# Chapter 5

# An additional regularity result

In this chapter we will prove a separation result for $u$ in **Theorem 2**. For this we will use the Moser iteration technique in the form of Alikakos [3] to establish the separation result. The key point is a proper choice of testfunctions.

**Theorem 3** *Suppose $f'(u_0) \in L^\infty(\Omega)$. Then $f'(u(t)) \in L^\infty(\Omega)$ for a.a. $t \in [0,T]$.*

**Remark 9** We get from **Theorem 3** that $0 < u(t,x) < 1$ for a.a. $(t,x) \in Q_T$, provided $0 < u_0(x) < 1$ for a.a. $x \in \Omega$.

*Proof.* We use here ideas close to [3] and [15]. Denote by

$$z := f'(u) = v - (w + \psi).$$

We introduce

$$\sigma(z) := u = \frac{1}{1 + \exp(-z)},$$

and have

$$\sigma'(z) = u(1-u) = \frac{\exp(-z)}{1+\exp(-z))^2} = \frac{1}{f''(u)},$$

$$\sigma''(z) = \frac{(\exp(-z) - 1)\exp(-z)}{(1+\exp(-z))^3}.$$

Becauce of **(A2)**, **(B4')** we have

$$\sigma''(z) \le 0 \quad \text{if } z \ge 0 \tag{5.1}$$

$$\sigma''(z) \ge 0 \quad \text{if } z \le 0 \tag{5.2}$$

Using (5.1) and testing (2.9) with (see [15])

$$\varphi = \frac{z_+^{2^k-1}}{\sigma'(z)}, \quad k \ge 1, \quad z_+ = \max(0,z),$$

and taking into account

$$\nabla\varphi = \frac{(2^k-1)z_+^{2^k}\nabla z}{\sigma'} - \frac{\sigma''}{\sigma'^{\,2}}\nabla z z_+^r = \frac{1}{\sigma'}\left\{(2^k-1)z_+^{2^k-2}\nabla z - \sigma''\varphi\nabla z\right\},$$
$$\frac{\partial\sigma(z)}{\partial t}\varphi = z_t z_+^{2^k-1} = \frac{1}{2^k}\frac{d}{dt}z_+^{2^k},$$

we get

$$\frac{1}{2^k}\frac{d}{dt}\int\limits_{\{z\geq 0\}} z_+^{2^k}dx + \int\limits_{\{z\geq 0\}} \nabla v\cdot\{(2^k-1)z_+^{2^k-2}\nabla z - \sigma''\varphi\nabla z\}dx = 0. \tag{5.3}$$

We expand the integrand of the second integral in the form

$$\begin{aligned} S =&[\nabla z + \nabla(w+\psi)]\cdot\left\{(2^k-1)z_+^{2^k-2}\nabla z - \sigma''\varphi\nabla z\right\} \\ =&(2^k-1)z_+^{2^k-2}\left\{|\nabla z|^2 + \nabla(w+\psi)\cdot\nabla z\right\} - \sigma''(z)\varphi\left\{|\nabla z|^2 + \nabla(w+\psi)\cdot\nabla z\right\}. \end{aligned}$$

Because of (5.1) we can estimate using Young's inequality

$$\begin{aligned} S \geq&(2^k-1)z_+^{2^k-2}\left\{|\nabla z|^2 - \frac{1}{2}(|\nabla(w+\psi)|^2 + |\nabla z|^2)\right\} \\ &- \sigma''(z)\varphi\left\{|\nabla z|^2 - \frac{1}{2}(k|\nabla(w+\psi)|^2 + \frac{1}{k}|\nabla z|^2)\right\}. \end{aligned}$$

We find with the choise $k = 1/2$

$$\begin{aligned} S \geq&\frac{1}{2}(2^k-1)z_+^{2^k-2}|\nabla z|^2 - \frac{1}{2}(2^k-1)z_+^{2^k-2}|\nabla(w+\psi)|^2 \\ &+ \frac{1}{4}\frac{\sigma''(z)}{\sigma'(z)}z_+^{2^k-1}|\nabla(w+\psi)|^2. \end{aligned}$$

Because of

$$-1 \leq \frac{\sigma''(z)}{\sigma'(z)} \leq 1,$$

we obtain

$$\begin{aligned} S \geq&\frac{1}{2}(2^k-1)z_+^{2^k-2}|\nabla z|^2 - \frac{1}{2}(2^k-1)z_+^{2^k-2}|\nabla(w+\psi)|^2 \\ &- \frac{1}{4}z_+^{2^k-1}|\nabla(w+\psi)|^2. \end{aligned}$$

Because of assumption **(A2)**, **(B4')** and (4.47) we obtain

$$S \geq \frac{1}{2}(2^k-1)z_+^{2^k-2}|\nabla z|^2 - \frac{C_{54}}{2}(2^k-1)z_+^{2^k-2} - \frac{C_{54}}{4}z_+^{2^k-1}.$$

Taking into account

$$z_+^{2^k-2}|\nabla z_+|^2 = \frac{4|\nabla(z_+^{2^{k-1}})|^2}{(2^k)^2},$$

we finally get from the identity (5.3)

$$\begin{aligned}\frac{1}{2^k}\frac{d}{dt}\int\limits_\Omega z_+^{2^k}dx \le -\frac{2(2^k-1)}{(2^k)^2}\int\limits_\Omega |\nabla(z_+^{2^{k-1}})|^2dx \\ \frac{C_{54}}{4}\int\limits_\Omega \{2(2^k-1)z_+^{2^k-2} + z_+^{2^k-1}\}dx.\end{aligned} \tag{5.4}$$

For $k=1$ we obtain from (5.4), the embedding $L^2 \subset L^1$ and by integration with respect to $t$

$$\frac{1}{2}\int\limits_\Omega z_+(t)^2dx + \frac{1}{2}\int\limits_\Omega |\nabla z_+(t)|^2dx \le \frac{1}{2}\int\limits_\Omega z_+(0)^2dx + \frac{C_{54}}{4}\left\{2|Q_t| + \int\limits_0^t\int\limits_\Omega z_+^2dx\right\}.$$

Recalling that $z_+(0) = \max(0, f'(u_0)) \in L^\infty(\Omega)$ we conclude from Gronwall's Lemma (A.4)

$$\|z_+\|_{L^\infty(0,T;L^2)} \le K,$$

where $K$ is a positive constant, which depends on $T$. Consequently we find

$$\|z_+\|_{L^\infty(0,T;L^1)} \le K. \tag{5.5}$$

We use the Moser iteration technique in the form of Alikakos ([3]). By Young's inequality we have

$$2(2^k-1)z_+^{2^k-2} + z_+^{2^k-1} \le \frac{2^k - 1(2(2^k-1)-1)}{2^k}z_+^{2^k} + \frac{4(2^k-1)+1}{2^k},$$

which transforms (5.4) into

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega z_+^{2^k}dx \le -\frac{2(2^k-1)}{(2^k)}\int\limits_\Omega |\nabla z_+^{2^{k-1}}|^2dx \\ \frac{C_{54}}{8}\int\limits_\Omega \{2^k - 1(2(2^k-1)-1)z_+^{2^k} + 4(2^k-1)+1\}dx.\end{aligned}$$

Take

$$z_+^* = z_+^{2^{k-1}}, \qquad k \ge 1.$$

Using $4(2^k-1)+1<8\cdot 2^k$ and $(2^k-1)(2(2^k-1)-1)<8\cdot 4^k$ it follows that

$$\frac{d}{dt}\int\limits_\Omega (z_+^*)^2 dx \quad \le \quad -\nu_k \int\limits_\Omega |\nabla(z_+^*)|^2 dx + a_k \int\limits_\Omega (z_+^*)^2 dx + b_k, \tag{5.6}$$

where we have used the natation

$$\nu_k := \frac{2(2^k-1)}{2^k}, \quad a_k := 4^k C_{54}, \quad b_k := 2^k C_{54}|\Omega|.$$

Next, recall the Gagliardo Nirenberg interpolation inequality (A.14) a speciall case of which is the inequality

$$\|\xi\|_{L^2} \le C\|\xi\|_{W^{1,2}}^{\theta}\|\xi\|_{L^1}^{1-\theta}, \qquad \theta = \frac{n}{n+2},$$

and from which we obtain with the help of Young's inequality

$$\|\xi\|_{L^2} \le \epsilon\|\xi\|_{W^{1,2}}^2 + C\epsilon^{-n/2}\|\xi\|_{L^1}^2,$$

whith an appropriate constant $C$ not depending on $\epsilon$.
Choosing $0<\epsilon<1/2$ it follows that

$$\|\xi\|_{L^2} \le \epsilon\|\nabla\xi\|_{L^2}^2 + C_\epsilon\|\xi\|_{L^1}^2, \qquad \text{where } C_\epsilon = C\epsilon^{-n/2}.$$

We take this inequality for $\xi = z_+^*$ and $\epsilon = \epsilon_k$ and we obtain after multiplying each side by $(a_k+\epsilon_k)$ and rearranging

$$\begin{aligned} -(a_k+\epsilon_k)\int\limits_\Omega (z_+^*)^2 dx + (a_k+\epsilon_k)\, C_{\epsilon_k}\left(\int\limits_\Omega z_+^* dx\right)^2 \\ \ge -(a_k+\epsilon_k)\,\epsilon_k \int\limits_\Omega |\nabla z_+^*|^2 dx \quad . \end{aligned} \tag{5.7}$$

Now choosing $\epsilon_k > 0$ so that

$$\epsilon_k(a_k+\epsilon_k) \le \nu_k, \tag{5.8}$$

we obtain from (5.7) and (5.8) that

$$\frac{d}{dt}\int\limits_\Omega (z_+^*)^2 dx \le -\epsilon_k \int\limits_\Omega (z_+^*)^2 dx + (a_k+\epsilon_k)\, C_{\epsilon_k}\left(\int\limits_\Omega z_+^* dx\right)^2 + b_k.$$

Obviously we can assume $(a_k+\epsilon_k)\, C_{\epsilon_k} \ge 1$. This is a differential inequality of the structure

$$\frac{dU}{dt} \le -\epsilon_k U + A_k + b_k, \qquad A_k = (a_k+\epsilon_k)\, C_{\epsilon_k}\left(\int\limits_\Omega z_+^* dx\right)^2,$$

which gives for $t \geq 0$ by variation of constants

$$U(t) \leq \exp(-\epsilon_k t)U_0 + \int_0^t e^{\epsilon_k(\tau - t)}(A_k + b_k)d\tau$$
$$\leq \left(U_0 + \frac{b_k}{\varepsilon_k}e^{\varepsilon_k t}\right) e^{-\epsilon_k t} + \frac{A_k}{\epsilon_k}(1 - e^{-\varepsilon_k t}),$$

and from which follows

$$U(t) \leq \max\left(U_0 + \frac{b_k}{\varepsilon_k}e^{\varepsilon_k T}, \frac{A_k}{\epsilon_k}\right), \tag{5.9}$$

with the initial $U_0 = U(0)$. We can estimate

$$U_0 = \int_\Omega z_+(0)^{2^k} dx \leq \|z_+(0)\|_{L^\infty}^{2^k} |\Omega|.$$

We choose $\epsilon_k$ of the order $1/4^k$ and obtain from (5.9)

$$\int_\Omega z_+^{2^k} dx \leq \max\left\{\frac{(a_k + \epsilon_k)\, C_{\epsilon_k}}{\epsilon_k}\left(\sup \int_\Omega z_+^{2^{k-1}} dx\right)^2, |\Omega|(\|z_+(0)\|_{L^\infty}^{2^k} + 8^k C_{54})\right\} \tag{5.10}$$

We continue the argument by ([3]). Without loss of generality we may assume that for all $k = 1, 2, ..$ $\frac{(a_k+\epsilon_k)\, C_{\epsilon_k}}{\epsilon_k} \geq 1$, this can be done by choosing $a_k$. We furthermore assume that

$$\|z_+(0)\|_{L^\infty} \leq K$$

where $K$ is defined by (5.5) We obtain from (5.10) the inequality

$$\int_\Omega z_+^{2^k} dx \leq \left(\frac{(a_k + \epsilon_k)\, C_{\epsilon_k}}{\epsilon_k}\right)^{2^0} \left(\frac{(a_{k-1} + \epsilon_{k-1})\, C_{\epsilon_{k-1}}}{\epsilon_{k-1}}\right)^{2^1} ... \tag{5.11}$$

$$\left(\frac{(a_1 + \epsilon_1)\, C_{\epsilon_1}}{\epsilon_1}\right)^{2^{k-1}} K^{2^k} \tag{5.12}$$

Due to [3] the right hand side of (5.10) behaves like (const.)$^{2^k}$. By taking the $1/2^k$ power of both sides we can pass to the limit and obtain the $L^\infty$ estimate for $z_+$. Analogously, from (5.2) we get an $L^\infty$ estimate for $z_-$ by using the testfunction

$$\varphi = \frac{z_-^{2^k-1}}{\sigma'(z)}, \quad k \geq 1, \quad z_- = -\min(0, z).$$