}. \end{aligned}$$

Proofs. 10 *If and only if the conditions of strong factorial invariance hold, the observed variables can be decomposed in the following way:*

$$\begin{aligned} Y_{tj11} &= S_{tj11} + E_{tj11} \\ Y_{tj1l} &= S_{tj1l} + E_{tj1l}, & \forall l > 1, \\ Y_{tjkl} &= \alpha_{tjkl} + \lambda_{Sijk} S_{tj11} + \lambda_{Mijk} M_{tjk1} + E_{tjkl}, & \forall k > 2, \\ Y_{tjkl} &= \alpha_{tjkl} + \lambda_{Sijk} S_{tj1l} + \lambda_{Mijk} M_{tjkl} + E_{tjkl}, & \forall k > 2, l > 1, \\ Y_{rtij21} &= \alpha_{tj2} + \lambda_{Sij2} S_{tj11} + \lambda_{CMij2} CM_{tj21} + \\ & \quad \lambda_{UMij2} UM_{rtj21} + E_{rtij21}, \\ Y_{rtij2l} &= \alpha_{tj2} + \lambda_{Sij2} S_{tj1l} + \lambda_{CMij2} CM_{tj2l} + \\ & \quad \lambda_{UMij2} UM_{rtj2l} + E_{rtij2l}, & \forall l > 1. \end{aligned}$$

The observed difference scores are then given by:

$$\begin{aligned}(Y_{tij1l} - Y_{tij11}) &= (S_{tij1l} + E_{tij1l}) - (S_{tij11} + E_{tij11}) \\ &= (S_{tij1l} - S_{tij11}) + E_{tij1l} - E_{tij11},\end{aligned}$$

$$\begin{aligned}(Y_{tijkl} - Y_{tijk1}) &= (\alpha_{tijk} + \lambda_{Sijk}S_{tij1l} + \lambda_{Mijk}M_{tjkl} + E_{tijkl}) - \\ &\quad (\alpha_{tijk} + \lambda_{Sijk}S_{tij11} + \lambda_{Mijk}M_{tjk1} + E_{tijk1}) \\ &= \lambda_{Sijk}(S_{tij1l} - S_{tij11}) + \lambda_{Mijk}(M_{tjkl} - M_{tjk1}) + \\ &\quad E_{tijkl} - E_{tijk1},\end{aligned}$$

$$\begin{aligned}(Y_{rtij2l} - Y_{rtij21}) &= (\alpha_{tij2} + \lambda_{Sij2}S_{tij1l} + \lambda_{CMij2}CM_{tj2l} + \lambda_{UMij2}UM_{rtj2l} + E_{rtij2l}) - \\ &\quad (\alpha_{tij2} + \lambda_{Sij2}S_{tij11} + \lambda_{CMij2}CM_{tj21} + \lambda_{UMij2}UM_{rtj21} + E_{rtij21}) \\ &= \lambda_{Sij2}(S_{tij1l} - S_{tij11}) + \lambda_{CMij2}(CM_{tj2l} - CM_{tj21}) + \\ &\quad \lambda_{UMij2}(UM_{rtj2l} - UM_{rtj21}) + E_{rtij2l} - E_{rtij21}.\end{aligned}$$

Let $(Y_{tijkl} - Y_{tijk1})$ be Y_{tijkl}^{BC} , $(Y_{rtij2l} - Y_{rtij21})$ be Y_{rtij2l}^{BC} and $(S_{tij1l} - S_{tij11})$ be S_{tij1l}^{BC} , $(M_{tjkl} - M_{tjk1})$ be M_{tjkl}^{BC} , $(CM_{tj2l} - CM_{tj21})$ be CM_{tj2l}^{BC} , and $(UM_{rtj2l} - UM_{rtj21})$ be UM_{rtj2l}^{BC} , then the equations yield:

$$Y_{tij1l}^{BC} = S_{tij1l}^{BC} + E_{tij1l} - E_{tij11}, \quad (3.98)$$

$$Y_{tijkl}^{BC} = \lambda_{Sijk}S_{tij1l}^{BC} + \lambda_{Mijk}M_{tjkl}^{BC} + E_{tijkl} - E_{tijk1}, \quad \forall k > 2, \quad (3.99)$$

$$Y_{rtij2l}^{BC} = \lambda_{Sij2}S_{tij1l}^{BC} + \lambda_{CMij2}CM_{tj2l}^{BC} + \lambda_{UMij2}UM_{rtj2l}^{BC} + E_{rtij2l} - E_{rtij21}. \quad (3.100)$$

If and only if strong measurement invariance holds, then Equations 3.98 to 3.100 follow, given that the intercepts α_{tijk} drop out and all variables on the right hand side of the equations are uncorrelated with each other.

□

3.9 Mean structure

With respect to longitudinal studies many researcher seek to investigate mean change over time. In this section, the latent variable mean structure of the LC-COM model is discussed. The following theorem shows the consequences of the model definition for the observed and latent variables.

Theorem 3.5 (Mean structure)

If $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathbf{S}_{rt}, \mathbf{S}_t, \mathbf{S}_t^{BC}, \mathbf{UM}_{rt}, \mathbf{UM}_{rt}^{BC}, \mathbf{CM}_t, \mathbf{CM}_t^{BC}, \mathbf{M}_t, \mathbf{M}_t^{BC}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_{tijk}, \lambda_{Sijk}, \lambda_{UMij2}, \lambda_{CMij2}, \lambda_{Mijk} \rangle$ is called an extended baseline LC-COM change model and without loss of generality, $k=1$ method is chosen as reference method, then the following mean structure holds for all $r \in R \equiv \{1, \dots, a\}$, $t \in T \equiv \{1, \dots, b\}$, $i \in I \equiv \{1, \dots, c\}$, $j \in J \equiv \{1, \dots, d\}$, $k \in K \equiv \{1, \dots, e\}$, $l \in L \equiv \{1, \dots, f\}$:

$$E(Y_{rtij2l}^{BC}) = \lambda_{tij2}E(S_{tij1l}^{BC}), \quad \forall l > 1, \quad (3.101)$$

$$E(Y_{tijkl}^{BC}) = \lambda_{tijk}E(S_{tij1l}^{BC}), \quad \forall k > 2, \wedge \forall l > 1, \quad (3.102)$$

$$E(S_{tij1l}^{BC}) = E(Y_{tij1l}^{BC}), \quad \forall l > 1, \quad (3.103)$$

$$E(CM_{tj2l}^{BC}) = 0, \quad \forall l > 1, \quad (3.104)$$

$$E(UM_{rtj2l}^{BC}) = 0, \quad \forall l > 1, \quad (3.105)$$

$$E(M_{tijkkl}^{BC}) = 0, \quad \forall k > 2, \wedge \forall l > 1, \quad (3.106)$$

$$E(E_{tijkkl}) = 0, \quad \forall k \neq 2, \quad (3.107)$$

$$E(E_{rtij2l}) = 0, \quad (3.108)$$

where $E(\cdot)$ denotes expected value (mean).

Proofs. 11 According to Equation 3.99 the measurement equation of the observed difference variables Y_{tijkkl}^{BC} is given by:

$$Y_{tijkkl}^{BC} = \lambda_{Sijk} S_{tij1l}^{BC} + \lambda_{Mijk} M_{tijkkl}^{BC} + E_{tijkkl} - E_{tijk1l}.$$

Therefore, the equation above can be rewritten as follows:

$$E(Y_{tijkkl}^{BC}) = E(\lambda_{Sijk} S_{tij1l}^{BC}) + E(\lambda_{Mijk} M_{tijkkl}^{BC}) + E(E_{tijkkl}) - E(E_{tijk1l}).$$

According to Equation 3.106 and 3.107, the expected values of the latent method change as well as the measurement error variables equal zero. This is a direct consequence of the definition of these variables, given that these variables (M_{tijkkl}^{BC} , E_{tijkkl} , and E_{tijk1l}) are defined as residuals and residual variables always have an expected value of zero (Steyer, 1989; Steyer & Eid, 2001). Hence, the equation above simplifies to Equation 3.102:

$$E(Y_{tijkkl}^{BC}) = E(\lambda_{Sijk} S_{tij1l}^{BC}) = \lambda_{Sijk} E(S_{tij1l}^{BC}).$$

Similarly, according to Equation 3.100 the measurement equation of the observed difference variables Y_{rtij2l}^{BC} is given by:

$$Y_{rtij2l}^{BC} = \lambda_{Sij2} S_{tij1l}^{BC} + \lambda_{CMij2} CM_{tj2l}^{BC} + \lambda_{UMij2} UM_{rtj2l}^{BC} + E_{rtij2l} - E_{rtij21l}.$$

Again, the above equation can be rewritten in terms of expected values as follows:

$$E(Y_{rtij2l}^{BC}) = E(\lambda_{Sij2} S_{tij1l}^{BC}) + E(\lambda_{CMij2} CM_{tj2l}^{BC}) + E(\lambda_{UMij2} UM_{rtj2l}^{BC}) + E(E_{rtij2l}) - E(E_{rtij21l}).$$

As a consequence of the definition of the latent variables (CM_{tj2l}^{BC} , UM_{rtj2l}^{BC} , E_{rtij2l} , and $E_{rtij21l}$) as latent residual variables, the conditions stated in Equation 3.104, 3.105, and 3.108 hold. Therefore, Equation 3.101 follows, given that

$$E(Y_{rtij2l}^{BC}) = E(\lambda_{Sij2} S_{tij1l}^{BC}) = \lambda_{Sij2} E(S_{tij1l}^{BC}).$$

Finally, according to Equation 3.98 the measurement equation of the observed difference variables Y_{tij1l}^{BC} is given by:

$$Y_{tij1l}^{BC} = S_{tij1l}^{BC} + E_{tij1l} - E_{tij11l}.$$

With respect to the definition of the latent variables, this equation can be rewritten as follows:

$$Y_{tij1l}^{BC} = E(S_{tij1l}^{BC}),$$

given that the measurement error variables are again defined as latent residuals, and thus have an expected value of zero (Steyer, 1989; Steyer & Eid, 2001).

3.10 Identifiability

As mentioned before, both change models (baseline LC-COM as well as extended baseline LC-COM model) represent an alternative parametrization of the LS-COM model (see Chapter 2). Thus, the parameters of any latent LC-COM model are identified for at least two indicators, one construct, two methods, and two occasions of measurement and correlated latent state as well as change method factors as described in Theorem 2.6 of Chapter 2.

Box 3.1 (Baseline latent change Model)*Latent variables of the baseline LS-COM model*

S_{tij1l}^{BC}	<i>target-specific latent change variables of the reference (here: structurally different, $k = 1$) method of construct j on measurement occasion l assessed by indicator i (e.g., “true” intraindividual change of the leadership quality of a particular target assessed by indicator i from the initial state to occasion of measurement l)</i>
UM_{rtij2l}^{BC}	<i>rater-target-specific latent unique method change variables of the non-reference method $k = 2$ of construct j on measurement occasion l assessed by indicator i (e.g., “true” change of the unique rater bias of the leadership quality assessed by indicator i from the initial state to occasion of measurement l)</i>
CM_{tij2l}^{BC}	<i>target-specific latent common method change variables of the non-reference method $k = 2$ of construct j on measurement occasion l assessed by indicator i (e.g., “true” change of the common rater bias of the leadership quality assessed by indicator i from the initial state to occasion of measurement l)</i>
M_{tijkl}^{BC}	<i>target-specific latent method change variables of other non-reference methods $k > 2$ of construct j on measurement occasion l assessed by indicator i (e.g., the “true” change of method bias of the leadership quality assessed by indicator i on occasion of measurement l)</i>

Chapter 4

Formal definition of the latent state-trait (LST-COM) model

4.1 A gentle introduction

In the following chapter, a longitudinal multitrait-multimethod (MTMM) latent state-trait model for the combination of structurally different and interchangeable methods is formally defined. The model will be abbreviated LST-COM model. Latent state-trait (LST) models (Eid, Schneider, & Schwenkmezger, 1999; Steyer et al., 1992) are commonly used to study “true” (i.e., measurement error free) stable interindividual differences, true occasion-specific interindividual differences, as well as occasion-specific influences due to measurement error (Geiser, 2008). The basic principle of LST-theory is the decomposition of the latent state variables S_{il} into a latent trait variables ξ_{il} and an occasion-specific residual variables ζ_{il} (Eid et al., 1999; Steyer et al., 1992). For the sake of simplicity, the general decomposition of the latent state variables of a LST model is presented for monoconstruct-monomethod measurement designs first. That is, only two indices are needed: i for indicator (item) and l for occasion of measurement. Later in this chapter, this general decomposition is extended to multiconstruct-multimethod measurement designs combining structurally different and interchangeable methods. The latent variables in a LST model are defined as follows (Courvoisier, 2006; Eid, 1995; Eid et al., 1999; Steyer et al., 1992):

$$\begin{aligned} S_{il} &\equiv E(Y_{il}|p_u, p_{Sit_l}), \\ E_{il} &\equiv Y_{il} - E(Y_{il}|p_u, p_{Sit_l}), \\ \xi_{il} &\equiv E[E(Y_{il}|p_u, p_{Sit_l})|p_u] = E(Y_{il}|p_u), \\ \zeta_{il} &\equiv E(Y_{il}|p_u, p_{Sit_l}) - E(Y_{il}|p_u). \end{aligned}$$

$E(\cdot)$ is the conditional expectation, (p_u) represents the projection into a set of persons and (p_{Sit_l}) denotes the projection into a set of situations. The total measurement equation of any observed variable in a LST model can be therefore written as:

$$\begin{aligned} Y_{il} &= E(Y_{il}|p_u) & + [E(Y_{il}|p_u, p_{Sit_l}) - E(Y_{il}|p_u)] & + [Y_{il} - E(Y_{il}|p_u, p_{Sit_l})] \\ &= \xi_{il} & + \zeta_{il} & + E_{il}. \end{aligned}$$

The latent state variables S_{il} are defined as conditional expectations of Y_{il} given the person (p_u) and the situation (p_{Sit_i}). The measurement error variables E_{il} are defined as differences between the observed variables Y_{il} and the latent state variables. The latent trait variables ξ_{il} are defined as conditional expectations of the latent state variable S_{il} (or the observed variables Y_{il}) given the person (p_u). Consequently, the latent trait variables reflect “true” person-specific influences. The latent state residual variables ζ_{il} are defined as differences between the latent state variables S_{il} and the latent trait variables ξ_{il} . Hence, the latent state residual variables represent measurement error free influences due to the situation (p_{Sit_i}) and/or due to the interaction between the person (p_u) and the situation (p_{Sit_i}). As a consequence of the definition of ζ_{il} as latent residuals with respect to ξ_{il} , both latent variables are uncorrelated with each other. For a detail description of the LST theory see Steyer et al. (1992) as well as Eid (1995). In the following sections, the definition of the latent variables of a LST model for multiconstruct-multimethod measurement designs combining structurally different and interchangeable methods is given. This model is called LST-COM model, given that it combines LST-theory and multiconstruct-multimethod analysis for a combination of different types of methods. Again, the abbreviation LST denotes that a LST model is defined. The abbreviation COM stands for combination of structurally different and interchangeable methods.

Step 1: Definition of the latent trait and state-residual variables

In a similar way, the latent variables (i.e., the latent state, trait, state-residuals, and measurement error variables) of the LST-COM model are defined:

$$\begin{aligned} S_{rtij2l} &\equiv E(Y_{rtij2l} | p_T, p_{TS_i}, p_R, p_{RS_i}), \\ S_{tijk} &\equiv E(Y_{tijk} | p_T, p_{TS_i}), \\ E_{rtij2l} &\equiv Y_{rtij2l} - E(Y_{rtij2l} | p_T, p_{TS_i}, p_R, p_{RS_i}), \\ E_{tijk} &\equiv Y_{tijk} - E(Y_{tijk} | p_T, p_{TS_i}), \\ \xi_{tijk} &\equiv E(S_{tijk} | p_T), \\ \zeta_{tijk} &\equiv S_{tijk} - E(S_{tijk} | p_T). \end{aligned}$$

Again, $E(\cdot)$ is the conditional expectation. However, (p_T) stands for the projection into a set of targets, (p_{TS_i}) represents the projection into a set of target-specific situations, (p_R) refers to the projection into a set of raters, (p_{RS_i}) denotes the projection into a set of rater-specific situations. The latent variables (S_{rtij2l} , E_{rtij2l}) represent the latent state variables as well as their corresponding measurement error variables on level-1 (rater-level), whereas the latent variables (S_{tijk} , ξ_{tijk} , ζ_{tijk} , E_{tijk}) denote the latent state, trait, state-residual and measurement error variables on level-2 (i.e., the target-level). Note that six indices are used in order to define a LST model for multitrait-multimethod measurement designs combining different types of methods: Again, $k = 2$ represents the non-reference interchangeable method (e.g., multiple student ratings for teaching quality). The index r indicates that multiple ratings for target t are measured on the rater-level. Conversely, any latent variable without the index r is measured on the target-level. The remaining indices (i, j, l) stand for $i =$ item or indicator, $j =$ construct, and $l =$ occasion of measurement.

In order to define level-2 (target-specific) latent state variables S_{tij2l} on the basis of the level-1 (rater-specific) latent state variables S_{rtij2l} pertaining to the interchangeable methods ($k = 2$), the target- and occasion-specific expectations of the level-1 latent state variables S_{rtij2l} are considered once again:

$$S_{tij2l} \equiv E(S_{rtij2l}|p_T, p_{TS_i}) \quad (4.1)$$

$$= E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_{TS_i}] \quad (4.2)$$

$$= E(Y_{rtij2l}|p_T, p_{TS_i}). \quad (4.3)$$

According to the above Equation 4.1, the latent state variables S_{tij2l} are defined as conditional expectations of S_{rtij2l} given the target (p_T) in a situation (p_{TS_i}). Hence, these latent variables may be interpreted as expected values of the interchangeable ratings S_{tij2l} for a particular target t on occasion of measurement l . The residuals of the latent regression analysis are defined as unique method variables UM_{rtij2l} :

$$UM_{rtij2l} \equiv S_{rtij2l} - E(S_{rtij2l}|p_T, p_{TS_i}).$$

The latent residual variables UM_{rtij2l} represent the occasion-specific unique method bias for a particular rater. In other words, the latent unique method variables UM_{rtij2l} reflect the true over- or underestimation of the true ratings of the interchangeable raters by a particular rater on occasion of measurement l . Given that these latent variables are defined as residuals and given the fact that the expectations of residual variables are always zero (Steyer, 1988, 1989; Steyer & Eid, 2001), the expectations of the UM_{rtij2l} residual variables are also zero by definition. Another consequence is that the latent unique method variables are uncorrelated with any other latent state variable S_{tij2l} on the target-level. In order to defined level-2 latent trait as well as level-2 latent state-residual variables on the basis of the latent state variables S_{tij2l} , the target- and occasion-specific latent state variables S_{tij2l} are decomposed as follows:

$$\xi_{tij2l} \equiv E(S_{tij2l}|p_T) \quad (4.4)$$

$$= E[E(Y_{rtij2l}|p_T, p_{TS_i})|p_T] \quad (4.5)$$

$$= E(Y_{rtij2l}|p_T). \quad (4.6)$$

With respect to the Equation 4.4, the latent trait variables ξ_{tij2l} for the non-reference interchangeable method represent the conditional expectations of the latent state variables S_{tij2l} given the target (p_T). Thus, the latent trait variables ξ_{tij2l} represent the “true” and consistent view of the interchangeable raters for a particular target. Note that the latent trait variables for the remaining methods ($k = 1$ and $k > 2$) can be defined in a similar way:

$$\xi_{tij1l} \equiv E(Y_{tij1l}|p_T),$$

$$\xi_{tijkl} \equiv E(Y_{tijkl}|p_T), \quad \forall k > 2.$$

However, the latent trait variables $(\xi_{tij1l}, \xi_{tijk1l})$ represent the “true” consistent view of the reference or non-reference method (e.g., teacher’s self-ratings, rating of the school principle). That means that the index k indicates whether the “true” and consistent view of the target’s behavior is measured with regard to the reference method ($k=1$; self-report), the non-reference method belonging to the interchangeable method ($k=2$; peer reports), or the non-reference method belonging to a structurally different method ($k>2$; parent report). The residuals of these latent regression analyses are once again defined as occasion-specific latent residual variables $(\zeta_{tij1l}, \zeta_{tij2l}, \zeta_{tijk1l})$:

$$\begin{aligned}\zeta_{tij1l} &\equiv S_{tij1l} - \xi_{tij1l}, \\ \zeta_{tij2l} &\equiv S_{tij2l} - \xi_{tijk2l}, \\ \zeta_{tijk1l} &\equiv S_{tijk1l} - \xi_{tijk1l}, \quad \forall k > 2.\end{aligned}$$

The latent state-residual variables $(\zeta_{tij1l}, \zeta_{tij2l}, \zeta_{tijk1l})$ are defined as difference between latent state variables and the latent trait variables. Again, these variables are defined as residual variables. Therefore, these variables have expectations of zero and are uncorrelated with their corresponding latent trait variables. The latent state-residual variables ζ_{tijk2l} represent the occasion-specific deviations of the “true” common view of the interchangeable methods from the “true” time-invariant (i.e., trait) common view of the interchangeable methods. In contrast to that, the latent state-residual variables ζ_{tijk1l} represent the occasion-specific deviations of the “true” rating of the reference method (e.g., teacher’s self-rating) from the occasion-unspecific (time-invariant) “true” rating of the reference method. The latent state-residual variables ζ_{tijk1l} represent the occasion-specific deviations of the “true” ratings of the structurally different non-reference method (e.g., ratings of the school principle) from the occasion-unspecific “true” rating of the structurally different non-reference method.

Step 2: Definition of the latent trait and state-residual method variables on the rater-level

One of the main advantages of the LST-COM model is that trait $(\xi_{rtij2l}^{UM}, \xi_{tijk2l}^{CM}, \xi_{tijk1l}^M)$ as well as state-residual $(\zeta_{rtij2l}^{UM}, \zeta_{tijk2l}^{CM}, \zeta_{tijk1l}^M)$ method variables can be defined. With respect to the definition of these latent method variables it is possible to analyze consistent as well as occasion-specific method bias on different measurement levels (rater- and target-level). With respect to the definition of the latent trait and state method variables $(\xi_{tijk2l}^{CM}, \xi_{tijk1l}^M, \zeta_{tijk2l}^{CM}, \zeta_{tijk1l}^M)$ it is also possible to investigate “pure” trait or state method effects (i.e., not shared with the reference method).

In the following, the latent trait unique method variables ξ_{rtij2l}^{UM} on the rater-level (level-1) are considered first. In the next step, the latent trait method variables ξ_{tijk2l}^{CM} and ξ_{tijk1l}^M on the target-level (level-2) are discussed. The latent trait unique method variables ξ_{rtij2l}^{UM} are defined as follows:

$$\xi_{rtij2l}^{UM} \equiv E(UM_{rtij2l} | p_T, p_R). \quad (4.7)$$

With respect to the Equation 4.7, it is not easy to see what the latent trait unique method variables ξ_{rtij2l}^{UM} mean, given that the occasion-specific unique method variables UM_{rtij2l} (on the right side of the equation) can be further decomposed. For the sake of clarity, this decomposition shall be

illustrated briefly:

$$\begin{aligned}
\xi_{rtij2l}^{UM} &\equiv E(UM_{rtij2l}|p_T, p_R) \\
&= E[(S_{rtij2l} - S_{tij2l})|p_T, p_R] \\
&= E(S_{rtij2l}|p_T, p_R) - E(S_{tij2l}|p_T, p_R) \\
&= E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_R] - E[E(Y_{rtij2l}|p_T, p_{TS_i})|p_T, p_R].
\end{aligned}$$

With respect to one additional assumption (called: conditional regressive independence with respect to S_{tij2l}), the equation $[E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_R] - E[E(Y_{rtij2l}|p_T, p_{TS_i})|p_T, p_R]]$ can be simplified to (see Proof 12):

$$\xi_{rtij2l}^{UM} \equiv E(Y_{rtij2l}|p_T, p_R) - E(Y_{rtij2l}|p_T).$$

Hence, if this additional assumption holds, the latent trait unique method variables ξ_{rtij2l}^{UM} can be interpreted as “true” and consistent unique method bias of a particular rater. The additional assumption assumes that the latent state variables S_{tij2l} are conditionally regressive independent from the raters (p_R) given the target (p_T). Another consequence of this additional assumption is that the latent state-residual unique method variables ζ_{rtij2l}^{UM} can be defined as residuals with respect to the latent trait unique method variables:

$$\zeta_{rtij2l}^{UM} \equiv UM_{rtij2l} - \xi_{rtij2l}^{UM}.$$

Given that the latent state-residual unique method variables ζ_{rtij2l}^{UM} are defined as residuals, they have expectations of zero and are uncorrelated with their corresponding ξ_{rtij2l}^{UM} variables. The latent state-residual unique method variables ζ_{rtij2l}^{UM} can be interpreted as “true” momentary rater-bias with respect to the interchangeable methods.

Step 3: Definition of the latent trait and state-residual method variables on the target-level

The consistent as well as occasion-specific method variables on the target-level can be defined with respect to the following latent regression analyses:

$$E(\xi_{tijkl}|\xi_{tij1l}) = \alpha_{tijkl} + \lambda_{\xi_{ijkl}}\xi_{tij1l}, \quad (4.8)$$

$$E(\zeta_{tijkl}|\zeta_{tij1l}) = \lambda_{\zeta_{ijkl}}\zeta_{tij1l}. \quad (4.9)$$

Note that this latent regression approach relates to the general CTC($M-1$) modeling framework (see Courvoisier, 2006; Eid, 2000; Eid et al., 2003). The LST-COM model therefore combines the advantages of the CTC($M-1$) modeling framework as well as the longitudinal modeling approach of LST models. The residuals of the regression above (see Equation 4.8 and 4.9) can be defined as

the latent trait (common) method effects (ξ_{tij2l}^{CM} , ξ_{tijk}^M):

$$\begin{aligned}\xi_{tij2l}^{CM} &\equiv \xi_{tij2l} - E(\xi_{tij2l} | \xi_{tij1l}), \\ \xi_{tijk}^M &\equiv \xi_{tijk} - E(\xi_{tijk} | \xi_{tij1l}), \quad \forall k > 2.\end{aligned}$$

The latent trait common method variables ξ_{tij2l}^{CM} represent the common and consistent (time-invariant) part of method bias pertaining to the interchangeable methods (e.g., peer ratings, colleague ratings) that is not shared with the reference method (e.g., target's self-report). The term "common" refers to the fact that this consistent method bias is common to all interchangeable raters and thereby reflects the consistent view of the interchangeable raters that is not shared with the reference method. The latent trait method variables ξ_{tijk}^M reflect the consistent deviation of a non-reference structurally different method (e.g., parent rating, supervisor rating) from the reference method (e.g., self-report). With respect to the correlations of these two latent trait method variables (ξ_{tij2l}^{CM} , ξ_{tijk}^M), the generalizability of consistent method biases across indicators, methods and/or constructs can be studied. As a consequence of the definition of the latent trait method variables (namely as latent residual variables), it follows that these variables have an expected value of zero and are uncorrelated with all latent trait variables pertaining to the same indicator and construct.

According to the Equation 4.9, the latent state-residual (common) method variables (ζ_{tij2l}^{CM} , ζ_{tijk}^M) can be also defined as latent residual variables:

$$\begin{aligned}\zeta_{tij2l}^{CM} &\equiv \zeta_{tij2l} - E(\zeta_{tij2l} | \zeta_{tij1l}), \\ \zeta_{tijk}^M &\equiv \zeta_{tijk} - E(\zeta_{tijk} | \zeta_{tij1l}), \quad \forall k > 2.\end{aligned}$$

The latent state-residual common method ζ_{tij2l}^{CM} variables represent the occasion-specific (not consistent or time-invariant) part of method bias of the interchangeable methods (e.g., peer ratings, colleague ratings) that is not shared with the consistent view of the reference method (e.g., target's self-report). In other words, the latent ζ_{tij2l}^{CM} variables capture the amount of occasion-specific and common method influences of the set of interchangeable methods/raters that is not shared with occasion-specific influences of the reference method (target's self-report). In contrast, the ζ_{tijk}^M variables represent the occasion-specific deviation of the non-reference structurally different method (e.g., parent rating, supervisor rating) from the occasion-specific view of the reference method (e.g., target's self-report). Again, due to the definition of these variables as latent residuals, the general properties of residuals apply as well (Steyer & Eid, 2001).

According to the LST-theory it is possible to define different variance coefficients such as consistency and specificity (Eid et al., 1999; Steyer et al., 1992). However, for simplicity reasons these coefficients are not discussed in this gentle introduction, but rather in Section 4.7.1 after the LST-COM model has been formally defined. Besides, in order to define all of these variance coefficients properly, additional assumptions have to be introduced (see Section 4.6.1). In the next step, the gentle introduction ends with the general measurement equations of the observed variables in the LST-COM model.

Step 4: Definition of common trait and method variables

In the fourth step, additional homogeneity assumptions are imposed with regard to the latent trait variables. These assumptions are important for defining latent trait factors (see also Courvoisier, 2006; Eid, 1995; Eid et al., 1999; Geiser, 2008; Steyer et al., 1992). For a detailed discussion of how these latent trait factors can be formally defined based on the homogeneity assumptions see Section 4.3.

The first homogeneity assumption concerns the latent trait variables ξ_{tij1l} as measured by the reference method. Specifically, it is assumed that the latent trait variables pertaining to the same indicator i , construct j , the reference method $k = 1$, but different occasions of measurement l and l' are homogeneous and only differ with respect to an additive α_{tij1l} as well as multiplicative $\lambda_{\xi_{tij1l}}$ constant. With respect to this assumption, common item- and construct-specific latent trait factors (i.e., item- and construct-specific latent trait factors) can be defined:

$$\xi_{tij1l} = \alpha_{tij1l} + \lambda_{\xi_{tij1l}}\xi_{tij1}.$$

Note that the index $k = 1$ with respect to the latent trait variable ξ_{tij1} may also be dropped, given that all the latent trait variables belonging to the non-reference methods ξ_{tijkl} are regressed on the reference latent trait variables ξ_{tij1l} and thus all latent trait variables are measured by a general latent trait factor ξ_{tij} . In a similar way, it is assumed that the latent trait method variables pertaining to the same indicator i , construct j , method k , but different occasion of measurement l and l' are homogeneous. Thus, latent trait method factor maybe defined as well:

$$\xi_{rtij2l}^{UM} = \lambda_{\xi_{ij2l}^{UM}}\xi_{rtij2}^{UM}, \quad (4.10)$$

$$\xi_{tij2l}^{CM} = \lambda_{\xi_{ij2l}^{CM}}\xi_{tij2}^{CM}, \quad (4.11)$$

$$\xi_{tijkl}^M = \lambda_{\xi_{ijkl}^M}\xi_{tijk}^M, \quad \forall k > 2. \quad (4.12)$$

$$(4.13)$$

Finally, it is assumed that the latent occasion-specific method variables pertaining to the same occasion of measurement l , construct j and method k , but different indicators i and i' are homogeneous and only differ by a multiplicative constant:

$$\zeta_{rtij2l}^{UM} = \lambda_{\zeta_{ij2l}^{UM}}\zeta_{rtij2l}^{UM}, \quad (4.14)$$

$$\zeta_{tij2l}^{CM} = \lambda_{\zeta_{ij2l}^{CM}}\zeta_{tij2l}^{CM}, \quad (4.15)$$

$$\zeta_{tijkkl}^M = \lambda_{\zeta_{ijkl}^M}\zeta_{tijkkl}^M, \quad \forall k > 2. \quad (4.16)$$

$$(4.17)$$

Again, with respect to the Equations 4.14 to 4.16, it is possible to construe latent trait as well as latent method factors. The demonstrations of the existence, uniqueness as well as admissible transformations of these common latent variables are provided in the following sections. Finally,

the complete measurement equation of the observed variables of the LST-COM model is given by:

$$Y_{tij1l} = \xi_{tij} + \zeta_{tijl} + E_{tij1l}, \quad (4.18)$$

$$Y_{tijkl} = \alpha_{tijkl} + \lambda_{\xi_{ijkl}} \xi_{tij} + \lambda_{\xi_{ijkl}}^M \xi_{tij}^M + \lambda_{\zeta_{ijkl}} \zeta_{tijl} + \lambda_{\zeta_{ijkl}}^M \zeta_{tijl}^M + E_{tijkl}, \quad \forall k > 2, \quad (4.19)$$

$$Y_{rtij2l} = \alpha_{rtij2l} + \lambda_{\xi_{ij2l}} \xi_{tij} + \lambda_{\xi_{ij2l}}^{CM} \xi_{tij}^{CM} + \lambda_{\xi_{ij2l}}^{UM} \xi_{rtij2}^{UM} + \lambda_{\zeta_{ij2l}} \zeta_{tijl} + \lambda_{\zeta_{ij2l}}^{CM} \zeta_{tij2l}^{CM} + \lambda_{\zeta_{ij2l}}^{UM} \zeta_{rtj2l}^{UM} + E_{rtij2l}. \quad (4.20)$$

Figure 4.1 illustrates the LST-COM model that was explained in this gentle introduction. However, due to the complexity of this model a more restrictive variant of the LST-COM model with common latent state, trait, and method factors is presented in Figure 4.1. Both models are presented for a complete measurement design of three indicators, two constructs, three methods (two structurally different and one set of interchangeable methods), and two occasions of measurement.

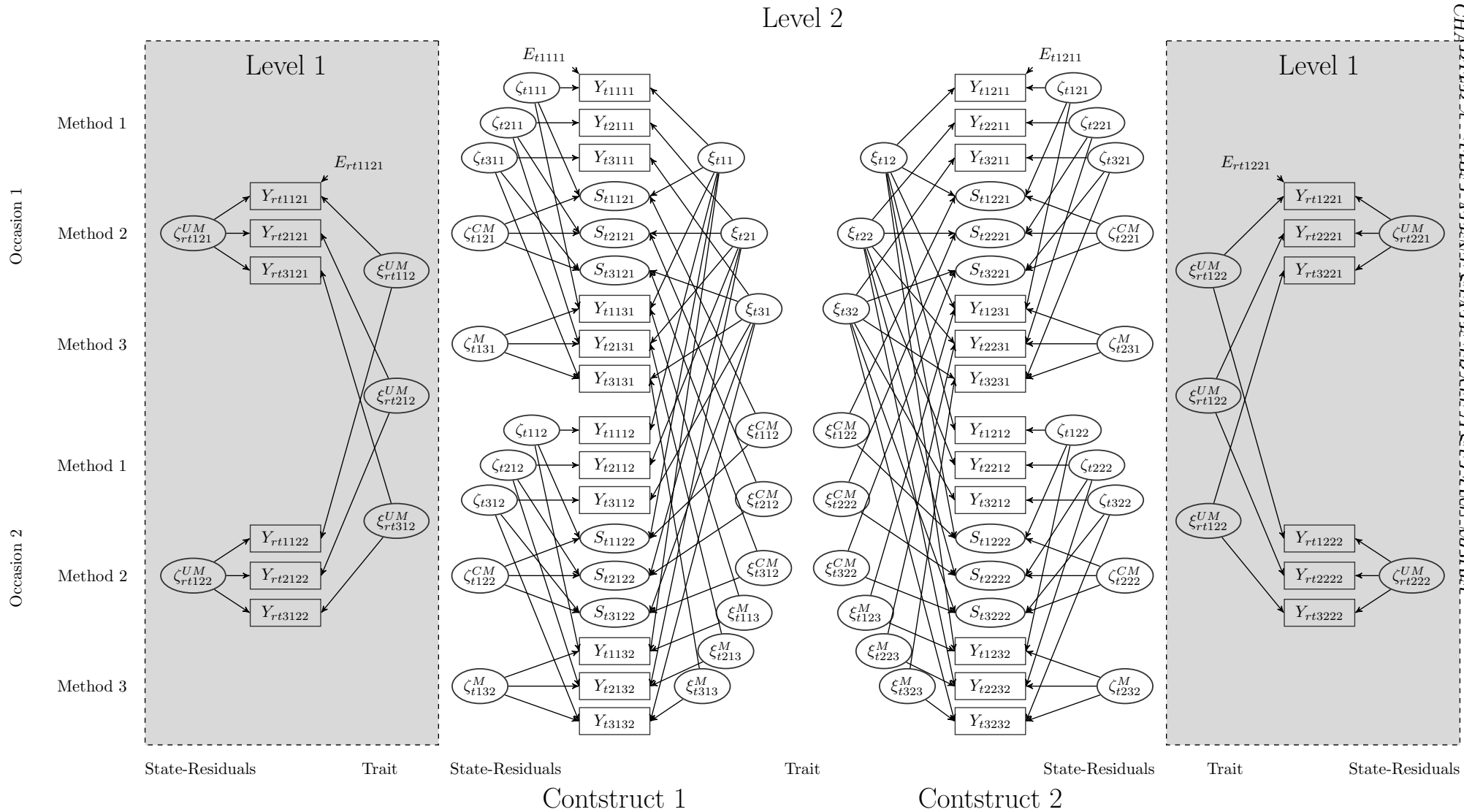


Figure 4.1: Path diagram of LST-COM model with indicator-specific latent trait factors.

Path diagram of LST-COM model for three indicators ($i=1,2,3$), two constructs ($j=1,2$), three methods ($k=1,2,3$) and two occasions of measurement ($l=1,2$). All correlations between latent variables as well as factor loading parameters were omitted for clarity. Measurement error variables E_{rtijkl} and E_{tijkl} are only depicted for the first indicator pertaining to method 1 and 2.

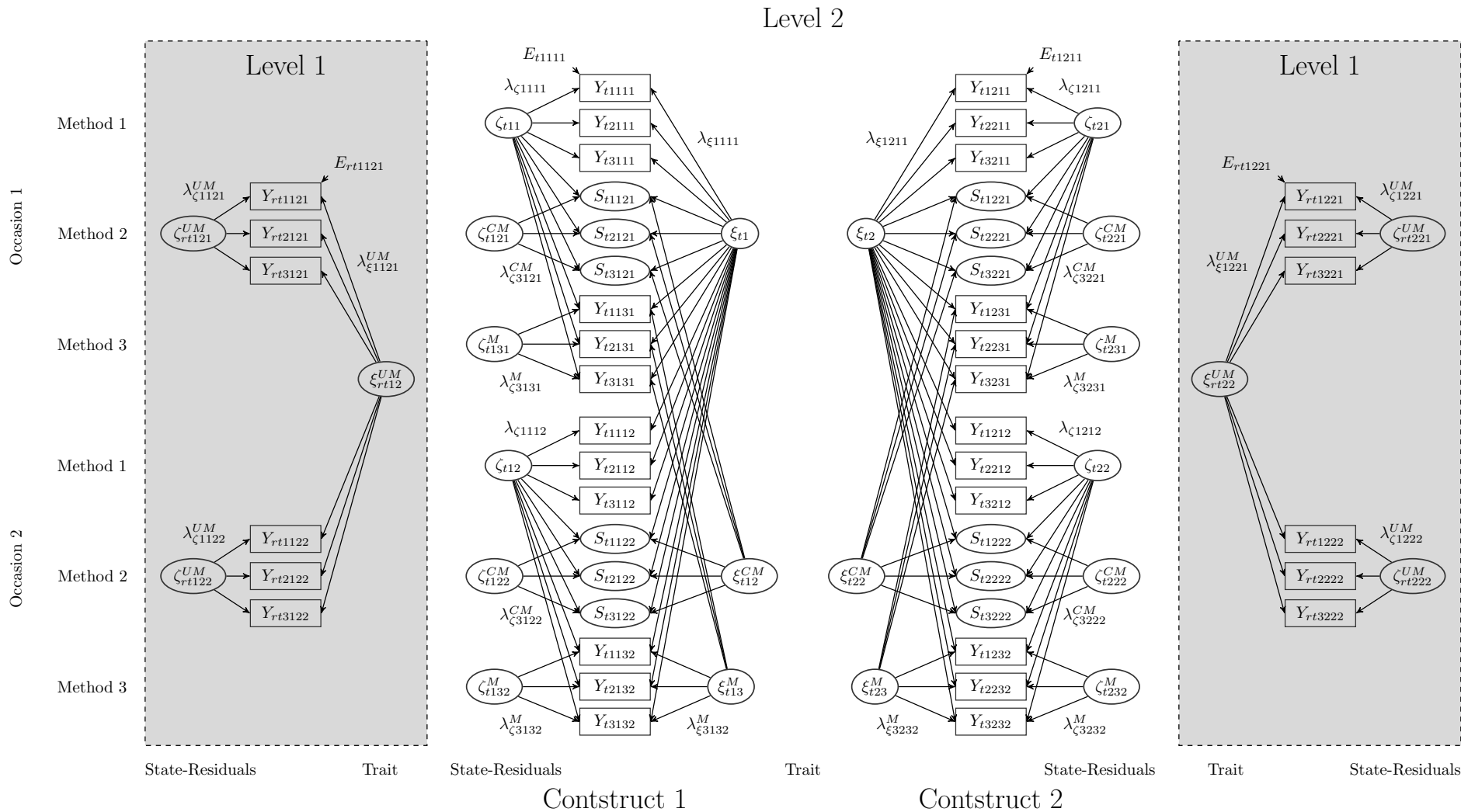


Figure 4.2: Path diagram of the LST-COM model with common latent trait factors.

Path diagram of the LST-COM model with common latent state, trait, and method factors incorporating for three indicators ($i=1,2,3$), two constructs ($j=1,2$), three methods ($k=1,2,3$) and two occasions of measurement ($l=1,2$). All correlations between latent variables were omitted for clarity. Measurement error variables E_{rtijkl} and E_{tijkl} are only depicted for the first indicator pertaining to method 1 and 2.

4.2 Definition of the LST-COM model

The following chapter the LST-COM model is formally defined based on the stochastic measurement theory by Steyer (1989) as well as Steyer and Eid (2001). Moreover, the LST-COM model incorporates the CTC($M-1$) modeling framework by Eid (2000) as well as Eid et al. (2003).

Definition 4.1 (LST-COM model)

The random variables $\{Y_{111111}, \dots, Y_{rtijkl}, \dots, Y_{abcdef}\}$ and $\{Y_{11111}, \dots, Y_{ijkl}, \dots, Y_{bcdef}\}$ on a probability space $(\Omega, \mathfrak{A}, P)$ are variables of a LST-COM model if and only if the conditions (a to e) of Definition 2.1 [i.e., LS-COM model] and the conditional regressive independence assumption made in Definition 2.2 hold:

(a) Then, the variables

Rater-level (level-1):

$$\xi_{rtij2l}^{UM} \equiv E(UM_{rtij2l} | p_T, p_R), \quad (4.21)$$

$$\zeta_{rtij2l}^{UM} \equiv UM_{rtij2l} - \xi_{rtij2l}^{UM}, \quad (4.22)$$

Target-level (level-2):

$$\xi_{tijk} \equiv E(S_{tijk} | p_T), \quad (4.23)$$

$$\zeta_{tijk} \equiv S_{tijk} - \xi_{tijk}, \quad (4.24)$$

$$\xi_{tij2l}^{CM} \equiv \xi_{tij2l} - E(\xi_{tij2l} | \xi_{tij1l}), \quad (4.25)$$

$$\xi_{tijk}^M \equiv \xi_{tijk} - E(\xi_{tijk} | \xi_{tij1l}), \quad \forall k > 2, \quad (4.26)$$

$$\zeta_{tij2l}^{CM} \equiv \zeta_{tij2l} - E(\zeta_{tij2l} | \zeta_{tij1l}), \quad (4.27)$$

$$\zeta_{tijk}^M \equiv \zeta_{tijk} - E(\zeta_{tijk} | \zeta_{tij1l}), \quad \forall k > 2, \quad (4.28)$$

$$(4.29)$$

are random variables on $(\Omega, \mathfrak{A}, P)$ with finite and positive variance.

(b) With respect to the same indicator i , same construct j , and same occasion of measurement l , it is assumed that the regression of the trait variable belonging to a non-reference method k on the latent trait variable belonging to the reference method ($k = 1$) is linear. For each construct j , measured by a non-reference method k on occasion of measurement l and for each pair $(i, i') \in I \times I'$, ($i \neq i'$) there is a constant $\alpha_{tijk} \in \mathbb{R}$ as well as a constant $\lambda_{\xi_{ijk}} \in \mathbb{R}_+$, such that

$$E(\xi_{tijk} | \xi_{tij1l}) = \alpha_{tijk} + \lambda_{\xi_{ijk}} \xi_{tij1l}. \quad (4.30)$$

(c) Definition of common trait variables. For each indicator i , construct j , measured by the reference method k ($k = 1$) and for each pair $(l, l') \in L \times L'$, ($l \neq l'$) there is a constant $\alpha_{tij1l'}$ as well as a constant $\lambda_{\xi_{ij1l'}}$, such that

$$\xi_{tij1l} = \alpha_{tij1l'} + \lambda_{\xi_{ij1l'}} \xi_{tij1l'}. \quad (4.31)$$

(d) For each indicator i , construct j , measured by a non-reference method k ($k \neq 1$) on occasion of measurement l there is a constant $\lambda_{\zeta_{ijk}} \in \mathbb{R}_+$, such that

$$E(\zeta_{tijk} | \zeta_{tij1l}) = \lambda_{\zeta_{ijk}} \zeta_{tij1l}. \quad (4.32)$$

(e) *Definition of common method trait variables.* For each indicator i , construct j , measured by the non-reference method k ($k \neq 1$) and for each pair $(l, l') \in L \times L'$, ($l \neq l'$) there are constants $\lambda_{\xi_{ij2l}^{CM}}, \lambda_{\xi_{ij2l'}^{UM}}$, as well as $\lambda_{\xi_{ijkl}^M}$, such that

$$\xi_{tij2l}^{CM} = \lambda_{\xi_{ij2l'}^{CM}} \xi_{tij2l'}^{CM}, \quad (4.33)$$

$$\xi_{rtij2l}^{UM} = \lambda_{\xi_{ij2l'}^{UM}} \xi_{rtij2l'}^{UM}, \quad (4.34)$$

$$\xi_{tijk}^M = \lambda_{\xi_{ijkl'}^M} \xi_{tijk'}^M, \quad \forall k > 2. \quad (4.35)$$

(f) *Definition of common method state residual variables.* For each construct j , measured by the non-reference method k ($k \neq 1$) and for each pair $(i, i') \in I \times I'$, ($i \neq i'$) there are constants $\lambda_{\zeta_{ii'j2l}^{CM}}, \lambda_{\zeta_{ii'j2l'}^{UM}}$, as well as $\lambda_{\zeta_{ii'jkl}^M}$, such that

$$\zeta_{tij2l}^{CM} = \lambda_{\zeta_{ii'j2l'}^{CM}} \zeta_{tii'j2l'}^{CM}, \quad (4.36)$$

$$\zeta_{rtij2l}^{UM} = \lambda_{\zeta_{ii'j2l'}^{UM}} \zeta_{rtii'j2l'}^{UM}, \quad (4.37)$$

$$\zeta_{tijk}^M = \lambda_{\zeta_{ii'jkl}^M} \zeta_{tii'jkl}^M, \quad \forall k > 2. \quad (4.38)$$

Remarks. According to the above definition the latent trait variables ξ_{tijk} in the LST-COM model are defined as conditional expectations of the latent state variables given the target $E(S_{tijk}|p_T)$. The latent trait variables are free of situational or rater-specific influences and are only due to target-specific influences. Note that Equation 4.23 also implies the formal definition of the (non)-reference latent trait variables, indicated by $k = 2$. The latent occasion-specific variables ζ_{tijk} (called: state-residuals) are defined as differences between the latent state and the latent trait variables (see Equation 4.24). The latent state-residual variables are defined as residuals with respect to the latent trait variables pertaining to the same indicator i , construct j , method k and occasion of measurement l . Consequently, both latent variables are uncorrelated by definition. According to Definition 4.1 the latent trait unique method variables ξ_{rtij2l}^{UM} are defined as conditional expectations of UM_{rtij2l} given (p_T) and (p_R) . The latent state-residual unique method variables are defined as difference between UM_{rtij2l} and ξ_{rtij2l}^{UM} . The latent trait (common) method variables (ξ_{tij2l}^{CM} and ξ_{tijk}^M) are defined as residuals with respect to the latent regression of the non-reference trait variables on the reference trait variables [see Equations 4.25-4.26]. Therefore, these latent variables reflect the consistent bias of the other ratings which is not shared with the consistent view of the target's self-perception [see Equations 4.25 and 4.26]. Hence, with respect to the latent trait method components it is possible to investigate the stability or consistency of method bias. The latent state method variables are again defined as difference between the latent method variables and the latent trait method variables [see Equations 4.28-4.22]. These latent variables represent the occasion-specific (momentary) rater bias which is not shared with the occasion-specific (momentary) view of the target (reference method). According to the Conditions (b) and (d) of the above definition latent linear regressions are assumed. With respect to the Conditions (e) and (f) in the above definition homogeneity assumptions regarding the latent trait and latent state method variables are postulated.

4.2.1 Definition of the LST-COM model with conditional regressive independent latent state variables

In order to interpret the unique trait variables ξ_{rtij2l}^{UM} as difference between the conditional expectations of Y_{rtij2l} given the target (p_T) and the rater (p_R) from the the conditional expectations of Y_{rtij2l} given the target (p_T) , it is necessary to impose an additional assumption. This assumption is called conditional regressive independence assumption with respect to the latent state variables S_{tij2l} . This assumption postulates that the level-2 latent state variables S_{tij2l} are conditionally regressive independent from (p_R) given (p_T) . LST-COM models that fulfill this assumption will

be called LST-COM model with CRI latent state variables S_{tij2l} .

Definition 4.2 (Conditional regressive independence of the latent state variables)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ be a LST-COM model according to the above Definition 4.1. If and only if,

$$E(S_{tij2l}|p_T, p_R) = E(S_{tij2l}|p_T) \quad (4.39)$$

holds, then \mathcal{M} is called LST-COM model with conditionally regressive independent S_{tij2l} -variables.

Remarks. In the above Definition 4.2, a LST-COM model with conditionally regressive independent S_{tij2l} -variables is formally defined. With respect to Equation 4.39 in Definition 4.2 it is assumed that the latent state variables S_{tij2l} on the target-level are conditionally independent from the rater p_R , given the target p_T . This additional assumption corresponds to the commonly known i.i.d. (independent and identically distributed random variables) assumption of level-1 residuals in multilevel regression analysis. Therefore, this additional assumption is relatively “weak” and common in multilevel analysis. However, this assumption might be violated in cross-classified data structures, where interchangeable raters evaluate multiple targets.

Theorem 4.1 (Consequences of the Definition 4.2)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ be a LST-COM model with conditional regressive independent latent state S_{tij2l} -variables according to Definition 4.2, then the ξ_{rtij2l}^{UM} variables can be defined as follows:

$$\xi_{rtij2l}^{UM} \equiv E(Y_{rtij2l}|p_T, p_R) - E(Y_{rtij2l}|p_T). \quad (4.40)$$

Remarks. According to Theorem 4.2 the latent trait unique method variables ξ_{rtij2l}^{UM} can be defined as the difference between the conditional expectation of Y_{rtij2l} given target and the rater (p_T, p_R) and the conditional expectation of Y_{rtij2l} given the target (p_T) . With respect to this definition, the latent trait unique method variables ξ_{rtij2l}^{UM} can be interpreted as consistent over- or underestimation of the trait of a target with respect to a particular rater. Hence, the latent trait unique method ξ_{rtij2l}^{UM} variables reflect the consistent bias of a particular rater that is free of measurement error and occasion-specific influences.

Proofs. 12 According to Definition 4.1 the latent trait unique method ξ_{rtij2l}^{UM} variables can be defined as follows:

$$\begin{aligned} \xi_{rtij2l}^{UM} &\equiv E(UM_{rtij2l}|p_T, p_R) \\ &= E[(S_{rtij2l} - S_{tij2l})|p_T, p_R] \\ &= E(S_{rtij2l}|p_T, p_R) - E(S_{tij2l}|p_T, p_R). \end{aligned}$$

If and only if, the statement in Definition 4.2 holds (see Equation 4.39), then $E(S_{rtij2l}|p_T, p_R) - E(S_{tij2l}|p_T, p_R)$ can be rewritten as follows:

$$\begin{aligned} &E(S_{rtij2l}|p_T, p_R) - E(S_{tij2l}|p_T) \\ &E[E(Y_{rtij2l}|p_T, p_{TS_i}, p_R, p_{RS_i})|p_T, p_R] - E[E(Y_{rtij2l}|p_T, p_{TS_i})|p_T] \\ &E(Y_{rtij2l}|p_T, p_R) - E(Y_{rtij2l}|p_T). \end{aligned}$$

□

4.3 Existence

Theorem 4.2 (Existence)

The random variables $\{Y_{111111}, \dots, Y_{rtijkl}, \dots, Y_{abcdef}\}$ and $\{Y_{111111}, \dots, Y_{tijk}, \dots, Y_{bcdef}\}$ are $(\xi_{tij}, \xi_{tij2}^{CM}, \xi_{rtij2}^{UM}, \xi_{tijk}^M, \zeta_{tijl}, \zeta_{tij2}^{CM}, \zeta_{rtj2l}^{UM}, \zeta_{tijk}^M)$ -congeneric variables of a LST-COM model with conditional regressive independent latent state variables if and only if the statements in Definition 4.2 hold and for each $r \in R, t \in T, i \in I, j \in J, k \in K, l \in L$, there are real-valued random variables $\xi_{tij}, \xi_{tij2}^{CM}, \xi_{rtij2}^{UM}, \xi_{tijk}^M, \zeta_{tijl}, \zeta_{tij2}^{CM}, \zeta_{rtj2l}^{UM}, \zeta_{tijk}^M$ on a probability space $(\Omega, \mathfrak{A}, P)$ and $(\alpha_{tij1l}, \lambda_{\xi_{ij1l}}, \lambda_{\xi_{ij2l}}^{CM}, \lambda_{\xi_{ij2l}}^{UM}, \lambda_{\xi_{ijkl}}^M, \lambda_{\zeta_{ij1l}}, \lambda_{\zeta_{ij2l}}^{CM}, \lambda_{\zeta_{ij2l}}^{UM}, \lambda_{\zeta_{ijkl}}^M) \in \mathbb{R}^+$, such that:

$$\xi_{tij1l} = \alpha_{tij1l} + \lambda_{\xi_{ij1l}} \xi_{tij}, \quad (4.41)$$

$$E(\xi_{tijk} | \xi_{tij1l}) = \alpha_{tijk} + \lambda_{\xi_{ijkl}} \xi_{tijk}, \quad (4.42)$$

$$\xi_{tij2l}^{CM} = \lambda_{\xi_{ij2l}}^{CM} \xi_{tij2}^{CM}, \quad (4.43)$$

$$\xi_{rtij2l}^{UM} = \lambda_{\xi_{ij2l}}^{UM} \xi_{rtij2}^{UM}, \quad (4.44)$$

$$\xi_{tijk}^M = \lambda_{\xi_{ijkl}}^M \xi_{tijk}^M, \quad \forall k > 2, \quad (4.45)$$

$$E(\zeta_{tijk} | \zeta_{tij1l}) = \lambda_{\zeta_{ij1l}} \zeta_{tijk}, \quad (4.46)$$

$$\zeta_{tij2l}^{CM} = \lambda_{\zeta_{ij2l}}^{CM} \zeta_{tij2}^{CM}, \quad (4.47)$$

$$\zeta_{rtij2l}^{UM} = \lambda_{\zeta_{ij2l}}^{UM} \zeta_{rtij2}^{UM}, \quad (4.48)$$

$$\zeta_{tijk}^M = \lambda_{\zeta_{ijkl}}^M \zeta_{tijk}^M, \quad \forall k > 2. \quad (4.49)$$

Proofs. 13 Existence of the latent variables.

4.42 Inserting Equation 4.31 into Equation 4.30 of the above definition, yields Equation 4.42:

$$E(\xi_{tijk} | \xi_{tij1l}) = \alpha_{tijk} + \lambda_{\xi_{ijkl}} (\alpha_{tij1l'} + \lambda_{\xi_{ij1l'}} \xi_{tij1l'}),$$

if α_{tijk} is defined as $\alpha_{tijk} + \lambda_{\xi_{ijkl}} \alpha_{tij1l'}$ and if $\lambda_{\xi_{ijkl}}$ is defined as $\lambda_{\xi_{ijkl}} \lambda_{\xi_{ij1l'}}$ hold. Similarly, according to Equation 4.41, two different latent trait variables ξ_{tij1l} and $\xi_{tij1l'}$ can be expressed as

$$\xi_{tij} = \frac{\xi_{tij1l} - \alpha_{tij1l}}{\lambda_{\xi_{ij1l}}}, \quad \text{and} \quad \xi_{tij} = \frac{\xi_{tij1l'} - \alpha_{tij1l'}}{\lambda_{\xi_{ij1l'}}}.$$

Setting both equations equal, yields:

$$\xi_{tij1l} = \alpha_{tij1l} + \frac{\lambda_{\xi_{ij1l}}}{\lambda_{\xi_{ij1l'}}} (\xi_{tij1l'} - \alpha_{tij1l'}).$$

Let $\alpha_{tij1l'} \equiv \alpha_{tij1l} - \alpha_{tij1l'} \left(\frac{\lambda_{\xi_{ij1l}}}{\lambda_{\xi_{ij1l'}}} \right)$ and $\lambda_{\xi_{ij1l'}} \equiv \frac{\lambda_{\xi_{ij1l}}}{\lambda_{\xi_{ij1l'}}}$, then Equation 4.31 is obtained:

$$\xi_{tij1l} = \alpha_{tij1l'} + \lambda_{\xi_{ij1l'}} \xi_{tij1l'} \text{ (repeated).}$$

4.43 For all i, j, l , assume that $\xi_{tij2}^{CM} \equiv \xi_{tijk}^{CM}$, as well as $\lambda_{\xi_{ij2l}}^{CM} \equiv \lambda_{\xi_{tijk}^{CM}}$. Inserting ξ_{tijk}^{CM} , as well as $\lambda_{\xi_{ij2l'}^{CM}}$ in Equation 4.33 of the above Definition 4.1, yields Equation 4.43:

$$\xi_{tijk}^{CM} = \lambda_{\xi_{ij2l}}^{CM} \xi_{tijk}^{CM} \text{ (repeated).}$$

According to Equation 4.43, ξ_{tijk}^{CM} can be expressed as

$$\xi_{tijk}^{CM} = \frac{\xi_{tijk}^{CM}}{\lambda_{\xi_{ij2l}}^{CM}}, \text{ as well as } \xi_{tijk}^{CM} = \frac{\xi_{tijk}^{CM}}{\lambda_{\xi_{ij2l'}}^{CM}}.$$

By setting both equations equal, it follows $\xi_{tijk}^{CM} = \frac{\lambda_{\xi_{ij2l'}}^{CM}}{\lambda_{\xi_{ij2l}}^{CM}} \xi_{tijk}^{CM}$. Let $\lambda_{\xi_{ij2l'}}^{CM} \equiv \frac{\lambda_{\xi_{ij2l'}}^{CM}}{\lambda_{\xi_{ij2l}}^{CM}}$, then Equation 4.33 is obtained:

$$\xi_{tijk}^{CM} = \lambda_{\xi_{ij2l'}}^{CM} \xi_{tijk}^{CM} \text{ (repeated).}$$

4.44 For all i, j, l , assume that $\xi_{rtij2}^{UM} \equiv \xi_{rtij2l'}^{UM}$ as well as $\lambda_{\xi_{ij2l}}^{UM} \equiv \lambda_{\xi_{ij2ll'}}^{UM}$. Inserting $\xi_{rtij2l'}^{UM}$ as well as $\lambda_{\xi_{ij2ll'}}^{UM}$ into Equation 4.34 of the above Definition 4.1, yields Equation 4.44:

$$\xi_{rtij2l}^{UM} = \lambda_{\xi_{ij2l}}^{UM} \xi_{rtij2}^{UM} \text{ (repeated).}$$

According to Equation 4.44, ξ_{rtij2}^{UM} can be expressed as

$$\xi_{rtij2}^{UM} = \frac{\xi_{rtij2l}^{UM}}{\lambda_{\xi_{ij2l}}^{UM}}, \text{ as well as } \xi_{rtij2}^{UM} = \frac{\xi_{rtij2l'}^{UM}}{\lambda_{\xi_{ij2l'}}^{UM}}.$$

By setting both equations equal, it follows

$$\xi_{rtij2l}^{UM} = \frac{\lambda_{\xi_{ij2l}}^{UM}}{\lambda_{\xi_{ij2l'}}^{UM}} \xi_{rtij2l'}^{UM}.$$

Let $\lambda_{\xi_{ij2ll'}}^{UM} \equiv \frac{\lambda_{\xi_{ij2l}}^{UM}}{\lambda_{\xi_{ij2l'}}^{UM}}$, then Equation 4.34 is obtained:

$$\xi_{rtij2l}^{UM} = \lambda_{\xi_{ij2ll'}}^{UM} \xi_{rtij2l'}^{UM} \text{ (repeated).}$$

4.45 For all $i, j, k > 2, l$, assume that $\xi_{tijk}^M \equiv \xi_{tijk l'}^M$ as well as $\lambda_{\xi_{ijkl}}^M \equiv \lambda_{\xi_{ijkl l'}}^M$. Inserting $\xi_{tijk l'}^M$ as well as $\lambda_{\xi_{ijkl l'}}^M$ in Equation 4.35 of the above Definition 4.1, yields Equation 4.45:

$$\xi_{tijk}^M = \lambda_{\xi_{ijkl}}^M \xi_{tijk}^M \text{ (repeated).}$$

According to Equation 4.45, ξ_{tijk}^M can be expressed as

$$\xi_{tijk}^M = \frac{\xi_{tijk l'}^M}{\lambda_{\xi_{ijkl}}^M}, \text{ as well as } \xi_{tijk}^M = \frac{\xi_{tijk l'}^M}{\lambda_{\xi_{ijkl l'}}^M}.$$

By setting both equations equal, it follows $\xi_{tijk l'}^M = \frac{\lambda_{\xi_{ijkl}}^M}{\lambda_{\xi_{ijkl l'}}^M} \xi_{tijk l'}^M$. Let $\lambda_{\xi_{ijkl l'}}^M \equiv \frac{\lambda_{\xi_{ijkl}}^M}{\lambda_{\xi_{ijkl l'}}^M}$, then Equation 4.35 is obtained:

$$\xi_{tijk l'}^M = \lambda_{\xi_{ijkl l'}}^M \xi_{tijk l'}^M \text{ (repeated).}$$

4.47 For all i, j, l , assume that $\zeta_{tj2l}^{CM} \equiv \zeta_{tj2l}^{CM}$ as well as $\lambda_{\zeta_{ii'j2l}}^{CM} \equiv \lambda_{\zeta_{ii'j2l}}^{CM}$. Inserting ζ_{tj2l}^{CM} as well as $\lambda_{\zeta_{ii'j2l}}^{CM}$ in Equation 4.36 of the above Definition 4.1, yields Equation 4.47:

$$\zeta_{tj2l}^{CM} = \lambda_{\zeta_{ii'j2l}}^{CM} \zeta_{tj2l}^{CM} \text{ (repeated).}$$

According to Equation 4.47, ζ_{tj2l}^{CM} can be expressed as

$$\zeta_{tj2l}^{CM} = \frac{\zeta_{tj2l}^{CM}}{\lambda_{\zeta_{ii'j2l}}^{CM}}, \text{ as well as } \zeta_{tj2l}^{CM} = \frac{\zeta_{tj2l}^{CM}}{\lambda_{\zeta_{ii'j2l}}^{CM}}.$$

By setting both equations equal, it follows $\zeta_{tj2l}^{CM} = \frac{\lambda_{\zeta_{ii'j2l}}^{CM}}{\lambda_{\zeta_{ii'j2l}}^{CM}} \zeta_{tj2l}^{CM}$. Let $\lambda_{\zeta_{ii'j2l}}^{CM} \equiv \frac{\lambda_{\zeta_{ii'j2l}}^{CM}}{\lambda_{\zeta_{ii'j2l}}^{CM}}$, then the Equation 4.36 is obtained:

$$\zeta_{tj2l}^{CM} = \lambda_{\zeta_{ii'j2l}}^{CM} \zeta_{tj2l}^{CM} \text{ (repeated).}$$

The demonstrations of Equations 4.48 and 4.49 follow the same principle and are straightforward. Therefore will be left to the reader.

Remarks. The above Theorem 4.2 shows that common (1) indicator-specific and occasion-unspecific latent trait factors ξ_{tij} , (2) common indicator-specific and occasion-unspecific latent trait method factors (ξ_{tij2}^{CM} , ξ_{rtij2}^{UM} , ξ_{tijk}^M), (3) common indicator- and occasion-specific state-residual factors (ζ_{tijl}), as well as (4) common indicator-unspecific and occasion-specific latent state-residual method factors (ζ_{tj2l}^{CM} , ζ_{rtj2l}^{UM} , ζ_{tijk}^M) exist.

4.4 Uniqueness

According to the statements in Theorem 4.2 it is clear that the latent factors (trait, trait-specific method, occasion-specific and occasion-specific method) are not uniquely defined in LST-COM models. If such models are defined with $(\xi_{tij}, \xi_{tij2}^{CM}, \xi_{rtij2}^{UM}, \xi_{tijk}^M, \zeta_{tijl}, \zeta_{tj2l}^{CM}, \zeta_{rtj2l}^{UM}, \zeta_{tjkl}^M)$ -congeneric variables, all of the parameters are defined up to positive linear or similarity transformation. The next theorem concerns the uniqueness and admissible transformations of the latent factors and their corresponding factor loadings in LST-COM models.

Theorem 4.3 (Admissible transformations & uniqueness)

1. Admissible transformations

$\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ is a LST-COM model with:

$$\xi_t \equiv (\xi_{111} \cdots \xi_{tij} \cdots \xi_{bcd})^T, \quad (4.50)$$

$$\xi_{rt}^{UM} \equiv (\xi_{11111}^{UM} \cdots \xi_{rtij2}^{UM} \cdots \xi_{abcd2}^{UM})^T, \quad (4.51)$$

$$\xi_t^{CM} \equiv (\xi_{1111}^{CM} \cdots \xi_{tij2}^{CM} \cdots \xi_{bcd2}^{CM})^T, \quad (4.52)$$

$$\xi_t^M \equiv (\xi_{1111}^M \cdots \xi_{tijk}^M \cdots \xi_{bcde}^M)^T, \quad (4.53)$$

$$\zeta_t \equiv (\zeta_{1111} \cdots \zeta_{tijl} \cdots \zeta_{bcdf})^T, \quad (4.54)$$

$$\zeta_{rt}^{UM} \equiv (\zeta_{11111}^{UM} \cdots \zeta_{rtj2l}^{UM} \cdots \zeta_{abcd2f}^{UM})^T, \quad (4.55)$$

$$\zeta_t^{CM} \equiv (\zeta_{1111}^{CM} \cdots \zeta_{tj2l}^{CM} \cdots \zeta_{bd2f}^{CM})^T, \quad (4.56)$$

$$\zeta_t^M \equiv (\zeta_{1111}^M \cdots \zeta_{tjkl}^M \cdots \zeta_{bdef}^M)^T, \quad (4.57)$$

$$\mathbf{E}_{rt} \equiv (E_{1111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^T, \quad (4.58)$$

$$\mathbf{E}_t \equiv (E_{111111} \cdots E_{tijkl} \cdots E_{bcdef})^T, \quad (4.59)$$

$$\alpha_t \equiv (\alpha_{1111} \cdots \alpha_{tijk} \cdots \alpha_{bcdef})^T, \quad (4.60)$$

$$\lambda_\xi \equiv (\lambda_{\xi 1111} \cdots \lambda_{\xi ijkl} \cdots \lambda_{\xi cdef})^T, \quad (4.61)$$

$$\lambda_\xi^{UM} \equiv (\lambda_{\xi 1111}^{UM} \cdots \lambda_{\xi ij2l}^{UM} \cdots \lambda_{\xi cd2f}^{UM})^T, \quad (4.62)$$

$$\lambda_\xi^{CM} \equiv (\lambda_{\xi 1111}^{CM} \cdots \lambda_{\xi ij2l}^{CM} \cdots \lambda_{\xi cd2f}^{CM})^T, \quad (4.63)$$

$$\lambda_\xi^M \equiv (\lambda_{\xi 1111}^M \cdots \lambda_{\xi ijkl}^M \cdots \lambda_{\xi cdef}^M)^T, \quad (4.64)$$

$$\lambda_\zeta \equiv (\lambda_{\zeta 1111} \cdots \lambda_{\zeta ijkl} \cdots \lambda_{\zeta cdef})^T, \quad (4.65)$$

$$\lambda_\zeta^{UM} \equiv (\lambda_{\zeta 1111}^{UM} \cdots \lambda_{\zeta ij2l}^{UM} \cdots \lambda_{\zeta cd2f}^{UM})^T, \quad (4.66)$$

$$\lambda_\zeta^{CM} \equiv (\lambda_{\zeta 1111}^{CM} \cdots \lambda_{\zeta ij2l}^{CM} \cdots \lambda_{\zeta cd2f}^{CM})^T, \quad (4.67)$$

$$\lambda_\zeta^M \equiv (\lambda_{\zeta 1111}^M \cdots \lambda_{\zeta ijkl}^M \cdots \lambda_{\zeta cdef}^M)^T, \quad (4.68)$$

and if for all $r \in R, t \in T, i \in I, j \in J, k \in K, l \in L$:

$$\xi'_{tij} = \gamma_{\xi ij} + \beta_{\xi ij} \xi_{tij}, \quad (4.69)$$

$$\xi_{rtij2}^{UM'} = \beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM}, \quad (4.70)$$

$$\xi_{tij2}^{CM'} = \beta_{\xi ij2}^{CM} \xi_{tij2}^{CM}, \quad (4.71)$$

$$\xi_{tijk}^{M'} = \beta_{\xi ijk}^M \xi_{tijk}^M, \quad \forall k > 2, \quad (4.72)$$

$$\zeta'_{tijl} = \beta_{\zeta ij} \zeta_{tijl}, \quad (4.73)$$

$$\zeta_{rtj2l}^{UM'} = \beta_{\zeta ij2l}^{UM} \zeta_{rtj2l}^{UM}, \quad (4.74)$$

$$\zeta_{tj2l}^{CM'} = \beta_{\zeta ij2l}^{CM} \zeta_{tj2l}^{CM}, \quad (4.75)$$

$$\zeta_{tjkl}^{M'} = \beta_{\zeta ijkl}^M \zeta_{tjkl}^M, \quad \forall k > 2, \quad (4.76)$$

$$\alpha'_{tijk} = \alpha_{tijk} - \gamma_{\xi ij} \frac{\lambda_{\xi ijkl}}{\beta_{\xi ij}}, \quad (4.77)$$

$$\lambda'_{\xi ijkl} = \lambda_{\xi ijkl} / \beta_{\xi ij}, \quad (4.78)$$

$$\lambda_{\xi ij2l}^{UM'} = \lambda_{\xi ij2l}^{UM} / \beta_{\xi ij2}, \quad (4.79)$$

$$\lambda_{\xi ij2l}^{CM'} = \lambda_{\xi ij2l}^{CM} / \beta_{\xi ij2}, \quad (4.80)$$

$$\lambda_{\xi ijkl}^{M'} = \lambda_{\xi ijkl}^M / \beta_{\xi ijkl}^M, \quad \forall k > 2, \quad (4.81)$$

$$\lambda'_{\zeta ijkl} = \lambda_{\zeta ijkl} / \beta_{\zeta ijkl}, \quad (4.82)$$

$$\lambda_{\zeta ij2l}^{UM'} = \lambda_{\zeta ij2l}^{UM} / \beta_{\zeta ij2l}^{UM}, \quad (4.83)$$

$$\lambda_{\zeta ij2l}^{CM'} = \lambda_{\zeta ij2l}^{CM} / \beta_{\zeta ij2l}^{CM}, \quad (4.84)$$

$$\lambda_{\zeta ijkl}^{M'} = \lambda_{\zeta ijkl}^M / \beta_{\zeta ijkl}^M, \quad \forall k > 2, \quad (4.85)$$

where $\gamma_{\xi ij}$, $\beta_{\xi ij}$, $\beta_{\xi ij2}^{UM}$, $\beta_{\xi ij2}^{CM}$, $\beta_{\xi ijkl}^M$, $\beta_{\zeta ijkl}$, $\beta_{\zeta ij2l}^{UM}$, $\beta_{\zeta ij2l}^{CM}$, and $\beta_{\zeta ijkl}^M \in \mathbb{R}_+$, ' then $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \xi'_t, \xi'_{rt}, \xi'_{rt}^{UM'}, \xi'_{rt}^{CM'}, \xi'_t, \xi'_t, \xi'_t, \xi'_{rt}^{UM'}, \xi'_{rt}^{CM'}, \xi'_t, \xi'_t, \mathbf{E}'_{rt}, \mathbf{E}'_t, \alpha'_t, \lambda'_\xi, \lambda_{\xi}^{UM'}, \lambda_{\xi}^{CM'}, \lambda_{\xi}^{M'}, \lambda'_{\zeta}, \lambda_{\zeta}^{UM'}, \lambda_{\zeta}^{CM'}, \lambda_{\zeta}^{M'} \rangle$ is a LST-COM model, too, with:

$$\xi'_t \equiv (\xi'_{111} \cdots \xi'_{tij} \cdots \xi'_{bcd})^T, \quad (4.86)$$

$$\xi'_{rt}^{UM'} \equiv (\xi'_{11111} \cdots \xi'_{rtij2} \cdots \xi'_{abcd2})^T, \quad (4.87)$$

$$\xi'_{rt}^{CM'} \equiv (\xi'_{11111} \cdots \xi'_{tij2} \cdots \xi'_{bcd2})^T, \quad (4.88)$$

$$\xi'_{rt}^{M'} \equiv (\xi'_{11111} \cdots \xi'_{tijk} \cdots \xi'_{bcde})^T, \quad (4.89)$$

$$\zeta'_t \equiv (\zeta'_{1111} \cdots \zeta'_{tijl} \cdots \zeta'_{bcdf})^T, \quad (4.90)$$

$$\zeta'_{rt}^{UM'} \equiv (\zeta'_{11111} \cdots \zeta'_{rtj2l} \cdots \zeta'_{abd2f})^T, \quad (4.91)$$

$$\zeta'_{rt}^{CM'} \equiv (\zeta'_{11111} \cdots \zeta'_{tj2l} \cdots \zeta'_{bd2f})^T, \quad (4.92)$$

$$\zeta'_{rt}^{M'} \equiv (\zeta'_{11111} \cdots \zeta'_{tjkl} \cdots \zeta'_{bdef})^T, \quad (4.93)$$

$$\mathbf{E}'_{rt} \equiv (E'_{111111} \cdots E'_{rtij2l} \cdots E'_{abcd2f})^T, \quad (4.94)$$

$$\mathbf{E}'_t \equiv (E'_{11111} \cdots E'_{tijkl} \cdots E'_{bcdef})^T, \quad (4.95)$$

$$\alpha'_t \equiv (\alpha'_{1111} \cdots \alpha'_{tijk} \cdots \alpha'_{bcdef})^T, \quad (4.96)$$

$$\lambda'_\xi \equiv (\lambda'_{\xi 1111} \cdots \lambda'_{\xi ijkl} \cdots \lambda'_{\xi cdef})^T, \quad (4.97)$$

$$\lambda_{\xi}^{UM'} \equiv (\lambda_{\xi 1111}^{UM'} \cdots \lambda_{\xi ij2l}^{UM'} \cdots \lambda_{\xi cd2f}^{UM'})^T, \quad (4.98)$$

$$\lambda_{\xi}^{CM'} \equiv (\lambda_{\xi 1111}^{CM'} \cdots \lambda_{\xi ij2l}^{CM'} \cdots \lambda_{\xi cd2f}^{CM'})^T, \quad (4.99)$$

$$\lambda_{\xi}^{M'} \equiv (\lambda_{\xi 1111}^{M'} \cdots \lambda_{\xi ijkl}^{M'} \cdots \lambda_{\xi cdef}^{M'})^T, \quad (4.100)$$

$$\lambda'_{\zeta} \equiv (\lambda'_{\zeta 1111} \cdots \lambda'_{\zeta ijkl} \cdots \lambda'_{\zeta cdef})^T, \quad (4.101)$$

$$\lambda_{\zeta}^{UM'} \equiv (\lambda_{\zeta 1111}^{UM'} \cdots \lambda_{\zeta ij2l}^{UM'} \cdots \lambda_{\zeta cd2f}^{UM'})^T, \quad (4.102)$$

$$\lambda_{\zeta}^{CM'} \equiv (\lambda_{\zeta 1111}^{CM'} \cdots \lambda_{\zeta ij2l}^{CM'} \cdots \lambda_{\zeta cd2f}^{CM'})^T, \quad (4.103)$$

$$\lambda_{\zeta}^{M'} \equiv (\lambda_{\zeta 1111}^{M'} \cdots \lambda_{\zeta ijkl}^{M'} \cdots \lambda_{\zeta cdef}^{M'})^T. \quad (4.104)$$

2. Uniqueness

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_{rt}^{CM}, \xi_t^M, \xi_t^M, \xi_t, \xi_{rt}^{UM}, \xi_{rt}^{CM}, \xi_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_{\xi}^{UM}, \lambda_{\xi}^{CM}, \lambda_{\xi}^M, \lambda'_{\zeta}, \lambda_{\zeta}^{UM}, \lambda_{\zeta}^{CM}, \lambda_{\zeta}^M \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \xi'_t, \xi'_{rt}, \xi'_{rt}^{UM'}, \xi'_{rt}^{CM'}, \xi'_t, \xi'_t, \xi'_t, \xi'_{rt}^{UM'}, \xi'_{rt}^{CM'}, \xi'_t, \xi'_t, \mathbf{E}'_{rt}, \mathbf{E}'_t, \alpha'_t, \lambda'_\xi, \lambda_{\xi}^{UM'}, \lambda_{\xi}^{CM'}, \lambda_{\xi}^{M'}, \lambda'_{\zeta}, \lambda_{\zeta}^{UM'}, \lambda_{\zeta}^{CM'}, \lambda_{\zeta}^{M'} \rangle$ are LST-COM models, then there are for each $i \in I$, $j \in J$, $k \in K$, $l \in L$, $\gamma_{\xi ij}$, $\beta_{\xi ij}$, $\beta_{\xi ij2}^{UM}$, $\beta_{\xi ij2}^{CM}$, $\beta_{\xi ijkl}^M$, $\beta_{\zeta ijkl}$, $\beta_{\zeta ij2l}^{UM}$, $\beta_{\zeta ij2l}^{CM}$, and $\beta_{\zeta ijkl}^M \in \mathbb{R}_+$ such that Equations 4.69 to 4.104 hold.

Proofs. 14 Admissible Transformations & Uniqueness

A. Latent trait variables

A1. Admissible transformations

If $\xi'_{tij} \equiv \gamma_{\xi ij} + \beta_{\xi ij} \xi_{tij}$, $\alpha'_{tijk} \equiv \alpha_{tijk} - \gamma_{\xi ij} \frac{\lambda_{\xi ijkl}}{\beta_{\xi ij}}$, and $\lambda'_{\xi ijkl} = \lambda_{\xi ijkl} / \beta_{\xi ij}$, then $(\xi_{tij} = \xi'_{tij} - \gamma_{\xi ij}) / \beta_{\xi ij}$, $\alpha_{tijk} = \alpha'_{tijk} + \gamma_{\xi ij} \lambda'_{\xi ijkl}$, and $\lambda_{\xi ijkl} = \lambda'_{\xi ijkl} \beta_{\xi ij}$. Inserting these parameters in Equation 4.42, this yields $E(\xi_{tijk} | \xi_{tij1l}) = \alpha'_{tijk} + \lambda'_{\xi ijkl} \xi'_{tij}$.

A2. Uniqueness

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \xi'_t, \xi_{rt}^{UM'}, \xi_t^{CM'}, \xi_t^{M'}, \zeta'_t, \zeta_{rt}^{UM'}, \zeta_t^{CM'}, \zeta_t^{M'}, \mathbf{E}'_{rt}, \mathbf{E}'_t, \alpha'_t, \lambda'_\xi, \lambda_{\xi'}^{UM'}, \lambda_{\xi'}^{CM'}, \lambda_{\xi'}^M, \lambda'_\zeta, \lambda_{\zeta'}^{UM'}, \lambda_{\zeta'}^{CM'}, \lambda_{\zeta'}^M \rangle$ are LST-COM models, then $\alpha_{tijk} + \lambda_{\xi ijkl} \xi_{tij} = \alpha'_{tijk} + \lambda'_{\xi ijkl} \xi'_{tij}$. As a consequence, it follows for all $i \in I$, $j \in J$, $k \in K$, and $l \in L$:

$$\xi'_{ij} = \left[\frac{\alpha_{tijk} - \alpha'_{tijk}}{\lambda'_{\xi ijkl}} \right] + \frac{\lambda_{\xi ijkl}}{\lambda'_{\xi ijkl}} \xi_{tij}.$$

Given the fact that the ratio of the parameters $\lambda_{\xi ijkl}$ and $\lambda'_{\xi ijkl}$ as well as the term

$$\left[\frac{\alpha_{tijk} - \alpha'_{tijk}}{\lambda'_{\xi ijkl}} \right]$$

have to be the same real value for each $i \in I$, $j \in J$, $k \in K$, and $l \in L$, one can also define constants, namely:

$$\gamma_{\xi ij} \equiv \frac{\alpha_{tijk} - \alpha'_{tijk}}{\lambda'_{\xi ijkl}},$$

$$\beta_{\xi ij} \equiv \frac{\lambda_{\xi ijkl}}{\lambda'_{\xi ijkl}}.$$

B. Latent trait method variables

B1. Admissible transformations

Let

$$\begin{aligned} \xi_{rtij2}^{UM'} &\equiv \beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM}, & \lambda_{\xi ij2l}^{UM'} &\equiv \lambda_{\xi ij2l}^{UM} / \beta_{\xi ij2}^{UM}, \\ \xi_{tij2}^{CM'} &\equiv \beta_{\xi ij2}^{CM} \xi_{tij2}^{CM}, & \lambda_{\xi ij2l}^{CM'} &\equiv \lambda_{\xi ij2l}^{CM} / \beta_{\xi ij2}^{CM}, \\ \xi_{tijk}^{M'} &\equiv \beta_{\xi ijk}^M \xi_{tijk}^M, \quad \forall k > 2, & \lambda_{\xi ijkl}^{M'} &\equiv \lambda_{\xi ijkl}^M / \beta_{\xi ijk}^M, \quad \forall k > 2. \end{aligned}$$

By simple manipulation of these equations, it follows:

$$\begin{aligned} \xi_{rtij2}^{UM} &= \xi_{rtij2}^{UM'} / \beta_{\xi ij2}^{UM}, & \lambda_{\xi ij2l}^{UM} &= \lambda_{\xi ij2l}^{UM'} \beta_{\xi ij2}^{UM}, \\ \xi_{tij2}^{CM} &= \xi_{tij2}^{CM'} / \beta_{\xi ij2}^{CM}, & \lambda_{\xi ij2l}^{CM} &= \lambda_{\xi ij2l}^{CM'} \beta_{\xi ij2}^{CM}, \\ \xi_{tijk}^M &= \xi_{tijk}^{M'} / \beta_{\xi ijk}^M, \quad \forall k > 2, & \lambda_{\xi ijkl}^M &= \lambda_{\xi ijkl}^{M'} \beta_{\xi ijk}^M, \quad \forall k > 2. \end{aligned}$$

Inserting these parameters back into Equations 4.43, 4.44, and 4.45, the following equations are received:

$$\begin{aligned} \xi_{rtij2l}^{UM} &= \lambda_{\xi ij2l}^{UM'} \xi_{rtij2}^{UM'}, \\ \xi_{tij2l}^{CM} &= \lambda_{\xi ij2l}^{CM'} \xi_{tij2}^{CM'}, \\ \xi_{tijk}^M &= \lambda_{\xi ijkl}^{M'} \xi_{tijk}^{M'}, \quad \forall k > 2. \end{aligned}$$

B2. Uniqueness

Let both \mathcal{M} and \mathcal{M}' be LST-COM models. Logically, for all $i \in I, j \in J, k \in K, l \in L$

$$\begin{aligned} \lambda_{\xi_{ij2l}}^{UM} \xi_{rtij2}^{UM} = \lambda_{\xi_{ij2l}}^{UM'} \xi_{rtij2}^{UM'} &= \xi_{rtij2}^{UM'} = \frac{\lambda_{\xi_{ij2l}}^{UM}}{\lambda_{\xi_{ij2l}}^{UM'}} \xi_{rtij2}^{UM}, \\ \lambda_{\xi_{ij2l}}^{CM} \xi_{tij2}^{CM} = \lambda_{\xi_{ij2l}}^{CM'} \xi_{tij2}^{CM'} &= \xi_{tij2}^{CM'} = \frac{\lambda_{\xi_{ij2l}}^{CM}}{\lambda_{\xi_{ij2l}}^{CM'}} \xi_{tij2}^{CM}, \\ \lambda_{\xi_{ijkl}}^M \xi_{tijk}^M = \lambda_{\xi_{ijkl}}^{M'} \xi_{tijk}^{M'} &= \xi_{tijk}^{M'} = \frac{\lambda_{\xi_{ijkl}}^M}{\lambda_{\xi_{ijkl}}^{M'}} \xi_{tijk}^M, \quad \forall k > 2. \end{aligned}$$

Given that, for each $i \in I, j \in J, k \in K, l \in L$ the parameters of the three ratios $\frac{\lambda_{\xi_{ij2l}}^{UM}}{\lambda_{\xi_{ij2l}}^{UM'}}$, $\frac{\lambda_{\xi_{ij2l}}^{CM}}{\lambda_{\xi_{ij2l}}^{CM'}}$, and $\frac{\lambda_{\xi_{ijkl}}^M}{\lambda_{\xi_{ijkl}}^{M'}}$ must be the same real values, the following real constants can be defined for each $i \in I, j \in J, k \in K, l \in L$:

$$\begin{aligned} \beta_{\xi_{ij2l}}^{CM} &\equiv \frac{\lambda_{\xi_{ij2l}}^{CM}}{\lambda_{\xi_{ij2l}}^{CM'}}, \\ \beta_{\xi_{ij2l}}^{UM} &\equiv \frac{\lambda_{\xi_{ij2l}}^{UM}}{\lambda_{\xi_{ij2l}}^{UM'}}, \\ \beta_{\xi_{ijk}}^M &\equiv \frac{\lambda_{\xi_{ijkl}}^M}{\lambda_{\xi_{ijkl}}^{M'}}, \quad \forall k > 2. \end{aligned}$$

C. Latent state variables

C1. Admissible transformations

Let ζ'_{tijl} be equal to $\beta_{\zeta_{ijl}} \zeta_{tijl}$ as well as $\lambda'_{\zeta_{ijkl}}$ be equal to $\lambda_{\zeta_{ijkl}} / \beta_{\zeta_{ijl}}$. Then, $\zeta_{tijl} = \frac{\zeta'_{tijl}}{\beta_{\zeta_{ijl}}}$ and $\lambda_{\zeta_{ijkl}} = \lambda'_{\zeta_{ijkl}} \beta_{\zeta_{ijl}}$. Substituting both parameters in Equation 4.46 yields: $E(\zeta_{tijk} | \zeta_{tijk}) = \lambda'_{\zeta_{ijkl}} \zeta'_{tijk}$.

C2. Uniqueness

If both \mathcal{M} and \mathcal{M}' are LST-COM models, then $\lambda_{\zeta_{ijkl}} \zeta_{tijk} = \lambda'_{\zeta_{ijkl}} \zeta'_{tijk}$. Thus,

$$\zeta'_{tijk} = \frac{\lambda_{\zeta_{ijkl}}}{\lambda'_{\zeta_{ijkl}}} \zeta_{tijk}.$$

Given that the ratio of the parameters $\lambda_{\zeta_{ijkl}}$ and $\lambda'_{\zeta_{ijkl}}$ have to be the same for all $i \in I, j \in J, k \in K, l \in L$, a real constant can be defined for all $i \in I, j \in J, k \in K, l \in L$:

$$\beta_{\zeta_{ijl}} \equiv \frac{\lambda_{\zeta_{ijkl}}}{\lambda'_{\zeta_{ijkl}}}.$$

D. Latent state method variables

D1. Admissible transformations

Again, let

$$\begin{aligned} \zeta_{tj2l}^{CM'} &\equiv \beta_{\zeta_{ij2l}}^{CM} \zeta_{tj2l}^{CM}, & \lambda_{\zeta_{ij2l}}^{CM'} &\equiv \lambda_{\zeta_{ij2l}}^{CM} / \beta_{\zeta_{ij2l}}^{CM}, \\ \zeta_{rtj2l}^{UM'} &\equiv \beta_{\zeta_{ij2l}}^{UM} \zeta_{rtj2l}^{UM}, & \lambda_{\zeta_{ij2l}}^{UM'} &\equiv \lambda_{\zeta_{ij2l}}^{UM} / \beta_{\zeta_{ij2l}}^{UM}, \\ \zeta_{tijk}^{M'} &\equiv \beta_{\zeta_{ijkl}}^M \zeta_{tijk}^M, \quad \forall k > 2, & \lambda_{\zeta_{ijkl}}^{M'} &\equiv \lambda_{\zeta_{ijkl}}^M / \beta_{\zeta_{ijkl}}^M, \quad \forall k > 2. \end{aligned} \quad \text{and}$$

Then, by simple manipulation of the above equation:

$$\begin{aligned} \zeta_{tj2l}^{CM} &= \frac{\zeta_{tj2l}^{CM'}}{\beta_{\zeta_{ij2l}}^{CM}}, & \lambda_{\zeta_{ij2l}}^{CM} &= \lambda_{\zeta_{ij2l}}^{CM'} \beta_{\zeta_{ij2l}}^{CM}, \\ \zeta_{rtj2l}^{UM} &= \frac{\zeta_{rtj2l}^{UM'}}{\beta_{\zeta_{ij2l}}^{UM}}, & \lambda_{\zeta_{ij2l}}^{UM} &= \lambda_{\zeta_{ij2l}}^{UM'} \beta_{\zeta_{ij2l}}^{UM}, \\ \zeta_{tijk}^M &= \frac{\zeta_{tijk}^{M'}}{\beta_{\zeta_{ijkl}}^M}, \quad \forall k > 2, & \lambda_{\zeta_{ijkl}}^M &= \lambda_{\zeta_{ijkl}}^{M'} \beta_{\zeta_{ijkl}}^M, \quad \forall k > 2. \end{aligned} \quad \text{and}$$

Finally, by substituting these parameters back into Equations 4.47, 4.48, and 4.49, yields:

$$\begin{aligned}\zeta_{tij2l}^{CM} &= \lambda_{\zeta_{ij2l}}^{CM'} \zeta_{tij2l}^{CM'}, \\ \zeta_{rtij2l}^{UM} &= \lambda_{\zeta_{ij2l}}^{CM'} \zeta_{rtij2l}^{UM'}, \\ \zeta_{tijkl}^M &= \lambda_{\zeta_{ijkl}}^{M'} \zeta_{tijkl}^{M'}, \quad \forall k > 2.\end{aligned}$$

D2. Uniqueness

Again, let both \mathcal{M} and \mathcal{M}' be LST-COM models. Then, for all $i \in I$, $j \in J$, $k \in K$, $l \in L$:

$$\begin{aligned}\lambda_{\zeta_{ij2l}}^{CM} \zeta_{tij2l}^{CM} &= \lambda_{\zeta_{ij2l}}^{CM'} \zeta_{tij2l}^{CM'} &= \zeta_{tij2l}^{CM'} &= \frac{\lambda_{\zeta_{ij2l}}^{CM} \zeta_{tij2l}^{CM}}{\lambda_{\zeta_{ij2l}}^{CM'}}, \\ \lambda_{\zeta_{ij2l}}^{UM} \zeta_{rtij2l}^{UM} &= \lambda_{\zeta_{ij2l}}^{UM'} \zeta_{rtij2l}^{UM'} &= \zeta_{rtij2l}^{UM'} &= \frac{\lambda_{\zeta_{ij2l}}^{UM} \zeta_{rtij2l}^{UM}}{\lambda_{\zeta_{ij2l}}^{UM'}}, \\ \lambda_{\zeta_{ijkl}}^M \zeta_{tijkl}^M &= \lambda_{\zeta_{ijkl}}^{M'} \zeta_{tijkl}^{M'} &= \zeta_{tijkl}^{M'} &= \frac{\lambda_{\zeta_{ijkl}}^M \zeta_{tijkl}^M}{\lambda_{\zeta_{ijkl}}^{M'}}, \quad \forall k > 2.\end{aligned}$$

As stated before, the ratios of the parameters $\lambda_{\zeta_{ij2l}}^{CM}$ and $\lambda_{\zeta_{ij2l}}^{CM'}$, $\lambda_{\zeta_{ij2l}}^{UM}$ and $\lambda_{\zeta_{ij2l}}^{UM'}$, as well as $\lambda_{\zeta_{ijkl}}^M$ and $\lambda_{\zeta_{ijkl}}^{M'}$ must have the same real values for each $i \in I$, $j \in J$, $k \in K$, $l \in L$. Therefore, the following real constants can be defined for each $i \in I$, $j \in J$, $k \in K$, $l \in L$:

$$\begin{aligned}\beta_{\zeta_{ij2l}}^{CM} &\equiv \frac{\lambda_{\zeta_{ij2l}}^{CM}}{\lambda_{\zeta_{ij2l}}^{CM'}}, \\ \beta_{\zeta_{ij2l}}^{UM} &\equiv \frac{\lambda_{\zeta_{ij2l}}^{UM}}{\lambda_{\zeta_{ij2l}}^{UM'}}, \\ \beta_{\zeta_{ijkl}}^M &\equiv \frac{\lambda_{\zeta_{ijkl}}^M}{\lambda_{\zeta_{ijkl}}^{M'}}, \quad \forall k > 2.\end{aligned}$$

□

Remarks. The above Theorem 4.3 concerns the level of measurement of the latent variables in the LST-COM model. The latent trait variables are measured on an interval scale, whereas the latent trait method variables as well as the latent state variables and latent state method variables are measured on ratio scale. In other words, the latent trait variables are only uniquely defined up to linear transformation, whereas the remaining latent variables are only defined up to similarity transformations (i.e., multiplication with a real constant).

4.5 Meaningfulness

According to Theorem 4.3 it was shown that the parameters in the LST-COM model are only uniquely defined up to positive linear or similarity transformations. The following theorem addresses the question whether statements regarding LST-COM model parameters remain meaningful (true), if the particular parameter has been subject to one of the admissible transformations. The most important and meaningful statements of LST-COM model parameters are summarized in Theorem 4.4.

Theorem 4.4 (Meaningfulness)

If both $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ and $\mathcal{M}' \equiv \langle (\Omega, \mathfrak{A}, P), \xi'_t, \xi'_{rt}^{UM'}, \xi'_t{}^{CM'}, \xi'_t{}^M, \zeta'_t, \zeta'_{rt}{}^{UM'}, \zeta'_t{}^{CM'}, \zeta'_t{}^M, \mathbf{E}'_{rt}, \mathbf{E}'_t, \alpha'_t, \lambda'_\xi, \lambda'_\xi{}^{UM'}, \lambda'_\xi{}^{CM'}, \lambda'_\xi{}^M, \lambda'_\zeta, \lambda'_\zeta{}^{UM'}, \lambda'_\zeta{}^{CM'}, \lambda'_\zeta{}^M \rangle$

$\alpha'_t, \lambda'_\xi, \lambda_\xi^{UM'}, \lambda_\xi^{CM'}, \lambda_\xi^{M'}, \lambda'_\zeta, \lambda_\zeta^{UM'}, \lambda_\zeta^{CM'}, \lambda_\zeta^{M'}$ are LST-COM models, then for $\omega_1, \omega_2 \in \Omega$; $r, r' \in R, t, t' \in T, i, i' \in I, j, j' \in J, k, k' \in K$, and $l, l' \in L$:

$$\frac{\lambda_{\xi ijkl}}{\lambda_{\xi jkl'}} = \frac{\lambda'_{\xi ijkl}}{\lambda'_{\xi jkl'}}, \quad (4.105)$$

$$\frac{\lambda_{\zeta ijkl}}{\lambda_{\zeta' jkl}} = \frac{\lambda'_{\zeta ijkl}}{\lambda'_{\zeta' jkl}}, \quad (4.106)$$

$$\frac{\lambda_{\xi ij2l}^{UM}}{\lambda_{\xi ij2l'}^{UM}} = \frac{\lambda_{\xi ij2l}^{UM'}}{\lambda_{\xi ij2l'}^{UM'}}, \quad (4.107)$$

$$\frac{\lambda_{\xi ij2l}^{CM}}{\lambda_{\xi ij2l'}^{CM}} = \frac{\lambda_{\xi ij2l}^{CM'}}{\lambda_{\xi ij2l'}^{CM'}}, \quad (4.108)$$

$$\frac{\lambda_{\xi ijkl}^M}{\lambda_{\xi jkl'}^M} = \frac{\lambda_{\xi ijkl}^{M'}}{\lambda_{\xi jkl'}^{M'}}, \quad \forall k > 2, \quad (4.109)$$

$$\frac{\lambda_{\zeta ij2l}^{UM}}{\lambda_{\zeta' j2l}^{UM}} = \frac{\lambda_{\zeta ij2l}^{UM'}}{\lambda_{\zeta' j2l}^{UM'}}, \quad (4.110)$$

$$\frac{\lambda_{\zeta ij2l}^{CM}}{\lambda_{\zeta' j2l}^{CM}} = \frac{\lambda_{\zeta ij2l}^{CM'}}{\lambda_{\zeta' j2l}^{CM'}}, \quad (4.111)$$

$$\frac{\lambda_{\zeta ijkl}^M}{\lambda_{\zeta' jkl}^M} = \frac{\lambda_{\zeta ijkl}^{M'}}{\lambda_{\zeta' jkl}^{M'}}, \quad \forall k > 2, \quad (4.112)$$

$$\frac{\lambda_{\xi ij2l}^{UM}}{\lambda_{\xi' j2l}^{UM}} - \frac{\lambda_{\xi ij2l'}^{UM}}{\lambda_{\xi' j2l'}^{UM}} = \frac{\lambda_{\xi ij2l}^{UM'}}{\lambda_{\xi' j2l}^{UM'}} - \frac{\lambda_{\xi ij2l'}^{UM'}}{\lambda_{\xi' j2l'}^{UM'}}, \quad (4.113)$$

$$\frac{\lambda_{\xi ij2l}^{CM}}{\lambda_{\xi' j2l}^{CM}} - \frac{\lambda_{\xi ij2l'}^{CM}}{\lambda_{\xi' j2l'}^{CM}} = \frac{\lambda_{\xi ij2l}^{CM'}}{\lambda_{\xi' j2l}^{CM'}} - \frac{\lambda_{\xi ij2l'}^{CM'}}{\lambda_{\xi' j2l'}^{CM'}}, \quad (4.114)$$

$$\frac{\lambda_{\xi ijkl}^M}{\lambda_{\xi' jkl}^M} - \frac{\lambda_{\xi ijkl'}^M}{\lambda_{\xi' jkl'}^M} = \frac{\lambda_{\xi ijkl}^{M'}}{\lambda_{\xi' jkl}^{M'}} - \frac{\lambda_{\xi ijkl'}^{M'}}{\lambda_{\xi' jkl'}^{M'}}, \quad \forall k > 2, \quad (4.115)$$

$$\frac{\lambda_{\zeta ij2l}^{UM}}{\lambda_{\zeta' j2l}^{UM}} - \frac{\lambda_{\zeta ij2l'}^{UM}}{\lambda_{\zeta' j2l'}^{UM}} = \frac{\lambda_{\zeta ij2l}^{UM'}}{\lambda_{\zeta' j2l}^{UM'}} - \frac{\lambda_{\zeta ij2l'}^{UM'}}{\lambda_{\zeta' j2l'}^{UM'}}, \quad (4.116)$$

$$\frac{\lambda_{\zeta ij2l}^{CM}}{\lambda_{\zeta' j2l}^{CM}} - \frac{\lambda_{\zeta ij2l'}^{CM}}{\lambda_{\zeta' j2l'}^{CM}} = \frac{\lambda_{\zeta ij2l}^{CM'}}{\lambda_{\zeta' j2l}^{CM'}} - \frac{\lambda_{\zeta ij2l'}^{CM'}}{\lambda_{\zeta' j2l'}^{CM'}}, \quad (4.117)$$

$$\frac{\lambda_{\zeta ijkl}^M}{\lambda_{\zeta' jkl}^M} - \frac{\lambda_{\zeta ijkl'}^M}{\lambda_{\zeta' jkl'}^M} = \frac{\lambda_{\zeta ijkl}^{M'}}{\lambda_{\zeta' jkl}^{M'}} - \frac{\lambda_{\zeta ijkl'}^{M'}}{\lambda_{\zeta' jkl'}^{M'}}, \quad \forall k > 2, \quad (4.118)$$

$$\frac{\alpha_{tijk} - \alpha_{tijk'}}{\alpha_{tijk''} - \alpha_{tijk'''}} = \frac{\alpha'_{tijk} - \alpha'_{tijk'}}{\alpha'_{tijk''} - \alpha'_{tijk'''}}, \quad (4.119)$$

for $\alpha_{tijk''} - \alpha_{tijk'''}$ and $\alpha'_{tijk''} - \alpha'_{tijk'''} \neq 0$,

$$\frac{\zeta_{tijl}(\omega_1)}{\zeta_{tijl}(\omega_2)} = \frac{\zeta'_{tijl}(\omega_1)}{\zeta'_{tijl}(\omega_2)}, \quad (4.120)$$

for $\zeta_{tijl}(\omega_2)$ and $\zeta'_{tijl}(\omega_2) \neq 0$,

$$\frac{\xi_{rtij2}^{UM}(\omega_1)}{\xi_{rtij2}^{UM}(\omega_2)} = \frac{\xi_{rtij2}^{UM'}(\omega_1)}{\xi_{rtij2}^{UM'}(\omega_2)}, \quad (4.121)$$

for $\xi_{rtij2}^{UM}(\omega_2)$ and $\xi_{rtij2}^{UM'}(\omega_2) \neq 0$,

$$\frac{\xi_{tij2}^{CM}(\omega_1)}{\xi_{tij2}^{CM}(\omega_2)} = \frac{\xi_{tij2}^{CM'}(\omega_1)}{\xi_{tij2}^{CM'}(\omega_2)}, \quad (4.122)$$

for $\xi_{tij2}^{CM}(\omega_2)$ and $\xi_{tij2}^{CM'}(\omega_2) \neq 0$,

$$\frac{\xi_{tijk}^M(\omega_1)}{\xi_{tijk}^M(\omega_2)} = \frac{\xi_{tijk}^{M'}(\omega_1)}{\xi_{tijk}^{M'}(\omega_2)}, \quad \forall k > 2, \quad (4.123)$$

for $\xi_{tijk}^M(\omega_2)$ and $\xi_{tijk}^{M'}(\omega_2) \neq 0$,

$$\frac{\xi_{tij}(\omega_1) - \xi_{tij}(\omega_2)}{\xi_{tij}(\omega_3) - \xi_{tij}(\omega_4)} = \frac{\xi'_{tij}(\omega_1) - \xi'_{tij}(\omega_2)}{\xi'_{tij}(\omega_3) - \xi'_{tij}(\omega_4)}, \quad (4.124)$$

for $\xi_{tij}(\omega_3) - \xi_{tij}(\omega_4)$ and $\xi'_{tij}(\omega_3) - \xi'_{tij}(\omega_4) \neq 0$,

$$\frac{\zeta_{tijl}(\omega_1)}{\zeta_{tijl}(\omega_2)} - \frac{\zeta_{tijl'}(\omega_1)}{\zeta_{tijl'}(\omega_2)} = \frac{\zeta'_{tijl}(\omega_1)}{\zeta'_{tijl}(\omega_2)} - \frac{\zeta'_{tijl'}(\omega_1)}{\zeta'_{tijl'}(\omega_2)}, \quad (4.125)$$

for $\zeta_{tijl}(\omega_2)$, $\zeta_{tijl'}(\omega_2)$, $\zeta'_{tijl}(\omega_2)$, $\zeta'_{tijl'}(\omega_2) \neq 0$,

$$\frac{\xi_{rtij2}^{UM}(\omega_1)}{\xi_{rtij2}^{UM}(\omega_2)} - \frac{\xi_{rti'j2}^{UM}(\omega_1)}{\xi_{rti'j2}^{UM}(\omega_2)} = \frac{\xi_{rtij2}^{UM'}(\omega_1)}{\xi_{rtij2}^{UM'}(\omega_2)} - \frac{\xi_{rti'j2}^{UM'}(\omega_1)}{\xi_{rti'j2}^{UM'}(\omega_2)}, \quad (4.126)$$

for $\xi_{rtij2}^{UM}(\omega_2)$, $\xi_{rti'j2}^{UM}(\omega_2)$, $\xi_{rtij2}^{UM'}(\omega_2)$, $\xi_{rti'j2}^{UM'}(\omega_2) \neq 0$,

$$\frac{\xi_{tij2}^{CM}(\omega_1)}{\xi_{tij2}^{CM}(\omega_2)} - \frac{\xi_{ti'j2}^{CM}(\omega_1)}{\xi_{ti'j2}^{CM}(\omega_2)} = \frac{\xi_{tij2}^{CM'}(\omega_1)}{\xi_{tij2}^{CM'}(\omega_2)} - \frac{\xi_{ti'j2}^{CM'}(\omega_1)}{\xi_{ti'j2}^{CM'}(\omega_2)}, \quad (4.127)$$

for $\xi_{tij2}^{CM}(\omega_2)$, $\xi_{ti'j2}^{CM}(\omega_2)$, $\xi_{tij2}^{CM'}(\omega_2)$, $\xi_{ti'j2}^{CM'}(\omega_2) \neq 0$,

$$\frac{\xi_{tijk}^M(\omega_1)}{\xi_{tijk}^M(\omega_2)} - \frac{\xi_{ti'jk'}^M(\omega_1)}{\xi_{ti'jk'}^M(\omega_2)} = \frac{\xi_{tijk}^{M'}(\omega_1)}{\xi_{tijk}^{M'}(\omega_2)} - \frac{\xi_{ti'jk'}^{M'}(\omega_1)}{\xi_{ti'jk'}^{M'}(\omega_2)}, \quad \forall k > 2, \quad (4.128)$$

for $\xi_{tijk}^M(\omega_2)$, $\xi_{ti'jk'}^M(\omega_2)$, $\xi_{tijk}^{M'}(\omega_2)$, $\xi_{ti'jk'}^{M'}(\omega_2) \neq 0$,

$$\frac{\zeta_{rtij2l}^{UM}(\omega_1)}{\zeta_{rtij2l}^{UM}(\omega_2)} - \frac{\zeta_{rti'j2l}^{UM}(\omega_1)}{\zeta_{rti'j2l}^{UM}(\omega_2)} = \frac{\zeta_{rtij2l}^{UM'}(\omega_1)}{\zeta_{rtij2l}^{UM'}(\omega_2)} - \frac{\zeta_{rti'j2l}^{UM'}(\omega_1)}{\zeta_{rti'j2l}^{UM'}(\omega_2)}, \quad (4.129)$$

for $\zeta_{rtij2l}^{UM}(\omega_2)$, $\zeta_{rti'j2l}^{UM}(\omega_2)$, $\zeta_{rtij2l}^{UM'}(\omega_2)$, $\zeta_{rti'j2l}^{UM'}(\omega_2) \neq 0$,

$$\frac{\zeta_{tij2l}^{CM}(\omega_1)}{\zeta_{tij2l}^{CM}(\omega_2)} - \frac{\zeta_{ti'j2l}^{CM}(\omega_1)}{\zeta_{ti'j2l}^{CM}(\omega_2)} = \frac{\zeta_{tij2l}^{CM'}(\omega_1)}{\zeta_{tij2l}^{CM'}(\omega_2)} - \frac{\zeta_{ti'j2l}^{CM'}(\omega_1)}{\zeta_{ti'j2l}^{CM'}(\omega_2)}, \quad (4.130)$$

for $\zeta_{tij2l}^{CM}(\omega_2)$, $\zeta_{ti'j2l}^{CM}(\omega_2)$, $\zeta_{tij2l}^{CM'}(\omega_2)$, $\zeta_{ti'j2l}^{CM'}(\omega_2) \neq 0$,

$$\frac{\zeta_{tijk}^M(\omega_1)}{\zeta_{tijk}^M(\omega_2)} - \frac{\zeta_{ti'jk'l}^M(\omega_1)}{\zeta_{ti'jk'l}^M(\omega_2)} = \frac{\zeta_{tijk}^{M'}(\omega_1)}{\zeta_{tijk}^{M'}(\omega_2)} - \frac{\zeta_{ti'jk'l}^{M'}(\omega_1)}{\zeta_{ti'jk'l}^{M'}(\omega_2)}, \quad \forall k > 2, \quad (4.131)$$

for $\zeta_{tijk}^M(\omega_2)$, $\zeta_{ti'jk'l}^M(\omega_2)$, $\zeta_{tijk}^{M'}(\omega_2)$, $\zeta_{ti'jk'l}^{M'}(\omega_2) \neq 0$,

$$\lambda_{\xi_{ijkl}}^2 \text{Var}(\xi_{tij}) = \lambda_{\xi'_{ijkl}}^2 \text{Var}(\xi'_{tij}), \quad (4.132)$$

$$\lambda_{\zeta_{ijkl}}^2 \text{Var}(\zeta_{tijl}) = \lambda_{\zeta'_{ijkl}}^2 \text{Var}(\zeta'_{tijl}), \quad (4.133)$$

$$(\lambda_{\xi_{ij2l}}^{UM})^2 \text{Var}(\xi_{rtij2}^{UM}) = (\lambda_{\xi_{ij2l}}^{UM'})^2 \text{Var}(\xi_{rtij2}^{UM'}), \quad (4.134)$$

$$(\lambda_{\xi_{ij2l}}^{CM})^2 \text{Var}(\xi_{tij2}^{CM}) = (\lambda_{\xi_{ij2l}}^{CM'})^2 \text{Var}(\xi_{tij2}^{CM'}), \quad (4.135)$$

$$(\lambda_{\xi_{ijk}}^M)2Var(\xi_{ijk}^M) = (\lambda_{\xi_{ijk}}^{M'})^2Var(\xi_{ijk}^{M'}), \quad \forall k > 2, \quad (4.136)$$

$$(\lambda_{\zeta_{ij2l}}^{UM})^2Var(\zeta_{ij2l}^{UM}) = (\lambda_{\zeta_{ij2l}}^{UM'})^2Var(\zeta_{ij2l}^{UM'}), \quad (4.137)$$

$$(\lambda_{\zeta_{ij2l}}^{CM})^2Var(\zeta_{ij2l}^{CM}) = (\lambda_{\zeta_{ij2l}}^{CM'})^2Var(\zeta_{ij2l}^{CM'}), \quad (4.138)$$

$$(\lambda_{\zeta_{ijkl}}^M)2Var(\zeta_{ijkl}^M) = (\lambda_{\zeta_{ijkl}}^{M'})^2Var(\zeta_{ijkl}^{M'}), \quad \forall k > 2, \quad (4.139)$$

$$Corr(\xi_{tij}, \xi_{ti'j'}) = Corr(\xi'_{tij}, \xi'_{ti'j'}), \quad (4.140)$$

$$Corr(\zeta_{tijl}, \zeta_{ti'j'l}) = Corr(\zeta'_{tijl}, \zeta'_{ti'j'l}), \quad (4.141)$$

$$Corr(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM}) = Cor(\xi_{rtij2}^{UM'}, \xi_{rti'j'2}^{UM'}), \quad (4.142)$$

$$Corr(\xi_{tij2}^{CM}, \xi_{ti'j'2}^{CM}) = Cor(\xi_{tij2}^{CM'}, \xi_{ti'j'2}^{CM'}), \quad (4.143)$$

$$Corr(\xi_{ijk}^M, \xi_{ti'j'k'}) = Cor(\xi_{ijk}^{M'}, \xi_{ti'j'k'}), \quad \forall k > 2, \quad (4.144)$$

$$Corr(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l}^{UM}) = Cor(\zeta_{rtj2l}^{UM'}, \zeta_{rtj'2l}^{UM'}), \quad (4.145)$$

$$Corr(\zeta_{tjk2l}^{CM}, \zeta_{tj'2l}^{CM}) = Cor(\zeta_{tjk2l}^{CM'}, \zeta_{tj'2l}^{CM'}), \quad (4.146)$$

$$Corr(\zeta_{tjkl}^M, \zeta_{tj'k'l}^M) = Cor(\zeta_{tjkl}^{M'}, \zeta_{tj'k'l}^{M'}), \quad \forall k > 2, \quad (4.147)$$

where $Var(\cdot)$ denotes variance and $Corr(\cdot)$ denotes correlation.

Proofs. 15 Meaningfulness

The proofs for Equations 4.105, 4.113, 4.120, 4.121, 4.126, 4.132, 4.134, 4.140, 4.142 are provided as examples. The remaining proofs for the other statements follow the same principle and are straightforward. Therefore these proofs will not be reported here.

4.105 Replacing $\lambda_{\xi_{ijkl}}$ by $\left(\frac{\lambda'_{\xi_{ijkl}}}{\beta_{\xi_{ijkl}}}\right)$ and $\lambda_{\xi_{ijk'l'}}$ by $\left(\frac{\lambda'_{\xi_{ijk'l'}}}{\beta_{\xi_{ijk'l'}}}\right)$ in Equation 4.105 verifies the equality

$$\frac{\lambda_{\xi_{ijkl}}}{\lambda_{\xi_{ijk'l'}}} = \frac{\lambda'_{\xi_{ijkl}}}{\beta_{\xi_{ijkl}}} = \frac{\lambda'_{\xi_{ijk'l'}}}{\beta_{\xi_{ijk'l'}}}.$$

4.113 Replacing $\lambda_{\xi_{ij2l}}^{UM}$ by $\left(\frac{\lambda_{\xi_{ij2l}}^{UM'}}{\beta_{\xi_{ij2l}}^{UM'}}\right)$, $\lambda_{\xi_{i'j2l}}^{UM}$ by $\left(\frac{\lambda_{\xi_{i'j2l}}^{UM'}}{\beta_{\xi_{i'j2l}}^{UM'}}\right)$, $\lambda_{\xi_{ij2l'}}^{UM}$ by $\left(\frac{\lambda_{\xi_{ij2l'}}^{UM'}}{\beta_{\xi_{ij2l'}}^{UM'}}\right)$ and $\lambda_{\xi_{i'j2l'}}^{UM}$ by $\left(\frac{\lambda_{\xi_{i'j2l'}}^{UM'}}{\beta_{\xi_{i'j2l'}}^{UM'}}\right)$ verifies the equality

$$\frac{\lambda_{\xi_{ij2l}}^{UM}}{\lambda_{\xi_{i'j2l}}^{UM}} = \frac{\lambda_{\xi_{ij2l'}}^{UM}}{\lambda_{\xi_{i'j2l'}}^{UM}} = \frac{\lambda_{\xi_{ij2l}}^{UM'}}{\beta_{\xi_{ij2l}}^{UM'}} - \frac{\lambda_{\xi_{i'j2l}}^{UM'}}{\beta_{\xi_{i'j2l}}^{UM'}} = \frac{\lambda_{\xi_{ij2l'}}^{UM'}}{\beta_{\xi_{ij2l'}}^{UM'}} - \frac{\lambda_{\xi_{i'j2l'}}^{UM'}}{\beta_{\xi_{i'j2l'}}^{UM'}}.$$

4.120 Replacing ζ_{tijl} by $\beta_{\zeta_{ijl}}\zeta'_{tijl}$ in Equation 4.120 verifies the equality

$$\frac{\zeta_{tijl}(\omega_1)}{\zeta_{tijl}(\omega_2)} = \frac{\beta_{\zeta_{ijl}}\zeta'_{tijl}(\omega_1)}{\beta_{\zeta_{ijl}}\zeta'_{tijl}(\omega_2)} = \frac{\zeta'_{tijl}(\omega_1)}{\zeta'_{tijl}(\omega_2)}.$$

4.121 Replacing ξ_{rtij2}^{UM} by $\beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM'}$ in Equation 4.121 verifies the equality

$$\frac{\xi_{rtij2}^{UM}(\omega_1)}{\xi_{rtij2}^{UM}(\omega_2)} = \frac{\beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM'}(\omega_1)}{\beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM'}(\omega_2)} = \frac{\xi_{rtij2}^{UM'}(\omega_1)}{\xi_{rtij2}^{UM'}(\omega_2)}.$$

4.126 Replacing ξ_{rtij2}^{UM} by $\beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM'}$ and $\xi_{rti'j2}^{UM}$ by $\beta_{\xi i'j2}^{UM} \xi_{rti'j2}^{UM'}$ in Equation 4.126 verifies the equality

$$\begin{aligned} \frac{\xi_{rtij2}^{UM}(\omega_1)}{\xi_{rtij2}^{UM}(\omega_2)} - \frac{\xi_{rti'j2}^{UM}(\omega_1)}{\xi_{rti'j2}^{UM}(\omega_2)} &= \frac{\beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM'}(\omega_1)}{\beta_{\xi ij2}^{UM} \xi_{rtij2}^{UM'}(\omega_2)} - \frac{\beta_{\xi i'j2}^{UM} \xi_{rti'j2}^{UM'}(\omega_1)}{\beta_{\xi i'j2}^{UM} \xi_{rti'j2}^{UM'}(\omega_2)} \\ &= \frac{\xi_{rtij2}^{UM'}(\omega_1)}{\xi_{rtij2}^{UM'}(\omega_2)} - \frac{\xi_{rti'j2}^{UM'}(\omega_1)}{\xi_{rti'j2}^{UM'}(\omega_2)}. \end{aligned}$$

4.132 Replacing $\lambda_{\xi ijkl}^2$ by $\lambda_{\xi ijkl}'^2 \beta_{\xi ij}^2$ and $\text{Var}(\xi_{tij})$ by $\text{Var}\left(\frac{\xi'_{tij} - \gamma_{\xi ij}}{\beta_{\xi ij}}\right)$ in Equation 4.132 verifies the equality

$$\begin{aligned} \lambda_{\xi ijkl}^2 \text{Var}(\xi_{tij}) &= \lambda_{\xi ijkl}'^2 \beta_{\xi ij}^2 \text{Var}\left(\frac{\xi'_{tij} - \gamma_{\xi ij}}{\beta_{\xi ij}}\right) \\ &= \lambda_{\xi ijkl}'^2 \beta_{\xi ij}^2 \text{Var}\left(\frac{\xi'_{tij}}{\beta_{\xi ij}} - \frac{\gamma_{\xi ij}}{\beta_{\xi ij}}\right) \\ &= \lambda_{\xi ijkl}'^2 \beta_{\xi ij}^2 \frac{1}{\beta_{\xi ij}^2} \text{Var}(\xi'_{tij} - \gamma_{\xi ij}) \\ &= \lambda_{\xi ijkl}'^2 \text{Var}(\xi'_{tij}). \end{aligned}$$

4.134 Replacing $(\lambda_{\xi ij2l}^{UM})^2$ by $(\lambda_{\xi UMij2l}^{UM'})^2 (\beta_{\xi ij2l}^{UM})^2$ and $\text{Var}(\xi_{rtij2}^{UM})$ by $\text{Var}\left(\frac{\xi_{rtij2}^{UM'}}{(\beta_{\xi ij2l}^{UM})^2}\right)$ in Equation 4.132 verifies the equality

$$\begin{aligned} (\lambda_{\xi ij2l}^{UM})^2 \text{Var}(\xi_{rtij2}^{UM}) &= (\lambda_{\xi UMij2l}^{UM'})^2 (\beta_{\xi ij2l}^{UM})^2 \text{Var}\left(\frac{\xi_{rtij2}^{UM'}}{(\beta_{\xi ij2l}^{UM})^2}\right) \\ &= (\lambda_{\xi ij2l}^{UM'})^2 (\beta_{\xi ij2l}^{UM})^2 \frac{1}{(\beta_{\xi ij2l}^{UM})^2} \text{Var}(\xi_{rtij2}^{UM'}) \\ &= (\lambda_{\xi ij2l}^{UM'})^2 \text{Var}(\xi_{rtij2}^{UM'}). \end{aligned}$$

4.140 Replacing ξ_{tij} by $\left(\frac{\xi'_{tij} - \gamma_{\xi ij}}{\beta_{\xi ij}}\right)$ and $\xi_{ti'j'}$ by $\left(\frac{\xi'_{ti'j'} - \gamma_{\xi ij}}{\beta_{\xi ij}}\right)$ in Equation 4.140 verifies the equality

$$\begin{aligned} \text{Corr}(\xi_{tij}, \xi_{ti'j'}) &= \text{Corr}\left(\frac{\xi'_{tij} - \gamma_{\xi ij}}{\beta_{\xi ij}}, \frac{\xi'_{ti'j'} - \gamma_{\xi ij}}{\beta_{\xi ij}}\right) \\ &= \text{Corr}\left(\frac{\xi'_{tij}}{\beta_{\xi ij}} - \frac{\gamma_{\xi ij}}{\beta_{\xi ij}}, \frac{\xi'_{ti'j'}}{\beta_{\xi ij}} - \frac{\gamma_{\xi ij}}{\beta_{\xi ij}}\right) \\ &= \text{Corr}\left(\frac{\xi'_{tij}}{\beta_{\xi ij}}, \frac{\xi'_{ti'j'}}{\beta_{\xi ij}}\right) \\ &= \text{Corr}(\xi'_{tij}, \xi'_{ti'j'}). \end{aligned}$$

4.142 Replacing ξ_{rtij2}^{UM} by $\left(\frac{\xi_{rtij2}^{UM}}{\beta_{\xi ij2l}^{UM}}\right)$ and $\xi_{rti'j'2}^{UM}$ by $\left(\frac{\xi_{rti'j'2}^{UM}}{\beta_{\xi ij2l}^{UM}}\right)$ in Equation 4.142 verifies the equality

$$\begin{aligned} \text{Corr}(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM}) &= \text{Corr}\left(\frac{\xi_{rtij2}^{UM}}{\beta_{\xi ij2l}^{UM}}, \frac{\xi_{rti'j'2}^{UM}}{\beta_{\xi ij2l}^{UM}}\right) \\ &= \text{Corr}(\xi_{rtij2}^{UM'}, \xi_{rti'j'2}^{UM'}). \end{aligned}$$

□

Remarks. According to the Theorem 4.4, statements regarding the ratios of differences regarding the values of the common latent trait variables as well as the ratios regarding the values of the common latent state variables are meaningful. In contrast, statements regarding the absolute values of the LST-COM model parameters (such as values of the latent trait or state variables) are not meaningful. For example, it is meaningful to say that the difference of the latent trait values of two targets t_1 and t_2 is n -times the difference between the values of two other targets on the same latent trait variable. Similarly, it is meaningful to say that the latent state value of a target t_1 is n -times greater or smaller than the latent state value of a target t_2 at the same occasion of measurement (Courvoisier, 2006). However, statements regarding the change of the targets' latent state values from occasion of measurement l to l' are only meaningful, if these statements refer to the ratio of the differences of the latent state values. Moreover, statements regarding (i) the ratio of factor loadings, (ii) the variance components defined above as well as (iii) the permissible correlations between latent variables are meaningful. In the next section permissible as well as non-permissible covariances and correlations among the latent variables of the LST-COM model are described in detail.

4.6 Testability

In order to derive testable consequences for the covariance structure of the LST-COM model, it is necessary to introduce an additional assumption. Again, with respect to this assumption a more restrictive variant of the LST-COM model is defined. LST-COM models that fulfill this assumption will be called LST-COM model with conditional regressive independence (CRI). Based on the additional assumption, it is possible to demonstrate that not all covariances between latent variables in the LST-COM are permissible. These non-permissible (zero) covariances between the latent variables of the LST-COM are discussed in Theorem 4.5. The total variance-covariance structure of the LST-COM model is provided in Section 4.6.2. In addition, permissible covariances that should be fixed to zero for parsimony are discussed. Finally, the interpretations of the admissible (freely estimated) covariances of the latent variables in the LST-COM model are given.

Definition 4.3 (The LST-COM model with conditional regressive independence)
 $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{\text{UM}}, \xi_t^{\text{CM}}, \xi_t^{\text{M}}, \zeta_t, \zeta_{rt}^{\text{UM}}, \zeta_t^{\text{CM}}, \zeta_t^{\text{M}}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{\text{UM}}, \lambda_\xi^{\text{CM}}, \lambda_\xi^{\text{M}}, \lambda_\zeta, \lambda_\zeta^{\text{UM}}, \lambda_\zeta^{\text{CM}}, \lambda_\zeta^{\text{M}} \rangle$ is called a LST-COM model of $(\xi_{tij}, \xi_{tij2}^{\text{UM}}, \xi_{tij2}^{\text{CM}}, \xi_{tij2}^{\text{M}}, \zeta_{tij}, \zeta_{rtij2l}^{\text{UM}}, \zeta_{tij2l}^{\text{CM}}, \zeta_{tij2l}^{\text{M}})$ -congeneric variables with conditional regressive independence if and only if Definition 4.2 and Theorem 4.2 hold and

$$E(Y_{tijk} | p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'})) = E(Y_{tijk} | p_T, p_{TS_i}), \quad (4.148)$$

$$\begin{aligned} E(Y_{rtij2l} | p_T, p_{TS_1}, \dots, p_{TS_S}, p_R, p_{RS_1}, \dots, p_{RS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'})) \\ = E(Y_{rtij2l} | p_T, p_{TS_i}, p_R, p_{RS_i}), \end{aligned} \quad (4.149)$$

$$\begin{aligned} E(Y_{rtij2l} | p_T, p_{TS_1}, \dots, p_{TS_S}, (Y_{t(ijkl)'}, (Y_{rt(ij2l)'})) \\ = E(Y_{rtij2l} | p_T, p_{TS_i}), \end{aligned} \quad (4.150)$$

$$E(S_{tijk} | p_T, p_{TS_1}, \dots, p_{TS_{S-1}}, p_{TS_{S+1}}, \dots, p_{TS_S}) = E(S_{tijk} | p_T), \quad (4.151)$$

$$\begin{aligned} E(S_{rtijkl} | p_T, p_{TS_1}, \dots, p_{TS_{S-1}}, p_{TS_{S+1}}, \dots, p_{TS_S}, p_R, p_{RS_1}, \dots, p_{RS_{S-1}}, p_{RS_{S+1}}, \dots, p_{RS_S}) \\ = E(S_{rtijkl} | p_T, p_R), \end{aligned} \quad (4.152)$$

where $(i, j, k, l)' \neq (i, j, k, l)$.

Remarks. The assumptions made in the above theorem (see Equations 4.148 to 4.150) can be interpreted in the same way as the assumptions made in Theorem 2.2 in Chapter 2. The two additional assumptions (see Equations 4.151-4.152) have important consequences for the uncorrelatedness of the latent state-residual variables on both levels (rater- and target-level). According to Equation 4.151, it is assumed that the target-specific latent state variables S_{tijk} are conditionally independent from other target-situations on different occasions of measurement $(p_{TS_1}, \dots, p_{TS_{S-1}}, p_{TS_{S+1}}, \dots, p_{TS_S})$ given the target (p_T) . In other words, different target-situations that could be realized on other measurement occasions do not contain any additional informations with respect to the expectations of the latent state variables S_{tijk} above the target (p_T) itself (see also Eid, 1995; Steyer, 1988). Similarly, it is assumed that the rater-specific latent state variables S_{rtijkl} are conditionally independent from other rater- or target-situation on different occasions of measurement, given the target (p_T) and the rater (p_R) . The assumptions stated in Equation 4.151 and 4.152 have important consequences for the independence of the latent state-residual variables measured on different occasions of measurement l and l' (see also Eid, 1995; Steyer, 1988).

4.6.1 Zero covariances based on model definition

By definition of the LST-COM model the following covariances are zero. Note that these covariances/correlations must be fixed to zero in empirical applications.

Theorem 4.5 (Testability: consequences of model definition)

If $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{\text{UM}}, \xi_t^{\text{CM}}, \xi_t^{\text{M}}, \zeta_t, \zeta_{rt}^{\text{UM}}, \zeta_t^{\text{CM}}, \zeta_t^{\text{M}}, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{\text{UM}}, \lambda_\xi^{\text{CM}}, \lambda_\xi^{\text{M}}, \lambda_\zeta, \lambda_\zeta^{\text{UM}}, \lambda_\zeta^{\text{CM}}, \lambda_\zeta^{\text{M}} \rangle$ is called a LST-COM model with CRI, then for $r \in R$, $t \in T$, $i, i' \in I$, $j, j' \in J$, $k, k' \in K$, $l, l' \in L$ where i can be equal to i' , j to j' , k to k' and l to l' but $(ijkl) \neq (ijkl)'$.

Uncorrelatedness of latent residual variables:

$$\text{Cov}(E_{rt(ij2l)}, E_{rt(ij2l)'}) = 0, \quad (4.153)$$

$$\text{Cov}(E_{t(ijkl)}, E_{t(ijkl)'}) = 0, \quad (4.154)$$

$$\text{Cov}(E_{rt(ij2l)}, E_{t(ijkl)'}) = 0. \quad (4.155)$$

Uncorrelateness of latent variables and latent residual variables:

$$\text{Cov}(\xi_{ti'j'}, E_{(r)tijkkl}) = 0, \quad (4.156)$$

$$\text{Cov}(\xi_{rti'j'2}, E_{(r)tijkkl}) = 0, \quad (4.157)$$

$$\text{Cov}(\xi_{ti'j'2}^{CM}, E_{(r)tijkkl}) = 0, \quad (4.158)$$

$$\text{Cov}(\xi_{ti'j'k'}^M, E_{(r)tijkkl}) = 0, \quad (4.159)$$

$$\text{Cov}(\zeta_{ti'j'l'}, E_{(r)tijkkl}) = 0, \quad (4.160)$$

$$\text{Cov}(\zeta_{rtj'2l'}, E_{(r)tijkkl}) = 0, \quad (4.161)$$

$$\text{Cov}(\zeta_{tj'2l'}^{CM}, E_{(r)tijkkl}) = 0, \quad (4.162)$$

$$\text{Cov}(\zeta_{tj'k'l'}^M, E_{(r)tijkkl}) = 0. \quad (4.163)$$

Uncorrelateness of latent trait variables and latent trait method variables:

$$\text{Cov}(\xi_{tij}, \xi_{rti'j'2}^{UM}) = 0, \quad (4.164)$$

$$\text{Cov}(\xi_{tij}, \xi_{tij2}^{CM}) = 0, \quad (4.165)$$

$$\text{Cov}(\xi_{tij}, \xi_{tijk}^M) = 0. \quad (4.166)$$

Uncorrelateness of latent trait variables and latent state (method) variables:

$$\text{Cov}(\xi_{tij}, \zeta_{ti'j'l'}) = 0, \quad (4.167)$$

$$\text{Cov}(\xi_{tij}, \zeta_{rtj'2l'}^{UM}) = 0, \quad (4.168)$$

$$\text{Cov}(\xi_{tij}, \zeta_{tj'2l'}^{CM}) = 0, \quad (4.169)$$

$$\text{Cov}(\xi_{tij}, \zeta_{tj'k'l'}^M) = 0. \quad (4.170)$$

Uncorrelateness of latent trait method variables and latent state (method) variables:

$$\text{Cov}(\xi_{tijk}^M, \zeta_{ti'j'l'}) = 0, \quad (4.171)$$

$$\text{Cov}(\xi_{tijk}^M, \zeta_{tj'k'l'}^M) = 0, \quad (4.172)$$

$$\text{Cov}(\xi_{tijk}^M, \zeta_{tj'2l'}^{CM}) = 0, \quad (4.173)$$

$$\text{Cov}(\xi_{tijk}^M, \zeta_{rtj'2l'}^{UM}) = 0, \quad (4.174)$$

$$\text{Cov}(\xi_{tij2}^{CM}, \zeta_{ti'j'l'}) = 0, \quad (4.175)$$

$$\text{Cov}(\xi_{tij2}^{CM}, \zeta_{tj'k'l'}^M) = 0, \quad (4.176)$$

$$\text{Cov}(\xi_{tij2}^{CM}, \zeta_{tj'2l'}^{CM}) = 0, \quad (4.177)$$

$$\text{Cov}(\xi_{tij2}^{CM}, \zeta_{rtj'2l'}^{UM}) = 0, \quad (4.178)$$

$$\text{Cov}(\xi_{rtij2}^{UM}, \zeta_{ti'j'l'}) = 0, \quad (4.179)$$

$$\text{Cov}(\xi_{rtij2}^{UM}, \zeta_{tj'k'l'}^M) = 0, \quad (4.180)$$

$$\text{Cov}(\xi_{rtij2}^{UM}, \zeta_{tj'2l'}^{CM}) = 0, \quad (4.181)$$

$$\text{Cov}(\xi_{rtij2}^{UM}, \zeta_{rtj'2l'}^{UM}) = 0. \quad (4.182)$$

Uncorrelateness of latent trait method variables:

$$\text{Cov}(\xi_{tijk}^M, \xi_{rti'j'2}^{UM}) = 0, \quad (4.183)$$

$$\text{Cov}(\xi_{tijk}^{CM}, \xi_{rti'j'2}^{UM}) = 0. \quad (4.184)$$

Uncorrelateness of latent state and state method variables:

$$\text{Cov}(\zeta_{tijl}, \zeta_{tjkl}^M) = 0, \quad (4.185)$$

$$\text{Cov}(\zeta_{tijl}, \zeta_{tj2l'}^M) = 0, \quad (4.186)$$

$$\text{Cov}(\zeta_{tijl}, \zeta_{rtj'2l'}^{UM}) = 0. \quad (4.187)$$

Uncorrelateness of latent state method variables:

$$\text{Cov}(\zeta_{tijl}, \zeta_{tijl'}) = 0, \quad \forall l \neq l', \quad (4.188)$$

$$\text{Cov}(\zeta_{tjkl}^M, \zeta_{tjkl'}^M) = 0, \quad \forall l \neq l', \quad (4.189)$$

$$\text{Cov}(\zeta_{tj2l}^{CM}, \zeta_{tj2l'}^{CM}) = 0, \quad \forall l \neq l', \quad (4.190)$$

$$\text{Cov}(\zeta_{rtj2l}^{UM}, \zeta_{rtj2l'}^{UM}) = 0, \quad \forall l \neq l'. \quad (4.191)$$

Remarks. Note that in Equations 4.188 to 4.191 of the above theorem it was necessary to indicate that both latent state-residual variables ζ_{tijl} and $\zeta_{tijl'}$ were measured on different occasions l and l' . Hence, latent state-residual variables are only uncorrelated with each other if they pertain to the same indicator i , same construct j , but different occasions of measurement l and l' . For example, latent state-residual variables may be correlated, if they belong to same indicator i and same occasion of measurement l , but different constructs j and j' . In contrast to that, latent state-residual variables are uncorrelated with any latent state-residual (common or unique) method variable belonging to the same construct j , regardless whether or not both latent variables were measured on the same or different occasions of measurement (see Equation 4.185 to 4.187).

Proofs. 16 Testability

Again, the following proofs are based on the above Definition 4.3 as well as the principle properties of residual variables, namely that any expression of the form $\text{Cov}[f(X), f(Y - E(Y|X))]$ equals zero (see Steyer, 1988, 1989; Steyer & Eid, 2001; Steyer et al., in press).

4.153-4.153 The uncorrelateness of the latent residual variables has been already demonstrated in Chapter 2.6. Thus, the proofs will not be repeated again.

4.156-4.163 In Section 2.6 it was demonstrated that the latent state variables (S_{tijkl} and $S_{rtij'2l}$) are uncorrelated with any latent error variable ($E_{ti'j'k'l'}$ and $E_{rti'j'2l'}$). By definition of the LST-COM, the following non-error variables (i.e., ξ_{tijkl} , ξ_{tij2l} , $\xi_{rtij'2l}^{UM}$, $\xi_{tij'2l}^{CM}$, ξ_{tjkl}^M , ζ_{tijkl} , ζ_{tij2l} , $\zeta_{rtij'2l}^{UM}$, $\zeta_{tij'2l}^{CM}$, ζ_{tjkl}^M) are functions of their corresponding latent state variables pertaining to the same indicator i , construct j , method k , and occasion of measurement l (see Definition 4.1). Consequently, these latent non-error variables of the LST-COM model are also uncorrelated with any latent error variable ($E_{ti'j'k'l'}$ and $E_{rti'j'2l'}$). A similar proof of the uncorrelatedness between measurement error and latent non-error variables is shown by Steyer (1988).

4.164 (a) The latent trait variables measured by the reference method ξ_{tij} are functions of ξ_{tij1l} , given that:

$$\xi_{tij} = \frac{\xi_{tij1l} - \alpha_{tijkl}}{\lambda_{\xi_{ijkl}}}.$$

(b) The latent trait unique method variables $\xi_{rti'j'2}^{UM}$ are functions of $\xi_{rti'j'2l'}^{UM}$, given that:

$$\xi_{rti'j'2}^{UM} = \frac{\xi_{rti'j'2l'}^{UM}}{\lambda_{\xi_{i'j'2l'}^{UM}}}.$$

(c) Hence, $\text{Cov}(\xi_{tij}, \xi_{rti'j'2}^{UM})$ equals zero, if $\text{Cov}(\xi_{tij1l}, \xi_{rti'j'2l'}^{UM})$ equals zero.

(d) The covariance $\text{Cov}(\xi_{tij1l}, \xi_{rti'j'2l'}^{UM})$ can be expressed as follows:

$$\begin{aligned} \text{Cov}(\xi_{tij1l}, \xi_{rti'j'2l'}^{UM}) &= \text{Cov} [E(Y_{tij1l}|p_T, p_{TSi})|p_T], E(Y_{rti'j'2l'}|p_T, p_R) - E(Y_{rti'j'2l'}|p_T)] \\ &= \text{Cov} [E(Y_{tij1l}|p_T), E(Y_{rti'j'2l'}|p_T, p_R) - E(Y_{rti'j'2l'}|p_T)]. \end{aligned}$$

(e) Therefore, ξ_{tij1l} is a (p_T) -measurable function and $\xi_{rti'j'2l'}$ is a residual with respect to the regressor p_T . Given that residuals are always uncorrelated with their regressors (see Steyer, 1988; Steyer & Eid, 2001), it follows that ξ_{tij1l} and $\xi_{rti'j'2l'}$ are uncorrelated.

4.165-4.166 (a) Similarly, the latent trait method variables (i.e., ξ_{tij2}^{CM} and ξ_{tijk}^M) are functions of ξ_{tij2l}^{CM} and ξ_{tijkl}^M , given that:

$$\xi_{tij2}^{CM} = \frac{\xi_{tij2l}^{CM}}{\lambda_{\xi_{tij2l}^{CM}}}, \quad \xi_{tijk}^M = \frac{\xi_{tijkkl}^M}{\lambda_{\xi_{tijkkl}^M}}.$$

(b) Thus, $Cov(\xi_{tij}, \xi_{tij2}^{CM})$ and $Cov(\xi_{tij}, \xi_{tijk}^M)$ equal zero, if $Cov(\xi_{tij1l}, \xi_{tij2l}^{CM})$ and $Cov(\xi_{tij1l}, \xi_{tijkkl}^M)$ equal zero.

(c) Given that ξ_{tij2l}^{CM} and ξ_{tijkkl}^M are defined as

$$\begin{aligned} \xi_{tij2l}^{CM} &\equiv \xi_{tij2l} - E(\xi_{tij2l} | \xi_{tij1l}) = E(Y_{rtij2l} | p_T) - E[E(Y_{rtij2l} | p_T) | E(Y_{tij1l} | p_T)], \\ \xi_{tijkkl}^M &\equiv \xi_{tijkkl} - E(\xi_{tijkkl} | \xi_{tij1l}) = E(Y_{tijkkl} | p_T) - E[E(Y_{tijkkl} | p_T) | E(Y_{tij1l} | p_T)], \end{aligned}$$

and the latent trait variables ξ_{tij1l} are defined as $E(Y_{tij1l} | p_T)$, it is clear that the latent trait method variables ξ_{tij2l}^{CM} as well as ξ_{tijkkl}^M are defined as residuals with respect to $E(Y_{tij1l} | p_T)$. Therefore, the covariances $Cov(\xi_{tij}, \xi_{tij2}^{CM})$ and $Cov(\xi_{tij}, \xi_{tijk}^M)$ equal zero.

4.167 (a) Again, ξ_{tij} is a function of ξ_{tij1l} (see proofs above).

(b) The latent state-residual variables $\zeta_{ti'j'k'l'}$ are functions of $\zeta_{ti'j'k'l'}$, given that

$$\zeta_{ti'j'k'l'} = \frac{\zeta_{ti'j'k'l'}}{\lambda_{\zeta_{ti'j'k'l'}}}.$$

(c) Thus, $Cov(\xi_{tij}, \zeta_{ti'j'k'l'})$ equal zero, if $Cov(\xi_{tij1l}, \zeta_{ti'j'k'l'})$ equal zero.

(d) Again, ξ_{tij1l} is a (p_T) -measurable function (as explained in Proof 4.164).

(e) $\zeta_{ti'j'k'l'}$ is defined as residual with respect to p_T , given that

$$\begin{aligned} \zeta_{ti'j'k'l'} &\equiv S_{ti'j'k'l'} - \xi_{ti'j'k'l'} \\ &= E(Y_{ti'j'k'l'} | p_T, p_{TS_{i'}}) - E(Y_{ti'j'k'l'} | p_T). \end{aligned}$$

(f) Hence, the latent state-residual variables $\zeta_{ti'j'k'l'}$ are defined as residuals with respect to p_T and ξ_{tij1l} is a (p_T) -measurable function. Given that residuals are always uncorrelated with their regressors as well as with numerically measurable functions of their regressors (see Steyer, 1988; Steyer & Eid, 2001), it follows that $\zeta_{ti'j'k'l'}$ and ξ_{tij1l} are uncorrelated.

4.168 (a) ξ_{tij} is a function of ξ_{tij1l} and $\zeta_{rtj'2l'}^{UM}$ is a function of $\zeta_{rti'j'2l'}^{UM}$, given that

$$\zeta_{rtj'2l'}^{UM} = \frac{\zeta_{rti'j'2l'}^{UM}}{\lambda_{\zeta_{rti'j'2l'}^{UM}}}.$$

(b) Thus, $Cov(\xi_{tij}, \zeta_{rtj'2l'}^{UM})$ equal zero, if $Cov(\xi_{tij1l}, \zeta_{rti'j'2l'}^{UM})$ equal zero.

(c) The covariance $Cov(\xi_{tij1l}, \zeta_{rti'j'2l'}^{UM})$ can be rewritten as follows

$$\begin{aligned} &Cov(\xi_{tij1l}, UM_{rti'j'2l'} - \xi_{rti'j'2l'}^{UM}) \\ &Cov(\xi_{tij1l}, UM_{rti'j'2l'} - Cov(\xi_{tij1l}, \xi_{rti'j'2l'}^{UM})). \end{aligned}$$

(d) As explained above (see Proofs 4.156-4.163), ξ_{tij1l} is a function of S_{tij1l} and it has been already shown that S_{tij1l} is uncorrelated with all unique method variables $UM_{rti'j'2l'}^{UM}$ (see Proof 2.94). In addition, it has been demonstrated that ξ_{tij1l} is uncorrelated with all latent unique method variables $\xi_{rti'j'2l'}^{UM}$ (see Proof 4.164). According to Proof 4.164 it has also been shown that $Cov(\xi_{tij1l}, \xi_{rti'j'2l'}^{UM}) = 0$. Therefore, it follows that $\zeta_{rti'j'2l'}^{UM}$ is also uncorrelated with ξ_{tij1l} .

4.169-4.170 (a) ξ_{tij} is a function of ξ_{tij1l} and $\zeta_{tj'2l'}^{CM}$, as well as $\zeta_{tj'k'l'}^M$ are functions of

$$\zeta_{tj'2l'}^{CM} = \frac{\zeta_{ti'j'2l'}^{CM}}{\lambda_{\zeta_{ti'j'2l'}^{CM}}}, \quad \zeta_{tj'k'l'}^M = \frac{\zeta_{ti'j'k'l'}^M}{\lambda_{\zeta_{ti'j'k'l'}^M}}.$$

(b) Therefore, $Cov(\xi_{tij}, \zeta_{tj'2l'}^{CM})$ and $Cov(\xi_{tij}, \zeta_{tj'k'l'}^M)$ equal zero, if $Cov(\xi_{tij1l}, \zeta_{ti'j'2l'}^{CM})$ and $Cov(\xi_{tij1l}, \zeta_{ti'j'k'l'}^M)$ equal zero.

(c) The latent variables $\zeta_{ti'j'2l'}^{CM}$ and $\zeta_{ti'j'k'l'}^M$ are defined by:

$$\begin{aligned}\zeta_{ti'j'2l'}^{CM} &\equiv \zeta_{ti'j'2l'} - E(\zeta_{ti'j'2l'} | \zeta_{ti'j'1l'}), \\ \zeta_{ti'j'k'l'}^M &\equiv \zeta_{ti'j'k'l'} - E(\zeta_{ti'j'k'l'} | \zeta_{ti'j'1l'}).\end{aligned}$$

(d) Therefore, $\zeta_{ti'j'2l'}^{CM}$ and $\zeta_{ti'j'k'l'}^M$ are functions of $\zeta_{ti'j'2l'}$ and $\zeta_{ti'j'k'l'}$, respectively. Thus, the covariances $Cov(\xi_{tij1l}, \zeta_{ti'j'2l'}^{CM})$ and $Cov(\xi_{tij1l}, \zeta_{ti'j'k'l'}^M)$ equal zero, if $Cov(\xi_{tij1l}, \zeta_{ti'j'2l'})$ and $Cov(\xi_{tij1l}, \zeta_{ti'j'k'l'})$ equal zero.

(e) According to Proof 4.167, it has already been shown that the covariances $Cov(\xi_{tij1l}, \zeta_{ti'j'2l'})$ and $Cov(\xi_{tij1l}, \zeta_{ti'j'k'l'})$ must equal zero.

(f) Hence, the statements $Cov(\xi_{tij}, \xi_{tij2}^{CM}) = 0$ and $Cov(\xi_{tij}, \xi_{tijk}^M) = 0$ are true as well.

4.171 (a) ξ_{tijk}^M is a function of ξ_{tijk}^M , given that

$$\xi_{tijk}^M = \frac{\xi_{tijk}^M}{\lambda_{\xi_{tijk}^M}^M}.$$

(b) $\zeta_{ti'j'l'}$ is a function of $\zeta_{ti'j'k'l'}$ (see proofs above).

(c) Therefore, $Cov(\xi_{tijk}^M, \zeta_{ti'j'l'})$ equals zero, if and only if $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'})$ equals zero.

(d) The covariance $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'})$ can be rewritten as follows:

$$Cov[\xi_{tijk} - E(\xi_{tijk} | \xi_{tij1l}), S_{ti'j'k'l'} - \xi_{ti'j'k'l'}].$$

(e) Given that ξ_{tijk}^M is a function of ξ_{tijk} , the covariance $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'})$ is zero, if

$$\begin{aligned}Cov(\xi_{tijk}, S_{ti'j'k'l'} - \xi_{ti'j'k'l'}) &= 0, \\ Cov[E(Y_{tijk} | p_T), E(Y_{ti'j'k'l'} | p_T, p_{TS_{l'}}) - E(Y_{ti'j'k'l'} | p_T)] &= 0.\end{aligned}$$

(f) Given that the latent trait variable ξ_{tijk} is defined as (p_T) -measurable function and the latent state-residual variables $\zeta_{ti'j'k'l'}$ are defined as residuals with respect to the regressor p_T , it follows that ξ_{tijk}^M is also uncorrelated with $\zeta_{ti'j'k'l'}$. The proofs for Equations 4.175 and 4.179 follow the same principle and thus will not be demonstrated here.

4.172 (a) Again, ξ_{tijk}^M is a function of ξ_{tijk}^M and $\zeta_{ti'j'k'l'}^M$ is a function of $\zeta_{ti'j'k'l'}$ (see proofs above).

(b) Therefore, $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'}^M)$ equals zero, if $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'})$.

(c) Again, ξ_{tijk}^M is defined as $\xi_{tijk} - E(\xi_{tijk} | \xi_{tij1l})$, and $\zeta_{ti'j'k'l'}^M$ is defined as $\zeta_{ti'j'k'l'} - E(\zeta_{ti'j'k'l'} | \zeta_{ti'j'1l'})$.

(d) That means that ξ_{tijk}^M is a function of ξ_{tijk} . In addition, $\zeta_{ti'j'k'l'}^M$ is a function of $\zeta_{ti'j'1l'}$:

$$\begin{aligned}\xi_{tijk} &= E(\xi_{tijk} | \xi_{tij1l}) + \xi_{tijk}^M \\ \zeta_{ti'j'1l'} &= E(\zeta_{ti'j'k'l'} | \zeta_{ti'j'1l'}) + \zeta_{ti'j'k'l'}^M.\end{aligned}$$

(e) Hence, the covariance $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'}^M)$ is zero, if the covariance $Cov(\xi_{tijk}, \zeta_{ti'j'k'l'})$ is zero. In Proof 4.167 it has been already been shown that the covariance $Cov(\xi_{tijk}, \zeta_{ti'j'k'l'})$ must equal zero. It follows that the $Cov(\xi_{tijk}^M, \zeta_{ti'j'k'l'}^M)$ must also equal zero. Hence, Equation 4.172 holds. The proofs for Equation 4.173, 4.174, 4.176, 4.177, 4.178, 4.180, 4.181, and 4.182 follow the same principle and thus are not demonstrated here.

4.183 (a) ξ_{tijk}^M is a functions of ξ_{tijk}^M and $\xi_{rti'j'2}^{UM}$ is a function of $\xi_{rti'j'2l'}^{UM}$ (see proofs above).

(b) Accordingly, $Cov(\xi_{tijk}^M, \xi_{rti'j'2}^{UM})$ equals zero, if $Cov(\xi_{tijk}^M, \xi_{rti'j'2l'}^{UM})$ equals zero. The covariance $Cov(\xi_{tijk}^M, \xi_{rti'j'2l'}^{UM})$ can be expressed as follows:

$$Cov\{\xi_{tijk} - E(\xi_{tijk} | \xi_{tij1l}), \xi_{rti'j'2l'}^{UM}\}$$

(c) According to Equation 4.30 in Definition 4.1, it is possible to replace $E(\xi_{tijk}| \xi_{tij1})$ by

$$\alpha_{tijk} + \lambda_{\xi_{tijk}} \xi_{tij1}.$$

(d) Given that the covariances $Cov(\xi_{tijk}, \xi_{rtij'2l}^{UM})$ as well as $Cov(\xi_{tij1}, \xi_{rtij'2l'}^{UM})$ must equal zero, according to Proof 4.164, it follows that the latent covariances $Cov(\xi_{tijk}^M, \xi_{rtij'2l'}^{UM})$ must also be zero. The proof for Equation 4.184 follow the same principles and is straightforward. Hence, this proof will not be demonstrated here.

4.188 (a) For all $l \neq l'$, ζ_{tijl} is a function of ζ_{tijk} and $\zeta_{tijl'}$ is a function of $\zeta_{tijk'}$ (see proofs above).

(b) Therefore, for all $l \neq l'$ $Cov(\zeta_{tijl}, \zeta_{tijl'}) = 0$, if $Cov(\zeta_{tijk}, \zeta_{tijk'}) = 0$.

(c) The latent variables ζ_{tijk} and $\zeta_{tijk'}$ are defined as follows:

$$\begin{aligned}\zeta_{tijk} &\equiv S_{tijk} - E(S_{tijk}|p_T), \\ \zeta_{tijk'} &\equiv S_{tijk'} - E(S_{tijk'}|p_T).\end{aligned}$$

(d) According to Equation 4.151 of the above Definition 4.3, it is possible to replace $E(S_{tijk}|p_T)$ by:

$$E(S_{tijk}|p_T, p_{TS_1}, \dots, p_{TS_{S-1}}, p_{TS_{S+1}}, \dots, p_{TS_l}).$$

(e) Therefore, the latent state-residual variables ζ_{tijk} and $\zeta_{tijk'}$ are defined as residual variables with respect to the same regressor p_T (see also Steyer, 1988, p. 403). Thus, ζ_{tijk} and $\zeta_{tijk'}$ are uncorrelated.

4.189 (a) For all $l \neq l'$, ζ_{tijk}^M is a function of ζ_{tijk} and $\zeta_{tijk'}^M$ is a function of $\zeta_{tijk'}$ (see proofs above).

(b) Therefore, for all $l \neq l'$ $Cov(\zeta_{tijk}^M, \zeta_{tijk'}^M) = 0$, if $Cov(\zeta_{tijk}, \zeta_{tijk'}) = 0$.

(c) The latent variables ζ_{tijk}^M and $\zeta_{tijk'}^M$ are defined as follows:

$$\begin{aligned}\zeta_{tijk}^M &\equiv \zeta_{tijk} - E(\zeta_{tijk}|\zeta_{tij1}), \\ \zeta_{tijk'}^M &\equiv \zeta_{tijk'} - E(\zeta_{tijk'}|\zeta_{tij1'}).\end{aligned}$$

(d) Given that ζ_{tijk}^M is a function of ζ_{tijk} and $\zeta_{tijk'}^M$ is a function of $\zeta_{tijk'}$, the latent state-residual method variables ζ_{tijk}^M and $\zeta_{tijk'}^M$ are uncorrelated with each other (for all $l \neq l'$), if the latent state-residual variables ζ_{tijk} and $\zeta_{tijk'}$ uncorrelated with each other (for all $l \neq l'$).

(e) In the above Proof 4.188 it has been already been shown that the latent state-residual variables ζ_{tijk} and $\zeta_{tijk'}$ pertaining to different occasions of measurements l and l' are uncorrelated with each other. Thus, the latent state-residual method variables ζ_{tijk}^M and $\zeta_{tijk'}^M$ are also uncorrelated with each other for all $l \neq l'$.

The Proof for Equation 4.190 follow the same principle and is straightforward. Thus, this proof is not demonstrated here.

4.191 (a) For all $l \neq l'$, ζ_{rtij2l}^{UM} is a function of ζ_{rtij2l} and $\zeta_{rtij2l'}^{UM}$ is a function of $\zeta_{rtij2l'}$ (see proofs above).

(b) Therefore, for all $l \neq l'$ $Cov(\zeta_{rtij2l}^{UM}, \zeta_{rtij2l'}^{UM}) = 0$, if $Cov(\zeta_{rtij2l}, \zeta_{rtij2l'}) = 0$.

(c) The latent variables ζ_{rtij2l}^{UM} and $\zeta_{rtij2l'}^{UM}$ are defined as follows:

$$\begin{aligned}\zeta_{rtij2l}^{UM} &\equiv UM_{rtij2l} - \xi_{rtij2l}^{UM}, \\ \zeta_{rtij2l'}^{UM} &\equiv UM_{rtij2l'} - \xi_{rtij2l'}^{UM}.\end{aligned}$$

(d) Hence, the latent variables ζ_{rtij2l}^{UM} and $\zeta_{rtij2l'}^{UM}$ are defined as residuals with respect to the latent variables ξ_{rtij2l}^{UM} and $\xi_{rtij2l'}^{UM}$, respectively. These regressor variables can be defined as follows:

$$\begin{aligned}\xi_{rtij2l}^{UM} &\equiv E(S_{rtij2l}|p_T, p_R) - E(S_{tij2l}|p_T), \\ \xi_{rtij2l'}^{UM} &\equiv E(S_{rtij2l'}|p_T, p_R) - E(S_{tij2l'}|p_T).\end{aligned}$$

(e) According to Equation 4.151 of the above Definition 4.3, it is possible to replace $E(S_{tijk|p_T})$ by:

$$E(S_{tijk|p_T, p_{TS_1}, \dots, p_{TS_{S-1}}, p_{TS_{S+1}}, \dots, p_{TS_S}).$$

(f) Similarly, according to Equation 4.152 of the above Definition 4.3, it is possible to replace $E(S_{rtij2l|p_T, p_R})$ by:

$$E(S_{rtij2l|p_T, p_{TS_1}, \dots, p_{TS_{S-1}}, p_{TS_{S+1}}, \dots, p_{TS_S}, p_R, p_{RS_1}, \dots, p_{RS_{S-1}}, p_{RS_{S+1}}, \dots, p_{RS_S}).$$

(g) Therefore, the latent variables ζ_{rtij2l}^{UM} and $\zeta_{rtij2l'}^{UM}$ are defined as residual variables with respect to the same regressor ξ_{rtij2l}^{UM} (see also Steyer, 1988, p. 403). Thus, Equation 4.191 holds as well.

4.185 (a) For all $l \neq l'$, ζ_{tijl} is a function of ζ_{tijk} and $\zeta_{tjl'}$ is a function of $\zeta_{tjkl'}$ (see proofs above).

(b) Therefore, for all $l \neq l'$, $Cov(\zeta_{tijk}, \zeta_{tjkl'}) = 0$, if $Cov(\zeta_{tijk}, \zeta_{tjkl'}) = 0$.

(c) The latent variables ζ_{tijk} and $\zeta_{tjkl'}$ are defined as follows:

$$\begin{aligned}\zeta_{tijk} &\equiv S_{tijk} - \xi_{tijk}, \\ \zeta_{tjkl'} &\equiv \zeta_{tjkl'} - E(\zeta_{tjkl'} | \zeta_{tijl'}).\end{aligned}$$

(d) Given that $\zeta_{tjkl'}$ is a function of ζ_{tijk}

$$\zeta_{tjkl'} = E(\zeta_{tjkl'} | \zeta_{tijl'}) + \zeta_{tjkl'}^M,$$

it follows the latent state-residual method variables $\zeta_{tjkl'}^M$ are uncorrelated with all latent variables ζ_{tijk} pertaining to different measurement occasions (l and l'), if the latent state-residual variables ζ_{tijk} are uncorrelated with the latent variables $\zeta_{tjkl'}$.

(e) According to the above Proof 4.188, it has already been shown that the latent state-residual variables ζ_{tijk} and $\zeta_{tjkl'}$ pertaining to different occasions of measurements l and l' are uncorrelated with each other. Thus, for all $l \neq l'$, the latent variables $\zeta_{tjkl'}^M$ and ζ_{tijk} are also uncorrelated with each other.

The Proofs for Equations 4.186 and 4.187 follow the same principle and are straightforward. Thus, these proofs are not shown here.

□

4.6.2 Covariance structure: LST-COM model with conditional regressive independence

In the following section the total variance-covariance matrix of the LST-COM model for three indicators \times two traits \times two methods \times three occasions of measurements is described. Similar to the previous chapters, the total covariance matrix $\sum_{\mathbf{T}}$ of size 36×36 (i.e., $ijkl \times ijkl$) can be decomposed into a within $\sum_{\mathbf{W}}$ and a between $\sum_{\mathbf{B}}$ matrix:

$$\sum_{\mathbf{T}} = \sum_{\mathbf{W}} + \sum_{\mathbf{B}}.$$

As a consequence of the definition of the model, each of these matrices $\sum_{\mathbf{W}}$ and $\sum_{\mathbf{B}}$ can be further decomposed into a trait, state and residual matrix. This decomposition follows directly, given that latent trait variables are uncorrelated with latent state-residual variables (see above Theorem 4.5). Thus, the within $\sum_{\mathbf{W}}$ and between $\sum_{\mathbf{B}}$ variance-covariance matrices may be represented as

$$\sum_{\mathbf{W}} = \sum_{\xi_{\mathbf{W}}} + \sum_{\zeta_{\mathbf{W}}} + \sum_{\theta_{\mathbf{W}}}, \quad \text{and} \quad \sum_{\mathbf{B}} = \sum_{\xi_{\mathbf{B}}} + \sum_{\zeta_{\mathbf{B}}} + \sum_{\theta_{\mathbf{B}}}.$$

$\sum_{\xi\mathbf{W}}$ refers to the within trait matrix, $\sum_{\zeta\mathbf{W}}$ refers to the within state matrix, $\sum_{\theta\mathbf{W}}$ refers to the within residual matrix, $\sum_{\xi\mathbf{B}}$ refers to the between trait matrix, $\sum_{\zeta\mathbf{B}}$ refers to the between state matrix, and $\sum_{\theta\mathbf{B}}$ is the between residual matrix. The within and between residual matrices $\sum_{\theta\mathbf{W}}$ and $\sum_{\theta\mathbf{B}}$ are structurally equivalent to the residual matrices of the LS-COM and LC-COM model. Therefore, the residual matrices of the LST-COM model are not represented in this section. The within and between trait and state matrices $\sum_{\xi\mathbf{W}}$, $\sum_{\zeta\mathbf{W}}$, $\sum_{\xi\mathbf{B}}$, and $\sum_{\zeta\mathbf{B}}$ are then further decomposed into:

$$\begin{aligned} \sum_{\xi\mathbf{W}} &= \Lambda_{\xi\mathbf{W}} \Phi_{\xi\mathbf{W}} \Lambda_{\xi\mathbf{W}}^{\mathbf{T}}, & \text{and} & & \sum_{\zeta\mathbf{W}} &= \Lambda_{\zeta\mathbf{W}} \Phi_{\zeta\mathbf{W}} \Lambda_{\zeta\mathbf{W}}^{\mathbf{T}}, \\ \sum_{\xi\mathbf{B}} &= \Lambda_{\xi\mathbf{B}} \Phi_{\xi\mathbf{B}} \Lambda_{\xi\mathbf{B}}^{\mathbf{T}}, & \text{and} & & \sum_{\zeta\mathbf{B}} &= \Lambda_{\zeta\mathbf{B}} \Phi_{\zeta\mathbf{B}} \Lambda_{\zeta\mathbf{B}}^{\mathbf{T}}. \end{aligned}$$

$\Lambda_{\xi\mathbf{W}}$ refers to the factor loading matrix for the trait-specific latent variables on the within level, with $\Lambda_{\xi\mathbf{W}}^{\mathbf{T}}$ being its transpose, $\Phi_{\xi\mathbf{W}}$ is the variance and covariance matrix of the latent trait-specific variables on the within level, $\Lambda_{\zeta\mathbf{W}}$ is the factor loading matrix for the latent state-residual variables on the within level, with $\Lambda_{\zeta\mathbf{W}}^{\mathbf{T}}$ being the transposed matrix, $\Phi_{\zeta\mathbf{W}}$ is the variance and covariance matrix of the latent state-residual variables on the within level. In a similar way, the target-level matrices are denoted by the subscript \mathbf{B} for between level. Again, a two-dimensional index (j, l) is defined similarly as described in Section 2.6.2. The index can take the following values in the given ordering $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)$. In addition, the function $Pos((j, l))$ which maps the two-dimensional index (j, l) on its position \mathbf{p} is defined. Then, the matrix $\Lambda_{\xi\mathbf{W}}$ of size 36×6 (i.e., $ijkl \times ij$) containing the factor loadings of the latent trait unique method variables ξUM_{rtij2} is given by:

$$\Lambda_{\xi\mathbf{W}} = \sum_{\mathbf{p}=1}^6 \mathbf{I}_{\Lambda_{\xi}}^{\mathbf{p}} \otimes \Lambda_{\xi\mathbf{W}_{\mathbf{p}}},$$

$\sum_{\mathbf{p}=1}^6$ refers to the sum over all constructs j and measurement occasions l . $\mathbf{I}_{\Lambda_{\xi}}^{\mathbf{p}}$ is a contrast or dummy matrix for a particular combination of construct and occasion of measurement (e.g., $j = 1$ and $l = 1$). \otimes is the Kronecker product and $\Lambda_{\xi\mathbf{W}_{\mathbf{p}}}$ is the within trait unique method factor loading matrix of size 6×3 (i.e., $ik \times i$). The contrast matrix $\mathbf{I}_{\Lambda_{\xi}}^{\mathbf{p}}$, where $\mathbf{p} \in \mathbb{N} = \{1, \dots, 6\}$ is defined as 6×2 matrix (i.e., $jl \times j$):

$$\mathbf{I}_{\Lambda_{\xi}}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{I}_{\Lambda_{\xi}}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{I}_{\Lambda_{\xi}}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{I}_{\Lambda_\xi}^4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{I}_{\Lambda_\xi}^5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{I}_{\Lambda_\xi}^6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, the within trait unique method factor loading matrix $\Lambda_{\xi\mathbf{w}_p}$ of size 6×3 (i.e., $ik \times i$), where the elements $\lambda_{\xi 1j2l}^{UM}$, $\lambda_{\xi 2j2l}^{UM}$, $\lambda_{\xi 3j2l}^{UM} > 0$ and all other elements are zero, is given by:

$$\Lambda_{\xi\mathbf{w}_p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_{\xi 1j2l}^{UM} & 0 & 0 \\ 0 & \lambda_{\xi 2j2l}^{UM} & 0 \\ 0 & 0 & \lambda_{\xi 3j2l}^{UM} \end{pmatrix}.$$

Similarly, the within state unique method factor loading matrix $\Lambda_{\zeta\mathbf{w}}$ of size 36×6 (i.e., $ijkl \times jl$) can be defined:

$$\Lambda_{\zeta\mathbf{w}} = \sum_{p=1}^6 \mathbf{I}_{\Lambda_\zeta}^p \otimes \Lambda_{\xi\mathbf{w}_p}.$$

$\mathbf{I}_{\Lambda_\zeta}^p$ refers to a contrast matrix of size 6×6 (i.e., $jl \times jl$) where $\mathbf{p} \in \mathbb{N} = \{1, \dots, 6\}$ with a one on the p^{th} diagonal element and zeros elsewhere, e.g. for $\mathbf{p}=2$:

$$\mathbf{I}_{\Lambda_\zeta}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, the within state unique method factor loading vector $\Lambda_{\zeta\mathbf{w}_p}$ of size 6×1 (i.e., $ik \times 1$) is given by

$$\Lambda_{\zeta\mathbf{w}_p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda_{\zeta 1j2l}^{UM} \\ \lambda_{\zeta 2j2l}^{UM} \\ \lambda_{\zeta 3j2l}^{UM} \end{pmatrix}.$$

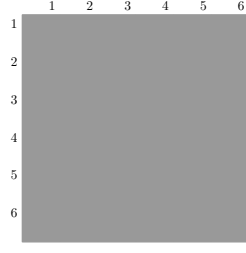


Figure 4.3: Within variance-covariance matrix $\Phi_{\xi\mathbf{W}}$ of the LST-COM, where $1=\xi_{rt112}^{UM}$, $2=\xi_{rt212}^{UM}$, $3=\xi_{rt312}^{UM}$, $4=\xi_{rt122}^{UM}$, $5=\xi_{rt222}^{UM}$, $6=\xi_{rt322}^{UM}$. Cells colored in dark gray indicate permissible and interpretable variances and covariances among the latent variables.

The complete within covariance matrix of the latent trait variables $\Phi_{\xi\mathbf{W}}$ of size 6×6 (i.e., $ij \times ij$) can be represented as follows (see Figure 4.3):

$$\Phi_{\xi\mathbf{W}} = \mathbf{E} [(\mathbf{V}_{\Phi_{\xi\mathbf{W}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\xi\mathbf{W}}}])(\mathbf{V}_{\Phi_{\xi\mathbf{W}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\xi\mathbf{W}}}]^{\mathbf{T}})],$$

where $\mathbf{V}_{\Phi_{\xi\mathbf{W}}}$ refers to the vector of size 6×1 (i.e., $ij \times 1$) including all latent trait unique method factors on the within level, namely $(\xi_{rt112}^{UM}, \xi_{rt212}^{UM}, \xi_{rt312}^{UM}, \xi_{rt122}^{UM}, \xi_{rt222}^{UM}, \xi_{rt322}^{UM})^{\mathbf{T}}$. Note that all covariances and correlations between latent trait unique method variables are permissible (see Theorem 4.5). Consequently, $\Phi_{\xi\mathbf{W}}$ does not contain zero-elements. In a similar way, $\Phi_{\zeta\mathbf{W}}$ is given by:

$$\Phi_{\zeta\mathbf{W}} = \mathbf{E} [(\mathbf{V}_{\Phi_{\zeta\mathbf{W}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\zeta\mathbf{W}}}])(\mathbf{V}_{\Phi_{\zeta\mathbf{W}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\zeta\mathbf{W}}}]^{\mathbf{T}})],$$

where $\mathbf{V}_{\Phi_{\zeta\mathbf{W}}}$ refers to the vector of size 6×1 (i.e., $jl \times 1$) including all latent state unique method factors on the within level, namely $(\zeta_{rt121}^{UM}, \zeta_{rt221}^{UM}, \zeta_{rt122}^{UM}, \zeta_{rt222}^{UM}, \zeta_{rt123}^{UM}, \zeta_{rt223}^{UM})^{\mathbf{T}}$. Note that ζ_{rtij2l}^{UM} are assumed to be homogeneous across items, therefore the index i was dropped. In contrast to $\Phi_{\xi\mathbf{W}}$, the within variance and covariance matrix $\Phi_{\zeta\mathbf{W}}$ of the latent state-residual variables ζ_{rtj2l}^{UM} of size 6×6 (i.e., $jl \times jl$) contains zero-elements. The zero-elements (see Theorem 4.5) refer to the correlations among the latent state unique method variables pertaining to the same construct j , but different occasions of measurement l and l' , that is $Cov(\zeta_{rtj2l}^{UM}, \zeta_{rtj2l'}^{UM}) = 0, \forall l \neq l'$ (see white cells in Figure 4.4). Furthermore, it is also recommended to fix all of the following correlations referring to associations between latent state unique method factors pertaining to different constructs $j \neq j'$ and different occasions of measurement $l \neq l'$ to zero as well: $Cov(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l'}^{UM}) = 0, \forall j, l \neq j', l'$ (see light gray cells in Figure 4.4). In most empirical applications these correlations will be close to zero, and therefore may be fixed to zero for parsimony.

The target-level matrices can be defined following a similar logic. First, the between latent trait factor loadings matrix $\Lambda_{\xi\mathbf{B}}$ of size 36×12 (i.e., $ijkl \times jkl$) containing the latent factor loading onto the latent trait variables ξ_{tij} and ξ_{tij2}^{CM} is given by:

$$\Lambda_{\xi\mathbf{B}} = \sum_{\mathbf{p}=1}^6 \mathbf{I}_{\Lambda_{\xi}}^{\mathbf{p}} \otimes \Lambda_{\xi\mathbf{B}_{\mathbf{p}}},$$

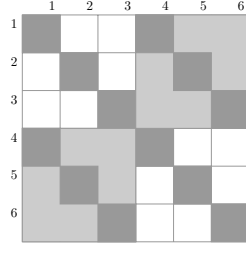


Figure 4.4: Within variance-covariance matrix $\Phi_{\zeta_{\mathbf{W}}}$ of the LST-COM, where $1=\zeta_{rt121}^{UM}$, $2=\zeta_{rt221}^{UM}$, $3=\zeta_{rt122}^{UM}$, $4=\zeta_{rt222}^{UM}$, $5=\zeta_{rt123}^{UM}$, $6=\zeta_{rt223}^{UM}$. Cells colored in dark gray indicate permissible and interpretable variances and covariances among the latent variables. Cells colored in light gray refer to covariances that can be fixed to zero for parsimony. White cells refer to non-permissible covariances among the latent variables.

for which the elements $\lambda_{\xi 1j1l}, \lambda_{\xi 2j1l}, \lambda_{\xi 3j1l}, \lambda_{\xi 1j2l}, \lambda_{\xi 2j2l}, \lambda_{\xi 3j2l}, \lambda_{\xi 1j2l}^{CM}, \lambda_{\xi 2j2l}^{CM}, \lambda_{\xi 3j2l}^{CM} > 0$ and all other elements are necessarily zero. $\mathbf{I}_{\Lambda_{\xi}}^{\mathbf{P}}$ refers to a contrast matrix of size 6×2 (i.e., $jl \times j$) described above. Then, $\Lambda_{\xi \mathbf{B}_p}$ is the matrix of the between factor loadings of size 6×6 (i.e., $ik \times ik$) which is given by:

$$\Lambda_{\xi \mathbf{B}_p} = \begin{pmatrix} \lambda_{\xi 1j1l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\xi 2j1l} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\xi 3j1l} & 0 & 0 & 0 \\ \lambda_{\xi 1j2l} & 0 & 0 & \lambda_{\xi 1j2l}^{CM} & 0 & 0 \\ 0 & \lambda_{\xi 2j2l} & 0 & 0 & \lambda_{\xi 2j2l}^{CM} & 0 \\ 0 & 0 & \lambda_{\xi 3j2l} & 0 & 0 & \lambda_{\xi 3j2l}^{CM} \end{pmatrix}.$$

In a similar way, the matrix $\Lambda_{\zeta \mathbf{B}}$ of size 36×12 (i.e., $ijkl \times jkl$) containing the between latent state factor loadings of the common latent state variables ζ_{tjl} and ζ_{tj2l}^{CM} is given by:

$$\Lambda_{\zeta \mathbf{B}} = \sum_{p=1}^6 \mathbf{I}_{\Lambda_{\zeta}}^{\mathbf{P}} \otimes \Lambda_{\zeta \mathbf{B}_p}.$$

Again, $\mathbf{I}_{\Lambda_{\zeta}}^{\mathbf{P}}$ refers to the contrast matrix of size 6×6 (i.e., $jl \times jl$) described above and $\Lambda_{\zeta \mathbf{B}_p}$ is the between factor loadings matrix¹, represented by:

$$\Lambda_{\zeta \mathbf{B}_p} = \begin{pmatrix} \lambda_{\zeta 1j1l} & 0 \\ \lambda_{\zeta 2j1l} & 0 \\ \lambda_{\zeta 3j1l} & 0 \\ \lambda_{\zeta 1j2l} & \lambda_{\zeta 1j2l}^{CM} \\ \lambda_{\zeta 2j2l} & \lambda_{\zeta 2j2l}^{CM} \\ \lambda_{\zeta 3j2l} & \lambda_{\zeta 3j2l}^{CM} \end{pmatrix}.$$

¹Note that for the sake of simplicity, it is assumed that the latent state-residual variables ζ_{tj1l} are homogeneous across items. Hence, it is assumed that the latent state-residual variables ζ_{tj1l} are measured by a common latent state-residual factor ζ_{tj1l} . The matrix $\Lambda_{\zeta \mathbf{B}_p}$ refers therefore to the factor loading matrix of common ζ_{tj1l} and ζ_{tj2l}^{CM} variables. Note that this model differs slightly from the model in Definition 4.1.

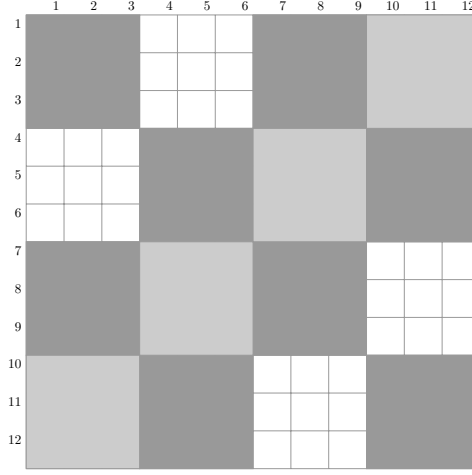


Figure 4.5: Between variance-covariance matrix $\Phi_{\xi\mathbf{B}}$ of the LST-COM model, where $1=\xi_{t11}$, $2=\xi_{t21}$, $3=\xi_{t31}$, $4=\xi_{t11}^{CM}$, $5=\xi_{t21}^{CM}$, $6=\xi_{t31}^{CM}$, $7=\xi_{t12}$, $8=\xi_{t22}$, $9=\xi_{t32}$, $10=\xi_{t12}^{CM}$, $11=\xi_{t22}^{CM}$, $12=\xi_{t32}^{CM}$. Cells colored in white indicate zero-covariances, cells colored in gray indicate permissible and interpretable variances and covariances. Cells in light gray indicate covariances that should be fixed to zero for parsimony.

The between variance and covariance matrix of the latent trait variables $\Phi_{\xi\mathbf{B}}$ of size 12×12 (i.e., $ijk \times ijk$) is given by:

$$\Phi_{\xi\mathbf{B}} = \mathbf{E} [(\mathbf{V}_{\Phi_{\xi\mathbf{B}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\xi\mathbf{B}}}])(\mathbf{V}_{\Phi_{\xi\mathbf{B}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\xi\mathbf{B}}}]^{\mathbf{T}}),$$

where $\mathbf{V}_{\Phi_{\xi\mathbf{B}}}$ refers to the vector of size 12×1 including all latent trait unique method factors on the between level, namely $(\xi_{t11}, \xi_{t21}, \xi_{t31}, \xi_{t11}^{CM}, \xi_{t21}^{CM}, \xi_{t31}^{CM}, \xi_{t12}, \xi_{t22}, \xi_{t32}, \xi_{t12}^{CM}, \xi_{t22}^{CM}, \xi_{t32}^{CM})^{\mathbf{T}}$. As a consequence of the definition of the model, all elements referring to $Cov(\xi_{tij}, \xi_{tj2}^{CM}) = 0$ are zero-elements. For parsimony reasons, it is recommended to also fix the elements referring to $Cov(\xi_{ti'j'}, \xi_{tj'2}^{CM}) = 0, \forall i, j \neq i', j'$ to zero. In Figure 4.5 the structure of the variance-covariance matrix $\Phi_{\xi\mathbf{B}}$ is depicted.

The between variance and covariance matrix of the latent state factors $\Phi_{\zeta\mathbf{B}}$ of size 12×12 (i.e., $jkl \times jkl$) is given by:

$$\Phi_{\zeta\mathbf{B}} = \mathbf{E} [(\mathbf{V}_{\Phi_{\zeta\mathbf{B}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\zeta\mathbf{B}}}])(\mathbf{V}_{\Phi_{\zeta\mathbf{B}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\zeta\mathbf{B}}}]^{\mathbf{T}}),$$

where $\mathbf{V}_{\Phi_{\zeta\mathbf{B}}}$ refers to the vector of size 12×1 including all latent state factors on the between level, namely $(\zeta_{t11}, \zeta_{t121}^{CM}, \zeta_{t21}, \zeta_{t221}^{CM}, \zeta_{t12}, \zeta_{t122}^{CM}, \zeta_{t22}, \zeta_{t222}^{CM}, \zeta_{t13}, \zeta_{t123}^{CM}, \zeta_{t23}, \zeta_{t223}^{CM})^{\mathbf{T}}$. By definition, all elements referring to $Cov(\zeta_{tjl}, \zeta_{tj'l'}) = Cov(\zeta_{tjl}, \zeta_{tj'2l'}) = Cov(\zeta_{tj2l}^{CM}, \zeta_{tj'2l'}^{CM}) = 0, \forall l \neq l'$ are zero elements. Again, for parsimony reasons, it is recommended to fix the elements referring to $Cov(\zeta_{tjl}, \zeta_{tj'2l'}^{CM}) \forall j, l \neq j', l'$ to zero as well. Figure 4.6 illustrates the complete between variance-covariance matrix for the latent state variables.

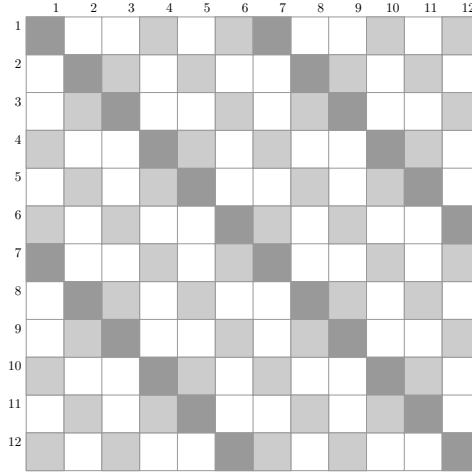


Figure 4.6: Between variance-covariance matrix $\Phi_{\zeta_{\mathbf{B}}}$ of the LST-COM model, where $1=\zeta_{t11}$, $2=\zeta_{t121}^{CM}$, $3=\zeta_{t21}$, $4=\zeta_{t221}^{CM}$, $5=\zeta_{t12}$, $6=\zeta_{t122}^{CM}$, $7=\zeta_{t22}$, $8=\zeta_{t222}^{CM}$, $9=\zeta_{t13}$, $10=\zeta_{t123}^{CM}$, $11=\zeta_{t23}$, $12=\zeta_{t223}^{CM}$. Cells colored in white indicate zero covariances, cells colored in gray indicate permissible and interpretable variances and covariances among the latent variables. Cells in light gray indicate covariances among the latent variables that should be fixed to zero for parsimony.

4.6.3 Interpretation of non-zero correlations and correlations

In the following section permissible covariances between latent variables in the LST-COM model with CRI are discussed. Some of the correlation coefficients may be interpreted as discriminant validity or the generalization of trait and/or state method effects. Therefore, these latent correlations are of practical significance. The interpretation of some of these correlations will be illustrated briefly in the next section. Note that the correlation coefficients described below refer to the LST-COM model illustrated in Figure 4.1. Hence, indicator-specific latent trait variables (ξ_{tij} , ξ_{tij2}^{CM} , ξ_{rtij2}^{UM} , and ξ_{tijk}^M) as well as indicator-specific latent state-residual (ζ_{tijl}) variables are assumed.

1. The correlations $Cor(\xi_{tij}, \xi_{ti'j})$ between latent trait factors belonging to the same trait j , but different indicators i and i' can be interpreted as degree of homogeneity of the indicators. If these correlations differ from 1, then it can be concluded that the items measure different facets or aspects of the construct. The correlations between the latent trait factors belonging to the same indicator, but different constructs j and j' indicate discriminant validity with respect to the reference method. Two different correlations can be distinguished: (A) The latent correlations $Cor(\xi_{tij}, \xi_{ti'j'})$ between trait factors of the reference method belonging to the same indicator i across different constructs j and j' . High correlations indicate low discriminant validity with respect to the reference method. (B) The correlations $Cor(\xi_{tij}, \xi_{ti'j'})$ between latent trait factors of the reference method belonging to different indicators i and i' as well as different constructs j and j' . These correlations can be interpreted as discriminant validity coefficients with respect to the reference method that are corrected for indicator-specific effects.
2. The correlations $Cor(\xi_{tij2}^{CM}, \xi_{ti'j2}^{CM})$ between the latent trait common method factors of the same construct j , but different indicators i and i' can be interpreted as generalization of

common rater bias with respect to different indicators. If these correlations are close to one, it is reasonable to define latent trait common method factors ξ_{tj2}^{CM} . The correlations $Cor(\xi_{tij2}^{CM}, \xi_{ti'j'2}^{CM})$ between the latent trait common method variables belonging to the same indicator i , but different constructs j and j' indicate to which extent the common trait bias of the interchangeable methods (that is not shared with the trait bias of the reference method) generalizes across different constructs. For example, it might be interesting to know whether or not peers consistently under- or overestimate the students self-ratings over time with regard to two different constructs (e.g., depression and anxiety). The correlations $Cor(\xi_{tij2}^{CM}, \xi_{ti'j'2}^{CM})$ between latent trait common method variables pertaining to different indicators i and i' as well as different constructs j and j' represent the generalization of the latent trait common method bias across different indicators and different constructs.

3. The correlations $Cor(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM})$, $Cor(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM})$, and $Cor(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM})$ between the latent trait unique method variables can be interpreted in a similar way as the correlations described before. However, these correlations reflect the generalization of stable unique rater bias (i.e., the consistent deviation of a particular interchangeable rater from the common view of the interchangeable raters) across different indicators, different constructs, or different indicators and different constructs.
4. If other structurally different methods (e.g., teacher or parent ratings) are used, then the generalization of stable common method effects (e.g., common peer bias) and stable method effects (e.g., parent or teacher rating) can be investigated with respect to the following correlations: $Cor(\xi_{tij2}^{CM}, \xi_{tik}^M)$, $Cor(\xi_{tij2}^{CM}, \xi_{ti'jk}^M)$, $Cor(\xi_{tij2}^{CM}, \xi_{ti'j'k}^M)$, and $Cor(\xi_{tij2}^{CM}, \xi_{ti'j'k}^M)$. For example, it might be interesting to know, whether or not teachers and peers consistently converge in their judgments, and whether or not, these rater agreement can be generalized across different constructs j and j' , and/or indicators i and i' .
5. The correlations [$Cor(\xi_{tij}, \xi_{tij'2}^{CM})$, $Cor(\xi_{tij}, \xi_{ti'j'k}^M)$, $Cor(\xi_{tij}, \xi_{ti'j'2}^{CM})$, and $Cor(\xi_{tij}, \xi_{ti'j'k}^M)$] between the latent trait variables and the latent trait (common) method variables are admissible by definition of the LST-COM model, if and only if both latent variables pertain to different constructs j and j' . Therefore, these correlations may be estimated in empirical applications. Nevertheless, these correlations are rather difficult to interpret and will be rather low in empirical applications. Therefore, the correlations should be set to zero if possible. With respect to the simulation study (see Chapter 6) these correlations were fixed to zero for parsimony.
6. The correlations between the latent state-residual variables may be investigated as well. The correlations $Cor(\zeta_{tijl}, \zeta_{ti'jl})$ between the latent state-residual variables pertaining to the same construct j , but different indicators i and i' , reflect the homogeneity of indicators with respect to occasion-specific (momentary) influences. If these correlations are close to 1, this indicates that the occasion-specific (momentary) influences are homogeneous across different indicators i and i' . Hence, a latent state-residual factor may be construed. For parsimony, latent state-residual factors were assumed for the simulation of this model. The correlations $Cor(\zeta_{tijl}, \zeta_{ti'j'l})$ between latent state-residual variables belonging to different constructs j

and j' indicate whether or not occasion-specific influences of the reference method can be generalized across different constructs. These correlations can also be interpreted as degree of discriminant validity on the state-level. For example, it may be interesting to know whether or not the child's self-rated depression level at the first occasion of measurement is associated with the self-rated anxiety level at the same occasion of measurement. The correlations $Cor(\zeta_{tijl}, \zeta_{ti'j'l})$ between latent state-residual variables of different indicators i and i' and different constructs j and j' represent to what extent occasion-specific (momentary) influences can be generalized across different indicators and different constructs.

7. The correlations [$Cor(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l}^{UM})$, $Cor(\zeta_{tj2l}^{CM}, \zeta_{tj'2l}^{CM})$, $Cor(\zeta_{tjkl}^M, \zeta_{tj'kl}^M)$, $Cor(\zeta_{tjkl}^M, \zeta_{tj'k'l}^M)$, and $Cor(\zeta_{tjkl}^M, \zeta_{tj'k'l}^M)$] between the occasion-specific and the method-specific effects might be studied as well. The correlations $Cor(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l}^{UM})$ between the latent state-residual unique method factors belonging to different constructs j and j' , indicate to what extent occasion-specific and rater-specific effects can be generalized across different constructs. For example, a particular peer may over- or underestimate a child's depression level rated by all peers on the first measurement occasion in a similar way as the child's anxiety level rated by all raters on the same occasion of measurement. In a similar way, the correlations $Cor(\zeta_{tj2l}^{CM}, \zeta_{tj'2l}^{CM})$ between latent state-residual common method factors pertaining to different constructs j and j' can be interpreted as generalization of the occasion-specific common view of the interchangeable raters across different constructs. Of course, these kind of correlations (see correlations above) may be also investigated with respect to structurally different methods ($k > 2$) of different constructs j and j' .
8. Moreover, the occasion-specific common method effects of a set of interchangeable methods may be related to the occasion-specific method effects of a structurally different method [i.e., $Cor(\zeta_{tj2l}^{CM}, \zeta_{tjkl}^M)$]. Again, these correlations between occasion-specific method variables pertaining to a set of interchangeable methods and occasion-specific method variables pertaining to structurally different methods may generalize across different constructs [i.e., $Cor(\zeta_{tj2l}^{CM}, \zeta_{tj'kl}^M)$].
9. Finally, the correlations [$Cor(\zeta_{tijl}, \zeta_{tj'2l}^{CM})$ and $Cor(\zeta_{tijl}, \zeta_{tj'kl}^M)$] between the occasion-specific effects of the reference method and occasion-specific effects of non-reference methods should be fixed to zero for parsimony. Again, these correlations are difficult to interpret and will often not differ significantly from zero. However, substantial correlations would indicate, for example, that the momentary self-rated anxiety level of a child is associated with the momentary peer-rated depression level of the child corrected for the self-reported momentary depression level.

4.7 General measurement equations and variance decompositions

In the following section the general measurement equations of the LST-COM models are derived. Based on the definition of the LST-COM model different variance coefficients can be calculated. In Theorem 4.4 it has been shown that these variance coefficients can be meaningfully interpreted. As already discussed in the previous chapters the covariance structure of the latent variables of the LST-COM model is essential for the decomposition of different variance components. In Theorem 4.5 non-permissible covariances that must be fixed to zero in empirical applications were discussed. Based on these conditions the measurement equations as well as the variance decomposition of LST-COM models are presented next.

Definition 4.4

$\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_{\mathbf{t}}, \xi_{\mathbf{rt}}^{\mathbf{UM}}, \xi_{\mathbf{t}}^{\mathbf{CM}}, \xi_{\mathbf{t}}^{\mathbf{M}}, \zeta_{\mathbf{t}}, \zeta_{\mathbf{rt}}^{\mathbf{UM}}, \zeta_{\mathbf{t}}^{\mathbf{CM}}, \zeta_{\mathbf{t}}^{\mathbf{M}}, \mathbf{E}_{\mathbf{rt}}, \mathbf{E}_{\mathbf{t}}, \alpha_{\mathbf{t}}, \lambda_{\xi}, \lambda_{\xi}^{\mathbf{UM}}, \lambda_{\xi}^{\mathbf{CM}}, \lambda_{\xi}^{\mathbf{M}}, \lambda_{\zeta}, \lambda_{\zeta}^{\mathbf{UM}}, \lambda_{\zeta}^{\mathbf{CM}}, \lambda_{\zeta}^{\mathbf{M}} \rangle$ is a LST-COM model with conditional regressive independent latent state variables if and only if the statements in Definition 4.2 as well as the statements 4.41 to 4.49 of Theorem 4.2 hold, and:

$$\begin{aligned}
\xi_{\mathbf{t}} &\equiv (\xi_{111} \cdots \xi_{tij} \cdots \xi_{bcd})^{\mathbf{T}}, \\
\xi_{\mathbf{rt}}^{\mathbf{UM}} &\equiv (\xi_{11111}^{\mathbf{UM}} \cdots \xi_{rtij2}^{\mathbf{UM}} \cdots \xi_{abcd2}^{\mathbf{UM}})^{\mathbf{T}}, \\
\xi_{\mathbf{t}}^{\mathbf{CM}} &\equiv (\xi_{1111}^{\mathbf{CM}} \cdots \xi_{tij2}^{\mathbf{CM}} \cdots \xi_{bde2}^{\mathbf{CM}})^{\mathbf{T}}, \\
\xi_{\mathbf{t}}^{\mathbf{M}} &\equiv (\xi_{1111}^{\mathbf{M}} \cdots \xi_{tijk}^{\mathbf{M}} \cdots \xi_{bcde}^{\mathbf{M}})^{\mathbf{T}}, \\
\zeta_{\mathbf{t}} &\equiv (\zeta_{1111} \cdots \zeta_{tjkl} \cdots \zeta_{bcdf})^{\mathbf{T}}, \\
\zeta_{\mathbf{rt}}^{\mathbf{UM}} &\equiv (\zeta_{11111}^{\mathbf{UM}} \cdots \zeta_{rtj2l}^{\mathbf{UM}} \cdots \zeta_{abd2f}^{\mathbf{UM}})^{\mathbf{T}}, \\
\zeta_{\mathbf{t}}^{\mathbf{CM}} &\equiv (\zeta_{1111}^{\mathbf{CM}} \cdots \zeta_{tj2l}^{\mathbf{CM}} \cdots \zeta_{bd2f}^{\mathbf{CM}})^{\mathbf{T}}, \\
\zeta_{\mathbf{t}}^{\mathbf{M}} &\equiv (\zeta_{1111}^{\mathbf{M}} \cdots \zeta_{tjkl}^{\mathbf{M}} \cdots \zeta_{bdef}^{\mathbf{M}})^{\mathbf{T}}, \\
\mathbf{E}_{\mathbf{rt}} &\equiv (E_{111111} \cdots E_{rtij2l} \cdots E_{abcd2f})^{\mathbf{T}}, \\
\mathbf{E}_{\mathbf{t}} &\equiv (E_{11111} \cdots E_{tijkkl} \cdots E_{bcdef})^{\mathbf{T}}, \\
\alpha_{\mathbf{t}} &\equiv (\alpha_{1111} \cdots \alpha_{tijkkl} \cdots \alpha_{bcdef})^{\mathbf{T}}, \\
\lambda_{\xi} &\equiv (\lambda_{\xi 1111} \cdots \lambda_{\xi ijkl} \cdots \lambda_{\xi cdef})^{\mathbf{T}}, \\
\lambda_{\xi}^{\mathbf{UM}} &\equiv (\lambda_{\xi 1111}^{\mathbf{UM}} \cdots \lambda_{\xi ij2l}^{\mathbf{UM}} \cdots \lambda_{\xi cd2f}^{\mathbf{UM}})^{\mathbf{T}}, \\
\lambda_{\xi}^{\mathbf{CM}} &\equiv (\lambda_{\xi 1111}^{\mathbf{CM}} \cdots \lambda_{\xi ij2l}^{\mathbf{CM}} \cdots \lambda_{\xi cd2f}^{\mathbf{CM}})^{\mathbf{T}}, \\
\lambda_{\xi}^{\mathbf{M}} &\equiv (\lambda_{\xi 1111}^{\mathbf{M}} \cdots \lambda_{\xi ijkl}^{\mathbf{M}} \cdots \lambda_{\xi cdef}^{\mathbf{M}})^{\mathbf{T}}, \\
\lambda_{\zeta} &\equiv (\lambda_{\zeta 1111} \cdots \lambda_{\zeta ijkl} \cdots \lambda_{\zeta cdef})^{\mathbf{T}}, \\
\lambda_{\zeta}^{\mathbf{UM}} &\equiv (\lambda_{\zeta 1111}^{\mathbf{UM}} \cdots \lambda_{\zeta ij2l}^{\mathbf{UM}} \cdots \lambda_{\zeta cd2f}^{\mathbf{UM}})^{\mathbf{T}}, \\
\lambda_{\zeta}^{\mathbf{CM}} &\equiv (\lambda_{\zeta 1111}^{\mathbf{CM}} \cdots \lambda_{\zeta ij2l}^{\mathbf{CM}} \cdots \lambda_{\zeta cd2f}^{\mathbf{CM}})^{\mathbf{T}}, \\
\lambda_{\zeta}^{\mathbf{M}} &\equiv (\lambda_{\zeta 1111}^{\mathbf{M}} \cdots \lambda_{\zeta ijkl}^{\mathbf{M}} \cdots \lambda_{\zeta cdef}^{\mathbf{M}})^{\mathbf{T}}.
\end{aligned}$$

Remarks. According to the above Definition 4.4, all indicators Y_{tij1l} belonging to the reference method ($k = 1$), the same construct j , and the same measurement occasion l measure a latent trait ξ_{tij} , a latent state residual ζ_{tijl} , and an occasion-specific measurement error E_{tij1l} . All indicators Y_{tijkkl} belonging to a non-reference method ($k > 2$) as well as construct j , and occasion of measurement l measure also a latent trait ξ_{tij} , a latent state residual ζ_{tijl} as well as an occasion-specific measurement error E_{tijkkl} . In addition, these variables also measure a latent trait method

factors ξ_{tijk}^M as well as a latent state method factors ζ_{tijk}^M . All indicators Y_{rtij2l} belonging to a non-reference method ($k = 2$) as well as to the same construct j , and the same measurement of occasion l measure a latent trait ξ_{tij} , a latent state residual ζ_{tijl} and an occasion-specific measurement error E_{rtij2l} . Moreover, these observed variables also measure two indicator- and construct-specific latent trait method factors, namely ξ_{tij2}^{CM} and ξ_{rtij2}^{UM} , as well as two construct- and occasion-specific latent state-residual method variables, namely ζ_{tj2l}^{CM} and ζ_{rtj2l}^{UM} . The measurement equations of the observed variables are given by:

$$Y_{tij1l} = \xi_{tij} + \zeta_{tijl} + E_{tij1l}, \quad (4.192)$$

$$Y_{tijk} = \alpha_{tijk} + \lambda_{\xi_{ijk}} \xi_{tij} + \lambda_{\xi_{ijk}}^M \xi_{tijk}^M + \lambda_{\zeta_{ijk}} \zeta_{tijl} + \lambda_{\zeta_{ijk}}^M \zeta_{tijk}^M + E_{tijk}, \quad \forall k > 2, \quad (4.193)$$

$$Y_{rtij2l} = \alpha_{rtij2l} + \lambda_{\xi_{ij2l}} \xi_{tij} + \lambda_{\xi_{ij2l}}^{CM} \xi_{tij2}^{CM} + \lambda_{\xi_{ij2l}}^{UM} \xi_{rtij2}^{UM} + \lambda_{\zeta_{ij2l}} \zeta_{tijl} + \lambda_{\zeta_{ij2l}}^{CM} \zeta_{tj2l}^{CM} + \lambda_{\zeta_{ij2l}}^{UM} \zeta_{rtj2l}^{UM} + E_{rtij2l}. \quad (4.194)$$

4.7.1 Variance decomposition

Based on the above Equations 4.192 and 4.194, the variance of the observed variables can be additively decomposed into the variance of the indicator-specific trait factors (ξ_{tij}), variance of the indicator- and trait-specific method factors (ξ_{rtij2}^{UM} , ξ_{tij2}^{CM} , ξ_{tijk}^M), variance of the indicator- and occasion-specific factors (ζ_{tijl}), variance of the common indicator-occasion-specific method factors (ξ_{rtij2}^{UM} , ζ_{tj2l}^{CM} , ζ_{tijk}^M) as well as the variance of the measurement error variables (E_{tijk} , E_{rtij2l}):

$$Var(Y_{tij1l}) = Var(\xi_{tij}) + Var(\zeta_{tijl}) + Var(E_{tij1l}), \quad (4.195)$$

$$Var(Y_{rtij2l}) = (\lambda_{\xi_{ij2l}})^2 Var(\xi_{tij}) + (\lambda_{\xi_{ij2l}}^{CM})^2 Var(\xi_{tij2}^{CM}) + (\lambda_{\xi_{ij2l}}^{UM})^2 Var(\xi_{rtij2}^{UM}) + (\lambda_{\zeta_{ij2l}})^2 Var(\zeta_{tijl}) + (\lambda_{\zeta_{ij2l}}^{CM})^2 Var(\zeta_{tj2l}^{CM}) + (\lambda_{\zeta_{ij2l}}^{UM})^2 Var(\zeta_{rtj2l}^{UM}) + Var(E_{rtij2l}), \quad (4.196)$$

$$Var(Y_{tijk}) = (\lambda_{\xi_{ijk}})^2 Var(\xi_{tij}) + (\lambda_{\xi_{ijk}}^M)^2 Var(\xi_{tijk}^M) + (\lambda_{\zeta_{ijk}})^2 Var(\zeta_{tijk}) + (\lambda_{\zeta_{ijk}}^M)^2 Var(\zeta_{tijk}^M) + Var(E_{tijk}), \quad \forall k > 2. \quad (4.197)$$

Moreover, it is possible to define different variance components. First, the true intraclass coefficient (ICC) can be defined on the basis of true-score variables pertaining to the interchangeable method (τ_{rtij2l}):

$$ICC(\tau_{rtij2l}) = \frac{(\lambda_{\xi_{ij2l}})^2 Var(\xi_{tij}) + (\lambda_{\xi_{ij2l}}^{CM})^2 Var(\xi_{tij2}^{CM}) + (\lambda_{\xi_{ij2l}}^{UM})^2 Var(\xi_{rtij2}^{UM}) + (\lambda_{\zeta_{ij2l}}^{CM})^2 Var(\zeta_{tj2l}^{CM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

The true ICC coefficient represents the proportion of true-score variance that is determined by consistent (stable) as well as momentary rater influences on the target-level. In other words, the true ICC coefficient reflects the amount of true (stable as well as momentary) rater-congruency on the target-level. That means that these true rater effects are free of single rater-specific influences. Hence, the true ICC coefficients can also be interpreted as true amount of rater-consistency on the target-level. Note that the true rater-consistency coefficient is calculated on the level of the true-score variables and corresponds to the classical intraclass correlation (Hox, 2010; Luke, 2004; Snijders & Bosker, 2011; Raudenbush & Bryk, 2002). The true rater-consistency coefficient on the

target-level can be further decomposed into a consistent [trait; $\xi RC(\tau_{rtij2l})$] as well as a momentary [state; $\zeta RC(\tau_{rtij2l})$] rater-consistency coefficient:

$$\xi RC(\tau_{rtij2l}) = \frac{(\lambda_{\xi ij2l})^2 Var(\xi_{tij}) + (\lambda_{\xi ij2l}^{CM})^2 Var(\xi_{tij}^{CM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})},$$

$$\zeta RC(\tau_{rtij2l}) = \frac{(\lambda_{\zeta ij2l})^2 Var(\zeta_{tijl}) + (\lambda_{\zeta ij2l}^{CM})^2 Var(\zeta_{tijl}^{CM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

The true trait (ξ) rater-consistency coefficient reflects the amount of true stable rater-consistency (i.e., free of rater-specific and occasion-specific influences). In contrast to that the true state (ζ) rater-consistency coefficient reflects the amount of true occasion-specific (momentary) rater-consistency (i.e., free of rater-specific and trait-specific influences). In addition, it is possible to define different trait or state specificity coefficients: (1) the true trait (ξ) specificity coefficient and (2) true state (ζ) specificity coefficient. The true trait (ξ) specificity coefficient represents the proportion of true-score variance that is determined by stable (not occasion-specific or momentary) influences and is given by:

$$\xi S(\tau_{tij1l}) = \frac{Var(\xi_{tij})}{Var(Y_{tij1l}) - Var(E_{tij1l})},$$

$$\xi S(\tau_{tijkl}) = \frac{(\lambda_{\xi ijkl})^2 Var(\xi_{tij}) + (\lambda_{\xi ijkl}^M)^2 Var(\xi_{tij}^M)}{Var(Y_{tijkl}) - Var(E_{tijkl})}, \quad \forall k > 2,$$

$$\xi S(\tau_{rtij2l}) = \frac{(\lambda_{\xi ij2l})^2 Var(\xi_{tij}) + (\lambda_{\xi ij2l}^{CM})^2 Var(\xi_{tij}^{CM}) + (\lambda_{\xi ij2l}^{UM})^2 Var(\xi_{rtij2l}^{UM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

These coefficients may be of particular interest for researchers who seek to determine how much variance of the true-score variables is due to stable (trait) influences. Conversely, researchers may calculate the true state (ζ) specificity coefficient in order to investigate how much true-score variance is due to occasion-specific or momentary influences:

$$\zeta S(\tau_{tij1l}) = \frac{Var(\zeta_{tijl})}{Var(Y_{tij1l}) - Var(E_{tij1l})},$$

$$\zeta S(\tau_{tijkl}) = \frac{(\lambda_{\zeta ijkl})^2 Var(\zeta_{tijl}) + (\lambda_{\zeta ijkl}^M)^2 Var(\zeta_{tijl}^M)}{Var(Y_{tijkl}) - Var(E_{tijkl})}, \quad \forall k > 2,$$

$$\zeta S(\tau_{rtij2l}) = \frac{(\lambda_{\zeta ij2l})^2 Var(\zeta_{tijl}) + (\lambda_{\zeta ij2l}^{CM})^2 Var(\zeta_{tijl}^{CM}) + (\lambda_{\zeta ij2l}^{UM})^2 Var(\zeta_{rtij2l}^{UM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

The true occasion-specific (ζ) specificity coefficient reflects the proportion of true-score variance that is due to occasion-specific or momentary influences. The true trait (ξ) specificity coefficients can be further decomposed into a trait consistency and a trait method specificity coefficient. In

a similar way, the true occasion-specific (ζ) specificity coefficients can be further decomposed into an occasion-specific consistency and an occasion-specific method specificity coefficient. With regard to the trait consistency coefficients it is possible to investigate the degree of true convergent validity on trait level, whereas the occasion-specific consistency coefficients reflects the degree of true convergent validity on state level. The true trait (ξ) consistency coefficients are given by:

$$\xi CON(\tau_{tijkkl}) = \frac{(\lambda_{\xi_{ijkl}})^2 Var(\xi_{tij})}{Var(Y_{tijkkl}) - Var(E_{tijkkl})}, \quad \forall k > 2,$$

$$\xi CON(\tau_{rtij2l}) = \frac{(\lambda_{\xi_{ij2l}})^2 Var(\xi_{tij})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

Note that the trait consistency coefficients are only shown for the true-score variables belonging to the non-reference methods. The square root of these true trait (ξ) consistency coefficients [$\sqrt{\xi CON(\cdot)}$] can be interpreted as degree of true convergent validity on trait level. In a similar way, the true occasion-specific (ζ) consistency coefficients are defined as follows:

$$\zeta CON(\tau_{tijkkl}) = \frac{(\lambda_{\zeta_{ijkl}})^2 Var(\zeta_{tijl})}{Var(Y_{tijkkl}) - Var(E_{tijkkl})}, \quad \forall k > 2,$$

$$\zeta CON(\tau_{rtij2l}) = \frac{(\lambda_{\zeta_{ij2l}})^2 Var(\zeta_{tijl})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

The square root of these true occasion-specific (ζ) consistency coefficients [$\sqrt{\zeta CON(\cdot)}$] represent the degree of true convergent validity on state level. The true trait (or occasion-specific) method specificity coefficients represent the true (measurement error free) stable (or momentary) amount of method bias. These coefficients are defined by:

$$\xi MS(\tau_{tijkkl}) = \frac{(\lambda_{\xi_{ijkl}}^M)^2 Var(\xi_{tijk}^M)}{Var(Y_{tijkkl}) - Var(E_{tijkkl})}, \quad \forall k > 2,$$

$$\xi CMS(\tau_{rtij2l}) = \frac{(\lambda_{\xi_{ij2l}}^{CM})^2 Var(\xi_{tij2}^{CM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})},$$

$$\xi UMS(\tau_{rtij2l}) = \frac{(\lambda_{\xi_{ij2l}}^{UM})^2 Var(\xi_{rtij2}^{UM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})},$$

$$\zeta MS(\tau_{tijkkl}) = \frac{(\lambda_{\zeta_{ijkl}}^M)^2 Var(\zeta_{tijk}^M)}{Var(Y_{tijkkl}) - Var(E_{tijkkl})}, \quad \forall k > 2,$$

$$\zeta CMS(\tau_{rtij2l}) = \frac{(\lambda_{\zeta_{ij2l}}^{CM})^2 Var(\zeta_{tj2l}^{CM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})},$$

$$\zeta UMS(\tau_{rtij2l}) = \frac{(\lambda_{\zeta_{ij2l}}^{UM})^2 Var(\zeta_{rtj2l}^{UM})}{Var(Y_{rtij2l}) - Var(E_{rtij2l})}.$$

$\xi MS(\tau_{tijkkl})$ represents the proportion of true-score variance of Y_{tijkkl} that is determined by consistent method-specific influences due to the non-reference method k . For example, with respect to this coefficient it is possible to investigate the proportion of true variance that is due to consistent (stable) over- or underestimations of the non-reference structurally different method (e.g.,

supervisor) with respect to the reference method (e.g., employee's self-report). $\xi CMS(\tau_{rtij2l})$ reflects the proportion of true-score variance of an observed variable Y_{rtij2l} that is determined by consistent method-specific influences common to the non-reference interchangeable methods $k = 2$. In empirical applications, $\xi CMS(\tau_{rtij2l})$ represents the amount of true consistent (stable) over- or underestimations of the general view of the interchangeable raters (e.g., colleagues) with respect to the target's self-report (reference method). In contrast, $\xi UMS(\tau_{rtij2l})$ denotes the proportion of true-score variance that is due to stable single rater-specific influences. That means that $\xi UMS(\tau_{rtij2l})$ represents the amount of true and consistent over- or underestimation of a particular rater (e.g., colleague A) with respect to the general and consistent view of all interchangeable raters (e.g., all colleagues for the particular target). This rater influence is unique (specific) to a particular rater, thus not shared with other raters. The coefficients $\zeta MS(\tau_{tijk})$, $\zeta CMS(\tau_{rtij2l})$, and $\zeta UMS(\tau_{rtij2l})$ can be calculated in an analogous way. These coefficients reflect pure measurement-error free occasion-specific method bias. Finally, the reliability coefficients of the observed variables are defined as follows:

$$\begin{aligned} Rel(Y_{tij1l}) &= 1 - \frac{Var(E_{tij1l})}{Var(Y_{tij1l})}, \\ Rel(Y_{tijk}) &= 1 - \frac{Var(E_{tijk})}{Var(Y_{tijk})}, & \forall k > 2, \\ Rel(Y_{rtij2l}) &= 1 - \frac{Var(E_{rtij2l})}{Var(Y_{rtij2l})}. \end{aligned}$$

Consequently, the unreliability coefficients are given by:

$$\begin{aligned} Unrel(Y_{tij1l}) &= \frac{Var(E_{tij1l})}{Var(Y_{tij1l})}, \\ Unrel(Y_{tijk}) &= \frac{Var(E_{tijk})}{Var(Y_{tijk})}, & \forall k > 2, \\ Unrel(Y_{rtij2l}) &= \frac{Var(E_{rtij2l})}{Var(Y_{rtij2l})}. \end{aligned}$$

4.8 Mean structure

This section concerns the latent variable mean structure of the LST-COM model. The following theorem shows the consequence of the model definition for the observed and latent variables.

Theorem 4.6 (Mean structure)

If $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ is a LST-COM model of $(\xi_{tij}, \xi_{tij2}^{UM}, \xi_{tij2}^{CM}, \xi_{tijk}^M, \zeta_{tijl}, \zeta_{rtij2l}^{UM}, \zeta_{rtij2l}^{CM}, \zeta_{tijk}^M)$ -congeneric variables and without loss of generality, $k = 1$ method is chosen as reference method, then the following mean structure holds for all $r \in R \equiv \{1, \dots, a\}$, $t \in T \equiv \{1, \dots, b\}$, $i \in I \equiv \{1, \dots, c\}$, $j \in J \equiv \{1, \dots, d\}$, $k \in K \equiv \{1, \dots, e\}$, $l \in L \equiv \{1, \dots, f\}$:

$$E(Y_{tijkkl}) = \alpha_{tijkkl} + \lambda_{\xi_{ijkl}} E(\xi_{tij}), \quad \forall k > 2, \quad (4.198)$$

$$E(Y_{rtij2l}) = \alpha_{rtij2l} + \lambda_{\xi_{ij2l}} E(\xi_{tij}). \quad (4.199)$$

$$E(Y_{tij1l}) = E(\xi_{tij}), \quad (4.200)$$

$$E(\zeta_{tijl}) = 0, \quad (4.201)$$

$$E(\xi_{rtij2}^{UM}) = 0, \quad (4.202)$$

$$E(\xi_{tij2}^{CM}) = 0, \quad (4.203)$$

$$E(\xi_{tijk}^M) = 0, \quad \forall k > 2, \quad (4.204)$$

$$E(\zeta_{rtj2l}^{UM}) = 0, \quad (4.205)$$

$$E(\zeta_{tj2l}^{CM}) = 0, \quad (4.206)$$

$$E(\zeta_{tjkl}^M) = 0, \quad \forall k > 2, \quad (4.207)$$

$$E(E_{tijkkl}) = 0, \quad \forall k \neq 2, \quad (4.208)$$

$$E(E_{rtij2l}) = 0, \quad (4.209)$$

where $E(\cdot)$ denotes expected value.

Proofs. 17 Mean structure

According to Equation 4.192, the observed variable Y_{tij1l} measured by the reference method is decomposed into:

$$Y_{tij1l} = \xi_{tij} + \zeta_{tijl} + E_{tij1l}.$$

The expected value of Y_{tij1l} is

$$E(Y_{tij1l}) = E(\xi_{tij}) + E(\zeta_{tijl}) + E(E_{tij1l}).$$

According to the Equations 4.201 and 4.208 in the above Theorem 4.6, it follows:

$$E(Y_{tij1l}) = E(\xi_{tij}).$$

Equations 4.201 and 4.208 in the above Theorem 4.6 state that the latent state-residual variables ζ_{tijl} as well as the measurement error variables E_{tij1l} are defined as residuals. As a consequence of this definition, it follows directly that these variables have expectations of zero (Steyer, 1989; Steyer & Eid, 2001). Similarly, according to Equation 4.193, the observed variable Y_{tijkkl} is decomposed into:

$$Y_{tijkkl} = \alpha_{tijkkl} + \lambda_{\xi_{ijkl}} \xi_{tij} + \lambda_{\xi_{ijk}^M} \xi_{tijk}^M + \lambda_{\zeta_{ijkl}} \zeta_{tijl} + \lambda_{\zeta_{ijk}^M} \zeta_{tijk}^M + E_{tijkkl}.$$

The expected value of Y_{tijkkl} is

$$\begin{aligned} E(Y_{tijkkl}) &= E(\alpha_{tijkkl}) + E(\lambda_{\xi_{ijkl}} \xi_{tij}) + E(\lambda_{\xi_{ijk}^M} \xi_{tijk}^M) \\ &\quad + E(\lambda_{\zeta_{ijkl}} \zeta_{tijl}) + E(\lambda_{\zeta_{ijk}^M} \zeta_{tijk}^M) + E(E_{tijkkl}). \end{aligned}$$

According to the Equations 4.201, 4.204, 4.207, and 4.208 of the above Theorem 4.6, it follows that the expected values of the residual variables ζ_{tijl} , ξ_{tijk}^M , ζ_{tijk}^M and E_{tijkkl} are zero. Thus, the above equation simplifies to (see Equation 4.198):

$$E(Y_{tijkkl}) = \alpha_{tijkkl} + \lambda_{\xi_{ijkl}} E(\xi_{tij}).$$

Equations 4.201, 4.204, 4.207 and 4.208 follow by definition, given that ζ_{tijl} , ξ_{tijk}^M , ζ_{tijk}^M and E_{tijkkl} are defined as residuals (Steyer, 1989; Steyer & Eid, 2001). In a similar way, according to Equation 4.194 the observed variable Y_{rtij2l} is

$$\begin{aligned} Y_{rtij2l} &= \alpha_{rtij2l} + \lambda_{\xi_{ij2l}} \xi_{tij} + \lambda_{\xi_{ij2l}^{CM}} \xi_{tij2}^{CM} + \lambda_{\xi_{ij2l}^{UM}} \xi_{rtij2}^{UM} \\ &\quad + \lambda_{\zeta_{ij2l}} \zeta_{tijl} + \lambda_{\zeta_{ij2l}^{CM}} \zeta_{tj2l}^{CM} + \lambda_{\zeta_{ij2l}^{UM}} \zeta_{rtj2l}^{UM} + E_{rtij2l}. \end{aligned}$$

Therefore, the expected value of Y_{rtij2l} is

$$E(Y_{rtij2l}) = E(\alpha_{tij2l}) + E(\lambda_{\xi_{ij2l}} \xi_{tij}) + E(\lambda_{\xi_{ij2l}}^{CM} \xi_{tij}^{CM}) + E(\lambda_{\xi_{ij2l}}^{UM} \xi_{rtij2l}^{UM}) + E(\lambda_{\zeta_{ij2l}} \zeta_{tijl}) + E(\lambda_{\zeta_{ij2l}}^{CM} \zeta_{tijl}^{CM}) + E(\lambda_{\zeta_{ij2l}}^{UM} \zeta_{rtj2l}^{UM}) + E(E_{rtij2l}).$$

Again, the expected values of the latent variables ξ_{tij2}^{CM} , ξ_{rtij2}^{UM} , ζ_{tijl} , ζ_{tj2l}^{CM} , ζ_{rtj2l}^{UM} , and E_{rtij2l} are zero with respect to the above theorem, then the Equation simplifies to (see Equation 4.199):

$$E(Y_{rtij2l}) = \alpha_{tij2l} + \lambda_{\xi_{ij2l}} E(\xi_{tij}).$$

Again, Equations 4.201, 4.203, 4.202, 4.206, 4.205 and 4.209 follow by definition, given that the latent variables ξ_{tij2}^{CM} , ξ_{rtij2}^{UM} , ζ_{tijl} , ζ_{tj2l}^{CM} , ζ_{rtj2l}^{UM} , and E_{rtij2l} are defined as residuals, and residuals have always an expected value of zero.

□

Remarks. Equations 4.198 and 4.199 clarify that the means of the observed variables are equal to $\alpha_{tijkl} + \lambda_{\xi_{ijkl}} E(\xi_{tij})$ and $\alpha_{tij2l} + \lambda_{\xi_{ij2l}} E(\xi_{tij})$, respectively. According to Equation 4.200, the mean of the latent trait variable is identical to the mean of the indicator pertaining to the reference method. Equations 4.201 to 4.207 reveal the latent state residuals as well as the trait-specific and state-specific method factors are defined as residuals and therefore have an expected value of zero. The same holds for the measurement error variables (see Equation 4.208 and 4.209).

4.9 Identifiability

According to the following theorem, the parameter of the LST-COM model are uniquely identified for at least two indicators, two traits, two methods, and three occasions of measurement (i.e., $2 \times 2 \times 2 \times 3$ measurement design). Again, the between covariance matrix of any LST-COM model is identical to a restricted covariance matrix of the MM-LST (Multitrait-Multimethod latent state-trait model by Courvoisier (2006) for the same dimension, and for which the measurement error variances of the observed variables pertaining to the second method Y_{tj2l} have been fixed to zero. Hence, the between variance-covariance matrix of the LST-COM model is a special case of the variance-covariance matrix of the MM-LST model by Courvoisier (2006). The minimal condition of parameter identification with respect to a MM-LST model is a $2 \times 2 \times 2 \times 3$ measurement design. Thus, the parameter of a LST-COM model are also uniquely identified for a $2 \times 2 \times 2 \times 3$ measurement design, if and only if the indicator specific latent state (method) variables on the rater- and target-level are correlated. In cases of two indicators, two constructs, two sets of methods, and two occasions of measurement ($2 \times 2 \times 2 \times 2$ measurement design) the model is not identified without further restrictions. As Courvoisier (2006) pointed out this model would be only identified if the factor loading parameters of the latent trait variables are fixed to one, and the latent state variables are homogeneous across items and substantially correlated.

Theorem 4.7 (Identification of the LST-COM covariance structure)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ be a LST-COM model of $(\xi_{tij}, \xi_{tij2}^{UM}, \xi_{tij2}^{CM}, \xi_{tij2}^M, \zeta_{tijl}, \zeta_{rtj2l}^{UM}, \zeta_{tj2l}^{CM}, \zeta_{tjkl}^M)$ -congeneric variables with conditional regressive independence, then the parameters of the matrices $\mu_{\xi B}$, $\Lambda_{\xi W}$, $\Phi_{\xi W}$, $\Lambda_{\zeta W}$, $\Phi_{\zeta W}$, $\sum_{\theta W}$, $\Lambda_{\xi B}$, $\Phi_{\xi B}$, $\Lambda_{\zeta B}$, $\Phi_{\zeta B}$, $\sum_{\theta W}$ are identified, if either one factor loading $\lambda_{\xi_{ijkl}}^M, \lambda_{\xi_{ij2l}}^{CM}, \lambda_{\xi_{ij2l}}^{UM}, \lambda_{\zeta_{ijkl}}^M, \lambda_{\zeta_{ij2l}}^{CM}, \lambda_{\zeta_{ij2l}}^{UM}$ for each factor $\xi_{tij}, \xi_{tij2}^{UM}, \xi_{rtij2}^{UM}, \zeta_{tijl}, \zeta_{tj2l}^{CM}, \zeta_{rtj2l}^{UM}$ or the variance of these factors are set to any real value larger than 0, and

- (a) iff $i = 2, j \geq 2, k \geq 2, l \geq 3$ and $\Phi_{\xi\mathbf{W}}, \Phi_{\zeta\mathbf{W}}, \Phi_{\xi\mathbf{B}}, \Phi_{\zeta\mathbf{B}}$ contain permissible intercorrelations among the latent variables (i.e., nonzero elements in the off-diagonal), otherwise
- (b) iff $i \geq 3, j \geq 1, k \geq 3, l \geq 3$.

Remarks. According to the above Theorem 4.7 the LST-COM model parameter are uniquely identified for the minimal condition of two indicators, two constructs, two sets of methods (one structurally different and one set of interchangeable methods) and three occasions of measurement. Given that the between covariance matrix $\Sigma_{\mathbf{B}}$ of the LST-COM can be seen as restrictive variant of the total covariance matrix of the MM-LST model by Courvoisier (2006), the model identification for the parameter with respect to the between covariance matrix $\Sigma_{\mathbf{B}}$ is shown by Courvoisier (2006, chapter 5.4.11). The identification for the parameters of the within covariance matrix $\Sigma_{\mathbf{W}}$ is demonstrated for the case of a $2 \times 2 \times 2 \times 3$ measurement design.

Proofs. 18 (Identification of $\Sigma_{\mathbf{W}}$) The following proofs concern the identification of the parameters of the within (rater-level) variance-covariance matrix $\Sigma_{\mathbf{W}}$. The identification of the LST-COM model parameters on the target-level is demonstrated by Courvoisier (2006). Therefore, it will be assumed that these parameters are known throughout the subsequent proofs. Note that the parameters measured on the target-level will be used for identification without replacing them by parameters of the observed variables. Moreover, parameters that are identified in previous identification steps will also not be replaced by parameters of the observed variables. As starting point for the identification of the within variance-covariance matrix $\Sigma_{\mathbf{W}}$, the measurement equation of any observed variable pertaining to the interchangeable method is considered:

$$\begin{aligned} Y_{rtij2l} = & \alpha_{tij2l} + \lambda_{\xi ij2l} \xi_{tij} + \lambda_{\xi ij2l}^{CM} \xi_{tij}^{CM} + \lambda_{\xi ij2l}^{UM} \xi_{rtij2}^{UM} + \\ & \lambda_{\zeta ij2l} \zeta_{tijl} + \lambda_{\zeta ij2l}^{CM} \zeta_{tj2l}^{CM} + \lambda_{\zeta ij2l}^{UM} \zeta_{rtj2l}^{UM} + \\ & E_{rtij2l}. \end{aligned}$$

For the subsequent proofs the zero-covariances among the latent variables of the LST-COM are used (see Theorem 4.5).

Identification of $\lambda_{\xi ij2l}^{UM}$:

For any observed variables pertaining to the set of non-reference (interchangeable) methods Y_{rtij2l} measured on the rater-level:

$$\begin{aligned} Cov(Y_{rtij21}, Y_{rtij2l'}) &= \lambda_{\xi ij21}^{UM} \lambda_{\xi ij2l'}^{UM} Var(\xi_{rtij2}^{UM}), \\ Cov(Y_{rtij2l}, Y_{rtij2l'}) &= \lambda_{\xi ij2l}^{UM} \lambda_{\xi ij2l'}^{UM} Var(\xi_{rtij2}^{UM}). \end{aligned}$$

According to Theorem 4.7 the first factor loadings of any latent factor in the LST-COM model is set to one for identification purposes (see also Bollen, 1989, 2002). Hence, $\lambda_{\xi ij21}^{UM} = 1$. Substituting this parameter value in the equations above as well as setting above equations equal, yields:

$$Cov(Y_{rtij21}, Y_{rtij2l'}) = \frac{Cov(Y_{rtij2l}, Y_{rtij2l'})}{\lambda_{\xi ij2l}^{UM}}.$$

The equation above can be reformulated as follows:

$$\lambda_{\xi ij2l}^{UM} = \frac{Cov(Y_{rtij2l}, Y_{rtij2l'})}{Cov(Y_{rtij21}, Y_{rtij2l'})}.$$

Identification of $Var(\xi_{rtij2}^{UM})$:

Given that, $\lambda_{\xi ij2l}^{UM} = \frac{Cov(Y_{rtij2l}, Y_{rtij2l'})}{Cov(Y_{rtij21}, Y_{rtij2l'})}$ and $\lambda_{\xi ij21}^{UM} = 1$ (see Theorem 4.7), it follows from that:

$$\begin{aligned} Cov(Y_{rtij21}, Y_{rtij2l}) &= \lambda_{\xi ij21}^{UM} \lambda_{\xi ij2l}^{UM} Var(\xi_{rtij2}^{UM}) \\ &= \frac{Cov(Y_{rtij2l}, Y_{rtij2l'})}{Cov(Y_{rtij21}, Y_{rtij2l'})} Var(\xi_{rtij2}^{UM}). \end{aligned}$$

Consequently,

$$\text{Var}(\xi_{rtij2}^{UM}) = \frac{\text{Cov}(Y_{rtij21}, Y_{rtij2l})\text{Cov}(Y_{rtij21}, Y_{rtij2l'})}{\text{Cov}(Y_{rtij2l}, Y_{rtij2l'})}.$$

Identification of $\text{Cov}(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM})$, where $(i, j) \neq (i', j')$:

Because of the zero-covariances between latent state-residual unique method variables ζ_{rtij2l}^{UM} and $\zeta_{rti'j'2l'}$ for all $(j, l) \neq (j', l')$, it follows that

$$\text{Cov}(Y_{rtij21}, Y_{rti'j'2l'}) = \lambda_{\xi ij21}^{UM} \lambda_{\xi i'j'2l'}^{UM} \text{Cov}(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM}).$$

Given that, $\lambda_{\xi j2l'}^{UM} = \frac{\text{Cov}(Y_{rtij2l'}, Y_{rtij2l''})}{\text{Cov}(Y_{rtij21}, Y_{rtij2l''})}$ and $\lambda_{\xi ij21}^{UM} = 1$ (see Theorem 4.7), it follows from that

$$\text{Cov}(\xi_{rtij2}^{UM}, \xi_{rti'j'2}^{UM}) = \frac{\text{Cov}(Y_{rtij21}, Y_{rti'j'2l'})\text{Cov}(Y_{rtij21}, Y_{rtij2l''})}{\text{Cov}(Y_{rtij2l'}, Y_{rtij2l''})}.$$

Identification of $\text{Cov}(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l}^{UM})$:

Given that $\lambda_{\zeta 1j2l}^{UM} = 1$, $\lambda_{\zeta 1j'2l}^{UM} = 1$, $\lambda_{\xi 1j2l}^{UM} = 1$, $\lambda_{\xi 1j'2l'}^{UM} = 1$ (see Theorem 4.7) and given that the covariance $\text{Cov}(\xi_{rt1j2}^{UM}, \xi_{rt1j'2}^{UM})$ has been identified (see previous steps), it follows from that:

$$\text{Cov}(Y_{rt1j2l}, Y_{rt1j'2l}) = \lambda_{\zeta 1j2l}^{UM} \lambda_{\zeta 1j'2l}^{UM} \text{Cov}(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l}^{UM}) + \lambda_{\xi 1j2l}^{UM} \lambda_{\xi 1j'2l'}^{UM} \text{Cov}(\xi_{rt1j2}^{UM}, \xi_{rt1j'2}^{UM}).$$

Rearrangement of the equation above, yields:

$$\text{Cov}(\zeta_{rtj2l}^{UM}, \zeta_{rtj'2l}^{UM}) = \text{Cov}(Y_{rt1j2l}, Y_{rt1j'2l}) - \text{Cov}(\xi_{rt1j2}^{UM}, \xi_{rt1j'2}^{UM}).$$

Identification of $\lambda_{\zeta ij2l}^{UM}$:

For two observed variables Y_{rt112l} and Y_{rtij2l} measured on the rater-level, it follows that:

$$\text{Cov}(Y_{rt112l}, Y_{rtij2l}) = \lambda_{\zeta 112l}^{UM} \lambda_{\zeta ij2l}^{UM} \text{Cov}(\zeta_{rt12l}^{UM}, \zeta_{rtj2l}^{UM}) + \lambda_{\xi 112l}^{UM} \lambda_{\xi ij2l}^{UM} \text{Cov}(\xi_{rt112}^{UM}, \xi_{rtij2}^{UM}).$$

Given that all parameters of the above equation are known, with the exception of $\lambda_{\zeta ij2l}^{UM}$, it follows that:

$$\begin{aligned} \text{Cov}(Y_{rt112l}, Y_{rtij2l}) &= \lambda_{\zeta ij2l}^{UM} \text{Cov}(\zeta_{rt12l}^{UM}, \zeta_{rtj2l}^{UM}) + \lambda_{\xi ij2l}^{UM} \text{Cov}(\xi_{rt112}^{UM}, \xi_{rtij2}^{UM}), \\ \lambda_{\zeta ij2l}^{UM} &= \frac{\text{Cov}(Y_{rt112l}, Y_{rtij2l}) - \lambda_{\xi ij2l}^{UM} \text{Cov}(\xi_{rt112}^{UM}, \xi_{rtij2}^{UM})}{\text{Cov}(\zeta_{rt12l}^{UM}, \zeta_{rtj2l}^{UM})}. \end{aligned}$$

Note that the parameters $\lambda_{\xi ij2l}^{UM}$, $\text{Cov}(\xi_{rt112}^{UM}, \xi_{rtij2}^{UM})$, and $\text{Cov}(\zeta_{rt12l}^{UM}, \zeta_{rtj2l}^{UM})$ have been already identified in the previous steps.

Identification of $\text{Var}(\zeta_{rtj2l}^{UM})$:

For two observed variables Y_{rt1j2l} and Y_{rtij2l} measured on the rater-level, it follows that:

$$\text{Cov}(Y_{rt1j2l}, Y_{rtij2l}) = \lambda_{\zeta 1j2l}^{UM} \lambda_{\zeta ij2l}^{UM} \text{Var}(\zeta_{rtj2l}^{UM}) + \lambda_{\xi 1j2l}^{UM} \lambda_{\xi ij2l}^{UM} \text{Cov}(\xi_{rt1j2}^{UM}, \xi_{rtij2}^{UM}).$$

Rearrangement of the above equation yields,

$$\text{Var}(\zeta_{rtj2l}^{UM}) = \frac{\text{Cov}(Y_{rt1j2l}, Y_{rtij2l}) - \lambda_{\xi ij2l}^{UM} \text{Cov}(\xi_{rt1j2}^{UM}, \xi_{rtij2}^{UM})}{\lambda_{\zeta ij2l}^{UM}}.$$

Note that the parameters $\lambda_{\xi ij2l}^{UM}$, $\text{Cov}(\xi_{rt1j2}^{UM}, \xi_{rtij2}^{UM})$, and $\lambda_{\zeta ij2l}^{UM}$ have been already identified in the previous steps.

Identification of $\text{Var}(E_{rtij2l})$:

For any observed variable Y_{rtij2l} ,

$$\begin{aligned} \text{Var}(Y_{rtij2l}) &= (\lambda_{\xi ij2l})^2 \text{Var}(\xi_{tij}) + (\lambda_{\xi ij2l}^{CM})^2 \text{Var}(\xi_{tij}^{CM}) + \\ &\quad (\lambda_{\xi ij2l}^{UM})^2 \text{Var}(\xi_{rtij2}^{UM}) + (\lambda_{\zeta ij2l})^2 \text{Var}(\zeta_{tijl}) + \\ &\quad (\lambda_{\zeta ij2l}^{CM})^2 \text{Var}(\zeta_{tj2l}^{CM}) + (\lambda_{\zeta ij2l}^{UM})^2 \text{Var}(\zeta_{rtj2l}^{UM}) + \\ &\quad \text{Var}(E_{rtij2l}). \end{aligned}$$

Therefore, $\text{Var}(E_{rtij2l})$ is identified by:

$$\begin{aligned} \text{Var}(E_{rtij2l}) = & \text{Var}(Y_{rtij2l}) - (\lambda_{\xi_{ij2l}})^2 \text{Var}(\xi_{tij}) - (\lambda_{\xi_{ij2l}}^{CM})^2 \text{Var}(\xi_{tij2}^{CM}) - \\ & (\lambda_{\xi_{ij2l}}^{UM})^2 \text{Var}(\xi_{rtij2}^{UM}) - (\lambda_{\zeta_{ij2l}})^2 \text{Var}(\zeta_{tijl}) - \\ & (\lambda_{\zeta_{ij2l}}^{CM})^2 \text{Var}(\zeta_{tj2l}^{CM}) - (\lambda_{\zeta_{ij2l}}^{UM})^2 \text{Var}(\zeta_{rtj2l}^{UM}), \end{aligned}$$

given that all other parameters are identified. □

4.10 Measurement Invariance

Testing measurement invariance is crucial when fitting LST models to empirical data (see Geiser, Keller, Lockhart, Eid, et al., 2012). In the next theorem, a LST-COM model with conditional regressive independence (RCI) and strong measurement invariance (MI) is therefore defined.

Definition 4.5 (LST-COM model with RCI and strong MI)

$\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ is called a LST-COM model of $(\xi_{tij}, \xi_{tij2}^{CM}, \xi_{rtij2}^{UM}, \xi_{tijk}^M, \zeta_{tijl}, \zeta_{tj2l}^{CM}, \zeta_{rtj2l}^{UM}, \zeta_{tjkl}^M)$ -congeneric variables with conditional regressive independence and with strong measurement invariance if and only if Definition 4.1, Theorem 4.2, Definition 4.3 hold and for each indicator i , construct j , method k and for each pair $(l, l') \in L \times L'$, $(l \neq l')$ there is a constant $\alpha_{tijk} \in \mathbb{R}$, a constant $\lambda_{\xi_{ijk}} \in \mathbb{R}_+$, a constant $\lambda_{\xi_{ij2}^{CM}} \in \mathbb{R}_+$, a constant $\lambda_{\xi_{ij2}^{UM}} \in \mathbb{R}_+$, a constant $\lambda_{\xi_{ijk}^M} \in \mathbb{R}_+$, a constant $\lambda_{\zeta_{ijk}} \in \mathbb{R}_+$, a constant $\lambda_{\zeta_{ij2}^{CM}} \in \mathbb{R}_+$, a constant $\lambda_{\zeta_{ij2}^{UM}} \in \mathbb{R}_+$, a constant $\lambda_{\zeta_{ijk}^M} \in \mathbb{R}_+$, such that

$$\alpha_{tijk} \equiv \alpha_{tijkl} = \alpha_{tijkl'}, \quad (4.210)$$

$$\lambda_{\xi_{ijk}} \equiv \lambda_{\xi_{ijkl}} = \lambda_{\xi_{ijkl'}}, \quad (4.211)$$

$$\lambda_{\xi_{ij2}^{CM}} \equiv \lambda_{\xi_{ij2l}^{CM}} = \lambda_{\xi_{ij2l'}^{CM}}, \quad (4.212)$$

$$\lambda_{\xi_{ij2}^{UM}} \equiv \lambda_{\xi_{ij2l}^{UM}} = \lambda_{\xi_{ij2l'}^{UM}}, \quad (4.213)$$

$$\lambda_{\xi_{ijk}^M} \equiv \lambda_{\xi_{ijkl}^M} = \lambda_{\xi_{ijkl'}^M}, \quad \forall k > 2, \quad (4.214)$$

$$\lambda_{\zeta_{ijk}} \equiv \lambda_{\zeta_{ijkl}} = \lambda_{\zeta_{ijkl'}}, \quad (4.215)$$

$$\lambda_{\zeta_{ij2}^{CM}} \equiv \lambda_{\zeta_{ij2l}^{CM}} = \lambda_{\zeta_{ij2l'}^{CM}}, \quad (4.216)$$

$$\lambda_{\zeta_{ij2}^{UM}} \equiv \lambda_{\zeta_{ij2l}^{UM}} = \lambda_{\zeta_{ij2l'}^{UM}}, \quad (4.217)$$

$$\lambda_{\zeta_{ijk}^M} \equiv \lambda_{\zeta_{ijkl}^M} = \lambda_{\zeta_{ijkl'}^M}, \quad \forall k > 2. \quad (4.218)$$

Box 4.1 (LST-COM Model)

ξ_{tijk1} target-specific latent trait variable of the (non-)reference (structurally different) method

ξ_{tij2l} rater-target-specific latent trait variable of the non-reference (interchangeable) method

ζ_{tijk1} target-specific latent state-residual variable of the (non-)reference (structurally different) method

ζ_{tij2l} target-specific latent state-residual variable of the non-reference (interchangeable) method

ξ_{rtij2l}^{UM} rater-target-specific latent trait unique method variable of the non-reference (interchangeable) method

ζ_{rtij2l}^{UM} rater-target-specific latent state-residual unique method variable of the non-reference (interchangeable) method

ξ_{tij2l}^{CM} target-specific latent trait common method variable of the non-reference (interchangeable) method

ζ_{tij2l}^{CM} target-specific latent state-residual common method variable of the non-reference (interchangeable) method

ξ_{tijk1}^M target-specific latent trait method variable of the non-reference (structurally different) method

ζ_{tijk1}^M target-specific latent state-residual method variable of the non-reference (structurally different) method

$$\xi_{tijk1} \equiv E(S_{tijk1}|p_T),$$

$$\zeta_{tijk1} \equiv S_{tijk1} - \xi_{tijk1},$$

$$\xi_{rtij2l}^{UM} \equiv E(UM_{rtij2l}|p_T, p_R),$$

$$\xi_{tij2l}^{CM} \equiv \xi_{tij2l} - E(\xi_{tij2l}|\xi_{tij1l}),$$

$$\xi_{tijk1}^M \equiv \xi_{tijk1} - E(\xi_{tijk1}|\xi_{tijk1l}),$$

$$\forall k > 2,$$

$$\zeta_{rtij2l}^{UM} \equiv UM_{rtij2l} - \xi_{rtij2l}^{UM},$$

$$\zeta_{tij2l}^{CM} \equiv \zeta_{tij2l} - E(\zeta_{tij2l}|\zeta_{tijk1l}),$$

$$\zeta_{tijk1}^M \equiv \zeta_{tijk1} - E(\zeta_{tijk1}|\zeta_{tijk1l}),$$

$$\forall k > 2.$$

Chapter 5

Formal definition of the latent growth curve (LGC-COM) model

5.1 A gentle introduction

Over the last decades, latent growth curve (LGC) models have been increasingly applied to social and behavioral data (Bollen & Curran, 2006; Ferrer, Balluerka, & Widaman, 2008; Hancock, Kuo, & Lawrence, 2001; McArdle & Epstein, 1987; McArdle, 1988; Meredith & Tisak, 1990). One of the main advantages of LGC models is that the shape of true intraindividual change can be directly modeled, rather than modeled indirectly, as for instance with respect to latent change models (Geiser, 2012). LGC models also allow relating true interindividual differences in intraindividual change to manifest or latent background variables (e.g., gender, age, treatment groups etc.) in order to explain interindividual differences in growth. Many researchers have noticed the methodological similarities between models for measuring change (LC models), growth (LGC models) and variability processes (LST models) in the past (Cole, Martin, & Steiger, 2005; Eid & Hoffmann, 1998; Geiser, 2012; Tisak & Tisak, 2000). However, the methodological link between LGC models and LST theory (Eid, 1995; Steyer et al., 1992) has just been recently formalized in a work by Geiser, Keller, and Lockhart (2012). In particular, Geiser, Keller, and Lockhart (2012) showed that first and second order LGC models represent a restrictive variant of LST change models, where the change of the latent trait variables is modeled by a linear or nonlinear function. Moreover, Geiser, Keller, and Lockhart (2012) showed analytically as well as empirically (with simulation studies) why second order LGC models often outperform first order LGC models. That is that in second order LGC models “true” change can be studied separately from “true” occasion-specific as well as measurement error influences, which is not possible with respect to first order LGC models (c.f Geiser, Keller, & Lockhart, 2012).

It is worth noting that the terms “first order LGC model” and “second order LGC model” can be quiet misleading. The reason for that is that first order LGC models are generally based on *one* single observed variable measured repeatedly over time (e.g., Hancock et al., 2001), whereas second order LGC models require *multiple* (at least two) observed variables measured repeatedly over time (e.g., Geiser, Keller, & Lockhart, 2012; Tisak & Tisak, 2000). Moreover, second order LGC models do not always have to be specified as second order CFA models. For example, in Figure 5.1, model F is graphically presented as first order CFA model. However, under specific conditions, model F

becomes equivalent to model D, a second order LGC models. Hence, in this work the distinction between single vs. multiple indicator LGC models is preferred rather than the general distinction between first vs. second order LGC models. Many researchers have emphasized the advantages of multiple indicator (or second order) LGC models (e.g., Chan, 1998; Geiser, Keller, & Lockhart, 2012; Ferrer et al., 2008; Leite, 2007; Murphy, Beretvas, & Pituch, 2011; von Oerzen, Hertzog, Lindenberger, & Ghisletta, 2010). According to Geiser, Keller, and Lockhart (2012, pp. 3-4) the main advantages of multiple indicator LGC models can be summarized as follows:

- Multiple indicator LGC models allow separating different variances components from one another. That is, multiple indicator LGC models allow separating measurement error variance from true change as well as reliable time-specific variance (see also Sayer & Cumsille, 2001).
- Multiple indicator LGC models allow testing crucial assumptions such as measurement invariance assumptions in longitudinal data analysis (see also Chan, 1998; Ferrer et al., 2008).
- Multiple indicator LGC models are more sensitive than single indicator LGC models for the investigation of individual differences in change (see also von Oerzen et al., 2010).
- Multiple indicator LGC models allow separating indicator-specific (or method) variance from construct variance.

In order to understand why multiple indicator LGC models often outperform single indicator LGC models, the methodological links between latent change (LC) models, latent state-trait (LST) models and latent growth curve (LGC) models are summarized again. The key steps for defining second order LGC models are depicted in Figure 5.1. Model A in Figure 5.1 represents a latent state model with strong measurement invariance, which is often used as baseline model. Strong measurement invariance requires equivalence restrictions on the intercepts α_{il} and factor loading parameters λ_{il} for each indicator belonging to different occasions of measurement (Meredith, 1993; Widaman & Reise, 1997). As already shown in Chapter 3, any latent state (LS) model with strong measurement invariance can be reparametrized into a latent change (LC) model. A data equivalent latent change model is given in B of Figure 5.1. Latent change models allow studying the true interindividual differences in intraindividual change with respect to the initial status. The formal tautological restatement of a latent state model into a latent baseline (BC) change model is given by the following equation:

$$S_2 = S_1 + (S_l - S_1) = S_1 + S_l^{BC}$$

Again, with respect the latent difference or change variables S_l^{BC} it is possible to study true interindividual differences in intraindividual change. The term “true” refers to the fact that the latent difference variables are free of measurement error influences. In order to define a latent growth curve model, it is necessary to assume that the latent change (difference) variables S_2^{BC} pertaining to different measurement occasions l and l' follow a particular linear or non-linear function. For example, if the true intraindividual change for each individual is assumed to be linear for each individual, the relationship between the latent change (difference) variables maybe rewritten as $S_l^{BC} = (l - 1) \cdot S_{(l-1)}^{BC}$. Hence, LGC models can also be seen as special case (or more

restrictive variant) of latent change (LC) models, where researchers specify a particular function for the true intraindividual change. The function of these individual growth curves can be linear ($l - 1$) or non-linear [e.g., quadratic $(l - 1)^2$ or cubic $(l - 1)^3$]. In order to define a second order LGC model, it is useful to consider the basic concept of LST theory. According to LST-theory (Eid, 1995; Steyer et al., 1992), latent state variables can be decomposed into a latent trait as well as a latent state-residual variable. Generally, this decomposition can be expressed as follows:

$$\begin{aligned} Y_{il} &= S_{il} + E_{il}, \\ S_{il} &= \xi_{il} + \zeta_{il}, \\ Y_{il} &= \xi_{il} + \zeta_{il} + E_{il}. \end{aligned}$$

The index i denotes the observed variables, whereas the index l refers to the occasion of measurement. Furthermore, ξ_{il} refers to the latent trait variables, ζ_{il} to the latent state-residual variables, and E_{il} the measurement error variables. The main advantage of LST models is that person-specific influences (ξ_{il}), occasion-specific or momentary (ζ_{il}) influences and measurement error influences (E_{il}) can be separated from one another. This is not possible with respect to latent state (LS) models, given that latent state variables in LS models consist of both stable as well as momentary influences. Of course, LST-models may be reformulated into latent state-trait change version:

$$\begin{aligned} Y_{il} &= S_{il} + E_{il}, \\ S_{il} &= \xi_{il} + \zeta_{il}, \\ \xi_{il} &= \xi_{i1} + (\xi_{il} - \xi_{i1}), \\ Y_{il} &= \xi_{i1} + (\xi_{il} - \xi_{i1}) + \zeta_{il} + E_{il}. \end{aligned}$$

With respect to latent state-trait change (LSTC) models it is possible to investigate true trait change with respect to initial trait, while accounting for occasion-specific and measurement error influences. In order to define a second order LGC model (that allows the separation of measurement error variance from true change and reliable time-specific variance), it is assumed that the trait change follows some linear or nonlinear function:

$$\begin{aligned} \xi_{il} &= \xi_{i1} + (\xi_{il} - \xi_{i1}), \\ (\xi_{il} - \xi_{i1}) &= (l - 1) \cdot (\xi_{i2} - \xi_{i1}), \\ Y_{il} &= \xi_{i1} + (l - 1) \cdot (\xi_{i2} - \xi_{i1}) + \zeta_{il} + E_{il}. \end{aligned}$$

The structural similarities of LGC and LST models become obvious, if ξ_{i1} is replaced by \mathcal{I}_{i1} (for intercept) and $(\xi_{i2} - \xi_{i1})$ is replaced by \mathcal{S}_{il} (for slope):

$$Y_{il} = \mathcal{I}_{i1} + (l - 1) \cdot \mathcal{S}_{il} + \zeta_{il} + E_{il}. \quad (5.1)$$

Again, the second order LGC model given in Equation 5.1 allows separating measurement error influence from true trait change and occasion-specific influences. In addition to that, it is possible

to test measurement invariance with χ^2 difference tests. In summary, the model above encompasses all advantages of a second order LGC model. Moreover, the model given in Equation 5.1 allows the specification of indicator-specific intercept and slope factors. This is not possible for the models C and D in Figure 5.1. In fact, these models implicitly make the rather restrictive assumption that the intercept and slope variables belonging to different indicators i and i' are perfectly correlated with each other. However, if these assumptions hold, one may derive a second order LGC model according to the following equations (see Geiser, Keller, & Lockhart, 2012):

$$\begin{aligned} Y_{il} &= \alpha_{il} + \lambda_{il}S_l + E_{il}, \\ S_l &= \xi_l + \zeta_l, \\ \xi_l &= \xi_1 + (\xi_l - \xi_1), \\ (\xi_l - \xi_1) &= (l - 1) \cdot (\xi_2 - \xi_1), \\ Y_{il} &= \alpha_{il} + \lambda_{il}[\xi_1 + (l - 1) \cdot (\xi_2 - \xi_1) + \zeta_l] + E_{il}. \end{aligned}$$

Therefore,

$$Y_{il} = \alpha_{il} + \lambda_{il}[\mathcal{I}_1 + (l - 1) \cdot \mathcal{S}_l + \zeta_l] + E_{il}. \quad (5.2)$$

Note that the index i has been dropped from the latent variables \mathcal{I}_1 , \mathcal{S}_l , and ζ_l in Equation 5.2 in order to express that these latent variables are unidimensional. The model given in Equation 5.2 is represented in model D of Figure 5.1. Again, it is important to note that the model F in Figure 5.1 is data equivalent to model D and therefore also implies that the latent intercept and slope variables pertaining to different indicators i and i' are linear functions of each other. Nevertheless, this restriction can be relaxed, which is not possible with respect to model D in Figure 5.1. In the following chapter, a latent growth curve model with indicator specific intercept and slope variables for longitudinal MTMM data combining structurally different and interchangeable methods is formally defined. This model will be called LGC-COM model and represents a restrictive variant of the LST-COM model discussed in Chapter 4.

Measurement invariance and other necessary restrictions

Again, measurement equivalence across time is a crucial prerequisite for the application of LGC models. If strong measurement invariance holds “true” interindividual differences in change can be investigated with respect to the same latent variables (i.e., ensuring no changes of the measurement with regard to the latent variables over time). The latent intercept variables in the LGC-COM model represent the “true” average of the measured attribute at the first occasion of measurement $l = 1$. The variance of the latent intercepts variables represents the amount of “true” interindividual differences with respect to the attribute measured at the first occasion of measurement $l = 1$. Analogously, the mean of the latent slope variables may be interpreted as general (average) growth of the measured attribute across different occasions of measurement. The variance of the latent slope variables indicate the degree of “true” interindividual differences

in intraindividual change of the measured attribute. Furthermore, the correlations between the latent intercept and slope variables may be investigated. In order to estimate the latent means of the intercept and slope variables, it is necessary to fix all intercepts of the observed variables α_{il} to zero and fix the latent factor parameters of the latent intercept variables $\lambda_{\mathcal{I}il}$ to one. Figure 5.1 shows a LGC-COM model with common latent intercept, slope, trait, and state (method) factors. Note that this figure does not incorporate indicator-specific latent variables for simplicity.

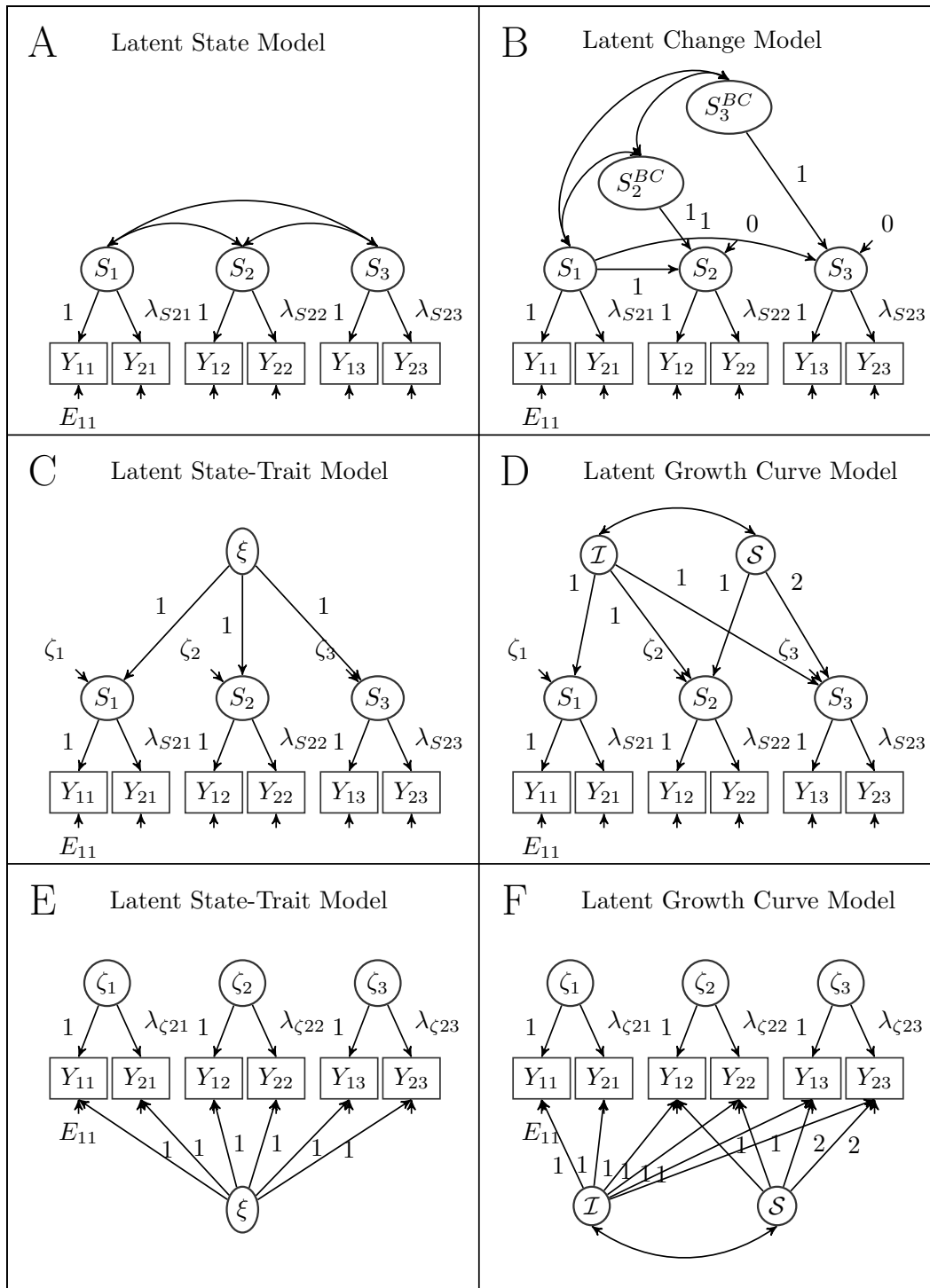


Figure 5.1: Possible ways of defining LGC models on the basis of multiple indicator LST and LC models. Y_{il} = observed variables (i = indicator, l = occasion of measurement). S_l = latent state variables, ξ = latent trait variable, ζ_l = latent state-residual variable, \mathcal{I} = latent intercept variable, \mathcal{S} = latent slope variable, E_{il} . For all model strong measurement invariance is assumed. Detailed explanations are given in the text.

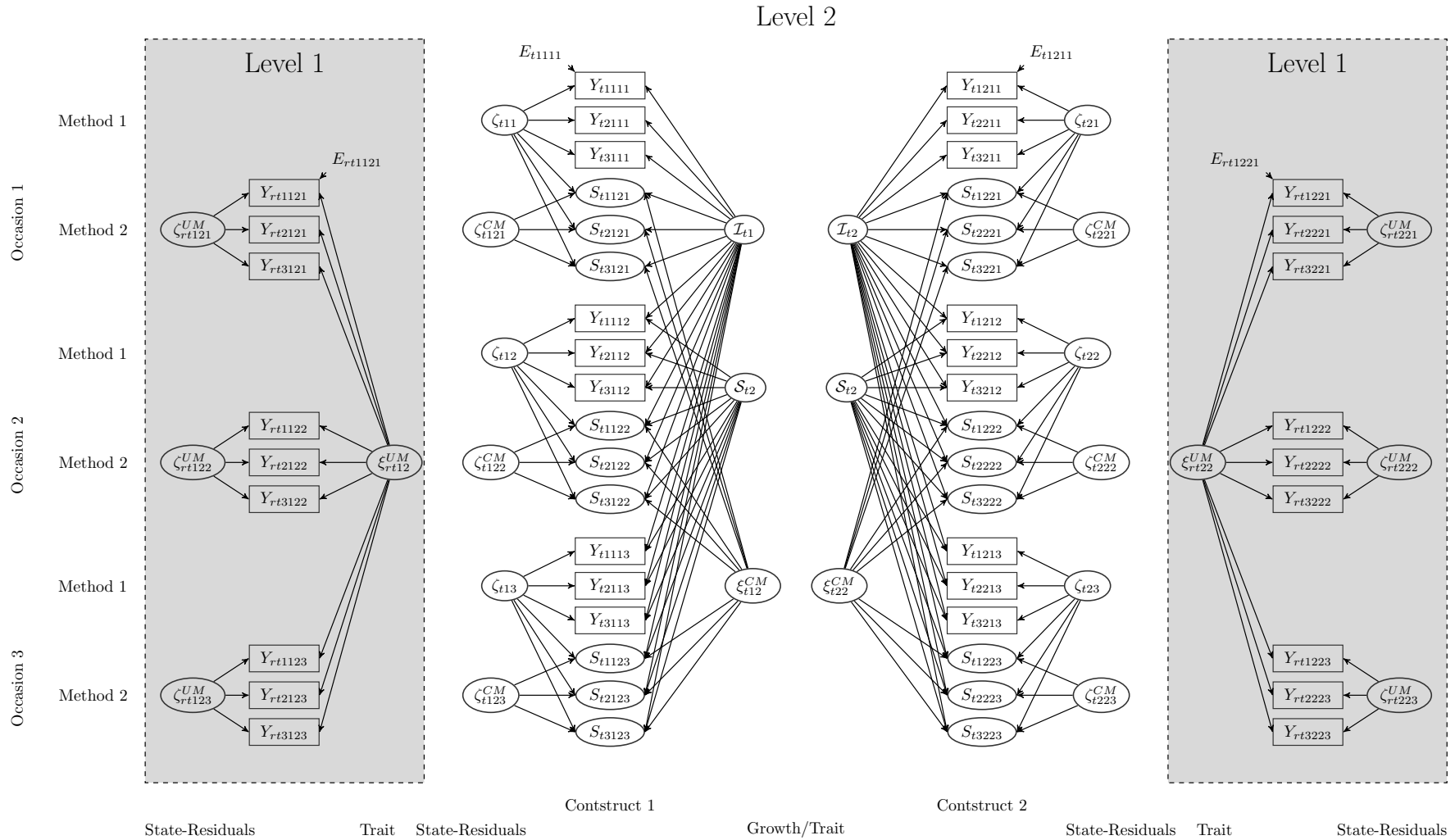


Figure 5.2: Path diagram of the LGC-COM model with common latent intercept, slope, trait, state (method) factors. The LGC-COM model with common latent factors incorporating three indicators ($i=1,2,3$), two constructs ($j=1,2$), two methods ($k=1,2$) and three occasions of measurement ($l=1,2,3$). For the sake of clarity, all latent variables of the model are represented by common latent factors. All factor loadings as well as correlations between latent variables were omitted for clarity. Measurement error variables E_{rtijkl} and E_{tijkl} are only depicted for the first indicator pertaining to method 1 and 2.

5.2 Definition of the LGC-COM model

Definition 5.1 (LGC-COM model)

The random variables $\{Y_{111111}, \dots, Y_{rtijkl}, \dots, Y_{abcdef}\}$ and $\{Y_{11111}, \dots, Y_{tijk}, \dots, Y_{bcdef}\}$ on a probability space $(\Omega, \mathfrak{A}, P)$ are variables of a LGC-COM model if and only if the conditions (a to f, except for c) of Definition 4.1 with conditional regressive independence (see Definition 4.3) and strong measurement invariance (see Definition 4.5) hold.

- (a) Then, without any loss of generality the latent trait variables ξ_{tij1l} pertaining to the reference method $k = 1$ and measurement occasion l , where $l > 0$ can be further decomposed into an initial trait variable ξ_{tij11} and a latent trait change variable $(\xi_{tij1l} - \xi_{tij11})$:

$$\xi_{tij1l} \equiv \xi_{tij11} + (\xi_{tij1l} - \xi_{tij11}), \quad (5.3)$$

which are also random variables on $(\Omega, \mathfrak{A}, P)$ with finite first- and second order moments.

- (b) For each indicator i , construct j , measured by method k and for each $l \in L$, where $l > 0$ there is a constant $\delta_{ijk(l-1)} = (l - 1)$, such that

$$(\xi_{tij1l} - \xi_{tij11}) \equiv \delta_{ijk(l-1)}(\xi_{tij12} - \xi_{tij11}), \quad (5.4)$$

and for all indicators pertaining to ξ_{tij1l} (see condition b in Definition 4.1), the intercepts α_{tijk} are constrained to zero and the factor loadings $\lambda_{\xi_{ijk}}$ are constrained to one.

- (c) Finally, let $\xi_{tij11} \equiv \mathcal{I}_{tij}$ and $(\xi_{tij12} - \xi_{tij11}) \equiv \mathcal{S}_{tij}$.

Remarks. According to the conditions made in Definition 5.1, it is clear that the LGC-COM model is defined as a restrictive variant of a LST-COM baseline change model. With respect to Equation 5.3 each latent trait variable can be decomposed into an initial latent trait variable ξ_{tij11} and a latent trait change or latent trait difference variable $(\xi_{tij1l} - \xi_{tij11})$. This tautological decomposition cannot be falsified empirically. With respect to Equation 5.4 it is assumed that any latent trait change variable $(\xi_{tij1l} - \xi_{tij11})$ is a linear function of $(\xi_{tij12} - \xi_{tij11})$. For example, the latent trait change from the first to the third measurement occasion is two times the latent trait change from the first to the second measurement occasion:

$$(\xi_{tij13} - \xi_{tij11}) = 2 \cdot (\xi_{tij12} - \xi_{tij11}).$$

For simplicity, it is assumed that the shape of the true intraindividual change $(\delta_{ijk(l-1)})$ is linear for the remaining chapter. However, non-linear (quadratic or cubic) growth functions may also be defined as follows:

$$\begin{aligned} (\xi_{tij1l} - \xi_{tij11}) &= \delta_{ijk(l-1)}(\xi_{tij12} - \xi_{tij11}) + \delta_{ijk(l-1)}^2(\xi_{tij12} - \xi_{tij11}), \\ (\xi_{tij1l} - \xi_{tij11}) &= \delta_{ijk(l-1)}(\xi_{tij12} - \xi_{tij11}) + \delta_{ijk(l-1)}^2(\xi_{tij12} - \xi_{tij11}) + \delta_{ijk(l-1)}^3(\xi_{tij12} - \xi_{tij11}). \end{aligned}$$

Moreover, it is important to constrain each intercept α_{tijk} to zero and each factor loading $\lambda_{\xi_{ijk}}$ to 1. As a direct consequence of these constraints, the latent regression simplifies as follows:

$$E(\xi_{tijk} | \xi_{tij1l}) = \alpha_{tijk} + \lambda_{\xi_{ijk}}[\xi_{tij11} + \delta_{ijk(l-1)}(\xi_{tij12} - \xi_{tij11})].$$

If $\alpha_{tijk} = 0$ and $\lambda_{\xi_{ijk}} = 1$ as well as $\xi_{tij11} \equiv \mathcal{I}_{tij}$ and $(\xi_{tij12} - \xi_{tij11}) \equiv \mathcal{S}_{tij}$, then

$$E(\xi_{tijk} | \xi_{tij1l}) = \mathcal{I}_{tij} + \delta_{ijk(l-1)}\mathcal{S}_{tij}.$$

Given that the LGC-COM model can be derived from the LST-COM baseline change model by imposing additional constraints with respect to the intercept and factor loadings, all psychometric statements with respect to existence, uniqueness, admissible transformations or meaningfulness follow directly from the definition of the LST-COM model.

5.3 Testability

As stated above, the LGC-COM model represents a restrictive variant of the LST-COM model with conditional regressive independence (CRI) and strong measurement invariance (MI) (see Definition 5.1). Therefore, the covariance structure of the LGC-COM model is almost equivalent to the covariance structure of the LST-COM model provided in Section 4.6.2. The only differences between the covariance structure of the LST-COM and the LGC-COM model refer to the additional restrictions made with respect to the covariance matrix $\sum_{\xi\mathbf{B}}$ in the LGC-COM model. The following theorem summarizes the additional restrictions of the LGC-COM model. Figure 5.3 illustrates the latent covariance matrix $\Phi_{\xi\mathbf{B}}$ of the LGC model. For simplicity, the following covariance matrix is solely discussed for LGC-COM models with two methods (i.e., a structurally different method $k = 1$ and a set of interchangeable methods $k = 2$). The extension of the LGC-COM models to three or more methods can be easily done as discussed in the previous chapters.

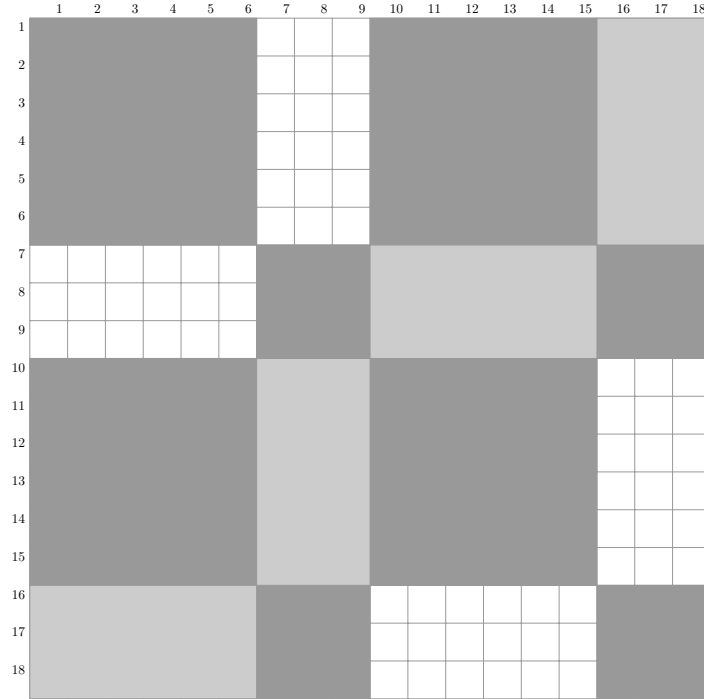


Figure 5.3: Between variance-covariance matrix $\Phi_{\xi\mathbf{B}}$ of the LGC-COM model, where $1=\mathcal{I}_{t11}$, $2=\mathcal{I}_{t21}$, $3=\mathcal{I}_{t31}$, $4=\mathcal{S}_{t11}$, $5=\mathcal{S}_{t21}$, $6=\mathcal{S}_{t31}$, $7=\xi_{t112}^{CM}$, $8=\xi_{t212}^{CM}$, $9=\xi_{t312}^{CM}$, $10=\mathcal{I}_{t12}$, $11=\mathcal{I}_{t22}$, $12=\mathcal{I}_{t32}$, $13=\mathcal{S}_{t12}$, $14=\mathcal{S}_{t22}$, $15=\mathcal{S}_{t32}$, $16=\xi_{t122}^{CM}$, $17=\xi_{t222}^{CM}$, $18=\xi_{t322}^{CM}$. Cells colored in white indicate zero correlations, cells colored in gray indicate permissible and interpretable correlations. Cells in light gray indicate correlations that should be fixed to zero for parsimony.

Theorem 5.1 (Covariance structure)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathcal{I}_t, \mathcal{S}_t, \xi_{\mathbf{rt}}^{\text{UM}}, \xi_{\mathbf{t}}^{\text{CM}}, \epsilon_{\mathbf{t}}^{\text{M}}, \zeta_{\mathbf{t}}, \zeta_{\mathbf{rt}}^{\text{UM}}, \zeta_{\mathbf{t}}^{\text{CM}}, \zeta_{\mathbf{t}}^{\text{M}}, \mathbf{E}_{\mathbf{rt}}, \mathbf{E}_{\mathbf{t}}, \delta, \lambda_{\xi}^{\text{UM}}, \lambda_{\xi}^{\text{CM}}, \lambda_{\xi}^{\text{M}}, \lambda_{\zeta}, \lambda_{\zeta}^{\text{UM}}, \lambda_{\zeta}^{\text{CM}}, \lambda_{\zeta}^{\text{M}} \rangle$ be a LGC-COM model of $(\xi_{tij2}^{\text{UM}}, \xi_{tij2}^{\text{CM}}, \xi_{tijk}^{\text{M}}, \zeta_{tijl}, \zeta_{rtj2l}^{\text{UM}}, \zeta_{tj2l}^{\text{CM}}, \zeta_{tjkl}^{\text{M}})$ -congeneric variables with conditional regressive independence and strong measurement invariance (see Definition 4.5). Then, with respect to a $3 \times 2 \times 2 \times 3$ measurement the total variance-covariance matrices of the LGC-COM model is equivalent to the variance-covariance matrices of the LST-COM model, except for the between trait matrix $\sum_{\xi\mathbf{B}}$, which is constraint such

that:

$$\sum_{\xi_{\mathbf{B}}} = \Lambda_{\xi_{\mathbf{B}}} \Phi_{\xi_{\mathbf{B}}} \Lambda_{\xi_{\mathbf{B}}}^{\mathbf{T}}.$$

$\Lambda_{\xi_{\mathbf{B}}}$ refers to the between factor loading matrix of size 36×18 , $\Lambda_{\xi_{\mathbf{B}}}^{\mathbf{T}}$ is the transposed matrix and $\Phi_{\xi_{\mathbf{B}}}$ refers to the between covariance matrix of size 18×18 . Furthermore, $\Lambda_{\xi_{\mathbf{B}}}$ is given by:

$$\Lambda_{\xi_{\mathbf{B}}} = \sum \mathbf{I}_{\Lambda_{\xi}}^{\mathbf{P}} \otimes \Lambda_{\xi_{\mathbf{B}}},$$

where \sum refers to the sum over all constructs j and measurement occasions l . $\mathbf{I}_{\Lambda_{\xi}}^{\mathbf{P}}$ refers to the contrast matrix of size 6×2 equivalent to the contrast matrix presented in Section 4.6.2, \otimes refers to the Kronecker product and $\Lambda_{\xi_{\mathbf{B}}}$ refers to the factor loading matrix of size 6×9 , which is given by

$$\Lambda_{\xi_{\mathbf{B}}} = \begin{pmatrix} 1 & 0 & 0 & \delta_{1j1(l-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta_{2j1(l-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \delta_{3j1(l-1)} & 0 & 0 & 0 \\ 1 & 0 & 0 & \delta_{1j2(l-1)} & 0 & 0 & \lambda_{\xi_{1j2l}}^{CM} & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta_{2j2(l-1)} & 0 & 0 & \lambda_{\xi_{2j2l}}^{CM} & 0 \\ 0 & 0 & 1 & 0 & 0 & \delta_{3j2(l-1)} & 0 & 0 & \lambda_{\xi_{3j2l}}^{CM} \end{pmatrix}.$$

The latent between variance-covariance matrix $\Phi_{\xi_{\mathbf{B}}}$ of size 18×18 is given by:

$$\Phi_{\xi_{\mathbf{B}}} = \mathbf{E} [(\mathbf{V}_{\Phi_{\xi_{\mathbf{B}}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\xi_{\mathbf{B}}}}])(\mathbf{V}_{\Phi_{\xi_{\mathbf{B}}}} - \mathbf{E}[\mathbf{V}_{\Phi_{\xi_{\mathbf{B}}}}])^{\mathbf{T}}],$$

where $\mathbf{V}_{\Phi_{\xi_{\mathbf{B}}}}$ refers to the vector of size 18×1 including all latent intercept, slope, and trait common method factors on the between level. Specifically, $\mathbf{V}_{\Phi_{\xi_{\mathbf{B}}}}$ is then given by:

$$\begin{pmatrix} \mathcal{I}_{t11}, \mathcal{I}_{t21}, \mathcal{I}_{t31}, \mathcal{S}_{t11}, \mathcal{S}_{t21}, \mathcal{S}_{t31}, \xi_{t112}^{CM}, \xi_{t212}^{CM}, \xi_{t312}^{CM}, \\ \mathcal{I}_{t12}, \mathcal{I}_{t22}, \mathcal{I}_{t32}, \mathcal{S}_{t12}, \mathcal{S}_{t22}, \mathcal{S}_{t32}, \xi_{t122}^{CM}, \xi_{t222}^{CM}, \xi_{t322}^{CM} \end{pmatrix}^{\mathbf{T}}.$$

Furthermore, $\forall r \in R, t \in T, i \in I, j \in J, k \in K$:

$$\text{Cov}(\mathcal{I}_{tij}, \xi_{tij2}^{CM}) = 0, \quad (5.5)$$

$$\text{Cov}(\mathcal{S}_{tij}, \xi_{tij2}^{CM}) = 0. \quad (5.6)$$

All other matrices of the LST-COM model (i.e., $\Lambda_{\xi_{\mathbf{W}}}$, $\Phi_{\xi_{\mathbf{W}}}$, $\Lambda_{\zeta_{\mathbf{W}}}$, $\sum_{\theta_{\mathbf{W}}}$, $\Lambda_{\zeta_{\mathbf{B}}}$, $\Phi_{\zeta_{\mathbf{B}}}$, $\sum_{\theta_{\mathbf{B}}}$) remain unchanged.

Remarks. According to the above theorem, the covariance structure of the LGC-COM model differs from the covariance structure of the LST-COM only with respect to the covariance matrix $\sum_{\xi_{\mathbf{B}}}$. Specifically, it is stated that the LST-COM model matrix of the loading parameters $\Lambda_{\xi_{\mathbf{B}}}$ is restricted in such a way that the latent trait factors ξ_{tij} are decomposed into a latent intercept factor \mathcal{I}_{t11} and a latent slope factor \mathcal{S}_{t11} . Note that the covariance matrix of the LGC-COM model is only represented for two method, not three. However, if another structurally different method $k = 3$ would be present, the covariance matrix of the LGC-COM model could be easily extended in a similar way as discussed in the previous chapters. The covariances (correlations) among the latent intercept and latent slope factors are permissible in the LGC-COM model. However, the covariances between the latent intercept and/or latent slope factors and the latent trait method factors ξ_{tij2}^{CM} pertaining to the same indicators and same constructs are not permissible (see Equation 5.5 and 5.6). In addition to that, the covariances between the latent intercept and slope factors and any other latent trait method factor ξ_{tijk}^M are zero as well. For parsimony, it is recommended to fix the covariances between the latent intercept and slope factors and the latent trait (common) method factors pertaining to different indicators i and i' as well as different constructs j and j' to zero as well.

Proofs. 19 *Testability: consequences of model definition*

5.5 Given that \mathcal{I}_{tij} is a function of ξ_{tij11} , and $Cov(\xi_{tij11}, \xi_{ti'j'2l}^{CM})$ is necessarily zero (see Proof 4.165), it follows that $Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM}) = 0$.

5.6 Similarly, given that \mathcal{S}_{tij} is a function of ξ_{tij11} , and $Cov(\xi_{tij11}, \xi_{ti'j'2l}^{CM})$ is necessarily zero (see Proof 4.165), it follows that $Cov(\mathcal{S}_{tij}, \xi_{ti'j'2}^{CM}) = 0$.

In a similar way, the proofs for $Cov(\mathcal{I}_{tij}, \xi_{ti'j'k}^M) = 0$ and $Cov(\mathcal{S}_{tij}, \xi_{ti'j'k'}^M) = 0$ can be shown, given that \mathcal{I}_{tij} as well as \mathcal{S}_{tij} are functions of ξ_{tij11} , and $Cov(\xi_{tij11}, \xi_{ti'j'k'}^M)$ is necessarily zero (see Proof 4.166).

5.3.1 Interpretation of non-zero covariances and correlations

In the following section, the permissible non-zero correlations in LGC-COM models are discussed.

1. The correlations $Cor(\mathcal{I}_{tij}, \mathcal{I}_{ti'j})$ between the latent intercept variables belonging to the same construct j , but different indicators i and i' can be interpreted as degree of homogeneity with respect to the indicators at the first measurement occasion (the initial status). If these correlations are close to one, a common intercept factor may be defined. The correlations between the latent intercept variables belonging to different constructs j and j' indicate discriminant validity with respect to the reference method at the first measurement occasion (initial status). Again, two different correlations can be distinguished: (A) The latent correlations $Cor(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'})$ between the latent intercept variables of the reference method belonging to the same indicator i across different constructs j and j' . And (B) the correlations $Cor(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'})$ between the latent intercept variables of the reference method belonging to different indicators i and i' as well as different constructs j and j' . Both correlations indicate the generalization of the true initial status as measured by the reference method across different items and/or constructs. A vivid example may be that the level of self-reported leadership quality measured on the first occasion of measurement may be significantly correlated with the level of self-reported communication skills on the first occasion of measurement. Thus, high leadership quality as measured by the reference method on the first occasion of measurement might be positively associated with high communication skills as measured by the reference method on the first occasion of measurement.
2. In a similar way, the correlations $Cor(\mathcal{S}_{tij}, \mathcal{S}_{ti'j})$ between latent slope variables belonging to the same trait j , but different indicators i and i' can be interpreted as degree of homogeneity with respect to the indicators. Again, correlations close to one indicate that a general slope factor may be specified instead for parsimony reasons. The correlations $Cor(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'})$ between latent slope variables belonging to different constructs j and j' indicate discriminant validity with respect to the reference method across different occasions of measurement. High positive correlations would indicate low discriminant validity in the growth among two constructs. For example, the linear (or non linear) growth of leadership quality as measured by the reference method might be positively related to the linear (or non linear) growth of

social competencies as measured by the reference method. Again, it is also possible that these correlations may generalize across different items and constructs [i.e., $Cor(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'})$].

3. The correlations between the latent intercept and slope variables $Cor(\mathcal{I}_{tij}, \mathcal{S}_{tij})$ of the same indicator and same construct can be interpreted as the strength of the association between the initial status at T1 as measured by the reference method and the linear (or non linear) growth of a construct across time measured by the reference method. For example, high positive correlations indicate that higher initial statuses are significantly associated with a stronger increases in the linear (or non linear) growth as measured by the reference method. Negative correlations indicate that higher initial statuses are associated with stronger declines in the linear growth as measured by the reference method. With respect to the previous example, it might be reasonable assuming that higher leadership quality measured by the reference method (e.g., self-report) at T1 are negatively associated with the linear (or non linear) growth of leadership quality measured by the reference method. Again, these correlations may be also be investigated for latent intercept and slope variables that pertain to different indicators, or different constructs, or both [i.e., $(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})$, $Cor(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})$, $Cor(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})$]. For example, it might be interesting investigating whether or not low communication skills measured by the reference method at T1 are on average significantly associated with a stronger increase of leadership quality as measured by the reference method (e.g., self-report).

5.4 General measurement equations and variance decompositions

In the following section the general measurement equations of LGC-COM models are discussed. Again, based on the definition of the LGC-COM model different variance coefficients can be defined. Given that the LST-COM model is a special case of the LGC-COM model the meaningfulness of these coefficients has already been demonstrated in Theorem 4.4. In addition, the independence of the LST-COM latent variables has already been shown in Theorem 4.5. In the following Definition 5.2 a LGC-COM model is defined based on the definition of the LST-COM model. Next, additional variance coefficients that could be studied by researchers are discussed.

Definition 5.2 (Definition 2)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \xi_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \alpha_t, \lambda_\xi, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ be a LST-COM model with conditional regressive independence, strong measurement invariance, and:

$$\begin{aligned} \mathcal{I}_t &\equiv (\mathcal{I}_{111} \cdots \mathcal{I}_{tij} \cdots \mathcal{I}_{bcd})^T, \\ \mathcal{S}_t &\equiv (\mathcal{S}_{111} \cdots \mathcal{S}_{tij} \cdots \mathcal{S}_{bcd})^T, \\ \delta &\equiv (\delta_{1111} \cdots \delta_{ijk(1-1)} \cdots \delta_{cde(f-1)})^T. \end{aligned}$$

Then, $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathcal{I}_t, \mathcal{S}_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \delta, \lambda_\xi^{UM}, \lambda_\xi^{CM}, \lambda_\xi^M, \lambda_\zeta, \lambda_\zeta^{UM}, \lambda_\zeta^{CM}, \lambda_\zeta^M \rangle$ is called a LGC-COM model, if and only if the statements a to c in

Definition 5.1 hold. Note that all other latent variables of the LST-COM model (see Definition 4.4) remain unaltered.

Remarks. According to the above Definition 5.2, all indicators Y_{tij1l} belonging to the reference method ($k = 1$), the same construct j , and measurement occasion l measure a latent intercept factor \mathcal{I}_{tij} , a latent slope factor \mathcal{S}_{tij} weighted by a constant $\delta_{ij1(l-1)}$, a latent state residual ζ_{tijl} and an occasion-specific measurement error E_{tij1l} . All indicators Y_{tijkkl} belonging to a non-reference method ($k > 2$), construct j , and occasion of measurement l measure also a latent intercept factor \mathcal{I}_{tij} , a latent slope factor \mathcal{S}_{tij} weighted by a constant $\delta_{ijk(l-1)}$, a latent state residual ζ_{tijkkl} as well as an occasion-specific measurement error E_{tijkkl} . Besides that, all of these indicators also measure a latent trait method factors ξ_{tijk}^M as well as a latent state method factors ζ_{tijkkl}^M . All indicators Y_{rtij2l} belonging to the non-reference method ($k = 2$) as well as to the same construct and same measurement of occasion l measure a latent intercept factor \mathcal{I}_{tij} , a latent slope factor \mathcal{S}_{tij} weighted by a constant $\delta_{ij2(l-1)}$, a latent state residual ζ_{rtij2l} and an occasion-specific measurement error E_{rtij2l} and above that, two indicator-and construct specific latent trait method factors, namely ξ_{rtij2}^{CM} and ξ_{rtij2}^{UM} , as well as two construct and occasion-specific latent method state variables, namely ζ_{rtij2l}^{CM} and ζ_{rtij2l}^{UM} . Therefore, the measurement equations of the observed variables are given by:

$$Y_{tij1l} = \mathcal{I}_{tij} + \delta_{ij1(l-1)}\mathcal{S}_{tij} + \zeta_{tijl} + E_{tij1l}, \quad (5.7)$$

$$Y_{tijkkl} = \mathcal{I}_{tij} + \delta_{ijk(l-1)}\mathcal{S}_{tij} + \lambda_{\xi Mijkkl}\xi_{tijk}^M + \lambda_{\zeta Mijkkl}\zeta_{tijkkl}^M + E_{tijkkl}, \quad \forall k > 2, \quad (5.8)$$

$$Y_{rtij2l} = \mathcal{I}_{tij} + \delta_{ij2(l-1)}\mathcal{S}_{tij} + \lambda_{\xi ij2l}^{CM}\xi_{rtij2}^{CM} + \lambda_{\xi ij2l}^{UM}\xi_{rtij2}^{UM} + \lambda_{\zeta ij2l}^{CM}\zeta_{rtij2l}^{CM} + \lambda_{\zeta ij2l}^{UM}\zeta_{rtij2l}^{UM} + E_{rtij2l}. \quad (5.9)$$

Note that $\delta_{ijk(l-1)}$ equals zero for indicators pertaining to the first measurement occasion.

5.4.1 Variance decomposition

According to the above measurement Equations (see Equation 5.7 to 5.9) as well as the statements in Theorem 4.6.1, the variance of the observed variables (indicators) can be decomposed as

follows:

$$\begin{aligned}
\text{Var}(Y_{tij1l}) = & \text{Var}(\mathcal{I}_{tij}) + (\delta_{ij1(l-1)})^2 \text{Var}(\mathcal{S}_{tij}) + \\
& 2(\delta_{ij1(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\
& \text{Var}(\zeta_{tijl}) + \text{Var}(E_{tij1l}),
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
\text{Var}(Y_{tijkl}) = & \text{Var}(\mathcal{I}_{tij}) + (\delta_{ijk(l-1)})^2 \text{Var}(\mathcal{S}_{tij}) + \\
& 2(\delta_{ijk(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\
& (\lambda_{\xi_{ijk}^M}^M)^2 \text{Var}(\xi_{ijk}^M) + \quad \forall k > 2, \\
& (\lambda_{\zeta_{ijk}^M}^M)^2 \text{Var}(\zeta_{ijk}^M) + (\lambda_{\zeta_{tjkl}^M}^M)^2 \text{Var}(\zeta_{tjkl}^M) + \\
& \text{Var}(E_{tijkl}),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\text{Var}(Y_{rtij2l}) = & \text{Var}(\mathcal{I}_{tij}) + (\delta_{ijk(l-1)})^2 \text{Var}(\mathcal{S}_{tij}) + \\
& 2(\delta_{ijk(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\
& (\lambda_{\xi_{ij2l}^{CM}}^{CM})^2 \text{Var}(\xi_{ij2l}^{CM}) + (\lambda_{\xi_{ij2l}^{UM}}^{UM})^2 \text{Var}(\xi_{rtij2l}^{UM}) + \\
& (\lambda_{\zeta_{ij2l}^{CM}}^{CM})^2 \text{Var}(\zeta_{ij2l}^{CM}) + (\lambda_{\zeta_{ij2l}^{CM}}^{CM})^2 \text{Var}(\zeta_{tj2l}^{CM}) + \\
& (\lambda_{\zeta_{ij2l}^{UM}}^{UM})^2 \text{Var}(\zeta_{rtij2l}^{UM}) + \text{Var}(E_{rtij2l}).
\end{aligned} \tag{5.12}$$

Similar to the coefficients proposed in Section 4.7.1 of the previous chapter, it is possible to define different variance components such as true ICC coefficients, reliability coefficients etc. In fact, most of the coefficients presented in the previous chapter remain unaltered, given that the LGC-COM model represents a restrictive variant of the LST-COM model. However, given that latent growth curve models assume that the variance of any observed variable increases in a non-linear form [due to the expression $(\delta_{ijk(l-1)})^2 \text{Var}(\mathcal{S}_{tij}) + 2(\delta_{ijk(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij})$], it is not recommended to compare different variance ratios across time points. For example, the reliability coefficients of an observed variable may increase over time, given that latent growth curve models implicitly assume that the interindividual differences in intraindividual change increase over time points. This can be seen by computing the variance of reference indicators pertaining to different time points, while holding the error variance of the indicators [e.g., $\text{Var}(E_{tij1l})$] constant:

$$\begin{aligned}
\text{Var}(Y_{tij11}) = & \text{Var}(\mathcal{I}_{tij}) + 0 \cdot \text{Var}(\mathcal{S}_{tij}) + (2 \cdot 0) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\
& \text{Var}(\zeta_{tijl}) + \text{Var}(E_{tij1l}),
\end{aligned}$$

$$\begin{aligned}
\text{Var}(Y_{tij12}) = & \text{Var}(\mathcal{I}_{tij}) + 1 \cdot \text{Var}(\mathcal{S}_{tij}) + (2 \cdot 1) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\
& \text{Var}(\zeta_{tij2}) + \text{Var}(E_{tij12}),
\end{aligned}$$

$$\begin{aligned} \text{Var}(Y_{tij13}) &= \text{Var}(\mathcal{I}_{tij}) + (2^2)\text{Var}(\mathcal{S}_{tij}) + (2 \cdot 2)\text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\ &\quad \text{Var}(\zeta_{tij3}) + \text{Var}(E_{tij13}). \end{aligned}$$

Researchers who are interested in investigating the psychometric properties of their measures across time points may rather compare the amount of residual variance of the observed variables [$\text{Var}(E_{tij1l})$, $\text{Var}(E_{tijkl})$, $\text{Var}(E_{rtij2l})$] than compute reliability coefficients.

Nevertheless, in addition to the coefficients proposed in the previous chapter, an additional coefficient for studying true consistency and true trait change is introduced:

$$CC(\tau_{tij1l}) = \frac{\text{Var}(\mathcal{I}_{tij}) + (\delta_{ij1(l-1)})^2 \text{Var}(\mathcal{S}_{tij}) + 2(\delta_{ij1(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij})}{\text{Var}(Y_{tij1l}) - \text{Var}(E_{tij1l})},$$

$$CC(\tau_{tijkl}) = \frac{\text{Var}(\mathcal{I}_{tij}) + \delta_{ijk(l-1)}^2 \text{Var}(\mathcal{S}_{tij}) + 2(\delta_{ijk(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij})}{\text{Var}(Y_{tijkl}) - \text{Var}(E_{tijkl})}, \quad \forall k > 2,$$

$$CC(\tau_{rtij2l}) = \frac{\text{Var}(\mathcal{I}_{tij}) + (\delta_{ij2(l-1)})^2 \text{Var}(\mathcal{S}_{tij}) + 2(\delta_{ij2(l-1)}) \text{Cov}(\mathcal{I}_{tij}, \mathcal{S}_{tij})}{\text{Var}(Y_{rtij2l}) - \text{Var}(E_{rtij2l})}.$$

The *consistency and trait change coefficient* $CC(Y_{tij1l})$, $CC(Y_{tijkl})$, and $CC(Y_{rtij2l})$ reflect the proportion of true-score variance that is accounted by trait and trait change effects.

5.5 Mean structure

This section concerns the latent variable mean structure of the LST-COM model. The following theorem shows the consequence of the model definition for the observed and latent variables.

Theorem 5.2 (Mean structure)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathcal{I}_t, \mathcal{S}_t, \xi_{\text{rt}}^{\text{UM}}, \xi_t^{\text{CM}}, \xi_t^{\text{M}}, \zeta_t, \zeta_{\text{rt}}^{\text{UM}}, \zeta_t^{\text{CM}}, \zeta_t^{\text{M}}, \mathbf{E}_{\text{rt}}, \mathbf{E}_t, \delta, \lambda_\xi^{\text{UM}}, \lambda_\xi^{\text{CM}}, \lambda_\xi^{\text{M}}, \lambda_\zeta, \lambda_\zeta^{\text{UM}}, \lambda_\zeta^{\text{CM}}, \lambda_\zeta^{\text{M}} \rangle$ be a LGC-COM model. Then the mean structure of the LGC-COM holds for all $r \in R \equiv \{1, \dots, a\}$, $t \in T \equiv \{1, \dots, b\}$, $i \in I \equiv \{1, \dots, c\}$, $j \in J \equiv \{1, \dots, d\}$, $k \in K \equiv \{1, \dots, e\}$, $l \in L \equiv \{1, \dots, f\}$:

$$E(Y_{tij1l}) = E(\mathcal{I}_{tij}) + \delta_{ij1(l-1)} E(\mathcal{S}_{tij}), \quad \text{for } k = 1, \quad (5.13)$$

$$E(Y_{tijkl}) = E(\mathcal{I}_{tij}) + \delta_{ijk(l-1)} E(\mathcal{S}_{tij}), \quad \forall k > 2, \quad (5.14)$$

$$E(Y_{rtij2l}) = E(\mathcal{I}_{tij}) + \delta_{ij2(l-1)} E(\mathcal{S}_{tij}), \quad (5.15)$$

$$E(Y_{tijk1}) = E(\mathcal{I}_{tij}), \quad (5.16)$$

$$E(\zeta_{tijl}) = 0, \quad (5.17)$$

$$E(\xi_{rtij2}^{UM}) = 0, \quad (5.18)$$

$$E(\xi_{tij2}^{CM}) = 0, \quad (5.19)$$

$$E(\xi_{tijk}^M) = 0, \quad \forall k > 2, \quad (5.20)$$

$$E(\zeta_{rtj2l}^{UM}) = 0, \quad (5.21)$$

$$E(\zeta_{tj2l}^{CM}) = 0, \quad (5.22)$$

$$E(\zeta_{tjkl}^M) = 0, \quad \forall k > 2, \quad (5.23)$$

$$E(E_{tijkkl}) = 0, \quad \forall k \neq 2, \quad (5.24)$$

$$E(E_{rtij2l}) = 0, \quad (5.25)$$

where $E(\cdot)$ denotes expected value.

Proofs. 20 Mean structure

Due to the measurement equations 5.7 to 5.9, it follows that:

$$\begin{aligned} E(Y_{tij1l}) &= E(\mathcal{I}_{tij}) + \delta_{ij1(l-1)}E(\mathcal{S}_{tij}) + E(\zeta_{tijl}) + E(E_{tij1l}), \\ E(Y_{tijkkl}) &= E(\mathcal{I}_{tij}) + \delta_{ijk(l-1)}E(\mathcal{S}_{tij}) + \lambda_{\xi M_{ijkkl}}E(\xi M_{tijk}) + \lambda_{\zeta_{ijkkl}}E(\zeta_{tijk}) + \lambda_{\zeta M_{ijkkl}}E(\zeta M_{tjkl}) + E(E_{tijkkl}), \quad \forall k > 2, \\ E(Y_{rtij2l}) &= E(\mathcal{I}_{tij}) + \delta_{ij2(l-1)}E(\mathcal{S}_{tij}) + \lambda_{\xi_{ij2l}^{CM}}E(\xi_{tij2}^{CM}) + \lambda_{\xi_{ij2l}^{UM}}E(\xi_{rtij2}^{UM}) + \lambda_{\zeta_{ij2l}}E(\zeta_{tijl}) + \lambda_{\zeta_{ij2l}^{CM}}E(\zeta_{tj2l}^{CM}) + \lambda_{\zeta_{ij2l}^{UM}}E(\zeta_{rtj2l}^{UM}) + E(E_{rtij2l}). \end{aligned}$$

According to the Equations 5.17 to 5.25 of the above Theorem 5.2, it follows that the expectations of the latent residual variables ζ_{tijl} , ξM_{tijk} , ζM_{tjkl} , E_{tijkkl} , $\xi_{CM_{tij2}}$, $\xi_{UM_{rtij2}}$, ζ_{tijl} , $\zeta_{CM_{tj2l}}$, $\zeta_{UM_{rtj2l}}$, and E_{rtij2l} are zero. These statements follow directly by the definition of these variables as latent residuals, and given the fact that residual variables always have expectations of zero (see e.g., Steyer & Eid, 2001). Researchers must therefore fix the expected values of these variables to zero in empirical applications. As a consequence of these zero expectations, the above Equations can be simplified as follows:

$$\begin{aligned} E(Y_{tij1l}) &= E(\mathcal{I}_{tij}) + \delta_{ij1(l-1)}E(\mathcal{S}_{tij}), \\ E(Y_{tijkkl}) &= E(\mathcal{I}_{tij}) + \delta_{ijk(l-1)}E(\mathcal{S}_{tij}), \quad \forall k > 2, \\ E(Y_{rtij2l}) &= E(\mathcal{I}_{tij}) + \delta_{ij2(l-1)}E(\mathcal{S}_{tij}). \end{aligned}$$

Because of $\delta_{ij1(l-1)} = \delta_{ijk(l-1)} = \delta_{ij2(l-1)} = 0$, for $l=1$ (see Equation 5.16), it follows

$$E(Y_{tijk1}) = E(\mathcal{I}_{tij}).$$

Given that $E(\mathcal{I}_{tij})$ can be identified with respect to the first occasion of measurement $E(Y_{tijk1})$, the expected values of the latent slope variables $E(\mathcal{S}_{tij})$ are identified for $l > 1$:

$$\begin{aligned} E(\mathcal{S}_{tij}) &= \frac{E(Y_{tij1l}) - E(Y_{tij11})}{\delta_{ij1(l-1)}}, \\ E(\mathcal{S}_{tij}) &= \frac{E(Y_{tijkkl}) - E(Y_{tijk1})}{\delta_{ijk(l-1)}}, \quad \forall k > 2, \\ E(\mathcal{S}_{tij}) &= \frac{E(Y_{rtij2l}) - E(Y_{rtij21})}{\delta_{ij2(l-1)}}. \end{aligned}$$

□

Remarks. Equations 5.14 and 5.15 clarify that the expected values of the observed variables are equal to the expected values of the latent trait variable ξ_{tij} . According to Equation 5.16, the

expected values of the latent trait variables are identical to the expected values of the indicators pertaining to the reference method. Equations 5.17 to 5.23 reveal that the latent state residuals as well as the trait-specific and state-specific method factors are defined as residuals and therefore the expected values of these latent residual variables are necessarily zero. The same holds for the measurement error variables (see Equation 5.24 and 5.25).

5.6 Identifiability

In this section, the identification of the LGC-COM model parameters is addressed. The identification of the mean structure of the latent variables in the LGC-COM model has been already demonstrated in Theorem 20 and will not be repeated again. This section only concerns the identification of the variance-covariance structure of the LGC-COM model. Given that the LGC-COM model is a special case (or restrictive variant) of the LST-COM model, the parameters of the LGC-COM model are identified whenever the parameters of the LST-COM model are identified. According to Theorem 4.7 the parameters of the LST-COM model are uniquely determined under the minimal condition of a $2 \times 2 \times 2 \times 3$ measurement design (see Courvoisier, 2006, p. 73-130). The following theorem states that this condition holds also for the identification of the LGC-COM model parameters. For the sake of simplicity, it will be assumed that all parameters of the LST-COM model are known and thereby only the identification of the remaining parameters of the LGC-COM model is shown. Note that the known parameters of the LST-COM will be used for identification without replacing them by parameters of the observed variables. Furthermore, conditions concerning the independence (uncorrelatedness) of the latent variables in LGC-COM model will be used for identification (see Theorem 5.1 and 4.5). By definition of the LGC-COM model, it is assumed that for all $t \in T$, $i \in I$, $j \in J$, $k \in K$, and $l \in L$ $\alpha_{tijk} = 0$ and $\delta_{ijk(l-1)} = (l-1)$.

Theorem 5.3 (Identification of the LGC-COM covariance structure)

Let $\mathcal{M} \equiv \langle (\Omega, \mathfrak{A}, P), \mathcal{I}_t, \mathcal{S}_t, \xi_{rt}^{UM}, \xi_t^{CM}, \xi_t^M, \zeta_t, \zeta_{rt}^{UM}, \zeta_t^{CM}, \zeta_t^M, \mathbf{E}_{rt}, \mathbf{E}_t, \lambda_{\mathcal{I}}, \delta, \lambda_{\xi}^{UM}, \lambda_{\xi}^{CM}, \lambda_{\xi}^M, \lambda_{\zeta}, \lambda_{\zeta}^{UM}, \lambda_{\zeta}^{CM}, \lambda_{\zeta}^M \rangle$ be a LGC-COM model of $(\xi_{tij2}^{UM}, \xi_{tij2}^{CM}, \xi_{tijk}^M, \zeta_{tijl}, \zeta_{rtj2l}^{UM}, \zeta_{tj2l}^{CM}, \zeta_{tjkl}^M)$ -congeneric variables with conditional regressive independence, then the parameters of the matrix $\Phi_{\xi B}$ are identified, if either one factor loading $\lambda_{\xi_{ijk}l}^M, \lambda_{\xi_{ij2l}^{CM}}, \lambda_{\xi_{ij2l}^{UM}}, \lambda_{\zeta_{ijk}l}^M, \lambda_{\zeta_{ij2l}^{CM}}, \lambda_{\zeta_{ij2l}^{UM}}$ for each factor, $\xi_{tij2}^{CM}, \xi_{rtij2}^{UM}, \zeta_{tijl}, \zeta_{tj2l}^{CM}, \zeta_{rtj2l}^{UM}$ or the variance of these factors are set to any real value larger than 0, and

- (a) iff $i = 2, j \geq 2, k \geq 2, l \geq 3$ and $\Phi_{\xi W}, \Phi_{\zeta W}, \Phi_{\xi B}, \Phi_{\zeta B}$ contain intercorrelations (i.e., nonzero elements in the off-diagonal).

Proofs. 21 (Identification) Assuming that all elements of the matrices $\Lambda_{\xi W}, \Phi_{\xi W}, \Lambda_{\zeta W}, \sum_{\theta W}, \Lambda_{\xi B}, \Lambda_{\zeta B}, \Phi_{\zeta B}, \sum_{\theta B}$ are identified and the parameters are known for a LST-COM model with 2 indicators, 2 constructs, 2 methods (one structurally different method and one set of interchangeable methods), and 3 occasions of measurement, except for the elements of $\Phi_{\xi B}$. Then, for any observed variable Y_{rtij2l} and Y_{tij1l} in a LGC-COM model with the same dimension (i.e., $2 \times 2 \times 2 \times 3$) the expected values are:

$$\begin{aligned} E(Y_{tij1l}) &= E(\mathcal{I}_{tij}) + \delta_{ij1(l-1)} E(\mathcal{S}_{tij}) + E(\zeta_{tijl}) + E(E_{tij1l}), \\ E(Y_{rtij2l}) &= E(\mathcal{I}_{tij}) + \delta_{ij2(l-1)} E(\mathcal{S}_{tij}) + \lambda_{\xi_{ij2l}^{CM}} E(\xi_{tij2}^{CM}) + \lambda_{\xi_{ij2l}^{UM}} E(\xi_{rtij2}^{UM}) + \\ &\quad \lambda_{\zeta_{ij2l}^{CM}} E(\zeta_{tijl}) + \lambda_{\zeta_{ij2l}^{UM}} E(\zeta_{tj2l}^{CM}) + \lambda_{\zeta_{ij2l}^{UM}} E(\zeta_{rtj2l}^{UM}) + E(E_{rtij2l}). \end{aligned}$$

Identification of $Var(\mathcal{I}_{tij})$:

$Var(\mathcal{I}_{tij})$ can be directly identified by the variance of the first indicator measured on the first measurement occasion.

$$\begin{aligned} Var(Y_{tij11}) &= Var(\mathcal{I}_{tij}) + Var(\zeta_{tij1}) - Var(E_{tij11}). \\ Var(\mathcal{I}_{tij}) &= Var(Y_{tij11}) - Var(\zeta_{tij1}) - Var(E_{tij11}). \end{aligned}$$

Presuming that $Var(\zeta_{tij1})$ and $Var(E_{tij11})$ are known.

Identification of $Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij})$:

$Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij})$ is identified with respect to the covariance of Y_{tij11} and Y_{tij12} .

$$\begin{aligned} Cov(Y_{tij11}, Y_{tij12}) &= Cov\{(\mathcal{I}_{tij} + \zeta_{tij1} + E_{tij11}), (\mathcal{I}_{tij} + \delta_{ij1(2-1)}\mathcal{S}_{tij} + \zeta_{tij2} + E_{tij12})\} \\ &= Cov(\mathcal{I}_{tij}, \mathcal{I}_{tij}) + Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) \\ &= Var(\mathcal{I}_{tij}) + Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}). \end{aligned}$$

Hence, $Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) = Cov(Y_{tij11}, Y_{tij12}) - Var(\mathcal{I}_{tij})$.

Identification of $Var(\mathcal{S}_{tij})$:

$Var(\mathcal{S}_{tij})$ is identified with respect to the covariance of Y_{tij1l} and Y_{rtij2l} , for $l > 1$.

$$Cov(Y_{tij1l}, Y_{rtij2l}) = Cov\left\{ \begin{array}{l} (\mathcal{I}_{tij} + \delta_{ij1(l-1)}\mathcal{S}_{tij} + \zeta_{tijl} + E_{tij1l}), \\ (\mathcal{I}_{tij} + \delta_{ij2(l-1)}\mathcal{S}_{tij} + \lambda_{\zeta_{ij2l}}^{CM}\epsilon_{\mathcal{S}_{tij2}}^{CM} + \lambda_{\zeta_{ij2l}}^{UM}\epsilon_{\mathcal{S}_{rtij2l}}^{UM} + \\ \lambda_{\zeta_{ij2l}}\zeta_{tijl} + \lambda_{\zeta_{ij2l}}^{CM}\zeta_{\mathcal{S}_{tj2l}}^{CM} + \lambda_{\zeta_{ij2l}}^{UM}\zeta_{\mathcal{S}_{rtj2l}}^{UM} + E_{rtij2l}) \end{array} \right\}.$$

$$\begin{aligned} Cov(Y_{tij1l}, Y_{rtij2l}) &= Var(\mathcal{I}_{tij}) + \delta_{ij1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \\ &\quad \delta_{ij2(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) + \delta_{ij1(l-1)}\delta_{ij2(l-1)}Var(\mathcal{S}_{tij}) + \\ &\quad \lambda_{\zeta_{ij2l}}^2Var(\zeta_{tijl}). \end{aligned}$$

$$Var(\mathcal{S}_{tij}) = \frac{\left\{ \begin{array}{l} Cov(Y_{tij1l}, Y_{rtij2l}) - Var(\mathcal{I}_{tij}) - \\ \delta_{ij1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) - \delta_{ij2(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) - \\ \lambda_{\zeta_{ij2l}}^2Var(\zeta_{tijl}) \end{array} \right\}}{\delta_{ij1(l-1)}\delta_{ij2(l-1)}}.$$

For example, with respect to measurement occasion $l = 2$ (i.e., $\delta_{ij1(l-1)}$ and $\delta_{ij2(l-1)}$ equal one), the variance of the latent slope variables is given by:

$$Var(\mathcal{S}_{tij}) = Cov(Y_{tij12}, Y_{rtij22}) - Var(\mathcal{I}_{tij}) - 2Cov(\mathcal{I}_{tij}, \mathcal{S}_{tij}) - \lambda_{\zeta_{ij2l}}^2Var(\zeta_{tij2}).$$

Identification of $Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'})$:

$Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'})$ is identified with respect to the covariance between Y_{tij11} and $Y_{ti'j'11}$, where $(i, j) \neq (i', j')$.

$$\begin{aligned} Cov(Y_{tij11}, Y_{ti'j'11}) &= Cov\left\{ \begin{array}{l} (\mathcal{I}_{tij} + \zeta_{tij1} + E_{tij11}), \\ (\mathcal{I}_{ti'j'} + \zeta_{ti'j'1} + E_{ti'j'11}) \end{array} \right\} \\ &= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) + Cov(\zeta_{tij1}, \zeta_{ti'j'1}). \end{aligned}$$

$$Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) = Cov(Y_{tij11}, Y_{ti'j'11}) - Cov(\zeta_{tij1}, \zeta_{ti'j'1}).$$

Identification of $Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})$:

$Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j})$ is identified with respect to the covariance between Y_{tij1l} and $Y_{ti'j1l}$, where $i \neq i'$ as well as $l > 1$.

$$\begin{aligned} Cov(Y_{tij1l}, Y_{ti'j1l}) &= Cov \left\{ \begin{array}{l} (\mathcal{I}_{tij} + \zeta_{tij1} + E_{tij1l}), \\ (\mathcal{I}_{ti'j} + \delta_{i'j1(l-1)}\mathcal{S}_{ti'j} + \zeta_{ti'j1} + E_{ti'j1l}) \end{array} \right\} \\ &= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j}) + \delta_{i'j1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j}). \\ Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j}) &= \frac{Cov(Y_{tij1l}, Y_{ti'j1l}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j})}{\delta_{i'j1(l-1)}}. \end{aligned}$$

Identification of $Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j})$:

$Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j})$ is identified with respect to the covariance between Y_{tij1l} and $Y_{ti'j1l}$, where $i \neq i'$ and $l > 1$.

$$\begin{aligned} Cov(Y_{tij1l}, Y_{ti'j1l}) &= Cov \left\{ \begin{array}{l} (\mathcal{I}_{tij} + \delta_{ij1(l-1)}\mathcal{S}_{tij} + \zeta_{tijl} + E_{tij1l}), \\ (\mathcal{I}_{ti'j} + \delta_{i'j1(l-1)}\mathcal{S}_{ti'j} + \zeta_{ti'jl} + E_{ti'j1l}) \end{array} \right\}. \\ Cov(Y_{tij1l}, Y_{ti'j1l}) &= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j}) + \delta_{ij1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j}) + \\ &= \delta_{i'j1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{I}_{ti'j}) + \delta_{ij1(l-1)}\delta_{i'j1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j}) + \\ &= Cov(\zeta_{tijl}, \zeta_{ti'jl}). \\ Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j}) &= \frac{\left\{ \begin{array}{l} Cov(Y_{tij1l}, Y_{ti'j1l}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j}) - \\ \delta_{ij1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j}) - \delta_{i'j1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{I}_{ti'j}) - \\ Cov(\zeta_{tijl}, \zeta_{ti'jl}) \end{array} \right\}}{\delta_{ij1(l-1)}\delta_{i'j1(l-1)}}. \end{aligned}$$

Identification of $Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})$:

$Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})$ is identified with respect to the covariance between Y_{tij1l} and $Y_{ti'j'1l}$, where $(i, j) \neq (i', j')$ and $l > 1$.

$$\begin{aligned} Cov(Y_{tij1l}, Y_{ti'j'1l}) &= Cov \left\{ \begin{array}{l} (\mathcal{I}_{tij} + \zeta_{tij1} + E_{tij1l}), \\ (\mathcal{I}_{ti'j'} + \delta_{i'j'1(l-1)}\mathcal{S}_{ti'j'} + \zeta_{ti'j'l} + E_{ti'j'1l}) \end{array} \right\} \\ &= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) + \delta_{i'j'1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}). \\ Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) &= \frac{Cov(Y_{tij1l}, Y_{ti'j'1l}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'})}{\delta_{i'j'1(l-1)}}. \end{aligned}$$

Identification of $Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'})$:

$Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'})$ is identified with respect to the covariance between Y_{tij1l} and $Y_{ti'j'1l}$, where (i, j)

$\neq (i', j')$ and $l > 1$.

$$\begin{aligned}
Cov(Y_{tij1l}, Y_{ti'j'1l}) &= Cov \left\{ \begin{array}{l} (\mathcal{I}_{tij} + \delta_{ij1(l-1)}\mathcal{S}_{tij} + \zeta_{tijl} + E_{tij1l}), \\ (\mathcal{I}_{ti'j'} + \delta_{i'j'1(l-1)}\mathcal{S}_{ti'j'} + \zeta_{ti'j'l} + E_{ti'j'1l}) \end{array} \right\} \\
&= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) + \delta_{i'j'1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) + \\
&\quad \delta_{ij1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{I}_{ti'j'}) + \delta_{ij1(l-1)}\delta_{i'j'1(l-1)}Cov(\mathcal{S}_{ti'j'}, \mathcal{S}_{tij}) + \\
&\quad Cov(\zeta_{tijl}, \zeta_{ti'j'l}). \\
Cov(\mathcal{S}_{ti'j'}, \mathcal{S}_{tij}) &= \frac{\left\{ \begin{array}{l} Cov(Y_{tij1l}, Y_{ti'j'1l}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) - \\ \delta_{i'j'1(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) - \delta_{ij1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{I}_{ti'j'}) - \\ Cov(\zeta_{tijl}, \zeta_{ti'j'l}) \end{array} \right\}}{\delta_{ij1(l-1)}\delta_{i'j'1(l-1)}}.
\end{aligned}$$

For the second measurement occasion $l = 2$, where $\delta_{ij1(2-1)} = \delta_{i'j'1(2-1)} = 1$ the equation above simplifies to:

$$\begin{aligned}
Cov(Y_{tij12}, Y_{ti'j'12}) &= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) + Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) + \\
&\quad Cov(\mathcal{I}_{ti'j'}, \mathcal{S}_{tij}) + Cov(\mathcal{S}_{ti'j'}, \mathcal{S}_{tij}) + \\
&\quad Cov(\zeta_{tij1}, \zeta_{ti'j'1}).
\end{aligned}$$

$$\begin{aligned}
Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'}) &= Cov(Y_{tij12}, Y_{ti'j'12}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) - \\
&\quad Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) - Cov(\mathcal{I}_{ti'j'}, \mathcal{S}_{tij}) - \\
&\quad Cov(\zeta_{tij2}, \zeta_{ti'j'2}).
\end{aligned}$$

Identification of $Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM})$:

$Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM})$ is identified with respect to the covariance between Y_{tij11} and $Y_{rti'j'2l}$, where $(i, j) \neq (i', j')$ and $l > 1$.

$$\begin{aligned}
Cov(Y_{tij11}, Y_{rti'j'2l}) &= Cov \left\{ \begin{array}{l} (\mathcal{I}_{tij} + \zeta_{tij1} + E_{tij11}), \\ (\mathcal{I}_{ti'j'} + \delta_{i'j'2(l-1)}\mathcal{S}_{ti'j'} + \lambda_{\xi_{i'j'2l}^{CM}}\xi_{ti'j'2}^{CM} + \lambda_{\xi_{i'j'2l}^{UM}}\xi_{rti'j'2}^{UM} + \\ \lambda_{\zeta_{i'j'2l}}\zeta_{ti'j'l} + \lambda_{\zeta_{i'j'2l}^{CM}}\zeta_{tj'2l}^{CM} + \lambda_{\zeta_{i'j'2l}^{UM}}\zeta_{rti'j'2l}^{UM} + E_{rti'j'2l}) \end{array} \right\} \\
&= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) + \delta_{i'j'2(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) + \lambda_{\xi_{i'j'2l}^{CM}}Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM}). \\
Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM}) &= \frac{Cov(Y_{tij11}, Y_{rti'j'2l}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) - \delta_{i'j'2(l-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'})}{\lambda_{\xi_{i'j'2l}^{CM}}}.
\end{aligned}$$

Identification of $Cov(\mathcal{S}_{tij}, \xi_{tij'2}^{CM})$:

Finally, $Cov(\mathcal{S}_{tij}, \xi_{tij'2}^{CM})$ is identified with respect to the covariance between Y_{tij1l} and $Y_{ti'j'2l'}$, where $(i, j, l) \neq (i', j', l')$ and $l, l' > 1$.

$$\begin{aligned}
Cov(Y_{tij1l}, Y_{rti'j'2l'}) &= Cov \left\{ \begin{array}{l} (\mathcal{I}_{tij} + \delta_{ij1(l-1)}\mathcal{S}_{tij} + \zeta_{tijl} + E_{tij1l}), \\ (\mathcal{I}_{ti'j'} + \delta_{i'j'2(l'-1)}\mathcal{S}_{ti'j'} + \lambda_{\xi_{i'j'2l'}^{CM}}\xi_{ti'j'2}^{CM} + \lambda_{\xi_{i'j'2l'}^{UM}}\xi_{rti'j'2}^{UM} + \\ \lambda_{\zeta_{i'j'2l'}^{CM}}\zeta_{ti'j'l'} + \lambda_{\zeta_{i'j'2l'}^{CM}}\zeta_{tj'2l'} + \lambda_{\zeta_{i'j'2l'}^{UM}}\zeta_{rtj'2l'} + E_{rti'j'2l'}) \end{array} \right\} \\
&= Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) + \delta_{i'j'2(l'-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) + \lambda_{\xi_{i'j'2l'}^{CM}}Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM}) + \\
&\quad \delta_{ij1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{I}_{ti'j'}) + \delta_{ij1(l-1)}\delta_{i'j'2(l'-1)}Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'}) + \\
&\quad \delta_{ij1(l-1)}\lambda_{\xi_{i'j'2l'}^{CM}}Cov(\mathcal{S}_{tij}, \xi_{ti'j'2}^{CM}). \\
Cov(\mathcal{S}_{tij}, \xi_{tij'2}^{CM}) &= \frac{\left\{ \begin{array}{l} Cov(Y_{tij1l}, Y_{rti'j'2l'}) - Cov(\mathcal{I}_{tij}, \mathcal{I}_{ti'j'}) - \\ \delta_{i'j'2(l'-1)}Cov(\mathcal{I}_{tij}, \mathcal{S}_{ti'j'}) - \lambda_{\xi_{i'j'2l'}^{CM}}Cov(\mathcal{I}_{tij}, \xi_{ti'j'2}^{CM}) - \\ \delta_{ij1(l-1)}Cov(\mathcal{S}_{tij}, \mathcal{I}_{ti'j'}) - \delta_{ij1(l-1)}\delta_{i'j'2(l'-1)}Cov(\mathcal{S}_{tij}, \mathcal{S}_{ti'j'}) \end{array} \right\}}{\delta_{ij1(l-1)}\lambda_{\xi_{i'j'2l'}^{CM}}}.
\end{aligned}$$

□

Box 5.1 (LGC-COM Model)

\mathcal{I}_{tij} target-specific latent intercept factor of the reference (structurally different) method

\mathcal{S}_{tij} target-specific latent slope factor of the reference (structurally different) method

$\delta_{ijk(l-1)}$ latent slope factor loading for modeling the shape of trait change

$$\begin{aligned}\xi_{tij11} &\equiv \mathcal{I}_{tij}, \\ (\xi_{tij12} - \xi_{tij11}) &\equiv \mathcal{S}_{tij}, \\ \delta_{ijk(l-1)} &\equiv (l-1).\end{aligned}$$

Part III

Monte Carlo Simulation Studies

Chapter 6

Rationale and Aims of the Monte Carlo simulation studies

Some statistical questions that cannot be answered analytically (e.g., questions concerning the limits of the applicability of a statistical model) may be better answered by conducting Monte Carlo simulation studies (Geiser, 2008; Harwell, Stone, Hsu, & Kirisci, 1996). There are numerous reasons for doing so. First, Monte Carlo (MC) simulation studies enable researchers to investigate the performance of a given statistical model under *experimental* conditions (see Li, Boos, & Gumpertz, 2001). For example, Monte Carlo simulation studies allow researchers to predetermine a set of true population parameters (i.e., population model), to randomly generate numerous data sets (usually 500 to 1000 MC replications) based on the population model, and to compare the true population parameters with the average parameter estimates from the MC samples after fitting the model (or even different models) of interest to the generated MC samples (Geiser, 2008). In real data applications, it is often not possible to achieve such controlled conditions, where only specific conditions (e.g., sample size, skewness, misspecification) are varied and others are held constant. Moreover, in real data applications, the true population parameters are usually unknown and large sample sizes are rather difficult to obtain. Therefore, MC simulation studies are more efficient than real data applications, given that they require less time, money, and man power. Second, numerous criteria can be used for evaluating the performance of a given model in MC simulation studies. For example, researchers may calculate the bias in parameter estimates (peb) and standard errors (seb) in order to scrutinize the consistency of parameter estimates. They can furthermore count the number of improper solutions (i.e., Heywood cases) as well as the number of convergence problems, and/or they can study the trustworthiness of fit statistics (for more details see Bandalos, 2006; L. K. Muthén & Muthén, 2002). If researchers are interested in investigating the minimal required sample size for valid parameter estimates, they may either perform a power analysis by conducting a MC simulation study (L. K. Muthén & Muthén, 2002), or relate different types of biases (e.g., peb or seb) to different MC conditions (e.g., different sample sizes, model misspecification etc.). Thereby, researchers may be able to identify favorable and unfavorable conditions for proper estimation of parameters, standard errors, fit statistics etc. (Geiser, 2008). For more details concerning the basic principles of MC simulation studies as well as their implementation see Bandalos (2006) as well as L. K. Muthén and Muthén (2002).

In the following chapters the results of four extensive simulation studies are presented. Each

simulation study refers to a different model presented in the previous chapter:

1. Simulation study I : latent state (LS-COM) model (see Chapter 7)
2. Simulation study II : latent change (LC-COM) model (see Chapter 8)
3. Simulation study III : latent state-trait (LST-COM) model (see Chapter 9)
4. Simulation study IV : latent growth curve (LGC-COM) model (see Chapter 10).

The results of the following simulation studies reveal new insights to multilevel structural equation modeling of complex MTMM-MO data. To my knowledge no simulation study has yet been conducted investigating the performance of such complex structural equation models (i.e., longitudinal multilevel MTMM-SEMs). However, several simulation studies have been carried out investigating the performance of less complex SEMs. For example, the performance of single level SEMs under various conditions has been scrutinized by numerous researchers (Chen, Bollen, Paxton, Curran, & Kirby, 2001; Beauducel & Herzberg, 2006; Boomsma, 1982; Gerbing & Anderson, 1985; Jackson, 2001; Marsh, Hau, Balla, & Grayson, 1998; MacKenzie, Podsakoff, & Jarvis, 2005). In general, the results of these simulation studies suggest that the parameter estimates in SEMs become more reliable with increasing sample size and an increasing number of unidimensional indicators per factor (Anderson & Gerbing, 1984; Boomsma, 1982; Marsh et al., 1998). For example, according to the results of simulation studies by Boomsma (1982) the minimal required sample size for proper parameter estimates in single level SEM is 100. According to Bentler and Chou (1987) the minimal required sample size depends on the ratio between the number of observations and the number of parameters should be above 5:1. Bollen (1989, 2002) recommends a ratio of 10:1 for more complex models instead. With respect to two-level SEMs, Julian (2001) suggests to sample at least 100 level-2 units (observations) for proper parameter estimates when using maximum likelihood estimation. In addition, simulation studies by Maas and Hox (2005) indicate that the sample size on the cluster (between) level is more important for proper parameter estimates than the sample size on the individual (within) level with respect to general multilevel models.

The performance of cross-sectional MTMM-SEMs has also been investigated by various simulation studies (Conway, Lievens, Scullen, & Lance, 2004; Marsh & Bailey, 1991; Nussbeck, Eid, & Lischetzke, 2006; Tomas, Hontangas, & Oliver, 2000). According to Nussbeck et al. (2006) even relatively complex MTMM-MO models with categorical items [CTC(M)-1 model] perform well using the WLSMV (weighted least square mean and variance adjusted) estimator implemented in *Mplus*. Important contributions to the field of longitudinal MTMM-SEMs are the simulation studies by Crayen (2008) and Geiser (2008). In both simulation studies, the authors investigated the performance of multiple indicator SEMs for longitudinal MTMM measurement designs. However, the authors focused on the performance of longitudinal MTMM-SEMs for structurally different methods, and *not* on the performance of longitudinal MTMM-SEMs for a combination of structurally different and interchangeable methods. The main findings of both simulation studies can be summarized as follows:

- Estimates of parameters as well as standard errors are well recovered even in small sample sizes ($N = 125$) for longitudinal CTC(M)-1 model models.

- Estimation of standard errors are more sensitive to bias than the estimations of parameters.
- Longitudinal CTC(M)-1 model models perform better (i.e., less bias, less improper solutions, higher convergence rates) with an increasing amount of empirical information (e.g., sample size, number of indicators per factor, number of occasions).
- An increase of method specificity (i.e., higher convergent validity) was related to an decrease in estimation accuracy.
- The empirical χ^2 -distribution was not well approximated for complex models. In general, an increase of type I error was found with increasing model complexity and decreasing small sample size. Only in extremely large data sets the χ^2 -distribution could be approximated correctly.

6.1 Aims of the simulation studies

The main goals of the simulation studies are investigating

- (a) the appropriateness of parameter and standard error estimates via coefficients of parameter estimate bias (peb) and standard error bias (seb),
- (b) the amount of improper solutions (estimation problems) with respect to the latent covariance matrix Ψ and latent error covariance matrix Θ (so called Heywood cases),
- (c) the robustness of the χ^2 -fit statistics,
- (d) the amount of convergence problems, and
- (e) the limits of the applicability of presented models by relating different types of biases (i.e., average peb and seb) to different MC conditions.

In the subsequent chapters, the terms “possible”, “actual” and “negligible” improper solution are distinguished. Throughout this thesis, “actual” improper solutions refer to out-of-range parameter estimates, also known as *Heywood cases* (Chen et al., 2001; Geiser, 2008). It is important to note that the term “actual” improper solution is only used to refer to negative variances (< 0) of the latent variables in the model and/or permissible correlations among the latent variables in the model that are greater than $|1|$. However, improper solutions referring to higher order (partial) correlations among the latent variables in the model (that have no substantive meaning with respect to the definition of the model, but will still count as improper solution in *Mplus* outputs) will be treated as “negligible” improper solutions, and *not* as “actual” or “real” improper solutions. In other words, “actual” or “real” improper solutions refer to out-of-range parameter estimates that can be investigated by the TECH4 output option in *Mplus*. The sum of “actual” and “negligible” improper solutions equals the amount of “possible” improper solutions. Therefore, the amount of “possible” improper solutions represents the total amount of *Mplus* warning messages. Again, it shall be re-emphasized that the total amount of *Mplus* warning messages does *not* necessarily corresponds to the total amount of “actual” improper solution or Heywood cases. With regard to the subsequent

simulation studies, models with “possible” improper solutions were first refitted to their specific MC sample(s). Afterwards the TECH4 outputs of the specific model(s) were investigated for “actual” improper solutions.

6.2 Simulation designs

The simulation designs of the latent state (LS-COM) and the latent change (LC-COM) model are given in Table 6.1. Note that the simulation designs for both models are identical. The maximum number of conditions for each of these models was 232 with 500 replications (56 conditions were not simulated due to identification problems¹). In total, 116,000 data sets were simulated and saved for LS-COM and LC-COM model simulations.

Table 6.1: Simulation design of simulation study I & II

		Multiconstruct Condition														
		High Consistency						Low Consistency								
nL1	nL2	N	k=2			k=3			k=2		k=3					
			l=2	l=3	l=4	l=2	l=3	l=4	l=2	l=3	l=4	l=2	l=3	l=4		
2	100	200	✓	n.s.	n.s.	n.s.	n.s.	n.s.	✓	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	
	250	500	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	500	1000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
5	100	500	✓	n.s.	n.s.	n.s.	n.s.	n.s.	✓	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	
	250	1250	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	500	2500	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
10	100	1000	✓	n.s.	n.s.	n.s.	n.s.	n.s.	✓	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	
	250	2500	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	500	5000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
20	100	2000	✓	n.s.	n.s.	n.s.	n.s.	n.s.	✓	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	
	250	5000	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	500	10000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
		Monoconstruct Condition														
2	100	200	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	250	500	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
	500	1000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
5	100	500	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	250	1250	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
	500	2500	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
10	100	1000	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	250	2500	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
	500	5000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
20	100	2000	✓	✓	✓	✓	✓	n.s.	✓	✓	✓	✓	✓	✓	n.s.	
	250	5000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
	500	10000	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	

Notes. Simulation design of the LS-COM and LC-COM model. The symbol n.s. refers to conditions that were not simulated and thus not part of the simulation design. The checkmark sign ✓ refers to conditions for which 500 data sets of a particular model were simulated. nL1 = number of raters per target (level-1 units); nL2 = number of targets (level-2 units).

Five conditions were manipulated in simulation study I and II (see Table 6.1):

¹The term “identification problems” refers to a general warning message in *Mplus* 6.1. Note that these warning messages do not indicate a problem of the model per se or that the model is not identified. Instead these error messages refer to the fact that the number of parameters to be estimated is larger than the number of observations on the between level. In many cases, these warning messages may be ignored and will not necessarily lead improper parameter estimates (L. K. Muthén & Muthén, 1998-2010). Given that the results of these MC conditions constitute another condition in the simulation design that was not of interest here, these conditions were not simulated in the first place.

1. the degree of convergent validity (i.e., High vs. Low Con),
2. the number of methods (i.e., $k = 2, 3$),
3. the number of occasions of measurement (i.e., $l = 2, 3$, and 4),
4. the number of level-1 units or the number of raters per target (i.e., $nL1 = 2, 5, 10$, or 20),
5. the number of level-2 units or the number of targets (i.e., $nL2 = 100, 250$, or 500).

The main reason for choosing small sample sizes ($N = 200$) was to investigate the model under minimal and realistic conditions. For example, previous simulation studies have shown that at least 100 level-2 units (here: targets) are required for valid parameter estimates (see Julian, 2001). Therefore, the minimum sample size in the simulation studies I and II was set to 200 observations ($nL1 = 2, nL2 = 100$). The simulation designs of study III and IV were similar to the simulation designs of the latent state and latent change model. However, given that the LST-COM and LGC-COM model more complex (i.e., more parameters) than the LS-COM model, the sample size on the between (target) level was increased to 350, 500, or 700 in the LST-COM simulation study, and to 400, 600, or 800 observations in the LGC-COM simulation study. With respect to these adjustments all cells in the simulation designs in Table 6.1 could be simulated. In total, 288 (232 + 56) conditions were simulated for the simulation study III and IV. Again, for each condition 500 MC sample were simulated ($500 \times 288 = 144,000$ data sets). Finally, the number of measurement occasions was set to 3, 4 or 5 in the LGC-COM model simulation, given that latent growth curve models require at least 3 occasions of measurement to be identified (see e.g., Geiser, 2012).

6.3 General procedure

Due to the complexity of the simulation designs and models, it was necessary to assign different MC conditions to multiple computers (max. 28 PCs). All simulations were carried out in the computer lab of the Freie Universität Berlin. By this procedure, it was possible to reduce the duration of estimation notably (approximately 6 days for each simulation study). Numerous automation syntaxes and R-functions were written. The R-functions were used for creating *Mplus* inputs as well as for extracting different simulation results after the simulation was done. All simulation results were then analyzed in R (see syntaxes in the appendix CD).

The general procedure for running one simulation study encompassed the following steps:

1. Set up the simulation design for a given model and predetermine the population parameters.
2. Specify the correlation matrix of the observed variables for the given model using `OpenMx` (Boker et al., 2011).
3. Check whether or not the correlation matrix and/or subsets of the correlation matrix are positive definite and invertible.
4. Transform the correlation matrix into a variance-covariance matrix of the observed variables using the R package `corcounts` (Erhardt, 2009) and use the unstandardized parameters in order to create *Mplus* input files with the true population model.

5. Create *Mplus* input templates for all conditions of the simulations design using the package `MplusAutomation` (Hallquist, 2011).
6. Run a test simulation and fix syntax errors, estimate the approximate duration of estimation, and organize the simulation study.
7. Run the simulation study and save all data files and outputs.
8. Write R functions and syntaxes to automate the extraction of the simulation results, (e.g., parameter estimation bias, standard error bias etc.). Extract all needed informations.
9. Re-run part of the simulation in order to calculate the number of “actual” Ψ -problems.
10. Analyze the results of the simulation study.

All models were simulated using *Mplus* 6.1 (L. K. Muthén & Muthén, 1998-2010), the free software R 2.14.0 (R Development Core Team, 2008), as well as various R packages such as `MplusAutomation` (Hallquist, 2011), `OpenMx` (Boker et al., 2011), and `corcounts` (Erhardt, 2009).

6.3.1 Estimators

The LS-COM and LC-COM model were estimated using two different estimators: 1) the maximum likelihood (ML) estimator and 2) the robust maximum likelihood (MLR) implemented in *Mplus* 6.1. Because of this additional condition, it was possible to compare the simulation results of both estimators. With respect to the LST-COM and LGC-COM model only one estimator, the robust maximum likelihood (MLR) estimator was used. This was done, because the MLR-estimator is the default estimator for multilevel analyses in *Mplus* and is also generally recommended for this type of analysis (L. K. Muthén & Muthén, 1998-2010). However the ML and MLR results should only differ to a small extent, given that the standard errors as well as the χ^2 fit statistics are adjusted under MLR (see Satorra & Bentler, 1994, 2001; Yuan & Bentler, 2000). The parameter estimates as well as the parameter estimate biases are unaffected and remain the same for both estimators.

6.3.2 Criteria for evaluating the performance of the models

The performance of the models was evaluated according to the similar criteria used by Crayen (2008) and Geiser (2008):

Rate of non-convergence

The term “convergence” refers to the ability of a particular SEM-software (e.g., *Mplus*) to find a unique solution for the parameters of the model after a certain number of iterations (Geiser, 2008). The default number of iterations in *Mplus* 6.1 is 1000. If a software (e.g., *Mplus*) is unable to find a unique solution for the parameter of the model after a certain number iterations, the estimation process has not converged. According to previous simulation studies complex models usually require a larger number of iterations (Geiser, 2008). It is important to distinguish between

the convergence of the H0 and the H1 model. The H0 model represents the specified model, whereas the H1 model represents the saturated model. In the current simulation studies the number of replications in which the H0 model (i.e., specified model) did not converge was recorded. According to Geiser (2008) the non-convergence rate should be below 1 %.

Improper solutions

Improper solutions indicate estimation problems of particular model parameters, for example the residual covariance matrix (so-called: Θ -problems) or the variance-covariance matrix of the latent variables (so-called Ψ -problems). In the current simulation studies the number of Θ -problems as well as the number of “actual” Ψ -problems was recorded. No more than 5 % of all replications should refer to “actual” improper solutions (Crayen, 2008; Geiser, 2008).

Relative parameter estimation bias (peb)

The accuracy of the parameter estimation was investigated according to the absolute value of the relative parameter estimation bias (peb). The peb coefficient can be seen as standardized indicator of parameter bias and is calculated by the following formula:

$$peb = \frac{|M_p - e_p|}{e_p}. \quad (6.1)$$

M_p is the average of the MC parameter estimates (over all replications) and e_p is the true population value. The absolute value of the standardized peb was taken in order to average across different sets of parameters (e.g., factor loadings, latent covariances, residual variances etc.). The parameter estimation bias should not exceed 10 percent of any parameter of the model.

Relative standard error bias (seb)

The accuracy of the standard errors was evaluated with respect to the absolute value of the relative standard error bias. The seb can be considered as standardized indicator of standard error bias. Significant standard error bias can lead to serious bias in significant testing (Geiser, 2008). The seb is calculated by the following formula:

$$seb = \frac{|M_{SE} - SD_p|}{SD_p}. \quad (6.2)$$

The cutoff value of the seb in the present simulation studies is .10. The absolute value of the relative seb was taken, in order to average across different sets of parameters.

χ^2 -Test

The adequacy of χ^2 fit statistics in multilevel structural equation modeling is a delicate and ongoing research topic. Recently, researchers provided methods in order to obtain level-specific χ^2 fit statistics for multilevel structural equation models (Yuan & Bentler, 2003, 2007; Ryu & West, 2009). Besides, the results of previous simulation studies suggests that the χ^2 fit statistic are not trustworthy for complex structural equation models (Crayen, 2008; Geiser, 2008). In the

present study, the adequacy of the χ^2 fit statistic was only evaluated for the LS-COM and LC-COM model, given that these models represent the least complex models (with respect to the number of parameters). This analysis was not repeated for the LST-COM and LGC-COM model, because the computing time for obtaining the additional χ^2 fit statistics would have increased to an unacceptable amount. By not computing the χ^2 fit statistics for these models the elapsed time of computation was reduced by 80 %. In order to evaluate the robustness of the χ^2 fit statistic the observed Monte Carlo χ^2 distribution was compared to the theoretical χ^2 distribution. A large discrepancy between both distributions would indicate a biased χ^2 fit statistic. The adequacy of the χ^2 fit statistic was evaluated according to the following criterion: The proportion of models that would be rejected at a nominal 5 % alpha level on the basis of the theoretical χ^2 distribution should not be larger than .10 according to the MC χ^2 distribution (see Crayen, 2008; Geiser, 2008).

6.4 General expectations

Simulation studies are mainly considered as an exploratory method. Therefore, it is not very common to explicitly formulate statistical hypothesis. However, in the next section a list of general expectations will be provided. These expectations can be seen as “working hypothesis” which may be formulated based on the findings of previous simulation studies.

Convergence

All specified models (i.e., LS-COM, LC-COM, LST-COM, LGC-COM) should converge. According to previous simulation studies, complex models require a higher number of iterations in order to converge (see Geiser, 2008). The main focus of the subsequent simulation studies concerns the convergence rate of the specified (H0) model, not of the saturated (H1) model.

Improper solutions

In general, the number of “actual” improper solutions should be low across all conditions and across all four simulated studies. However, it is assumed that more *Mplus* error messages (potential improper solutions) will be encountered for the Ψ matrix than for the Θ matrix. There are several reasons for that: First, *Mplus* error messages referring to Ψ -problems do not necessarily indicate “actual” Heywood cases (as indicated above), whereas Θ error messages generally do. Second, the Ψ -problems may result from higher order partial correlations among latent variables. Many previous simulation studies indicate that improper solutions with respect to variance-covariance matrix Ψ of the latent variables occur frequently in complex MTMM-SEM models (Crayen, 2008; Geiser, 2008; Lance, Noble, & Scullen, 2002; Marsh & Bailey, 1991; Marsh, Byrne, & Craven, 1992). It can be expected, that the amount of *Mplus* error messages referring to Ψ -problems increases with an increase of the model complexity (e.g., increasing number of constructs, items, methods) but decreases with an increase of the number of empirical information (e.g., sample size, number of measurement waves). Moreover, it can be assumed that the number of Ψ -problems is greater in the high consistency than in the low consistency conditions. The reason for this assumption is that the amount of method variance in the high consistency condition is very low compared to the

low consistency condition. In fact, a higher degree of convergent validity (i.e., high consistency) was chosen in the simulation studies than usually present in real data applications. This was done, to put the models through an endurance test and investigate the performance of the models under extreme (but rather unlikely) data conditions. Therefore, a greater number of “actual” improper solutions is expected in the high than in the low consistency condition.

Parameter estimate bias and standard error bias

Overall, the amount of bias of parameter estimates (peb) as well as standard errors (seb) should be low. With respect to findings of previous simulation studies it can be expected that the bias of parameter estimation is lower than the bias of standard errors. Moreover, it can be assumed that the amount of bias (peb or seb) increases with increasing model complexity (e.g., number of parameters or constructs) and decreases with an increasing number of empirical information (e.g., sample size, occasions of measurement). Furthermore, it is expected that bias of parameter estimates and standard errors occur more often in the high consistency condition than in the low consistency condition.

χ^2 -fit statistics

In general, it is expected that the observed χ^2 -distribution does not approximate well the theoretical χ^2 -distribution well for complex models or in conditions with few observations. Unfortunately, *Mplus* 6.1 only reports unscaled (uncorrected) χ^2 fit statistics (i.e., ML χ^2 fit statistics) in the Monte Carlo option, even when using the MLR estimator (L. K. Muthén & Muthén, 1998-2010). Hence, all χ^2 fit statistics will be based on maximum likelihood estimation.

ML vs. MLR estimator

Many researchers recommend to use maximum likelihood estimator with robust standard errors (MLR) instead of the regular maximum likelihood estimator (ML) when modeling multilevel (hierarchical) data structures (see L. K. Muthén & Muthén, 1998-2010; Satorra & Bentler, 1994, 2001; Yuan & Bentler, 2000). However, it is important to note that the MLR estimator is based on the regular ML estimator and solely adjusts the χ^2 fit-statistics and the standard errors for the parameter estimates (L. K. Muthén & Muthén, 1998-2010). It can be expected that both estimators (ML and MLR) yield similar results, if (a) the sample size is relatively large, (b) the data multivariate is normally distributed and (c) the multilevel structure is explicitly modeled (i.e., no additional clustering or dependency). With respect to the subsequent simulation studies, all of these assumptions are met. Therefore, it is assumed that the difference between the ML and the MLR estimates will be negligible. However, it is expected that the MLR estimator outperforms the regular ML estimator, when one or more of the assumptions above are not met.

6.5 Statistical analyses

In order to determine favorable or unfavorable conditions for the applicability of the models different types of biases (peb and seb) were related to different MC conditions (i.e., number of constructs, method, measurement occasions, sample size, consistency condition). It is very common to use analyses of variance (e.g., ANOVA or MANOVA) for analyzing simulation results. When conducting an analysis of variance (ANOVA) different MC conditions are regarded as fixed factors. However, sometimes it is more reasonable to consider different MC conditions as random factors. For example, in cases in which (a) the number of MC conditions is large (above 100), (b) the amount of bias varies across different conditions and/or different parameters (i.e., multiple crossed random effects), (c) the simulated models contain numerous parameters (e.g., covariances, variances, factor loadings etc.) with varying degree of bias, and (d) different MC conditions can be seen as random samples of data conditions which might occur in real data applications. In these cases, considering the different MC conditions as random may be beneficial, given that a multilevel model can be used for the analysis instead of a general ANOVA. In fact, multilevel analysis has many advantages, for instance less strict assumptions and higher flexibility, prevention of standard error bias, inflated type 1 error (see Geiser, 2012). For the analysis of the simulation studies in the present work, a multilevel model with two crossed random effects was specified. The reason for using multilevel analysis with crossed random effects was that both types of biases (peb and seb) varied across different conditions and different types of parameters (e.g., factor loadings, latent covariances etc.). With this procedure, it was possible investigating which MC conditions are most (un)favorable and which set of parameters (e.g., latent covariance, factor loadings, intercepts, etc.) is most sensitive to bias in a single analysis. Given that the absolute values of the peb and seb coefficients were calculated, the pep and seb (dependent variables) were extremely positive skewed. For that reason, the absolute peb and seb values were log-transformed before entering in the model. In addition to that, a small constant (.00001) was added to the peb and seb coefficients, given that some peb and seb values approached zero. This log-transformation was done according to the recommendations of Cohen, Cohen, West, and Aiken (2003, p. 235). The different types of MC conditions (e.g., consistency, sample size on level-1 and level-2, number of methods, number of constructs, number of occasions) were dummy coded and used as independent variables for the prediction of bias in parameter estimates (peb) and standard errors (seb). The unstandardized regression coefficients were then back-transformed by taking an exponential function. As a consequence of the exponential transformation, the unstandardized regression weights (β_i) of the dummy coded independent variables (C_i) can be interpreted as the percentage impact of C_i on Y (i.e., peb or seb) controlling for all other independent variables in the model (see Giles, 2011). The back-transformation formula can be expressed as follows: $100 * [exp(\beta_i) - 1]$. The analyses was performed in R using `lme4` (D. Bates, Maechler, & Bolker, 2011). For more information concerning multilevel regression with crossed random effects see D. M. Bates (in press).

Chapter 7

Simulation I : Latent state (LS-COM) model

7.1 Specification of the population model

Table 7.1 provides informations regarding the specification of the population LS-COM model. As indicated in Chapter 2 (see Figure 2.4), a LS-COM model with common latent state factors was simulated. In other words, it was assumed that the indicator-specific latent state variables are perfectly correlated. The reliability was set to .8 and portioned into the variance due to consistency and method specificity. In the low consistency (low convergent validity) condition the common and unique method specificity coefficient was set to .25, whereas the method specificity coefficient of the third method was set to .5. The consistency coefficient was set to .3. In empirical applications items do not often have equal reliability coefficients. In order to achieve most realistic conditions in the simulation studies, all model parameters varied across items per CMOU (construct-method-occasion unit). That is, for each item per CMOU a constant of .025 (.05) was either subtracted or added to the coefficients given in table 7.1. For example, the reliability coefficients for the indicators per construct-method-occasion-unit (CMOU) was .775, .8, and .825. Moreover, strong measurement invariance was assumed with respect to the simulation of the LS-COM model (Meredith, 1993; Widaman & Reise, 1997). Again, strong measurement invariance implies restrictions on the latent intercepts and latent factor loadings for each item belonging to the construct j , method k , but different occasions of measurement l and l' . Furthermore, the H1 iterations were set to 7500 for all models in order to provide goodness-of-fit statistics. Note that the H1 model refers to the saturated and not to the specified model. A complete *Mplus* Monte Carlo input of a LS-COM model is given on the appendix CD-ROM.

7.2 Results

7.2.1 Convergence

All models converged.

Table 7.1: Consistency, method specificity & reliability of the LS-COM model

	low consistency		high consistency	
consistency	.30	($\pm.025$)	.60	($\pm.025$)
unique method specificity	.25	($\pm.025$)	.10	($\pm.025$)
common method specificity	.25	($\pm.025$)	.10	($\pm.025$)
method specificity	.50	($\pm.050$)	.20	($\pm.050$)
reliability	.80	($\pm.025$)	.80	($\pm.025$)

Note. The values for the remaining items per CMOU varied by .025.

7.2.2 Improper solutions

Across all 232 conditions, 65 (28.0 %) conditions contained replications with improper solutions with respect to the latent covariance matrix Ψ . Most of these Ψ -warning messages were encountered in the multiconstruct conditions (37 out of 65 conditions, 56.9 %) compared to the monoconstruct conditions (28 out of 65 conditions, 43.0 %). In addition, more Ψ -warnings messages were encountered in high consistency (56 out of 65 conditions, 86.2 %) conditions (see Table 7.2) for both multiconstruct as well as monoconstruct measurement designs.

Table 7.2: Amount of Ψ -Problems in Multi- and Monoconstruct Designs for High and Low Consistency Condition.

	Multiconstruct		Monoconstruct		Total	
High consistency condition	32	(86.4 %)	24	(85.7 %)	56	(86.2 %)
Low consistency condition	5	(13.5 %)	4	(14.3 %)	9	(13.8 %)
Total	37	(56.9 %)	28	(43.0 %)	65	(100.0 %)

Note. Results do not represent the “actual” amount of Ψ -problems.

It is important to note that these results do not represent the amount of “actual” Ψ -problems, but rather the total amount of *Mplus* warning messages (i.e., “possible” Ψ -problems). In fact, only 69 (3.09 %) out of 2231 *Mplus* warning messages referred to “actual” Ψ -problems. Hence, the amount of “actual” Ψ -problems was below the cut-off value of 5 %. According to Figure 7.1, the “potential” Ψ -problems occurred more often in high than low consistency conditions and decreased substantially with increasing level-1 as well as level-2 units (number of raters and targets). Moreover, the number of level-1 units (i.e., number of raters per target) seems to be especially important for the reduction of “potential” Ψ - problems. For example, in Figure 7.1 it can be noticed that the amount of “potential” Ψ - problems decreases substantially with more than 2 raters per target. However, in Figure 7.1 it is not yet clear how many observations are sufficient for proper parameter estimates. In order to get a better understanding of how many observations are needed for reducing the amount of “potential” Ψ -problems, the ratio “observation per parameter” was calculated and related to the amount of Ψ -problems (see Bentler & Chou, 1987; Bollen, 1989, 2002).

Figure 7.2 shows the relationship between the amount of Ψ -problems and the amount of observations per parameter. According to the Figure 7.2 the amount of Ψ -problems decreased substantially as the amount of observations per parameter increased. As potential cutoff value, at least five (bet-

ter ten) observations per parameter are needed to reduce the amount of “potential” Ψ -problems notably. This recommendation is in line with the findings of previous simulation studies by Bentler and Chou (1987) (see straight line in Figure 7.2) and Bollen (1989, 2002) (see dashed line in Figure 7.2). Moreover, the Figure 7.2 reveals that more “potential” Ψ -problems are more likely to occur in the high consistency condition than in the low consistency condition. Most of these “potential” Ψ -problems refer to improper solutions with regard to the correlations among the latent method factors (common as well as unique method factors). This might be explained by the fact that a higher amount of consistency implies a lower amount of method variance (i.e., lower method bias). Therefore the LS-COM model is over-factorized in these MC conditions. However, if method bias exists (low consistency condition) almost no Ψ -problems occur. Overall, only 2 out of 232 conditions contained Θ -problems. Both of these errors occurred in conditions referring to LS-COM models in the high consistency condition incorporating 1 or 2 constructs, 2 methods, 2 occasions of measurement and a total sample size of 200 ($nL1 = 2$, $nL2 = 100$). Thus, the Θ -problems occurred only in conditions of low sample sizes on the rater- as well as target-level. These results show the importance of level-1 and level-2 units for the reduction of improper solutions.

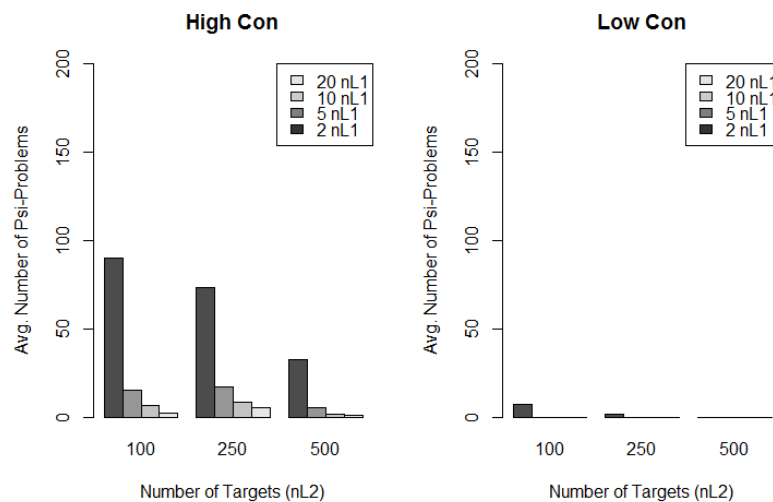


Figure 7.1: Average number of Ψ -problems in high and low consistency conditions. $nL1$ = number of level-1 units; $nL2$ = number of level-2 units.

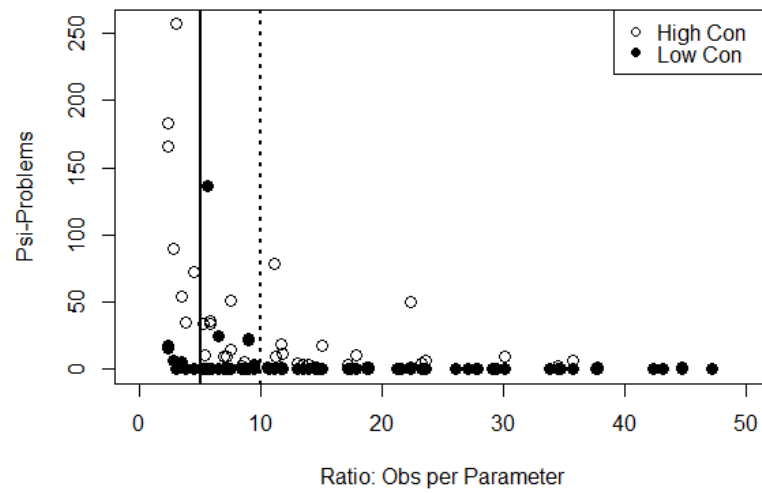


Figure 7.2: Relation between Ψ -problems and observations per number of parameters.

7.2.3 Bias of parameter estimates and standard errors

Across all 232 conditions the absolute parameter estimation bias (peb) was below .1 and thereby did not exceed the critical cutoff value of .1. However, the standard error bias (seb) exceeded the cutoff value of .1 in 21 out of 232 (9.1 %) conditions. Higher standard error biases (seb above .1) were more often encountered in the monoconstruct (14 out of 21 conditions, 66.7 %) than in the multiconstruct (7 out of 21 conditions, 33.3 %) condition. In addition, higher standard error biases (seb above .1) were more often found in the high than in the low consistency condition (see Table 7.3). These results indicate that bias of standard errors (seb) are more likely to occur if the amount of method bias is low (i.e. high convergent validity). Again, this might be partially explained by the fact that the LS-COM model implicitly assumed substantial method bias that can be modeled. If the different types of method (i.e., structurally as well as interchangeable raters) perfectly converge in their ratings, the LS-COM model would be over-factorized for the data.

Amount of seb in Multi- and Monoconstruct Designs for High and Low Consistency Conditions.

Table 7.3: Amount of seb in LS-COM model

	Multiconstruct		Monoconstruct		Total	
High consistency condition	4	(57.1 %)	10	(71.4 %)	14	(66.6 %)
Low consistency condition	3	(42.9 %)	4	(28.6 %)	7	(33.3 %)
Total	7	(33.3 %)	14	(66.6 %)	21	(100.0 %)

Figure 7.3 illustrates the average peb and seb values with regard to the high and low consistency condition. The two figures in the upper row refer to the average peb values in the high and low consistency condition, whereas the two figures in the lower half refer to the average seb values in the high and low consistency condition. According to the figures, peb as well as seb values are lower in the low consistency condition than in the high consistency condition (i.e., high convergent validity). Interestingly, the peb as well as seb values decrease substantially with increasing sample size. Again, the sample size on level-1 (raters per targets) seems to be important for the reduction of parameter as well as standard error bias. The major drop in the average seb can be noticed for five instead two raters per target. It can also be seen in Figure 7.3 that the average peb and seb values did not exceed the critical cut-off value of .1.

In order to investigate possible reasons for bias in parameter estimates (peb) as well as standard errors (seb), a multilevel regression analysis with two random effect terms (one for the parameters types and one for the condition types) was carried out. The two random effect terms were specified, given that the amount of bias in parameter estimates and standard errors varied across different types of parameters (e.g., factor loadings of latent factors, covariances among latent factors, variances of latent factors etc.) as well as across conditions (i.e., in total 232). The multilevel analysis with cross random effects was carried out using the package `lme4` (D. Bates et al., 2011). The results are given in Table 7.4. Note that the dependent variables (i.e., peb and seb values) were first log-transformed and afterwards back-transformed as explained in section 6.5. Therefore, the unstandardized regression weights may be interpreted as average percentage increase or decrease in bias in the given group (e.g., 500 targets) with respect to the reference group (e.g., 100 targets)

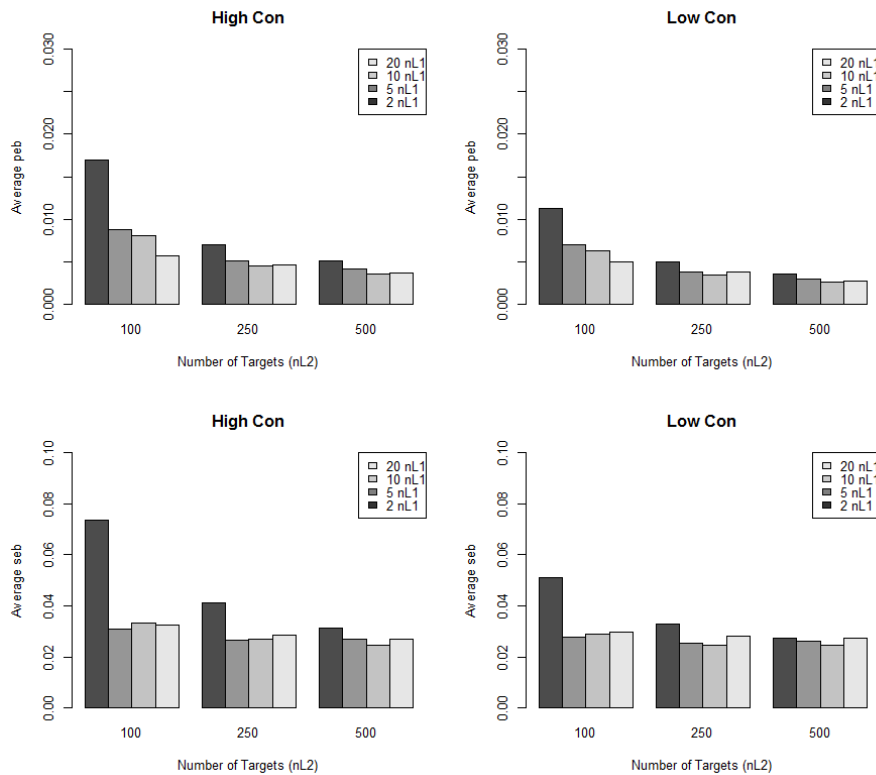


Figure 7.3: Average peb and seb values with respect to sample size in high and low consistency conditions in the LS-COM model. nL1 = number of level-1 units; nL2 = number of level-2 units.

holding everything else constant (*ceteris paribus*). According to the results given in Table 7.4, the bias in parameter estimates (peb, see model 1) increased significantly with an increase of the number of constructs (32 % increase of bias), but decreased significantly with an increase in the number of measurement occasions (8-10 % decrease of bias), number of raters (36-51 % decrease of bias) and targets (50-64 % decrease of bias). Again, bias in parameter estimates was rather found in high consistency than in low consistency conditions (21 % increase of bias). Interestingly, the number of methods was not significantly associated with bias in parameter estimates. This implies that an increase of methods (e.g., parents, teacher, self-ratings etc.) does not lead to more bias of the parameter estimates. According to the results regarding model 2, the standard error bias (seb) decreased significantly with increasing number of raters (20-28 % decrease of bias) and number of targets (21-29 % decrease of bias). The seb also increased with model complexity (i.e., number of constructs and measurement waves; 12-19 % increase of bias). Moreover, bias in standard error was significantly more often found in the high convergent validity condition than in the low convergent validity condition (6 % increase of bias). According to these results, the amount of parameter as well as standard error bias can be substantially reduced with increasing sample size on both levels.

With respect to the variability of the peb as well as seb-values across different types of parameters, it is possible to evaluate which class of parameter (e.g., factor loadings, latent variances, covariances) is most sensitive to bias. The variability of peb and seb-values across different types of parameters is displayed in Figure A.1 and A.2 in the appendix. The 95 % prediction intervals of the random effects confirm that the conditional distribution of the parameter estimation bias

Table 7.4: Estimates for the prediction of bias in parameter estimates (peb) and standard errors (seb) in the LS-COM model.

Parameter	Model1(peb)	Model2(seb)
-2*LL	6085.64	5560.64
AIC	6164.33	5643.85
BIC	6246.72	5726.23
Fixed effects		
Constant	0.01*** (0.22)	0.03*** (0.06)
Level-2 (conditions)		
Methods 3 vs 2	0.04 ^{ns} (0.03)	0.17*** (0.03)
Constructs 2 vs 1	0.32*** (0.03)	0.19*** (0.03)
Occasion 3 vs 2	-0.08* (0.03)	0.12*** (0.03)
Occasion 4 vs 2	-0.10** (0.04)	0.15*** (0.04)
Rater 5 vs 2	-0.36*** (0.04)	-0.23*** (0.04)
Rater 10 vs 2	-0.45*** (0.04)	-0.28*** (0.04)
Rater 20 vs 2	-0.51*** (0.04)	-0.20*** (0.04)
Target 250 vs 100	-0.50*** (0.04)	-0.21*** (0.04)
Target 500 vs 100	-0.64*** (0.04)	-0.29*** (0.04)
Cond low vs high	-0.21*** (0.03)	-0.06* (0.03)
Random effects		
Level-1		
σ_r^2	0.57	0.47
Level-2		
σ_{u01}^2 (con)	0.00	0.00
σ_{u02}^2 (par)	0.56	0.01

Note. Reference group is a LS-COM model with 3 indicators, 1 construct, 2 methods, 2 occasions of measurement, 2 raters per target, 100 targets in the high consistency condition. standard errors are in parentheses. peb= log-transformed parameter estimation bias; seb = log-transformed standard error bias; LL = log likelihood; N= 2656; (con) = condition type (232); (par) = parameter type (12).

* $p < .05$; *** $p < .001$; ^{ns} not significant.

for different parameters showed less variability than the conditional distribution of standard error bias of different parameters. Moreover, bias in parameter estimates seem to be more sensitive with respect to the estimation of latent covariances on the rater- and target-level, whereas the standard error bias was rather associated with the factor loadings of different kinds of method factors (λ_{CMij2l} , λ_{UMij2l} , λ_{Mijkl}) as well as the variance of the unique method factor [$Var(UM_{rtj2l})$].

7.2.4 Appropriateness of χ^2 fit statistics

In Table 7.5 and 7.6 the expected and observed proportions of the χ^2 fit-statistics for different sample sizes and different models are given. The results of Table 7.5 refer to the χ^2 fit statistics of a monotrait LS-COM model with 3 indicators, 1 construct, 2 methods and 2 occasions of measurement. The monotrait LS-COM model represents the least complex LS-COM model. The results given in Table 7.6 refer to the χ^2 fit statistics of a multitrait LS-COM model with 3 indicators, 2 constructs, 2 methods and 2 occasions of measurement. The values in the first column refer to the probability of observing a χ^2 -value greater than the corresponding percentile values from a theoretical χ^2 distribution with the degrees of freedom given by the model (L. K. Muthén & Muthén, 1998-2010). The bold values of .05 in the first columns of Table 7.5 and 7.6 indicate the nominal alpha level of 5 % for the theoretical χ^2 distribution with the degrees of freedom given by the model. The bold values (in columns 2 to 12) correspond to the observed values in the MC replications. According to the results given in Table 7.5 and 7.6 the theoretical χ^2 -values were well recovered by the observed χ^2 -values on a nominal alpha level of 5 % (see bold value in Table 7.5 and 7.6). The discrepancies between the observed and theoretical proportions on a nominal alpha level of 5 % vary between .01 und .05. However, in all of these cases, the observed χ^2 -values were lower than the theoretical χ^2 -values, implying a downward bias in asymptotic type I errors. This means that too many specified models would be accepted according to the observed χ^2 test statistics, which results in too liberal χ^2 model fit tests. As expected the χ^2 fit statistics were less downward biased in the monotrait condition (i.e., for less complex models) compared to the multitrait condition. Interestingly, there is no clear-cut interpretation with regard the relationship between χ^2 fit statistics and different samples sizes (cf. Crayen, 2008; Geiser, 2008). A graphical representation of these results is given in Figures B.1 and B.2 in the appendix). Given that these results refer to the general maximum likelihood χ^2 values, a scaling (correction) factor of the χ^2 values may be appropriate (see Yuan & Bentler, 2000).

Table 7.5: Expected and observed proportions of the χ^2 -statistic for different sample sizes in the low consistency condition for a monotrait LS-COM model.

Expected proportions	Observed proportions											
	2x100	2x250	2x500	5x100	5x250	5x500	10x100	10x250	10x500	20x100	20x250	20x500
.99	.97	.96	.95	.97	.96	.96	.96	.98	.97	.97	.96	.98
.98	.95	.94	.93	.94	.93	.93	.93	.94	.94	.94	.94	.96
.95	.87	.88	.84	.89	.87	.86	.87	.89	.88	.87	.87	.92
.90	.78	.79	.77	.78	.76	.77	.80	.79	.78	.78	.78	.86
.80	.62	.60	.65	.62	.61	.62	.67	.63	.64	.64	.67	.71
.70	.51	.50	.49	.49	.48	.51	.55	.50	.52	.52	.57	.60
.50	.33	.30	.30	.29	.30	.32	.36	.34	.30	.33	.37	.39
.30	.21	.16	.13	.14	.14	.15	.18	.17	.16	.18	.18	.18
.20	.13	.09	.07	.08	.08	.09	.09	.10	.08	.10	.11	.11
.10	.06	.03	.03	.04	.03	.04	.03	.04	.04	.04	.05	.07
.05	.04	.01	.01	.02	.01	.02	.01	.02	.01	.01	.02	.03
.02	.01	.00	.00	.00	.01	.01	.00	.00	.00	.00	.01	.01
.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00

Note. LS-COM model with one construct, two methods, two occasions of measurement, and three indicators per CMOU (3 x 1 x 2 x 2 version) and with 37 degrees of freedom; Expected proportions = proportions based on the theoretical chi-square distribution; 2 x 100, 10 x 250, etc. indicate the sample size on level-1 and level-2.

Table 7.6: Expected and observed proportions of the χ^2 -statistic for different sample sizes in the low consistency condition for a multitrait LS-COM model.

Expected proportions	Observed proportions											
	2x100	2x250	2x500	5x100	5x250	5x500	10x100	10x250	10x500	20x100	20x250	20x500
.99	.98	.93	.91	.94	.90	.95	.95	.93	.93	.95	.95	.95
.98	.95	.89	.87	.91	.88	.89	.91	.88	.89	.91	.91	.93
.95	.90	.83	.78	.82	.78	.78	.81	.79	.80	.82	.82	.85
.90	.79	.69	.68	.71	.68	.65	.71	.67	.69	.72	.71	.74
.80	.65	.51	.50	.56	.51	.50	.55	.49	.53	.55	.54	.58
.70	.52	.38	.36	.43	.35	.38	.42	.35	.40	.41	.41	.43
.50	.34	.23	.19	.23	.20	.20	.27	.20	.22	.20	.23	.21
.30	.18	.09	.08	.12	.09	.08	.12	.10	.10	.10	.09	.11
.20	.10	.06	.03	.06	.05	.04	.07	.06	.04	.05	.05	.05
.10	.04	.03	.02	.02	.01	.02	.03	.03	.02	.01	.02	.01
.05	.02	.00	.00	.01	.01	.01	.01	.01	.01	.01	.01	.00
.02	.01	.00	.00	.00	.00	.00	.01	.00	.00	.01	.00	.00
.01	.00	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00

Note. LS-COM model with two construct, two methods, two occasions of measurement, and three indicators per CMOU (3 x 2 x 2 x 2 version) and with 86 degrees of freedom; Expected proportions = proportions based on the theoretical chi-square distribution; 2x100, 10x250, etc. indicate the sample size on level-1 and level-2.

7.3 Results based on the MLR estimator

In line with the general expectations, the average parameter bias (peb) was identical for all sets of parameters regardless of the type of estimator (ML or MLR) used (see Table 7.7). Moreover, the average standard error bias differed negligibly (second and third decimal place) with respect to the different estimators used (see Table 7.8). In contrast to the general expectations, the average standard error bias was higher in the MLR condition than in the ML condition. This might be partially explained by the fact that the MLR estimator corrects the standard errors (by adding a constant) which may be not necessary for the simulated data structure, given that all assumptions of the general maximum likelihood estimator (e.g., multivariate normality, no additional clustering, sufficient sample size) were met. Finally, it is rather encouraging to see that the average peb and seb values are relatively small for all parameters across all MC conditions (see Table 7.7 and 7.8).

Table 7.7: Average absolute peb values for different LS-COM model parameters and different maximum likelihood estimators.

	Peb	ML	MLR
State factor loadings	0.001	0.001	0.001
Common method factor loadings	0.005	0.005	0.005
Between covariances	0.016	0.016	0.016
Between latent means	0.005	0.005	0.005
Between intercepts	0.002	0.002	0.002
Between variances	0.007	0.007	0.007
Between residuals	0.005	0.005	0.005
Unique method factor loadings	0.004	0.004	0.004
Within covariances	0.013	0.013	0.013
Within variances	0.003	0.003	0.003
Within residuals	0.003	0.003	0.003

Note. ML= maximum likelihood estimator;
MLR=robust maximum likelihood estimator;
peb= average parameter estimate bias.

Table 7.8: Average absolute seb values for different LS-COM model parameters and different maximum likelihood estimators.

	Seb	ML	MLR
State factor loadings	0.029	0.032	0.032
Common method factor loadings	0.048	0.055	0.055
Between covariances	0.029	0.032	0.032
Between latent means	0.026	0.026	0.026
Between intercepts	0.026	0.026	0.026
Between variances	0.031	0.035	0.035
Between residuals	0.029	0.033	0.033
Unique method factor loadings	0.036	0.038	0.038
Within covariances	0.031	0.033	0.033
Within variances	0.032	0.034	0.034
Within residuals	0.025	0.027	0.027

Note. ML= maximum likelihood estimator;
MLR=robust maximum likelihood estimator;
seb= average standard error bias.

Chapter 8

Simulation II : Latent change (LC-COM) model

8.1 Specification of the population model

The population parameters of the LC-COM model were set to the same values as the population parameters of the LS-COM model (see Table 7.1). For the sake of simplicity, a baseline LC-COM model with common latent change factors for the reference method, but without latent *method* change factors was specified. In other words, with respect to the simulated LC-COM model only “true” change with respect to the reference method was assumed. Given that the LS-COM and the baseline LC-COM are statistically equivalent, the results will be identical, except for variations due to the different Monte Carlo sampling processes. The results regarding the ML- χ^2 fit statistics are not presented here. Results with respect to the amount of improper solutions, bias in estimation of parameter and standard error etc. are provided for reasons of completeness and comparability of the LS-COM model. In addition, all parameter restrictions such as strong measurement invariance, number of H1 iterations (7,500) were set to the same values of the simulation study I (i.e., LS-COM model). An example of the *Mplus* input for the simplest LC-COM model is provided in the appendix CD-ROM.

8.2 Results

8.2.1 Convergence

All models converged.

8.2.2 Improper solutions

In total, 65 out 232 (28.0 %) conditions contained warning messages with respect to “possible” Ψ -problems, whereas only 3 (1.3 %) conditions contained error messages referring to Θ -problems. Again, more Ψ -warning messages were encountered in multi-construct designs than in

mono-constructs designs as well as in the high consistency condition compared to the low consistency condition (see Table 8.1).

Table 8.1: Amount of Ψ -problems in multi- and monoconstruct designs for high and low consistency condition.

	Multiconstruct		Monoconstruct		Total	
High consistency condition	32	(86.4 %)	24	(85.7 %)	56	(86.2 %)
Low consistency condition	5	(13.5 %)	4	(14.3 %)	9	(13.8 %)
Total	37	(56.9 %)	28	(43.0 %)	65	(100.0 %)

Note. Results do not represent the “actual” amount of Ψ -problems.

The maximum number of replications with warning messages was 333. This MC replication referred to LC-COM models in the high consistency condition with 2 constructs, 3 methods, 3 occasions of measurements and a sample size of $N = 500$ (2×250). The percentage of replications with “actual” Ψ -problems was below the 5 % cutoff value (3.0 %, 69 out of 2311).

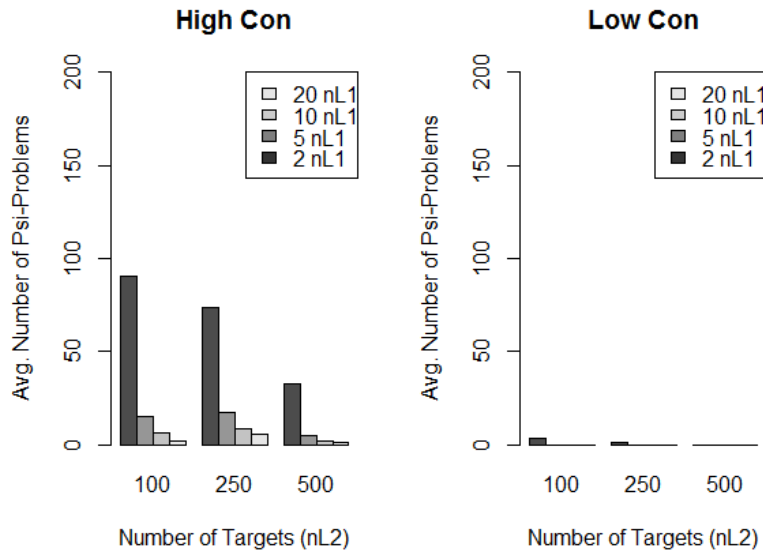


Figure 8.1: Average number of Ψ -problems in high and low consistency conditions. nL1 = number of level-1 units; nL2 = number of level-2 units.

Figure 8.1 confirms that the amount of “possible” Ψ -problems is higher in the high consistency conditions than in the low consistency condition. Moreover, the amount of “possible” Ψ -problems decreased substantially with increasing sample size on both levels. The majority of improper solutions referred to the estimation of the latent correlations among unique and common method factor (e.g., UM_{rtj2l} , CM_{tj2l}), indicating that these parameter estimates are most prone to improper solutions if the sample size on both levels is small and/or only a small amount of method bias is present. The amount of “possible” Ψ -problems decreased exponentially with an increasing number of observations per parameter (see Figure 8.2). Again, a ratio of five observations per parameter reduces the amount of improper solutions notably.

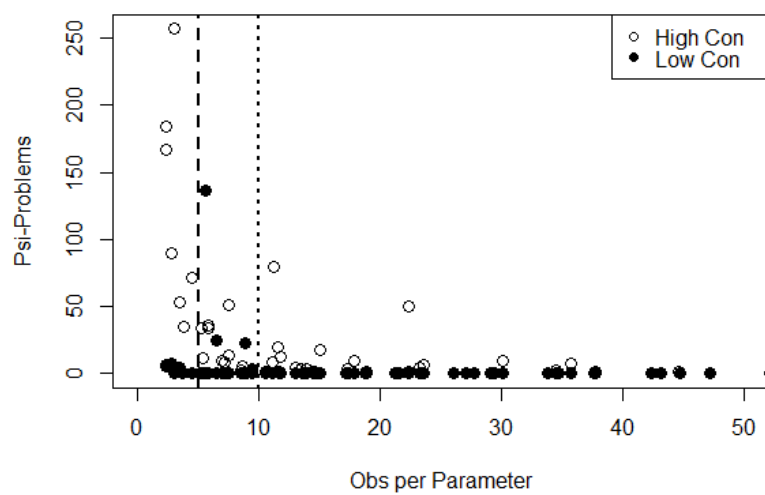


Figure 8.2: Relationship between Ψ -problems and observations per number of parameters.

8.2.3 Bias of parameter estimates and standard errors

Similar to the previous results, no bias of parameter estimates greater than .1 (10 %) was found across all 232 conditions. However, the standard error bias (seb) exceeded the critical cutoff value of .1 in 15 out of 232 (6.5 %) conditions (see Table 8.2). Therefore, the amount of bias with regard to standard errors greater than .1 occurred less frequently in the change than in the state parametrization of the model. However, similar to the LS-COM simulation, it was more likely to find higher standard error bias (seb) in the monoconstruct (11 out of 15, 73.3 %) than in the multiconstruct (4 out of 15, 26.7 %) condition. Besides that, more standard error bias (seb) above |1| were encountered in the high consistency (i.e., high convergent validity) condition.

Amount of seb in multi- and monoconstruct designs for high and low consistency condition.

Table 8.2: Amount of seb in LC-COM model.

	Multiconstruct	Monoconstruct	Total
High consistency condition	3 (75.0 %)	6 (54.5 %)	9 (60.0 %)
Low consistency condition	1 (25.0 %)	5 (45.5 %)	6 (40.0 %)
Total	4 (26.7 %)	11 (73.3 %)	15 (100.0 %)

The average peb and seb values with regard to the high and low consistency condition are illustrated in Figure 8.3. The two figures in the upper row show the average peb values in the high and low consistency condition, whereas the two figures in the bottom row present the average seb values in the high and low consistency condition. Similar to the previous results of the LS-COM model, the peb as well as seb values are lower in the low consistency condition than in the high consistency condition and decrease substantially with increasing sample size.

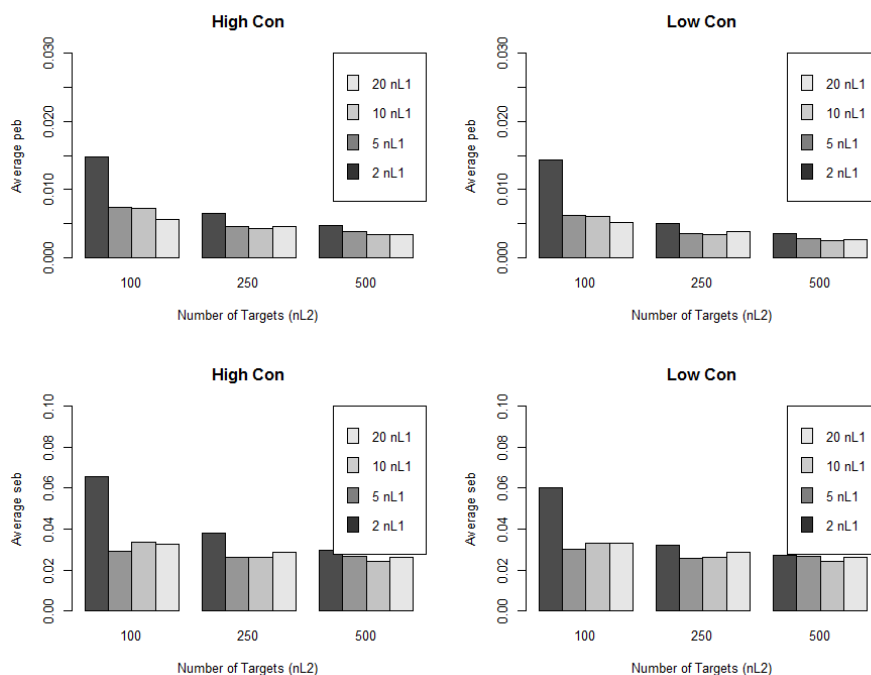


Figure 8.3: Average peb and seb values with respect to sample size in high and low consistency conditions in the LC-COM model. nL1 = number of level-1 units; nL2 = number of level-2 units.

The results of multilevel regression analysis with two crossed random effects are shown in Table 8.3. According to these results, the bias in parameter estimates (peb, see model 1) increases significantly with an increase of model complexity (i.e., number of constructs, 39 % increase of bias), but decreases significantly with increasing sample size (number of raters, 36-47 % and targets, 51-65 %). However, the amount of bias in parameter estimates was not significantly associated with the number of measurement occasions or methods. Again, the average amount peb decrease substantially in the low consistency conditions (10 % decrease of bias). The amount of standard error bias (seb) decreases significantly with increasing samples size (number of raters, 18-25 % and targets, 25-33 %). Similar to the previous results the number of constructs as well as methods was significantly associated with an increase of standard error bias (16 %). Interestingly, the average amount of standard error bias was not significantly associated with the consistency conditions (high vs. low).

The variability of peb and seb-values across different types of parameters is displayed in Figure A.3 and A.4 in the appendix. Again, the estimates of latent covariances on the within (rater) and between (target) level were most sensitive to parameter bias (peb) as well as standard error bias (seb).

Table 8.3: Estimates for the prediction of bias in parameter estimates (peb) and standard errors (seb) in the LC-COM model.

Parameter	<i>Model1(peb)</i>	<i>Model2(seb)</i>
-2*LL	6116.40	5206.02
AIC	6195.23	5290.88
BIC	6277.66	5373.31
Fixed effects		
Intercept	0.01*** (0.21)	0.03*** (0.05)
Level-2 (conditions)		
Methods 3 vs 2	0.04 ^{ns} (0.03)	0.16*** (0.03)
Constructs 2 vs 1	0.39*** (0.03)	0.16*** (0.03)
Occasion 3 vs 2	-0.02 ^{ns} (0.04)	0.09** (0.04)
Occasion 4 vs 2	-0.07 ^{ns} (0.04)	0.12*** (0.03)
Rater 5 vs 2:	-0.36*** (0.04)	-0.21*** (0.04)
Rater 10 vs 2	-0.44*** (0.04)	-0.25*** (0.04)
Rater 20 vs 2	-0.47*** (0.04)	-0.18*** (0.04)
Target 250 vs 100	-0.51*** (0.04)	-0.25*** (0.04)
Target 500 vs 100	-0.65*** (0.04)	-0.33*** (0.04)
Cond low vs high	-0.10** (0.03)	-0.01 ^{ns} (0.03)
Random effects		
Level-1		
σ_r^2	0.57	0.41
Level-2		
σ_{u01}^2 (con)	0.00	0.00
σ_{u02}^2 (par)	0.48	0.01

Note. Reference group is a LC-COM model with 3 indicators, 1 construct, 2 methods, 2 occasions of measurement, 2 raters per target, 250 targets in the high consistency condition. standard errors are in parentheses. peb= log-transformed parameter estimation bias; seb = log-transformed standard error bias; LL = log likelihood; N= 2664; (con) = condition type (232); (par) = parameter type (12).

* $p < .05$; *** $p < .001$; ^{ns} not significant.

Chapter 9

Simulation III : Latent state-trait (LST-COM) model

9.1 Specification of the population model

Table 9.1 summarizes the amount of variance due to trait/state specificity as well as trait/state method specificity that was manipulated in the study. In contrast to the LS-COM and LC-COM simulation studies, the values were not varied across items for reasons of simplicity. The amount of variance due to trait/occasion-specific (unique/common) method influences is rather low in the high consistency condition, varying between 4 and 6.25 %. Parameter restrictions with regard to strong measurement invariance were imposed (see Geiser, Keller, Lockhart, Eid, et al., 2012; Meredith, 1993; Widaman & Reise, 1997). The covariance matrix of the simulated LST-COM model is represented in Chapter 4.6.2. A *Mplus* input of the LST-COM model is provided on the appendix CD-ROM.

Table 9.1: Consistency, method specificity and reliability in the LST-COM population model.

	Low consistency	High consistency
Trait specificity coefficient		
Reference method	49.00	49.00
Non-reference method	12.25	30.25
Trait-specific unique method coefficient	16.00	06.25
Trait-specific common method coefficient	12.25	04.00
Trait-specific method coefficient	30.25	12.25
Occasion specificity coefficient		
Reference method	36.00	36.00
Non-reference method	12.25	30.25
Occasion-specific unique method coefficient	16.00	06.25
Occasion-specific common method coefficient	12.25	04.00
Occasion-specific method coefficient	30.25	12.25
Reliability coefficient		
Reliability (within)	85.00	85.00
Reliability (between)	81.00	81.00

Note. Exact values of the amount of trait/state (method) specificity and reliability in percent. The values did not vary across items.

9.2 Results

9.2.1 Convergence

All H0 models converged. However, the number of iterations for the H1 models was set to 1. Therefore, none of the H1 models converged and no χ^2 fit statistics were produced. This was done in order to reduce the computation time of these complex models (see Section 6.3). With respect to this restriction the elapsed time of the entire simulation could be reduced by 80 %. This also shows that the software *Mplus* needs a lot of time for the estimation of the H1 (saturated) model.

9.2.2 Improper solutions

The percentage of warning messages referring to “possible” improper solution was relatively low across all conditions. In total, 58 out of 288 conditions (20.1 %) contained replications with “possible” improper solutions concerning the latent covariance matrix Ψ . The majority of these warning messages were encountered in the multiconstruct condition (38 out of 58). Less warning messages were given in the monoconstruct condition (20 out of 58). Furthermore, more Ψ -warnings were found in the high consistency (convergent validity) condition (see Table 9.2.2). No warning messages were found concerning Θ -problems.

Table 9.2: Amount of “possible” Ψ -problems in multi- and monoconstruct designs for high and low consistency condition.

	Multiconstruct		Monoconstruct		Total	
High consistency condition	32	(84.2 %)	19	(95.0 %)	51	(87.9 %)
Low consistency condition	6	(15.8 %)	1	(5.0 %)	7	(12.1 %)
Total	38	(65.5 %)	20	(34.5 %)	58	(100.0 %)

Note. Results do not represent the “actual” amount of Ψ -problems.

It is important to note that these results do not represent the amount of “actual” Ψ -problems. In fact, only 379 (10.8 %) out of 3517 *Mplus* warning messages referred to “actual” Ψ -problems. Regarding the absolute number of replications (144,000), the percentage of real Ψ -problems was below 1 %.

More than 90 % of the “actual” Ψ -problems were encountered in high consistency conditions. According to Figure 9.1 the amount of “possible” Ψ -problems decreased substantially with increasing sample size on both levels (i.e., number of targets and raters). Again, a ratio of 5 (better 10) observations per parameter reduces the amount of Ψ -warning messages substantially in the low consistency condition (see Figure 9.2). Note that this ratio incorporates the total number of observations (targets and raters).

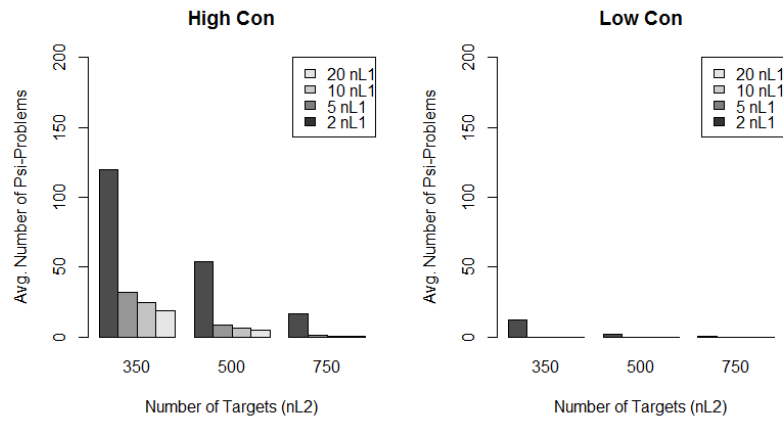


Figure 9.1: Average number of Ψ -problems in high and low consistency conditions. nL1 = number of level-1 units; nL2 = number of level-2 units.

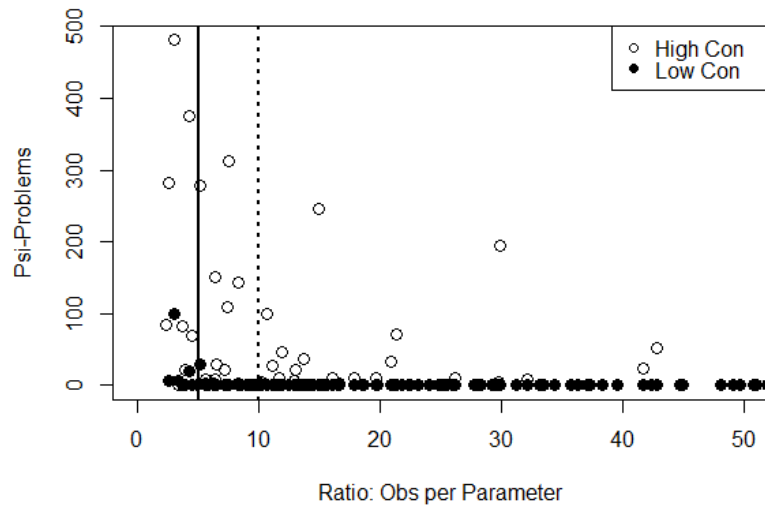


Figure 9.2: Relationship between Ψ -problems and observations per number of parameters.

9.2.3 Bias of parameter estimates and standard errors

The parameter estimation bias (peb) exceeded the cutoff value of .1 (10 %) in 1 out of 288 conditions. This condition referred to LST-COM models with 3 indicators, 2 constructs, 2 methods, 2 occasion of measurement in the high consistency condition with a sample size of 700 (350 targets and 2 raters per target). The peb value of .125 in this MC condition referred to the covariance among two latent unique state factors pertaining to the same occasion of measurement, but different constructs. This high parameter bias may result from the small sample size on level-1 (only 2 raters per target). The standard error bias (seb) exceeded the cutoff value in 2 out of 288 conditions. Both of these MC conditions referred to LST-COM models in the high consistency condition with also few level-1 observations (i.e., 2 raters per target). The increased seb values referred to the standard errors of the covariance between occasion-specific method factors on the between level and to the latent factor loadings of the occasion-specific common method factors. The maximum seb value was .109. As indicated above, bias of parameter estimates and/or standard errors were encountered solely in conditions with 2 rater per target. Figure 7.3 illustrates the average peb and seb values with regard to the high and the low consistency condition. Again, a multilevel analysis

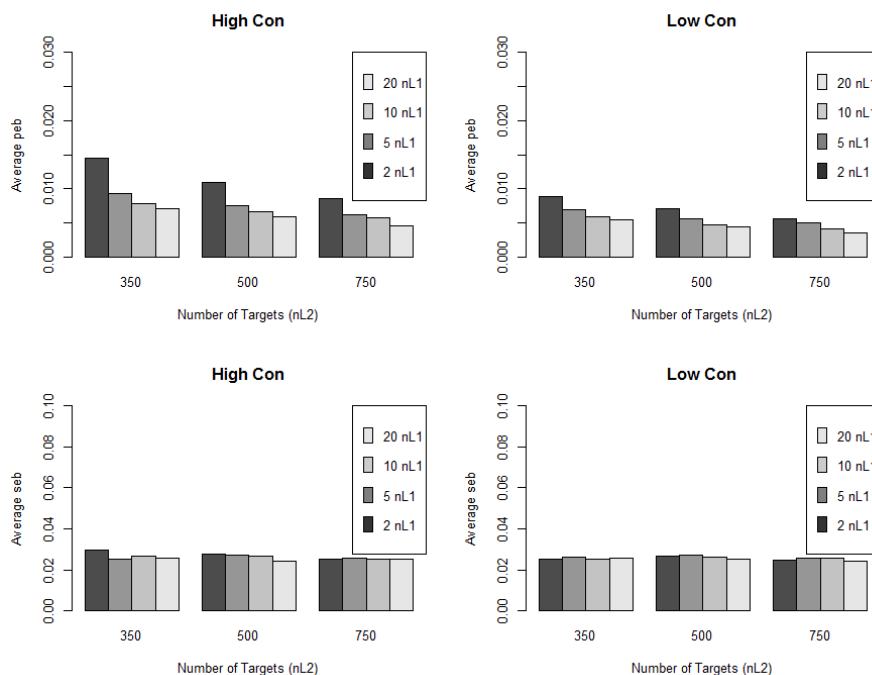


Figure 9.3: Average peb and seb values with respect to sample size in high and low consistency conditions in the LST-COM model. nL1 = number of level-1 units; nL2 = number of level-2 units.

with crossed random effects was carried out in order to investigate possible reasons for bias of parameter estimates as well as standard errors. The results of this analysis are given in Table 9.3. The bias of parameter estimates (see Model 1) decreased substantially with increasing sample size on level-1 (rater, 31-53 % decrease of bias) as well as level-2 (targets, 20-36 % decrease of bias). Moreover, the amount of peb decreased significantly with an increasing number of measurement occasions (25-31 % decrease of bias). This may be partially due to the fact that the number of observations as well as the number of measurement occasions constitute more empirical information.

Furthermore, LST-models are more restrictive than LS-models given that the covariances that would be freely estimated in LS-models are restricted in LST-models (Geiser, 2012). Therefore, additional occasions of measurement should not lead to more bias or improper solutions. Similar to the previous results, the amount of parameter bias increased significantly with an increasing number of constructs (25 % increase of bias) and methods (12 % increase of bias). Again, the amount of parameter bias was higher in the high convergent validity condition than in the low convergent validity condition (21 % of bias). The amount of standard error bias (seb) increased significantly with an increasing number of constructs (17 % increase of bias). However, in contrast to the previous results the amount of standard error bias was neither significantly associated with the number of measurement occasions, methods, targets nor the condition type (high vs. low). Nevertheless, the average amount of seb reduced significantly with an increasing number of raters per target (6-17 %).

The variability of peb and seb values across different types of parameters is displayed in Figure A.5 and A.6 in the appendix. The 95 % prediction intervals of the random effects confirm that the conditional distribution of the parameter estimation bias of different parameters has much less variability than the conditional distribution of the standard error bias of different parameters. Furthermore, bias in parameter estimates seem to be more sensitive to covariances among latent factors on the within (rater) and between (target) level. In particular, higher peb values are more often found with respect to the estimation of the covariance between latent trait method factors and the latent factor loading of the occasion-specific method factors. The standard error bias (seb) is rather associated with the latent covariance of the unique method state factors ζ_{rtj2l}^{UM} .

Table 9.3: Estimates for the prediction of bias in parameter estimates (peb) and standard errors (seb) in the LST-COM model.

Parameter	<i>Model1(peb)</i>	<i>Model2(seb)</i>
-2*LL	10920.67	11876.36
AIC	11009.00	11964.81
BIC	11100.94	12056.75
Fixed effects		
Intercept	0.01*** (0.18)	0.02*** (0.05)
Level 2 (conditions)		
methods 3 vs 2	0.12*** (0.02)	0.00 ^{ns} (0.02)
constructs 2 vs 1	0.25*** (0.02)	0.17*** (0.02)
occasion 3 vs 2	-0.27*** (0.02)	0.04 ^{ns} (0.03)
occasion 4 vs 2	-0.31*** (0.02)	0.04 ^{ns} (0.02)
rater 5 vs 2:	-0.31*** (0.03)	-0.03 ^{ns} (0.03)
rater 10 vs 2	-0.42*** (0.03)	-0.06* (0.03)
rater 20 vs 2	-0.53*** (0.03)	-0.17*** (0.03)
target 500 vs 350	-0.20*** (0.03)	-0.04 ^{ns} (0.03)
target 750 vs 350	-0.36*** (0.03)	-0.05 ^{ns} (0.03)
cond low vs high	-0.21*** (0.03)	0.02 ^{ns} (0.03)
Random effects		
Level 1		
σ_r^2	0.45	0.55
Level 2		
σ_{u01}^2 (con)	0.00	0.00
σ_{u02}^2 (par)	0.63	0.02

Note. Reference group is a LST-COM model with 3 indicators, 1 construct, 2 methods, 2 occasions of measurement, 2 raters per target, 350 targets in the high consistency condition. Standard errors are in parentheses. peb = log-transformed parameter estimation bias; seb = log-transformed standard error bias; LL = log likelihood; N = 5256; (con) = condition type (288); (par) = parameter type (21).

* $p < .05$; *** $p < .001$; ^{ns} not significant.

Chapter 10

Simulation IV : Latent growth curve (LGC-COM) model

10.1 Specification of the population model

The parameter specification of the population model is given in Table 10.1. Note that the LGC-COM is a special variant of the the LST-COM. Therefore, only the additional variance decomposition of the latent intercept and slope variables are provided in Table 10.1. Again, these values were not varied across items, and strong measurement invariance was assumed for all LGC-COM models. A *Mplus* input file can be found on the appendix CD-ROM.

Table 10.1: Variances of the intercept and slope factors in the LGC-COM population model.

	low consistency	high consistency
Intercept Variance		
Reference method indicator	.672	.672
Non-reference method indicator	.336	.528
Slope Variance		
Reference method indicator (linear growth)	.196	.196
Non-reference method indicator (linear growth)	.098	.154

Note. Exact values of the variance coefficients of the intercept and slope factors in the LGC-COM population model. The values did not vary across items. All other coefficients in the LGC-COM model were set to the same values as proposed for the LST-COM model.

10.2 Results

10.2.1 Convergence

All H0 models converged. However, the number of iterations for the H1 models was set to 1. Therefore, non of the H1 models converged and no χ^2 fit statistics were produced.

10.2.2 Improper solutions

Overall, 110 out of 288 conditions (38.2 %) contained replications with warning messages indicating possible estimation problems with respect to the latent covariance matrix Ψ . More warning messages were encountered in the multiconstruct condition (61, 55.5 %) than in the monoconstruct condition (49, 45.5 %). The amount of “possible” Ψ -problems was almost equally distributed across high and low consistency condition (see Table 10.2). In the worst case, 400 out of 500 replications entailed Ψ -warning messages. However, the amount of real Ψ -problems was relatively low (5237 of 144,000 replication, 3.6 %). That means that only 58.7 % (5237 out of 8922) of *Mplus* warning messages referred to “actual” Ψ -problems. The rest of the warning messages referred to estimation problems concerning higher order partial correlations. All of the 5237 “actual” Ψ -problems were associated with the estimation of between (target) level parameters.

Table 10.2: Amount of “possible” Ψ -problems in multi- and monoconstruct designs for high and low consistency condition.

	Multiconstruct		Monoconstruct		Total	
High consistency condition	32	(52.5 %)	25	(51.0 %)	57	(51.8 %)
Low consistency condition	29	(47.5 %)	24	(49.0 %)	53	(48.2 %)
Total	61	(55.5 %)	49	(45.5 %)	110	(100.0 %)

Note. Results do not represent the “actual” amount of Ψ -problems.

In Figure 10.1 the average amount of Ψ -warning messages for different samples sizes is given. As the figure illustrates, the average amount of Ψ -warning messages decreased with an increasing number of observations on both levels (number of targets, number of rater per targets). Note that the average amount of Ψ -warning messages is almost equal in the high and low consistency condition. This might be explained by the fact that the LGC-COM model implies the decomposition of an observed variable (of the interchangeable method) into eight different components (see Chapter 5). In addition to that, according to Figure 10.2 the average number of Ψ -warning messages decreases with an increasing number of observations per parameter. However, due to the complexity of the LGC-COM model, a higher ratio of observations per parameter is needed in order to reduce the amount of improper solutions. It can be seen from Figure 10.2 that even a ratio of 10:1 is not sufficient for reducing the amount of Ψ -warning messages. However, it is important to note that both Figures 10.1 and 10.2 refer to the total amount of *Mplus* warning messages (hence: “possible” improper solutions) and not to the amount of “actual” improper solutions.

Bias of parameter estimates and standard errors

The amount of parameter estimate bias and standard errors bias was relatively low. Across all 288 conditions the parameter estimation bias (peb) was below .1 and thereby did not exceed the critical cutoff value of .1. Nevertheless, the standard error bias (seb) exceeded the cutoff value in 4 cases of the low consistency condition and once in the high consistency conditions. In all of these MC conditions only 2 raters per target (few level-1 observation) were given. The maximum seb value of 1.88 was associated with the standard error estimation of slope covariances. Nonetheless, the average amount of bias (peb as well as seb) was relatively low across all simulation studies.

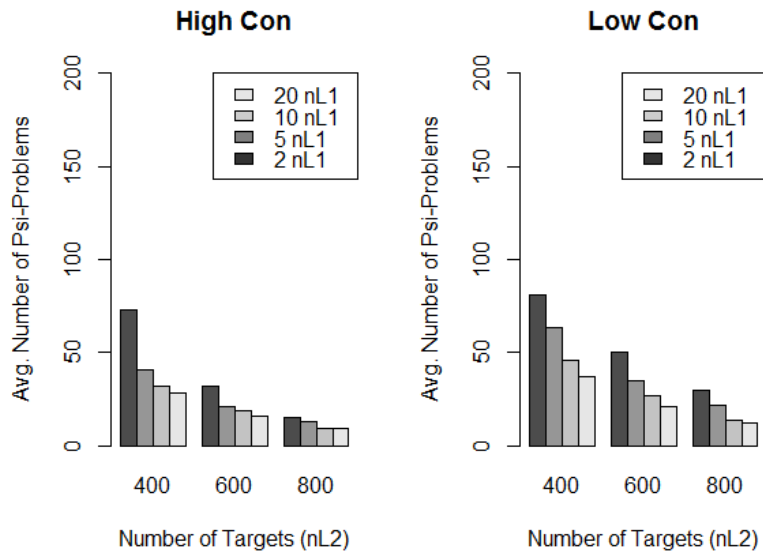


Figure 10.1: Average number of Ψ -problems in high and low consistency conditions. nL1 = number of level-1 units; nL2 = number of level-2 units.

According to the results of the multilevel analysis with crossed-random effects (see Table 10.3), the average amount of *peb* decreased significantly with an increasing number of measurement waves (13-27 % reduction), raters (25-45 %) as well as targets (19-30 %). Moreover, the average *peb* was higher in the high consistency condition than in the low consistency condition (11 %). Only with respect to the number of constructs (i.e., model complexity) the average amount of bias (*peb*) increased significantly (26 %). The average percentage increase or decrease of the standard error bias with respect the MC conditions was relatively low (1-13 %). The average amount standard error bias (*seb*) was related to an increase of the number of constructs (13 % increase of bias). In contrast to that, the number of rater (1-8 %) as well as targets (2-6 %) were negatively associated with the average amount of standard error bias.

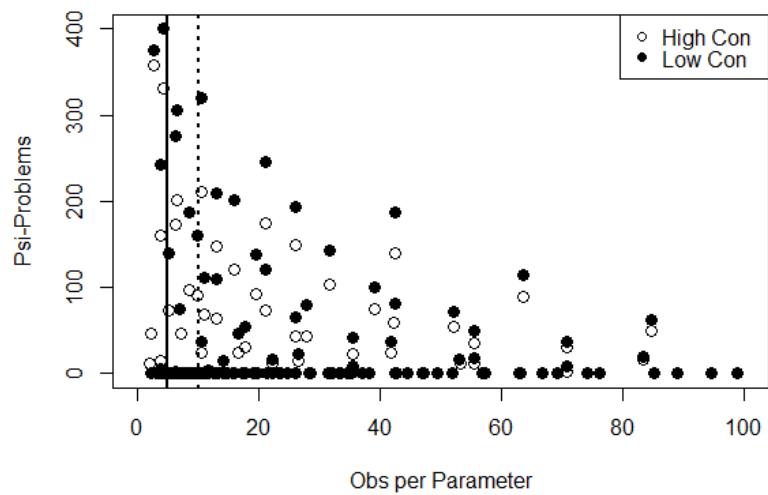


Figure 10.2: Relationship between Ψ -problems and observations per number of parameters.

Table 10.3: Estimates for the prediction of bias in parameter estimates (peb) and standard errors (seb) in the LGC-COM model.

Parameter	<i>Model1(peb)</i>	<i>Model2(seb)</i>
-2*LL	13592.21	13717.57
AIC	13683.17	13809.89
BIC	13778.50	13905.22
Fixed effects		
Intercept	0.01*** (0.21)	0.02*** (0.21)
Level 2 (conditions)		
Methods 3 vs 2	0.06*** (0.02)	0.02 ^{ns} (0.02)
Constructs 2 vs 1	0.26*** (0.02)	0.13*** (0.02)
Occasion 3 vs 2	-0.13*** (0.02)	0.07** (0.02)
Occasion 4 vs 2	-0.27*** (0.02)	0.02 ^{ns} (0.02)
Rater 5 vs 2	-0.25*** (0.02)	-0.01 ^{ns} (0.02)
Rater 10 vs 2	-0.29*** (0.02)	-0.07** (0.02)
Rater 20 vs 2	-0.45*** (0.02)	-0.08** (0.02)
Target 600 vs 400	-0.19*** (0.02)	-0.02 ^{ns} (0.02)
Target 800 vs 400	-0.30*** (0.02)	-0.06** (0.02)
Cond low vs high	-0.11*** (0.02)	0.05** (0.02)
Random effects		
Level 1		
σ_r^2	0.44	0.45
Level 2		
σ_{u01}^2 (con)	0.00	0.01
σ_{u02}^2 (par)	1.08	0.07

Note. Reference group is a LGC-COM model with 3 indicators, 1 construct, 2 methods, 2 occasions of measurement, 2 raters per target, 400 targets in the high consistency condition. Standard errors are in parentheses. peb = log-transformed parameter estimation bias; seb = log-transformed standard error bias; LL = log likelihood; N= 6696; (con) = condition type (288); (par) = parameter type (26).

* $p < .05$; *** $p < .001$; ^{ns} not significant.

10.3 Summary of the Monte Carlo simulation studies and general recommendations

The results of these four extensive simulation studies indicate that the presented models (LS-COM model, LC-COM model, LST-COM model, and LGC-COM model) perform well in general. No convergence problems with respect to the specified H0 model were encountered in any of the simulation studies. With respect to the H1 (saturated) model, researchers should increase the number of iterations to at least 7,500 for the LS-COM and LC-COM model and to 15,000 for the LST-COM and LGC-COM model in order to obtain χ^2 fit statistics. The convergence difficulties of the H1 model may be partially explained by the fact that *Mplus* tries to estimate the saturated model, which may be computationally extremely demanding for complex ML-SEM models when random MC sampling is involved. Nevertheless, further research needs to be done in order to scrutinize the convergence issues related to the estimation of complex ML-SEMs.

The amount of “actual” improper solutions (Ψ - and/or Θ -problems) was below 5 % in any of the MC simulation studies. Moreover, the average parameter bias (peb) as well as standard error bias (seb) was relatively small for most parameters and only exceeded the critical cutoff value of 10 % in rare cases. Therefore, it can be concluded that the model parameters are generally well recovered by the presented models. Moreover, the amount of improper solutions as well as bias can be substantially reduced by an increasing number of observations on both levels. Most sensitive to bias (peb and seb) as well as improper solutions were the latent covariances among the latent variables as well as the factor loadings of the latent factors. More specifically, the estimation problems involved common as well as unique method factors, especially in high consistency (i.e., low method variance) conditions. However, with respect to an increase of empirical information (e.g., sample size as well as occasions of measurement) the amount of bias and improper solutions can be significantly reduced. Based on these findings, at least five (better ten) observations per parameter are needed for proper parameter estimates. These recommendations are in line with the results of previous simulation studies (Bentler & Chou, 1987; Bollen, 1989, 2002). Moreover, the results of the simulation study suggest that the number of level-1 (rater per target) observations are important for proper parameter estimates. Specifically, the number of improper solutions as well as the amount of bias decreased significantly with increasing number of raters per target. Therefore, it is recommended to sample at least 5 raters per target.

High standard error bias (seb) occurred more often in conditions with few level-1 observations (e.g., 2 raters per target), low method bias (i.e., high consistency conditions) and high model complexity (i.e., number of parameters). In general, the standard error bias (seb) may be therefore reduced

- by increasing the number of level-1 units (more than 2 raters per target),
- by reducing the number of free estimated parameters (e.g., by reducing the number of constructs, inducing more restrictions), and/or
- by increasing the total number of observations per parameter.

As a rule of thumb, a minimal ratio of 5:1 (better 10:1) observations per parameter is recommended for proper standard errors. Across all simulation studies, the sample size on level-2 (target) and level-1 (rater) are equally important for proper parameter estimates. In line with the simulation study by Julian (2001) the number of level-2 units (number of targets) should be above 100. Based on the simulation results it is however recommended to sample at least as many targets (level-2 units) as parameters are estimated by the model, given that *Mplus* produces warning messages and it is yet not clear, whether or not parameters will be biased under these circumstances.

In contrast, to previous simulation studies on multilevel models (see e.g., Maas & Hox, 2005), the number of level-1 units (i.e., raters per target) is extremely important for valid parameter estimates. In regard to the χ^2 test of model fit, it was found that the observed χ^2 distribution approximates the theoretical χ^2 distribution well under maximum likelihood (ML) estimation. These results are quite interestingly, given that previous simulation studies show that the χ^2 fit statistics are often not trustworthy for complex MTMM-MO structural equation models (Crayen, 2008; Geiser, 2008). However, researcher should be aware that the observed χ^2 distribution is downward biased compared to the theoretical χ^2 distribution, meaning that the χ^2 model fit test is too liberal. Unfortunately, no clear cut trends with respect to sample size and the χ^2 test of model fit could be made with regard to the presented simulation studies. Consequently, more research is needed in order to scrutinize adequacy and robustness of χ^2 fit statistics in longitudinal multilevel MTMM-SEMs. Future research should especially focus on level-specific χ^2 fit statistics.

Part IV

Final discussion

Chapter 11

Final discussion

11.1 Practical guidelines for empirical applications

In the subsequent section, practical guidelines for empirical applications of the presented models are provided. The guidelines are divided into the following topic-related parts: (a) selection strategies for choosing an appropriate MTMM-MO model, (b) selection of the reference method, (c) selection of the indicators (e.g., items vs. item-parcels), (d) selection of the methods and dealing with complex hierarchical data structures (e.g., three-level data structures, multiple sets of interchangeable methods, cross-classification of raters etc.), and (e) choosing the optimal sample size.

11.1.1 Model selection

In the present work, four longitudinal multilevel structural equation models for complex MTMM measurement designs combining structurally different and interchangeable methods have been proposed. Specifically, a latent state (LS-COM) model, two latent baseline change (LC-COM) models, a latent state-trait (LST-COM) model, and a latent growth curve (LGC-COM) model have been formally defined. Depending on the substantive research questions that researchers may seek to answer, the presented models may be more or less useful. In the following section the strength and weaknesses of the presented models are briefly summarized in order to provide a basic guideline to researchers for choosing the appropriate MTMM-MO model.

The latent state (LS-COM) model can be used as baseline model for modeling complex MTMM-MO measurement designs combining structurally different and interchangeable methods, given that the LS-COM model implies less restrictions on the latent variance-covariance matrix than, for example, the LST-COM or the LGC-COM model. The LS-COM model can be used for testing crucial assumptions of longitudinal analysis such as the degree of measurement invariance (see also Geiser, 2008, 2012). By applying the LS-COM model it is also possible (a) to investigate the stability and change of construct as well as method effects across time, (b) to analyze the true convergent and discriminant validity of the given measures, and (c) study the latent mean structure. However, with LS-COM models it is only possible to investigate the stability and change of construct as well as method effects *indirectly* by examining the correlations among the latent variables. In other words, it is neither possible to model the stability nor the true change of method effects directly with respect to latent state models (see Geiser, 2012). Researchers who are interested in modeling true

interindividual differences in intraindividual change via latent difference variables, should therefore rather apply one of the LC-COM models or the LGC-COM model. The simple baseline LC-COM model is useful for studying true interindividual differences in intraindividual change with respect to the reference method. The simple baseline LC-COM model allows analyzing true change of construct effects, but not of method effects. The extended latent baseline LC-COM model also enables the investigation of true change in construct as well as method effects. Both LC-COM models are particularly useful for analyzing longitudinal MTMM intervention studies, given that it is possible to relate external variables (e.g., intervention group) to the latent differences variables. By including these additional variables into the LC-COM model, researcher may predict the true change of leadership quality as measured by the reference method. For example, researchers may explain why leadership quality is over- or underestimated by colleagues and/or the supervisor across time. Researchers can also conduct an intervention study in order to establish more congruency between the different raters (e.g., self-report, colleagues reports, supervisor reports). Geiser et al. (2010) provide a detailed description of how true change can be investigated via MTMM-MO-SEMs for structurally different methods.

The LGC-COM model can be used for modeling the shape of the true intraindividual change as measured by the reference method. That means, researchers can test whether or not the growth in leadership quality as measured by the reference method increases (or decreases) in a linear or in a non-linear form. In addition, researchers may also investigate whether or not the true interindividual differences in growth and the initial status as measured by the reference method can be predicted by external variables. The main advantage of the LGC-COM model is that growth can be studied free of stable method influences, occasion-specific as well as occasion-specific method influences, and measurement error influences. In other words, the LGC-COM model combines the advantages of multiple indicator latent growth curve models and MTMM modeling approaches (specifically the CT-C($M-1$) modeling approach).

The LST-COM model is useful for studying variability processes in MTMM-MO designs incorporating a combination of structurally different and interchangeable methods. Specifically, researchers can investigate (a) to which degree the constructs are stable or occasion-specific, (b) to which degree the method effects are stable or occasion-specific, (c) to which degree the consistency (congruency) between different methods (e.g., raters) are stable or occasion-specific. The latter allows investigating the convergent validity of different measures on trait as well as state level. Moreover, the LST-COM model allows researchers to examine (d) whether or not stable (trait) method effects generalize across different constructs and (e) whether or not momentary (occasion-specific) method effects generalize across different constructs.

Measurement designs do not always incorporate a combination of structurally different and interchangeable methods. Measurement designs that just use structurally different methods (e.g., self-report, parent report, and physiological measures), researchers can apply the models presented by Geiser (2008) or Courvoisier (2006). With respect to measurement designs that just incorporate interchangeable methods (e.g., multiple peer reports for teaching quality), researchers can apply a longitudinal version of the ML-CFA-MTMM model proposed by Eid et al. (2008).

In summary, researchers should be aware of the fact that substantive research questions as well

as the type of methods used in the MTMM measurement design should guide the model selection process (see Eid et al., 2008).

11.1.2 Choice of the reference method

The choice of the reference (gold standard) method is crucial for the interpretation of the model parameters. It is, therefore, strongly recommended selecting the reference method based on theoretical considerations (Geiser et al., 2008). In many cases, researchers will be able to select an appropriate reference method by considering their substantive research questions. For example, researchers may select the reference method considering whether the attribute of interest is observable (or properly measurable) by peer reports. Social competencies may be more closely linked to peer evaluations than to self-evaluations. Moreover, if researchers are interested in explaining why particular students over- or underestimate their level of social competencies with respect to the peer reports (here: gold standard) then the peer reports should be taken as the reference method. Pham et al. (2012) showed how the set of interchangeable methods may be used as a reference method for evaluating teaching quality (performance) of teachers rated by their corresponding students. In cases where researchers struggle with a theory-driven selection of the reference method, it may be appropriate to choose the most reliable method as the reference method or impose additional restrictions on the factor loading parameters so that the model fit of the specified CTC($M-1$) model is not affected by the choice of the reference method (Geiser et al., 2008; Geiser, Eid, West, Lischetzke, & Nussbeck, 2012).

11.1.3 Item selection

Researchers should use reliable (homogeneous) items (e.g., self-report, peer report, other report). First, it has been shown that the number of homogeneous (unidimensional) items are beneficial for the proper parameter estimates (Marsh et al., 1998). Second, if the items per factor are homogeneous (unidimensional), it is possible to specify common, instead of indicator-specific latent factors. Hence, homogeneous items are useful for specifying a more parsimonious model. If scales with numerous items are used (e.g., Large Scale Assessments), it is recommended to build item parcels following the recommendations by Little, Cunningham, Shahar, and Widaman (2002). By using item parcels (e.g., test halves) it is possible to reduce the complexity of the models (number of parameters) as well as the computational burden with respect to maximum likelihood estimation. For example, the computational burden for the estimation of complex multilevel structural equation models with categorical items increases exponentially with an increasing number of items (integration points) (L. K. Muthén & Muthén, 1998-2010). Therefore, it is strongly recommended to reduce the complexity of the given model as much as possible. In summary, this can be done by

- specifying common latent factors instead of item-specific latent factors,
- using item parcels in case of many categorical or not normally distributed items, and/or
- imposing as many permissible and reasonable restrictions as possible:
 - establishing the highest degree of measurement invariance,

- specifying latent factors that are common to all indicators
- fixing correlations to zero for parsimony

11.1.4 Complex hierarchical data structures

The presented models in this thesis can be considered as two-level longitudinal structural equation models, where interchangeable methods (e.g., raters) are modeled on the within-level and structurally different methods are modeled on the between-level. Researchers who seek to apply these models should follow the general guidelines of designing multilevel studies (e.g., Hox, 2010; Luke, 2004; Raudenbush & Bryk, 2002; Snijders & Bosker, 2011). Researchers should note that measurement designs with more complex hierarchical data structures, including:

- three-level multilevel structural equation models (ML-SEMs),
- ML-SEMs with multiple sets of interchangeable raters,
- ML-SEMs with cross-classified methods,

imply different random experiments. The random experiment as well as the probability spaces for the presented models have been explicitly characterized throughout this work. In order to extend these models to more complex multilevel measurement designs, additional research (e.g., simulation studies) is needed (see Section 11.4 for more details).

11.1.5 Optimal sample size

According to the extensive simulation designs in Chapter 7 to 10, the minimum required sample size depends on the particular model (LS-COM, LC-COM, LST-COM, LGC-COM) and the specific MTMM-MO measurement design (i.e., number of items, constructs, methods, occasions of measurement). Under realistic circumstances (i.e., low convergent validity), the parameter estimates will be well recovered by the models in samples with a ratio of five observations per parameter, which corresponds to the general recommendation of Bentler and Chou (1987). In addition, it is recommended to use more than two level-1 units (interchangeable raters per target), given that the results of the simulation studies indicate that the sample size on level-1 is extremely important for the reduction of parameter as well as standard error bias. As a rule of thumb, it is recommended to (a) sample at least as many level-2 units (i.e., targets) as parameters are estimated by the model, and to (b) multiply the number by five (for the number of raters per target) in order to obtain the total minimal sample size for proper parameter estimates. Given that the presented models can easily become quite complex (if multiple traits and multiple methods are used). Additionally, it is suggested to sample at least 100 level-2 units (i.e., targets). Under unrealistic circumstances (i.e., high convergent validity and low method variance) a larger sample size is required for proper parameter estimates. This is especially the case for the more complex models such as the LST-COM or LGC-COM model.

11.2 Advantages

The models presented in this thesis encompass many advantages. Most of all, the models (i.e., LS-COM model, LC-COM model, LST-COM model, LGC-COM model) combine the strengths and benefits of (a) longitudinal modeling approaches, (b) multimethod-multitrait (MTMM) modeling approaches, (c) multilevel modeling approaches and (d) structural equation modeling approaches. To my knowledge, the combination of all of these modeling approaches is unique. A similar general but different modeling framework represents the generalized linear latent and mixed modeling (GLLAMM) approach by Rabe-Hesketh and Skrondal (2004). However, in contrast to the GLLAMM approach, the presented models are formally defined based on stochastic measurement theory (Steyer, 1989; Steyer & Eid, 2001; Suppes & Zinnes, 1963; Zimmermann, 1975). By this mathematical formalization, the psychometric meaning as well as the psychometric properties of the latent variables (e.g., existence, uniqueness, admissible transformations, etc.) were clearly shown. The major advantage of these models is that they allow investigating complex (multilevel) MTMM-MO matrices by estimating one single model, instead of running multiple models separately. In other words, researchers will no longer have to aggregate the ratings per target or have to specify different MTMM models for different waves of measurement. As a consequence, research questions concerning the study of

- level-specific method bias at each occasion of measurement,
- the generalizability of method effects across constructs,
- the stability and/or occasion-specificity of method effects,
- the degree of true convergent and discriminant validity,
- true intraindividual change as well as the degree of stable or occasion-specific influences due to interindividual differences,
- potential causes for method bias or the change of method bias,

can be answered properly without losing any relevant information. In addition, the presented models are formulated based on four main longitudinal modeling frameworks:

- latent state modeling,
- latent difference/change variables modeling,
- latent state-trait modeling,
- latent growth curve modeling.

The methodological similarities of all of these frameworks have been explained and the appropriateness of each modeling framework for different substantive research questions has been discussed. In addition to that, the presented models may be considered as extensions of the models presented by Geiser (2008) and Courvoisier (2006) to measurement designs with structurally different and interchangeable methods. Therefore, the presented models will also encompass the advantages

of these models. Finally, important assumptions such as measurement invariance can be directly tested with χ^2 model fit statistics.

11.3 Limitations

Despite the numerous advantages, the presented models are limited in some aspects. First, the major limitations relate to the complexity of the models. For example, a general LS-COM model with 3 indicators, 2 methods, 2 constructs and 3 occasions of measurement incorporates 133 parameters in case of strong measurement invariance and 189 parameter estimates in case of configural measurement invariance (see Table 11.1). According to the general recommendation given above, at least five observations per parameter are needed for proper parameter estimates. Therefore, the required sample size for this 3x2x2x3 MTMM-MO measurement design ranges between 665 and 945 observations. Evidently, the presented models are not appropriate for small samples. However, there are many things that researchers can do in order to reduce the complexity of the models (see Section 11.1).

Table 11.1: Number of parameters of the LS-COM model depending on different degrees of measurement invariance and the dimension of the model.

Model dimension	configural MI	weak MI	strong MI	strict MI
2x1x2x2	35	30	27	23
3x1x2x2	51	42	37	31
2x1x3x2	56	48	43	37
3x1x3x2	80	66	58	49
2x1x2x3	57	47	41	33
3x1x2x3	81	63	53	41
2x1x3x3	93	77	67	55
3x1x3x3	129	101	85	67
2x2x2x2	82	72	66	58
3x2x2x2	114	96	86	74
2x2x3x2	136	120	110	98
3x2x3x2	184	156	140	122
2x2x2x3	141	121	109	93
3x2x2x3	189	153	133	109
2x2x3x3	240	208	188	164
3x2x3x3	312	256	224	188

Note. Model dimension = items×constructs×methods×occasions;
MI = measurement invariance.

First of all, researchers should impose as many restrictions as possible (e.g., strong measurement invariance restrictions, restrictions of theoretical meaningless factor correlations etc.). The additional restrictions are beneficial for reducing the model complexity and introducing additional hypotheses for a more stringent model test (Geiser, 2008). Second of all, homogeneous items should be selected in order to specify common latent factors instead of indicator-specific latent factors (Geiser, 2008). Researchers may also reduce the complexity of the model by specifying the models for each construct separately (i.e., monoconstruct measurement design). For example, with regard to the LS-COM model mentioned above, the required sample size reduces to 265 (strong MI) and 465 (configural MI) observations in the monoconstruct measurement design (see Table 11.1). This

corresponds to a reduction of the model complexity by 51% to 60%. Note that complex structural equation models do not always lead to more problems in SEMs (Geiser, 2008; Marsh et al., 1998). For example, according to the results of simulation studies a relatively small sample size ($2 \times 100 = 200$) maybe sometimes beneficial for the appropriateness of the χ^2 fit statistics (see Table 7.5 and 7.6 in Section 7.2.4). Finally, researchers may use parcels when it is reasonable to reduce the number of indicators per CMOU. Generally, it is recommended to use three rather than two indicators (items or parcels) per CMOU for model identification reasons.

Another limitation of the model concerns the scale level of the items. All of the models are defined using continuous observed variables. Even though it is straightforward to define the models for ordered categorical observed variables, it may be computationally demanding to estimate these models. Researchers who want to apply these models to categorical observed variables should conduct additional simulation studies in order to investigate efficient ways with respect to the estimation and the required samples size for such models. An appropriate estimator for SEMs with categorical observed variables is the WLSMV (weighted least square mean and variance adjusted) or MLR (maximum likelihood robust) estimator implemented in Mplus (L. K. Muthén & Muthén, 1998-2010). Given that both estimators require numerical integration, which is computationally demanding, researchers should specify a sufficiently large number of Monte Carlo integration points in order to receive proper parameter estimates. An alternative estimation procedure may involve Bayesian estimation techniques (Asparouhov & Muthén, 2010b; B. Muthén & Asparouhov, 2012; Asparouhov & Muthén, 2010a; B. O. Muthén, 2010). In general, additional research is needed in order to develop appropriate estimation techniques and algorithms for complex structural equation models.

The presented models are also limited by the fact that only one set of interchangeable methods can be modeled. However, it is possible to extend the models to multiple sets of interchangeable methods. In a recent work by Mahlke et al. (2012) it is shown how multilevel SEMs with multiple sets of interchangeable methods can be specified for cross-sectional MTMM data. The presented models also may be extended to three-level clustered data. With respect to the new version of Mplus (version 7.0) it will be possible to directly specify three-level SEMs. The new version of Mplus also allows one to model cross-classified multilevel data structures. Modeling cross-classified multilevel(-rater) data is especially interesting, given that in many empirical applications it is neither realistic nor feasible to sample different raters per target. Note that these extensions to categorical, three-level, or cross-classified data will necessarily lead to more complex models. I therefore hope that the above mentioned limitations will inspire researchers to overcome these limitations.

11.4 Future research

In this section, possible directions for future research are discussed. An important area of research concerns the extension of the presented models to ordered categorical (ordinal) response variables. This area of research is essential as well as challenging. It is essential given that the scale level of many response variables in the behavioral sciences (e.g., items of a given questionnaire) is not continuous, but rather (ordered) categorical (Agresti, 2007). It is also often not feasible or

advisable to use item parcels instead of the raw items (Crayen et al., 2011; Little et al., 2002). Extending the presented models to categorical (ordinal) observed variables is also very challenging. The main difficulties associated with this extension concern the general estimation of such models rather than their psychometric formalization. With respect to the work by Eid (1995), the mathematical formalization and extension of the presented models to ordered categorical indicators is straightforward. However, relatively few attempts have been made to estimate complex multilevel structural equation models with categorical items (Rabe-Hesketh & Skrondal, 2004). One possible explanation may be that the estimation process is extremely computationally demanding (L. K. Muthén & Muthén, 1998-2010). Therefore, future research is needed to develop appropriate and efficient estimation techniques for complex multilevel SEMs with ordered categorical (ordinal) items. With respect to Bayesian estimation techniques, the estimation of complex multilevel SEMs with categorical indicators can be done (Asparouhov & Muthén, 2010b; B. Muthén & Asparouhov, 2012; Asparouhov & Muthén, 2010a; B. O. Muthén, 2010). In addition to that, Bayesian estimation techniques may also improve the applicability of complex multilevel SEMs to small sample sizes.

Additional research is needed for developing adequate, robust and level-specific model fit statistics for complex multilevel structural equation models. Important contributions to this field have been made by Yuan and Bentler (2003), Yuan and Bentler (2007) as well as Ryu and West (2009). However, many of the presented solutions for producing level-specific and unbiased χ^2 fit statistics are cumbersome and require more observations than necessary for the actual model identification. Correct fit statistics are important for testing specific model restrictions (e.g., measurement invariance) and for comparing alternative models. Again, Bayesian fit indices represent an alternative way for calculating adequate fit statistics (Levy, 2011).

In many empirical applications of longitudinal multirater designs, multiple targets often are rated by the same raters. These so called cross-classified data structures violate the assumptions of uncorrelated error terms in general multilevel analyses. Further research is needed to investigate the statistical consequences of ignoring the additional dependencies in such data structures. Moreover, additional psychometric work is required for defining appropriate multilevel MTMM-MO-SEMs with crossed classified interchangeable raters. With respect to the new developments in Mplus (version 7.0), it will be possible to model and estimate the effects of such cross-classifications.

Another important research direction concerns the statistical examination of the interchangeability of different raters per target. Nussbeck et al. (2009) showed how the interchangeability assumption for an equal number of raters per target can be tested empirically by introducing additional restrictions on the parameters. However, additional research is needed for testing the interchangeability of methods when the number of raters per target differs. It may also be interesting to study why particular raters do not fulfill the interchangeability assumption. Alternative ways for scrutinizing the interchangeability of raters may be latent mixture models (e.g., latent class analysis) or the specification of random latent factors in multilevel SEMs. Finally, future research may be directed to the calculation of confidence intervals for the variance coefficients proposed throughout this thesis or the estimation of latent mediation and moderation effects in complex multilevel MTMM-SEMs.

11.5 Summary and Conclusion

In this thesis the flexibility, versatility, and advantages of longitudinal modeling, multitrait-multimethod modeling, structural equation modeling, and multilevel modeling have been combined to one general modeling approach. In particular, four new multilevel structural equation models for complex MTMM longitudinal data have been proposed. In the first part of this thesis, the formal (psychometric) soundness of the models was demonstrated. All of the presented models were defined based on the stochastic measurement theory (Steyer, 1989; Steyer & Eid, 2001; Suppes & Zinnes, 1963; Zimmermann, 1975). The meaning and the level of measurement (i.e., scale level) of the latent variables was shown. Meaningful statements with regard to the model parameters were discussed. Finally, it was shown under which conditions the parameter of the models are identified. In the second part of this thesis, the empirical applicability of the models was scrutinized. As the results of four extensive simulation studies reveal, the model parameters were generally well recovered by the models, the amount of actual improper solutions (Heywood cases) were low, and the standard error bias decreased notably with increasing sample size. Based on the results of these simulation studies, practical guidelines have been provided for how to design and model complex longitudinal MTMM data. Finally, the advantages and limitations of the models have been discussed. Many of the listed limitations lead to new directions for future research and may be temporal as the development of new estimation techniques and software packages increases. The main purpose of this thesis was to provide appropriate structural equation models for multilevel longitudinal MTMM data that are flexible, general, and powerful for complex empirical applications. Complex measurement designs are increasingly found in educational, developmental, and organizational research.

References

- Agresti, A. (2007). *An introduction to categorical data analysis* (2nd ed.). Florida, FL: Wiley.
- Anderson, J., & Gerbing, D. (1984). The effect of sampling error on convergence, improper solutions, and goodness-of-fit indices for maximum likelihood confirmatory factor analysis. *Psychometrika*, *49*, 155-173.
- Asparouhov, T., & Muthén, B. (2010a). *Bayesian Analysis Using Mplus : Technical Implementation*. Retrieved from <http://www.statmodel.com/download/Bayes3.pdf>
- Asparouhov, T., & Muthén, B. (2010b). *Bayesian Analysis of Latent Variable Models using Mplus*. Retrieved from <http://www.statmodel2.com/download/BayesAdvantages18.pdf>
- Bandalos, D. L. (2006). The use of monte carlo studies in structural equation modeling research. In G. R. Hancock & R. O. Mueller (Eds.), *Structural equation modeling: A second course* (pp. 385-426). Greenwich, CT: Information Age Publishing.
- Bates, D., Maechler, M., & Bolker, B. (2011). *lme4: Linear mixed-effects models using s4 classes*. Retrieved from <http://cran.r-project.org/web/packages/lme4/index.html>
- Bates, D. M. (2010, January 11). *lme4: Mixed-effects modeling with R*. Retrieved from https://r-forge.r-project.org/scm/viewvc.php/*checkout*/www/lmmwR/lrgprt.pdf?revision=600&root=lme4
- Beauducel, A., & Herzberg, P. Y. (2006). On the performance of maximum likelihood versus means and variance adjusted weighted least squares estimation in CFA. *Structural Equation Modeling*, *13*(2), 186-203.
- Behrends, E. (2013). *Elementare Stochastik: Ein Lernbuch - von Studierenden mitentwickelt. [Elementary stochastic: A textbook - developed with the help of students]*. Berlin: Springer.
- Bentler, P., & Chou, C.-P. (1987). Practical issues in structural modeling. *Sociological Methods & Research*, *17*, 78-117.
- Biesanz, J. C., West, S. G., & Kwok, O.-M. (2003). Personality over time: Methodological approaches to the study of short-term and long-term development and change. *Journal of Personality*, *71*(6), 905-941.
- Boker, S., Neale, M., Maes, H., Wilde, M., Spiegel, M., Brick, T., . . . Fox, J. (2011). OpenMx: An open source extended structural equation modeling framework. *Psychometrika*, *76*(2), 306-317.
- Bollen, K. A. (1989). *Structural equations with latent variables*. New York: John Wiley & Sons.
- Bollen, K. A. (2002). Latent variables in psychology and the social sciences. *Annual Review of Psychology*, *53*(1), 605-634.
- Bollen, K. A., & Curran, P. J. (2006). *Latent curve models: A structural equation perspective*. New York: Wiley.
- Boomsma, A. (1982). The robustness of LISREL against small sample sizes in factor analysis models. In K. G. Jöreskog & H. Wold (Eds.), *Systems under indirect observation: causality, structure, prediction*. (pp. 149-173). Amsterdam: North-Holland.
- Borsboom, D., Mellenbergh, G. J., & van Heerden, J. (2004). The concept of validity. *Psychological Review*, *111*(4), 1061-1071.

- Burns, G. L., & Haynes, S. N. (2006). Clinical psychology: construct validation with multiple sources of information and multiple settings. In M. Eid & E. Diener (Eds.), *Handbook of multimethod measurement in psychology* (pp. 401–418). Washington, DC: APA Publications.
- Burns, G. L., Walsh, J. A., & Gomez, R. (2003). Convergent and discriminant validity of trait and source effects in ADHD-inattention and hyperactivity/Impulsivity measures across a 3-month Interval. *Journal of Abnormal Child Psychology: An official publication of the International Society for Research in Child and Adolescent Psychopathology*, *31*(5), 529-541.
- Campbell, D. T., & Fiske, D. W. (1959). Convergent and discriminant validation by the multitrait-multimethod matrix. *Psychological Bulletin*, *56*(2), 81-105.
- Carretero-Dios, H., Eid, M., & Ruch, W. (2011). Analyzing multitrait-multimethod data with multilevel confirmatory factor analysis: An application to the validation of the State-Trait Cheerfulness Inventory. *Journal of Research in Personality*, *45*(2), 153-164.
- Chan, D. (1998). The conceptualization and analysis of change over time: An integrative approach incorporating longitudinal mean and covariance structures analysis (lmacs) and multiple indicator latent growth modeling (mlgm). *Organizational Research Methods*, *1*, 421-483.
- Chen, F., Bollen, K. A., Paxton, P., Curran, P. J., & Kirby, J. B. (2001). Improper solutions in structural equation models: Causes, consequences, and strategies. *Sociological Methods & Research*, *29*(4), 468-508.
- Ciarrochi, J., & Heaven, P. (2009). A longitudinal study into the link between adolescent personality and peer-rated likeability and adjustment: Evidence of gender differences. *Journal of Research in Personality*, *43*(6), 978–986.
- Cohen, J., Cohen, P., West, S. G., & Aiken, L. S. (2003). *Applied multiple regression/correlational analysis for the behavioral sciences*. Mahwah, NJ: Erlbaum.
- Cole, D. A., Martin, N. M., & Steiger, J. H. (2005). Empirical and conceptual problems with longitudinal trait-state models: Introducing a trait-state-occasion model. *Psychological Methods*, *10*, 3-20.
- Cole, D. A., & Maxwell, S. E. (2003). Testing mediational models with longitudinal data: Questions and tips in the use of structural equation modeling. *Journal of Abnormal Psychology*, *112*(4), 558–577.
- Conway, J. M., Lievens, F., Scullen, S. E., & Lance, C. E. (2004). Bias in the correlated uniqueness model for MTMM data. *Structural Equation Modeling*, *11*(4), 535-559.
- Courvoisier, D. S. (2006). *Unfolding the constituents of psychological scores: Development and application of mixture and multitrait-multimethod LST models*. (Unpublished doctoral dissertation). University of Geneva, Switzerland. Retrieved from <http://archive-ouverte.unige.ch/downloader/vital/pdf/tmp/7j3rqacbpmv5mmla2073892gg6/out.pdf>
- Courvoisier, D. S., Nussbeck, F. W., Eid, M., Geiser, C., & Cole, D. A. (2008). Analyzing the convergent and discriminant validity of states and traits: Development and applications of multimethod latent state-trait models. *Psychological Assessment*, *20*(3), 270–280.
- Crayen, C. (2008). *Eine Monte-Carlo Simulationsstudie zur Untersuchung der Anwendbarkeit von Strukturgleichungsmodellen für Multitrait-Multimethod-Multioccasion Daten*. [A simulation study for investigating the applicability of structural equation models for multitrait-

- multimethod-multioccasion data*]. (Unpublished diploma thesis). Freie Universität Berlin, Germany.
- Crayen, C., Geiser, C., Scheithauer, H., & Eid, M. (2011). Evaluating interventions with multimethod data: a structural equation modeling approach. *Structural Equation Modeling: A Multidisciplinary Journal*, *18*(4), 497-524.
- Cronbach, L. J., & Meehl, P. E. (1955). Construct validity in psychological tests. *Psychological Bulletin*, *52*(4), 281-303.
- Dai, G., De Meuse, K. P., & Peterson, C. (2010). Impact of multi-source feedback on leadership competency development: a longitudinal field study. *Journal of Managerial Issues*, *22*(2), 197-219.
- Denissen, J. J. A., Schönbrodt, F. D., van Zalk, M., Meeus, W. H. J., & van Aken, M. A. G. (2011). Antecedents and consequences of peer-rated intelligence. *European Journal of Personality*, *25*(2), 108-119.
- Dumenci, L. (2000). Multitrait-multimethod analysis. In S. D. Brown & H. E. Tinsley (Eds.), *Handbook of applied multivariate statistics and mathematical modeling* (pp. 583-611). Academic Press.
- Eid, M. (1995). *Modelle der Messung von Personen in Situationen [Models for measuring persons in situations]*. Weinheim: Psychologie Verlags Union.
- Eid, M. (2000). A multitrait-multimethod model with minimal assumptions. *Psychometrika*, *65*(2), 241-261.
- Eid, M. (2006). Methodological approaches for analyzing multimethod data. In M. Eid & E. Diener (Eds.), *Handbook of psychological measurement: A multimethod perspective* (pp. 223-230). Washington, DC: American Psychological Association.
- Eid, M. (2010). Multitrait-multimethod-matrix. In N. Salkind (Ed.), *Encyclopedia of research design* (Vol. 1, pp. 850-855). Thousand Oaks: Sage Publications.
- Eid, M., & Diener, E. (2006). Introduction: The need for multimethod measurement in psychology. In M. Eid & E. Diener (Eds.), *Handbook of psychological measurement: A multimethod perspective* (pp. 3-8). Washington, DC: American Psychological Association.
- Eid, M., & Hoffmann, L. (1998). Measuring variability and change with an item response model for polytomous variables. *Journal of Educational and Behavioral Statistics*, *23*, 193-215.
- Eid, M., Lischetzke, T., & Nussbeck, F. (2006). Structural equation models for multitrait-multimethod data. In M. Eid & E. Diener (Eds.), *Handbook of multimethod measurement in psychology* (pp. 289-299). American Psychological Association.
- Eid, M., Lischetzke, T., Nussbeck, F. W., & Trierweiler, L. I. (2003). Separating trait effects from trait-specific method effects in multitrait-multimethod models: A multiple-indicator CT-C(M-1) model. *Psychological Methods*, *8*(1), 38-60.
- Eid, M., & Nussbeck, F. W. (2009). The multitrait-multimethod matrix at 50! *Methodology: European Journal of Research Methods for the Behavioral and Social Sciences*, *5*(3), 71.
- Eid, M., Nussbeck, F. W., Geiser, C., Cole, D. A., Gollwitzer, M., & Lischetzke, T. (2008). Structural equation modeling of multitrait-multimethod data: Different models for different types of methods. *Psychological Methods*, *13*(3), 230-253.

- Eid, M., & Schmidt, K. (in press). *Testtheorie [test theory]*. Beltz.
- Eid, M., Schneider, C., & Schwenkmezger, P. (1999). Do you feel better or worse? The validity of perceived deviations of mood states from mood traits. *European Journal of Personality*, *13*(4), 283-306.
- Erhardt, V. (2009). *Corcounts: Generate correlated count random variables*. Retrieved from <http://cran.r-project.org/web/packages/corcounts/index.html>
- Ferrer, E., Balluerka, N., & Widaman, K. F. (2008). Factorial invariance and the specification of second-order latent growth models. *Methodology: European Journal of Research Methods for the Behavioral and Social Sciences*, *4*(1), 22-36.
- Geiser, C. (2008). *Structural equation modeling of multitrait-multimethod-multioccasion data*. (Unpublished doctoral dissertation). Freie Universität Berlin, Germany. Retrieved from http://www.diss.fu-berlin.de/diss/servlets/MCRFileNodeServlet/FUDISS_derivate_000000004268/Dissertation_Christian_Geiser.pdf?hosts=
- Geiser, C. (2009). *Multitrait-multimethod-multioccasion modeling*. München: Akademischer Verlag München.
- Geiser, C. (2012). *Data analysis with Mplus*. New York: Guilford Press.
- Geiser, C., Eid, M., & Nussbeck, F. W. (2008). On the meaning of the latent variables in the CT-C(M-1) model: A comment on Maydeu-Olivares and Coffman (2006). *Psychological Methods*, *13*(1), 49-57.
- Geiser, C., Eid, M., Nussbeck, F. W., Courvoisier, D. S., & Cole, D. A. (2010). Analyzing true change in longitudinal multitrait-multimethod studies: Application of a multimethod change model to depression and anxiety in children. *Developmental Psychology*, *46*(1), 29-45.
- Geiser, C., Eid, M., West, S. G., Lischetzke, T., & Nussbeck, F. W. (2012). A Comparison of Method Effects in Two Confirmatory Factor Models for Structurally Different Methods. *Structural Equation Modeling*, *19*(3), 409-436.
- Geiser, C., Keller, B., & Lockhart, G. (2012). First versus second order latent growth curve models: Some insights from latent state-trait theory. *Structural Equation Modeling*, *17*.
- Geiser, C., Keller, B., Lockhart, G., Eid, M., Cole, D., & Koch, T. (2012). When variability models measure change: The role of measurement (non)invariance in latent state-trait analyses. (Manuscript submitted for publication.)
- Gerbing, D. W., & Anderson, J. C. (1985). The effects of sampling error and model characteristics on parameter estimation for maximum likelihood confirmatory factor analysis. *Multivariate Behavioral Research*, *20*(3), 255-271.
- Giles, D. (2011). *Dummies for dummies*. Retrieved from <http://davegiles.blogspot.de/2011/03/dummies-for-dummies.html>
- Grimm, K. J., Pianta, R. C., & Konold, T. (2009). Longitudinal multitrait-multimethod models for developmental research. *Multivariate Behavioral Research*, *44*(2), 233-258.
- Hallquist, M. (2011). *MplusAutomation: Automating Mplus model estimation and interpretation*. Retrieved from <http://cran.r-project.org/web/packages/MplusAutomation/index.html>
- Hancock, G. R., Kuo, W.-L., & Lawrence, F. R. (2001). An illustration of second-order latent

- growth models. *Structural Equation Modeling*, 8(3), 470-489.
- Harwell, M. R., Stone, C. A., Hsu, T.-C., & Kirisci, L. (1996). Monte Carlo studies in Item Response Theory. *Applied Psychological Measurement*, 20, 101-125.
- Hays, W. L. (1994). *Statistics* (5th ed.). Orlando, FL: Harcourt Brace College.
- Ho, D. Y. F. (2010). Pooled peer ratings, self-ratings, and estimated ratings of therapeutic communication and popularity: a relational analysis. *Humanistic Psychologist*, 38(4), 317-335.
- Hox, J. J. (2010). *Multilevel analysis techniques and applications* (2nd ed.). New York: Routledge.
- Jackson, D. L. (2001). Sample size and number of parameter estimates in maximum likelihood confirmatory factor analysis: a monte carlo investigation. *Structural Equation Modeling*, 8(2), 205-223.
- Jöreskog, K. G. (1979). Statistical models and methods for analysis of longitudinal data. In K. G. Jöreskog & D. Sörbom (Eds.), *Advances in factor analysis and structural equation models* (pp. 129-169). Cambridge, MA: Abt Books.
- Julian, M. W. (2001). The consequences of ignoring multilevel data structures in nonhierarchical covariance modeling. *Structural Equation Modeling*, 8(3), 325-352.
- Kenny, D. A. (1995). The multitrait-multimethod matrix: Design, analysis, and conceptual issues. In P. Shrout & S. T. Fiske (Eds.), *Personality research, methods, and theory: A festschrift honoring Donald W. Fiske* (pp. 111-124). Hillsdale, NJ: Lawrence Erlbaum.
- Kenny, D. A., & Kashy, D. A. (1992). Analysis of the multitrait-multimethod matrix by confirmatory factor analysis. *Psychological Bulletin*, 112(1), 165-172.
- Khoo, S.-T., West, S. G., Wu, W., & Kwok, O.-M. (2006). Longitudinal methods. In M. Eid & E. Diener (Eds.), *Handbook of psychological measurement: A multimethod perspective* (pp. 301-317). Washington, DC: American Psychological Association.
- Lance, C. E., Noble, C. L., & Scullen, S. E. (2002). A critique of the correlated trait-correlated method and correlated uniqueness models for multitrait-multimethod data. *Psychological Methods*, 7(2), 228-244.
- Leite, W. L. (2007). A comparison of latent growth models for constructs measured by multiple items. *Structural Equation Modeling*, 14, 581-610.
- Levy, R. (2011). Bayesian Data-Model Fit Assessment for Structural Equation Modeling. *Structural Equation Modeling*, 18(4), 663-685.
- Li, E., Boos, D., & Gumpertz, M. (2001). *Simulation study in statistics*. Retrieved from <http://www4.stat.ncsu.edu/~davidian/st810a/simulationinstatistics.pdf>
- Little, T. D., Cunningham, W. A., Shahar, G., & Widaman, K. F. (2002). To parcel or not to parcel: Exploring the question, weighing the merits. *Structural Equation Modeling*, 9(2), 151-173.
- Luke, D. A. (2004). *Multilevel modeling*. Thousand Oaks: Sage.
- Maas, C. J. M., & Hox, J. J. (2005). Sufficient Sample Sizes for Multilevel Modeling. *Methodology: European Journal of Research Methods for the Behavioral and Social Sciences*, 1, 85-91.
- MacKenzie, S. B., Podsakoff, P. M., & Jarvis, C. B. (2005). The problem of measurement model misspecification in behavioral and organizational research and some recommended solutions.

- Journal of Applied Psychology*, 90(4), 710–730.
- Mahlke, J., Schultze, M., Koch, T., Eid, M., Eckert, R., & Brodbeck, F. (2012). Validation of 360 degree feedback instruments: A multilevel structural equation model for structurally different and multiple sets of interchangeable methods.
(Manuscript submitted for publication)
- Marsh, H. W. (1993). Multitrait-multimethod analyses: Inferring each trait-method combination with multiple indicators. *Applied Measurement in Education*, 6(1), 49–81.
- Marsh, H. W., & Bailey, M. (1991). Confirmatory factor analyses of multitrait-multimethod data: A comparison of alternative models. *Applied Psychological Measurement*, 15(1), 47–70.
- Marsh, H. W., Byrne, B. M., & Craven, R. (1992). Overcoming problems in confirmatory factor analyses of MTMM data: The correlated uniqueness model and factorial invariance. *Multivariate Behavioral Research*, 27(4), 489–507.
- Marsh, H. W., & Grayson, D. (1994). Longitudinal confirmatory factor analysis: Common, time-specific, item-specific, and residual-error components of variance. *Structural Equation Modeling*, 1, 116–145.
- Marsh, H. W., Hau, K.-T., Balla, J. R., & Grayson, D. (1998). Is more ever too much? The number of indicators per factor in confirmatory factor analysis. *Multivariate Behavioral Research*, 33(2), 181–220.
- Marsh, H. W., & Hocevar, D. (1988). A new, more powerful approach to multitrait-multimethod analyses: Application of second-order confirmatory factor analysis. *Journal of Applied Psychology*, 73(1), 107–117.
- McArdle, J. J. (1988). Dynamic but structural equation modeling of repeated measures data. In R. B. Cattell & J. Nesselroade (Eds.), *Handbook of multivariate experimental psychology* (2nd ed., pp. 561–614). New York: Plenum Press.
- McArdle, J. J., & Epstein, D. (1987). Latent growth curves within developmental structural equation models. *Child Development*, 58, 110–133.
- McArdle, J. J., & Hamagami, F. (2001). Latent difference score structural models for linear dynamic analyses with incomplete longitudinal data. In L. M. Collins & A. G. Sayer (Eds.), *New methods for the analysis of change* (pp. 139–175). Washington, DC: American Psychological Association.
- Meredith, W. (1993). Measurement invariance, factor analysis and factorial invariance. *Psychometrika*, 58(4), 525–543.
- Meredith, W., & Horn, J. (2001). The role of factorial invariance in modeling growth and change. In L. Collins & A. Sayer (Eds.), *New methods for the analysis of change* (pp. 203–240). Washington, DC: American Psychological Association.
- Meredith, W., & Tisak, J. (1990). Latent curve analysis. *Psychometrika*, 55(1), 107–122.
- Messick, S. (1980). Test validity and the ethics of assessment. *American Psychologist*, 35(11), 1012–1027.
- Messick, S. (1989). Validity. In R. L. Linn (Ed.), *Educational measurement* (3rd ed., pp. 13–103). New York: Macmillan Publishing Co.
- Messick, S. (1995). Validity of psychological assessment: Validation of inferences from persons'

- responses and performances as scientific inquiry into score meaning. *American Psychologist*, *50*(9), 741-749.
- Murphy, D. L., Beretvas, S. N., & Pituch, K. A. (2011). The effects of autocorrelation on the curve-of-factors growth model. *Structural Equation Modeling*, *18*, 430-448.
- Muthén, B., & Asparouhov, T. (2012). Bayesian structural equation modeling: A more flexible representation of substantive theory. *Psychological Methods*, *17*, 313-335.
- Muthén, B. O. (2010, September 29). *Bayesian analysis in Mplus : A brief introduction*. Retrieved from <http://www.statmodel.com/download/IntroBayesVersion%203.pdf>
- Muthén, L. K., & Muthén, B. O. (1998-2010). *Mplus user's guide*. (6th ed.). Los Angeles, CA: Muthén & Muthén.
- Muthén, L. K., & Muthén, B. O. (2002). How to use a monte carlo study to decide on sample size and determine power. *Structural Equation Modeling*, *9*(4), 599-620.
- Nussbeck, F. W. (2008). *Multirater agreement models. analyzing the convergent and discriminant validity of categorical ratings*. (Unpublished doctoral dissertation). Freie Universität Berlin, Germany.
- Nussbeck, F. W., Eid, M., Geiser, C., Courvoisier, D. S., & Lischetzke, T. (2009). A CTC(M-1) model for different types of raters. *Methodology: European Journal of Research Methods for the Behavioral and Social Sciences*, *5*(3), 88-98.
- Nussbeck, F. W., Eid, M., & Lischetzke, T. (2006). Analysing multitrait-multimethod data with structural equation models for ordinal variables applying the WLSMV estimator: What sample size is needed for valid results? *British Journal of Mathematical and Statistical Psychology*, *59*(1), 195-213.
- Pham, G., Koch, T., Helmke, A., Schrader, F.-W., Helmke, T., & Eid, M. (2012). Do teachers know how their teaching is perceived by their pupils? *Procedia - Social and Behavioral Sciences*, 3368-3374.
- Podsakoff, P. M., MacKenzie, S. B., Lee, J.-Y., & Podsakoff, N. P. (2003). Common method biases in behavioral research: A critical review of the literature and recommended remedies. *Journal of Applied Psychology*, *88*(5), 879-903.
- Pohl, S., & Steyer, R. (2010). Modeling common traits and method effects in multitrait-multimethod analysis. *Multivariate Behavioral Research*, *45*(1), 45-72.
- R Development Core Team. (2008). *R: A language and environment for statistical computing*. Vienna, Austria: R Foundation for Statistical Computing. Retrieved from <http://www.R-project.org>
- Rabe-Hesketh, S., & Skrondal, A. (2004). Generalized multilevel structural equation modeling. *Psychometrika*, *69*, 167-190.
- Raudenbush, S. W., & Bryk, A. S. (2002). *Hierarchical linear models*. Thousand Oaks, CA: Sage.
- Ryu, E., & West, S. G. (2009). Level-specific evaluation of model fit in multilevel structural equation modeling. *Structural Equation Modeling*, *16*(4), 583-601.
- Satorra, A., & Bentler, P. M. (1994). Corrections to test statistics and standard errors in covariance structure analysis. In A. von Eye & C. Clogg (Eds.), *Latent variables analysis: Applications for developmental research* (pp. 399-419). Thousand Oaks, CA: Sage.

- Satorra, A., & Bentler, P. M. (2001). A scaled difference chi-square test statistic for moment structure analysis. *Psychometrika*, *66*(4), 507-514.
- Sayer, A. G., & Cumsille, P. E. (2001). Second-order latent growth models. In *New methods for the analysis of change* (pp. 179-200). Washington, DC: American Psychological Association.
- Scherpenzeel, A., & Saris, W. (2007). Multitrait-multimethod models for longitudinal research. In K. van Montfort, J. Oud, & A. Satorra (Eds.), *Longitudinal models in the behavioral and related sciences* (pp. 381-401). Mahwah, NJ: Lawrence Erlbaum.
- Shadish, W. R., Cook, T. D., & Campbell, D. T. (2002). *Experimental and quasi-experimental designs for generalized causal inference*. Boston, MA: Houghton Mifflin Co.
- Shrout, P. E. (1995). Measuring the degree of consensus in personality judgments. In P. E. Shrout & S. T. Fiske (Eds.), *Personality research, methods and theory: A festschrift honoring Donald W. Fiske* (pp. 79-92). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Snijders, T. A. B., & Bosker, R. J. (2011). *Multilevel analysis: An introduction to basic and advanced multilevel modeling*. London: Sage.
- Sternberg, R. J. (1992). Psychological bulletin's top 10 'hit parade'. *Psychological Bulletin*, *112*(3), 387-388.
- Steyer, R. (1988). *Experiment, Regression und Kausalität. Die logische Struktur kausaler Regressionsmodelle [Experiment, regression, and causality. The logical structure of causal regression models]*. (Unpublished habilitation thesis). Universität Trier, Germany.
- Steyer, R. (1989). Models of classical psychometric test theory as stochastic measurement models: Representation, uniqueness, meaningfulness, identifiability, and testability. *Methodika*, *3*, 25-60.
- Steyer, R., & Eid, M. (2001). *Messen und Testen. [Measurement and testing.]* (2nd ed.). Heidelberg: Springer.
- Steyer, R., Eid, M., & Schwenkmezger, P. (1997). Modeling true intraindividual change: True change as a latent variable. *Methods of Psychological Research*, *2*(1), 21-33.
- Steyer, R., Ferring, D., & Schmitt, M. J. (1992). States and traits in psychological assessment. *European Journal of Psychological Assessment*, *8*(2), 79-98.
- Steyer, R., Nagel, W., Partchev, I., & Mayer, A. (in press). *Measure, probability, and regression*. Springer.
- Steyer, R., Partchev, I., & Shanahan, M. J. (2000). Modeling true intraindividual change in structural equation models: The case of poverty and children's psychosocial adjustment. In T. D. Little, K. U. Schnabel, & J. Baumert (Eds.), *Modeling longitudinal and multilevel data: Practical issues, applied approaches, and specific examples* (pp. 109-126). Mahwah, NJ: Lawrence Erlbaum.
- Steyer, R., Schmitt, M., & Eid, M. (1999). Latent state-trait theory and research in personality and individual differences. *European Journal of Personality*, *13*(5), 389-408.
- Suppes, P., & Zinnes, J. L. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of mathematical psychology* (Vol. 1, pp. 1-76). New York: Wiley.
- Tisak, J., & Tisak, M. S. (2000). Permanency and ephemerality of psychological measures with

- application to organizational commitment. *Psychological Methods*, 5(2), 175-198.
- Tomas, J. M., Hontangas, P. M., & Oliver, A. (2000). Linear confirmatory factor models to evaluate multitrait-multimethod matrices: The effects of number of indicators and correlation among methods. *Multivariate Behavioral Research*, 35(4), 469-499.
- Violato, C., Lockyer, J. M., & Fidler, H. (2008). Changes in performance: a 5-year longitudinal study of participants in a multi-source feedback programme. *Medical Education*, 42(10), 1007-1013.
- von Oerzen, T., Hertzog, C., Lindenberger, U., & Ghisletta, P. (2010). The effect of multiple indicators on the power to detect inter-individual differences in change. *British Journal of Mathematical and Statistical Psychology*, 63, 627-646.
- Widaman, K. F. (1985). Hierarchically nested covariance structure models for multitrait-multimethod data. *Applied Psychological Measurement*, 9(1), 1-26.
- Widaman, K. F., & Reise, S. P. (1997). Exploring the measurement invariance of psychological instruments: Applications in the substance use domain. In K. J. Bryant, M. Windle, & S. G. West (Eds.), *The science of prevention: Methodological advances from alcohol and substance abuse research* (pp. 281-324). Washington, DC: American Psychological Association.
- Wothke, W. (1995). Covariance components analysis of the multitrait-multimethod matrix. In P. E. Shrout & S. T. Fiske (Eds.), *Personality research, methods, and theory: A festschrift honoring Donald W. Fiske* (pp. 125-144). Hillsdale, NJ: Lawrence Erlbaum.
- Yuan, K.-H., & Bentler, P. M. (2000). Robust mean and covariance structure analysis through iteratively reweighted least squares. *Psychometrika*, 65, 43-58.
- Yuan, K.-H., & Bentler, P. M. (2003). Eight test statistics for multilevel structural equation models. *Computational Statistics & Data Analysis*, 44(1/2), 89-107.
- Yuan, K.-H., & Bentler, P. M. (2007). Multilevel covariance structure analysis by fitting multiple single-level models. *Sociological Methodology*, 37(1), 53-82.
- Zimmermann, D. W. (1975). Probability spaces, Hilbert spaces, and the axioms of test theory. *Psychometrika*, 40, 395-412.

Chapter 12

Appendix

Appendix A

Dotplots for the random effects

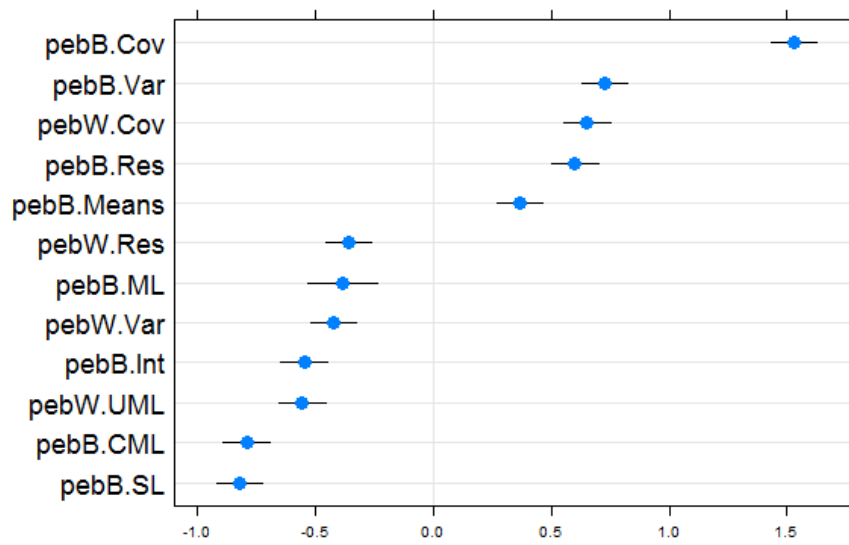


Figure A.1: 95 % prediction intervals of random effects for different parameters of model 1 (peb) for the LS-COM model.

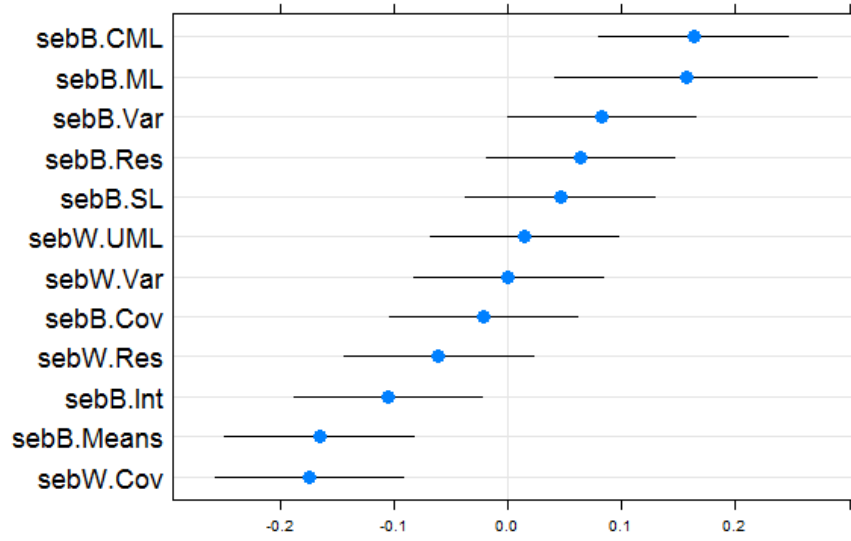


Figure A.2: 95 % prediction intervals of random effects for different parameters of model 2 (seb) for the LS-COM model.

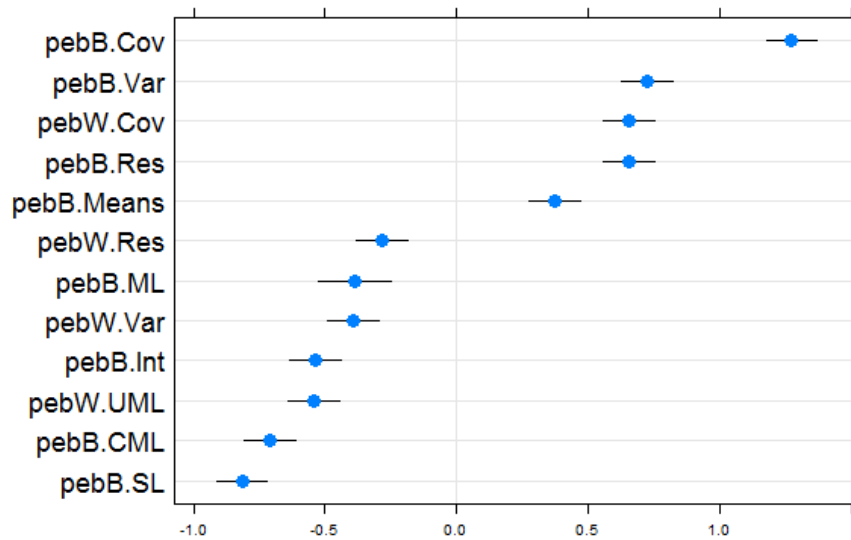


Figure A.3: 95 % prediction intervals on the random effects for different parameters of model 1 (peb) for the LC-COM model.

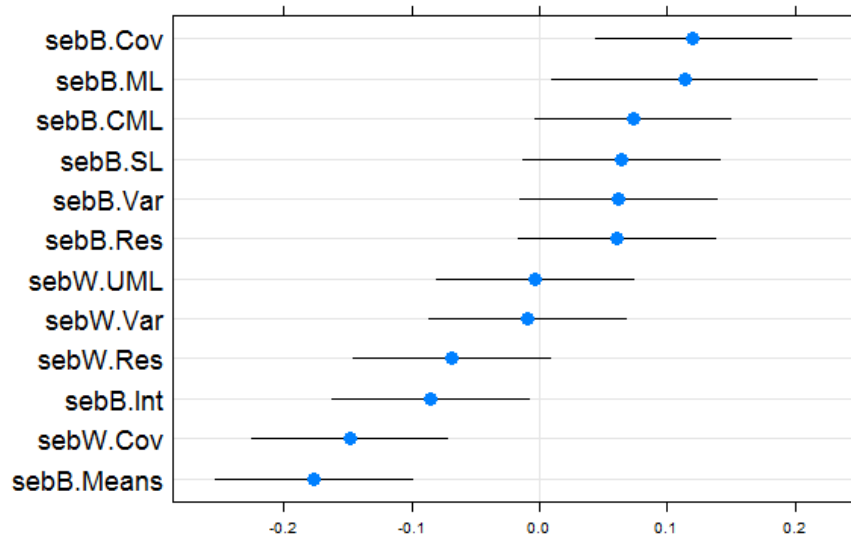


Figure A.4: 95 % prediction intervals on the random effects for different parameters of model 2 (seb) for the LC-COM model.

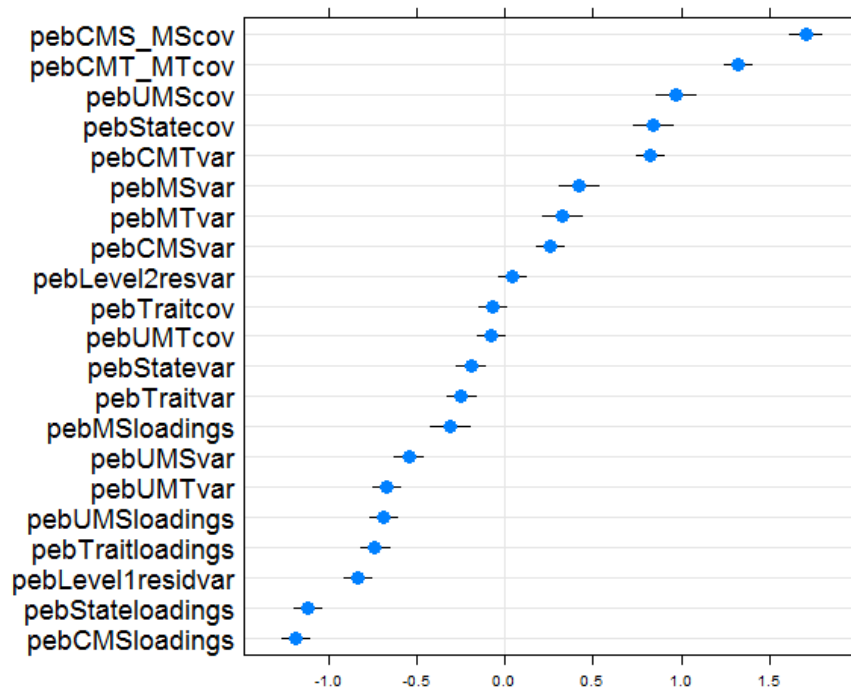


Figure A.5: 95 % prediction intervals of the random effects for different parameters of model 1 (peb) for the LST-COM model.

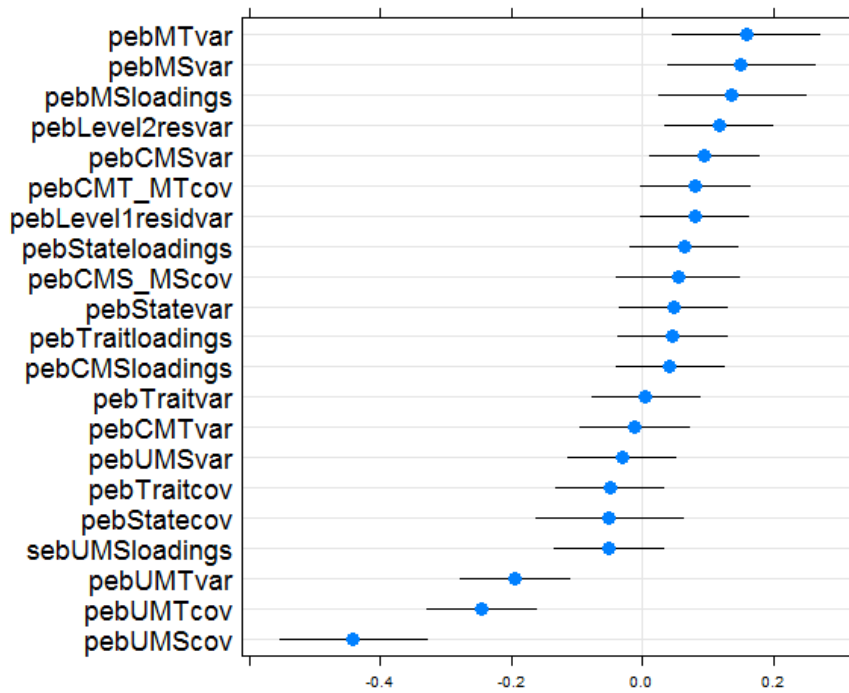


Figure A.6: 95 % prediction intervals of the random effects for different parameters of model 2 (seb) for the LST-COM model.

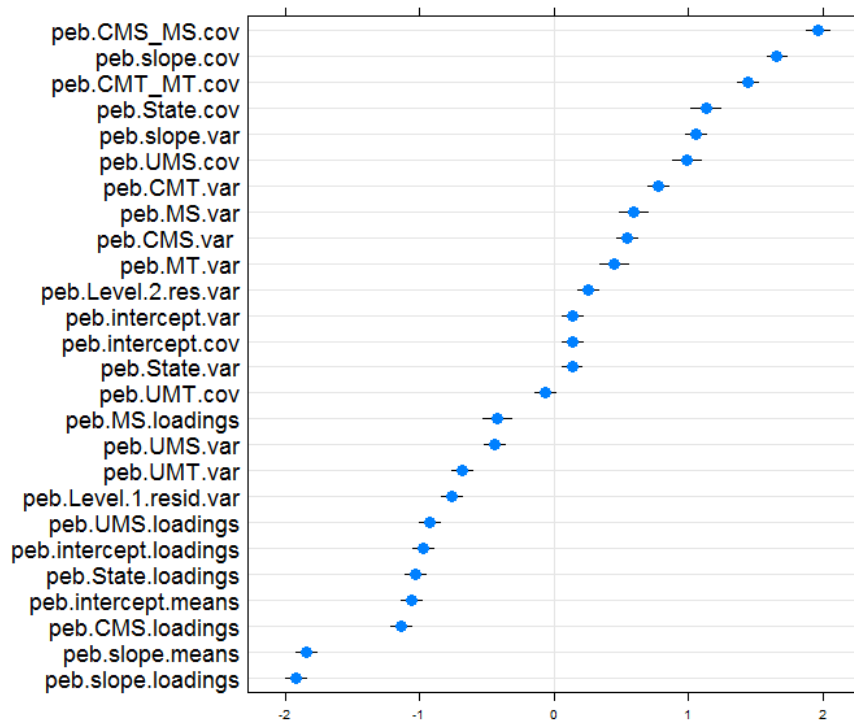


Figure A.7: 95 % prediction intervals of the random effects for different parameters of model 1 (peb) for the LGC-COM model.

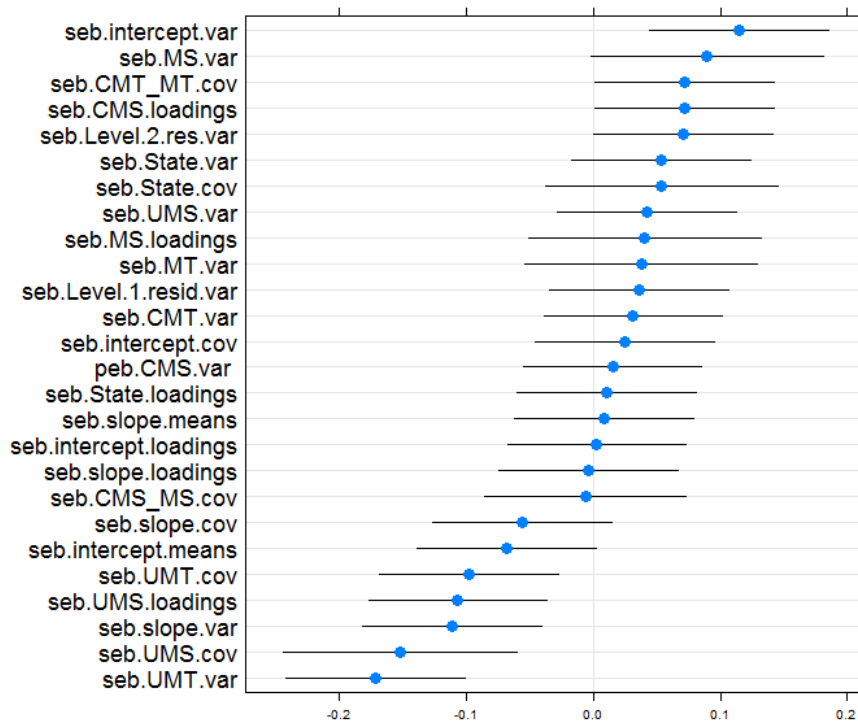


Figure A.8: 95 % prediction intervals of the random effects for different parameters of model 2 (seb) for the LGC-COM model.

Appendix B

χ^2 -Approximation

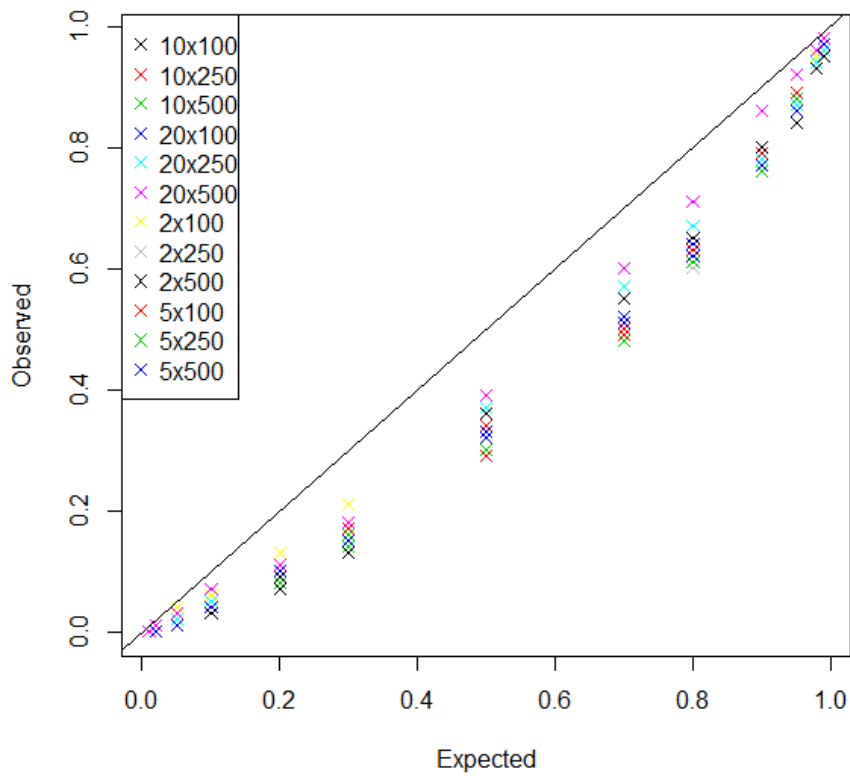


Figure B.1: PP-plot of the observed and theoretical proportions of the χ^2 values for the monotrait LS-COM model.

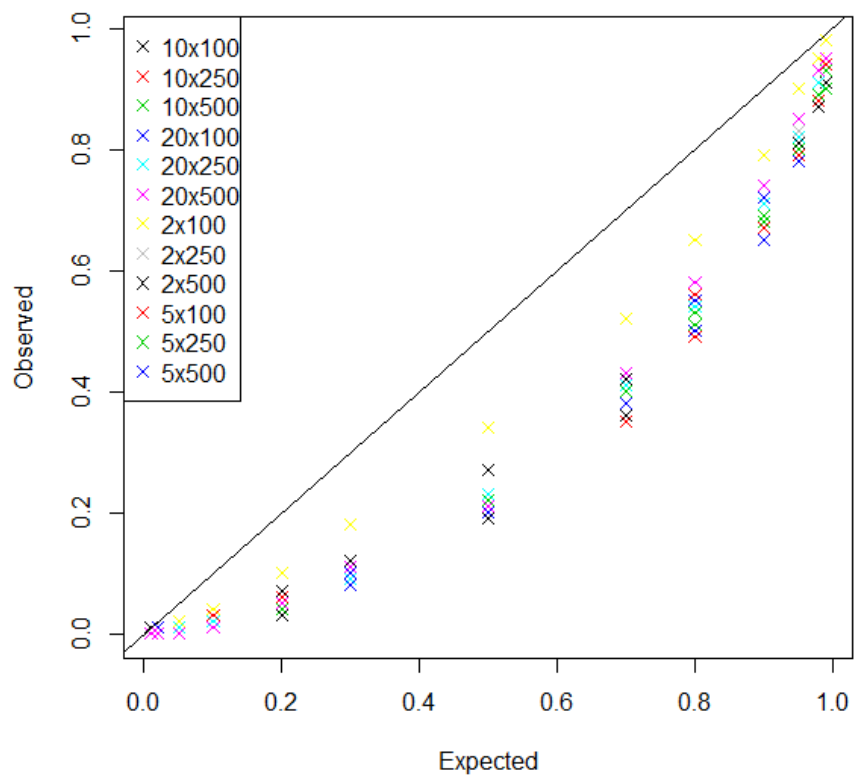


Figure B.2: PP-plot of the observed and theoretical proportions of the χ^2 values for the multitrait LS-COM model.

Appendix C

Appendix CD-ROM

All files of the appendix CD-ROM can be downloaded from the following website:

<http://www.ewi-psy.fu-berlin.de/einrichtungen/arbeitsbereiche/psymeth/mitarbeiter/tkoch/index.html>

Appendix D

German Appendix (Anhang in deutscher Sprache)

D.1 Zusammenfassung in deutscher Sprache

In der vorliegenden Arbeit werden insgesamt vier Mehrebenen-Strukturgleichungsmodelle für multimethodale Längsschnittsuntersuchungen (multitrait-multimethod-multioccasion, MTMM-MO Designs) vorgestellt. Insbesondere werden in dieser Arbeit folgende Modelle vorgestellt:

- Multimethod-Latent-State-Modelle (LS-COM-Modelle),
- Multimethod-Latent-Change-Modelle (LC-COM-Modelle),
- Multimethod-Latent-State-Trait-Modelle (LST-COM-Modelle) und
- Multimethod-Latent-Growth-Curve-Modelle (LGC-COM-Modelle).

Ein wesentlicher Vorteil der neu definierten Modelle ist, dass sie für die Analyse von MTMM-MO Forschungsdesigns mit einer *Kombination* von strukturell unterschiedlichen und austauschbaren Methoden eingesetzt werden können. Gemäß Eid et al. (2008) sind austauschbare Methoden (z.B. Peer-Berichte) Methoden, die zufällig aus einer Population gleichartiger Methoden gezogen werden können. Im Gegensatz dazu, können strukturell unterschiedliche Methoden (z.B. Eltern- und Schülerberichte) nicht zufällig aus einer Population von gleichartigen Methoden gezogen werden, sondern liegen a priori für den zu Beurteilenden (Target) fest (Eid et al., 2008). So liegt der Elternbericht beispielsweise für jeden Schüler und jede Schülerin fest. Ferner werden die in dieser Arbeit vorgestellten Modelle für unterschiedliche Längsschnittanalysen definiert. Das bedeutet, dass Forscher die neuen Modelle für Latent-State-, Latent-Change-, Latent-State-Trait-, oder Latent-Growth-Curve-Modellierungen verwenden können. Die Definitionen der Modelle erfolgen dabei auf der Basis der stochastischen Messtheorie (Steyer, 1989; Steyer & Eid, 2001; Suppes & Zinnes, 1963; Zimmermann, 1975). Das heißt, es wurden in dieser Arbeit die wesentlichen psychometrischen Eigenschaften der Modelle (Fragen bzgl. der Existenz, Eindeutigkeit, Bedeutsamkeit, Identifizierbarkeit) im Detail geklärt. Darüber hinaus, wurden alle Modelle mittels aufwendigen Monte-Carlo-Simulationsstudien in Hinblick auf ihre statistische Performanz untersucht. Im Rahmen der Simulationsstudien zeigte sich, dass die neu definierten Modelle generell für komplexe MTMM-Modellierungen geeignet sind und zu zuverlässigen Parameter- sowie Standardfehlerschätzungen

führen. Ebenso war die Anzahl von wahren unzulässigen Parameterschätzungen (sog. Heywood cases) gering. Außerdem zeigte sich, dass mit zunehmender Anzahl von Level-1 Einheiten (d.h. Anzahl der austauschbaren Rater pro Target) der Standardfehlerbias sowie die Anzahl von wahren unzulässigen Parameterschätzungen deutlich reduziert werden kann. Eine Erhöhung der Anzahl der Level-2 Einheiten (d.h. Anzahl von Targets) zeigte ähnliche, aber weniger stark ausgeprägte Ergebnisse in Bezug auf die Reduktion von Standardfehlerverzerrungen.

Die beiliegende Arbeit ist wie folgt gegliedert: Im ersten Teil der Arbeit wird eine generelle Einführung zum Thema MTMM-Analysen gegeben, wobei die zentralen Vorteile von Strukturgleichungsmodellen für die Analyse von MTMM-Daten hervorgehoben werden. Ferner werden die längsschnittlichen MTMM-Modelle von Geiser (2008) sowie von (Courvoisier, 2006) vorgestellt, da diese als Spezialfälle aus den vorgestellten Modellen hervorgehen. Im zweiten Teil der Arbeit werden die neu vorgestellten Modelle in einzelnen Kapiteln formal definiert und deren psychometrischen Eigenschaften mathematisch bewiesen. Im dritten Teil der Arbeit werden die Ergebnisse der Monte-Carlo-Simulationsstudien vorgestellt. Abschließend werden die Ergebnisse dieser Arbeit im vierten Teil nochmals zusammengefasst und diskutiert.

D.2 Erklärung

Hiermit versichere ich, dass ich die vorgelegte Arbeit selbständig verfasst habe. Andere als die angegebenen Hilfsmittel habe ich nicht verwendet. Die Arbeit ist in keinem früheren Promotionsverfahren angenommen oder abgelehnt worden.

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