Appendix A

Some inequalities

Young's Inequality *Suppose that* $1 < p, q < +\infty$ *and* $1/p + 1/q = 1$. *Then*

$$|ab| \leq \frac{1}{p}\epsilon^p |a|^p + \frac{1}{q}\epsilon^{-q}|b|^q, \quad \forall a, b \in \mathbb{R}, \forall \epsilon > 0. \tag{A.1}$$

Hölder's Inequality *Suppose that* $1 < p, q < +\infty$ *and* $1/p + 1/q = 1$. *Then*

$$|xy| \leq \|x\|_p \|y\|_q, \quad \forall x, y \in \mathbb{R}^n. \tag{A.2}$$

Gronwall's Lemma *Let* $c \in L^\infty(0,T)$ *and* $a \in L^1(0,T)$ *denote non-negative functions. If a function* $u \in L^\infty(0,T)$ *satisfies*

$$0 \leq u(t) \leq c(t) + \int_0^t a(s)u(s)ds, \quad \textit{a.e. in } (0,T), \tag{A.3}$$

then

$$0 \leq u(t) \leq c(t) + \int_0^t c(c)a(s)\left(\int_s^t a(\tau)d\tau\right)ds, \quad \textit{a.e. in } (0,T), \tag{A.4}$$

In particular, if $c(t) = c$ *and* $a(t) = a$ *for almost every* $t \in (0,T)$, *then*

$$0 \leq u(t) \leq c\exp(at), \quad \textit{a.e. in } (0,T). \tag{A.5}$$

Embedding Theorems *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$, *with a piecewise smooth boundary, i.e.,* $\partial\Omega \in C^{0,1}$. *Assume* $u \in W^{k,p}(\Omega)$. *(i) If*

$$k < \frac{n}{p}, \tag{A.6}$$

then $u \in L^q(\Omega)$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}. \tag{A.7}$$

We have in addition the estimate

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)}, \tag{A.8}$$

the constant C depending only on k, p, n and Ω. (ii) If

$$k > \frac{n}{p}, \tag{A.9}$$

then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{\Omega})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer },\\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases} \tag{A.10}$$

We have in addition the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{\Omega})} \le C\|u\|_{W^{k,p}(\Omega)}, \tag{A.11}$$

the constant C depending only on k, p, n, γ and Ω.
Rellich-Kondrachov Compactness Theorem *Assume Ω is a bounded open subset of $\mathbb{R}^n$, and $\partial\Omega$ is C^1. Suppose $1 \le p < n$. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega), \tag{A.12}$$

for each $1 \le q < p^*$.
Generalized Poincaré Inequality *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a piecewise smooth boundary, i.e., $\partial\Omega \in C^{0,1}$. Then there exists a constant c_p depending only on Ω such that*

$$\|u\|_{L^2(\Omega)} \le c_p(\Omega)\left\{\|\nabla u\|_{L^2(\Omega)} + \left|\int_\Omega u(x)dx\right|\right\}, \quad \forall u \in H^1(\Omega). \tag{A.13}$$

Gagliardo-Nirenberg Inequality *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^m and let $u \in W^{m,r}(\Omega) \cap L^p(\Omega)$ where $1 \le r, q \le \infty$. For any integer $j, 0 \le j < m$ and any $j/m \le \vartheta \le 1$ we have*

$$\|D^j u\|_{0,p} \le C_g \|u\|_{m,r}^{\vartheta} \|u\|_{0,q}^{1-\vartheta}, \tag{A.14}$$

provided that

$$\frac{1}{p} = \frac{j}{n} + \vartheta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\vartheta)\frac{1}{q}, \tag{A.15}$$

and $m - j - n/r$ is not a nonnegative integer. If $m - j - n/r$ is a nonnegative integer (A.14) holds with $\vartheta = j/m$.