Chapter 6

Global behaviour

In this chapter we study the global behaviour of the solution (2.9)-(2.11) for $T \to \infty$. Our main tool is the energy estimate. Because of **Theorem 3** and **(A2)** $f'(u) + w + \psi =: v \in L^\infty(0,T; L^\infty(\Omega))$, thus an admissible testfunction in (2.9). We get

$$\frac{\gamma}{2} \sup_{t \geq 0} \int\limits_\Omega |\nabla \psi(t)|^2 dx + \int\limits_0^\infty \int\limits_\Omega |\psi|^2 dxdt + \int\limits_0^\infty \int\limits_\Omega \mu(u) |\nabla v|^2 dxdt \leq C_{55} < \infty. \tag{6.1}$$

Theorem 4 *Let (u, w, ψ) be a solution of (2.9)-(2.11). Then there exist a sequence $\{t_k : k = 1, 2, \cdots\}$ with $t_k \to \infty$ for $k \to \infty$ and a triplet (u^*, w^*, ψ^*) such that $u_k = u(t_k), w_k = w(t_k), \psi_k = \psi(t_k)$ satisfy*

$$\begin{array}{llll} u_k & \to & u^* & \textit{strongly in } L^2 \textit{ and weakly in } H^1, \\ w_k & \to & w^* & \textit{strongly in } H^1, \\ \psi_k & \to & 0 & \textit{strongly in } L^2, \\ \nabla\psi_k & \to & \nabla\psi^* & \textit{strongly in } L^2, \end{array} \tag{6.2}$$

and

$$\arctan(e^{-v_k/2}) \to \arctan(e^{-v^*/2}) \textit{ strongly in } H^1, v^* = \textit{ const. } . \tag{6.3}$$

Moreover, the following relations hold:

$$w^* = \int\limits_\Omega \mathcal{K}(|x-y|)(1 - 2u^*(y))dy, \quad \bar{u}^* = \bar{u}_0, \tag{6.4}$$

$$u^* = \frac{1}{1 + \exp(w^* - v^*)}, \quad v^* = \textit{const.} \tag{6.5}$$

Proof. 1. Because of (6.1) there exists a sequence $t_j \in [j, j+1]$, $j = 1, 2, \cdots$, such that $u_j = u(t_j), w_j = w(t_j), \psi_j = \psi(t_j)$ and $v_j = v(t_j)$ satisfy

$$\lim_{j \to \infty} \int\limits_\Omega \mu(u_j) |\nabla v_j|^2 dx = 0, \tag{6.6}$$

and

$$\lim_{j\to\infty} \int_\Omega |\psi_j|^2 dx = 0.$$

From (6.1) we get

$$\int_\Omega |\nabla \psi_j|^2 dx \le C_{56}. \tag{6.7}$$

The chemical potential $v = f'(u) + w + \psi$ gives $\nabla v = f''(u)\nabla u + \nabla(w+\psi)$ and consequently

$$\nabla u = \frac{\nabla v - \nabla(w+\psi)}{f''(u)} \quad \text{and } \mu(u) = u(1-u) \le \frac{1}{4} \text{ for } 0 \le u \le 1.$$

We get by **(A2)**, **(B4')**, (6.6) and (6.7)

$$\begin{aligned}
\int_\Omega |\nabla u_j|^2 dx &= \int_\Omega \left| \frac{\nabla v_j - \nabla(w_j + \psi_j)}{f''(u_j)} \right|^2 dx \le 2 \int_\Omega \frac{|\nabla v_j|^2 + |\nabla(w_j + \psi_j)|^2}{(f''(u_j))^2} dx \\
&\le \frac{1}{2} \int_\Omega \frac{|\nabla v_j|^2}{f''(u_j)} dx + \frac{1}{4} \int_\Omega |\nabla w_j|^2 dx + \frac{1}{4} \int_\Omega |\nabla \psi_j|^2 dx \\
&\le \varepsilon + \frac{r_2^2 |\Omega|}{4} + C_{56} = C_{57}.
\end{aligned}$$

Moreover we know from **Theorem 2** that $0 \le u_j \le 1$, so that the generalized Poincaré inequality (A.13) gives

$$\|u_j\|_{H^1(\Omega)} \le C_{58}.$$

Hence by the compactness of the embedding $H^1 \subset L^2$ (A.12), there exists a subsequence $\{t_k\} \subset \{t_j\}$ such that (6.2) holds. By assumption **(A2)**, **(B4')** and **Theorem 1** it follows the estimate

$$\|w_k\|_{L^\infty(\Omega)} \le C_{59}.$$

Furthermore, we get from (6.6)

$$\int_\Omega |\nabla \arctan(e^{-v_k/2})|^2 dx = \frac{1}{4} \int_\Omega \frac{|\nabla v_k|^2 e^{-v_k}}{(1+e^{-v_k})^2} dx \le C \int_\Omega \frac{|\nabla v_k|^2}{f''(u_k)} dx \longrightarrow 0.$$

Finally, (6.2)-(6.3) together with (2.6),(2.7),(2.8) give (6.4)-(6.5).