Boundary Value Problems for Complex Partial Differential Equations in a Ring Domain

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Dedicated to the memory of my Mother Natal'ya Vaitsiakhovich
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CONTENTS

1 Introduction

The theory of boundary value problems for complex partial differential equations combines knowledges and methods from many fields of mathematics, i.e. complex analysis, partial differential equations, functional analysis, equations of mathematical physics etc. Initiated by B. Riemann and D. Hilbert the theory develops up to nowadays involving different research groups all over the world.

One of the main aim of the theory of complex boundary value problems is to obtain solutions in analytic or closed form. Many results in this directions are known for for special kinds of equations, namely, for the Cauchy-Riemann equation, the Beltrami equation, for elliptic equations with constant or analytic coefficients etc., and are connected with such names as N.I. Muskhelishvili, I.N. Vekua, F.D. Gakhov, J. Garnett, G.-C. Wen, A. Dzhuraev, H. Begehr, see [5], [15], [30], [33], [34], [40], [49], [51], [52]. Boundary value problems were mostly considered in simply connected domains, i.e. unit disk, a half plane, a corner etc.

The intention to investigate boundary value problems for multiply connected domains gives rise to many additional difficulties even in the simplest case dealing with analytic functions. There are just few results known on this subject. Among them is the celebrated Villat's formula for the solution of Schwarz problem in a concentric ring domain, see e.g. [3], [39], expressed in terms of Weierstrass ζ-function; and the formulas obtained by V.V. Mityushev and S.V. Rogosin [39] as well for the Schwarz and the Riemann-Hilbert problems as for the Riemann problem of linear conjugacy for a multiply connected circular domain in form of series with respect to the elements of a special Schottky symmetry group. The main difficulty appeared is connected with the single validness of solutions while passing for simply to multiply connected domains.

The present thesis contributes to the research subject initiated by Prof. Dr. H. Begehr and developed by his students and collaborates [1], [5]–[21], [29], [32], [37], [47], [48].

A systematic investigation of boundary value problems for complex partial differential equations of arbitrary order on the base of integral representation formulas was initiated by H. Begehr. To start with, the basic boundary value problems for model equations are observed. The differential operator of a model equation consists of a product of powers of the complex Cauchy-

Riemann operator $\partial_{\overline{z}}$ and its complex conjugate ∂_z. The main methods of the theory will be pointed on now.

The complex form of the Gauss theorem for a regular domain D on the complex plane $\mathbb{C}$ and an arbitrary function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ leads to the Cauchy-Pompeiu representation formula [7], [5]. This formula is a generalization of the Cauchy formula for analytic functions. The area integral appearing in the Cauchy-Pompeiu formula is called the Pompeiu operator. It plays an important role in treating boundary value problems for complex partial differential equations. The properties of the Pompeiu operator were studied by I.N. Vekua [49]. If f belongs to $L_p(D;\mathbb{C})$, $p > 1$, then Tf possesses weak derivatives with respect to z and $\overline{z}$, moreover $\partial_{\overline{z}}Tf = f$, $\partial_z Tf =: \Pi f$, where Π is a singular integral being understood in the principle value sense. Integrals of such type are investigated in [27].

From the Cauchy-Pompeiu representation formula it follows that any function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ can be found by known values on the boundary and values of a first order derivative inside of the domain. On the other hand, for given $f \in L_p(D;\mathbb{C})$, $p > 1$, and $\gamma \in C(\partial D;\mathbb{C})$ a new function

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} \gamma(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{1.1}$$

can be constructed according to this formula. The boundary integral is an analytic function, while the area integral represents the Pompeiu operator. Using the properties of the Pompeiu operator, the function w is seen to be a solution of the differential equation $w_{\overline{z}} = f$ in D, being understood in the weak sense. But the boundary values of the function defined in (1.1) in general differ from γ. Therefore the function w given by (1.1) is not the solution of the problem

$$w_{\overline{z}} = f \text{ in } D, \quad w = \gamma \text{ on } \partial D,$$

which is called the Dirichlet boundary value problem for the inhomogeneous Cauchy-Riemann equation. This fact leads to the idea that the Cauchy-Pompeiu representation formula has to be modified in a proper way for being useful to treat boundary value problems.

There are three basic boundary value problems for complex partial differential equations, namely, Schwarz, Dirichlet and Neumann problems. To find the solutions in explicit form they are investigated in particular domains, i.e. the unit disk, half planes, quarter planes, etc. For the unit disk the modified Cauchy-Pompeiu formula, which is known as Cauchy-Pompeiu-Schwarz-Poisson formula (or Schwarz-Poisson formula in the case of analytic functions), serves as the starting point in [7], where the solutions of the basic boundary value problems to first order equations are given.

To solve boundary value problems for the inhomogeneous Cauchy-Riemann equation $w_{\overline{z}} = f$ the idea of I.N. Vekua is exploited, who suggested [49] to represent the solution of these problems in the form $w = \varphi + Tf$, with φ being an analytic function. By using the properties of the Pompeiu operator the boundary value problems for the inhomogeneous Cauchy-Riemann equation are reduced to homogeneous case (see, e.g. [7]).

Besides the three main boundary value problems listed above the Robin boundary value problem should be mentioned. This problem is the combination of Dirichlet and Neumann ones. The solutions of the particular Robin boundary value problem to the Cauchy-Riemann operator is given in [16].

On the next step boundary value problems for second order model equations are investigated. There are two main differential operators of second order, namely, the Laplace operator $\partial_z \partial_{\overline{z}}$ and the Bitsadze operator $\partial_{\overline{z}}^2$, which produce the model equations of second order: the Laplace, the Poisson (or inhomogeneous Laplace), homogeneous and inhomogeneous Bitsadze equations.

The Dirichlet problem for the Laplace operator (or which is equivalent, for harmonic functions) in the unit disk

$$w_{z\overline{z}} = 0 \text{ in } \mathbb{D}, \quad w = 0 \text{ on } \partial\mathbb{D}$$

has only the trivial solution, see e.g. [7]. As A.V. Bitsadze has shown [25], this is failed for the Dirichlet problem

$$w_{\overline{z}\,\overline{z}} = 0 \text{ in } \mathbb{D}, \quad w = 0 \text{ in } \partial\mathbb{D}, \tag{1.2}$$

having constructed the infinite set of linearly independent solutions $w_k(z) = (1 - |z|^2)z^k$, $k \in \mathbb{N}$. The complex differential equation in (1.2) is called the homogeneous Bitsadze equation.

The classical way to solve the Dirichlet and Neumann boundary value problems for the Poisson equation is by using the representation formulas via the Green and Neumann (or the Green function of second kind) functions. These functions are the fundamental solutions for the Laplace operator. There are several ways to construct them for a particular domain. In the case of the unit disk they can be found directly, taking into account those properties, which they have to satisfy by definition. On the other hand, the Green and Neumann functions arise in a natural way while modifying the Cauchy-Pompeiu representation formula of second order, see [7]. As it was mentioned above, this formula can not be used directly to solve boundary value problems. To get the desired representation formula e.g. for the Dirichlet problem the first order derivatives have to be excluded. For this aim the Gauss theorem is used, see e.g. [7], [5], [10].

In [8] different boundary value problems including those with mixed boundary conditions are solved for complex partial differential equations of second order in the unit disk. The solution of some problems is based on the idea of reducing a problem to a system of boundary value problems for first order equations, for which the results of [7] are used.

The solution of the Robin boundary value problem for the Poisson equation is found in [18].

The theory is extended by investigating boundary value problems for higher order complex model equations. In particular the homogeneous and inhomogeneous polyanalytic and polyharmonic equations are considered. They consist correspondingly of powers of the Cauchy-Riemann operator, i.e. $\partial_{\overline{z}}^k$, and of powers of the Laplace operator, $(\partial_z \partial_{\overline{z}})^k$, $k \geq 2$. Different types of boundary conditions are prescribed for these equations. Some boundary value problems for the polyanalytic equation are solved in [20], [8] in the case of the unit disk. To treat boundary value problems to the polyharmonic equation the concept of polyharmonic Green functions is developed. They are certain fundamental solutions for the polyharmonic operator (or k- Laplace operator). One of the possible ways to construct a polyharmonic Green function is to represent it as a convolution of two polyharmonic Green functions of lower order. The convolution of Green functions of different kinds defines a hybrid Green function (the denotation was introduced by H. Begehr) [11], [12]. The results concerning boundary value problems for the polyharmonic equation are given in [22],[23] for the unit disk, and in [14] for the upper half plane.

The differential operator of an arbitrary model equation of higher order can be represented as a product of polyanalytic and polyharmonic operators. The important step in investigating boundary value problems for higher order model equations was made by H. Begehr and G.N. Hile [19] (see also [6]), who constructed the hierarchy of higher order Pompeiu operators $T_{m,n}$, defined for any pair (m, n) of integer numbers m, n such that $m+n \geq 0$. Operators

$T_{m,n}$ act on the space $L_p(D;\mathbb{C})$, $p > 1$, $T_{0,1}$ coincides with the Pompeiu operator T, $T_{-1,1}$ is the Π operator, while $T_{0,0}$ is the identical operator. In [19] (see also [10]) the Cauchy-Pompeiu formulas of higher order are obtained. They provide the representation of a function $w \in C^{m+n}(\mathrm{cl}\, D;\mathbb{C})$ via the area integral $T_{m,n}\left(\partial^{m+n}w/\partial z^m\partial\overline{z}^n\right)$ and some boundary integrals of the lower order derivatives. In the same way as the classical Cauchy-Pompeiu representation formulas are used to solve boundary value problems for the first order equations, the Cauchy-Pompeiu formulas of higher order serve to solve boundary value problems for higher order model equations.

The theory of boundary value problems for complex partial differential equations is far from being complete. Besides the working group at Free University, Berlin (Germany) there are working groups in Ankara (Turkey), Astana and Almaty (Kazakhstan), Caracas (Venesolana), Delhi (India), Yerevan (Armenia), Minsk (Belarus) involved in the research. In the last years several research works appeared concerning boundary value problems for non-regular domains, such as the upper half plane [32], a quarter plane [1], [17].

In this thesis the main boundary value problems, namely, the Schwarz, the Dirichlet, the Neumann and the Robin boundary value problems are studied for the homogeneous and inhomogeneous Cauchy-Riemann equation in a circular ring domain. The main tool to treat the boundary value problems for the Poisson equation is related Green functions. Besides the known Green function for a circular ring domain [3] connected with the Dirichlet problem, the Neumann (Green function of second order) and the Robin (Green function of third order) functions are found here for a circular ring. On their basis the integral representation formulas for the solutions of the Neumann and Robin problem for the Poisson equation are derived. The Dirichlet and Schwarz problem for the Bitsadze equation are also solved explicitly. At last, a biharmonic Green function is constructed for a circular ring domain leading to the solution of the respective Dirichlet problem for the biharmonic equation. The obtained biharmonic Green function differs from the Green function constructed in [31] for the bi-Laplace operator with a particular Dirichlet boundary condition.

Concerning with applications, the case of multiply connected domains represents a special interest as it appears naturally in many problems of mechanics, especially in the filtration theory [35], [42]–[44], in hydrodynamics, in the theory of composite materials [2], [39], etc.

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2 Boundary Value Problems for First Order Complex Partial Differential Equations in a Ring Domain

Four basic boundary value problems, namely, the Schwarz, the Dirichlet, the Neumann, the Robin problems for analytic functions and more generally for the inhomogeneous Cauchy-Riemann equation are investigated in a concentric ring domain. The representations for the solutions and solvability conditions are given in explicit form.

2.1 Notations and technical preliminaries

Let $\mathbb{C}$ be the complex plane of the variable $z = x + iy,\ x, y \in \mathbb{R}$. The extended complex plane is denoted by $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The complex number $\overline{z} = x - iy$ is called the conjugate number to z. By $\operatorname{Re} z,\ \operatorname{Im} z$ the real and imaginary part of z are denoted.

The complex partial differential operators of first order are defined by

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y),\ \ \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{2.1.1}$$

A complex-valued function $w = u + iv$ is given by a couple of real-valued functions $u = u(x, y)$, $v = v(x, y)$ and being a function of the two variables z and $\overline{z}$. In the case, when u and v are differentiable and w is independent of $\overline{z}$ in an open set of the complex plane, the function w is said to be analytic in the set; the functions u, v then satisfy the Cauchy-Riemann system of partial differential equations

$$\partial_x u = \partial_y v,\ \ \partial_y u = -\partial_x v,$$

which is equivalent to the complex homogeneous Cauchy-Riemann equation

$$\partial_{\overline{z}} w = 0.$$

For analytic functions the Cauchy theorem is valid.

Theorem 2.1.1. *Let w be an analytic function in a simply connected domain $D \subset \mathbb{C}$ and let Γ be a simple closed smooth curve, $\Gamma \subset D$. Then*

$$\int_\Gamma w(z)dz = 0.$$

From the Cauchy theorem the representation of an analytic function via the Cauchy type integral is deduced. A simple closed smooth curve Γ on the complex plane divides the plane in to two parts $\operatorname{int}\Gamma$, $\operatorname{ext}\Gamma$, which are internal and external domains with respect to Γ. If w is analytic in $\operatorname{int}\Gamma$ and continuous in $\overline{\operatorname{int}\Gamma}$, then

$$\frac{1}{2\pi i}\int\limits_{\Gamma} w(\zeta)\frac{d\zeta}{\zeta - z} = \begin{cases} w(z), & z \in \operatorname{int}\Gamma, \\ 0, & z \in \operatorname{ext}\Gamma. \end{cases}$$

If w is analytic in $\operatorname{ext}\Gamma$ and continuous in $\overline{\operatorname{ext}\Gamma}$, then

$$\frac{1}{2\pi i}\int\limits_{\Gamma} w(\zeta)\frac{d\zeta}{\zeta - z} = \begin{cases} w(\infty), & z \in \operatorname{int}\Gamma, \\ -w(z) + w(\infty), & z \in \operatorname{ext}\Gamma. \end{cases}$$

A domain D on the complex plane is said to be regular if it is bounded and its boundary ∂D is smooth.

The fundamental tools for solving boundary value problems for complex first order partial differential equations are the Gauss theorem and the Cauchy-Pompeiu representation formulas.

Theorem 2.1.2. (Gauss Theorem, complex form) [7] *Let $D \subset \mathbb{C}$ be a regular domain, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, $z = x + iy$, then*

$$\int\limits_{D} w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int\limits_{\partial D} w(z)dz \qquad (2.1.2)$$

and

$$\int\limits_{D} w_{z}(z)dxdy = -\frac{1}{2i}\int\limits_{\partial D} w(z)d\overline{z}. \qquad (2.1.3)$$

From the Gauss theorem the Cauchy-Pompeiu representation formulas can be derived.

Theorem 2.1.3. (Cauchy-Pompeiu representations) [7] *Let $D \subset \mathbb{C}$ be a regular domain of $\mathbb{C}$, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, $\zeta = \xi + i\eta$. Then*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} \qquad (2.1.4)$$

and

$$w(z) = -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_{D} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \tag{2.1.5}$$

hold for all $z \in D$.

Let us define the integral operator, which is used to solve boundary value problems for the inhomogeneous Cauchy-Riemann equation.

Definition 2.1.1. [49] For $f \in L_1(D;\mathbb{C})$ the integral operator

$$Tf(z) = -\frac{1}{\pi}\int\limits_{D} f(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \mathbb{C}, \tag{2.1.6}$$

is called Pompeiu operator.

The Pompeiu operator possesses some important properties listed below.

Theorem 2.1.4. [49] *Let* $D \subset \mathbb{C}$ *be a bounded domain. If* $f \in L_1(D;\mathbb{C})$, *then* Tf *is analytic in* $\mathbb{C} \setminus \overline{D}$, *vanishing at infinity.*

Theorem 2.1.5. [49] *Let* $D \subset \mathbb{C}$ *be a bounded domain. If* $f \in L_1(D;\mathbb{C})$, *then* Tf, *regarded as a function of the point* z *of the domain* D, *exists almost everywhere and belongs to an arbitrary class* $L_p(D^*;\mathbb{C})$, *where* p *is an arbitrary number satisfying the condition* $1 \le p < 2$, *and* D^* *is an arbitrary bounded domain of the complex plane.*

Theorem 2.1.6. [49] *If* $f \in L_1(D;\mathbb{C})$, *then*

$$\int\limits_{D} Tf(z)\varphi_{\overline{z}}(z)dxdy + \int\limits_{D} f(z)\varphi(z)dxdy = 0, \tag{2.1.7}$$

where φ *is an arbitrary complex-valued function in* D *being continuously differentiable and having compact support in* D.

In other words, **Theorem 2.1.6** states that Tf is differentiable in distributional sense with respect to $\overline{z}$ if $f \in L_1(D;\mathbb{C})$, moreover

$$\partial_{\overline{z}}Tf = f \tag{2.1.8}$$

in D.

In the case, when $f \in L_p(D;\mathbb{C})$, $p > 1$, the Pompeiu operator Tf is differentiable in distributional sense with respect to z and

$$\partial_z Tf(z) =: \Pi f(z) = -\frac{1}{\pi}\int\limits_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}. \tag{2.1.9}$$

It is a singular integral operator being understood in the Cauchy principal sense.

Theorem 2.1.7. [49] *Let $D \subset \mathbb{C}$ be a bounded domain. If $f \in L_p(D;\mathbb{C})$, $p > 2$, then Tf is a linear completely continuous operator mapping $L_p(D;\mathbb{C})$ onto $C^{\alpha}(\overline{D};\mathbb{C})$, $\alpha = \dfrac{p-2}{p}$.*

Theorem 2.1.8. [49] *Let $D \in C^{m+1,\alpha}$, $f \in C^{m,\alpha}(\overline{D};\mathbb{C})$, $0 < \alpha < 1$, $m \geq 0$. Then Tf belongs to the class $C^{m+1,\alpha}(\overline{D};\mathbb{C})$ and Tf is a completely continuous operator in $C^{m,\alpha}(\overline{D};\mathbb{C})$. Moreover, Πf exists in the sense of Cauchy principal value and belongs to the class $C^{m,\alpha}(\overline{D};\mathbb{C})$. Besides, Πf represents a linear bounded operator in $C^{m,\alpha}(\overline{D};\mathbb{C})$ mapping this space onto itself.*

The space of functions with generalized z-derivatives ($\overline{z}$-derivatives) of order m in $L_p(D;\mathbb{C})$ are denoted by $W_z^{m,p}(D;\mathbb{C})$ ($W_{\overline{z}}^{m,p}(D;\mathbb{C})$), while $W_z^{m+\alpha}(D;\mathbb{C})$ ($W_{\overline{z}}^{m+\alpha}(D;\mathbb{C})$) is a space of functions with generalized Hölder continuous z-derivatives ($\overline{z}$-derivatives) of order m in D.

Let $\mathbb{D}$ be the unit disk with the center at the origin on the complex plane, i.e. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The kernel

$$\frac{\zeta + z}{\zeta - z}, \quad z \in \mathbb{D}, \quad \zeta \in \partial\mathbb{D},$$

is called the Schwarz kernel for the unit disk, while its real part

$$\operatorname{Re}\frac{\zeta + z}{\zeta - z} = \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1, \quad z \in \mathbb{D}, \quad \zeta \in \partial\mathbb{D},$$

is the Poisson kernel for $\mathbb{D}$.

In 1872 A.H. Schwarz proved [45] that for $\gamma \in C(\partial\mathbb{D};\mathbb{R})$, the Poisson kernel for the unit disk possesses the property

$$\lim_{|z|\to 1, |z|<1}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big]\frac{d\zeta}{\zeta} = \gamma(z). \tag{2.1.10}$$

From (2.1.10) it follows that for $\gamma \in C(\partial\mathbb{D}_r;\mathbb{R})$, $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, the equality

$$\lim_{|z|\to r, |z|>r} \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta)\Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big]\frac{d\zeta}{\zeta} = -\gamma(z) \tag{2.1.11}$$

holds. The minus sign appears because of the changing of orientation.

For the unit disk the modification of the Cauchy-Pompeiu formula is useful, which is obtained from (2.1.4) by using the Gauss theorem.

Theorem 2.1.9. [7] *Any function* $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\mathrm{cl}\mathbb{D};\mathbb{C})$, $\zeta = \xi + i\eta$, $z \in \mathbb{D}$, *can be represented by*

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\, w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Im}\, w(\zeta)\frac{d\zeta}{\zeta} - \\ & - \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big[\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} + \frac{z w_{\overline{\zeta}}(\zeta)}{1 - z\overline{\zeta}}\Big] d\xi d\eta. \end{aligned} \tag{2.1.12}$$

The representation (2.1.12) is called the Cauchy-Pompeiu-Schwarz-Poisson formula.

For analytic functions in $\mathbb{D}$, continuous up to the boundary the formula (2.1.12) transforms into

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\, w(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + i\,\mathrm{Im}\, w(0), \tag{2.1.13}$$

and is called Schwarz-Poisson formula.

The formula (2.1.13) is the starting point in the investigation of boundary value problems for the unit disk in [7].

2.2 Boundary value problems for analytic functions

Let $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ be the concentric ring domain with the center at the origin.

To solve boundary value problems for analytic functions in R the following representation formula, analogous to (2.1.13) for the unit disk, is important.

Theorem 2.2.10. *Let w be an analytic function in R, continuous on $\overline{R}$. Then the representation formula*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \operatorname{Re} w(z)\Big[\frac{\zeta+z}{\zeta-z} + 2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta} - \\ - \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \overline{w(\zeta)}\,\frac{d\zeta}{\zeta} \tag{2.2.1}$$

holds.

Proof. By the Cauchy theorem we have for any fixed $z \in R$ and any $n \in \mathbb{N}$

$$\frac{1}{2\pi i}\int\limits_{\partial R} w(z)\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\Big)\frac{d\zeta}{\zeta} = 0, \tag{2.2.2}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R} w(z)\Big[\frac{\overline{z}\zeta}{1-\overline{z}\zeta} - \sum_{n=1}^{\infty}\Big(\frac{r^{2n}\overline{z}\zeta}{r^{2n}\overline{z}\zeta - 1} + \frac{r^{2n}}{\overline{z}\zeta - r^{2n}}\Big)\Big]\frac{d\zeta}{\zeta} = 0. \tag{2.2.3}$$

By adding (2.2.2) and the complex conjugate of (2.2.3) to the right-hand side of the Cauchy formula, applied to w in R, we get

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial R} w(\zeta)\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\Big)\frac{d\zeta}{\zeta} + \\ + \frac{1}{2\pi i}\int\limits_{\partial R} \overline{w(\zeta)}\,\Big[\frac{z\overline{\zeta}}{1 - z\overline{\zeta}} - \sum_{n=1}^{\infty}\Big(\frac{r^{2n}z\overline{\zeta}}{r^{2n}z\overline{\zeta} - 1} + \frac{r^{2n}}{z\overline{\zeta} - r^{2n}}\Big)\Big]\frac{d\zeta}{\zeta}$$

or

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \operatorname{Re} w(\zeta)\Big[\frac{\zeta}{\zeta - z} + \frac{z|\zeta|^2}{\zeta - z|\zeta|^2} + \\ + \sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} - \frac{r^{2n}z|\zeta|^2}{r^{2n}z|\zeta|^2 - \zeta} - \frac{r^{2n}\zeta}{z|\zeta|^2 - r^{2n}\zeta}\Big)\Big]\frac{d\zeta}{\zeta} + \\ + \frac{1}{2\pi i}\int\limits_{\partial R} \operatorname{Im} w(\zeta)\Big[\frac{\zeta}{\zeta - z} - \frac{z|\zeta|^2}{\zeta - z|\zeta|^2} + \\ + \sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} + \frac{r^{2n}z|\zeta|^2}{r^{2n}z|\zeta|^2 - \zeta} + \frac{r^{2n}\zeta}{z|\zeta|^2 - r^{2n}\zeta}\Big)\Big]\frac{d\zeta}{\zeta}.$$

Dividing the boundary into the two components and performing some simplifications lead to

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\operatorname{Re}w(\zeta)\Big[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta}-$$
$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\operatorname{Re}w(\zeta)\Big[\frac{\zeta+z}{\zeta-z}+1+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta}+$$
$$+\frac{1}{2\pi}\int\limits_{|\zeta|=1}\operatorname{Im}w(\zeta)\frac{d\zeta}{\zeta},$$

which is equivalent to

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial R}\operatorname{Re}w(\zeta)\Big[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta}+$$
$$+\frac{1}{2\pi}\int\limits_{\partial R}\operatorname{Im}w(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\overline{w(\zeta)}\,\frac{d\zeta}{\zeta}.$$

Then the result follows if one takes into account that $\frac{1}{2\pi i}\int\limits_{\partial R}w(\zeta)\frac{d\zeta}{\zeta}=0$ for w being analytic in R and therefore $\frac{1}{2\pi}\int\limits_{\partial R}\operatorname{Im}w(\zeta)\frac{d\zeta}{\zeta}=0$. □

Corollary 2.2.1. *The real part of an analytic function w in R, continuous on $\overline{R}$ can be represented in the form*

$$\operatorname{Re}w(z)=\frac{1}{2\pi i}\int\limits_{\partial D}\operatorname{Re}w(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1+\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}\overline{\zeta}}{\overline{r^{2n}\zeta-z}}+$$
$$+\frac{r^{2n}z}{\zeta-r^{2n}z}+\frac{r^{2n}\overline{z}}{\overline{\zeta-r^{2n}z}}\Big)\Big]\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\operatorname{Re}w(\zeta)\frac{d\zeta}{\zeta}. \tag{2.2.4}$$

Schwarz boundary value problem. Find an analytic function w in R, i.e. a solution for the homogeneous Cauchy-Riemann equation

$$w_{\overline{z}}=0 \text{ in } R,$$

continuous on $\overline{R}$, satisfying

$$\operatorname{Re} w = \gamma \ \text{ on } \partial R, \quad \frac{1}{2\pi i} \int\limits_{|\zeta|=\rho} \operatorname{Im} w(\zeta) \frac{d\zeta}{\zeta} = c, \tag{2.2.5}$$

for $\gamma \in C(\partial R; \mathbb{R})$, $c \in \mathbb{R}$ given, and arbitrary ρ, $r < \rho < 1$.

Theorem 2.2.11. *The Schwarz problem* (2.2.5) *is uniquely solvable in the class of analytic functions in* R *with continuous real part on* $\overline{R}$ *if and only if*

$$\frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{d\zeta}{\zeta} = 0. \tag{2.2.6}$$

Then the solution is unique and given by

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \Big[\frac{\zeta + z}{\zeta - z} + 2 \sum_{n=1}^{\infty} \Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z} \Big) \Big] \frac{d\zeta}{\zeta} - \\ & - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} + ic. \end{aligned} \tag{2.2.7}$$

Proof. The formula (2.2.7) provides an analytic function in R and gives

$$\begin{aligned} \operatorname{Re} w(z) = {} & \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \sum_{n=1}^{\infty} \Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z} + \\ & + \frac{r^{2n} z|\zeta|^2}{r^{2n} z|\zeta|^2 - |z|^2\zeta} + \frac{r^{2n}|z|^2\zeta}{z|\zeta|^2 - r^{2n}|z|^2\zeta} \Big) \Big] \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta}. \end{aligned}$$

Then

$$\lim_{|z|\to 1,\, z\in R} \operatorname{Re} w(z) = \lim_{|z|\to 1,\, z\in R} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \Big] \frac{d\zeta}{\zeta}, \tag{2.2.8}$$

$$\begin{aligned} \lim_{|z|\to r,\, z\in R} \operatorname{Re} w(z) = {} & - \lim_{|z|\to r,\, z\in R} \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \Big] \frac{d\zeta}{\zeta} + \\ & + \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{d\zeta}{\zeta}. \end{aligned} \tag{2.2.9}$$

It can be seen that (2.2.6) represents the necessary and sufficient conditions for the function (2.2.7) to satisfy the first condition in (2.2.5). The second one, which is called a normalization condition, is also valid since

$$\frac{1}{2\pi i}\int\limits_{|z|=\rho} w(z)\frac{dz}{z} = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=\rho}\Big[\frac{\zeta+z}{\zeta-z}+ \\ +2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]\frac{dz}{z}\frac{d\zeta}{\zeta}- \\ -\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta}+ic = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta}+ic.$$

Passing to the imaginary part verifies the normalization condition.

The uniqueness of the solution follows from the fact that the corresponding homogeneous problem

$$w_{\overline{z}}=0 \text{ in } R,\quad \operatorname{Re} w=0 \text{ on } \partial R,\quad \frac{1}{2\pi i}\int\limits_{|z|=\rho}\operatorname{Im} w(z)\frac{dz}{z}=0,$$

has only the trivial solution. Indeed, from $w_{\overline{z}} = 0$ in R, $\operatorname{Re} w = 0$ on ∂R, one obtains that $w(z) = ic$, $c \in \mathbb{R}$. Evaluating the integral $\frac{1}{2\pi i}\int\limits_{|z|=\rho} w(z)\frac{dz}{z} = ic$ provides $c = 0$ according to the normalization condition. □

Remark 2.2.1. The equalities (2.2.8), (2.2.9) are obtained taking into account the possibility to change as well the order of passing to the limit and integration, as passing to the limit and summation.

Remark 2.2.2. The formula (2.2.7) follows from the representation (2.2.1).

Remark 2.2.3. This result differs from the one for simply connected domains (see [7]) as a solvability condition appears. It excludes functions which are not determined in a unique way by their respective boundary data. Let us consider, for instance, the function $\log z$. It is an analytic function, having vanishing Schwarz data on $|z| = 1$ and the data $\log r$ on $|z| = r$. The solvability condition is not satisfied. The function $\log z$ is multi valued with single valued real part. One can not expect this function to be determined by its real part on the boundary.

Dirichlet boundary value problem. *Find an analytic function w in R, i.e. a solution to the equation*

$$w_{\overline{z}} = 0 \ \ in \ R,$$

continuous on $\overline{R}$, satisfying

$$w = \gamma \ \ on \ \partial R \tag{2.2.10}$$

for a given $\gamma \in C(\partial R; \mathbb{C})$.

Theorem 2.2.12. *The Dirichlet problem* (2.2.10) *is solvable in the class of analytic functions in R, continuous on $\overline{R}$ if and only if for $z \in R$*

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\zeta = 0, \ \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}}{r^2-\overline{z}\zeta}d\zeta = 0. \tag{2.2.11}$$

Then the solution is unique and given by the Cauchy type integral

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta - z}. \tag{2.2.12}$$

Proof. Let w defined by (2.2.12) be a solution to the Dirichlet problem. Then the equality

$$\lim_{z\to\zeta,\, z\in R} w(\zeta) = \gamma(\zeta), \ \forall\, \zeta \in \partial R, \tag{2.2.13}$$

holds. The Cauchy type integral provides an analytic function in R and $\widehat{\mathbb{C}} \setminus \overline{R}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Consider for $|z| < 1$ the function

$$w\Big(\frac{1}{\overline{z}}\Big) = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{\overline{z}\zeta - 1} = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}}{\overline{z-\zeta}}\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{|z|^2\zeta}{|z|^2\zeta - z}\frac{d\zeta}{\zeta}.$$

From

$$w(z) - w\Big(\frac{1}{\overline{z}}\Big) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta} -$$
$$- \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} - \frac{|z|^2\zeta}{|z|^2\zeta - z}\Big)\frac{d\zeta}{\zeta},$$

(2.2.13) and the property of the Poisson kernel (2.1.10), the existence of $\lim\limits_{z\to\zeta,\,|z|<1} w\Big(\dfrac{1}{\overline{z}}\Big)$ and the equality

$$\lim_{z\to\zeta,\,|z|<1} w\Big(\frac{1}{\overline{z}}\Big) = 0$$

follow.

Because $\overline{w(1/\overline{z})}$ is analytic in $|z| < 1$, the maximum principle for analytic functions states that $w\Big(\dfrac{1}{\overline{z}}\Big) \equiv 0$ for $|z| < 1$, i.e. the first condition in (2.2.11) holds.

Consider for $|z| > r$ the function

$$w\Big(\frac{r^2}{\overline{z}}\Big) = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{\overline{z}\zeta - r^2} = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{|z|^2\zeta}{|z|^2\zeta - z}\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{\overline{z}}{\overline{\overline{\zeta - z}}}\frac{d\zeta}{\zeta},$$

and the difference

$$w(z) - w\Big(\frac{r^2}{\overline{z}}\Big) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} - \frac{|z|^2\zeta}{|z|^2\zeta - r^2 z}\Big)\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\overline{\zeta - z}}} - 1\Big)\frac{d\zeta}{\zeta}.$$

Using the properties of the Poisson kernel (2.1.11), the equality

$$\lim_{z\to\zeta,\,|z|>r} w\Big(\frac{r^2}{\overline{z}}\Big) = 0$$

follows. This implies

$$w\Big(\frac{r^2}{\overline{z}}\Big) \equiv 0 \text{ for } |z| > r,$$

due to the statement of maximum principle for analytic functions. Thus the second condition in (2.2.11) is valid. It completes the proof for (2.2.11) to be necessary. Sufficiency of (2.2.11) is obvious. □

Remark 2.2.4. The Dirichlet problem for analytic functions in more general domains is handled via the Plemelj-Sokhotsky formula. For this purpose the boundary data required to be Hölder continuous. The reason for Hölder continuity to be not necessary for circular domains is that for these domains the Cauchy kernel is replaced by the Poisson kernel according to solvability conditions.

To formulate the third boundary value problem we need to define the outward normal derivative at the boundary of a regular domain. The direction of this derivative on the unit circle coincides with the direction of the radius vector, and on the circle $|z| = r$ is opposite to this direction, more exactly on the boundary of the concentric annulus R the normal derivative is given by the formulas

$$\partial_{\nu_z} = \begin{cases} z\,\partial_z + \overline{z}\,\partial_{\overline{z}}, & |z| = 1, \\ -\dfrac{z}{r}\,\partial_z - \dfrac{\overline{z}}{r}\,\partial_{\overline{z}}, & |z| = r. \end{cases} \tag{2.2.14}$$

Neumann boundary value problem. *Find an analytic function w in R, i.e. a solution to the equation*

$$w_{\overline{z}} = 0 \ \text{ in } R,$$

continuously differentiable on $\overline{R}$, satisfying

$$\lambda|z|\partial_{\nu_z} w = \gamma \ \text{ on } \partial R, \quad w(z_{\text{fix}}) = c, \quad \lambda = \begin{cases} 1, & |z| = 1, \\ -1, & |z| = r, \end{cases} \tag{2.2.15}$$

for given $\gamma \in C(\partial R; \mathbb{C})$, $c \in \mathbb{C}$, $z_{\text{fix}} \in R$.

For w being an analytic function in R, the boundary condition can be rewritten in the form

$$z w_z = \gamma \ \text{ on } \partial R. \tag{2.2.16}$$

Theorem 2.2.13. *The Neumann problem*

$$w_{\overline{z}} = 0 \ \text{ in } R, \quad z w_z = \gamma \ \text{ on } \partial R, \quad w(z_{fix}) = c \tag{2.2.17}$$

is solvable in the class of analytic functions in R, continuously differentiable on $\overline{R}$ if and only if for $\gamma \in C(\partial R; \mathbb{C})$, $c \in \mathbb{C}$, $z_{fix} \in R$ given the conditions

$$\frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} = 0, \tag{2.2.18}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=0, \tag{2.2.19}$$

are satisfied. Moreover if γ *satisfies*

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta}=0, \tag{2.2.20}$$

then the solution is a unique, single valued function given by

$$w(z)=c-\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma(\zeta)\log\Big(\frac{1-z\overline{\zeta}}{1-z_{fix}\overline{\zeta}}\Big)\frac{d\zeta}{\zeta}+\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\log\Big(\frac{z\overline{\zeta}-r^2}{z_{fix}\overline{\zeta}-r^2}\Big)\frac{d\zeta}{\zeta}. \tag{2.2.21}$$

Proof. Let us introduce the new function $\varphi=zw_z$. For w being analytic, the function φ is also analytic.

The Neumann problem (2.2.17) with respect to w is equivalent to the Dirichlet problem with respect to the analytic functions φ. Hence from the preceding result

$$\varphi(z)=\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta-z}$$

if and only if for $z\in R$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=0.$$

Then

$$\begin{aligned}w_z(z)=\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{d\zeta}{z(\zeta-z)}=\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma(\zeta)\Big(\frac{\overline{\zeta}}{1-z\overline{\zeta}}+\frac{1}{z}\Big)\frac{d\zeta}{\zeta}-\\ -\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma(\zeta)\Big(\frac{\overline{\zeta}}{r^2-z\overline{\zeta}}+\frac{1}{z}\Big)\frac{d\zeta}{\zeta}.\end{aligned} \tag{2.2.22}$$

The primitive of the function in (2.2.22) is

$$
\begin{aligned}
w(z) = c_0 - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log(1 - z\overline{\zeta}) \frac{d\zeta}{\zeta} + \\
+ \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \log(z\overline{\zeta} - r^2) \frac{d\zeta}{\zeta} + \frac{\log z}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{d\zeta}{\zeta},
\end{aligned}
\tag{2.2.23}
$$

where $c_0 \in \mathbb{C}$.

As for any ρ, $r < |\rho| < 1$,

$$
\begin{aligned}
\frac{1}{2\pi i} \int\limits_{|z|=\rho} dw(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{1}{2\pi i} \int\limits_{|z|=\rho} \frac{\overline{\zeta} dz}{1 - z\overline{\zeta}} \frac{d\zeta}{\zeta} + \\
+ \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{1}{2\pi i} \int\limits_{|z|=\rho} \frac{\overline{\zeta} dz}{z\overline{\zeta} - r^2} \frac{d\zeta}{\zeta} + \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{1}{2\pi i} \int\limits_{|z|=\rho} \frac{dz}{z} \frac{d\zeta}{\zeta} = \\
= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta},
\end{aligned}
$$

then the function defined in (2.2.23) is single valued if and only if (2.2.20) is satisfied. The solution to the Neumann problem (2.2.17) has the form (2.2.21), if one defines c_0 by

$$
c_0 = c + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log(1 - z_{\text{fix}}\overline{\zeta}) \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \log(z_{\text{fix}}\overline{\zeta} - r^2) \frac{d\zeta}{\zeta}.
$$

Remark 2.2.5. The solution to the Neumann problem (2.2.17) can be written in the form

$$
w(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log\left| \frac{1 - z\overline{\zeta}}{1 - z_{\text{fix}}\overline{\zeta}} \right|^2 \frac{d\zeta}{\zeta} + \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \log\left| \frac{z\overline{\zeta} - r^2}{z_{\text{fix}}\overline{\zeta} - r^2} \right|^2 \frac{d\zeta}{\zeta},
\tag{2.2.24}
$$

if the conditions

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=$$
$$=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}, \tag{2.2.25}$$
$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta}=0,$$

are satisfied.

Indeed, differentiating (2.2.24) with respect to $\overline{z}$ and using one part of (2.2.25) shows

$$w_{\overline{z}}(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{1-\overline{z}\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{r^2-\overline{z}\zeta}=0.$$

That (2.2.24) satisfies the boundary condition follows as before, because (2.2.18), (2.2.19) are included in (2.2.25).

Robin boundary value problem. *Find an analytic function w in R, i.e. a solution to the equation*

$$w_{\overline{z}}=0 \ \text{ in } R,$$

continuously differentiable on $\overline{R}$, satisfying the boundary condition

$$w+\lambda|z|\partial_{\nu_z}w=\gamma \ \text{ on } \partial R, \quad \lambda=\begin{cases} 1, & |z|=1, \\ -1, & |z|=r, \end{cases} \tag{2.2.26}$$

for $\gamma\in C(\partial R;\mathbb{C})$ given.

Note, that the boundary condition (2.2.26) can be rewritten in the form

$$w+zw_z=\gamma \ \text{ on } \partial R$$

in the case of w being an analytic function in R.

Introducing the new function

$$\varphi=w+zw_z, \tag{2.2.27}$$

the boundary problem is reduced to the following Dirichlet problem

$$\varphi_{\overline{z}}=0 \ \text{ in } R, \quad \varphi=\gamma \ \text{ on } \partial R, \tag{2.2.28}$$

which has to be solved in the class of analytic functions in R, continuously differentiable in $\overline{R}$, represented by formula (2.2.27) with some analytic function w.

According to **Theorem 2.2.12**, problem (2.2.28) is uniquely solvable if and only if for $z \in R$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=0. \qquad (2.2.29)$$

The unique solution is then given by the Cauchy type integral

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta-z}. \qquad (2.2.30)$$

Any analytic function in R is uniquely representable by a convergent Laurent series

$$w(z)=\sum_{n=-\infty}^{\infty}c_n z^n. \qquad (2.2.31)$$

Then

$$\varphi(z)\equiv w(z)+zw_z(z)=\sum_{n=-\infty}^{\infty}(n+1)c_n z^n \qquad (2.2.32)$$

holds, and together with (2.2.30) it leads to the equation

$$\sum_{n=-\infty}^{\infty}(n+1)c_n z^n=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}z^n+\sum_{n=-\infty}^{-1}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}z^n \qquad (2.2.33)$$

with respect to c_n, $n=0,\pm1,\pm2,...$

Comparing both sides of (2.2.33), equalities

$$c_n=\frac{1}{n+1}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}},\ n=0,1,2,...;$$

$$c_n=\frac{1}{n+1}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}},\ n=...,-3,-2,$$

and the condition

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)d\zeta=0 \qquad (2.2.34)$$

are deduced, while the coefficient c_{-1} may take arbitrary values from $\mathbb{C}$. So $w(z)$ has the form

$$w(z) = \sum_{n=0}^{\infty} \frac{1}{n+1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\Big)z^n + \frac{c_{-1}}{z} +$$

$$+ \sum_{n=-\infty}^{-2} \frac{1}{n+1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\Big)z^n =$$

$$= -\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\log(1 - z\overline{\zeta})d\zeta + \frac{c_{-1}}{z} + \frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\log(1 - \frac{\zeta}{z})d\zeta$$

For the uniqueness of the solution one has to pose an additional condition, e.g. of the type

$$z_{\text{fix}} w(z_{\text{fix}}) = c$$

for some fixed point $z_{\text{fix}} \in R$, $c \in \mathbb{C}$. So the following theorem has been proved.

Theorem 2.2.14. *The Robin boundary value problem*

$$w_{\overline{z}} = 0 \ \text{ in } R, \quad w + z w_z = \gamma \ \text{ on } \partial R, \quad z_{fix} w(z_{fix}) = c$$

for $\gamma \in C(\partial R; \mathbb{C})$, $c \in \mathbb{C}$, $z_{fix} \in R$ *given is solvable in the class of analytic functions in* R, *continuously differentiable on* $\overline{R}$ *if and only if conditions* (2.2.29), (2.2.34) *are satisfied. Then the solution is unique and given by*

$$w(z) = \frac{c}{z} - \frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\log(1 - z\overline{\zeta})d\zeta + \frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\log(1 - \frac{\zeta}{z})d\zeta +$$

$$+\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\log(1 - z_{fix}\overline{\zeta})d\zeta - \frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\log(1 - \frac{\zeta}{z_{fix}})d\zeta. \tag{2.2.35}$$

Remark 2.2.6. The general Robin boundary value problem for analytic functions in R, continuously differentiable on $\overline{R}$, deals with the more complicated boundary condition $\alpha\, w + z w_z = \gamma$ on ∂R, where $\alpha \in C(\partial R; \mathbb{C})$. Here only the above special kind of Robin problem has been considered.

2.3 Boundary value problems for the inhomogeneous Cauchy-Riemann equation

The Schwarz, the Dirichlet, the Neumann, the Robin boundary value problems for the inhomogeneous Cauchy-Riemann equation in the concentric ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ are solved in this section. Using the definition and properties of the Pompeiu operator, the problems are reduced to the homogeneous case, or which is equivalent, to the boundary value problems for analytic functions.

Theorem 2.3.15. *The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in* R

$$w_{\overline{z}} = f \ \text{ in } R, \quad \operatorname{Re} w = \gamma \ \text{ on } \partial R, \quad \frac{1}{2\pi i}\int\limits_{|\zeta|=\rho} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} = c, \tag{2.3.1}$$

for $f \in L_p(R;\mathbb{C})$, $p > 2$, $\gamma \in C(\partial R;\mathbb{R})$, $c \in \mathbb{R}$, $r < \rho < 1$ *given is solvable by a function from* $W_{\overline{z}}^{1,p}(R;\mathbb{C})$ *with continuous real part on* $\overline{R}$ *if and only if*

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma(\zeta)\frac{d\zeta}{\zeta} = \frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)d\xi d\eta. \tag{2.3.2}$$

If the condition (2.3.2) *is satisfied, then the solution to the problem* (2.3.1) *is unique and given by*

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\Big[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\Big]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta} - \\ & - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\Big[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\Big]d\xi d\eta - \\ & - \frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} + K_2(z,\zeta)\Big]d\xi d\eta + \\ & - \frac{1}{2\pi}\int\limits_{r<|\zeta|<\rho}\Big(\frac{f(\zeta)}{\zeta} - \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)d\xi d\eta + ic, \end{aligned} \tag{2.3.3}$$

where

$$K_1(z,\zeta) = 2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\Big), \tag{2.3.4}$$

$$K_2(z,\zeta) = 2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}} + \frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\Big). \tag{2.3.5}$$

Proof. Let us introduce the new unknown function $\varphi = w - Tf$, where Tf is the Pompeiu operator. Then (see (2.1.8))

$$\varphi_{\overline{z}} = 0 \text{ in } R, \quad \operatorname{Re}\varphi = \gamma - \operatorname{Re}Tf \text{ on } \partial R, \quad \frac{1}{2\pi i}\int\limits_{|\zeta|=\rho} \operatorname{Im}[\varphi(\zeta)+Tf(\zeta)]\frac{d\zeta}{\zeta} = c,$$

i.e. φ is a solution to the Schwarz boundary value problem for analytic functions. By **Theorem 2.2.11**,

$$\begin{aligned}\varphi(z) = \frac{1}{2\pi i}\int\limits_{\partial R}(\gamma(\zeta) - \operatorname{Re}Tf(\zeta))\Big[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\Big]\frac{d\zeta}{\zeta} - \\ - \frac{1}{2\pi i}\int\limits_{|\zeta|=r}(\gamma(\zeta) - \operatorname{Re}Tf(\zeta))\frac{d\zeta}{\zeta} + i\widehat{c}\end{aligned} \tag{2.3.6}$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial R}(\gamma(\zeta) - \operatorname{Re}Tf(\zeta))\frac{d\zeta}{\zeta} = 0, \tag{2.3.7}$$

where $\widehat{c} = \dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=\rho} \operatorname{Im}\varphi(\zeta)\frac{d\zeta}{\zeta}$.

Calculating the integral

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\operatorname{Re}Tf(\zeta)\frac{d\zeta}{\zeta} &= \frac{1}{2\pi i}\int\limits_{\partial R}\Big(\frac{1}{2\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{\zeta-\widetilde{\zeta}} + \frac{1}{2\pi}\int\limits_{R}\overline{f(\widetilde{\zeta})}\frac{d\widetilde{\xi}d\widetilde{\eta}}{\overline{\zeta-\widetilde{\zeta}}}\Big)\frac{d\zeta}{\zeta} = \\ &= \frac{1}{2\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{d\zeta}{\zeta(\zeta-\widetilde{\zeta})}d\widetilde{\xi}d\widetilde{\eta} + \frac{1}{2\pi}\int\limits_{R}\overline{f(\widetilde{\zeta})}\frac{1}{2\pi i}\int\limits_{\partial R}\frac{d\zeta}{\zeta(\overline{\zeta-\widetilde{\zeta}})}d\widetilde{\xi}d\widetilde{\eta} = \\ &= \frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\widetilde{\zeta})}{\widetilde{\zeta}} + \frac{\overline{f(\widetilde{\zeta})}}{\overline{\widetilde{\zeta}}}\Big)d\widetilde{\xi}d\widetilde{\eta},\end{aligned}$$

shows that the solvability condition (2.3.7) get the form (2.3.2).

In the same way, one can find that

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=\rho}\operatorname{Im}Tf(\zeta)\frac{d\zeta}{\zeta} = -\frac{1}{2\pi i}\int\limits_{\rho<|\zeta|<1}\Big(\frac{f(\zeta)}{\zeta} - \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)d\xi d\eta,$$

therefore

$$\widehat{c} = c + \frac{1}{2\pi i}\int\limits_{\rho<|\zeta|<1}\Big(\frac{f(\zeta)}{\zeta} - \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)d\xi d\eta.$$

Applying the result of the following calculations

$$\frac{1}{2\pi i}\int\limits_{\partial R}\operatorname{Re}Tf(\zeta)\Big[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\Big]\frac{d\zeta}{\zeta}=$$

$$=\frac{1}{2\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{1}{\zeta(\zeta-\widetilde{\zeta})}\Big[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\Big]d\zeta d\widetilde{\xi}d\widetilde{\eta}+$$

$$+\frac{1}{2\pi}\int\limits_{R}\overline{f(\widetilde{\zeta})}\frac{1}{2\pi i}\int\limits_{\partial R}\frac{1}{\overline{\zeta(\zeta-\widetilde{\zeta})}}\Big[\frac{\zeta+z}{\zeta-z}+K_1(z,\zeta)\Big]d\zeta d\widetilde{\xi}d\widetilde{\eta}=$$

$$=\frac{1}{2\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{1}{\zeta-\widetilde{\zeta}}\Big[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta}d\widetilde{\xi}d\widetilde{\eta}+$$

$$+\frac{1}{2\pi}\int\limits_{R}\overline{f(\widetilde{\zeta})}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{1}{1-\zeta\overline{\widetilde{\zeta}}}\Big[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]d\zeta-$$

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{1}{r^2-\zeta\overline{\widetilde{\zeta}}}\Big[\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big)\Big]d\zeta\Big)d\widetilde{\xi}d\widetilde{\eta}=$$

$$=\frac{1}{2\pi}\int\limits_{R}\frac{f(\widetilde{\zeta})}{\widetilde{\zeta}}\Big[\frac{\widetilde{\zeta}+z}{\widetilde{\zeta}-z}+K_1(z,\widetilde{\zeta})\Big]d\widetilde{\xi}d\widetilde{\eta}+$$

$$+\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\widetilde{\zeta})}}{\overline{\widetilde{\zeta}}}\Big[\frac{1+z\overline{\widetilde{\zeta}}}{1-z\overline{\widetilde{\zeta}}}+K_2(z,\widetilde{\zeta})\Big]d\widetilde{\xi}d\widetilde{\eta}-$$

$$-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{\widetilde{\zeta}-z}-\frac{1}{\pi}\int\limits_{R}\frac{\overline{f(\widetilde{\zeta})}}{\overline{\widetilde{\zeta}}}d\widetilde{\xi}d\widetilde{\eta},$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\operatorname{Re}Tf(\zeta)\frac{d\zeta}{\zeta}=\frac{1}{2\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{d\zeta}{\zeta(\zeta-\widetilde{\zeta})}d\widetilde{\xi}d\widetilde{\eta}-$$

$$-\frac{1}{2\pi}\int\limits_{R}\overline{f(\widetilde{\zeta})}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{d\zeta}{\zeta\overline{\widetilde{\zeta}}-r^2}d\widetilde{\xi}d\widetilde{\eta}=-\frac{1}{2\pi}\int\limits_{R}\frac{f(\widetilde{\zeta})}{\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta}-\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\widetilde{\zeta})}}{\overline{\widetilde{\zeta}}}d\widetilde{\xi}d\widetilde{\eta},$$

and the expression for $\widehat{c}$ to the formula (2.3.6) and using then the definition of the Pompeiu operator in the formula $w = \varphi + Tf$, the solution is obtained in the form (2.3.3).

If the solvability condition (2.3.2) is satisfied, then (2.3.3) gives the solution to (2.3.1), since

$$\operatorname{Re} w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + \sum_{n=1}^{\infty} \Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z} +$$

$$+ \frac{r^{2n} z}{r^{2n} z - |z|^2 \zeta} + \frac{r^{2n}|z|^2\zeta}{z - r^{2n}|z|^2\zeta} \Big) \Big] \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 +$$

$$+ \sum_{n=1}^{\infty} \Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z} + \frac{r^{2(n+1)} z}{r^{2(n+1)} z - |z|^2\zeta} + \frac{r^{2(n-1)}|z|^2\zeta}{z - r^{2(n-1)}|z|^2\zeta} \Big) \Big] \frac{d\zeta}{\zeta} -$$

$$- \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{4\pi} \int\limits_{R} \frac{f(\zeta)}{\zeta} \Big[\frac{\zeta + z}{\zeta - z} + \frac{z + |z|^2\zeta}{z - |z|^2\zeta} +$$

$$+ 2 \sum_{n=1}^{\infty} \Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z} + \frac{r^{2n} z}{r^{2n} z - |z|^2\zeta} + \frac{r^{2n}|z|^2\zeta}{z - r^{2n}|z|^2\zeta} \Big) \Big] d\xi d\eta -$$

$$- \frac{1}{4\pi} \int\limits_{R} \frac{\overline{f(\zeta)}}{\overline{\zeta}} \Big[\frac{z\overline{\zeta} + |z|^2}{z\overline{\zeta} - |z|^2} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} +$$

$$+ 2 \sum_{n=1}^{\infty} \Big(\frac{r^{2n} z\overline{\zeta}}{r^{2n} z\overline{\zeta} - |z|^2} + \frac{r^{2n}|z|^2}{z\overline{\zeta} - r^{2n}|z|^2} + \frac{r^{2n}}{r^{2n} - z\overline{\zeta}} + \frac{r^{2n} z\overline{\zeta}}{1 - r^{2n} z\overline{\zeta}} \Big) \Big] d\xi d\eta.$$

Then $\operatorname{Re} w(z)$ tends to $\gamma(z)$ as $|z| \to 1$, $z \in R$, due to the property of Poisson kernel for the circle $|\zeta| = 1$, and $\operatorname{Re} w(z)$ tends to $\gamma(z)$ as $|z| \to r$, $z \in R$, because of the property of the Poisson kernel for the circle $|\zeta| = r$ and the solvability condition (2.3.2).

The uniqueness of the solution follows from **Theorem 2.2.11**. □

Theorem 2.3.16. *The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in* R

$$w_{\overline{z}} = f \ \text{ in } R, \quad w = \gamma \ \text{ on } \partial R, \tag{2.3.8}$$

for $f \in L_p(R;\mathbb{C})$, $p > 2$, $\gamma \in C(\partial R;\mathbb{R})$ *given is solvable in the class* $W^{1,p}_{\overline{z}}(R;\mathbb{C}) \cap C(\overline{R};\mathbb{C})$ *if and only if for* $z \in R$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta, \tag{2.3.9}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{z}}{r^2-\overline{z}\zeta}d\xi d\eta. \tag{2.3.10}$$

Then the solution is unique and expressed by

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{d\xi d\eta}{\zeta - z}. \tag{2.3.11}$$

Proof. If the problem is solvable, then the formula (2.3.11) follows from the Cauchy-Pompeiu representation (2.1.4). The uniqueness of the solution is a consequence of **Theorem 2.2.12**.

Introducing the new unknown function

$$\varphi = w - Tf$$

we arrive at the following boundary value problem

$$\varphi_{\overline{z}} = 0 \ \text{ in } R, \quad \varphi = \gamma - Tf \ \text{ on } \partial R, \tag{2.3.12}$$

equivalent to (2.3.8).

By **Theorem 2.2.12** the solvability conditions for (2.3.12) are

$$\frac{1}{2\pi i}\int\limits_{\partial R}(\gamma(\zeta) - Tf(\zeta))\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = 0,$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}(\gamma(\zeta) - Tf(\zeta))\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} = 0.$$

They coincide with (2.3.9), (2.3.10), if one takes into account that

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}Tf(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} &= \frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\overline{z}}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta-\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta} = \\ &= \frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{\overline{z}}{1-\overline{z}\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta},\end{aligned} \tag{2.3.13}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R} Tf(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\overline{z}}{r^2-\overline{z}\zeta}\frac{d\zeta}{\zeta-\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta}$$
$$= \frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{\overline{z}}{r^2-\overline{z}\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta}. \tag{2.3.14}$$

The formula (2.3.11) determines the solution to (2.3.8) under the solvability conditions (2.3.9), (2.3.10) due to the properties of the Pompeiu operator and

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\Big]\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\Big[\frac{1}{\zeta-z}+\frac{\overline{z}}{1-\overline{z}\zeta}\Big]d\zeta-$$

$$-\frac{1}{\pi}\int\limits_{R} f(\zeta)\Big[\frac{1}{\zeta-z}+\frac{\overline{z}}{1-\overline{z}\zeta}\Big]d\xi d\eta,$$

which tends to $\gamma(z)$ as $|z|\to 1,\ z\in R,$ or

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\Big[\frac{1}{\zeta-z}+\frac{\overline{z}}{r^2-\overline{z}\zeta}\Big]d\zeta-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\Big]\frac{d\zeta}{\zeta}-$$

$$-\frac{1}{\pi}\int\limits_{R} f(\zeta)\Big[\frac{1}{\zeta-z}+\frac{\overline{z}}{r^2-\overline{z}\zeta}\Big]d\xi d\eta,$$

which also tends to $\gamma(z)$ as $|z|\to r,\ z\in R.$ □

For the proof of the next theorem the following lemma is used.

Lemma 2.3.1. *For* $|z|,|\widetilde{\zeta}|>r$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log(\overline{z}\zeta-r^2)}{(r^2-\zeta\overline{\widetilde{\zeta}})^2}d\zeta = 0.$$

Proof. Consider the integral

$$I(\overline{z},\overline{\widetilde{\zeta}}) := \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log(\overline{z}\zeta-r^2)}{(r^2-\zeta\overline{\widetilde{\zeta}})^2}d\zeta$$

as a function of two parameters $\overline{z}, \overline{\widetilde{\zeta}}$, $|z|, |\widetilde{\zeta}| > r$. Differentiation with respect to $\overline{z}$ gives

$$\partial_{\overline{z}}\, I(\overline{z}, \overline{\widetilde{\zeta}}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{\zeta d\zeta}{(\overline{z}\zeta - r^2)(r^2 - \zeta\overline{\widetilde{\zeta}})^2} =$$

$$= \partial_{\overline{\widetilde{\zeta}}} \Big(\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{d\zeta}{(\overline{z}\zeta - r^2)(\zeta\overline{\widetilde{\zeta}} - r^2)} \Big) = 0,$$

hence

$$I(\overline{z}, \overline{\widetilde{\zeta}}) = I(\overline{\widetilde{\zeta}}, \overline{\widetilde{\zeta}}). \qquad (2.3.15)$$

Let us consider a new function of one parameter $\overline{z}$

$$\widehat{I}(\overline{z}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{\log(\overline{z}\zeta - r^2)}{\overline{z}\zeta - r^2} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i r^2} \int\limits_{|\zeta|=r} \log(\overline{z}\zeta - r^2) \Big(\frac{\overline{z}}{\overline{z}\zeta - r^2} - \frac{1}{\zeta} \Big) d\zeta =$$

$$= \frac{1}{2\pi i r^2} \int\limits_{|\zeta|=r} (\log \overline{z} + \log(\zeta - \zeta_0)) \Big(\frac{1}{\zeta - \zeta_0} - \frac{1}{\zeta} \Big) d\zeta =$$

$$= \frac{1}{2\pi i r^2} \int\limits_{|\zeta|=r} \log(\zeta - \zeta_0) \Big(\frac{1}{\zeta - \zeta_0} - \frac{1}{\zeta} \Big) d\zeta,$$

where $\zeta_0 = \dfrac{r^2}{\overline{z}}$, $|\zeta_0| < r$.

Evaluating

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \log(\zeta - \zeta_0) \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} [\log \zeta + \log\Big(1 - \frac{\zeta_0}{\zeta}\Big)] \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \log \zeta \frac{d\zeta}{\zeta}$$

as

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \log\Big(1 - \frac{\zeta_0}{\zeta}\Big) \frac{d\zeta}{\zeta} = -\sum_{k=1}^{\infty} \frac{\zeta_0^k}{k} \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{d\zeta}{\zeta^{k+1}} = 0,$$

and furthermore

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\log\zeta\frac{d\zeta}{\zeta}=\frac{1}{2\pi i}\int\limits_{0}^{2\pi}\log(re^{i(\varphi+2k\pi)})id\varphi=\log r+i(2k+1)\pi,$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\log(\zeta-\zeta_0)\frac{d\zeta}{\zeta-\zeta_0}=\frac{1}{2\pi i}\int\limits_{t_0}^{t_0+2\pi}\log(r(t)e^{i(t+2k\pi)})\Big(\frac{dr(t)}{r(t)}+idt\Big)=$$

$$=\frac{1}{2}\left[\log^2 r(t_0+2\pi)-\log^2 r(t_0)\right]+$$

$$+\frac{1}{2\pi}\int\limits_{t_0}^{t_0+2\pi}\Big(\log r(t)dt+(t+2k\pi)\frac{dr(t)}{r(t)}+i(t+2k\pi)dt\Big)=$$

$$=\log r+i(2k+1)\pi,$$

the equality $\widehat{I}(\overline{z})=0$ is valid for any $z,\ |z|>r$.

Observing that

$$\widehat{I_{\overline{z}}}(\overline{z})=\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Big[\frac{1}{(\overline{z}\zeta-r^2)^2}-\frac{\log(\overline{z}\zeta-r^2)}{(\overline{z}\zeta-r^2)^2}\Big]d\zeta=$$

$$=-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log(\overline{z}\zeta-r^2)}{(\overline{z}\zeta-r^2)^2}d\zeta=-I(\overline{z},\overline{z}),$$

from $\widehat{I_{\overline{z}}}(\overline{z})=-I(\overline{z},\overline{z}),\ \widehat{I}(\overline{z})=0$ the identity $I(\overline{z},\overline{z})=0$ is obtained. This completes the proof due to (2.3.15). □

Theorem 2.3.17. *The Neumann problem for the inhomogeneous Cauchy-Riemann equation in* R

$$w_{\overline{z}}=f\ \text{ in } R,\quad \lambda|z|\partial_{\nu_z}w=\gamma\ \text{ on }\partial R,\ \ w(z_{\mathit{fix}})=c,\ \ \lambda=\left\{\begin{array}{rl}1, & |z|=1,\\ -1, & |z|=r,\end{array}\right. \tag{2.3.16}$$

for $f\in C^{\alpha}(\overline{R};\mathbb{C}),\ 0<\alpha<1,\ \gamma\in C(\partial R;\mathbb{C}),\ c\in\mathbb{C},\ z_{\mathit{fix}}\in R$ *given is solvable by a function from* $W^{1+\alpha}_{\overline{z}}(\overline{R};\mathbb{C})$ *with continuous weak* z*-derivative* *on* $\overline{R}$ *if and only if for* $z \in R$

$$\frac{1}{2\pi i}\int_{\partial R}[\gamma(\zeta)-\zeta f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta}+\frac{1}{\pi}\int_{R} f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2}=0, \tag{2.3.17}$$

$$\frac{1}{2\pi i}\int_{\partial R}[\gamma(\zeta)-\zeta f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta}+\frac{r^2}{\pi}\int_{R} f(\zeta)\frac{d\xi d\eta}{(r^2-\overline{z}\zeta)^2}=0. \tag{2.3.18}$$

Moreover if γ *and* f *satisfy the condition*

$$\frac{1}{2\pi i}\int_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta}=0 \tag{2.3.19}$$

then the solution is a unique, single valued function represented by

$$w(z)=c-\frac{1}{2\pi i}\int_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log\Big(\frac{1-z\overline{\zeta}}{1-z_{fix}\overline{\zeta}}\Big)\frac{d\zeta}{\zeta}+ \tag{2.3.20}$$

$$+\frac{1}{2\pi i}\int_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log\Big(\frac{z\overline{\zeta}-r^2}{z_{fix}\overline{\zeta}-r^2}\Big)\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int_{R} f(\zeta)\frac{z-z_{fix}}{(\zeta-z_{fix})(\zeta-z)}d\xi d\eta.$$

Proof. To reduce the problem to the homogeneous case the new function

$$\varphi = w - Tf$$

is introduced. It satisfies

$$\varphi_{\overline{z}}=0, \text{ in } R, \quad z\varphi_z=\gamma-z\Pi f-\overline{z}f \text{ on } \partial R, \ \varphi(z_{\text{fix}})=c-Tf(z_{\text{fix}}). \tag{2.3.21}$$

For $f \in C^\alpha(\overline{R};\mathbb{C})$, $\Pi f \in C^\alpha(\overline{R};\mathbb{C})$ (see [49]). Therefore the right-hand side of the boundary condition in (2.3.21) is a continuous function. **Theorem 2.2.13**, applied to (2.3.21), gives

$$\varphi(z)=c-Tf(z_{\text{fix}})-\frac{1}{2\pi i}\int_{|\zeta|=1}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\log\Big(\frac{1-z\overline{\zeta}}{1-z_{\text{fix}}\overline{\zeta}}\Big)\frac{d\zeta}{\zeta}+ \tag{2.3.22}$$

$$+\frac{1}{2\pi i}\int_{|\zeta|=r}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\log\Big(\frac{z\overline{\zeta}-r^2}{z_{\text{fix}}\overline{\zeta}-r^2}\Big)\frac{d\zeta}{\zeta}$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta}=0, \tag{2.3.23}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta}=0, \tag{2.3.24}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta}=0. \tag{2.3.25}$$

From the relations

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Pi f(\zeta)\log(1-z\overline{\zeta})d\zeta=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\,\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\overline{\zeta})d\zeta}{(\zeta-\widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta}= \tag{2.3.26}$$

$$=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\,\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\overline{z}\zeta)d\zeta}{(1-\zeta\overline{\widetilde{\zeta}})^2}}d\widetilde{\xi}d\widetilde{\eta}=0,$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Pi f(\zeta)\log(z\overline{\zeta}-r^2)d\zeta=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\,\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log(z\overline{\zeta}-r^2)d\zeta}{(\zeta-\widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta}= \tag{2.3.27}$$

$$=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\,\overline{\frac{r^2}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log(\overline{z}\zeta-r^2)d\zeta}{(r^2-\zeta\overline{\widetilde{\zeta}})^2}}d\widetilde{\xi}d\widetilde{\eta}=0,$$

where the result of the previous lemma has been used, the formula (2.3.22) and the definition of the Pompeiu operator, it follows that (2.3.20) gives the solution to the problem (2.3.16).

One can deduce conditions (2.3.17)–(2.3.19) from (2.3.23)–(2.3.25), since

$$\frac{1}{2\pi i}\int\limits_{\partial R}\zeta\Pi f(\zeta)\frac{d\zeta}{1-\overline{z}\zeta}=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\zeta d\zeta}{(\zeta-\widetilde{\zeta})^2(1-\overline{z}\zeta)}d\widetilde{\xi}d\widetilde{\eta}=$$

$$=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{(1-\overline{z}\widetilde{\zeta})^2}, \tag{2.3.28}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\zeta\Pi f(\zeta)\frac{d\zeta}{r^2-\overline{z}\zeta} = -\frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\zeta d\zeta}{(\zeta-\widetilde{\zeta})^2(r^2-\overline{z}\zeta)}d\widetilde{\xi}d\widetilde{\eta} =$$

$$= -\frac{r^2}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{(r^2-\overline{z}\widetilde{\zeta})^2}, \tag{2.3.29}$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Pi f(\zeta)d\zeta = -\frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(\zeta-\widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta} = 0. \tag{2.3.30}$$

□

Remark 2.3.7. The solution to the Neumann problem (2.3.16) can be written in the form

$$\begin{aligned}
w(z) = c &- \frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log\left|\frac{1-z\overline{\zeta}}{1-z_{\text{fix}}\overline{\zeta}}\right|^2\frac{d\zeta}{\zeta}+ \\
&+ \frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log\left|\frac{z\overline{\zeta}-r^2}{z_{\text{fix}}\overline{\zeta}-r^2}\right|^2\frac{d\zeta}{\zeta}+ \\
&+ \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{(z-z_{\text{fix}})}d\xi d\eta}{(1-\overline{z}\zeta)(1-\overline{z}_{\text{fix}}\zeta)}- \\
&- \frac{r^2}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{(z-z_{\text{fix}})}d\xi d\eta}{(r^2-\overline{z}\zeta)(r^2-\overline{z}_{\text{fix}}\zeta)} - \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{(z-z_{\text{fix}})d\xi d\eta}{(\zeta-z)(\zeta-z_{\text{fix}})}
\end{aligned} \tag{2.3.31}$$

if for $z \in R$ the conditions

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta} &+ \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2} = \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta} = \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta} = \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta} - \frac{r^2}{\pi}\int\limits_{R} f(\zeta)\frac{d\xi d\eta}{(r^2-\overline{z}\zeta)^2},
\end{aligned} \tag{2.3.32}$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta}=0 \tag{2.3.33}$$

are satisfied.

As before, the problem (2.3.16) is reduced to the homogeneous case. One can apply the formula (2.2.24) to solve the corresponding problem (2.3.21).

Then the solution to the problem (2.3.21) is given by

$$\begin{aligned}\varphi(z)=c-Tf(z_{\text{fix}})-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\log\Big|\frac{1-z\overline{\zeta}}{1-z_{\text{fix}}\overline{\zeta}}\Big|^2\frac{d\zeta}{\zeta}+\\ +\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\log\Big|\frac{z\overline{\zeta}-r^2}{z_{\text{fix}}\overline{\zeta}-r^2}\Big|^2\frac{d\zeta}{\zeta}\end{aligned} \tag{2.3.34}$$

under the solvability conditions

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta}=\\ &=\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta}=\\ &=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta}=\\ &=\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta},\end{aligned} \tag{2.3.35}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta)-\zeta\Pi f(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta}=0. \tag{2.3.36}$$

The formula (2.3.31) follows from (2.3.34), using (2.3.26), (2.3.27), if one takes into account that

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Pi f(\zeta)\log(1-\overline{z}\zeta)d\zeta=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\overline{z}\zeta)d\zeta}{(\zeta-\widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta}=\\ &=-\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\partial_\zeta(\log(1-\overline{z}\zeta))\big|_{\zeta=\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta}=\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{\overline{z}}{1-\overline{z}\widetilde{\zeta}}d\widetilde{\xi}d\widetilde{\eta},\end{aligned}$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Pi f(\zeta)\log|z\overline{\zeta}-r^2|^2 d\zeta = -\frac{1}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log|z\overline{\zeta}-r^2|^2 d\zeta}{(\zeta-\widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta} =$$

$$= -\frac{1}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{(\log|z-\zeta|^2+\log r^2)d\zeta}{(\zeta-\widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta} =$$

$$= -\frac{1}{\pi}\int\limits_R f(\widetilde{\zeta})\overline{\frac{r^2}{2\pi i}\int\limits_{|\zeta|=r}\frac{\log(z-\zeta)d\zeta}{(r^2-\zeta\overline{\widetilde{\zeta}})^2}}d\widetilde{\xi}d\widetilde{\eta} = -\frac{r^2}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{\overline{\widetilde{\zeta}}(r^2-\overline{z}\widetilde{\zeta})}.$$

Solvability conditions (2.3.32), (2.3.33) are obtained from (2.3.35), (2.3.36), where

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\zeta\Pi f(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} = -\frac{1}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta d\zeta}{(\zeta-\widetilde{\zeta})^2(1-\overline{z}\zeta)}d\widetilde{\xi}d\widetilde{\eta} =$$

$$= -\frac{1}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{(1-\overline{z}\widetilde{\zeta})^2},$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\zeta\Pi f(\zeta)\frac{d\zeta}{r^2-\overline{z}\zeta} = -\frac{1}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\frac{\zeta d\zeta}{(\zeta-\widetilde{\zeta})^2(r^2-\overline{z}\zeta)}d\widetilde{\xi}d\widetilde{\eta} =$$

$$= \frac{r^2}{\pi}\int\limits_R f(\widetilde{\zeta})\frac{d\widetilde{\xi}d\widetilde{\eta}}{(r^2-\overline{z}\widetilde{\zeta})^2},$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\zeta\Pi f(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} = 0,\quad \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\zeta\Pi f(\zeta)\frac{d\zeta}{r^2-\overline{z}\zeta} = 0.$$

Theorem 2.3.18. *The Robin boundary value problem for the inhomogeneous Cauchy-Riemann equation in R*

$$w_{\overline{z}} = f \text{ in } R,\ w+\lambda|z|\partial_{\nu_z}w = \gamma \text{ on } \partial R,\ z_{fix}w(z_{fix}) = c,\ \lambda = \begin{cases} 1, & |z|=1, \\ -1, & |z|=r, \end{cases} \tag{2.3.37}$$

for $f \in C^\alpha(\overline{R};\mathbb{C})$, $0<\alpha<1$, $\gamma \in C(\partial R;\mathbb{C})$, $c \in \mathbb{C}$, $z_{fix} \in R$ given is solvable by a function from $W_{\overline{z}}^{1+\alpha}(\overline{R};\mathbb{C})$ with continuous weak z-derivative on $\overline{R}$ if and only if for $z \in R$ the conditions

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{1-\overline{z}\zeta} + \frac{1}{\pi}\int\limits_R f(\zeta)\frac{\overline{z}\zeta d\zeta}{(1-\overline{z}\zeta)^2}d\xi d\eta = 0, \tag{2.3.38}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{r^2-\overline{z}\zeta}+\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{\overline{z}\zeta d\zeta}{(r^2-\overline{z}\zeta)^2}d\xi d\eta=0, \tag{2.3.39}$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]d\zeta=0, \tag{2.3.40}$$

are satisfied. Then the solution is unique and given by

$$\begin{aligned}
w(z)=&\frac{c}{z}-\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log(1-z\overline{\zeta})d\zeta+\\
&+\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log(1-\frac{\zeta}{z})d\zeta+\\
&+\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log(1-z_{fix}\overline{\zeta})d\zeta-\\
&-\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)](\zeta)\log(1-\frac{\zeta}{z_{fix}})d\zeta-\\
&-\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{z-z_{fix}}{(\zeta-z)(\zeta-z_{fix})}d\xi d\eta.
\end{aligned} \tag{2.3.41}$$

Proof. By introducing the new function

$$\varphi=w-Tf, \tag{2.3.42}$$

the problem (2.3.37) is reduced to

$$\varphi_{\overline{z}}=0 \text{ in } R,\ \varphi+z\varphi_z=\gamma-Tf-z\Pi f-\overline{z}f \text{ on } \partial R,\ z_{\text{fix}}\varphi(z_{\text{fix}})=c-Tf(z_{\text{fix}}). \tag{2.3.43}$$

The function φ is analytic in R, hence it is represented by the Laurent series

$$\varphi(z)=\sum_{n=-\infty}^{\infty}c_nz^n,\ c_n\in\mathbb{C}.$$

Consider the function $\varphi+z\varphi_z$. It is also analytic in R with

$$\varphi(z)+z\varphi_z(z)=\sum_{n=-\infty}^{\infty}(n+1)c_nz^n. \tag{2.3.44}$$

On the other side, any analytic function in R can be represented by the Cauchy integral according to its boundary values, i.e.

$$\varphi(z) + z\varphi_z(z) = \frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta) - Tf(\zeta) - \zeta\Pi f(\zeta) - \overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta - z},$$

under the conditions

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta) - Tf(\zeta) - \zeta\Pi f(\zeta) - \overline{\zeta}f(\zeta)]\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} = 0,$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta) - Tf(\zeta) - \zeta\Pi f(\zeta) - \overline{\zeta}f(\zeta)]\frac{\overline{z}d\zeta}{r^2 - \overline{z}\zeta} = 0,$$

(see Theorem 3.2). Using (2.3.13), (2.3.14), (2.3.28), (2.3.29) these conditions can be rewritten in the form

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta) - \overline{\zeta}f(\zeta)]\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} + \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{z}^2\zeta d\zeta}{(1 - \overline{z}\zeta)^2}d\xi d\eta = 0,$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta) - \overline{\zeta}f(\zeta)]\frac{\overline{z}d\zeta}{r^2 - \overline{z}\zeta} + \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{z}^2\zeta d\zeta}{(r^2 - \overline{z}\zeta)^2}d\xi d\eta = 0,$$

what is equivalent to (2.3.38), (2.3.39).

Evaluating

$$\frac{1}{2\pi i}\int\limits_{\partial R} Tf(\zeta)\frac{d\zeta}{\zeta - z} = \frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{d\zeta}{(\zeta - z)(\zeta - \widetilde{\zeta})}d\widetilde{\xi}d\widetilde{\eta} = 0,$$

$$\frac{1}{2\pi i}\int\limits_{\partial R} \zeta\Pi f(\zeta)\frac{d\zeta}{\zeta - z} = -\frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{2\pi i}\int\limits_{\partial R}\frac{\zeta d\zeta}{(\zeta - z)(\zeta - \widetilde{\zeta})^2}d\widetilde{\xi}d\widetilde{\eta} = 0$$

one obtains

$$\varphi(z) + z\varphi_z(z) = \frac{1}{2\pi i}\int\limits_{\partial R}[\gamma(\zeta) - \overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta - z}. \tag{2.3.45}$$

To find the constants c_n, $n = 0, \pm1, \pm2, ...,$ we have to solve the equation

$$\sum_{n=-\infty}^{\infty}(n+1)c_n z^n = \sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\Big)z^n + \sum_{n=-\infty}^{-1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\Big)z^n,$$

which is the result of comparing (2.3.44) with (2.3.45). Then

$$c_n = \frac{1}{n+1}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta^{n+1}},\ n=0,1,2,..., \tag{2.3.46}$$

$$c_n = \frac{1}{n+1}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta^{n+1}},\ n=...,-3,-2, \tag{2.3.47}$$

and the condition

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]d\zeta = 0$$

are obtained, while c_{-1} remains undetermined. To find it the last relation in (2.3.37) is used. Inserting (2.3.46), (2.3.47) into (2.3.44) we get the function φ in the form

$$\begin{aligned}\varphi(z) &= \sum_{n=0}^{\infty}\frac{1}{n+1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta^{n+1}}\Big)z^n+\frac{c_{-1}}{z}+\\ &+\sum_{n=-\infty}^{-2}\frac{1}{n+1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\frac{d\zeta}{\zeta^{n+1}}\Big)z^n=\\ &=-\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log(1-z\overline{\zeta})d\zeta+\frac{c_{-1}}{z}+\\ &+\frac{1}{z}\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma(\zeta)-\overline{\zeta}f(\zeta)]\log\Big(1-\frac{\zeta}{z}\Big)d\zeta.\end{aligned}$$

To find c_{-1} the last condition in (2.3.43) is exploited. Thus the function φ is defined. Using the definition of the Pompeiu operator and equality (2.3.42), one gets the initially unknown function in the form (2.3.41). □

3 Boundary Value Problems for Second Order Complex Partial Differential Equations in a Ring Domain

There are two basic second order partial differential operators: the Laplace operator $\partial_z\partial_{\overline{z}}$ and the Bitsadze operator $\partial_{\overline{z}}^2$. The third one ∂_z^2 is the complex conjugate to the Bitsadze operator. All results for this operator are derived from the corresponding results for the Bitsadze operator.

In order to treat boundary value problems for second order complex partial differential equations special kernel functions have to be constructed. The most important among them are Green, Neumann and Robin functions. All of them are certain fundamental solutions to the Laplace operator. The Green, the Neumann and the Robin functions are used to solve the Dirichlet, the Neumann and the Robin boundary value problems for the Poisson equation via corresponding integral representation formulas for solutions.

3.1 Green function for a circular ring domain

Definition 3.1.1. [5] A real-valued function $G(z,\zeta) = \frac{1}{2}G_1(z,\zeta)$ in a regular domain $D \subset \mathbb{C}$ is called the Green function of D, or more exactly the Green function of D for the Laplace operator, if it possesses for any fixed $\zeta \in D$ as a function of z the following properties:

1°. $G(z,\zeta)$ is harmonic in $D \setminus \{\zeta\}$,
2°. $G(z,\zeta) + \log|\zeta - z|$ is harmonic in D,
3°. $\lim\limits_{z\to\partial D} G(z,\zeta) = 0$.

The Green function has the additional properties [5]:

4°. $G(z,\zeta) > 0$,
5°. $G(z,\zeta) = G(\zeta, z)$.
6°. It is uniquely determined by $1° - 3°$.

Not any domain in the complex plane has a Green function. The existence of the Green function for a given domain $D \subset \mathbb{C}$ can be proved in the case when the Dirichlet problem for harmonic functions is solvable for D (see, e.g. [5]).

The procedure of constructing the Green function for a circular ring $\{z \in \mathbb{C} : 0 < 1/r < |z| < r\}$ is described in [28]. In our case, dealing with the circular ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$, it is given below (see also [3]).

Any harmonic function can be represented as the real part of an analytic function. The desired Green function is sought in the form

$$-\frac{1}{2}\mathrm{Re}\,\{\log f(z)\} = -\frac{1}{2}\log|f(z)|$$

with the function $f(z)$ being analytic in R, having a simple zero in, say c, with modulus 1 on the boundary. Without loss of generality, c is assumed to be real and positive. In order to find enough function-theoretical properties to construct $f(z)$ explicitly, it is extended beyond the two circles $|z| = 1$, $|z| = r$ by setting $f(z) = \overline{f(z^*)}$, where $z^* = \dfrac{1}{\overline{z}}$, for $|z| > 1$ or $z^* = \dfrac{r^2}{\overline{z}}$ for $|z| < r$.

If z approaches a boundary point, the corresponding point z^* does approach it also. The function $f(z)$ may be considered as a real function taking complex conjugate values at complex conjugate points. This implies that $f(z)f\left(\dfrac{1}{z}\right)$ or $f(z)f\left(\dfrac{r^2}{z}\right)$ approaches the positive real value $|f(z_0)|^2$, whenever z tends to $z_0 \in \partial R$. On the other hand, $f(z)$ has modulus 1 on the boundary. Thus the functional equations

$$f(z)f\left(\frac{1}{z}\right) = 1, \tag{3.1.1}$$

$$f(z)f\left(\frac{r^2}{z}\right) = 1, \tag{3.1.2}$$

hold identically.

Since c is a simple zero of $f(z)$, applying (3.1.1), (3.1.2) successively, shows that the function $f(z)$ has simple zeroes at the points

$$c,\ cr^2,\ \frac{c}{r^2},\ cr^4, \ldots, cr^{2k},\ \frac{c}{r^{2k}}, \ldots,$$

and simple poles at the points

$$\frac{1}{c}, \frac{r^2}{c}, \frac{1}{cr^2}, \frac{r^4}{c}, \frac{1}{cr^4}, \ldots, \frac{r^{2k}}{c}, \frac{1}{r^{2k}c}, \ldots$$

Thus it coincides in its zeroes and poles with the function

$$F(z) = \frac{c - z}{1 - cz}\prod_{k=1}^{\infty}\frac{(z - r^{2k}c)(c - r^{2k}z)}{(cz - r^{2k})(1 - r^{2k}cz)}. \tag{3.1.3}$$

The relation

$$F(z)F\Big(\frac{1}{z}\Big) = 1,$$

is valid.

From the other side,

$$F(z)F\Big(\frac{r^2}{z}\Big) = \frac{1}{1-cz}\prod_{k=1}^{\infty}\frac{1-r^{2(k-1)}cz}{1-r^{2k}cz}\,\frac{1}{z-r^2c}\prod_{k=1}^{\infty}\frac{z-r^{2k}c}{z-r^{2(k+1)}c}\times$$

$$\times(cz-r^2)\prod_{k=1}^{\infty}\frac{cz-r^{2(k+1)}}{cz-r^{2k}}\,(c-z)\prod_{k=1}^{\infty}\frac{c-r^{2k}z}{c-r^{2(k-1)}z} =$$

$$= \lim_{k\to\infty}\frac{(cz-r^{2(k+1)})(c-r^{2k}z)}{(1-r^{2k}cz)(z-r^{2(k+1)}c)} = c^2.$$

The function $F(z)$ does not satisfy (3.1.2).

Let us define

$$f(z) = az^bF(z),$$

$a, b \in \mathbb{R}$. Equations (3.1.1),(3.1.2) determine a system of equations with respect to a and b. Its solutions are $a = \pm 1$, $b = -\dfrac{\log c}{\log r}$. Let us choose $a = 1$. Then

$$f(z) = z^{-\log c/\log r}F(z).$$

Now instead of the real $c \in R$ one can take an arbitrary $\zeta \in R$ (see [28]). Then the Green function for R is represented by $G(z,\zeta) = -\log|f(z)| = \dfrac{1}{2}G_1(z,\zeta)$ with

$$G_1(z,\zeta) = \frac{\log|z|^2\log|\zeta|^2}{\log r^2} - \log\Big|\frac{\zeta-z}{1-z\overline{\zeta}}\prod_{k=1}^{\infty}\frac{(z-r^{2k}\zeta)(\zeta-r^{2k}z)}{(z\overline{\zeta}-r^{2k})(1-r^{2k}\overline{z}\zeta)}\Big|^2. \quad (3.1.4)$$

The Green function, defined in (3.1.4), satisfies clearly the properties 1°, 2°, 5°. Its boundary behavior is

$$\lim_{|z|\to 1,\, z\in R} G_1(z,\zeta) = -\lim_{|z|\to 1,\, z\in R}\log\Big|\frac{\overline{z}\zeta-|z|^2}{\overline{z}^2(1-\overline{z}\zeta)}\prod_{k=1}^{\infty}\frac{|z|^2-r^{2k}\overline{z}\zeta}{z\overline{\zeta}-r^{2k}}\times$$

$$\times\frac{\overline{z}\zeta-r^{2k}|z|^2}{1-r^{2k}z\overline{\zeta}}\Big|^2 = 0,$$

$$\lim_{|z|\to r,\, z\in R} G_1(z,\zeta) = \log|\zeta|^2 - \lim_{n\to\infty} \lim_{|z|\to r,\, z\in R} \log\Big|(\zeta - z)\prod_{k=1}^{n} \frac{|z|^2 - r^{2k}\overline{z}\zeta}{1 - r^{2k}z\overline{\zeta}}\times$$

$$\times\frac{1}{1-\overline{z}\zeta}\prod_{k=1}^{n}\frac{\zeta - r^{2k}z}{|z|^2\overline{\zeta} - r^{2k}z}\Big|^2 = \log|\zeta|^2 - \lim_{n\to\infty}\log\Big|\frac{\zeta - r^{2n}z}{1 - r^{2n}\overline{z}\zeta}\Big|^2 = 0.$$

The next theorem gives the representation formula for a certain class of functions via Green function and is used to solve the Dirichlet boundary value problem for the Poisson equation.

Theorem 3.1.1. *Any function* $w \in C^2(R;\mathbb{C}) \cap C^1(\overline{R};\mathbb{C})$ *can be represented by*

$$w(z) = -\frac{1}{4\pi i}\int_{\partial R} |\zeta|\partial_{\nu_\zeta} G_1(z,\zeta) w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_R w_{\zeta\overline{\zeta}}(\zeta) G_1(z,\zeta)\, d\xi d\eta. \quad (3.1.5)$$

Proof. Let $z \in R$ and $\varepsilon > 0$ be so small that $\overline{B_\varepsilon(z)} \subset R$,

$$B_\varepsilon(z) = \{\zeta \in \mathbb{C} : |\zeta - z| < \varepsilon\}.$$

Let us denote $R_\varepsilon = R \setminus \overline{B_\varepsilon(z)}$ and consider

$$\frac{1}{\pi}\int_{R_\varepsilon} w_{\zeta\overline{\zeta}}(\zeta) G_1(z,\zeta)\, d\xi d\eta =$$

$$= \frac{1}{2\pi}\int_{R_\varepsilon}\Big[\partial_{\overline{\zeta}}\left[w_\zeta(\zeta)G_1(z,\zeta)\right] + \partial_\zeta\left[w_{\overline{\zeta}}(\zeta)G_1(z,\zeta)\right] -$$

$$- w_\zeta(\zeta)G_{1\overline{\zeta}}(z,\zeta) - w_{\overline{\zeta}}(\zeta)G_{1\zeta}(z,\zeta)\Big] d\xi d\eta =$$

$$= \frac{1}{4\pi i}\int_{\partial R_\varepsilon} G_1(z,\zeta)[w_\zeta(\zeta)d\zeta - w_{\overline{\zeta}}(\zeta)d\overline{\zeta}] -$$

$$- \frac{1}{2\pi}\int_{R_\varepsilon}\Big[\partial_\zeta[w(\zeta)G_{1\overline{\zeta}}(z,\zeta)] + \partial_{\overline{\zeta}}[w(\zeta)G_{1\zeta}(z,\zeta)] -$$

$$- 2\partial_{\zeta\overline{\zeta}}\, G_1(z,\zeta)w(\zeta)\Big] d\xi d\eta =$$

$$= -\frac{1}{4\pi i}\int_{|\zeta - z| = \varepsilon} G_1(z,\zeta)[w_\zeta(\zeta)d\zeta - w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta}\,] -$$

$$-\frac{1}{4\pi i}\int_{\partial R_\varepsilon} w(\zeta)\Big[G_{1\zeta}(z,\zeta)d\zeta - G_{1\overline{\zeta}}(z,\zeta)d\overline{\zeta}\Big] =$$

$$= -\frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon} G_1(z,\zeta)\Big[(\zeta-z)w_\zeta(\zeta) + (\overline{\zeta-z})w_{\overline{\zeta}}(\zeta)\Big]\frac{d\zeta}{\zeta-z} -$$

$$-\frac{1}{4\pi i}\int_{\partial R} w(\zeta)|\zeta|\Big(\frac{\zeta}{|\zeta|}\partial_\zeta + \frac{\overline{\zeta}}{|\zeta|}\partial_{\overline{\zeta}}\Big)G_1(z,\zeta)\frac{d\zeta}{\zeta} +$$

$$+\frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon}\Big[(\zeta-z)G_{1\zeta}(z,\zeta) + (\overline{\zeta-z})G_{1\overline{\zeta}}(z,\zeta)\Big]w(\zeta)\frac{d\zeta}{\zeta-z}.$$

Introducing polar coordinates $\zeta = z + \varepsilon e^{i\theta}$ leads to

$$\frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon} G_1(z,\zeta)\Big[(\zeta-z)w_\zeta(\zeta) - (\overline{\zeta-z})w_{\overline{\zeta}}(\zeta)\Big]\frac{d\zeta}{\zeta-z} =$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \varepsilon[e^{i\theta}w_\zeta(\zeta) + e^{-i\theta}w_{\overline{\zeta}}(\zeta)]G_1(z,\zeta)d\theta$$

which tends to zero as $\varepsilon \to 0$.

From the property 2° of the Green function the representation

$$G_1(z,\zeta) = -\log|\zeta - z|^2 + h_1(z,\zeta)$$

holds, with $h_1(z,\zeta)$ being harmonic in R as a function of z for any $\zeta \in R$. Using this formula and polar coordinates gives

$$\frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon}\Big[(\zeta-z)G_{1\zeta}(z,\zeta) + (\overline{\zeta-z})G_{1\overline{\zeta}}(z,\zeta)\Big]w(\zeta)\frac{d\zeta}{\zeta-z} =$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\Big[\varepsilon[e^{i\theta}\partial_\zeta + e^{-i\theta}\partial_{\overline{\zeta}}]\,h_1(z, z+\varepsilon e^{i\theta}) - 2\Big]w(z+\varepsilon e^{i\theta})d\theta.$$

It tends to $-w(z)$ as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{R_\varepsilon} w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta = -\frac{1}{4\pi i}\int_{\partial R}|\zeta|\,\partial_{\nu_\zeta}G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} - w(\zeta).$$

This proves (3.1.5). □

3.2 Neumann function for a circular ring domain

Definition 3.2.2. [5] A real-valued function $N(z,\zeta) = \frac{1}{2} N_1(z,\zeta)$ in a regular domain $D \subset \mathbb{C}$ is called Neumann function of D (for the Laplace operator) if it satisfies as a function of z the following properties:

1°. $N(z,\zeta)$ is harmonic in $D \setminus \{\zeta\}$,

2°. $N(z,\zeta) + \log|\zeta - z|$ is harmonic in D,

3°. $\partial_{\nu_z} N(z,\zeta)$ is constant on any boundary component of D for any $\zeta \in D$.

Remark 3.2.1. The Neumann function is not uniquely defined by $1^\circ - 3^\circ$ (see, e.g., [5]).

Lemma 3.2.1. *Let D be a regular domain in the complex plane $\mathbb{C}$, $\partial D = \bigcup_{j=0}^{n} \Gamma_j$, $\Gamma_j \subset \operatorname{int}\Gamma_0$, $1 \le j \le n$, $n \in \mathbb{N} \cup \{0\}$ fixed. The Neumann function $N(z,\zeta)$ of D is uniquely defined by asking*

$$\frac{1}{2\pi} \int_{\Gamma_0} N(z,\zeta) ds_z = 0, \textit{ for all} \zeta \in D,\ ds_z = |dz|. \tag{3.2.1}$$

Proof. Let $N_{(1)}(z,\zeta), N_{(2)}(z,\zeta)$ be two Neumann functions of D. Consider

$$U(z) := N_{(1)}(z,\zeta) - N_{(2)}(z,\zeta) \text{ for any fixed } \zeta \in D.$$

Using the properties $1^\circ - 3^\circ$ of the Neumann function, it follows that $U(z)$ is harmonic in D and $\partial_{\nu_z} U(z) = 0$.

Applying the Green formula (real form)

$$\int_D (u(x,y) \triangle v(x,y) + <\operatorname{grad} u(x,y), \operatorname{grad} v(x,y)>) dxdy = \int_{\partial D} u(x,y) \partial_\nu v(x,y) ds$$

for $u(x,y) \equiv v(x,y) \equiv U(x,y)$, where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, one obtains

$$\int_D \left| \operatorname{grad} U(x,y) \right|^2 dxdy = 0,$$

hence $U(x,y) \equiv \text{const}$.

The formula (3.2.1) immediately provides this constant to be zero. $\square$

Proposition 3.2.1. *The Neumann function of the Laplace operator for the ring domain R is given by* $N(z,\zeta)=\dfrac{1}{2}N_1(z,\zeta)$ *with*

$$N_1(z,\zeta)=-\log\Big|(\zeta-z)(1-z\overline{\zeta})\times \\ \times\prod_{k=1}^{\infty}\frac{(z-r^{2k}\zeta)(z\overline{\zeta}-r^{2k})(\zeta-r^{2k}z)(1-r^{2k}z\overline{\zeta})}{|z|^2|\zeta|^2}\Big|^2. \tag{3.2.2}$$

Proof. The properties $1^\circ-3^\circ$ of the Neumann function have to be checked. The function $N(z,\zeta)$ is harmonic and satisfies 2°. The boundary behavior is observed from

$$\partial_{\nu_z}N_1(z,\zeta)=\begin{cases} zN_{1\,z}(z,\zeta)+\overline{z}N_{1\,\overline{z}}(z,\zeta), & |z|=1,\\ -\dfrac{z}{r}N_{1\,z}(z,\zeta)-\dfrac{\overline{z}}{r}N_{1\,\overline{z}}(z,\zeta), & |z|=r,\end{cases}$$

where

$$zN_{1\,z}(z,\zeta)=\frac{z}{\zeta-z}+\frac{z\overline{\zeta}}{1-z\overline{\zeta}}- \\ -\sum_{k=1}^{\infty}\Big(\frac{z}{z-r^{2k}\zeta}+\frac{z\overline{\zeta}}{z\overline{\zeta}-r^{2k}}-\frac{r^{2k}z}{\zeta-r^{2k}z}-\frac{r^{2k}z\overline{\zeta}}{1-r^{2k}z\overline{\zeta}}-2\Big). \tag{3.2.3}$$

Then for all $\zeta\in R$

$$\partial_{\nu_z}N_1(z,\zeta)=-2, \ \text{ on } |z|=1, \tag{3.2.4}$$

$$\partial_{\nu_z}N_1(z,\zeta)=0, \ \text{ on } |z|=r. \tag{3.2.5}$$

The normalization condition (3.2.1) reads for R

$$\frac{1}{2\pi i}\int_{|z|=1}N_1(z,\zeta)\frac{dz}{z}=0. \tag{3.2.6}$$

In order to check it let us consider

$$\frac{1}{2\pi i}\int_{|z|=1}N_1(z,\zeta)\frac{dz}{z}=-\frac{1}{2\pi i}\int_{|z|=1}\Big[\log|\zeta-z|^2+\log|1-z\overline{\zeta}|^2+ \\ +\sum_{k=1}^{\infty}\log\Big|\frac{(z-r^{2k}\zeta)(z\overline{\zeta}-r^{2k})(\zeta-r^{2k}z)(1-r^{2k}z\overline{\zeta})}{|z|^2|\zeta|^2}\Big|^2\Big]\frac{dz}{z}=$$

$$= -\frac{1}{2\pi i}\int\limits_{|z|=1}\Big[2\log|1-z\overline{\zeta}|^2+\sum_{k=1}^{\infty}\Big(\log|z-r^{2k}\zeta|^2+\log|z\overline{\zeta}-r^{2k}|^2+$$

$$+\log|\zeta-r^{2k}z|^2+\log|1-r^{2k}z\overline{\zeta}|^2-2\log|\zeta|^2\Big)\Big]\frac{dz}{z},$$

with

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\log|1-z\overline{\zeta}|^2\frac{dz}{z}=2\mathrm{Re}\Big[\frac{1}{2\pi i}\int\limits_{|z|=1}\log(1-z\overline{\zeta})\frac{dz}{z}\Big]=0,$$

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\log|z-r^{2k}\zeta|^2\frac{dz}{z}=\frac{1}{2\pi i}\int\limits_{|z|=1}\log|1-r^{2k}z\overline{\zeta}|^2\frac{dz}{z}=0,$$

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\log|z\overline{\zeta}-r^{2k}|^2\frac{dz}{z}=\frac{1}{2\pi i}\int\limits_{|z|=1}\log|\zeta-r^{2k}z|^2\frac{dz}{z}=\log|\zeta|^2.$$

This completes the proof. □

There is an analog of the representation formula (3.1.5), which involves the Neumann function instead of the Green function.

Theorem 3.2.2. *Any function* $w\in C^2(R;\mathbb{C})\cap C^1(\overline{R};\mathbb{C})$ *can be represented by the formula*

$$w(z)=-\frac{1}{4\pi i}\int\limits_{\partial R}|\zeta|\partial_{\nu_\zeta}N_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{4\pi i}\int\limits_{\partial R}|\zeta|\,\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\frac{d\zeta}{\zeta}-$$

$$-\frac{1}{\pi}\int\limits_{R}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta. \qquad (3.2.7)$$

Proof. Let $z\in R$ and $\varepsilon>0$ be so small that $\overline{B_\varepsilon(z)}\subset R$,

$$B_\varepsilon(z)=\{\zeta\in\mathbb{C}:|\zeta-z|<\varepsilon\}.$$

We denote $R_\varepsilon=R\setminus\overline{B_\varepsilon(z)}$ and consider

$$\frac{1}{\pi}\int\limits_{R_\varepsilon}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta=$$

$$=\frac{1}{2\pi}\int\limits_{R_\varepsilon}\Big[\partial_{\overline{\zeta}}\left[w_\zeta(\zeta)N_1(z,\zeta)\right]+\partial_\zeta\left[w_{\overline{\zeta}}(\zeta)N_1(z,\zeta)\right]-$$

$$- w_\zeta(\zeta) N_{1\overline{\zeta}}(z,\zeta) - w_{\overline{\zeta}}(\zeta) N_{1\zeta}(z,\zeta)\Big] d\xi d\eta =$$

$$= \frac{1}{4\pi i} \int\limits_{\partial R_\varepsilon} N_1(z,\zeta)[w_\zeta(\zeta) d\zeta - w_{\overline{\zeta}}(\zeta) d\overline{\zeta}\,] -$$

$$- \frac{1}{2\pi} \int\limits_{R_\varepsilon} \Big[\partial_\zeta [w(\zeta) N_{1\overline{\zeta}}(z,\zeta)] + \partial_{\overline{\zeta}} [w(\zeta) N_{1\zeta}(z,\zeta)] -$$

$$- 2\partial_{\zeta\overline{\zeta}} N_1(z,\zeta) w(\zeta)\Big] d\xi d\eta =$$

$$= \frac{1}{4\pi i} \int\limits_{\partial R} N_1(z,\zeta)[w_\zeta(\zeta) d\zeta - w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta}\,] -$$

$$- \frac{1}{4\pi i} \int\limits_{|\zeta - z| = \varepsilon} N_1(z,\zeta)[w_\zeta(\zeta) d\zeta - w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta}\,] -$$

$$- \frac{1}{4\pi i} \int\limits_{\partial R_\varepsilon} w(\zeta) \Big[N_{1\zeta}(z,\zeta) d\zeta - N_{1\overline{\zeta}}(z,\zeta) d\overline{\zeta}\Big] =$$

$$= \frac{1}{4\pi i} \int\limits_{\partial R} N_1(z,\zeta) |\zeta| \Big(\frac{\zeta}{|\zeta|}\partial_\zeta + \frac{\overline{\zeta}}{|\zeta|}\partial_{\overline{\zeta}}\Big) w(\zeta) \frac{d\zeta}{\zeta} -$$

$$- \frac{1}{4\pi i} \int\limits_{|\zeta - z| = \varepsilon} N_1(z,\zeta) \Big[(\zeta - z) w_\zeta(\zeta) + (\overline{\zeta - z}) w_{\overline{\zeta}}(\zeta)\Big] \frac{d\zeta}{\zeta - z} -$$

$$- \frac{1}{4\pi i} \int\limits_{\partial R} w(\zeta) |\zeta| \Big(\frac{\zeta}{|\zeta|}\partial_\zeta + \frac{\overline{\zeta}}{|\zeta|}\partial_{\overline{\zeta}}\Big) N_1(z,\zeta) \frac{d\zeta}{\zeta} +$$

$$+ \frac{1}{4\pi i} \int\limits_{|\zeta - z| = \varepsilon} \Big[(\zeta - z) N_{1\zeta}(z,\zeta) + (\overline{\zeta - z}) N_{1\overline{\zeta}}(z,\zeta)\Big] w(\zeta) \frac{d\zeta}{\zeta - z}.$$

This gives formula (3.2.7), letting ε tend to zero, by the same arguments as have been used in the proof of **Theorem 3.1.1**. □

Integral representation formula (3.2.7) is used to solve the Neumann problem for the Poisson equation.

3.3 Poisson and Bergman kernels for a circular ring domain

Definition 3.3.3. Let D be a regular domain with a Green function $G(z,\zeta) = \frac{1}{2}\,G_1(z,\zeta)$. The Poisson kernel for D is defined by

$$g_1(z,\zeta) = -\frac{1}{2}\,\partial_{\nu_\zeta} G_1(z,\zeta),\ z\in D,\ \zeta\in\partial D. \tag{3.3.1}$$

The Poisson kernel for R can be evaluated explicitly and has the form

$$g_1(z,\zeta) = \begin{cases} -\mathrm{Re}\,\widehat{g}_1(z,\zeta), & |\zeta| = 1,\\ \frac{1}{r}\,\mathrm{Re}\,\widehat{g}_1(z,\zeta), & |\zeta| = r, \end{cases} \tag{3.3.2}$$

with

$$\begin{aligned}\widehat{g}_1(z,\zeta) :=& \frac{\log|z|^2}{\log r^2} - \frac{\zeta}{\zeta - z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta} +\\ &+\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\overline{z}\zeta}{r^{2k}\overline{z}\zeta - 1} + \frac{\overline{z}\zeta}{\overline{z}\zeta - r^{2k}} - \frac{\zeta}{\zeta - r^{2k}z} - \frac{r^{2k}\zeta}{r^{2k}\zeta - z}\Big).\end{aligned}$$

Using the Poisson kernel the representation formula (3.1.5) can be rewritten in the form

$$w(z) = \frac{1}{2\pi i}\int_{\partial R} |\zeta| g_1(z,\zeta) w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_R w_{\zeta\overline{\zeta}}(\zeta)\,G_1(z,\zeta)d\xi d\eta. \tag{3.3.3}$$

Lemma 3.3.2. *Let* $G(z,\zeta) = \frac{1}{2}\,G_1(z,\zeta)$ *be the Green function for* R, *defined by* (3.1.4). *Then the formulas*

$$G_{1\zeta}(z,\zeta) = \frac{\log|z|^2}{\log r^2}\frac{1}{\zeta} - K(z,\zeta) + K\Big(\frac{1}{\overline{z}},\zeta\Big), \tag{3.3.4}$$

$$G_{1\zeta}(z,\zeta) = \Big(\frac{\log|z|^2}{\log r^2} - 1\Big)\frac{1}{\zeta} - K(z,\zeta) + K\Big(\frac{r^2}{\overline{z}},\zeta\Big), \tag{3.3.5}$$

hold, where

$$K(z,\zeta) = \frac{1}{\zeta - z} + \sum_{k=1}^{\infty}\Big(\frac{r^{2k}}{r^{2k}\zeta - z} + \frac{r^{2k}z}{\zeta(\zeta - r^{2k}z)}\Big) = \frac{1}{\zeta - z} + \widehat{K}(z,\zeta).$$

Proof. Formulas (3.3.4), (3.3.5) are proved by direct calculation. □

The Poisson kernel for R possesses the following important property.

Theorem 3.3.3. *Let $g_1(z,\zeta)$ be the Poisson kernel for R, defined by* (3.3.2). *For any $w \in C(\partial R;\mathbb{C})$*

$$\lim_{z\to z_0,\, z\in R} \frac{1}{2\pi i}\int_{\partial R} g_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} = w(z_0),\ z_0 \in \partial R. \tag{3.3.6}$$

Proof. From the definition of the Poisson kernel and the normal derivative the equalities

$$\frac{1}{2\pi i}\int_{\partial R} |\zeta| g_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} = -\frac{1}{4\pi i}\int_{\partial R}(\zeta\partial_\zeta + \overline{\zeta}\partial_{\overline{\zeta}})G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} =$$

$$= -\frac{1}{2}\Big[\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} + \overline{\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)\overline{w(\zeta)}\frac{d\zeta}{\zeta}}\Big]$$

follow.

Transformations of the integral

$$\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int_{\partial R}\Big[\frac{\log|z|^2}{\log r^2} - \frac{\zeta}{\zeta - z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta} - \zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}(\frac{1}{\overline{z}},\zeta)\Big]w(\zeta)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\Big[-\Big(\frac{\zeta}{\zeta-z} + \overline{\frac{\zeta}{\zeta-z}} - 1\Big) + \frac{\log|z|^2}{\log r^2} - \zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}(\frac{1}{\overline{z}},\zeta)\Big]w(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int_{|\zeta|=r}\Big[\frac{\log|z|^2}{\log r^2}\frac{1}{\zeta} - \frac{1}{\zeta-z} + \frac{|z|^2}{|z|^2\zeta - z} - \widehat{K}(z,\overline{\zeta}) + \widehat{K}(\frac{1}{\overline{z}},\zeta)\Big]w(\zeta)d\zeta \tag{3.3.7}$$

or

$$\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int_{\partial R}\Big[\Big(\frac{\log|z|^2}{\log r^2} - 1\Big) - \frac{\zeta}{\zeta-z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta} -$$

$$- \zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}(\frac{r^2}{\overline{z}},\zeta)\Big]w(\zeta)\frac{d\zeta}{\zeta} =$$

$$
\begin{aligned}
&= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Big[\Big(\frac{\log|z|^2}{\log r^2} - 1 \Big) - \frac{\overline{z}\zeta}{\overline{z}\zeta - |z|^2} + \frac{\overline{z}\zeta}{\overline{z}\zeta - r^2} - \\
&- \zeta \widehat{K}(z, \overline{\zeta}) + \zeta \widehat{K}(\frac{r^2}{\overline{z}}, \zeta) \Big] w(\zeta) \frac{d\zeta}{\zeta} - \\
&- \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \Big[\Big(\frac{\log|z|^2}{\log r^2} - 1 \Big) - \Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \Big) - \\
&- \zeta \widehat{K}(z, \overline{\zeta}) + \zeta \widehat{K}(\frac{r^2}{\overline{z}}, \zeta) \Big] w(\zeta) \frac{d\zeta}{\zeta}
\end{aligned}
\tag{3.3.8}
$$

are the results of applying formulas (3.3.4), (3.3.5) respectively.

The function $\widehat{K}(z, \zeta)$ is uniformly continuous on z, $|z| > r$ for any fixed $\zeta \in \partial R$, what is observed from the estimations

$$
\Big| \widehat{K}(z_1, \zeta) - \widehat{K}(z_2, \zeta) \Big| = \Big| \sum_{k=1}^{\infty} r^{2k} \Big(\tfrac{z_1 - z_2}{(r^{2k}\zeta - z_1)(r^{2k}\zeta - z_2)} + \tfrac{z_1 - z_2}{(\zeta - r^{2k} z_1)(\zeta - r^{2k} z_1)} \Big) \Big| \leq
$$

$$
\leq \begin{cases} |z_1 - z_2| \sum\limits_{k=1}^{\infty} r^{2k} \Big(\tfrac{1}{(|z_1| - r^{2k})(|z_2| - r^{2k})} + \tfrac{1}{(1 - r^{2k}|z_1|)(1 - r^{2k}|z_2|)} \Big), |\zeta| = 1, \\ |z_1 - z_2| \sum\limits_{k=1}^{\infty} r^{2k} \Big(\tfrac{1}{(|z_1| - r^{2k+1})(|z_2| - r^{2k+1})} + \tfrac{1}{r^2(1 - r^{2k-1}|z_1|)(1 - r^{2k-1}|z_2|)} \Big), |\zeta| = r, \end{cases}
$$

$$
\leq \Big(\frac{1}{(1-r)^2} + \frac{1}{(1-r^2)^2} \Big) \frac{|z_1 - z_2|}{1 - r^2}, \ \forall\, z_1,\ z_2 \in R.
$$

Hence

$$
\lim_{z \to z_0,\, z \in R} \widehat{K}(z, \zeta) - \widehat{K}(\frac{1}{\overline{z}}, \zeta) = 0, \ |z_0| = 1, \tag{3.3.9}
$$

$$
\lim_{z \to z_0,\, z \in R} \widehat{K}(z, \zeta) - \widehat{K}(\frac{r^2}{\overline{z}}, \zeta) = 0, \ |z_0| = r, \tag{3.3.10}
$$

as $\Big| z - \frac{1}{\overline{z}} \Big|$ tends to zero, when $z \to z_0$, $|z_0| = 1$, and $\Big| z - \frac{r^2}{\overline{z}} \Big|$ approaches zero, when $z \to z_0$, $|z_0| = r$.

The statement of the theorem follows from (3.3.9), (3.3.10) and the properties of the Poisson kernel (2.1.10), (2.1.11) taking the limit of (3.3.7), when z tends to z_0, $|z_0| = 1$ and the limit of (3.3.8), when z approaches z_0, $|z_0| = r$. $\square$

Definition 3.3.4. [5] For a domain $D \subset \mathbb{C}$ with Green function $G(z,\zeta)$ the function

$$k(z,\overline{\zeta}) = -\frac{2}{\pi} G_{z\overline{\zeta}}(z,\zeta)$$

is called the Bergman kernel function of D.

The Bergman kernel $k(z,\overline{\zeta})$ is analytic in z and $\overline{\zeta}$ for $z,\, \zeta \in D$, having a singularity at $z=\zeta$.

From the symmetry of the Green function the property of the Bergman kernel

$$k(z,\overline{\zeta}) = \overline{k(\zeta,\overline{z})}$$

follows immediately. The Bergman kernel possesses the so-called reproducing property in the space of bounded analytic function

Theorem 3.3.4. [5] *For any bounded analytic function w in D the representation formula*

$$w(z) = \int_D w(\zeta) k(z,\overline{\zeta}) d\xi d\eta,\ z \in D,$$

holds.

Proposition 3.3.2. *The Bergman kernel for the ring R is expressed by*

$$k(z,\zeta) = -\frac{1}{\pi}\Big[\frac{1}{z\overline{\zeta}\log r^2} - \frac{1}{(1-z\overline{\zeta})^2} - \sum_{k=1}^{\infty}\Big(\frac{r^{2k}}{(1-r^{2k}z\overline{\zeta})^2} + \frac{r^{2k}}{(r^{2k}-z\overline{\zeta})^2}\Big)\Big].$$

3.4 Robin function for a circular ring domain

Definition 3.4.5. Let $D \subset \mathbb{C}$ be a (regular) domain of the complex plane. Then a function $\mathcal{R}_{\alpha,\beta}(z,\zeta)$ from $D \times D$ into $\mathbb{R} \cup \{+\infty\}$ is called Robin function for D with the parameters α and β if for any $\zeta \in D$

1°. $\mathcal{R}_{\alpha,\beta}(\cdot,\zeta)$ is harmonic in $D \setminus \{\zeta\}$, continuously differentiable in $\overline{D} \setminus \{\zeta\}$,

2°. $\mathcal{R}_{\alpha,\beta}(z,\zeta) + \log|\zeta - z|$ is harmonic in z in the neighborhood of ζ,

3°. $\alpha\mathcal{R}_{\alpha,\beta}(z,\zeta) + \beta\partial_{\nu_z}\mathcal{R}_{\alpha,\beta}(z,\zeta) = 0$ for $z \in \partial D$.

Parameters α and β may be given continuous real functions on ∂D, not vanishing simultaneously. Here $\alpha, \beta \in \mathbb{R}$ is assumed. If $\alpha = 0$ and the third condition is replaced by

$$\partial_{\nu_z}\mathcal{R}_{1,0}(z,\zeta) = -\sigma(s) \ \text{ for } \ z = z(s) \in \partial D,$$

where s is the arc length parameter of ∂D and σ is a real valued function of s with finite mass $\int\limits_{\partial D} \sigma(s)ds$, then $\mathcal{R}_{0,1}(z,\zeta)$ is the Neumann function $N(z,\zeta)$. If $\beta = 0$, then $\mathcal{R}_{1,0}(z,\zeta)$ is the Green function $G(z,\zeta)$. Instead of $\mathcal{R}_{\alpha,\beta}(z,\zeta)$ the notation $\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) = 2\mathcal{R}_{\alpha,\beta}(z,\zeta)$ is used as in the cases of Green and Neumann functions.

The representation formula via the Robin function of an arbitrary regular domain D for a proper class of functions is given in the following theorem.

Theorem 3.4.5. *For $\alpha \neq 0$ any $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ can be represented as*

$$\begin{aligned} w(z) = &-\frac{1}{4\pi\alpha}\int\limits_{\partial D}[\alpha w(\zeta) + \beta\partial_{\nu_\zeta} w(\zeta)]\partial_{\nu_\zeta}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)ds_\zeta - \\ &-\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta. \end{aligned} \tag{3.4.1}$$

Proof. Writing

$$\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) = -\log|\zeta - z|^2 + h_1(z,\zeta) \tag{3.4.2}$$

with a harmonic function $h_1(\,\cdot\,,\zeta)$ satisfying

$\alpha h_1(z,\zeta) + \beta\partial_{\nu_z} h_1(z,\zeta) = \alpha\log|\zeta - z|^2 + \beta\partial_{\nu_z}\log|\zeta - z|^2$ for $z \in \partial D$, $\zeta \in D$,

and applying the complex Gauss theorem (**Theorem 2.1.2**) for $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ then as $\partial_\zeta\partial_{\overline{\zeta}} h_1(z,\zeta) = 0$ it follows

$$\begin{aligned} &\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta = \\ &= \frac{1}{2\pi}\int\limits_D \Big(\partial_{\overline{\zeta}}\Big[w_\zeta(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\Big] + \partial_\zeta\Big[w_{\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\Big] + \\ &+ w_\zeta(\zeta)\Big[\frac{1}{\overline{\zeta - z}} - \partial_{\overline{\zeta}} h_1(z,\zeta)\Big] + w_{\overline{\zeta}}(\zeta)\Big[\frac{1}{\zeta - z} - \partial_\zeta h_1(z,\zeta)\Big]\Big)d\xi d\eta = \\ &= \frac{1}{4\pi i}\int\limits_{\partial D}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)[w_\zeta(\zeta)d\zeta - w_{\overline{\zeta}}(\zeta)d\overline{\zeta}] + \\ &+ \frac{1}{2\pi}\int\limits_D\Big(\frac{w_\zeta(\zeta)}{\overline{\zeta - z}} + \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} - \partial_{\overline{\zeta}}[w(\zeta)\partial_\zeta h_1(z,\zeta)] - \partial_\zeta[w(\zeta)\partial_{\overline{\zeta}} h_1(z,\zeta)]\Big)d\xi d\eta. \end{aligned}$$

Using the Cauchy-Pompeiu formulas (2.1.4), (2.1.5) and the Gauss theorem again gives

$$\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta = -w(z) + \frac{1}{4\pi}\int\limits_{\partial D}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\partial_{\nu_\zeta}w(\zeta)ds_\zeta +$$
$$+\frac{1}{4\pi i}\int\limits_{\partial D} w(\zeta)\Big(\Big[\frac{1}{\zeta - z} - \partial_\zeta h_1(z,\zeta)\Big]d\zeta - \Big[\frac{1}{\overline{\zeta - z}} - \partial_{\overline{\zeta}} h_1(z,\zeta)\Big]d\overline{\zeta}\Big).$$

Thus

$$w(z) = \frac{1}{4\pi}\int\limits_{\partial D}\Big[\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\partial_{\nu_\zeta}w(\zeta) - \partial_{\nu_\zeta}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)w(\zeta)\Big]ds_\zeta - \\ -\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta. \tag{3.4.3}$$

For $\alpha \neq 0$ this is

$$w(z) = -\frac{1}{4\pi\alpha}\int\limits_{\partial D}\Big[\alpha w(\zeta) + \beta\partial_{\nu_\zeta}w(\zeta)\Big]\partial_{\nu_\zeta}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)ds_\zeta - \\ -\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta,$$

and for $\beta \neq 0$

$$w(z) = \frac{1}{4\pi\beta}\int\limits_{\partial D}\Big[\alpha w(\zeta) + \beta\partial_{\nu_\zeta}w(\zeta)\Big]\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)ds_\zeta - \\ -\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta. \tag{3.4.4}$$

Hence, if $\alpha \neq 0,\ \beta \neq 0$

$$w(z) = \frac{1}{8\pi\alpha\beta}\int\limits_{\partial D}[\alpha w(\zeta) + \beta\partial_{\nu_\zeta}w(\zeta)]\Big[\alpha R_{1;\,\alpha,\beta}(z,\zeta) - \beta\partial_{\nu_\zeta}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\Big]ds_\zeta -$$

$$-\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)d\xi d\eta. \tag{3.4.5}$$

□

The representation (3.4.1) suggests the solution to the Robin problem for the Poisson equation, which will be considered in the next section.

The Robin function can be shown to be symmetric by using its three main properties from the definition.

Proposition 3.4.3. *The Robin function of a regular domain D possesses the symmetry property*

$$\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) = \mathcal{R}_{1;\,\alpha,\beta}(\zeta,z),$$

if it is assumed to be also harmonic in $\zeta \in D \setminus \{z\}$.

Proof. The difference

$$h(z,\zeta) = \mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) - \mathcal{R}_{1;\,\alpha,\beta}(\zeta,z)$$

is a harmonic function in both variables. It satisfies for $z \in \partial D$

$$\alpha h(z,\zeta) + \beta \partial_{\nu_z} h(z,\zeta) = -\alpha \mathcal{R}_{1;\,\alpha,\beta}(\zeta,z) - \beta \partial_{\nu_z} \mathcal{R}_{1;\,\alpha,\beta}(\zeta,z),$$

and for $\zeta \in \partial D$

$$\alpha h(z,\zeta) + \beta \partial_{\nu_\zeta} h(z,\zeta) = \alpha \mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) + \beta \partial_{\nu_\zeta} \mathcal{R}_{1;\,\alpha,\beta}(z,\zeta).$$

As the Green function is known to be symmetric, $\beta \neq 0$ may be assumed. By formula (3.4.4) the function $h(z,\zeta)$ can be represented as

$$h(z,\zeta) = -\frac{1}{4\pi\beta} \int\limits_{\partial D} \Big[\alpha \mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta}) + \beta \partial_{\nu_{\widetilde{\zeta}}} \mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta})\Big] \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) ds_{\widetilde{\zeta}}$$

and

$$h(z,\zeta) = \frac{1}{4\pi\beta} \int\limits_{\partial D} \Big[\alpha \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) + \beta \partial_{\nu_{\widetilde{\zeta}}} \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta})\Big] \mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta}) ds_{\widetilde{\zeta}}.$$

Adding them gives

$$\begin{aligned} h(z,\zeta) &= \frac{1}{8\pi} \int\limits_{\partial D} \Big[\mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta}) \partial_{\nu_{\widetilde{\zeta}}} \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) - \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) \partial_{\nu_{\widetilde{\zeta}}} \mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta})\Big] ds_{\widetilde{\zeta}} \\ &= \frac{\alpha}{8\pi\beta} \int\limits_{\partial D} \Big[\mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) \mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta}) - \mathcal{R}_{1;\,\alpha,\beta}(\zeta,\widetilde{\zeta}) \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta})\Big] ds_{\widetilde{\zeta}} = \\ &= 0. \end{aligned}$$

□

To find the Robin function for the ring $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ the Schwarz problem for analytic functions needs to be solved, i.e. the result of **Theorem 2.2.10** is applied for this aim.

Theorem 3.4.6. *The Robin function for the circular ring R in case $\alpha, \beta \neq 0$ is*

$$\begin{aligned}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) &= G_1(z,\zeta)+\\ &+2\beta\sum_{k=1}^{\infty}\Big[\frac{(z\overline{\zeta})^k+(\overline{z}\zeta)^k}{\alpha+\beta k}+\frac{1}{\alpha-\beta k}\Big(\frac{r^{2k}}{(z\zeta)^k}+\frac{r^{2k}}{(\overline{z}\overline{\zeta})^k}\Big)\Big]\frac{1}{1-r^{2k}}+\\ &+\frac{2\beta}{\alpha}+\frac{2\beta}{\alpha}\frac{1}{\log r^2}\Big[\frac{2\beta}{\alpha}-\log|z|^2-\log|\zeta|^2\Big],\ \ \text{if}\ \frac{\alpha}{\beta}\notin\mathbb{Z};\end{aligned} \tag{3.4.6}$$

$$\begin{aligned}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) &= G_1(z,\zeta)+2\beta\sum_{\substack{k=-\infty,\\ k\neq k_0}}^{+\infty}\frac{\operatorname{sgn}k}{1-r^{2k}}\frac{(z\overline{\zeta})^k+(\overline{z}\zeta)^k}{\alpha+\beta k}+\\ &+\frac{2\operatorname{sgn}k_0}{(1-r^{2k_0})}\Big[(z\overline{\zeta})^{k_0}\log(z\overline{\zeta})+(\overline{z}\zeta)^{k_0}\log(\overline{z}\zeta)\Big]+\\ &+\frac{2\beta}{\alpha}+\frac{2\beta}{\alpha}\frac{1}{\log r^2}\Big[\frac{2\beta}{\alpha}-\log|z|^2-\log|\zeta|^2\Big],\ \ \text{if}\ \alpha+\beta k_0=0,\ k_0\in\mathbb{Z},\end{aligned} \tag{3.4.7}$$

where $\mathbb{Z}$ denotes the set of integer numbers. For $\alpha = 0$ there is no Robin function. For $\beta = 0$ the Robin function coincides with the Green function.

Proof. The harmonic function

$$h_1(z,\zeta) = \mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) + \log|\zeta - z|^2$$

satisfies for $z \in \partial R,\ \zeta \in R$

$$\alpha h_1(z,\zeta) + \partial_{\nu_z} h_1(z,\zeta) = \alpha\log|\zeta - z|^2 - \frac{z}{\zeta - z} - \frac{\overline{z}}{\overline{\zeta - z}}. \tag{3.4.8}$$

It is representable as the real part of an analytic function in R. This analytic function may be multi valued as long as its real part is single valued. Instead of looking for such a proper multi valued analytic function and because of symmetry it is appropriate to find $h_1(z,\zeta)$ in the form

$$h_1(z,\zeta) = a\log|z|^2 + a\log|\zeta|^2 + b\log|z|^2\log|\zeta|^2 + \operatorname{Re}\varphi(z,\zeta)$$

with a single valued analytic $\varphi(z,\zeta)$ in R. Then the boundary condition (3.4.8) on ∂R turns into the Schwarz problem

$$\mathrm{Re}\Big(\alpha\varphi(z,\zeta)+\beta z\varphi_z(z,\zeta)\Big) = -\alpha a\log|z|^2-\alpha a\log|\zeta|^2-\alpha b\log|z|^2\log|\zeta|^2$$

$$-2\beta a-2\beta b\log|\zeta|^2+\alpha\log|\zeta-z|^2-\beta\Big[\frac{z}{\zeta-z}+\frac{\overline{z}}{\overline{\zeta-z}}\Big] \text{ for } z\in\partial R.$$

According to **Theorem 2.2.10** this problem is uniquely solvable if

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(z,\zeta)\frac{dz}{z}=0,$$

where $\gamma(z,\zeta)$ denotes its right-hand sides.

As

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(z,\zeta)\frac{dz}{z}=\alpha a\log r^2+\alpha b\log r^2\log|\zeta|^2-\alpha\log|\zeta|^2+2\beta \quad (3.4.9)$$

this implies

$$\alpha a\log r^2+2\beta=0,\ \ b\log r^2-1=0,$$

and

$$\mathrm{Re}\Big(\alpha\varphi(z,\zeta)+\beta z\varphi_z(z,\zeta)\Big) =\frac{4\beta^2}{\alpha\log r^2}+2\beta\frac{\log|z|^2}{\log r^2}-\alpha\frac{\log|z|^2\log|\zeta|^2}{\log r^2}$$

$$+\alpha\log|\zeta-z|^2+\beta\Big[\frac{z}{\zeta-z}+\frac{\overline{z}}{\overline{\zeta-z}}\Big] \text{ for } z\in\partial R.$$

Hence, see (2.2.7),

$$\alpha\varphi(z,\zeta)+\beta z\varphi_z(z,\zeta)=\frac{1}{2\pi i}\int\limits_{\partial\mathcal{R}}\gamma(\widetilde{\zeta},\zeta)S(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}$$

$$-\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=r}\gamma(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} \text{ for } z,\zeta\in R,$$

where the imaginary additive constant is chosen as 0, and the function

$$S(z,\zeta)=\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\Big[\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\Big] \quad (3.4.10)$$

represents the Schwarz kernel for R.

From

$$\frac{1}{2\pi i}\int\limits_{\partial\mathcal{R}} S(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = 2,$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathcal{R}} \log|\widetilde{\zeta}|^2 S(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = \log r^2,$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathcal{R}} \log|\zeta-\widetilde{\zeta}|^2 S(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = 2\log(1-z\widetilde{\zeta}) + \log|\zeta|^2 +$$

$$+\,2\sum_{n=1}^{\infty}\Big[\log(1-r^{2n}z\overline{\zeta}) - \log\Big(1-\frac{r^{2n}\zeta}{z}\Big) - \log\Big(1-\frac{r^{2n}z}{\zeta}\Big) + \log\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)\Big],$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathcal{R}} \Big[\frac{\widetilde{\zeta}}{\widetilde{\zeta}-\zeta} + \frac{\overline{\widetilde{\zeta}}}{\overline{\widetilde{\zeta}-\zeta}}\Big] S(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = \frac{2}{1-z\overline{\zeta}} +$$

$$+\,2\sum_{n=1}^{\infty}\Big[\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} + \frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}} + \frac{r^{2n}}{r^{2n}-z\overline{\zeta}}\Big],$$

$$\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=r} \frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = 1, \quad \frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=r} \log|\widetilde{\zeta}-\zeta|^2\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = \log|\zeta|^2,$$

$$\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=r} \Big[\frac{\widetilde{\zeta}}{\widetilde{\zeta}-\zeta} + \frac{\overline{\widetilde{\zeta}}}{\overline{\widetilde{\zeta}-\zeta}}\Big]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} = 0,$$

finally

$$\alpha\varphi(z,\zeta) + \beta z\varphi_z(z,\zeta) = \frac{4\beta^2}{\alpha\log r^2} + 2\alpha\log(1-z\overline{\zeta}) + \frac{2\beta}{1-z\overline{\zeta}} +$$

$$+\,2\alpha\sum_{n=1}^{\infty}\Big[\log(1-r^{2n}z\overline{\zeta}) - \log\Big(1-\frac{r^{2n}\zeta}{z}\Big) + \log\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big) - \log\Big(1-\frac{r^{2n}z}{\zeta}\Big)\Big]$$

$$+\,2\beta\sum_{n=1}^{\infty}\Big[\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} + \frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}} + \frac{r^{2n}}{r^{2n}-z\overline{\zeta}}\Big] =$$

$$= \frac{4\beta^2}{\alpha\log r^2} + 2\beta + 2\sum_{k=1}^{\infty}\Big[\frac{\alpha+k\beta}{k}\frac{r^{2k}}{1-r^{2k}}\frac{1}{\zeta^k} - \frac{\alpha-k\beta}{k}\frac{\overline{\zeta}^{\,k}}{1-r^{2k}}\Big]z^k -$$

$$-\,2\sum_{k=1}^{\infty}\Big[\frac{\alpha+k\beta}{k}\frac{1}{\overline{\zeta}^{\,k}} - \frac{\alpha-k\beta}{k}\zeta^k\Big]\frac{r^{2k}}{1-r^{2k}}\,z^{-k}.$$

As an analytic function in R being single valued, $\varphi(z,\zeta)$ is representable as a Laurent series

$$\varphi(z,\zeta) = \sum_{k=-\infty}^{\infty} c_k z^k,$$

which converges in R. Thus

$$\alpha\varphi(z,\zeta) + \beta z\varphi_z(z,\zeta) = \sum_{k=-\infty}^{\infty} (\alpha+\beta k)c_k z^k.$$

Comparing coefficients shows

$$\begin{aligned}
\alpha c_0 &= 2\beta + 4\frac{\beta^2}{\alpha\log r^2},\\
(\alpha+\beta k)c_k &= 2\Big[\frac{\alpha+\beta k}{k}\frac{r^{2k}}{\zeta^k} - \frac{\alpha-\beta k}{k}\overline{\zeta}^{\,k}\Big]\frac{1}{1-r^{2k}} \quad \text{for } 1\le k,\\
(\alpha+\beta k)c_k &= \frac{2r^{-2k}}{1-r^{-2k}}\Big[\frac{\alpha-\beta k}{k}\overline{\zeta}^{\,k} - \frac{\alpha+\beta k}{k}\frac{1}{\zeta^k}\Big] \quad \text{for } k\le -1,
\end{aligned}$$

as long as $\alpha+\beta k \notin \mathbb{Z}$. Hence, then

$$\varphi(z,\zeta) = \frac{2\beta}{\alpha} + \Big(\frac{2\beta}{\alpha}\Big)^2\frac{1}{\log r^2} + 2\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-r^{2k}}\Big[\frac{r^{2k}z^k}{\zeta^k} + \frac{r^{2k}\zeta^k}{z^k} - (z\overline{\zeta})^k - \frac{r^{2k}}{(z\overline{\zeta})^k}\Big]$$

$$+4\beta\sum_{k=1}^{\infty}\frac{1}{1-r^{2k}}\Big[\frac{(z\overline{\zeta})^k}{\alpha+\beta k} + \frac{r^{2k}}{(\alpha-\beta k)(z\overline{\zeta})^k}\Big]. \qquad (3.4.11)$$

Therefore

$$\begin{aligned}
h_1(z,\zeta) &= -\frac{2\beta}{\alpha}\frac{\log|z|^2+\log|\zeta|^2}{\log r^2} + \frac{\log|z|^2\log|\zeta|^2}{\log r^2} + \frac{2\beta}{\alpha} + \Big(\frac{2\beta}{\alpha}\Big)^2\frac{1}{\log r^2} +\\
&+ \sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{1-r^{2k}}\Big[\frac{r^{2k}z^k}{\zeta^k} + \frac{r^{2k}\overline{z}^{\,k}}{\overline{\zeta}^{\,k}} + \frac{r^{2k}\zeta^k}{z^k} + \frac{r^{2k}\overline{\zeta}^{\,k}}{\overline{z}^{\,k}} -\\
&- (z\overline{\zeta})^k - (\overline{z}\zeta)^k - \frac{r^{2k}}{(z\overline{\zeta})^k} - \frac{r^{2k}}{(\overline{z}\zeta)^k}\Big] +\\
&+ 2\beta\sum_{k=1}^{\infty}\frac{1}{1-r^{2k}}\Big[\frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k}{\alpha+\beta k} + \frac{r^{2k}}{(\alpha-\beta k)}\Big(\frac{1}{(z\overline{\zeta})^k} + \frac{1}{(\overline{z}\zeta)^k}\Big)\Big] =\\
&= -\frac{2\beta}{\alpha}\frac{\log|z|^2+\log|\zeta|^2}{\log r^2} + \frac{\log|z|^2\log|\zeta|^2}{\log r^2} + \frac{2\beta}{\alpha} + \Big(\frac{2\beta}{\alpha}\Big)^2\frac{1}{\log r^2} +
\end{aligned}$$

$$+\log|1-z\zeta|^2-\log\prod_{n=1}^{\infty}\Big|\frac{(\zeta-r^{2n}z)(z-r^{2n}\zeta)}{(1-r^{2n}z\overline{\zeta})(z\overline{\zeta}-r^{2n})}\Big|^2+$$

$$+2\beta\sum_{n=1}^{\infty}\frac{1}{1-r^{2n}}\Big[\frac{(z\overline{\zeta})^n+(\overline{z}\zeta)^n}{\alpha+\beta n}+\frac{r^{2n}[(z\overline{\zeta})^{-n}+(\overline{z}\zeta)^{-n}]}{\alpha-\beta n}\Big],$$

so that

$$\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)=G_1(z,\zeta)+\frac{2\beta}{\alpha}\Big[1-\frac{\log|z|^2+\log|\zeta|^2}{\log r^2}\Big]+\Big(\frac{2\beta}{\alpha}\Big)^2\frac{1}{\log r^2}$$

$$+2\beta\sum_{n=1}^{\infty}\frac{1}{1-r^{2n}}\Big[\frac{(z\overline{\zeta})^n+(\overline{z}\zeta)^n}{\alpha+\beta n}+\frac{r^{2n}[(z\overline{\zeta})^{-n}+(\overline{z}\zeta)^{-n}]}{\alpha-\beta n}\Big].$$

If $\alpha+\beta k_0=0$ for some $k_0\in\mathbb{Z}$, then $\varphi(z,\zeta)$ is looked for in the form

$$\varphi(z,\zeta)=\sum_{k=-\infty}^{\infty}c_kz^k+c_{k_0}z^{k_0}\log(z\overline{\zeta}),$$

where c_{k_0} turns out to be determined as

$$c_{k_0}=\frac{4}{(1-r^{2k_0})}\,\overline{\zeta}^{\,k_0}.$$

Then

$$\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)=G_1(z,\zeta)+\frac{2\beta}{\alpha}\Big[1-\frac{\log|z|^2+\log|\zeta|^2}{\log r^2}\Big]+\Big(\frac{2\beta}{\alpha}\Big)^2\frac{1}{\log r^2}+$$

$$+2\beta\sum_{\substack{n=-\infty,\\ \alpha+\beta n\neq 0}}^{\infty}\frac{\operatorname{sgn} n}{1-r^{2n}}\frac{(z\overline{\zeta})^n+(\overline{z}\zeta)^n}{\alpha+\beta n}+$$

$$+\frac{2\operatorname{sgn} k_0}{(1-r^{2k_0})}\Big[(z\overline{\zeta})^{k_0}\log(z\overline{\zeta})+(\overline{z}\zeta)^{k_0}\log(\overline{z}\zeta)\Big].$$

□

Remark 3.4.2. In the case $\alpha+\beta k_0=0$ the function $\varphi(z,\zeta)$ is multi valued. But as z^{k_0} is a solution to the equation $\alpha\varphi(z,\zeta)+\beta z\varphi_z(z,\zeta)=0$, this multi validness does not influence the boundary behavior of $\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)$. Moreover, $\varphi(z,\zeta)$ can be altered adding an arbitrary multiple of z^{k_0} leading to another Robin function with the same properties. The Robin function then is multi valued and not unique as multiples of $[(z\overline{\zeta})^{k_0}+\ (\overline{z}\zeta)^{k_0}]$ may be added.

For $\alpha=0$ the solvability condition (3.4.9) is not valid as $\beta\neq 0$. Thus in this case there is no Robin function. But as mentioned before the Neumann function exists.

3.5 Boundary value problems for second order equations

The second order differential operators, i.e. the Laplace operator $\partial_z\partial_{\overline{z}}$ and the Bitsadze operator $\partial_{\overline{z}}^2$, determine second order partial differential equations, namely, the Laplace, the Poisson (i.e. inhomogeneous Laplace), the Bitsadze and inhomogeneous Bitsadze equations. Different boundary value problems can be formulated for these equations.

3.5.1 Dirichlet, Neumann and Robin problems for the Poisson equation

The Dirichlet and Neumann problems for the Poisson equation are well-known in the literature, see e.g. [36], [50]. Here explicit solutions for these problems are specified for a ring domain. A Robin problem for the Poisson equation is posed and its solution is found in explicit form.

Theorem 3.5.7. *The Dirichlet problem for the Poisson equation in R*

$$w_{z\overline{z}} = f \text{ in } R, \quad w = \gamma \text{ on } \partial R, \tag{3.5.1}$$

for $f \in L_2(R;\mathbb{C}) \cap C(R;\mathbb{C})$, $\gamma \in C(\partial R;\mathbb{C})$ given is solvable in the space $C^2(R;\mathbb{C}) \cap C(\overline{R};\mathbb{C})$. The solution is unique and expressed by

$$w(z) = -\frac{1}{4\pi i}\int_{\partial R} |\zeta|\partial_{\nu_\zeta} G_1(z,\zeta)\gamma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_R f(\zeta)G_1(z,\zeta)d\xi d\eta. \tag{3.5.2}$$

Proof. The representation formula from **Theorem 3.1.1**, the properties of the Poisson kernel (**Theorem 3.3.3**) and the Green function immediately provide the solution to the Dirichlet problem (3.5.1), expressed by (3.5.2), see e.g. [36, p. 84], [50, p. 324]. □

Theorem 3.5.8. *The Neumann problem for the Poisson equation in the ring R*

$$w_{z\overline{z}} = f \text{ in } R, \quad \lambda|z|\partial_{\nu_z} w(z) = \gamma \text{ on } \partial R, \quad \lambda = \begin{cases} 1, & |z| = 1, \\ -1, & |z| = r, \end{cases} \tag{3.5.3}$$

$$\frac{1}{2\pi i}\int_{|z|=1} w(z)\frac{dz}{z} = c, \tag{3.5.4}$$

for $f \in L_2(R;\mathbb{C}) \cap C(R;\mathbb{C})$, $\gamma \in C(\partial R;\mathbb{C})$, $c \in \mathbb{C}$ *given is solvable in the space* $C^2(R;\mathbb{C}) \cap C^1(\overline{R};\mathbb{C})$ *if and only if*

$$\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta} = \frac{2}{\pi}\int_R f(\zeta)d\xi d\eta. \tag{3.5.5}$$

Then the solution is unique and given by

$$w(z) = c + \frac{1}{4\pi i}\int_{\partial R}\gamma(\zeta)N_1(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_R f(\zeta)N_1(z,\zeta)d\xi d\eta. \tag{3.5.6}$$

Proof. If w is a solution to (3.5.3), then formula (3.5.6) is obtained from the representation (3.2.7), where (3.2.4), (3.2.5) are used.

That (3.5.6) is the solution to the Poisson equation follows from the properties of the Neumann function.

To see that (3.5.6) satisfies the differential equation in (3.5.3), the boundary integral is observed to be harmonic, while the area integral is a particular solution to the inhomogeneous equation, see e.g. [36], [50].

The boundary and normalization conditions have to be checked.

Let us find

$$zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta) = 2\mathrm{Re}\Big[\frac{z}{\zeta - z} + \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} -$$

$$-\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\zeta}{z - r^{2k}\zeta} + \frac{r^{2k}}{z\overline{\zeta} - r^{2k}} - \frac{r^{2k}z}{\zeta - r^{2k}z} - \frac{r^{2k}z\overline{\zeta}}{1 - r^{2k}z\overline{\zeta}}\Big)\Big],$$

for $z, \zeta \in R$.

From (3.5.6) it follows that

$$zw_z(z) + \overline{z}w_{\overline{z}}(z) = \frac{1}{4\pi i}\int_{\partial R}\gamma(\zeta)[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)]\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int_R f(\zeta)[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)]d\xi d\eta.$$

Then

$$\lim_{|z|\to 1,\, z\in R}[zw_z(z) + \overline{z}w_{\overline{z}}(z)] =$$

$$= \lim_{|z|\to 1,\, z\in R} \frac{1}{4\pi i} \int\limits_{\partial R} \gamma(\zeta)[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)]\frac{d\zeta}{\zeta} -$$

$$- \lim_{|z|\to 1,\, z\in R} \frac{1}{\pi} \int\limits_{R} f(\zeta)[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)]d\xi d\eta,$$

what is for $|z| = 1$ equivalent to

$$|z|\partial_{\nu_z} w(z) = \lim_{|z|\to 1,\, z\in R} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right)\frac{d\zeta}{\zeta} - \tag{3.5.7}$$
$$- \frac{1}{2\pi i}\int\limits_{\partial R} \gamma(\zeta)\frac{d\zeta}{\zeta} + \frac{2}{\pi}\int\limits_{R} f(\zeta)d\xi d\eta,$$

taking into account (3.2.4) and that for $|\zeta| = 1$

$$\lim_{|z|\to 1,\, z\in R}\left[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta) - 2\left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right)\right] =$$

$$= \lim_{|z|\to 1,\, z\in R} 2\mathrm{Re}\left[zN_{1z}(z,\zeta) - \frac{2\zeta}{\zeta - z} + 1\right] =$$

$$= - \lim_{|z|\to 1,\, z\in R} 2\mathrm{Re}\left[\frac{\zeta}{\zeta - z} - \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} + \right.$$
$$\left. + \sum_{k=1}^{\infty}\left(\frac{r^{2k}\zeta}{z - r^{2k}\zeta} + \frac{r^{2k}}{z\overline{\zeta} - r^{2k}} - \frac{r^{2k}z}{\zeta - r^{2k}z} - \frac{r^{2k}z\overline{\zeta}}{1 - r^{2k}z\overline{\zeta}}\right)\right] = -2,$$

and for $|\zeta| = r$

$$\lim_{|z|\to 1,\, z\in R}[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)] = -2.$$

The equality (3.5.7) shows that the boundary condition is valid on $|z| = 1$ if and only if the solvability condition (3.5.5) holds.

Let us consider

$$\lim_{|z|\to r,\, z\in R}[zw_z(z) + \overline{z}w_{\overline{z}}(z)] =$$

$$= \lim_{|z|\to r,\, z\in R} \frac{1}{4\pi i} \int\limits_{\partial R} \gamma(\zeta)[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)]\frac{d\zeta}{\zeta} -$$

$$- \lim_{|z|\to r,\, z\in R} \frac{1}{\pi} \int\limits_{R} f(\zeta)[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)]d\xi d\eta,$$

what for $|z| = r$ transforms into

$$|z|\partial_{\nu_z} w(z) = \lim_{|z|\to r,\, z\in R} \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right) \frac{d\zeta}{\zeta}, \tag{3.5.8}$$

by using (3.2.5) and observing that for $|\zeta| = 1$

$$\lim_{|z|\to r,\, z\in R} [zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta)] = 0,$$

and for $|\zeta| = r$

$$\lim_{|z|\to r,\, z\in R} \left[zN_{1z}(z,\zeta) + \overline{z}N_{1\overline{z}}(z,\zeta) - 2\left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 \right) \right] =$$

$$= \lim_{|z|\to r,\, z\in R} 2\mathrm{Re} \left[zN_{1z}(z,\zeta) - \frac{2\zeta}{\zeta - z} + 1 \right] =$$

$$= - \lim_{|z|\to r,\, z\in R} 2\mathrm{Re} \left[\frac{\zeta}{\zeta - z} - \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} + \right.$$

$$\left. + \sum_{k=1}^{\infty} \Big(\frac{r^{2k}\zeta}{z - r^{2k}\zeta} + \frac{r^{2k}}{z\overline{\zeta} - r^{2k}} - \frac{r^{2k}z}{\zeta - r^{2k}z} - \frac{r^{2k}z\overline{\zeta}}{1 - r^{2k}z\overline{\zeta}} \Big) \right] = 0.$$

From (3.5.8) the validity of the boundary condition on $|z| = r$ follows immediately.

Evaluating

$$\frac{1}{2\pi i} \int\limits_{|z|=1} w(z) \frac{dz}{z} = c + \frac{1}{4\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{1}{2\pi i} \int\limits_{|z|=1} N_1(z,\zeta) \frac{dz}{z} \frac{d\zeta}{\zeta} -$$

$$- \frac{1}{\pi} \int\limits_{R} f(\zeta) \frac{1}{2\pi i} \int\limits_{|z|=1} N_1(z,\zeta) \frac{dz}{z} d\xi d\eta,$$

the normalization condition is checked to be true due to (3.2.6). □

Theorem 3.5.9. *The Robin problem for the Poisson equation in* R

$$w_{z\overline{z}} = f \text{ in } R, \ \ \alpha w + \beta \partial_{\nu_z} w = \gamma \text{ on } \partial R, \tag{3.5.9}$$

for $f \in L_2(R;\mathbb{C})$ $\gamma \in C(\partial R;\mathbb{C})$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ *given is solvable in the space* $C^2(R;\mathbb{C}) \cap C^1(\overline{R};\mathbb{C})$. *The solution is unique and given by*

$$w(z) = -\frac{1}{4\pi i \alpha} \int\limits_{\partial R} \gamma(\zeta) \partial_{\nu_\zeta} \mathcal{R}_{1;\alpha,\beta}(z,\zeta) ds_\zeta - \frac{1}{\pi} \int\limits_{R} f(\zeta) \mathcal{R}_{1;\alpha,\beta}(z,\zeta) d\xi d\eta. \tag{3.5.10}$$

Proof. Uniqueness follows right away from the representation (3.4.1). Thus existence has to be shown. Obviously, the boundary integral in formula (3.5.10) provides a harmonic function. Decomposing $R_{1;\,\alpha,\beta}(z,\zeta)$ via (3.4.2) and observing $h_1(\,\cdot\,,\zeta)$ being harmonic shows

$$\partial_z\partial_{\overline{z}}w(z) = \partial_z\partial_{\overline{z}}\Big[\frac{1}{\pi}\int\limits_R \log|\zeta - z|^2 f(\zeta)\Big]d\xi d\eta.$$

According to, e.g., [36], [50], $w(z)$ is a solution to the differential equation in (3.5.9).

At last the boundary behavior has to be verified. From the property of the Green function $G_1(z,\zeta) = 0$ for $\zeta \in \partial R$ it follows $\partial_{\nu_z}G_1(z,\zeta) = 0$ for $\zeta \in \partial R$. Also $\widehat{h}_1(z,\zeta) = R_{1;\,\alpha,\beta}(z,\zeta) - G_1(z,\zeta)$ satisfies $(\partial_{\nu_\zeta} - \partial_{\nu_z})\widehat{h}_1(z,\zeta) = 0$ for z and ζ on ∂R, see **Lemma 3.5.3** below. Thus for z and ζ on ∂R

$$\begin{aligned}\partial_{\nu_\zeta}\Big[\alpha\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) + \beta\partial_{\nu_z}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\Big] &=\\ &= \alpha\Big[\partial_{\nu_\zeta}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) - \partial_{\nu_z}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)\Big]\\ &= \alpha\Big[\partial_{\nu_\zeta}G_1(z,\zeta) + \partial_{\nu_\zeta}\widehat{h}_1(z,\zeta) - \partial_{\nu_z}\widehat{h}_1(z,\zeta)\Big] = \alpha\partial_{\nu_\zeta}G_1(z,\zeta).\end{aligned}$$

This implies for $z_0 \in \partial R$

$$\begin{aligned}\alpha w(z_0) + \beta\partial_\nu w(z_0) &= \lim_{z\to z_0}\Big[-\frac{1}{4\pi i}\int\limits_{\partial R}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)\frac{d\zeta}{\zeta}\\ &\quad - \frac{1}{\pi}\int\limits_R f(\zeta)[\alpha\mathcal{R}_{1;\,z,\zeta}(z,\zeta) + \beta\partial_{\nu_z}\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)]d\xi d\eta\Big]\\ &= \gamma(z_0).\end{aligned}$$

□

Lemma 3.5.3. *For $z,\zeta \in \partial R$ the harmonic function*

$$\widehat{h}_1(z,\zeta) = \mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) - G_1(z,\zeta)$$

satisfies

$$\partial_{\nu_\zeta}\widehat{h}_1(z,\zeta) = \partial_{\nu_z}\widehat{h}_1(z,\zeta).$$

Proof. The function $\widehat{h}_1(z,\zeta)$ is a harmonic function of z in R for any $\zeta \in \overline{R}$ satisfying as well the Dirichlet condition

$$\widehat{h}_1(z,\zeta) = \mathcal{R}_{1;\,\alpha,\beta}(z,\zeta) \text{ for } z \in \partial R,\ \zeta \in \overline{R},$$

as the Robin condition

$$\alpha \widehat{h}_1(z,\zeta) + \beta \partial_{\nu_z} \widehat{h}_1(z,\zeta) = -\beta \partial_{\nu_z} G_1(z,\zeta) \text{ for } z \in \partial R,\ \zeta \in \overline{R}.$$

Hence as well

$$\widehat{h}_1(z,\zeta) = -\frac{1}{4\pi i} \int\limits_{\partial R} \mathcal{R}_{1;\,\alpha,\beta}(\widetilde{\zeta},\zeta) \partial_{\nu_{\widetilde{\zeta}}} G_1(\widetilde{\zeta},z) \frac{d\zeta}{\zeta}$$

as, according to (3.4.4),

$$\widehat{h}_1(z,\zeta) = -\frac{1}{4\pi i} \int\limits_{\partial R} \partial_{\nu_{\widetilde{\zeta}}} G_1(\widetilde{\zeta},\zeta) \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) \frac{d\zeta}{\zeta}$$

hold. Applying ∂_{ν_ζ} to the first and ∂_{ν_z} to the second expression on ∂R and observing the boundary behavior of $\mathcal{R}_{1;\,\alpha,\beta}(z,\zeta)$ shows for z and ζ on ∂R

$$\begin{aligned}
&(\partial_{\nu_z} - \partial_{\nu_\zeta}) \widehat{h}_1(z,\zeta) \\
&= \frac{1}{4\pi i} \int\limits_{\partial R} \Big(\partial_{\nu_\zeta} \mathcal{R}_{1;\,\alpha,\beta}(\widetilde{\zeta},\zeta) \partial_{\nu_{\widetilde{\zeta}}} G_1(\widetilde{\zeta},z) - \partial_{\nu_z} \mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) \partial_{\nu_{\widetilde{\zeta}}} G_1(\widetilde{\zeta},\zeta) \Big) \frac{d\zeta}{\zeta} \\
&= \frac{\alpha}{4\pi i \beta} \int\limits_{\partial R} \Big(\mathcal{R}_{1;\,\alpha,\beta}(z,\widetilde{\zeta}) \partial_{\nu_{\widetilde{\zeta}}} G_1(\widetilde{\zeta},\zeta) - \mathcal{R}_{1;\,\alpha,\beta}(\widetilde{\zeta},\zeta) \partial_{\nu_{\widetilde{\zeta}}} G_1(\widetilde{\zeta},z) \Big) \frac{d\zeta}{\zeta} = 0. \quad \Box
\end{aligned}$$

Remark 3.5.3. In the proofs of **Theorem 3.5.9** and **Lemma 3.5.3** the special configuration of the domain being a circular ring is not used. The same results hold for any regular domain. The only attention has to be turned on the classes of initial data of the problem (3.5.9). Namely, for a general regular domain D the function γ has to be taken from the space $C^\lambda(\partial D; \mathbb{C})$, $0 < \lambda$.

3.5.2 Dirichlet and Schwarz problems for the Bitsadze equation

The Dirichlet problem for the Bitsadze equation is different from the one for the Poisson equation. It is taken in such a formulation that it can be treated by reducing it to a system of first order partial differential equations.

Theorem 3.5.10. *The Dirichlet problem for the inhomogeneous Bitsadze equation in R*

$$w_{\overline{z}\,\overline{z}} = f \text{ in } R, \quad w = \gamma_0, \; w_{\overline{z}} = \gamma_1 \text{ on } \partial R, \tag{3.5.1}$$

for $f \in L_p(R;\mathbb{C})$, $p > 2$, γ_0, $\gamma_1 \in C(\partial R;\mathbb{C})$ given is solvable by a function from $W^{2,p}_{\overline{z}}(R;\mathbb{C}) \cap C(\overline{R};\mathbb{C})$ with continuous weak $\overline{z}$-derivative on $\overline{R}$ if and only if

$$\begin{aligned}\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} = \frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{1-\overline{z}\zeta}d\zeta - \frac{r^2\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{d\zeta}{\zeta} - \\ - \frac{\overline{z}}{\pi}\int_R f(\zeta)\frac{\overline{\zeta - z}}{1-\overline{z}\zeta}d\xi d\eta + \frac{r^2\overline{z}}{\pi}\int_R f(\zeta)\frac{d\xi d\eta}{\zeta},\end{aligned} \tag{3.5.2}$$

$$\begin{aligned}\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{r^2-\overline{z}\zeta} = \frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{r^2-\overline{z}\zeta}d\zeta - \frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{d\zeta}{\zeta} - \\ - \frac{\overline{z}}{\pi}\int_R f(\zeta)\frac{\overline{\zeta - z}}{r^2-\overline{z}\zeta}d\xi d\eta + \frac{\overline{z}}{\pi}\int_R f(\zeta)\frac{d\xi d\eta}{\zeta},\end{aligned} \tag{3.5.3}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi}\int_R f(\zeta)\frac{\overline{z}d\xi d\eta}{1-\overline{z}\zeta}, \tag{3.5.4}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} = \frac{1}{\pi}\int_R f(\zeta)\frac{\overline{z}d\xi d\eta}{r^2-\overline{z}\zeta}. \tag{3.5.5}$$

The solution is unique and expressed by

$$\begin{aligned}w(z) = \frac{1}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta - \frac{r^2}{2\pi i z}\int_{\partial R}\gamma_1(\zeta)\frac{d\zeta}{\zeta} + \\ + \frac{1}{\pi}\int_R f(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta + \frac{r^2}{\pi z}\int_R f(\zeta)\frac{d\xi d\eta}{\zeta}.\end{aligned} \tag{3.5.6}$$

Proof. The problem is reduced to the system

$$w_{\overline{z}} = \varphi \text{ in } R, \ \ w = \gamma_0, \text{ on } \partial R, \tag{3.5.7}$$

$$\varphi_{\overline{z}} = f \text{ in } R, \ \ \varphi = \gamma_1, \text{ on } \ \partial R \tag{3.5.8}$$

of Dirichlet problems for the inhomogeneous Cauchy-Riemann equation. Using the result of **Theorem 2.3.16**, one obtains

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{R} \varphi(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{3.5.9}$$

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \gamma_1(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{3.5.10}$$

under the solvability conditions

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma_0(\zeta)\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} \varphi(\zeta)\frac{\overline{z}d\xi d\eta}{1 - \overline{z}\zeta}, \tag{3.5.11}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma_0(\zeta)\frac{\overline{z}d\zeta}{r^2 - \overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} \varphi(\zeta)\frac{\overline{z}d\xi d\eta}{r^2 - \overline{z}\zeta}, \tag{3.5.12}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma_1(\zeta)\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{z}d\xi d\eta}{1 - \overline{z}\zeta}, \tag{3.5.13}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma_1(\zeta)\frac{\overline{z}d\zeta}{r^2 - \overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{\overline{z}d\xi d\eta}{r^2 - \overline{z}\zeta}. \tag{3.5.14}$$

Inserting (3.5.10) into (3.5.9) gives

$$\begin{aligned} w(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \gamma_0(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial R} \gamma_1(\widetilde{\zeta})\frac{1}{\pi}\int\limits_{R} \frac{d\xi d\eta}{(\zeta - \widetilde{\zeta})(\zeta - z)} d\widetilde{\zeta} - \\ - \frac{1}{\pi}\int\limits_{R} f(\widetilde{\zeta})\frac{1}{\pi}\int\limits_{R} \frac{d\xi d\eta}{(\zeta - \widetilde{\zeta})(\zeta - z)}\, d\widetilde{\xi} d\widetilde{\eta}, \end{aligned} \tag{3.5.15}$$

with

$$\frac{1}{\pi}\int\limits_{R} \frac{d\xi d\eta}{(\zeta - \widetilde{\zeta})(\zeta - z)} = \frac{1}{\widetilde{\zeta} - z}\Big[\frac{1}{\pi}\int\limits_{R} \frac{d\xi d\eta}{\zeta - \widetilde{\zeta}} - \frac{1}{\pi}\int\limits_{R} \frac{d\xi d\eta}{\zeta - z}\Big]. \tag{3.5.16}$$

The Cauchy-Pompeiu integral representation (2.1.4) is used to evaluate the integral

$$\frac{1}{\pi}\int_R \frac{d\xi d\eta}{\zeta - z} = -\overline{z} + \frac{1}{2\pi i}\int_{\partial R} \frac{\overline{\zeta} d\zeta}{\zeta - z} = -\overline{z} + \frac{r^2}{z}.$$

Then the integral (3.5.16) can be easily found. Substituting it to (3.5.15), the formula (3.5.6) for the solution is obtained.

The function $\varphi(z)$ is plugged into (3.5.11), (3.5.12) to get the solvability conditions. This leads to

$$\frac{1}{2\pi i}\int_{\partial R} \gamma_0(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = -\frac{1}{2\pi i}\int_{\partial R} \gamma_1(\widetilde{\zeta})\frac{1}{\pi}\int_R \frac{\overline{z}d\xi d\eta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)}\, d\widetilde{\zeta}+$$
$$+\frac{1}{\pi}\int_R f(\widetilde{\zeta})\frac{1}{\pi}\int_R \frac{d\xi d\eta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)}\, d\widetilde{\xi} d\widetilde{\eta}$$

and

$$\frac{1}{2\pi i}\int_{\partial R} \gamma_0(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} = -\frac{1}{2\pi i}\int_{\partial R} \gamma_1(\widetilde{\zeta})\frac{1}{\pi}\int_R \frac{\overline{z}d\xi d\eta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)}\, d\widetilde{\zeta}+$$
$$+\frac{1}{\pi}\int_R f(\widetilde{\zeta})\frac{1}{\pi}\int_R \frac{d\xi d\eta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)}\, d\widetilde{\xi} d\widetilde{\eta}$$

with

$$\frac{1}{\pi}\int_R \frac{d\xi d\eta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)} = -\frac{\overline{\widetilde{\zeta}-z}}{1-\overline{z}\widetilde{\zeta}} + \frac{1}{2\pi i}\int_{\partial R} \frac{(\overline{\zeta-z})d\zeta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)} =$$
$$= -\frac{\overline{\widetilde{\zeta}-z}}{1-\overline{z}\widetilde{\zeta}} + \frac{1}{2\pi i}\int_{\partial R} \frac{(|\zeta|^2-\overline{z}\zeta)d\zeta}{\zeta(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)} = \frac{r^2}{\widetilde{\zeta}} - \frac{\overline{\widetilde{\zeta}-z}}{1-\overline{z}\widetilde{\zeta}},$$

$$\frac{1}{\pi}\int_R \frac{d\xi d\eta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)} = -\frac{\overline{\widetilde{\zeta}-z}}{r^2-\overline{z}\widetilde{\zeta}} + \frac{1}{2\pi i}\int_{\partial R} \frac{(\overline{\zeta-z})d\zeta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)} =$$
$$= -\frac{\overline{\widetilde{\zeta}-z}}{r^2-\overline{z}\widetilde{\zeta}} + \frac{1}{2\pi i}\int_{\partial R} \frac{(|\zeta|^2-\overline{z}\zeta)d\zeta}{\zeta(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)} = \frac{1}{\widetilde{\zeta}} - \frac{\overline{\widetilde{\zeta}-z}}{r^2-\overline{z}\widetilde{\zeta}},$$

from what solvability conditions (3.5.2), (3.5.3) follow. □

Theorem 3.5.11. *The Schwarz problem for the inhomogeneous Bitsadze equation in* R

$$w_{\overline{z}\,\overline{z}} = f \text{ in } R,\ \operatorname{Re} w = \gamma_0,\ \operatorname{Re} w_{\overline{z}} = \gamma_1 \text{ on } \partial R, \tag{3.5.17}$$

$$\frac{1}{2\pi i}\int\limits_{|z|=\rho} \operatorname{Im} w(z)\frac{dz}{z} = c_0,\ \frac{1}{2\pi i}\int\limits_{|z|=\rho} \operatorname{Im} w_{\overline{z}}(z)\frac{dz}{z} = c_1 \tag{3.5.18}$$

for $f \in L_p(R;\mathbb{C})$, γ_0, $\gamma_1 \in C(\partial R;\mathbb{C})$, c_0, $c_1 \in \mathbb{R}$, *given is solvable by a function from* $W^{2,p}_{\overline{z}}(R;\mathbb{C})$ *with continuously differentiable real part in* $\overline{R}$ *if and only if*

$$\frac{1}{2\pi i}\int\limits_{\partial R} \gamma_1(\zeta)\frac{d\zeta}{\zeta} = \frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)d\xi d\eta, \tag{3.5.19}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{d\zeta}{\zeta} = -\frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta+\overline{\zeta})d\xi d\eta. \tag{3.5.20}$$

Then the solution is unique and expressed by

$$\begin{aligned}
w(z) &= ic_0 + i(z+\overline{z})c_1 + \\
&+ \frac{1}{2\pi i}\int\limits_{\partial R}[\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta - z + \overline{\zeta - z})]\left[\frac{\zeta + z}{\zeta - z} + K_1(z,\zeta)\right]\frac{d\zeta}{\zeta} - \\
&- \frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta - z + \overline{\zeta - z})]\frac{d\zeta}{\zeta} + \\
&+ \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\right](\zeta - z + \overline{\zeta - z})d\xi d\eta + \\
&+ \frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} + K_2(z,\zeta)\right](\zeta - z + \overline{\zeta - z})d\xi d\eta + \\
&+ \frac{1}{2\pi}\int\limits_{r<|\zeta|<\rho}\left[\frac{f(\zeta)}{\zeta} - \frac{\overline{f(\zeta)}}{\overline{\zeta}}\right](\zeta - z + \overline{\zeta - z})d\xi d\eta + \\
&+ \frac{r^2+\rho^2}{2\pi(1-r^2)}\int\limits_{|\zeta|=1}\gamma_1(\zeta)(\zeta - \overline{\zeta})\frac{d\zeta}{\zeta} - \\
&- \frac{1+\rho^2}{2\pi(1-r^2)}\int\limits_{|\zeta|=r}\gamma_1(\zeta)(\zeta-\overline{\zeta})\frac{d\zeta}{\zeta} -
\end{aligned} \tag{3.5.21}$$

$$- \frac{r^2+\rho^2}{2\pi i(1-r^2)} \int\limits_{\rho<|\zeta|<1} [f(\zeta) - \overline{f(\zeta)}]d\xi d\eta -$$

$$- \frac{1+\rho^2}{2\pi i(1-r^2)} \int\limits_{r<|\zeta|<\rho} [f(\zeta) - \overline{f(\zeta)}]d\xi d\eta +$$

$$+ \frac{r^2+\rho^2}{2\pi i(1-r^2)} \int\limits_{\rho<|\zeta|<1} \left[\frac{f(\zeta)}{\zeta^2} - \frac{\overline{f(\zeta)}}{\overline{\zeta}^2}\right] d\xi d\eta +$$

$$+ \frac{r^2(1+\rho^2)}{2\pi i(1-r^2)} \int\limits_{r<|\zeta|<\rho} \left[\frac{f(\zeta)}{\zeta^2} - \frac{\overline{f(\zeta)}}{\overline{\zeta}^2}\right] d\xi d\eta,$$

where

$$K_1(z,\zeta) = 2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z}\Big),$$

$$K_2(z,\zeta) = 2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}}{r^{2n} - z\overline{\zeta}} + \frac{r^{2n}z\overline{\zeta}}{1 - r^{2n}z\overline{\zeta}}\Big).$$

Proof. Observing the formula for the solution to the Schwarz problem for the Bitsadze equation (see [7], theorem 14) we note that an extra term in the form of a series appears when we pass from the problem in the unit disk to one in a circular ring domain. Formula (3.5.21) represents a generalization of the mentioned result in the case of R. Therefore it is sufficient to show that under conditions (3.5.19), (3.5.20) formula (3.5.21) gives the solution to (3.5.17), (3.5.18) and to see the uniqueness of such a solution.

That (3.5.21) is a solution to the Bitsadze equation is seen from

$$w_{\overline{z}}(z) = \frac{1}{2\pi i}\int\limits_{\partial R} \gamma_1(\zeta)\left[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\right]\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma_1(\zeta)\frac{d\zeta}{\zeta} -$$

$$- \frac{1}{2\pi}\int\limits_{R} \frac{f(\zeta)}{\zeta}\left[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\right] d\xi d\eta -$$

$$- \frac{1}{2\pi}\int\limits_{R} \frac{\overline{f(\zeta)}}{\overline{\zeta}}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} + K_2(z,\zeta)\right] d\xi d\eta - \tag{3.5.22}$$

$$- \frac{1}{2\pi}\int\limits_{r<|\zeta|<\rho} \left[\frac{f(\zeta)}{\zeta} - \frac{\overline{f(\zeta)}}{\overline{\zeta}}\right] d\xi d\eta + ic_1,$$

by taking $\overline{z}$-derivative once again and using the properties of the Pompeiu operator.

Let us verify the first boundary condition. Using the expression

$$\operatorname{Re} w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} [\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta - z + \overline{\zeta - z})]\Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 +$$
$$+ \sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} - \frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} - \frac{r^{2n}z}{|z|^2\zeta - r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta} -$$
$$- \frac{1}{2\pi i} \int\limits_{|\zeta|=r} [\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta - z + \overline{\zeta - z})]\Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 +$$
$$+ \sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} - \frac{r^{2(n-1)}|z|^2\zeta}{r^{2(n-1)}|z|^2\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} - \frac{r^{2(n+1)}z}{|z|^2\zeta - r^{2(n+1)}z}\Big)\Big]\frac{d\zeta}{\zeta} -$$
$$- \frac{1}{2\pi i} \int\limits_{|\zeta|=r} [\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta - z + \overline{\zeta - z})]\frac{d\zeta}{\zeta} +$$
$$+ \frac{1}{4\pi} \int\limits_{R} \frac{f(\zeta)}{\zeta} \left[\frac{\zeta + z}{\zeta - z} + \frac{z + |z|^2\zeta}{z - |z|^2\zeta} + 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n}z}{\zeta - r^{2n}z} +\right.\right.$$
$$\left.\left. + \frac{r^{2n}z}{r^{2n}z - |z|^2\zeta} + \frac{r^{2n}|z|^2\zeta}{z - r^{2n}|z|^2\zeta}\right)\right] (\zeta - z + \overline{\zeta - z}) d\xi d\eta +$$
$$+ \frac{1}{4\pi} \int\limits_{R} \frac{\overline{f(\zeta)}}{\overline{\zeta}} \left[\frac{z\overline{\zeta} + |z|^2}{z\overline{\zeta} - |z|^2} + \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} + 2\sum_{n=1}^{\infty}\left(\frac{r^{2n}z\overline{\zeta}}{r^{2n}z\overline{\zeta} - |z|^2} + \frac{r^{2n}|z|^2}{z\overline{\zeta} - r^{2n}|z|^2} +\right.\right.$$
$$\left.\left. + \frac{r^{2n}}{r^{2n} - z\overline{\zeta}} + \frac{r^{2n}z\overline{\zeta}}{1 - r^{2n}z\overline{\zeta}}\right)\right] (\zeta - z + \overline{\zeta - z}) d\xi d\eta,$$

one can find that

$$\lim_{|z|\to 1,\, z\in R} \operatorname{Re} w(z) = \gamma_0(z)$$

and

$$\lim_{|z|\to r,\, z\in R} \operatorname{Re} w(z) = \gamma_0(z) + \frac{1}{2\pi i} \int\limits_{\partial R} [\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta - z + \overline{\zeta - z})]\frac{d\zeta}{\zeta} +$$
$$+ \frac{1}{2\pi} \int\limits_{R} \Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta - z + \overline{\zeta - z}) d\xi d\eta.$$

Hence, the first boundary condition is satisfied if and only if

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\partial R}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})]\frac{d\zeta}{\zeta}=\\&=-\frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})d\xi d\eta\end{aligned}\tag{3.5.23}$$

holds. In the same way

$$\begin{aligned}\operatorname{Re} w_{\overline{z}}(z)=&\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\overline{\zeta-z}}}-1+\\&+\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\Big)\Big]\frac{d\zeta}{\zeta}-\\&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\overline{\zeta-z}}}-1+\\&+\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2(n-1)}|z|^2\zeta}{r^{2(n-1)}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2(n+1)}z}{|z|^2\zeta-r^{2(n+1)}z}\Big)\Big]\frac{d\zeta}{\zeta}\\&-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{d\zeta}{\zeta}\\&-\frac{1}{4\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\Big[\frac{\zeta+z}{\zeta-z}+\frac{z+|z|^2\zeta}{z-|z|^2\zeta}+\\&+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}+\frac{r^{2n}z}{r^{2n}z-|z|^2\zeta}+\frac{r^{2n}|z|^2\zeta}{z-r^{2n}|z|^2\zeta}\Big)\Big]d\xi d\eta-\\&-\frac{1}{4\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big[\frac{z\overline{\zeta}+|z|^2}{z\overline{\zeta}-|z|^2}+\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+\\&+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}z\overline{\zeta}}{r^{2n}z\overline{\zeta}-|z|^2}+\frac{r^{2n}|z|^2}{z\overline{\zeta}-r^{2n}|z|^2}+\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\Big)\Big]d\xi d\eta.\end{aligned}$$

Hence

$$\lim_{|z|\to 1,\, z\in R}\operatorname{Re} w_{\overline{z}}(z)=\gamma_1(z)$$

and

$$\lim_{|z|\to r,\, z\in R} \operatorname{Re} w_{\overline{z}}(z) = \gamma_1(z) + \frac{1}{2\pi i}\int_{\partial R} \gamma_1(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int_R \Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big) d\xi d\eta$$

are valid. This shows that the second boundary condition is satisfied if and only if (3.5.19) holds.

Condition (3.5.23) can be rewritten in the form

$$\frac{1}{2\pi i}\int_{\partial R} [\gamma_0(\zeta) - \gamma_1(\zeta)(\zeta + \overline{\zeta})]\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int_R \Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta + \overline{\zeta}) d\xi d\eta -$$

$$-(z+\overline{z})\Big[\frac{1}{2\pi i}\int_{\partial R} \gamma_1(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int_R \Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big) d\xi d\eta\Big] = 0,$$

what is equivalent to (3.5.20), due to (3.5.19).

To show that the function $w(z)$ defined by (3.5.21) satisfies the normalization conditions (3.5.18) the integrals $\frac{1}{2\pi i}\int_{|z|=\rho} w(z)\frac{dz}{z}$, $\frac{1}{2\pi i}\int_{|z|=\rho} w_{\overline{z}}(z)\frac{dz}{z}$, have to be evaluated. For the evaluation the equalities

$$\frac{1}{2\pi i}\int_{|z|=\rho} \Big[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\Big]\frac{dz}{z} = \begin{cases} 1, & \rho < |\zeta| \le 1, \\ -1, & r \le |\zeta| < \rho, \end{cases}$$

$$\frac{1}{2\pi i}\int_{|z|=\rho} \Big[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\Big] dz = \begin{cases} -\dfrac{2r^2\zeta}{1-r^2}, & \rho < |\zeta| \le 1, \\ -2\zeta - \dfrac{2r^2\zeta}{1-r^2}, & r \le |\zeta| < \rho, \end{cases}$$

$$\frac{1}{2\pi i}\int_{|z|=\rho} \Big[\frac{\zeta+z}{\zeta-z} + K_1(z,\zeta)\Big]\frac{dz}{z^2} = \begin{cases} \dfrac{2}{\zeta} + \dfrac{2r^2}{(1-r^2)\zeta}, & \rho < |\zeta| \le 1, \\ \dfrac{2r^2}{(1-r^2)\zeta}, & r \le |\zeta| < \rho, \end{cases}$$

$$\frac{1}{2\pi i}\int_{|z|=\rho} \Big[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} + K_2(z,\zeta)\Big]\frac{dz}{z} = 1 \ \text{ for } \ \zeta \in R,$$

$$\frac{1}{2\pi i}\int_{|z|=\rho} \Big[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}} + K_2(z,\zeta)\Big] dz = -\frac{2r^2}{(1-r^2)\overline{\zeta}} \ \text{ for } \ \zeta \in R,$$

$$\frac{1}{2\pi i}\int\limits_{|z|=\rho}\left[\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+K_2(z,\zeta)\right]\frac{dz}{z^2}=2\overline{\zeta}+\frac{2r^2\overline{\zeta}}{1-r^2}\ \text{ for }\ \zeta\in R,$$

are used. Then passing to the imaginary part verifies the normalization conditions (3.5.18).

To show the uniqueness of the solution (3.5.21), the Schwarz problem for the homogeneous Bitsadze equation with homogeneous boundary and normalization conditions is considered, i.e.

$$w_{\overline{z}\,\overline{z}}=0 \text{ in } R,\ \operatorname{Re} w=0,\ \operatorname{Re} w_{\overline{z}}=0 \text{ on } \partial R,$$

$$\frac{1}{2\pi i}\int\limits_{|z|=\rho}\operatorname{Im} w(z)\frac{dz}{z}=0,\ \frac{1}{2\pi i}\int\limits_{|z|=\rho}\operatorname{Im} w_{\overline{z}}(z)\frac{dz}{z}=0.$$

It is decomposed to a system of Schwarz problems for inhomogeneous and homogeneous Cauchy-Riemann equation

$$w_{\overline{z}}=\varphi \text{ in } R,\ \operatorname{Re} w=0,\ \text{on } \partial R,\ \frac{1}{2\pi i}\int\limits_{|z|=\rho}\operatorname{Im} w(z)\frac{dz}{z}=0,$$

$$\varphi_{\overline{z}}=0 \text{ in } R,\ \operatorname{Re}\varphi=0,\ \text{on } \partial R,\ \frac{1}{2\pi i}\int\limits_{|z|=\rho}\operatorname{Im}\varphi(z)\frac{dz}{z}=0.$$

which have only the trivial solutions, if one takes into account the results of **Theorem 2.2.11** and **Theorem 2.3.15**. The proof is completed. □

Remark 3.5.4. This result differs from the one for simply connected domains (see [7]) as solvability conditions appear. They exclude functions which are not determined in a unique way by their respective boundary data. Let us consider, for instance, the function $\log z$. It is the solution for the Bitsadze equation, having vanishing Schwarz data on $|z|=1$ and the data $\log r$ on $|z|=r$, while its derivative with respect to $\overline{z}$ is identically zero. The second condition in (3.5.20) is not valid.

The solution to the Dirichlet problem for the inhomogeneous Bitsadze equation in a circular ring domain are found here by the decomposition method. It should be emphasized that in principal this method can be used to solve other boundary value problems, but it is appeared not to be always effective. In the case of a circular ring domain such procedure leads to highly complicated analysis and it is difficult to get the final result (for instance, it happens when solving the Schwarz problem for the Poisson equation or the Neumann problem for the inhomogeneous Bitsadze equation). Thus the solution of certain problems needs some other approaches.

One possible way to handle such problems is to generalize the corresponding formulas for the unit disk. By using this technique the Schwarz problem for the inhomogeneous Bitsadze equation is solved here.

4 Biharmonic Green Function for a Ring Domain

This chapter is devoted to the construction of the Green function for the biharmonic operator $(\partial_z \partial_{\overline{z}})^2$ in a circular ring domain $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$.

There are two main approaches to determine a Green function for the biharmonic operator. The first one is to define the biharmonic Green function as a convolution of the harmonic Green function with itself, see e.g. [11], [12]. This leads to an equivalent definition of the biharmonic Green function as a solution of a certain Dirichlet problem for the Poisson equation.

The second approach is connected with the name of Almansi, who introduced a concept of polyharmonic Green functions for the unit disk [4] (see also [11], [12], [10], [13]). The Almansi biharmonic Green function is different from the one mentioned above in its boundary behavior.

The complex analytic approach gives the possibility to construct Green functions in explicit form for special domains. It is used to obtain the biharmonic Green functions in the case of the unit disk in [11].

In this chapter the biharmonic Green function defined by convolution of harmonic Green function with itself is constructed for R. It has appeared that the method of the direct evaluation of the biharmonic Green function, used in [11] for the unit disk, is not effective in the case of the circular ring as it leads to complicated calculations. From the definition the representation of the biharmonic Green function in the form

$$\widehat{G}_2(z,\zeta) = |\zeta - z|^2 G_1(z,\zeta) + \widehat{h}_2(z,\zeta)$$

follows, where $G_1(z,\zeta)$ is the harmonic Green function for R and $\widehat{h}_2(z,\zeta)$ is a certain biharmonic function, which has to be defined.

4.1 Preliminaries

Definition 4.1.1. A complex-valued function w, satisfying

$$(\partial_z \partial_{\overline{z}})^2 w = 0$$

in an open domain $D \subset \mathbb{C}$ is called a biharmonic function in D.

The operator $(\partial_z\partial_{\overline{z}})^2$ is referred to as the complex bi-Laplace or biharmonic operator. The harmonic Green function serves to solve the Dirichlet problem for the Poisson equation

$$\partial_z\partial_{\overline{z}}\, w = f \text{ in } D,\; w = \gamma \text{ on } \partial D, \tag{4.1.1}$$

where $f \in L_2(D;\mathbb{C})$, $\gamma \in C(\partial D;\mathbb{C})$ are given. The solution of the problem (4.1.1) is unique and expressed by, see e.g. [8], [21],

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial D} \partial_{\nu_\zeta} G_1(z,\zeta)\gamma(\zeta)ds_\zeta - \frac{1}{\pi}\int\limits_{D} G_1(z,\zeta)f(\zeta)d\xi d\eta. \tag{4.1.2}$$

Let us insert the Green function $G_1(z,\zeta)$ in (4.1.2) instead of $f(\zeta)$ and denote

$$\widehat{G}_2(z,\zeta) := -\frac{1}{\pi}\int\limits_{D} G_1(z,\widetilde{\zeta})G_1(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}. \tag{4.1.3}$$

Comparing this with formula (4.1.2), one observes that the introduced function $\widehat{G}_2(\,\cdot\,,\zeta)$ is the solution to the Dirichlet problem

$$\partial_z\partial_{\overline{z}}\, \widehat{G}_2(z,\zeta) = G_1(z,\zeta) \text{ in } D,\; \widehat{G}_2(z,\zeta) = 0, \text{ on } \partial D \tag{4.1.4}$$

for any $\zeta \in D$.

The function $\widehat{G}_2(z,\zeta)$ defined in (4.1.3) is called the biharmonic Green function of a domain D (see [12]). It satisfies for any fixed $\zeta \in D$ as a function of z the following properties

1°. $\widehat{G}_2(z,\zeta)$ is biharmonic in $D\setminus\{\zeta\}$,

2°. $\widehat{G}_2(z,\zeta) + |\zeta - z|^2\log|\zeta - z|^2$ is biharmonic in D,

3°. $\widehat{G}_2(z,\zeta) = 0$, $\partial_z\partial_{\overline{z}}\widehat{G}_2(z,\zeta) = 0$ for $z \in \partial D$,

4°. $\widehat{G}_2(z,\zeta) = \widehat{G}_2(\zeta,z)$ for $z \neq \zeta$.

Contrary to $\widehat{G}_2(z,\zeta)$ the mentioned biharmonic Green-Almansi function $G_2(z,\zeta)$ is not a primitive of $G_1(z,\zeta)$ with respect to the Laplace operator. It satisfies the properties $1°, 2°, 4°$, but has different boundary behavior, namely

$$G_2(z,\zeta) = 0,\; \partial_{\nu_\zeta} G_2(z,\zeta) = 0 \text{ for } z \in \partial D, \zeta \in D.$$

In the case of the unit disk the explicit form for $\widehat{G}_2(z,\zeta)$ is given in [11], [12], [21], for $G_2(z,\zeta)$ it is presented in [4] (see also [11], [12], [21]).

4.2 Construction of the biharmonic Green function of a circular ring domain

The biharmonic Green function $\widehat{G}_2(z,\zeta)$ of a domain D can be represented in the form

$$\widehat{G}_2(z,\zeta) = |\zeta - z|^2 G_1(z,\zeta) + \widehat{h}_2(z,\zeta), \tag{4.2.1}$$

with $\widehat{h}_2(z,\zeta)$ being a biharmonic function in D. This representation follows from the second property of $\widehat{G}_2(z,\zeta)$, and the second property of $G_1(z,\zeta)$, according to which

$$G_1(z,\zeta) = -\log|\zeta - z|^2 + h_1(z,\zeta)$$

where $h_1(z,\zeta)$ is a harmonic function in D. Thus to get the expression for $\widehat{G}_2(z,\zeta)$ the function $\widehat{h}_2(z,\zeta)$ has to be found.

As $\widehat{G}_2(z,\zeta)$ is the solution to the Dirichlet problem (4.1.4), it follows that the function $\widehat{h}_2(z,\zeta)$ is the solution to the Dirichlet problem

$$\partial_z\partial_{\overline{z}}\widehat{h}_2(z,\zeta) = 2\mathrm{Re}\,[(\zeta - z)\partial_z G_1(z,\zeta)] \text{ in } D, \;\; \widehat{h}_2(z,\zeta) = 0 \text{ on } \partial D, \tag{4.2.2}$$

for any $\zeta \in D$. To find the solution of this problem formula (4.1.2) is used. According to it

$$\widehat{h}_2(z,\zeta) = -\frac{2}{\pi}\int_D \mathrm{Re}\,[(\zeta - \widetilde{\zeta})\partial_{\widetilde{\zeta}} G_1(\widetilde{\zeta},\zeta)]G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta},$$

or equivalently

$$\widehat{h}_2(z,\zeta) = 2\mathrm{Re}\Big[\frac{1}{\pi}\int_D (\widetilde{\zeta} - \zeta)\partial_{\widetilde{\zeta}} G_1(\widetilde{\zeta},\zeta)G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big], \tag{4.2.3}$$

as the harmonic Green function $G_1(z,\zeta)$ is real-valued.

By using the explicit form of the harmonic Green function given in (3.1.4) for a circular ring domain R, the formula (4.2.3) can be rewritten as

$$\widehat{h}_2(z,\zeta) = 2\mathrm{Re}\left[\frac{1}{\pi}\int_R (\widetilde{\zeta} - \zeta)\Big[\frac{\log|\zeta|^2}{\log|r|^2}\frac{1}{\widetilde{\zeta}} - \frac{1}{\widetilde{\zeta} - \zeta} + \frac{\overline{\zeta}}{\widetilde{\zeta}\overline{\zeta} - 1} + \right.$$

$$\left. + \sum_{n=1}^{\infty}\Big(\frac{r^{2n}\overline{\zeta}}{r^{2n}\widetilde{\zeta}\overline{\zeta} - 1} + \frac{\overline{\zeta}}{\widetilde{\zeta}\overline{\zeta} - r^{2n}} - \frac{1}{\widetilde{\zeta} - r^{2n}\zeta} - \frac{r^{2n}}{r^{2n}\widetilde{\zeta} - \zeta}\Big)\Big]G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\right] =$$

$$= 2\mathrm{Re}\left[\frac{1}{\pi}\int\limits_R \Big[\frac{\log|\zeta|^2}{\log|r|^2}\frac{\widetilde{\zeta}-\zeta}{\widetilde{\zeta}} + \frac{1-|\zeta|^2}{\widetilde{\zeta}\overline{\zeta}-1} + \right. \tag{4.2.4}$$

$$\left. + \sum_{n=1}^{\infty}\Big(\frac{1-r^{2n}|\zeta|^2}{r^{2n}\widetilde{\zeta}\overline{\zeta}-1} - \frac{r^{2n}-|\zeta|^2}{r^{2n}-\widetilde{\zeta}\overline{\zeta}} + \frac{(1-r^{2n})\zeta}{\widetilde{\zeta}-r^{2n}\zeta} - \frac{(1-r^{2n})\zeta}{r^{2n}\widetilde{\zeta}-\zeta}\Big)\Big] G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\right],$$

according to (3.1.4).

The evaluation of the integral (4.2.4) is proceeded in five steps. The calculations are represented here just briefly. The main facts which are used for getting the final results of the evaluations are Gauss theorem, Cauchy theorem and formula (3.1.5).

1. Let us consider the integral

$$I_1(z,\zeta) := 2\mathrm{Re}\Big[\frac{1}{\pi}\int\limits_R \frac{\widetilde{\zeta}-\zeta}{\widetilde{\zeta}} G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big] = \frac{1}{\pi}\int\limits_R \Big[\frac{\widetilde{\zeta}-\zeta}{\widetilde{\zeta}} + \frac{\overline{\widetilde{\zeta}-\zeta}}{\overline{\widetilde{\zeta}}}\Big] G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta},$$

which is related to the first part in (4.2.4).

Since

$$\partial_z\partial_{\overline{z}}\Big[2|z|^2 - (z\overline{\zeta}+\overline{z}\zeta)\log|z|^2\Big] = \frac{z-\zeta}{z} + \frac{\overline{z-\zeta}}{\overline{z}},$$

by formula (3.1.5) we get

$$I_1(z,\zeta) = (z\overline{\zeta}+\overline{z}\zeta)\log|z|^2 - 2|z|^2 - \widehat{I}_1(z,\zeta),$$

where

$$\widehat{I}_1(z,\zeta) := \frac{1}{4\pi i}\int\limits_{\partial R}\Big[2|\widetilde{\zeta}|^2 - (\widetilde{\zeta}\overline{\zeta}+\overline{\widetilde{\zeta}}\zeta)\log|\widetilde{\zeta}|^2\Big]|\widetilde{\zeta}|\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} =$$

$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int\limits_{\partial R}[2|\widetilde{\zeta}|^2 - (\widetilde{\zeta}\overline{\zeta}+\overline{\widetilde{\zeta}}\zeta)\log|\widetilde{\zeta}|^2]\Big[\frac{\log|z|^2}{\log r^2}\frac{1}{\widetilde{\zeta}} - \frac{1}{\widetilde{\zeta}-z} - \frac{\overline{z}}{1-\overline{z}\widetilde{\zeta}} + \right.$$

$$\left. + \sum_{k=1}^{\infty}\Big(\frac{r^{2k}\overline{z}}{r^{2k}\overline{z}\widetilde{\zeta}-1} + \frac{\overline{z}}{\overline{z}\widetilde{\zeta}-r^{2k}} - \frac{1}{\widetilde{\zeta}-r^{2k}z} - \frac{r^{2k}}{r^{2k}\widetilde{\zeta}-z}\Big)\Big]d\widetilde{\zeta}\right] =$$

$$= 2\mathrm{Re}\Big[(1-r^2)\frac{\log|z|^2}{\log r^2} + \frac{r^2}{1-r^2}\log r^2\Big(\frac{\overline{\zeta}}{\overline{z}} - z\overline{\zeta}\Big)\Big].$$

Then the first part in (4.2.4) is

$$2\mathrm{Re}\Big[\frac{\log|\zeta|^2}{\log r^2}\frac{1}{\pi}\int\limits_R \frac{\widetilde{\zeta}-\zeta}{\widetilde{\zeta}}\, G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big] = \frac{\log|\zeta|^2}{\log r^2} I_1(z,\zeta) =$$

$$= 2\mathrm{Re}\left[\frac{\log|\zeta|^2}{\log r^2}\Big(1-|z|^2+z\overline{\zeta}\log|z|^2-(1-r^2)\frac{\log|z|^2}{\log r^2}- \right. \tag{4.2.5}$$

$$\left. -\frac{r^2}{1-r^2}\log r^2\Big(\frac{\overline{\zeta}}{\overline{z}}-z\overline{\zeta}\Big)\Big)\right].$$

2. In the next step we evaluate the integral

$$I_2(z,\zeta) := 2\mathrm{Re}\Big[\frac{1}{\pi}\int\limits_R \frac{1}{r^{2n}\widetilde{\zeta}\overline{\zeta}-1}\, G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big] =$$

$$= \frac{1}{\pi}\int\limits_R \Big[\frac{1}{r^{2n}\widetilde{\zeta}\overline{\zeta}-1}+\frac{1}{r^{2n}\overline{\widetilde{\zeta}}\zeta-1}\Big]G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}$$

for any fixed $n\in\mathbb{N}$.

Observing that

$$\partial_z\partial_{\overline{z}}\left[\frac{1}{r^{2n}}\Big(\frac{z}{\zeta}+\frac{\overline{z}}{\overline{\zeta}}\Big)\log|1-r^{2n}z\overline{\zeta}|^2\right] = \frac{1}{r^{2n}z\overline{\zeta}-1}+\frac{1}{r^{2n}\overline{z}\zeta-1},$$

by formula (3.1.5) one gets

$$I_2(z,\zeta) = -\frac{1}{r^{2n}}\Big(\frac{z}{\zeta}+\frac{\overline{z}}{\overline{\zeta}}\Big)\log|1-r^{2n}z\overline{\zeta}|^2 - \frac{1}{r^{2n}}\widehat{I_2}(z,\zeta),$$

with

$$\widehat{I_2}(z,\zeta) := \frac{1}{4\pi i}\int\limits_{\partial R}\Big(\frac{\widetilde{\zeta}}{\zeta}+\frac{\overline{\widetilde{\zeta}}}{\overline{\zeta}}\Big)\log|1-r^{2n}\widetilde{\zeta}\overline{\zeta}|^2|\widetilde{\zeta}|\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} =$$

$$= \mathrm{Re}\left[\frac{1}{2\pi i}\int\limits_{\partial R}\widetilde{\zeta}\Big(\frac{\widetilde{\zeta}}{\zeta}+\frac{\overline{\widetilde{\zeta}}}{\overline{\zeta}}\Big)\Big[\frac{\log|z|^2}{\log r^2}\frac{1}{\widetilde{\zeta}}-\frac{1}{\widetilde{\zeta}-z}-\frac{\overline{z}}{1-\overline{z}\widetilde{\zeta}}+\right.$$

$$\left.+\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\overline{z}}{r^{2k}\overline{z}\widetilde{\zeta}-1}+\frac{\overline{z}}{\overline{z}\widetilde{\zeta}-r^{2k}}-\frac{1}{\widetilde{\zeta}-r^{2k}z}-\frac{r^{2k}}{r^{2k}\widetilde{\zeta}-z}\Big)\Big]\log|1-r^{2n}\widetilde{\zeta}\overline{\zeta}|^2\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\right] =$$

$$= -2\mathrm{Re}\Big[(1-r^2)r^{2n}\frac{\log|z|^2}{\log r^2}+\Big(\frac{\overline{z}}{\overline{\zeta}}+\frac{1}{z\overline{\zeta}}\Big)\log(1-r^{2n}z\overline{\zeta})-$$

$$-(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{z}{r^{2k}\zeta}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big)-\frac{1}{r^{2k}z\overline{\zeta}}\log(1-r^{2(n+k)}z\overline{\zeta})\Big)\Big].$$

Then

$$2\mathrm{Re}\Big[(1-r^{2n}|\zeta|^2)\frac{1}{\pi}\int\limits_R \frac{1}{r^{2n}\widetilde{\zeta}\overline{\widetilde{\zeta}}-1}\,G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big]=(1-r^{2n}|\zeta|^2)I_2(z,\zeta)=$$

$$=-\frac{2(1-r^{2n}|\zeta|^2)}{r^{2n}}\mathrm{Re}\,\Big[\Big(\frac{\overline{z}}{\overline{\zeta}}-\frac{1}{z\overline{\zeta}}\Big)\log(1-r^{2n}z\overline{\zeta})-(1-r^2)r^{2n}\frac{\log|z|^2}{\log|r|^2}+ \tag{4.2.6}$$

$$+(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{z}{r^{2k}\zeta}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big)-\frac{1}{r^{2k}z\overline{\zeta}}\log(1-r^{2(n+k)}z\overline{\zeta})\Big)\Big].$$

3. The third integral is

$$I_3(z,\zeta):=2\mathrm{Re}\Big[\frac{1}{\pi}\int\limits_R\frac{1}{r^{2n}-\widetilde{\zeta}\overline{\zeta}}\,G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big]=$$

$$=\frac{1}{\pi}\int\limits_R\Big[\frac{1}{r^{2n}-\overline{\widetilde{\zeta}}\zeta}+\frac{1}{r^{2n}-\widetilde{\zeta}\overline{\zeta}}\Big]G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}$$

where $n\in\mathbb{N}$ is fixed.

As

$$\partial_z\partial_{\overline{z}}\,\Big[\Big(\frac{z}{\zeta}+\frac{\overline{z}}{\overline{\zeta}}\Big)\log|z\overline{\zeta}-r^{2n}|^2\Big]=-\Big(\frac{1}{r^{2n}-z\overline{\zeta}}+\frac{1}{r^{2n}-\overline{z}\zeta}\Big),$$

by formula (3.1.5)

$$I_3(z,\zeta)=\Big(\frac{z}{\zeta}+\frac{\overline{z}}{\overline{\zeta}}\Big)\log|z\overline{\zeta}-r^{2n}|^2+\widehat{I}_3(z,\zeta),$$

where

$$\widehat{I}_3(z,\zeta):=\frac{1}{4\pi i}\int\limits_{\partial R}\Big(\frac{\widetilde{\zeta}}{\zeta}+\frac{\overline{\widetilde{\zeta}}}{\overline{\zeta}}\Big)\log|\widetilde{\zeta}\overline{\zeta}-r^{2n}|^2|\widetilde{\zeta}|\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}=$$

$$=\mathrm{Re}\,\Bigg[\frac{1}{2\pi i}\int\limits_{\partial R}\widetilde{\zeta}\Big(\frac{\widetilde{\zeta}}{\zeta}+\frac{\overline{\widetilde{\zeta}}}{\overline{\zeta}}\Big)\Big(\log\Big||\widetilde{\zeta}|^2-\frac{r^{2n}\widetilde{\zeta}}{\zeta}\Big|^2+\log|\zeta|^2-\log|\widetilde{\zeta}|^2\Big)\times$$

$$\times\Big[\frac{\log|z|^2}{\log r^2}\frac{1}{\widetilde{\zeta}}-\frac{1}{\widetilde{\zeta}-z}-\frac{\overline{z}}{1-\overline{z}\widetilde{\zeta}}+$$

$$+\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\overline{z}}{r^{2k}\overline{z}\widetilde{\zeta}-1}+\frac{\overline{z}}{\overline{z}\widetilde{\zeta}-r^{2k}}-\frac{1}{\widetilde{\zeta}-r^{2k}z}-\frac{r^{2k}}{r^{2k}\widetilde{\zeta}-z}\Big)\Big]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\Bigg]=$$

$$= -2\mathrm{Re}\,\Bigg[(1-r^2)\frac{z}{\zeta}\log\Big(1-\frac{r^{2n}z}{\zeta}\Big)+\Big(\frac{r^2}{z\overline{\zeta}}+\frac{\overline{z}}{\overline{\zeta}}\Big)\log\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)+$$

$$+\frac{z}{\zeta}\log|\zeta|^2+\frac{r^2}{1-r^2}\Big(\frac{1}{z\overline{\zeta}}-\frac{\overline{z}}{\overline{\zeta}}\Big)\log r^2-$$

$$-(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{r^{2k}}{z\overline{\zeta}}\log\Big(1-\frac{r^{2(n+k)}}{z\overline{\zeta}}\Big)-\frac{r^{2k}z}{\zeta}\log\Big(1-\frac{r^{2(n+k)}z}{\zeta}\Big)\Big)\Bigg].$$

Then

$$2\mathrm{Re}\Big[(r^{2n}-|\zeta|^2)\frac{1}{\pi}\int_R\frac{1}{r^{2n}-\widetilde{\zeta}\overline{\zeta}}\,G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big]=(r^{2n}-|\zeta|^2)I_3(z,\zeta)=$$

$$=2(r^{2n}-|\zeta|^2)\mathrm{Re}\,\Bigg[\Big(\frac{\overline{z}}{\overline{\zeta}}-\frac{r^2}{z\overline{\zeta}}\Big)\log\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)-(1-r^2)\frac{z}{\zeta}\log\Big(1-\frac{r^{2n}z}{\zeta}\Big)+$$

$$+\frac{z}{\zeta}\log|z|^2+\frac{r^2}{1-r^2}\Big(\frac{\overline{z}}{\overline{\zeta}}-\frac{1}{z\overline{\zeta}}\Big)\log r^2+$$

$$+(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{r^{2k}}{z\overline{\zeta}}\log\Big(1-\frac{r^{2(n+k)}}{z\overline{\zeta}}\Big)-\frac{r^{2k}z}{\zeta}\log\Big(1-\frac{r^{2(n+k)}z}{\zeta}\Big)\Big)\Bigg]. \tag{4.2.7}$$

4. Now the integral

$$I_4(z,\zeta):=2\mathrm{Re}\Big[\frac{1}{\pi}\int_R\frac{1}{\widetilde{\zeta}-r^{2n}\zeta}\,G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big]=$$

$$=\frac{1}{\pi}\int_R\Big[\frac{1}{\widetilde{\zeta}-r^{2n}\zeta}+\frac{1}{\overline{\widetilde{\zeta}-r^{2n}\zeta}}\Big]G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta},$$

has to be found with $n\in\mathbb{N}$ fixed. Observing that

$$\partial_z\partial_{\overline{z}}\Big[(z+\overline{z})\log|z-r^{2n}\zeta|^2\Big]=\frac{1}{z-r^{2n}\zeta}+\frac{1}{\overline{z-r^{2n}\zeta}},$$

by formula (3.1.5) the equality

$$I_4(z,\zeta)=-(z+\overline{z})\log|z-r^{2n}\zeta|^2-\widehat{I}_4(z,\zeta),$$

is obtained, with

$$\widehat{I}_4(z,\zeta) := \frac{1}{4\pi i}\int\limits_{\partial R}(\widetilde{\zeta}+\overline{\widetilde{\zeta}})\log|\widetilde{\zeta}-r^{2n}\zeta|^2|\widetilde{\zeta}|\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} =$$

$$= \operatorname{Re}\left[\frac{1}{2\pi i}\int\limits_{\partial R}\widetilde{\zeta}(\widetilde{\zeta}+\overline{\widetilde{\zeta}})\Big(\log\Big||\widetilde{\zeta}|^2-r^{2n}\widetilde{\zeta}\overline{\zeta}\Big|^2-\log|\widetilde{\zeta}|^2\Big)\times\right.$$

$$\times\Big[\frac{\log|z|^2}{\log r^2}\frac{1}{\widetilde{\zeta}}-\frac{1}{\widetilde{\zeta}-z}-\frac{\overline{z}}{1-\overline{z}\widetilde{\zeta}}+$$

$$\left.+\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\overline{z}}{r^{2k}\overline{z}\widetilde{\zeta}-1}+\frac{\overline{z}}{\overline{z}\widetilde{\zeta}-r^{2k}}-\frac{1}{\widetilde{\zeta}-r^{2k}z}-\frac{r^{2k}}{r^{2k}\widetilde{\zeta}-z}\Big)\Big]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\right] =$$

$$= -2\operatorname{Re}\left[\Big(z+\frac{1}{\overline{z}}\Big)\log(1-r^{2n}z\overline{\zeta})+\Big(z+\frac{r^2}{\overline{z}}\Big)\log\Big(1-\frac{r^{2n}\zeta}{z}\Big)-\right.$$

$$-\Big(r^2z+\frac{1}{\overline{z}}\Big)\log(1-r^{2n}z\overline{\zeta})-\frac{r^2}{1-r^2}\Big(z-\frac{1}{\overline{z}}\Big)\log r^2-$$

$$\left.-(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{r^{2k}}{z}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big)-r^{2k}z\log(1-r^{2(n+k)}z\overline{\zeta})\Big)\right].$$

Then

$$2\operatorname{Re}\Big[(1-r^{2n})\zeta\frac{1}{\pi}\int\limits_R\frac{1}{\widetilde{\zeta}-r^{2n}\zeta}G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big] = -2(1-r^{2n})\operatorname{Re}\Big[z\overline{\zeta}\log|z|^2-$$

$$-(1-r^2)z\overline{\zeta}\log(1-r^{2n}z\overline{\zeta})+\Big(z\overline{\zeta}-\frac{r^2\overline{\zeta}}{\overline{z}}\Big)\log\Big(1-\frac{r^{2n}\zeta}{z}\Big)+\frac{r^2}{1-r^2}\Big(z\overline{\zeta}-\frac{\overline{\zeta}}{\overline{z}}\Big)\log r^2+$$

$$+(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\zeta}{z}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big)-r^{2k}z\overline{\zeta}\log(1-r^{2(n+k)}z\overline{\zeta})\Big)\Big]. \tag{4.2.8}$$

5. In the last step we evaluate

$$I_5(z,\zeta) := 2\operatorname{Re}\Big[\frac{1}{\pi}\int\limits_R\frac{1}{r^{2n}\widetilde{\zeta}-\zeta}G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta}\Big] =$$

$$= \frac{1}{\pi}\int\limits_R\Big[\frac{1}{r^{2n}\widetilde{\zeta}-\zeta}+\frac{1}{\overline{r^{2n}\widetilde{\zeta}-\zeta}}\Big]G_1(z,\widetilde{\zeta})d\widetilde{\xi}d\widetilde{\eta},$$

with $n \in \mathbb{N}$ fixed. As

$$\partial_z \partial_{\overline{z}} \left[\frac{1}{r^{2n}} (z + \overline{z}) \log |r^{2n} z - \zeta|^2 \right] = \frac{1}{r^{2n} z - \zeta} + \frac{1}{\overline{r^{2n} z - \zeta}},$$

using again the formula (3.1.5), one gets

$$I_5(z,\zeta) = -\frac{1}{r^{2n}} (z + \overline{z}) \log |r^{2n} z - \zeta|^2 - \frac{1}{r^{2n}} \widehat{I}_5(z,\zeta),$$

where

$$\widehat{I}_5(z,\zeta) := \frac{1}{4\pi i} \int\limits_{\partial R} (\widetilde{\zeta} + \overline{\widetilde{\zeta}}) \log |r^{2n} \widetilde{\zeta} - \zeta|^2 |\widetilde{\zeta}| \partial_{\nu_{\widetilde{\zeta}}} G_1(z, \widetilde{\zeta}) \frac{d\widetilde{\zeta}}{\widetilde{\zeta}} =$$

$$= \mathrm{Re} \left[\frac{1}{2\pi i} \int\limits_{\partial R} \widetilde{\zeta} (\widetilde{\zeta} + \overline{\widetilde{\zeta}}) \left(\log \left| 1 - \frac{r^{2n} \widetilde{\zeta}}{\zeta} \right|^2 + \log |\zeta|^2 \right) \times \right.$$

$$\times \Big[\frac{\log |z|^2}{\log r^2} \frac{1}{\widetilde{\zeta}} - \frac{1}{\widetilde{\zeta} - z} - \frac{\overline{z}}{1 - \overline{z} \widetilde{\zeta}} +$$

$$\left. + \sum_{k=1}^{\infty} \left(\frac{r^{2k} \overline{z}}{r^{2k} \overline{z} \widetilde{\zeta} - 1} + \frac{\overline{z}}{\overline{z} \widetilde{\zeta} - r^{2k}} - \frac{1}{\widetilde{\zeta} - r^{2k} z} - \frac{r^{2k}}{r^{2k} \widetilde{\zeta} - z} \right) \Big] \frac{d\widetilde{\zeta}}{\widetilde{\zeta}} \right] =$$

$$= -2\mathrm{Re} \left[z \log |\zeta|^2 + \left(z + \frac{1}{\overline{z}} \right) \log \left(1 - \frac{r^{2n} z}{\zeta} \right) + (1 - r^2) \frac{r^{2n}}{\zeta} \frac{\log |z|^2}{\log r^2} - \right.$$

$$\left. -(1 - r^2) \sum_{k=1}^{\infty} \left(\frac{z}{r^{2k}} \log \left(1 - \frac{r^{2(n+k)}}{z \overline{\zeta}} \right) - \frac{1}{r^{2k} z} \log \left(1 - \frac{r^{2(n+k)} z}{\zeta} \right) \right) \right].$$

Hence

$$2\mathrm{Re} \Big[(1 - r^{2n}) \zeta \frac{1}{\pi} \int\limits_R \frac{1}{r^{2n} \widetilde{\zeta} - \zeta} G_1(z, \widetilde{\zeta}) d\widetilde{\xi} d\widetilde{\eta} \Big] =$$

$$= 2 \frac{(1 - r^{2n})}{r^{2n}} \mathrm{Re} \left[\left(\frac{\overline{\zeta}}{\overline{z}} - z \overline{\zeta} \right) \log \left(1 - \frac{r^{2n} z}{\zeta} \right) + (1 - r^2) r^{2n} \frac{\log |z|^2}{\log r^2} - \right. \tag{4.2.9}$$

$$\left. -(1 - r^2) \sum_{k=1}^{\infty} \left(\frac{z \overline{\zeta}}{r^{2k}} \log \left(1 - \frac{r^{2(n+k)}}{z \overline{\zeta}} \right) - \frac{\zeta}{r^{2k} z} \log \left(1 - \frac{r^{2(n+k)} z}{\zeta} \right) \right) \right].$$

Combining the results of all evaluations (formulas (4.2.5)–(4.2.9)), one obtains

$$\widehat{h}_2(z,\zeta) = 2\text{Re}\ \Bigg[\frac{\log|\zeta|^2}{\log r^2}\Big(1-|z|^2+z\overline{\zeta}\log|z|^2-(1-r^2)\frac{\log|z|^2}{\log r^2}-$$

$$-\frac{r^2}{1-r^2}\log r^2\Big(\frac{\overline{\zeta}}{\overline{z}}-z\overline{\zeta}\Big)\Big)+$$

$$+(1-|\zeta|^2)\Big((1-r^2)\frac{\log|z|^2}{\log r^2}+\Big(\frac{1}{z\overline{\zeta}}-\frac{\overline{z}}{\overline{\zeta}}\Big)\log(1-z\overline{\zeta})-$$

$$-(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{z}{r^{2k}\zeta}\log\Big(1-\frac{r^{2k}\zeta}{z}\Big)-\frac{1}{r^{2k}z\overline{\zeta}}\log(1-r^{2k}z\overline{\zeta})\Big)\Big)-$$

$$-\sum_{n=1}^{\infty}\Big[\frac{(r^{2n}|\zeta|^2-1)}{r^{2n}}\Big((1-r^2)r^{2n}\frac{\log|z|^2}{\log|r|^2}+\Big(\frac{1}{z\overline{\zeta}}-\frac{\overline{z}}{\overline{\zeta}}\Big)\log(1-r^{2n}z\overline{\zeta})-$$

$$-(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{z}{r^{2k}\zeta}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big)-\frac{1}{r^{2k}z\overline{\zeta}}\log(1-r^{2(n+k)}z\overline{\zeta})\Big)\Big)+$$

$$+(r^{2n}-|\zeta|^2)\Big(\frac{z}{\zeta}\log|z|^2+\frac{r^2\log r^2}{1-r^2}\Big(\frac{\overline{z}}{\overline{\zeta}}-\frac{1}{z\overline{\zeta}}\Big)+\Big(\frac{\overline{z}}{\overline{\zeta}}-\frac{r^2}{z\overline{\zeta}}\Big)\log\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big)-$$

$$-(1-r^2)\frac{z}{\zeta}\log\Big(1-\frac{r^{2n}z}{\zeta}\Big)+$$

$$+(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{r^{2k}}{z\overline{\zeta}}\log\Big(1-\frac{r^{2(n+k)}}{z\overline{\zeta}}\Big)-\frac{r^{2k}z}{\zeta}\log\Big(1-\frac{r^{2(n+k)}z}{\zeta}\Big)\Big)\Big)+$$

$$+(1-r^{2n})\Big(z\overline{\zeta}\log|z|^2-\frac{r^2\log r^2}{1-r^2}\Big(\frac{\overline{\zeta}}{\overline{z}}-z\overline{\zeta}\Big)-$$

$$-(1-r^2)z\overline{\zeta}\log(1-r^{2n}z\overline{\zeta})+\Big(z\overline{\zeta}-\frac{r^2\overline{\zeta}}{\overline{z}}\Big)\log\Big(1-\frac{r^{2n}\zeta}{z}\Big)+$$

$$+(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{r^{2k}\zeta}{z}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big)-r^{2k}z\overline{\zeta}\log(1-r^{2(n+k)}z\overline{\zeta})\Big)\Big)-$$

$$-\frac{(1-r^{2n})}{r^{2n}}\Big((1-r^2)r^{2n}\frac{\log|z|^2}{\log r^2}+\Big(\frac{\overline{\zeta}}{\overline{z}}-z\overline{\zeta}\Big)\log\Big(1-\frac{r^{2n}z}{\zeta}\Big)-$$

$$-(1-r^2)\sum_{k=1}^{\infty}\Big(\frac{z\overline{\zeta}}{r^{2k}}\log\Big(1-\frac{r^{2(n+k)}}{z\overline{\zeta}}\Big)-\frac{\zeta}{r^{2k}z}\log\Big(1-\frac{r^{2(n+k)}z}{\zeta}\Big)\Big)\Big)\Big]\Bigg].$$

Simplifying the expression for $\widehat{h}_2(z,\zeta)$, using the representation (4.2.1) and formula (3.1.4), the following main formula for the biharmonic Green function of the circular ring domain R

$$
\begin{aligned}
&\widehat{G}_2(z,\zeta) = |\zeta - z|^2\times \\
&\times \left[\frac{\log|z|^2\log|\zeta|^2}{\log r^2} - \log\left|\frac{\zeta - z}{1 - z\overline{\zeta}}\prod_{k=1}^{\infty}\frac{(z - r^{2k}\zeta)(\zeta - r^{2k}z)}{(z\overline{\zeta} - r^{2k})(1 - r^{2k}\overline{z}\zeta)}\right|^2\right] + \\
&+ 2\mathrm{Re}\left[\frac{(1-|z|^2)\log|\zeta|^2}{\log r^2} + \frac{(1-|\zeta|^2)\log|z|^2}{\log r^2} - \right.\\
&- \frac{r^2(1-|z|^2)}{1-r^2}\frac{\zeta}{z}\log|\zeta|^2 - \frac{r^2(1-|\zeta|^2)}{1-r^2}\frac{z}{\zeta}\log|z|^2 + \\
&+ \frac{z\overline{\zeta}\log|z|^2\log|\zeta|^2}{\log r^2} - \frac{(1-r^2)\log|z|^2\log|\zeta|^2}{(\log r^2)^2} + \\
&+ \left(\frac{r^2}{1-r^2}\right)^2\frac{(1-|z|^2)(1-|\zeta|^2)\log r^2}{z\overline{\zeta}} + \\
&+ (1-|\zeta|^2)\Big[\frac{1-|z|^2}{z\overline{\zeta}}\log(1-z\overline{\zeta}) - \\
&- (1-r^2)\sum_{n=1}^{\infty}\Big(\frac{z}{r^{2n}\zeta}\log\Big(1-\frac{r^{2n}\zeta}{z}\Big) - \frac{1}{r^{2n}z\overline{\zeta}}\log(1-r^{2n}z\overline{\zeta})\Big)\Big] - \\
&- \sum_{n=1}^{\infty}\frac{(r^{2n}|\zeta|^2-1)}{r^{2n}}\frac{(1-|z|^2)}{z\overline{\zeta}}\log(1-r^{2n}z\overline{\zeta}) - \\
&- \sum_{n=1}^{\infty}(r^{2n}-|\zeta|^2)\left[\frac{|z|^2-r^2}{z\overline{\zeta}}\log\Big(1-\frac{r^{2n}}{z\overline{\zeta}}\Big) - (1-r^2)\frac{z}{\zeta}\log\Big(1-\frac{r^{2n}z}{\zeta}\Big)\right] - \\
&- \sum_{n=1}^{\infty}(1-r^{2n})\left[\frac{(|z|^2-r^2)\zeta}{z}\log\Big(1-\frac{r^{2n}\zeta}{z}\Big) - (1-r^2)z\overline{\zeta}\log(1-r^{2n}z\overline{\zeta})\right] - \\
&- \sum_{n=1}^{\infty}\frac{(1-r^{2n})}{r^{2n}}\frac{(1-|z|^2)\zeta}{z}\log\Big(1-\frac{r^{2n}z}{\zeta}\Big) + (1-r^2)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\times \\
&\times\left\{\frac{r^{2n}|\zeta|^2-1}{r^{2n}}\left[\frac{z}{r^{2k}\zeta}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big) - \frac{1}{r^{2k}z\overline{\zeta}}\log(1-r^{2(n+k)}z\overline{\zeta})\right] - \right.\\
&- (r^{2n}-|\zeta|^2)\left[\frac{r^{2k}}{z\overline{\zeta}}\log\Big(1-\frac{r^{2(n+k)}}{z\overline{\zeta}}\Big) - \frac{r^{2k}z}{\zeta}\log\Big(1-\frac{r^{2(n+k)}z}{\zeta}\Big)\right] - \\
&- (1-r^{2n})\left[\frac{r^{2k}\zeta}{z}\log\Big(1-\frac{r^{2(n+k)}\zeta}{z}\Big) - r^{2k}z\overline{\zeta}\log(1-r^{2(n+k)}z\overline{\zeta})\right] + \\
&\left. + \frac{1-r^{2n}}{r^{2n}}\left[(\frac{z\overline{\zeta}}{r^{2k}}\log\Big(1-\frac{r^{2(n+k)}}{z\overline{\zeta}}\Big) - \frac{\zeta}{r^{2k}z}\log\Big(1-\frac{r^{2(n+k)}z}{\zeta}\Big)\right]\right]\right\}
\end{aligned}
\tag{4.2.10}
$$

is obtained.

Remark 4.2.1. All series in (4.2.10) are convergent because of the properties of the logarithmic function.

In the same way as it is with the harmonic Green function, the biharmonic Green function provides a representation formula for a proper class of functions. Namely, the following result holds.

Theorem 4.2.1. [21] *Let $D \subset \mathbb{C}$ be a regular domain. Any $w \in C^4(D;\mathbb{C}) \cap C^3(\overline{D};\mathbb{C})$ can be represented as*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial D} \partial_{\nu_\zeta} G_1(z,\zeta) w(\zeta) ds_\zeta - \frac{1}{4\pi}\int\limits_{\partial D} \partial_{\nu_\zeta} \widehat{G}_2(z,\zeta) \partial_\zeta \partial_{\overline{\zeta}} w(\zeta) ds_\zeta -$$

$$-\frac{1}{\pi}\int\limits_{D} \widehat{G}_2(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^2 w(\zeta) d\xi d\eta,$$

with $G_1(z,\zeta)$, $\widehat{G}_2(z,\zeta)$ being the harmonic and the biharmonic Green functions of the domain D, correspondingly.

The respective result for the circular ring domain R is given below.

Corollary 4.2.1. *Any $w \in C^4(R;\mathbb{C}) \cap C^3(\overline{R};\mathbb{C})$ can be represented by*

$$w(z) = -\frac{1}{4\pi i}\int\limits_{\partial R} |\zeta| \partial_{\nu_\zeta} G_1(z,\zeta) w(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{4\pi i}\int\limits_{\partial R} |\zeta| \partial_{\nu_\zeta} \widehat{G}_2(z,\zeta) \partial_\zeta \partial_{\overline{\zeta}} w(\zeta) \frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{R} \widehat{G}_2(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^2 w(\zeta) d\xi d\eta,$$

where $G_1(z,\zeta)$ is given in (3.1.4) *and $\widehat{G}_2(z,\zeta)$ is expressed by* (4.2.10).

These integral representation formulas provides a solution of the Dirichlet problem for the bi-Poisson equation in corresponding domains.

Theorem 4.2.2. [21] *Let $D \subset \mathbb{C}$ be a regular domain. The solution of the Dirichlet problem to the bi-Poisson equation in D*

$$(\partial_z \partial_{\overline{z}})^2 w = f \ \text{ in } D, \ \ w = \gamma_0, \ \partial_z \partial_{\overline{z}} w = \gamma_1 \ \text{ on } \partial D,$$

where $f \in L_p(D;\mathbb{C})$, $p > 2$, $\gamma_0, \gamma_1 \in C(\partial D;\mathbb{C})$ are given, is unique and represented by

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial D} \partial_{\nu_\zeta} G_1(z,\zeta)\gamma_0(\zeta)ds_\zeta - \frac{1}{4\pi}\int\limits_{\partial D} \partial_{\nu_\zeta} \widehat{G}_2(z,\zeta)\gamma_1(\zeta)ds_\zeta -$$

$$-\frac{1}{\pi}\int\limits_{D} \widehat{G}_2(z,\zeta)f(\zeta)d\xi d\eta.$$

Corollary 4.2.2. *The solution of the Dirichlet problem to the bi-Poisson equation in* R

$$(\partial_z\partial_{\overline{z}})^2 w = f \ \text{ in } R, \ \ w = \gamma_0, \ \partial_z\partial_{\overline{z}} w = \gamma_1 \ \text{ on } \partial R,$$

where $f \in L_p(D;\mathbb{C})$, $p > 2$, $\gamma_0, \gamma_1 \in C(\partial D;\mathbb{C})$ *are given, is unique and expressed by*

$$w(z) = -\frac{1}{4\pi i}\int\limits_{\partial R} |\zeta|\partial_{\nu_\zeta} G_1(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{4\pi i}\int\limits_{\partial R} |\zeta|\partial_{\nu_\zeta} \widehat{G}_2(z,\zeta)\gamma_1(\zeta)\frac{d\zeta}{\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{R} \widehat{G}_2(z,\zeta)f(\zeta)d\xi d\eta.$$

Remark 4.2.2. It should be noted that the obtained biharmonic Green function for the ring domain R contains only elementary functions. It differs from the Green function constructed in [31] for the bi-Laplace operator with particular Dirichlet boundary condition in terms of new transcendental functions, which generalize the Weierstrass ζ-function.

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Zusammenfassung

Die Theorie von komplexen Randwertproblemen ist wegen ihrer Bedeutung für ebene Probleme in Physik und Technik vor allem in einfach zusammenhängenden Gebieten sehr eingehend untersucht. Vor allem sind es das Riemannsche Kopplungsproblem, wo zu vorgegebener Randfunktion Funktionen bestimmter Klassen – im einfachsten Fall analytische Funktionen – zu finden sind, die auf dem Rand den vorgegebenen Sprung aufweisen, und das Riemann-Hilbertsche Randwertproblem, dessen einfachster Fall – das Schwarz Problem – darin besteht, zu am Rand vorgegebenem Realteil eine analytische Funktion zu finden, deren Realteil am Rand diese Werte erreicht.

In dieser Arbeit werden Randwertprobleme in einem speziellen zweifach zusammenhängenden Gebiet, einem konzentrischen Kreisring, untersucht. Zunächst werden vier grundlegende Randwertprobleme für analytische Funktionen auf Grundlage einer modifizierten Cauchyschen Formel explizit gelöst, das Schwarzsche, das Dirichletsche, das Neumannsche und ein Robinsches Randwertproblem. Neben den Lösungen werden Lösbarkeitsbedingungen angegeben, die auch sichern, dass die Lösungen einwertig sind, was für mehrfach zusammenhängende Gebiete im Allgemeinen nicht zutrifft. Anschliessend werden inhomogene Cauchy-Riemannsche Gleichungen untersucht und die Ergebnisse auf diese erweitert. Schliesslich werden im Hinblick auf die Poissonsche Gleichung harmonische Green, Neumann und Robin Funktionen explizit für den Ring konstruiert, von denen nur die Greenfunktion in modifizierter Form in der Literatur bekannt ist. Diese Fundamentallösungen dienen dazu, die entsprechenden Randwertprobleme für die Poisson Gleichung im Ring zu lösen und gegebenenfalls Lösbarkeitsbedingungen zu finden. Ein Dirichlet und das Schwarz Problem werden auch für die Bitsadze Gleichung behandelt. Dies ist eine weitere Modellgleichung zweiter Ordnung, im Gegensatz zum Laplace Operator ist der Bitsadze Operator aber nur schwach elliptisch, was Konsequenzen für Randwertprobleme hat.

Als Letztes wird eine biharmonische Greensche Funktion explizit konstruiert, die sich als Faltung der harmonischen Greenschen Funktion mit sich selbst ergibt. Diese Funktion ist eine weitere biharmonische Greenfunktion, die sich durch ihr Randverhalten von einer kürzlich für den Ring konstruierten anderen biharmonischen Greenschen Funktion unterscheidet.