Chapter 4

Proof of Theorem 2

In this chapter we prove the existence and uniqueness for a solution to the problem in **Theorem 2**.

4.1 Existence

Unlike the proof of **Theorem 1** we here will not only apply the regularization and truncation (3.1) and (3.3), but also we will use a *biharmonic regularization* of the ψ-equation.

4.1.1 Regularized problems

For the system (2.9)-(2.11) we consider for $(\varepsilon, \delta > 0)$ the regularized system

$$\int_0^T\int_\Omega u_t\varphi dxdt + \int_0^T\int_\Omega (\nabla u + \mu_\varepsilon \nabla(w+\psi))\cdot\nabla\varphi dxdt = 0, \quad \forall\varphi\in L^2(0,T;H^2_\bullet(\Omega)), \tag{4.1}$$

$$\delta\int_0^T\int_\Omega \Delta\psi\Delta\varphi dxdt + \gamma\int_0^T\int_\Omega \nabla\psi_t\cdot\nabla\varphi dt + \int_0^T\int_\Omega \psi\varphi dxdt = \int_0^T\int_\Omega u_t\varphi dxdt, \tag{4.2}$$

$$w = P(1-2\Pi u) \text{ a.e. } (t,x)\in Q_T. \tag{4.3}$$

where $H^2_\bullet(\Omega) := \{\varphi\in H^2(\Omega)| \quad \nu\cdot\nabla\varphi = 0 \text{ on } \partial\Omega\}$ a dense subset of $H^2(\Omega)$, so that the choice of the testfunction space is consistent if we take $\delta\searrow 0$.

Existence result for the regularized problem Like in chapter 3 we employ a semidiscrete scheme to (4.1)-(4.3). To this end, let $M\in N$ be given and $h := T/M$. For $1\le m\le M$, we consider the semidiscrete problem on the time level $t := mh$ for the unknown functions

$u^m, w^m, \psi^m : \Omega \to \mathbb{R}$ given by

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\varphi dx + \int_\Omega \left[\nabla u^m + \mu_\varepsilon(u^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right]\cdot\nabla\varphi dx = 0, \quad \forall\varphi \in H^2_\bullet(\Omega), \tag{4.4}$$

$$\delta\int_\Omega \Delta\psi^m\Delta\varphi dx + \frac{\gamma}{h}\int_\Omega \nabla(\psi^m - \psi^{m-1})\cdot\nabla\varphi dx + \int_\Omega \psi^m\varphi dx = \frac{1}{h}\int_\Omega (u^m - u^{m-1})\varphi dx, \quad \forall\varphi \in H^2_\bullet(\Omega), \tag{4.5}$$

$$w^m = P(1 - 2\Pi u^m) \text{ a.e. } x \in \Omega. \tag{4.6}$$

For $1 \le m \le M$ (4.4)-(4.6) is a nonlinear elliptic system. Note that $u^0 = u_0, \psi^0 = \psi_0$. We prove existence via Schauder's fixed-point principle.

Lemma 6 *Suppose that the assumptions* **(A1)** *to* **(A4)** *and* **(B1')** *to* **(B4')** *hold. Then for every* $m \in \{1, ..., M\}$ *there exists a triple of functions* $(u^m, w^m, \psi^m) \in H^2(\Omega) \times H^{2,\infty}(\Omega) \times H^4(\Omega)$ *satifying (4.4)-(4.6).*

Proof. 1. Again our proof is based on the application of Schauder's fixed-point principle. Here we only give the new steps, which differ from the proof of **Lemma 1**. The other steps remain true. Let $m \in \{1, ..., M\}$ be fixed but arbitrary. For a given $u^m \in H^1$ we consider the *auxiliary linear problems*

$$\int_\Omega \nabla(\mathcal{T}_m u^m)\cdot\nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_m u^m\varphi dx = \int_\Omega g_2\varphi dx, \quad \forall\varphi \in H^2_\bullet(\Omega), \tag{4.7}$$

$$\int_\Omega \Delta\psi^m\Delta\varphi dx + \frac{\gamma}{\delta h}\int_\Omega \nabla\psi^m\cdot\nabla\varphi dx + \frac{1}{\delta}\int_\Omega \psi^m\varphi dx = \int_\Omega g_1\varphi dx, \quad \forall\varphi \in H^2_\bullet(\Omega), \tag{4.8}$$

where

$$\int_\Omega g_1\varphi dx := \frac{1}{\delta h}\int_\Omega (u^m - u^{m-1})\varphi dx + \frac{\gamma}{\delta h}\int_\Omega \nabla\psi^{m-1}\cdot\nabla\varphi dx, \quad \forall\varphi \in H^2_\bullet(\Omega), \tag{4.9}$$

$$\int_\Omega g_2\varphi dx := \int_\Omega \mu'_\varepsilon(u^m)\nabla u^m\cdot\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\varphi dx + \int_\Omega \mu_\varepsilon(u^m)\Delta\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_{m-1}u^{m-1}\varphi dx, \quad \forall\varphi \in H^2_\bullet(\Omega). \tag{4.10}$$

Here the existence and uniqueness theory of (4.7)-(4.10) is take from [20]. The strategy is to get coercive biliearforms and use the Lax-Milgram Theorem. From ([20], Lemma 7.1.1 and Theorem 7.1.2), respectively, we find that for a given $u^m \in H^1$ and consequently a given $g_1 \in H^1(\Omega)$ in (4.9) the linear equation (4.8) admits a unique solution $\psi^m \in H^4(\Omega)$. Setting $w^m = P(1 - 2\Pi u^m) \in H^{2,\infty}(\Omega)$, we find (4.6). Finally again from ([20], Corollary 2.2.2.4), respectively, we conclude that for a given $g_2 \in L^2(\Omega)$ (4.7) admits a unique solution $\mathcal{T}_m u^m \in H^2(\Omega)$.

2. Thus, we have properly defined a fixed-point operator $\mathcal{T}_m : H^1(\Omega) \longrightarrow H^1(\Omega)$. We can apply Schauder's theorem, if we are able to prove, that $\mathcal{T}_m : H^1(\Omega) \longrightarrow H^1(\Omega)$ is completely continuous and $\mathcal{T}_m[\mathcal{B}] \subset \mathcal{B}$ hold true for a closed ball $\mathcal{B} \subset H^1(\Omega)$ with a radius depending only on the data of the problem. The concrete steps in this part can be taken from the proof of **Lemma 1**.

3. Here we have $\mathcal{T}_m[H^1(\Omega)] \in H^2(\Omega)$ and because of the comlpetely continuous embedding of $H^2(\Omega)$ into $H^1(\Omega)$, the fixed-point map $\mathcal{T}_m : H^1(\Omega) \longrightarrow H^1(\Omega)$ is completely continuous. Having in mind the first step of the proof, Schauder's fixed-point theorem yields a solution $u^m \in H^2(\Omega) \cap \mathcal{B}$ of the equation $\mathcal{T}_m u^m = u^m$. Setting $w^m = P(1 - u^m) \in H^{2,\infty}(\Omega)$, we have found a solution $(u^m, w^m, \psi^m) \in H^2(\Omega) \times H^{2,\infty}(\Omega) \times H^4(\Omega)$ of the problem (4.7)-(4.10).

Lemma 7 (Discrete energy estimate) *Let (u^m, w^m, ψ^m) be solution of (4.4)-(4.6) for every $m \in \{1, ..., M\}$. Then*

$$\begin{aligned}\delta \sum_{m=1}^{M} h \|\Delta \psi^m\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \sum_{m=1}^{M} \|\nabla(\psi^m - \psi^{m-1})\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \max_{1 \le k \le M} \|\nabla \psi^k\|_{L^2(\Omega)}^2 \\ + \sum_{m=1}^{M} h \|\psi^m\|_{L^2(\Omega)}^2 + \sum_{m=1}^{M} h \int_\Omega \mu_\varepsilon(u^m) |\nabla v^m|^2 dx \le C_{21},\end{aligned} \tag{4.11}$$

and

$$\sum_{m=1}^{M} h \int_\Omega |\nabla u^m|^2 dx \le C_{22}. \tag{4.12}$$

Proof. Similar to the proof of **Lemma 2**.

Lemma 8 *Let (u^m, w^m, ψ^m) be solution of (4.4)-(4.6) for every $m \in \{1, ..., M\}$. Then*

$$\max_{1 \le k \le M} \|\Delta \psi^k\|_{L^2(\Omega)}^2 \le C_{23}. \tag{4.13}$$

Proof. 1. Similar to the proof of **Lemma 3**; we use the admissible testfuctions $\Delta\psi^m$ and $-u^m/\gamma$ to get

$$\begin{aligned}
&\frac{2\delta h}{\gamma}\|\nabla\Delta\psi^m\|^2_{L^2(\Omega)} + \|\Delta\psi^m - \Delta\psi^{m-1}\|^2_{L^2(\Omega)} + \|\Delta\psi^m\|^2_{L^2(\Omega)} + \frac{2h}{\gamma}\|\nabla\psi^m\|^2_{L^2(\Omega)} \\
&+ \frac{1}{\gamma^2}\|u^m - u^{m-1}\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|u^m\|^2_{L^2(\Omega)} \\
&= \|\Delta\psi^{m-1}\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|u^{m-1}\|^2_{L^2(\Omega)} + \frac{2\delta h}{\gamma^2}\int_\Omega \nabla\Delta\psi^m\cdot\nabla u^m dx + \frac{2h}{\gamma^2}\int_\Omega \psi^m u^m dx \\
&\quad - \frac{2}{\gamma}\int_\Omega \left(u^m\Delta\psi^m - u^{m-1}\Delta\psi^{m-1}\right)dx - \frac{2}{\gamma}\int_\Omega (u^m - u^{m-1})(\Delta\psi^m - \Delta\psi^{m-1})dx.
\end{aligned} \tag{4.14}$$

2. Using Young's inequality in the following way

$$\frac{2}{\gamma}\int_\Omega (u^m - u^{m-1})(\Delta\psi^m - \Delta\psi^{m-1})dx \le \frac{1}{\gamma^2}\|u^m - u^{m-1}\|^2_{L^2(\Omega)} + \|\Delta\psi^m - \Delta\psi^{m-1}\|^2_{L^2(\Omega)},$$

$$\frac{2\delta h}{\gamma^2}\int_\Omega \nabla\Delta\psi^m\cdot\nabla u^m dx \le \frac{2\delta h}{\gamma}\|\nabla\Delta\psi^m\|^2_{L^2(\Omega)} + \frac{\delta h}{2\gamma^3}\|\nabla u^m\|^2_{L^2(\Omega)},$$

$$\frac{2}{\gamma}\int_\Omega u^m\Delta\psi^m dx \le \frac{2}{\gamma^2}\|u^m\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta\psi^m\|^2_{L^2(\Omega)},$$

we get from (4.14)

$$\begin{aligned}
\|\Delta\psi^m\|^2_{L^2(\Omega)} + \frac{4h}{\gamma}\|\nabla\psi^m\|^2_{L^2(\Omega)} \le& 3\|\Delta\psi^{m-1}\|^2_{L^2(\Omega)} + \frac{6}{\gamma^2}\|u^{m-1}\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\|u^m\|^2_{L^2(\Omega)} \\
&+ \frac{\delta h}{\gamma^3}\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{2h}{\gamma^2}\|\psi^m\|^2_{L^2(\Omega)} + \frac{2h}{\gamma^2}\|u^m\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.15}$$

We sum in (4.15) both sides from $m = 1$ to $m = k$, where $1 \le k \le M$, and find

$$\begin{aligned}
\|\Delta\psi^k\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\sum_{m=1}^k h\|\nabla\psi^m\|^2_{L^2(\Omega)} \le& 3\|\Delta\psi^0\|^2_{L^2(\Omega)} + \frac{6}{\gamma^2}\|u^0\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\|u^k\|^2_{L^2(\Omega)} \\
&+ \frac{\delta}{\gamma^3}\sum_{m=1}^k h\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\sum_{m=1}^k h\|\psi^m\|^2_{L^2(\Omega)} \\
&+ \frac{2}{\gamma^2}\sum_{m=1}^k h\|u^m\|^2_{L^2(\Omega)}.
\end{aligned}$$

The discrete energy estimate (4.11) and (4.12) finally give (4.13).

Lemma 9 *Let (u^m, w^m, ψ^m) be solution of (4.4)-(4.6) for every $m \in \{1, ..., M\}$. Then*

$$\sum_{m=1}^M h\|\Delta u^m\|^2 \le C_{29} \tag{4.16}$$

Proof. 1. Because of **Lemma 6** $-\Delta u^m$ exists and is an admissible testfunction in (4.4)

$$\begin{aligned} \frac{1}{h}\int_\Omega \nabla(u^m - u^{m-1})\cdot\nabla u^m dx + \int_\Omega |\Delta u^m|^2 dx &= \int_\Omega \mu'(u^m)\nabla u^m\cdot\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\Delta u^m dx \\ &\quad + \int_\Omega \mu(u^m)\Delta\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\Delta u^m dx \\ &=: I_{11} + I_{12}. \end{aligned} \tag{4.17}$$

2. Using Hölder's inequality we have

$$\begin{aligned} I_{11} &\leq \|\nabla u^m\|_{L^3(\Omega)}\left\|\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\right\|_{L^6(\Omega)}\|\Delta u^m\|_{L^2(\Omega)} \\ &\leq \|\nabla u^m\|_{L^3(\Omega)}\left\|\frac{w^m+w^{m-1}}{2}+\psi^m\right\|_{W^{1,6}(\Omega)}\|\Delta u^m\|_{L^2(\Omega)}. \end{aligned} \tag{4.18}$$

By the embedding $H^2(\Omega)\subseteq H^{1,6}(\Omega)$ we get

$$I_{11} \leq C_{24}\|\nabla u^m\|_{L^3(\Omega)}\left\|\frac{w^m+w^{m-1}}{2}+\psi^m\right\|_{H^2(\Omega)}\|\Delta u^m\|_{L^2(\Omega)}.$$

The Gagliardo-Nirenberg inequality (A.14) for $\dim(\Omega)=3$

$$\|\nabla u^m\|_{L^3(\Omega)} \leq C_g\|\nabla u^m\|^{1/2}_{L^2(\Omega)}\|\Delta u^m\|^{1/2}_{L^2(\Omega)},$$

gives

$$I_{11} \leq C_{25}\|\nabla u^m\|^{1/2}_{L^2(\Omega)}\left\|\frac{w^m+w^{m-1}}{2}+\psi^m\right\|_{H^2(\Omega)}\|\Delta u^m\|^{3/2}_{L^2(\Omega)}.$$

We get using **(A2)**, **(B4')**, (4.11) and (4.13)

$$I_{11} \leq C_{26}\|\nabla u^m\|^{1/2}_{L^2(\Omega)}\|\Delta u^m\|^{3/2}_{L^2(\Omega)}.$$

Young's inequality gives

$$I_{11} \leq 3C_{26}^2\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{1}{4^2}\|\Delta u^m\|^2_{L^2(\Omega)}.$$

3. For the second term in (4.17) we find that

$$\begin{aligned} I_{12} &\leq \left\|\Delta\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\right\|^2_{L^2(\Omega)} + \frac{1}{4^2}\|\Delta u^m\|^2_{L^2(\Omega)} \\ &\leq C_{27} + \frac{1}{4^2}\|\Delta u^m\|^2_{L^2(\Omega)}, \end{aligned} \tag{4.19}$$

where we have used **(A2)**, **(B4')** and (4.13) for the last step in (4.19).

4. We multiply both sides of (4.17) with h and sum from $m = 1$ to $m = k$, where $1 \le k \le M$ to get

$$\frac{4}{7}\sum_{m=1}^{k}\|\nabla(u^m - u^{m-1})\|^2_{L^2(\Omega)} + \frac{4}{7}\|\nabla u^k\|^2_{L^2(\Omega)} - \frac{4}{7}\|\nabla u^0\|^2_{L^2(\Omega)} + \sum_{m=1}^{k} h\|\Delta u^m\|^2_{L^2(\Omega)} \le C_{28}\sum_{m=1}^{k} h\|\nabla u^m\|^2_{L^2(\Omega)}.$$

Using the estimate (4.12) we find (4.16).

Lemma 10 *Let (u^m, w^m, ψ^m) be solution of (4.4)-(4.6) for every $m \in \{1, ..., M\}$. Then*

$$\sum_{m=1}^{M} h\left\|\frac{u^m - u^{m-1}}{h}\right\|^2_{L^2(\Omega)} \le C_{35}. \tag{4.20}$$

Proof. 1. Using $(u^m - u^{m-1})$ as a testfuction in (4.4) we obtain

$$\begin{aligned}
\frac{1}{h}\|u^m - u^{m-1}\|^2 + &\int_\Omega \nabla u^m \cdot \nabla(u^m - u^{m-1})\,dx \\
&= \int_\Omega \mu_\varepsilon'(u^m)\nabla u^m \nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)(u^m - u^{m-1})\,dx \\
&\quad + \int_\Omega \mu_\varepsilon(u^m)\Delta\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)(u^m - u^{m-1})\,dx \\
&\equiv I_{13} + I_{14}.
\end{aligned}$$

2. Using Hölder's inequality we get

$$\begin{aligned}
I_{13} \le &C_{30} h\left(\int_\Omega \nabla u^m \cdot \nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\left(\frac{u^m - u^{m-1}}{h}\right)dx\right) \\
&\le C_{30} h\left(\|\nabla u^m\|_{L^3(\Omega)}\left\|\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right\|_{L^6(\Omega)}\left\|\frac{u^m - u^{m-1}}{h}\right\|_{L^2(\Omega)}\right) \\
&\le C_{30} h\left(\|\nabla u^m\|_{L^3(\Omega)}\left\|\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right\|_{W^{1,6}(\Omega)}\left\|\frac{u^m - u^{m-1}}{h}\right\|_{L^2(\Omega)}\right).
\end{aligned}$$

Again the embedding $H^2(\Omega) \subseteq H^{1,p}(\Omega)$, $p \in [1, 6]$, gives

$$I_{13} \le C_{31} h\left(\|u^m\|_{H^2(\Omega)}\left\|\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right\|_{H^2(\Omega)}\left\|\frac{u^m - u^{m-1}}{h}\right\|_{L^2(\Omega)}\right).$$

We obtain by assumption **(A2)**, **(B4)** and (4.16)

$$I_{13} \leq C_{32} h \left(\|u^m\|_{H^2(\Omega)} \left\| \frac{u^m - u^{m-1}}{h} \right\|_{L^2(\Omega)} \right).$$

Young's inequality gives

$$\begin{aligned} I_{13} &\leq C_{32}^2 h \|u^m\|_{H^2(\Omega)}^2 + \frac{h}{4} \left\| \frac{u^m - u^{m-1}}{h} \right\|_{L^2(\Omega)}^2 \\ &\leq hC_{33} + \frac{h}{4} \left\| \frac{u^m - u^{m-1}}{h} \right\|_{L^2(\Omega)}^2, \end{aligned}$$

where we have used (4.11) and (4.16).

3. We apply Young's inequality, **(A2)**, **(B4')** and (4.13) for the second term

$$\begin{aligned} I_{14} \leq & h \left\| \Delta \left(\frac{w^m + w^{m-1}}{2} + \psi^m \right) \right\|_{L^2(\Omega)}^2 + \frac{h}{4} \left\| \frac{u^m - u^{m-1}}{h} \right\|_{L^2(\Omega)}^2 \\ \leq & hC_{34} + \frac{h}{4} \left\| \frac{u^m - u^{m-1}}{h} \right\|_{L^2(\Omega)}^2 . \end{aligned}$$

We sum from $m = 1$ to $m = M$ and conclude

$$\sum_{m=1}^{M} h \left\| \frac{u^m - u^{m-1}}{h} \right\|_{L^2(\Omega)}^2 + \sum_{m=1}^{M} \|\nabla(u^m - u^{m-1})\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u^k\|_{L^2(\Omega)}^2 \leq C_{35}.$$

Lemma 11 *Let (u^m, w^m, ψ^m) be solution of (4.4)-(4.6) for every $m \in \{1, ..., M\}$. Then*

$$\sum_{m=1}^{k} h \left\| \nabla \left(\frac{\psi^m - \psi^{m-1}}{h} \right) \right\|_{L^2(\Omega)}^2 \leq C_{36}. \tag{4.21}$$

Proof. We use $(\psi^m - \psi^{m-1})$ as a testfunction in (4.5):

$$\begin{aligned} \delta \int_\Omega \Delta\psi^m \Delta(\psi^m - \psi^{m-1}) dx + \frac{\gamma}{h} \int_\Omega |\nabla(\psi^m - \psi^{m-1})|^2 dx + \int_\Omega \psi^m (\psi^m - \psi^{m-1}) dx \\ = \frac{1}{h} \int_\Omega (u^m - u^{m-1})(\psi^m - \psi^{m-1})) dx. \end{aligned}$$

Using Young's inequality and the Poincaré inequality for the right hand side we get

$$\begin{aligned} \delta \int_\Omega \Delta\psi^m \Delta(\psi^m - \psi^{m-1}) dx + \frac{\gamma}{2h} \int_\Omega |\nabla(\psi^m - \psi^{m-1})|^2 dx + \int_\Omega \psi^m (\psi^m - \psi^{m-1}) dx \\ \leq \frac{c_p}{2\gamma h} \|u^m - u^{m-1}\|_{L^2(\Omega)}^2, \end{aligned}$$

where c_p is the Poincaré constant. We sum from $m=1$ to $m=k$, where $1 \leq k \leq M$ and find

$$\frac{\delta}{\gamma}\sum_{m=1}^{k}\|\Delta(\psi^m-\psi^{m-1})\|^2_{L^2(\Omega)}+\frac{\delta}{\gamma}\|\Delta\psi^k\|^2_{L^2(\Omega)}-\frac{\delta}{\gamma}\|\Delta\psi^0\|^2_{L^2(\Omega)}+\sum_{m=1}^{k}h\left\|\nabla\left(\frac{\psi^m-\psi^{m-1}}{h}\right)\right\|^2_{L^2(\Omega)}$$
$$+\frac{1}{\gamma}\sum_{m=1}^{k}\|\psi^m-\psi^{m-1}\|^2_{L^2(\Omega)}+\frac{1}{\gamma}\|\psi^k\|^2-\frac{1}{\gamma}\|\psi^0\|^2_{L^2(\Omega)}\leq\frac{c_p}{\gamma^2}\sum_{m=1}^{k}h\left\|\frac{u^m-u^{m-1}}{h}\right\|^2_{L^2(\Omega)}.$$

We finally get (4.21) by (4.20).

Lemma 12 *Let (u^m, w^m, ψ^m) be solution of (4.4)-(4.6) for every $m \in \{1,...,M\}$. Then*

$$\begin{aligned}&\delta\sum_{m=1}^{M}h\|\Delta^2\psi^m\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\sum_{m=1}^{M}\|\nabla\Delta\psi^m-\nabla\Delta\psi^{m-1}\|^2_{L^2(\Omega)}\\&+\frac{\gamma}{2}\max_{1\leq k\leq M}\|\nabla\Delta\psi^k\|^2_{L^2(\Omega)}+2\sum_{m=1}^{M}h\|\Delta\psi^m\|^2_{L^2(\Omega)}\leq C_{37}.\end{aligned}\tag{4.22}$$

Proof. 1. Because of **Lemma 6** $\Delta^2\psi^m$ exists; thus an admissible testfunction in (4.5):

$$\begin{aligned}&\delta\int_\Omega|\Delta^2\psi^m|^2dx+\frac{\gamma}{h}\int_\Omega(\nabla\Delta\psi^m-\nabla\Delta\psi^{m-1})\cdot\nabla\Delta\psi^m dx+\int_\Omega|\Delta\psi^m|^2dx\\&=\frac{1}{h}\int_\Omega(u^m-u^{m-1})\Delta^2\psi^m dx.\end{aligned}\tag{4.23}$$

For the right handside of (4.23) we have

$$\begin{aligned}\frac{1}{h}\int_\Omega(u^m-u^{m-1})\Delta^2\psi^m dx=&\frac{1}{h}\int_\Omega\left(u^m\Delta^2\psi^m-u^{m-1}\Delta^2\psi^{m-1}\right)dx\\&+\frac{1}{h}\int_\Omega(u^m-u^{m-1})(\Delta^2\psi^m-\Delta^2\psi^{m-1})dx\\&-\frac{1}{h}\int_\Omega u^m(\Delta^2\psi^m-\Delta^2\psi^{m-1})dx.\end{aligned}$$

2. Because of **Lemma 6** $\Delta u^m/\gamma$ exists; thus an admissible testfunction in (4.5):

$$\begin{aligned}\frac{\delta}{\gamma}\int_\Omega\Delta^2\psi^m\Delta u^m dx-\frac{1}{h}\int_\Omega\Delta^2(\psi^m-\psi^{m-1})u^m dx+\frac{1}{\gamma h}(\nabla u^m-\nabla u^{m-1})\cdot\nabla u^m dx\\=\frac{1}{\gamma}\int_\Omega\nabla\psi^m\cdot\nabla u^m dx.\end{aligned}\tag{4.24}$$

We get by (4.23) and (4.24) the following estimate

$$\begin{aligned}
&\delta\|\Delta^2\psi^m\|^2_{L^2(\Omega)} + \frac{\gamma}{2h}\|\nabla\Delta\psi^m - \nabla\Delta\psi^{m-1}\|^2_{L^2(\Omega)} + \frac{\gamma}{2h}\|\nabla\Delta\psi^m\|^2_{L^2(\Omega)} - \frac{\gamma}{2h}\|\nabla\Delta\psi^{m-1}\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2\gamma h}\|\nabla u^m - \nabla u^{m-1}\|^2_{L^2(\Omega)} + \frac{1}{2\gamma h}\|\nabla u^m\|^2_{L^2(\Omega)} - \frac{1}{2\gamma h}\|\nabla u^{m-1}\|^2_{L^2(\Omega)} + \|\Delta\psi^m\|^2_{L^2(\Omega)} \\
&= \frac{1}{h}\int_\Omega (\nabla u^m \nabla\Delta\psi^m - \nabla u^{m-1}\nabla\Delta\psi^{m-1})dx \qquad (4.25) \\
&+ \frac{1}{h}\int_\Omega \nabla(u^m - u^{m-1})\cdot(\nabla\Delta\psi^m - \nabla\Delta\psi^{m-1})dx + \frac{\delta}{\gamma}\int_\Omega \Delta^2\psi^m \Delta u^m dx - \frac{1}{\gamma}\int_\Omega \nabla\psi^m\cdot\nabla u^m dx.
\end{aligned}$$

We multiply (4.25) by h and sum from $m = 1$ to $m = k$, where $1 \le k \le M$. By using Young's inequality in the form

$$\begin{aligned}
&\sum_{m=1}^{k}\int_\Omega \nabla(u^m - u^{m-1})\cdot(\nabla\Delta\psi^m - \nabla\Delta\psi^{m-1})dx \\
&\qquad \le \frac{1}{\gamma}\sum_{m=1}^{k}\|\nabla(u^m - u^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{4}\sum_{m=1}^{k}\|(\nabla\Delta\psi^m - \nabla\Delta\psi^{m-1})\|^2_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\delta}{\gamma}\sum_{m=1}^{k} h\int_\Omega \Delta^2\psi^m\Delta u^m dx &\le \frac{\delta}{2}\sum_{m=1}^{k} h\|\Delta^2\psi^m\|^2_{L^2(\Omega)} + \frac{\delta}{2\gamma^2}\sum_{m=1}^{k} h\|\Delta u^m\|^2_{L^2(\Omega)}, \\
\int_\Omega \nabla u^k\cdot\nabla\Delta\psi^k dx &\le \frac{1}{\gamma}\|\nabla u^k\|^2_{L^2(\Omega)} + \frac{\gamma}{4}\|\nabla\Delta\psi^k\|^2_{L^2(\Omega)},
\end{aligned}$$

we obtain the estimate

$$\begin{aligned}
&\delta\sum_{m=1}^{k} h\|\Delta^2\psi^m\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\sum_{m=1}^{k}\|\nabla\Delta\psi^m - \nabla\Delta\psi^{m-1}\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\nabla\Delta\psi^k\|^2_{L^2(\Omega)} + 2\sum_{m=1}^{k} h\|\Delta\psi^m\|^2_{L^2(\Omega)} \\
&\quad \le 2\gamma\|\nabla\Delta\psi^0\|^2 + \frac{2}{\gamma}\|\nabla u^0\|^2_{L^2(\Omega)} + +\frac{1}{\gamma}\|\nabla u^k\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\sum_{m=1}^{k}\|\nabla u^m - \nabla u^{m-1}\|^2_{L^2(\Omega)} \\
&\qquad + \frac{\delta}{\gamma^2}\sum_{m=1}^{k} h\|\Delta u^m\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\sum_{m=1}^{k} h\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\sum_{m=1}^{k} h\|\nabla\psi^m\|^2_{L^2(\Omega)}.
\end{aligned}$$

The energy estimate (4.11) and (4.12) together with (4.16) give (4.22). Again like in **Chapter 3** we denote for any $M \in \mathbb{N}$ the solutions of (4.4)-(4.6) by (u^m_M, w^m_M, ψ^m_M). We define piecewise linear and constant interpolates like in (3.32)-(3.36). for $1 \le m \le M$.

With these notations, the variational equations (4.4)-(4.6) can be written as

$$\begin{aligned}
&\int_0^T\int_\Omega \hat{u}_{M,t}\varphi\,dxdt + \int_0^T\int_\Omega (\nabla\bar{u}_M + \mu_\varepsilon(\bar{u}_M)\nabla(\bar{w}_M + \bar{\psi}_M))\cdot\nabla\varphi\,dxdt = 0,\\
&\delta\int_0^T\int_\Omega \Delta\bar{\psi}_M\Delta\varphi dxdt + \gamma\int_0^T\int_\Omega \nabla\hat{\psi}_{M,t}\cdot\nabla\varphi dx\,dt \qquad (4.26)\\
&+\int_0^T\int_\Omega \bar{\psi}_M\varphi\,dxdt = \int_0^T\int_\Omega \hat{u}_{M,t}\varphi dxdt, \quad \forall\varphi\in L^2(0,T;H^2_\bullet(\Omega)),
\end{aligned}$$

By virtue of the energy estimate (4.11),

$$\begin{aligned}
&\delta\int_0^T \|\Delta\bar{\psi}_M\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\sum_{m=1}^M \|\nabla(\psi^m - \psi^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\sup_{0\le t\le T}\|\nabla\bar{\psi}_M(t)\|^2_{L^2(\Omega)}\\
&\qquad + \int_0^T \|\bar{\psi}_M\|^2_{L^2(\Omega)}dt + \int_0^T\int_\Omega \mu_\varepsilon(\bar{u}_M)|\nabla\bar{v}_M|^2\,dxdt \le C_{21}, \qquad (4.27)
\end{aligned}$$

where $\bar{v}_M := f'_\varepsilon(\bar{u}_M) + \bar{w}_M + \bar{\psi}_M$. We also find from (4.12) that

$$\int_0^T\int_\Omega |\nabla\bar{u}_M|^2 dxdt \le C_{22}.$$

Moreover by (4.13), (4.16), (4.20), (4.21) and (4.22) we get

$$\begin{aligned}
&\sup_{0\le t\le T}\|\Delta\bar{\psi}_M(t)\|^2_{L^2(\Omega)} \le C_{23}, && \sup_{0\le t\le T}\|\nabla\Delta\bar{\psi}_M(t)\|^2_{L^2(\Omega)} \le C_{37},\\
&\int_0^T \|\Delta^2\bar{\psi}_M\|^2_{L^2(\Omega)}dt \le C_{37}, && \int_0^T \|\nabla\hat{\psi}_{M,t}\|^2_{L^2(\Omega)}dt \le C_{36},\\
&\int_0^T \|\Delta\bar{u}_M\|^2_{L^2(\Omega)}dt \le C_{29}, && \int_0^T \|\hat{u}_{M,t}\|^2_{L^2(\Omega)}dt \le C_{35}.
\end{aligned}$$

In addition, (4.16) imply that, as $M\to\infty$,

$$\|\Delta\bar{u}_M - \Delta\hat{u}_M\|_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^M \|\Delta u_M^m - \Delta u_M^{m-1}\|_{L^2(\Omega)} \to 0, \qquad (4.28)$$

and we obtain using (3.42) and (4.28)

$$\|\bar{u}_M - \hat{u}_M\|_{L^2(0,T;H^2(\Omega))} \to 0. \tag{4.29}$$

Moreover we have by (4.11), (4.13) and (4.22)

$$\|\nabla\bar{\psi}_M - \nabla\hat{\psi}_M\|_{L^\infty(0,T;H^2(\Omega))} = \max_{0\le t\le T} \|\nabla\psi_M^m - \nabla\psi_M^{m-1}\|_{H^2(\Omega)} \to 0. \tag{4.30}$$

By (4.22) we get in addition

$$\|\Delta^2\bar{\psi}_M - \Delta^2\hat{\psi}_M\|_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^{M} \|\Delta^2\psi_M^m - \Delta^2\psi_M^{m-1}\|_{L^2(\Omega)} \to 0. \tag{4.31}$$

In conclusion, there are functions $\bar{u}, \hat{u}_t, \bar{\psi}, \hat{\psi}_t$, such that for $M \to \infty$, possibly after selecting subsequences,

$$\begin{array}{llllll}
\bar{u}_M & \longrightarrow & \bar{u} & \text{weakly} & \text{in} & L^2(0,T;H^2(\Omega)), \\
\hat{u}_{M,t} & \longrightarrow & \hat{u}_t & \text{weakly} & \text{in} & L^2(0,T;L^2(\Omega)), \\
\bar{\psi}_M & \longrightarrow & \bar{\psi} & \text{weakly} & \text{in} & L^2(0,T;L^2(\Omega)), \\
\nabla\bar{\psi}_M & \longrightarrow & \nabla\bar{\psi} & \text{weakly-star} & \text{in} & L^\infty(0,T;H^2(\Omega)), \\
\Delta^2\bar{\psi}_M & \longrightarrow & \Delta^2\bar{\psi} & \text{weakly} & \text{in} & L^2(0,T;L^2(\Omega)), \\
\nabla\hat{\psi}_{M,t} & \longrightarrow & \nabla\hat{\psi}_t & \text{weakly} & \text{in} & L^2(0,T;L^2(\Omega)).
\end{array} \tag{4.32}$$

Taking into account (4.29), (4.30) and (4.31), we see that $\bar{u} = \hat{u}$ and $\bar{\psi} = \hat{\psi}$. It follows >from (4.32) that we may pass to the limit as $M \to \infty$ in (4.26). The passage to the limit can be done in the same way like in **Chapter 3**, so we here skip the concrete proof. We denote the solution of the regularized problem (4.1)-(4.3) by $(u_{\delta,\varepsilon}, w_{\delta,\varepsilon}, \psi_{\delta,\varepsilon})$. Next we are going to prove some a priori estimates, which will allow us to pass to the limit in (4.1)-(4.3). We can state the following energy estimate.

Lemma 13 (Energy estimate) *There exists an ε_0 (see **Remark 6**) such that for all $0 < \varepsilon \le \varepsilon_0$ and for all $\delta > 0$ the following estimate holds with constants C_{38}, C_{39} independent of ε and δ:*

$$\begin{aligned}\delta\int_0^T \|\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt + \frac{\gamma}{2}\max_{0\le t\le T}\|\nabla\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} + \int_0^T \|\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt \\ + \int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})|\nabla v_{\delta,\varepsilon}|^2 dxdt \le C_{38},\end{aligned} \tag{4.33}$$

and

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 dxdt \le C_{39}, \tag{4.34}$$

where $v_{\delta,\varepsilon} := f'_\varepsilon(u_{\delta,\varepsilon}) + w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}$.

Proof. Like in **Chapter 3** the function $v_{\delta,\varepsilon} \in L^2(0,T;H^2(\Omega))$ is a valid testfunction in (4.1). The proof is similar to the proof of **Lemma 4**. Because of sufficient regularity we here don't make use of steklov averaging.

Lemma 14 *There exists an* ε_0 *(see* **Remark 6***) such that for all* $0 < \varepsilon \le \varepsilon_0$ *and for all* $\delta > 0$ *the following estimate holds with a constant* C_{40} *independent of* ε *and* δ*:*

$$\max_{0\le t\le T} \|\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} \le C_{40}. \tag{4.35}$$

Proof. Similar to the proof of **Lemma 5** without steklov averaging.

Lemma 15 *There exists an* ε_0 *(see* **Remark 6***) such that for all* $0 < \varepsilon \le \varepsilon_0$ *and for all* $\delta > 0$ *the following estimate holds with a constant* C_{46} *independent of* ε *and* δ*:*

$$\int_0^T \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \le C_{46}. \tag{4.36}$$

Proof. 1. Because of the compactness results (4.32) $-\Delta u_\varepsilon$ is an admissible testfunction in (4.1). Using the chain rule we get after partial integration in (4.1)

$$\begin{aligned} \|\nabla u_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \|\nabla u_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \int_0^T \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \\ = -\int_0^T\int_\Omega \mu'_\varepsilon(u_{\delta,\varepsilon})\nabla u_{\delta,\varepsilon}\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\Delta u_{\delta,\varepsilon}dxdt \\ -\int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\Delta(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\Delta u_{\delta,\varepsilon}dxdt \\ =: I_{15} + I_{16}. \end{aligned} \tag{4.37}$$

2. We estimate each summand like in (4.17) and (4.18): Using Hölder's inequality we get

$$\begin{aligned} I_{15} &\le \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)}\ \|\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\|_{L^6(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt \\ &\le \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)}\ \|w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}\|_{W^{1,6}(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt \\ &\le C_{41}\int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)}\ \|w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}\|_{H^2(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt, \end{aligned}$$

where we have used the embedding $H^2(\Omega) \subseteq H^{1,6}(\Omega)$ in the last step. The Gagliardo-Nirenberg inequality (A.14) for $\dim(\Omega) = 3$

$$\|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)} \leq C_g \|\nabla u_{\delta,\varepsilon}\|_{L^2(\Omega)}^{1/2} \, \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}^{1/2}$$

gives

$$I_{15} \leq C_{42} \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^2(\Omega)}^{1/2} \, \|w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}\|_{H^2(\Omega)} \, \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}^{3/2} dt.$$

We obtain using **(A2)**, **(B4')** and (4.35)

$$I_{15} \leq C_{43} \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^2(\Omega)}^{1/2} \, \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}^{3/2} dt.$$

Young's inequality together with (4.34) gives

$$I_{15} \leq C_{44} + \frac{1}{4} \int_0^T \|\Delta u_{\delta,\varepsilon}\|_2^2 \, dt.$$

3. For the second term in (4.37) we get

$$\begin{aligned} I_{16} &\leq \frac{1}{8} \int_0^T \|\Delta(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})\|_{L^2(\Omega)}^2 + \frac{1}{8} \int_0^T \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}^2 dt \\ &\leq C_{45} + \frac{1}{8} \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}^2, \end{aligned}$$

where we have used **(A2)**, **(B4')** and (4.35).

Lemma 16 *There exists an ε_0 (see* **Remark 6***) such that for all $0 < \varepsilon \leq \varepsilon_0$ and for all $\delta > 0$ the following estimate holds with a constant C_{51} independent of ε and δ:*

$$\int_0^T \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)}^2 dt \leq C_{51}. \tag{4.38}$$

Proof. 1. Because of the compactness results (4.32) $\partial_t u_{\delta,\varepsilon} \in L^2(0,T;L^2(\Omega))$, thus an admissible testfuction in (4.1). Again using the chain rule we get after partial integration

$$\begin{aligned}\int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt + \|\nabla u_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \|\nabla u_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} \\ = \int_0^T \int_\Omega \mu'_\varepsilon(u_{\delta,\varepsilon}) \nabla u_{\delta,\varepsilon} \nabla (w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}) \partial_t u_{\delta,\varepsilon} dx dt \\ + \int_0^T \int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon}) \Delta (w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}) \partial_t u_{\delta,\varepsilon} dx dt \\ \equiv I_{17} + I_{18}.\end{aligned} \tag{4.39}$$

2. Using Hölder's inequality we get

$$\begin{aligned} I_{17} &\leq \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)} \ \|\nabla(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})\|_{L^6(\Omega)} \ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)} dt \\ &\leq \int_0^T \|u_{\delta,\varepsilon}\|_{W^{1,3}(\Omega)} \ \|w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}\|_{W^{1,6}(\Omega)} \ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)} dt. \end{aligned}$$

We again make use of the embedding $H^2(\Omega) \subseteq H^{1,p}(\Omega)$, $p \in [1,6]$, and get

$$I_{17} \leq C_{47} \int_0^T \|u_{\delta,\varepsilon}\|_{H^2(\Omega)} \ \|w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}\|_{H^2(\Omega)} \ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)} dt.$$

Assumption **(A2)**, **(B4')** and (4.37) gives

$$I_{17} \leq C_{48} \int_0^T \|u_{\delta,\varepsilon}\|_{H^2(\Omega)} \ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)} \, dt.$$

Applying Young's inequality together with (4.36) we find that

$$I_{17} \leq C_{49} + \frac{1}{4} \int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt.$$

3. Young's inequality gives for the second right term of (4.39)

$$I_{18} \le \int\limits_0^T \|\Delta(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})\|^2_{L^2(\Omega)} dt + \frac{1}{4}\int\limits_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt$$
$$\le \int\limits_0^T \|(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})\|^2_{H^2(\Omega)} dt + \frac{1}{4}\int\limits_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt$$

By **(A2)**, **(B4')** and (4.35) we get

$$I_{18} \le C_{50} + \frac{1}{4}\int\limits_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt.$$

Lemma 17 *There exists an ε_0 (see* **Remark 6***) such that for all $0 < \varepsilon \le \varepsilon_0$ and for all $\delta > 0$ the following estimate holds with a constant C_{52} independent of ε and δ:*

$$\int\limits_0^T \|\nabla\partial_t \psi_{\delta,\varepsilon}\|^2 dt \le C_{52}. \tag{4.40}$$

Proof. 1. Testing (4.2) by the admissible testfunction $\partial_t \psi_{\delta,\varepsilon}$ we find

$$\begin{aligned}\frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \gamma\int\limits_0^T \|\nabla\partial_t\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \\ + \frac{1}{2}\|\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} = \int\limits_0^T\int\limits_\Omega \partial_t u_{\delta,\varepsilon}\partial_t\psi_{\delta,\varepsilon} dx dt.\end{aligned} \tag{4.41}$$

2. Using Young's inequality together with Poincaré inequality for the right hand side of (4.41) we get

$$\frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\int\limits_0^T \|\nabla\partial_t\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt$$
$$+ \frac{1}{2}\|\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} \le \frac{1}{2}\int\limits_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt.$$

Using (4.38) we get (4.40)

Lemma 18 *There exists an* ε_0 *(see* **Remark 6***) such that for all* $0 < \varepsilon \le \varepsilon_0$ *and for all* $\delta > 0$ *the following estimate holds with a constant* C_{53} *independent of* ε *and* δ*:*

$$\max_{0\le t\le T} \|\nabla\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} \le C_{53}. \tag{4.42}$$

Proof. 1. To prove this we use again the steklov averaging technique (3.49). We use the admissible testfunction $-\Delta^2\psi_{\delta,\varepsilon h}$ in (4.2) and get after partial integration:

$$\begin{aligned}&\delta\int_0^t \|\Delta^2\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt + \frac{\gamma}{2}\|\nabla\Delta\psi_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt \\ &= \int_0^t\int_\Omega \partial_t u_{\delta,\varepsilon h}\Delta^2\psi_{\delta,\varepsilon h}dxdt + \frac{\gamma}{2}\|\nabla\Delta\psi_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)},\end{aligned} \tag{4.43}$$

for all $t \in [0, T]$.
2. We obtain applying the testfunction $\Delta u_{\delta,\varepsilon h}/\gamma$ in (4.2)

$$\begin{aligned}\frac{\delta}{\gamma}\int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt - \int_0^t\int_\Omega \Delta\partial_t\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt + \frac{1}{\gamma}\int_0^t\int_\Omega \psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt \\ = \int_0^t\int_\Omega \partial_t u_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt,\end{aligned} \tag{4.44}$$

for all $t \in [0, T]$.
3. Because of the steklov averaging $\Delta^2\partial_t\psi_{\delta,\varepsilon h} \in L^2(0, T; L^2(\Omega))$ exists and by partial integration we get for (4.44)

$$\begin{aligned}\frac{\delta}{\gamma}\int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt - \int_0^t\int_\Omega \Delta^2\partial_t\psi_{\delta,\varepsilon h}u_{\delta,\varepsilon h}dxdt + \frac{1}{\gamma}\int_0^t\int_\Omega \nabla\psi_{\delta,\varepsilon h}\cdot\nabla u_{\delta,\varepsilon h}dxdt \\ = \int_0^t\int_\Omega \partial_t u_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt,\end{aligned} \tag{4.45}$$

for all $t \in [0, T]$. Using the formula for partial integration in time

$$\begin{aligned}\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}(t)u_{\delta,\varepsilon h}(t)dx - \int_\Omega \Delta^2\psi_{\delta,\varepsilon h}(0)u_{\delta,\varepsilon h}(0)dx = &\int_0^t\int_\Omega \Delta^2\partial_t\psi_{\delta,\varepsilon h}u_{\delta,\varepsilon h}dxdt \\ &+ \int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\partial_t u_{\delta,\varepsilon h}dxdt,\end{aligned}$$

for all $t \in [0,T]$, we find

$$\begin{aligned}
&\delta \int_0^t \|\Delta^2 \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{\gamma}{2} \|\nabla \Delta \psi_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{1}{\gamma} \|\nabla u_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} \\
&= + \frac{\gamma}{2} \|\nabla \Delta \psi_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma} \|\nabla u_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} - \int_\Omega \nabla \Delta \psi_{\delta,\varepsilon h}(t) \cdot \nabla u_{\delta,\varepsilon h}(t) dx \qquad (4.46) \\
&\quad + \int_\Omega \nabla \Delta \psi_{\delta,\varepsilon h}(0) \cdot \nabla u_{\delta,\varepsilon h}(0) dx - \frac{\delta}{\gamma} \int_0^t \int_\Omega \Delta^2 \psi_{\delta,\varepsilon h} \Delta u_{\delta,\varepsilon h} dx dt + \frac{1}{\gamma} \int_0^t \int_\Omega \nabla \psi_{\delta,\varepsilon h} \cdot \nabla u_{\delta,\varepsilon h} dx dt,
\end{aligned}$$

for all $t \in [0,T]$. Using Young's inequality in the form

$$\begin{aligned}
&\frac{\delta}{\gamma} \int_0^t \int_\Omega \Delta^2 \psi_{\delta,\varepsilon h} \Delta u_{\delta,\varepsilon h} dx dt \le \delta \int_0^t \|\Delta^2 \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{\delta}{\gamma^2} \int_0^t \|\Delta u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt, \\
&\int_\Omega \nabla \Delta \psi_{\delta,\varepsilon h}(t) \cdot \nabla u_{\delta,\varepsilon h}(t) dx \le \frac{\gamma}{4} \|\nabla \Delta \psi_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} + \frac{1}{\gamma} \|\nabla u_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)}, \\
&\frac{1}{\gamma} \int_0^t \int_\Omega \nabla \psi_{\delta,\varepsilon h} \cdot \nabla u_{\delta,\varepsilon h} dx dt \le \frac{1}{2\gamma} \int_0^t \|\nabla \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} + \frac{1}{2\gamma} \int_0^t \|\nabla u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)},
\end{aligned}$$

for all $t \in [0,T]$, we get from (4.46)

$$\begin{aligned}
\|\nabla \Delta \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} + \frac{4}{\gamma} \int_0^t \|\Delta \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt =& 3 \|\nabla \Delta \psi_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2} \|\nabla u_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} \\
&+ \frac{4\delta}{\gamma^3} \int_0^t \|\Delta u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt \\
&+ \frac{2}{\gamma^2} \int_0^t \|\nabla \psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{2}{\gamma^2} \int_0^t \|\nabla u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt,
\end{aligned}$$

for all $t \in [0,T]$. Passing to the limit ($h \searrow 0$) we obtain

$$\begin{aligned}\|\nabla\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\int_0^t \|\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt =& 3\|\nabla\Delta\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|\nabla u_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} \\ &+ \frac{4\delta}{\gamma^3}\int_0^t \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt \\ &+ \frac{2}{\gamma^2}\int_0^t \|\nabla\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt + \frac{2}{\gamma^2}\int_0^t \|\nabla u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt,\end{aligned}$$

for all $t \in [0,T]$. The estimate (4.11), (4.12) and (4.36) give (4.42).

Remark 8 Because of (4.42) we have

$$\max_{0\le t\le T} \|\nabla\psi_{\delta,\varepsilon}\|^2_{H^2(\Omega)} \le C_{53}.$$

and by the Sobolev embedding Theorem (A.11) for $\dim(\Omega) = 3$ we get

$$\max_{0\le t\le T} \|\nabla\psi_{\delta,\varepsilon}\|_{L^\infty(\Omega)} \le \tilde{C}_{53}. \tag{4.47}$$

We find following standard compactness properties

$$\begin{array}{lllll} u_{\delta,\varepsilon} & \longrightarrow & u & \text{weakly} & \text{in } L^2(0,T;H^2(\Omega)), \\ \partial_t u_{\delta,\varepsilon} & \longrightarrow & \partial_t u & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\ \psi_{\delta,\varepsilon} & \longrightarrow & \psi & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\ \nabla\psi_{\delta,\varepsilon} & \longrightarrow & \nabla\psi & \text{weakly-star} & \text{in } L^\infty(0,T;H^2(\Omega)), \\ \nabla\partial_t\psi_{\delta,\varepsilon} & \longrightarrow & \nabla\psi_t & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)). \end{array} \tag{4.48}$$

so that as $\delta, \varepsilon \longrightarrow 0$ we may pass to the limit in (4.1)-(4.3). The convergence of the linear terms in (4.1)-(4.3) are standard. We take a closer look on the convergence of the nonlinear term. First we prove as $\delta \longrightarrow 0$ the passage to the limit of the nonlinear term

$$\begin{aligned}&\int_0^T\int_\Omega (\mu_\varepsilon(u_\varepsilon)\nabla(w_\varepsilon + \psi_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon})\nabla(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})) \cdot \nabla\varphi\, dxdt \\ &\qquad = \int_0^T\int_\Omega (\mu_\varepsilon(u_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon}))\nabla(w_\varepsilon + \psi_\varepsilon) \cdot \nabla\varphi\, dxdt \\ &\qquad + \int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\nabla\left[(w_\varepsilon - w_{\delta,\varepsilon}) + (\psi_\varepsilon - \psi_{\delta,\varepsilon})\right] \cdot \nabla\varphi\, dxdt.\end{aligned}$$

We follow the same argument as in (3.45) and skip here the details of the proof. Now we are able to prove the passage to the limit as $\varepsilon \to 0$

$$\int_0^T \int_\Omega (\mu_\varepsilon(u_\varepsilon)\nabla(w_\varepsilon + \psi_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon})\nabla(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})) \cdot \nabla\varphi \, dxdt$$
$$= \int_0^T \int_\Omega (\mu_\varepsilon(u_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon}))\nabla(w_\varepsilon + \psi_\varepsilon) \cdot \nabla\varphi \, dxdt$$
$$+ \int_0^T \int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\nabla\left[(w_\varepsilon - w_{\delta,\varepsilon}) + (\psi_\varepsilon - \psi_{\delta,\varepsilon})\right] \cdot \nabla\varphi \, dxdt.$$

Here we use the argument as in (3.56) to justify this passage. The rest of the proof goes like in **Chapter 3**. We also skip here the proof of the uniqueness. It is similar to the corresponding proof in **Chapter 3**.