Chapter 3

Proof of Theorem 1

In this chapter we prove the existence and uniqueness for a solution to the problem in **Theorem 1**.

3.1 Existence

The idea of existence proof is as follows: we construct a regularized nonlinearity, and get rid off the degeneracy. We then replace the regularized problem by a semi-discrete approximation, which we solve by Schauder's fixed-point principle. After constructing suitable a priori estimates and compactness we can converge from the semi-discrete approximation to the regularized problem. The similar procedure we repeat for regularized problem to get uniform a priori estimates and compactness results, which finally give convergence to the original problem. We devide our existence proof into a sequence of steps.

3.1.1 Regularized problems

At first we modify the moblity. We introduce a positive mobility μ_ε as

$$\mu_\varepsilon(u) := \begin{cases} \mu(\varepsilon) & \text{for} \quad u \leq \varepsilon, \\ \mu(u) & \text{for} \quad \varepsilon < u \leq 1-\varepsilon, \\ \mu(1-\varepsilon) & \text{for} \quad u > 1-\varepsilon. \end{cases} \tag{3.1}$$

This means that we symmetrically cut the mobility and constantly extend it to all of $\mathbb{R}$. Because of (2.5) we also do this for $f''(u)$ in the same manner. In the case of $f'(u)$ we use the corresponding linear extension and for $f(u)$ we give the following quadratic extension:

$$f_\varepsilon(u) := \begin{cases} \frac{1}{2}f''(\varepsilon)u^2 + b_1 u + c_1 & \text{for} \quad u \leq \varepsilon, \\ f(u) & \text{for} \quad \varepsilon < u \leq 1-\varepsilon, \\ \frac{1}{2}f''(1-\varepsilon)u^2 + b_2 u + c_2 & \text{for} \quad u > 1-\varepsilon, \end{cases} \tag{3.2}$$

where

$$b_1 := \log\left(\frac{\varepsilon}{1-\varepsilon}\right) + f''(\varepsilon)\varepsilon, \quad c_1 := \log(1-\varepsilon) + \frac{1}{2}f''(\varepsilon)\varepsilon^2,$$

$$b_2 := \log\left(\frac{1-\varepsilon}{\varepsilon}\right) + f''(1-\varepsilon)(1-\varepsilon), \quad c_2 := \log\varepsilon + \frac{1}{2}f''(1-\varepsilon)(1-\varepsilon)^2.$$

Furthermore we introduce the truncation

$$\Pi u := \begin{cases} 1 & \text{for} \quad u \geq 1, \\ u & \text{for} \quad 0 < u < 1, \\ 0 & \text{for} \quad u \leq 0. \end{cases} \tag{3.3}$$

For the system (2.6)-(2.8) we get by (3.1) the *regularized system:*

$$\int_0^T \langle u_t, \varphi\rangle\, dt + \int_0^T \int_\Omega (\nabla u + \mu_\varepsilon \nabla(w+\psi)) \cdot \nabla\varphi dx dt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{3.4}$$

$$\gamma \int_0^T \langle \nabla\psi_t, \nabla\varphi\rangle\, dt + \int_0^T \int_\Omega \psi\varphi dx dt = \int_0^T \langle u_t, \varphi\rangle\, dt, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{3.5}$$

$$w = P(1 - 2\Pi u) \text{ a.e. } (t,x) \in Q_T. \tag{3.6}$$

Remark 5 We have by **(A2)** and **(A4)**

$$\|w\|^2_{H^1(\Omega)} \leq r_2^2 \|1 - 2\Pi u\|_{L^2(\Omega)} \leq r_2^2 |\Omega|. \tag{3.7}$$

Remark 6 $\exists \varepsilon_0 := \varepsilon_0(w)$ so that $\forall \varepsilon \leq \varepsilon_0$

$$F_{NL,\varepsilon}(u) := \int_\Omega \left(f_\varepsilon(u) + \frac{1}{2}uw\right) dx \geq -C_F,$$

where $C_F > 0$.

Proof. Using **(A1)**, (3.2) and (3.6) we see that it depends on the choice of ε to ensure that $f_\varepsilon(u)$ dominates $\frac{1}{2}uw$. Thus, there exists an $\varepsilon_0 = \varepsilon_0(w)$ so that $\forall \varepsilon \leq \varepsilon_0$ this is true.

<u>Existence result for the regularized problem</u> We will denote the solution to the regularized system (3.4)-(3.6) by $(u_\varepsilon, w_\varepsilon, \psi_\varepsilon)$. The strategy of constructing solutions to (3.4)-(3.6) is to employ a semi-discrete approximation. To this end, let $M \in N$ be given and $h := T/M$. In the sequel, we will denote by $C_i, i \in \mathbb{N}$, positive constants that may depend on Ω, T and the initial data, but not on M or $m \in \{1, \ldots, M\}$.

For $1 \leq m \leq M$, we consider the *semi-discrete problem* on the time level $t := mh$ for the unknown functions $u^m, w^m, \psi^m : \Omega \to \mathbb{R}$ given by

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\varphi dx + \int_\Omega \left[\nabla u^m + \mu_\varepsilon(u^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right]\cdot\nabla\varphi dx = 0, \quad (3.8)$$

$$\frac{\gamma}{h}\int_\Omega \nabla(\psi^m - \psi^{m-1})\cdot\nabla\varphi dx + \int_\Omega \psi^m\varphi dx = \frac{1}{h}\int_\Omega (u^m - u^{m-1})\varphi dx, \quad \forall\varphi\in H^1(\Omega), \quad (3.9)$$

$$w^m = P(1 - 2\Pi u^m) \text{ a.e. } x\in\Omega. \quad (3.10)$$

For $1 \leq m \leq M$ (3.8)-(3.10) is a nonlinear elliptic system. Note that $u^0 = u_0, \psi^0 = \psi_0$.

Remark 7 For $\varphi = 1$ we get from (3.8)-(3.10)

$$\int_\Omega u^m dx = \int_\Omega u^{m-1}dx = \cdots = \int_\Omega u_0 dx = \bar{u}_0|\Omega|, \quad (3.11)$$

$$\int_\Omega \psi^m = 0. \quad (3.12)$$

We prove existence of approximate solutions step by step via Schauder's fixed-point principle.

Lemma 1 *Suppose that the assumptions* **(A1)** *to* **(A4)** *and* **(B1)** *to* **(B4)** *hold. Then for every* $m \in \{1, ..., M\}$ *there exists a triple of functions* $(u^m, w^m, \psi^m) \in H^1(\Omega)\times H^{1,\infty}(\Omega)\times H^2(\Omega)$ *satifying (3.8)-(3.10).*

Proof. 1. Our proof is based on the application of Schauder's fixed-point principle. Let $m \in \{1, ..., M\}$ be fixed but arbitrary. For a given $u^m \in L^2$ we consider the *auxiliary linear problems*

$$\int_\Omega \nabla(\mathcal{T}_m u^m)\cdot\nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_m u^m\varphi dx = \int_\Omega g_2\varphi dx, \quad \forall\varphi\in H^1(\Omega), \quad (3.13)$$

$$\int_\Omega \nabla\psi^m\cdot\nabla\varphi dx + \frac{h}{\gamma}\int_\Omega \psi^m\varphi dx = \int_\Omega g_1\varphi dx, \quad \forall\varphi\in H^1(\Omega), \quad (3.14)$$

where

$$\begin{aligned}\int_\Omega g_1\varphi dx &:= \frac{1}{\gamma}\int_\Omega (u^m - u^{m-1})\varphi dx + \int_\Omega \nabla\psi^{m-1}\cdot\nabla\varphi dx, \quad \forall\varphi\in H^1(\Omega),\\ \int_\Omega g_2\varphi dx &:= \int_\Omega \mu_\varepsilon(u^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\cdot\nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_{m-1}u^{m-1}\varphi dx.\end{aligned} \quad (3.15)$$

The existence and uniqueness theory of (3.13)-(3.14) is standard and can be found in [13]. The strategy is to convert the integral expression in (3.13)-(3.15) into linear- and bilinear-forms and use the Lax-Milgram Theorem. From ([20], Corollary 2.2.2.4), respectively, we find that for a given $u^m \in L^2$ and consequently a given $g_1 \in L^2(\Omega)$ in (3.15) the linear equation (3.14) admits a unique solution $\psi^m \in H^2(\Omega)$. Setting $w^m = P(1 - 2\Pi u^m) \in H^{1,\infty}(\Omega)$, we find (3.10). Finally again from ([20], Corollary 2.2.2.4), respectively, we conclude that for a given $g_2 \in L^2(\Omega)$ (3.13) admits a unique solution $\mathcal{T}_m u^m \in H^1(\Omega)$.

2. Thus, we have properly defined a fixed-point operator $\mathcal{T}_m : L^2(\Omega) \longrightarrow L^2(\Omega)$. We can apply Schauder's theorem, if we are able to prove, that $\mathcal{T}_m : L^2(\Omega) \longrightarrow L^2(\Omega)$ is completely continuous and $\mathcal{T}_m[\mathcal{B}] \subset \mathcal{B}$ hold true for a closed ball $\mathcal{B} \subset L^2(\Omega)$ with a radius depending only on the data of the problem.

3. Let $\psi^m \in H^2(\Omega)$ be a solution of (3.14). We obtain using ψ^m as a testfunction in (3.14)

$$\int_\Omega |\nabla\psi^m|^2 dx + \frac{h}{\gamma}\int_\Omega |\psi^m|^2 dx \le \frac{1}{\gamma}\int_\Omega (u^m - u^{m-1})\psi^m dx + \int_\Omega \nabla\psi^{m-1}\cdot\nabla\psi^m dx.$$

Applying Young's inequality in the form

$$\frac{1}{\gamma}\int_\Omega (u^m - u^{m-1})\psi^m dx \le \frac{\epsilon}{2\gamma}\int_\Omega |u^m - u^{m-1}|^2 dx + \frac{1}{2\gamma\epsilon}\int_\Omega |\psi^m|^2 dx,$$

we get

$$\int_\Omega |\nabla\psi^m|^2 dx + \frac{2h}{\gamma}\int_\Omega |\psi^m|^2 dx \le \frac{\epsilon}{\gamma}\int_\Omega |u^m - u^{m-1}|^2 dx + \int_\Omega |\nabla\psi^{m-1}|^2 dx + \frac{1}{\gamma\epsilon}\int_\Omega |\psi^m|^2 dx.$$

Using the Poincaré inequality for the last term we get

$$\left(1 - \frac{c_p}{\gamma\epsilon}\right)\int_\Omega |\nabla\psi^m|^2 dx + \frac{2h}{\gamma}\int_\Omega |\psi^m|^2 dx \le \frac{\epsilon}{\gamma}\int_\Omega |u^m - u^{m-1}|^2 dx + \int_\Omega |\nabla\psi^{m-1}|^2 dx.$$

Choosing $\epsilon = 2c_p/\gamma$ we finally conclude

$$\int_\Omega |\nabla\psi^m|^2 dx + \frac{4h}{\gamma}\int_\Omega |\psi^m|^2 dx \le \frac{4c_p}{\gamma^2}\int_\Omega |u^m - u^{m-1}|^2 dx + 2\int_\Omega |\nabla\psi^{m-1}|^2 dx. \tag{3.16}$$

4. Let $\mathcal{T}_m u^m \in H^1(\Omega)$ be a solution of (3.13). Applying the admissible test function $\varphi = \mathcal{T}_m u^m \in H^1(\Omega)$ in (3.13) and Young's inequality we get the estimate

$$\begin{aligned}&\int_\Omega |\nabla(\mathcal{T}_m u^m)|^2 dx + \frac{1}{h}\int_\Omega |\mathcal{T}_m u^m|^2 dx = \\ &\quad\int_\Omega \mu_\varepsilon(u^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\cdot\nabla(\mathcal{T}_m u^m) dx + \frac{1}{h}\int_\Omega (\mathcal{T}_{m-1}u^{m-1})(\mathcal{T}_m u^m) dx.\end{aligned} \tag{3.17}$$

Young's inequality

$$\int_\Omega \mu_\varepsilon(u^m)\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\cdot\nabla(\mathcal{T}_m u^m)dx$$
$$\leq \frac{1}{4}\int_\Omega \left|\mu_\varepsilon(u^m)\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\right|^2 dx + \int_\Omega |\nabla(\mathcal{T}_m u^m)|^2 dx,$$

gives for (3.17) the estimate

$$\begin{aligned}\frac{1}{2h}\int_\Omega |\mathcal{T}_m u^m|^2 dx &\leq \frac{1}{4}\int_\Omega \left|\mu_\varepsilon(u^m)\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\right|^2 dx + \frac{1}{2h}\int_\Omega |\mathcal{T}_{m-1}u^{m-1}|^2 dx\\ &\leq \frac{2}{4^3}\int_\Omega \left|\nabla\left(\frac{w^m+w^{m-1}}{2}\right)\right|^2 dx + \frac{2}{4^3}\int_\Omega |\nabla\psi^m|^2\, dx\\ &\quad+\frac{1}{2h}\int_\Omega |\mathcal{T}_{m-1}u^{m-1}|^2 dx. \end{aligned}\tag{3.18}$$

We obtain by the estimates (3.16), (3.18) and (3.7)

$$\begin{aligned}\|\mathcal{T}_m u^m\|^2_{L^2(\Omega)} \leq& \frac{hr_2^2|\Omega|}{4^2} + \frac{2h}{4^2}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)} + \|\mathcal{T}_{m-1}u^{m-1}\|^2_{L^2(\Omega)}\\ &+ \frac{2h}{4^2}\|u^{m-1}\|^2_{L^2(\Omega)} + \frac{2h}{4^2}\|u^m\|^2_{L^2(\Omega)}.\end{aligned}$$

That means, we have $\|\mathcal{T}_m u^m\|^2_{L^2(\Omega)} \leq \lambda^2$ for all $u^m \in L^2(\Omega)$, if we choose h so that $1-h/8 =: 1/\beta > 0$ and fix radius $\lambda > 0$ by

$$\lambda \equiv \frac{h\beta r_2^2|\Omega|}{4^2} + \frac{2h\beta}{4^2}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)} + \beta\|\mathcal{T}_{m-1}u^{m-1}\|^2_{L^2(\Omega)} + \frac{2h\beta}{4^2}\|u^{m-1}\|^2_{L^2(\Omega)}.$$

Hence, we get $\mathcal{T}_m[\mathcal{B}] \subset \mathcal{B}$ for a closed ball $\mathcal{B} := \{u^m \in L^2(\Omega) : \|u^m\|_{L^2(\Omega)} \leq \lambda\}$.

5. To show the continuity of $\mathcal{T}_m$, let $\{u_i^m\}_{i\in\mathbb{N}} \subset L^2(\Omega)$ be a sequence such that $\lim_{i\to\infty}\|u_i^m - u^m\|_{L^2(\Omega)} = 0$. For every $i \in \mathbb{N}$ there exists a uniquely determined solution $\mathcal{T}_m u_i^m \in H^1(\Omega)$ of the problem

$$\int_\Omega \nabla(\mathcal{T}_m u_i^m)\cdot\nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_m u_i^m \varphi dx = \int_\Omega g_{i,2}\varphi dx, \quad \forall\varphi \in H^1(\Omega),$$
$$\int_\Omega \nabla\psi_i^m\cdot\nabla\varphi dx + \frac{h}{\gamma}\int_\Omega \psi_i^m\varphi dx = \int_\Omega g_{1,i}\varphi dx, \quad \forall\varphi \in H^1(\Omega),$$

where

$$\int_\Omega g_{i,1}\varphi dx := \frac{1}{\gamma}\int_\Omega (u_i^m - u_i^{m-1})\varphi dx + \int_\Omega \nabla\psi_i^{m-1}\cdot\nabla\varphi dx, \quad \forall\varphi\in H^1(\Omega),$$

$$\int_\Omega g_{i,2}\varphi dx := \int_\Omega \mu_\varepsilon(u_i^m)\nabla\left(\frac{w_i^m + w_i^{m-1}}{2} + \psi_i^m\right)\cdot\nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_{m-1}u_i^{m-1}\varphi dx.$$

Because $\mathcal{T}_m u^m \in H^1(\Omega)$ is a solution of the problem (3.13), for every $i\in\mathbb{N}$ it follows

$$\int_\Omega \nabla(\mathcal{T}_m u_i^m - \mathcal{T}_m u^m)\cdot\nabla\varphi dx + \frac{1}{h}\int_\Omega (\mathcal{T}_m u_i^m - \mathcal{T}_m u^m)\varphi dx = \int_\Omega (g_{i,2} - g_2)\varphi dx, \tag{3.19}$$

$$\int_\Omega \nabla(\psi_i^m - \psi^m)\cdot\nabla\varphi dx + \frac{h}{\gamma}\int_\Omega (\psi_i^m - \psi^m)\varphi dx = \int_\Omega (g_{1,i} - g_1)\varphi dx, \quad \forall\varphi\in H^1(\Omega), \tag{3.20}$$

where

$$\begin{aligned}\int_\Omega (g_{i,1} - g_1)\varphi dx \;\; &:= \;\; \frac{1}{\gamma}\int_\Omega (u_i^m - u^m)\varphi dx - \frac{1}{\gamma}\int_\Omega (u_i^{m-1} - u^{m-1})\varphi dx \\ &\quad + \int_\Omega \nabla(\psi_i^{m-1} - \psi^{m-1})\cdot\nabla\varphi dx, \quad \forall\varphi\in H^1(\Omega),\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\int_\Omega (g_{i,2} - g_2)\varphi dx \;\; &:= \;\; \int_\Omega (\mu_\varepsilon(u^m) - \mu_\varepsilon(u_i^m))\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\cdot\nabla\varphi dx \\ &\quad + \int_\Omega \mu_\varepsilon(u_i^m)\nabla(w_i^m - w^m)\cdot\nabla\varphi dx \\ &\quad + \int_\Omega \mu_\varepsilon(u_i^m)\nabla(w_i^{m-1} - w^{m-1})\cdot\nabla\varphi dx \\ &\quad + \int_\Omega \mu_\varepsilon(u_i^m)\nabla(\psi_i^m - \psi^m)\cdot\nabla\varphi dx \\ &\quad + \frac{1}{h}\int_\Omega (\mathcal{T}_{m-1}u_i^{m-1} - \mathcal{T}_{m-1}u^{m-1})\varphi dx, \quad \forall\varphi\in H^1(\Omega).\end{aligned} \tag{3.22}$$

where we have made use of the equivalence in (3.22)

$$\begin{aligned}&\left[\mu_\varepsilon(u_i^m)\nabla\left(\frac{w_i^m + w_i^{m-1}}{2} + \psi_i^m\right) - \mu_\varepsilon(u^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right] \\ &= (\mu_\varepsilon(u^m) - \mu_\varepsilon(u_i^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right) + \mu_\varepsilon(u_i^m)\nabla(w_i^m - w^m) \\ &+ \mu_\varepsilon(u_i^m)\nabla(w_i^{m-1} - w^{m-1}) + \mu_\varepsilon(u_i^m)\nabla(\psi_i^m - \psi^m).\end{aligned}$$

Using in (3.20) the testfunction $\varphi = (\psi_i^m - \psi^m)$ we find that

$$\int_\Omega |\nabla(\psi_i^m - \psi^m)|^2 dx + \frac{h}{\gamma} \int_\Omega |\psi_i^m - \psi^m|^2 dx = \int_\Omega (g_{i,1} - g_1)(\psi_i^m - \psi^m) dx.$$

Similar calculations like in (3.16) give

$$\begin{aligned} \|\nabla(\psi_i^m - \psi^m)\|^2_{L^2(\Omega)} \leq & \frac{8c_p}{\gamma^2} \|u_i^m - u^m\|^2_{L^2(\Omega)} + \frac{8c_p}{\gamma^2} \|u_i^{m-1} - u^{m-1}\|^2_{L^2(\Omega)} \\ & + 2\|\nabla(\psi_i^{m-1} - \psi^{m-1})\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.23}$$

where c_p is the Poincaré constant.
Applying $\varphi = \mathcal{T}_m u_i^m - \mathcal{T}_m u^m \in H^1(\Omega)$ as a testfuntion in (3.19) we get

$$\begin{aligned} \int_\Omega |\nabla(\mathcal{T}_m u_i^m - \mathcal{T}_m u^m)|^2 dx + \frac{1}{h} \int_\Omega |\mathcal{T}_m u_i^m - \mathcal{T}_m u^m|^2 dx \\ = \int_\Omega (g_{i,2} - g_2)(\mathcal{T}_m u_i^m - \mathcal{T}_m u^m) dx. \end{aligned}$$

Young's inequality gives

$$\begin{aligned} \frac{1}{h} \int_\Omega |\mathcal{T}_m u_i^m - \mathcal{T}_m u^m|^2 dx \leq & 4 \int_\Omega \left| (\mu_\varepsilon(u^m) - \mu_\varepsilon(u_i^m)) \nabla \left(\frac{w^m + w^{m-1}}{2} + \psi^m \right) \right|^2 dx \\ & + 8 \int_\Omega |\mu_\varepsilon(u_i^m) \nabla (w_i^m - w^m)|^2 dx \\ & + 4^2 \int_\Omega |\mu_\varepsilon(u_i^m) \nabla (w_i^{m-1} - w^{m-1})|^2 dx \\ & + 4^2 \int_\Omega |\mu_\varepsilon(u_i^m) \nabla (\psi_i^m - \psi^m)|^2 dx \\ & =: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We will treat each summand separately: Because of the continuity of Π we get

$$I_2 \leq 8\|w_i^m - w^m\|^2_{H^1(\Omega)} \leq 2\|P(\Pi u_i^m - \Pi u^m)\|^2_{H^1(\Omega)} \leq 2Cr_2^2 \|u_i^m - u^m\|^2_{L^2(\Omega)}.$$

The summand I_3 can be trated in a similar way like I_2. For I_4 we use the estimate (3.23). In the limit process $i \longrightarrow \infty$ the expression $(\mu_\varepsilon(u^m) - \mu_\varepsilon(u_i^m))$ tends pointwise to zero, because of the Lipschitz continuity of $u^m \mapsto \mu_\varepsilon(u^m)$ and the convergence $\lim_{i\to\infty} \|u_i^m - u^m\|_{L^2(\Omega)} = 0$. Hence applying Lebesgue's theorem I_1 tends to zero. The convergence of I_2-I_4 follows from the boundedness of $\mu_\varepsilon(u_i^m)$ and the convergence $\lim_{i\to\infty} \|u_i^m - u^m\|_{L^2(\Omega)} = 0$.

6. Bacause of $\mathcal{T}_m[L^2(\Omega)] \in H^1(\Omega)$ and the completely continuous embedding of $H^1(\Omega)$ into $L^2(\Omega)$ (A.8), the fixed-point map $\mathcal{T}_m : L^2(\Omega) \longrightarrow L^2(\Omega)$ is completely continuous. Having in mind the first step of the proof, Schauder's fixed-point theorem yields a solution $u^m \in H^1(\Omega)\cap\mathcal{B}$ of the equation $\mathcal{T}u^m = u^m$. Setting $w^m = P(1-\Pi u^m) \in H^{1,\infty}(\Omega)$, we have found a solution $(u^m, w^m, \psi^m) \in H^1(\Omega) \times H^{1,\infty}(\Omega) \times H^2(\Omega)$ of the problem (3.8)-(3.10).

Now we have to derive discrete a priori estimates

Lemma 2 (Discrete energy estimate) *Let (u^m, w^m, ψ^m) be solution of (3.8)-(3.10) for every $m \in \{1, ..., M\}$. Then*

$$\frac{\gamma}{2}\sum_{m=1}^{M} \|\nabla(\psi^m - \psi^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\max_{1\le k\le M}\|\nabla\psi^k\|^2_{L^2(\Omega)} + \sum_{m=1}^{M} h\|\psi^m\|^2_{L^2(\Omega)}, + \sum_{m=1}^{M} h\int_\Omega \mu_\varepsilon(u^m)|\nabla v^m|^2 dx \le C_1 \tag{3.24}$$

and

$$\sum_{m=1}^{M} h\int_\Omega |\nabla u^m|^2 dx \le C_2(T). \tag{3.25}$$

Proof. 1. Because of **Lemma 1** $\varphi = v^m = f'_\varepsilon(u^m) + \frac{w^m+w^{m-1}}{2} + \psi^m \in H^1(\Omega)$ is an admissible testfunction in (3.8) and we find

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\left(f'_\varepsilon(u^m) + \frac{w^m + w^{m-1}}{2} + \psi^m\right) dx + \int_\Omega \mu_\varepsilon(u^m)|\nabla v^m|^2 dx = 0.$$

We will estimate the first summand term by term.

2. The first term can be estimated as follows

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})f'_\varepsilon(u^m)dx \ge \frac{1}{h}\int_\Omega f_\varepsilon(u^m) - f_\varepsilon(u^{m-1})dx,$$

where we have used the convexity of $f_\varepsilon(u)$, (see **(A1)**).

3. In order to estimate the second term we use the symmetry of P (see **Remark 1**).

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\left(\frac{w^m + w^{m-1}}{2}\right) dx = \frac{1}{h}\int_\Omega \frac{1}{4}\{(u^m + u^{m-1})(w^m - w^{m-1}) + (u^m - u^{m-1})(w^m + w^{m-1})\}dx = \frac{1}{h}\int_\Omega \frac{1}{2}\{u^m w^m - u^{m-1}w^{m-1}\}dx.$$

4. For the third term we use the testfunction $\varphi = \psi^m$ in (3.9) to get

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\psi^m dx = \frac{\gamma}{h}\int_\Omega \nabla(\psi^m - \psi^{m-1})\cdot\nabla\psi^m dx + \int_\Omega |\psi^m|^2 dx.$$

The above estimates give

$$\frac{\gamma}{h}\int_\Omega \nabla(\psi^m-\psi^{m-1})\cdot\nabla\psi^m dx+\int_\Omega|\psi^m|^2dx+\int_\Omega\mu_\varepsilon(u^m)|\nabla v^m|^2dx \\ +\frac{1}{h}\left[F_{NL,\varepsilon}(u^m)-F_{NL,\varepsilon}(u^{m-1})\right]\le 0. \tag{3.26}$$

Note that

$$\int_\Omega \nabla(\psi^m-\psi^{m-1})\cdot\nabla\psi^m dx=\frac{1}{2}\|\nabla(\psi^m-\psi^{m-1})\|^2_{L^2(\Omega)}+\frac{1}{2}\|\nabla\psi^m\|^2_{L^2(\Omega)} \\ -\frac{1}{2}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)}. \tag{3.27}$$

Applying the trick (3.27) for (3.26) we get

$$\frac{\gamma}{2h}\|\nabla(\psi^m-\psi^{m-1})\|^2_{L^2(\Omega)}+\frac{\gamma}{2h}\|\nabla\psi^m\|^2_{L^2(\Omega)}-\frac{\gamma}{2h}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)}+\|\psi^m\|^2_{L^2(\Omega)} \\ +\int_\Omega\mu_\varepsilon(u^m)|\nabla v^m|^2dx+\frac{1}{h}\left[F_{NL,\varepsilon}(u^m)-F_{NL,\varepsilon}(u^{m-1})\right]\le 0. \tag{3.28}$$

We multiply (3.28) by h and sum both sides from $m=1$ to $m=k$, where $1\le k\le M$. We find that

$$\frac{\gamma}{2}\sum_{m=1}^k\|\nabla(\psi^m-\psi^{m-1})\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\|\nabla\psi^k\|^2_{L^2(\Omega)}+\sum_{m=1}^k h\|\psi^m\|^2_{L^2(\Omega)} \\ +\sum_{m=1}^k h\int_\Omega\mu_\varepsilon(u^m)|\nabla v^m|^2dx\le F_{NL}(u_0)-F_{NL,\varepsilon}(u^k)+\frac{\gamma}{2}\|\nabla\psi^0\|^2_{L^2(\Omega)}.$$

Using **Remark 6** we conclude

$$\frac{\gamma}{2}\sum_{m=1}^M\|\nabla(\psi^m-\psi^{m-1})\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\max_{1\le k\le M}\|\nabla\psi^k\|^2_{L^2(\Omega)}+\sum_{m=1}^M h\|\psi^m\|^2_{L^2(\Omega)} \\ +\sum_{m=1}^M h\int_\Omega\mu_\varepsilon(u^m)|\nabla v^m|^2dx\le F_{NL}(u_0)-F_{NL,\varepsilon}(u^M)+\frac{\gamma}{2}\|\nabla\psi^0\|^2_{L^2(\Omega)}=:C_1.$$

Defining $\tilde{w}^m:=\frac{w^m+w^{m-1}}{2}+\psi^m$ we have the following estimate

$$\begin{aligned}\int_\Omega\mu_\varepsilon(u^m)|\nabla(f'_\varepsilon(u^m)+\tilde{w}^m)|^2dx&=\int_\Omega\left(f''_\varepsilon(u^m)|\nabla u^m|^2+2\nabla u^m\cdot\nabla\tilde{w}^m+\frac{|\nabla\tilde{w}^m|^2}{f''_\varepsilon(u^m)}\right)dx\\&\ge\int_\Omega\left(\frac{f''_\varepsilon(u^m)}{2}|\nabla u^m|^2-\frac{|\nabla\tilde{w}^m|^2}{f''_\varepsilon(u^m)}\right)dx\\&\ge\int_\Omega\left(2|\nabla u^m|^2-\frac{1}{4}|\nabla\tilde{w}^m|^2\right)dx,\end{aligned} \tag{3.29}$$

where we have used Young's inequality and the fact that $f''_\varepsilon(u^m) \geq 4$. We multiply (3.29) by h and sum both sides from $m = 1$ to $m = k$, where $1 \leq k \leq M$.

$$C_1 \geq \sum_{m=1}^{k} h \int_\Omega \left(2|\nabla u^m|^2 - \frac{1}{4}|\nabla \tilde{w}^m|^2 \right) dx,$$

where C_1 is the constant in (3.24). The definition of $\tilde{w}^m$ and **(A2)** and **(B4)** give

$$\sum_{m=1}^{k} h \int_\Omega |\nabla u^m|^2 dx \leq \frac{1}{4} \sum_{m=1}^{k} h \left\| \nabla \frac{w^m + w^{m-1}}{2} \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \sum_{m=1}^{k} h \|\nabla \psi^m\|_{L^2(\Omega)}^2 + \frac{C_1}{2}.$$

Finally we obtain using (3.24) and (3.7)

$$\begin{aligned}\sum_{m=1}^{M} h \int_\Omega |\nabla u^m|^2 dx &\leq T \left\{ r_2^2|\Omega| + 4r_2^2 + \frac{1}{4} \max_{1\leq k\leq M} \|\nabla \psi^k\|_{L^2(\Omega)}^2 \right\} + \frac{C_1}{2} \\ &\leq C_2(T).\end{aligned}$$

Lemma 3 *Let (u^m, w^m, ψ^m) be solution of (3.8)-(3.10) for every $m \in \{1, ..., M\}$. Then*

$$\max_{1\leq k\leq M} \|\Delta \psi^k\|_{L^2(\Omega)}^2 \leq C_3. \tag{3.30}$$

Proof. 1. Because of **Lemma 1** $\Delta \psi^m \in L^2(\Omega)$ exists, thus an admissible testfunction in (3.9)

$$\frac{\gamma}{h} \int_\Omega \Delta(\psi^m - \psi^{m-1}) \Delta \psi^m dx + \int_\Omega |\nabla \psi^m|^2 dx + \frac{1}{h} \int_\Omega (u^m - u^{m-1}) \Delta \psi^m dx = 0.$$

2. Applying the testfunction $-u^m/\gamma$ in (3.9) we have

$$\frac{1}{h} \int_\Omega \Delta(\psi^m - \psi^{m-1}) u^m dx - \frac{1}{\gamma} \int_\Omega \psi^m u^m dx + \frac{1}{\gamma h} \int_\Omega (u^m - u^{m-1}) u^m dx = 0.$$

3. We use the same trick as in (3.27) and the identity

$$\begin{aligned}&\int_\Omega \left((u^m - u^{m-1}) \Delta \psi^m + (\Delta \psi^m - \Delta \psi^{m-1}) u^m \right) dx \\ &= \int\limits_\Omega u^m \Delta \psi^m dx - \int\limits_\Omega u^{m-1} \Delta \psi^{m-1} dx + \int_\Omega \left((u^m - u^{m-1})(\Delta \psi^m - \Delta \psi^{m-1}) \right) dx,\end{aligned}$$

to get

$$\begin{aligned}&\|\Delta \psi^m - \Delta \psi^{m-1}\|_{L^2(\Omega)}^2 + \|\Delta \psi^m\|_{L^2(\Omega)}^2 + \frac{2h}{\gamma} \|\nabla \psi^m\|_{L^2(\Omega)}^2 + \frac{1}{\gamma^2} \|u^m - u^{m-1}\|_{L^2(\Omega)}^2 + \frac{1}{\gamma^2} \|u^m\|_{L^2(\Omega)}^2 \\ &= \|\Delta \psi^{m-1}\|_{L^2(\Omega)}^2 + \frac{1}{\gamma^2} \|u^{m-1}\|_{L^2(\Omega)}^2 - \frac{2h}{\gamma^2} \int_\Omega \psi^m u^m dx \\ &- \frac{2}{\gamma} \int_\Omega \left(u^m \Delta \psi^m - u^{m-1} \Delta \psi^{m-1} \right) - \frac{2}{\gamma} \int_\Omega (u^m - u^{m-1})(\Delta \psi^m - \Delta \psi^{m-1}) dx.\end{aligned} \tag{3.31}$$

4. Using Young's inequality in the following way

$$\frac{2}{\gamma}\int_\Omega (u^m - u^{m-1})(\Delta\psi^m - \Delta\psi^{m-1})dx \le \frac{1}{\gamma^2}\|u^m - u^{m-1}\|^2_{L^2(\Omega)} + \|\Delta\psi^m - \Delta\psi^{m-1}\|^2_{L^2(\Omega)},$$

$$\frac{2}{\gamma}\int_\Omega u^m \Delta\psi^m dx \le \frac{2}{\gamma^2}\|u^m\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta\psi^m\|^2_{L^2(\Omega)},$$

we get from (3.31)

$$\begin{aligned}\|\Delta\psi^m\|^2_{L^2(\Omega)} + \frac{4h}{\gamma}\|\nabla\psi^m\|^2_{L^2(\Omega)} =& 3\|\Delta\psi^{m-1}\|^2_{L^2(\Omega)} + \frac{6}{\gamma^2}\|u^{m-1}\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\|u^m\|^2_{L^2(\Omega)} \\ &+ \frac{h}{\gamma^3}\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{2h}{\gamma^2}\|\psi^m\|^2_{L^2(\Omega)} + \frac{2h}{\gamma^2}\|u^m\|^2_{L^2(\Omega)}.\end{aligned}$$

We sum both sides from $m = 1$ to $m = k$, where $1 \le k \le M$, and find

$$\begin{aligned}\|\Delta\psi^k\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\sum_{m=1}^{k} h\|\nabla\psi^m\|^2_{L^2(\Omega)} =& 3\|\Delta\psi^0\|^2_{L^2(\Omega)} + \frac{6}{\gamma^2}\|u^0\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\|u^k\|^2_{L^2(\Omega)} \\ &+ \frac{1}{\gamma^3}\sum_{m=1}^{k} h\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\sum_{m=1}^{k} h\|\psi^m\|^2_{L^2(\Omega)} \\ &+ \frac{2}{\gamma^2}\sum_{m=1}^{k} h\|u^m\|^2_{L^2(\Omega)}.\end{aligned}$$

The energy estimate (3.24) and (3.25) give (3.30). To indicate the dependence on M, we denote for any $M \in \mathbb{N}$ the solutions of (3.8)-(3.10) by (u^m_M, w^m_M, ψ^m_M). We define the piecewise linear

$$\hat{u}_M(x,t) = u^m + \frac{t - mh}{h}(u^m - u^{m-1}) \text{ for } t \in [(m-1)h, mh], \tag{3.32}$$

$$\hat{\psi}_M(x,t) = \psi^m + \frac{t - mh}{h}(\psi^m - \psi^{m-1}) \text{ for } t \in [(m-1)h, mh], \tag{3.33}$$

as well as the constant interpolates

$$\bar{u}_M(x,t) = u^m \quad \text{for} \quad t \in [(m-1)h, mh], \tag{3.34}$$

$$\bar{w}_M(x,t) = \frac{w^m + w^{m-1}}{2} \quad \text{for} \quad t \in [(m-1)h, mh], \tag{3.35}$$

$$\bar{\psi}_M(x,t) = \psi^m \quad \text{for} \quad t \in [(m-1)h, mh], \tag{3.36}$$

for $1 \le m \le M$. With these notations, the variational equations (3.8)-(3.10) can be written as

$$\int_\Omega \hat{u}_{M,t}(x,t)\varphi(x)dx + \int_\Omega (\nabla\bar{u}_M(x,t) + \mu_\varepsilon(\bar{u}_M)\nabla(\bar{w}_M(x,t) + \bar{\psi}_M(x,t)))\cdot\nabla\varphi(x)\,dx = 0,$$

$$\gamma\int_\Omega \nabla\hat{\psi}_{M,t}(x,t)\cdot\nabla\varphi dx + \int_\Omega \bar{\psi}_M(x,t)h(x)\,dx = \int_\Omega \hat{u}_{M,t}(x,t)\varphi(x)dx, \quad \forall\varphi \in H^1(\Omega),$$

for almost every $t \in (0,T)$. Consequently,

$$\begin{aligned}&\int_0^T\int_\Omega \hat{u}_{M,t}\varphi\,dxdt + \int_0^T\int_\Omega (\nabla \bar{u}_M + \mu_\varepsilon(\bar{u}_M)\nabla(\bar{w}_M + \bar{\psi}_M))\cdot\nabla\varphi\,dxdt = 0,\\ &\gamma\int_0^T\int_\Omega \nabla\hat{\psi}_{M,t}\cdot\nabla\varphi dx\,dt + \int_0^T\int_\Omega \bar{\psi}_M\varphi\,dxdt = \int_0^T\int_\Omega \hat{u}_{M,t}\varphi dxdt, \quad \forall\varphi \in L^2(0,T;H^1(\Omega)).\end{aligned} \tag{3.37}$$

We again get like in **Remark 4** using $\varphi = 1$

$$\begin{aligned}&\int_\Omega \bar{u}(t,x)dx = \int_\Omega u_0(x)dx = \text{const.},\\ &\int_0^T\int_\Omega \bar{\psi}(x)dxdt = 0.\end{aligned} \tag{3.38}$$

By virtue of the energy estimate (3.24),

$$\begin{aligned}\frac{\gamma}{2}\sum_{m=1}^M \|\nabla(\psi^m - \psi^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\sup_{0\le t\le T}\|\nabla\bar{\psi}_M(t)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\bar{\psi}_M|^2 dxdt\\ + \int_0^T\int_\Omega \mu_\varepsilon(\bar{u}_M)|\nabla\bar{v}_M|^2\,dxdt \le C_1,\end{aligned} \tag{3.39}$$

where $\bar{v}_M := f'_\varepsilon(\bar{u}_M) + \bar{w}_M + \bar{\psi}_M$. We also find from (3.25) that

$$\int_0^T\int_\Omega |\nabla\bar{u}_M|^2 dxdt \le C_2(T).$$

Using (3.38) and the generalized Poincaré inequality (A.13) we get

$$\|\bar{u}_M\|_{L^2(0,T;H^1(\Omega))} \le C_4(\sqrt{T}).$$

Moreover we find from (3.37) and (3.39)

$$\begin{aligned}\left|\int_0^T\int_\Omega \hat{u}_{M,t}\varphi dxdt\right| &\leq \left|\int_0^T\int_\Omega \mu_\varepsilon(\bar{u}_M)\nabla\bar{v}_M\cdot\nabla\varphi\, dxdt\right| \\ &\leq \left(\int_0^T\int_\Omega |\mu_\varepsilon(\bar{u}_M)\nabla\bar{v}_M|^2\, dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |\nabla\varphi|^2 dxdt\right)^{1/2} \\ &\leq \frac{1}{2}\left(\int_0^T\int_\Omega \mu_\varepsilon(\bar{u}_M)\,|\nabla\bar{v}_M|^2\, dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |\nabla\varphi|^2 dxdt\right)^{1/2} \\ &\leq \frac{C_1}{2}\left(\int_0^T\int_\Omega |\nabla\varphi|^2 dxdt\right)^{1/2}\end{aligned} \tag{3.40}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$. We get

$$\|\hat{u}_{M,t}\|_{L^2(0,T;H^1(\Omega)^*)} = \sup_{\varphi\in L^2(0,T;H^1(\Omega))}\frac{|\int_0^T\int_\Omega \hat{u}_{M,t}(x,t)\varphi dxdt|}{\|\varphi\|_{L^2(0,T;H^1(\Omega))}} \leq C_5.$$

Thus, we find

$$\begin{aligned}\left|\int_0^T\int_\Omega \nabla\hat{\psi}_{M,t}\cdot\nabla\varphi dxdt\right| \leq& \left|\int_0^T\int_\Omega \hat{u}_{M,t}\varphi dxdt\right| \\ &+\left(\int_0^T\int_\Omega |\psi^m|^2 dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |\varphi|^2 dxdt\right)^{1/2} \\ \leq& C_6\|\varphi\|^2_{L^2(0,T;H^1(\Omega))},\end{aligned} \tag{3.41}$$

and we find

$$\|\nabla\hat{\psi}_{M,t}\|_{L^2(0,T;H^1(\Omega)^*)} \leq C_6.$$

Becauce of (3.31) we get

$$\sup_{0\leq t\leq T}\|\Delta\bar{\psi}_M(t)\|^2_{L^2(\Omega)} \leq C_3(T).$$

In addition, (3.25), (3.32) and (3.34) imply that, as $M\to\infty$,

$$\|\nabla\bar{u}_M - \nabla\hat{u}_M\|_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^M \|\nabla u_M^m - \nabla u_M^{m-1}\|_{L^2(\Omega)} \to 0.$$

We also get from (3.32), (3.34) and (3.11) that, as $M \to \infty$

$$\|\bar{u}_M - \hat{u}_M\|_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^{M} \|u_M^m - u_M^{m-1}\|_{L^2(\Omega)} \to 0.$$

We obtain using the generalized Poincaré inequality (A.13) the following convergence

$$\|\bar{u}_M - \hat{u}_M\|_{L^2(0,T;H^1(\Omega))} \to 0. \tag{3.42}$$

Moreover we have with (3.12)

$$\|\bar{\psi}_M - \hat{\psi}_M\|_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^{M} \|\psi_M^m - \psi_M^{m-1}\|_{L^2(\Omega)} = 0,$$

and by (3.33), (3.36) and (3.24), as $M \to \infty$

$$\|\nabla\bar{\psi}_M - \nabla\hat{\psi}_M\|_{L^\infty(0,T;L^2(\Omega))} = \max_{0\le t\le T} \|\nabla\psi_M^m - \nabla\psi_M^{m-1}\|_{L^2(\Omega)} \to 0. \tag{3.43}$$

In conclusion, there are functions $\bar{u}, \hat{u}_t, \bar{\psi}, \hat{\psi}_t$, such that for $M \to \infty$, possibly after selecting subsequences,

$$\begin{array}{lllll}
\bar{u}_M & \longrightarrow & \bar{u} & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)), \\
\hat{u}_{M,t} & \longrightarrow & \hat{u}_t & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)^*), \\
\bar{\psi}_M & \longrightarrow & \bar{\psi} & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\
\nabla\bar{\psi}_M & \longrightarrow & \nabla\bar{\psi} & \text{weakly-star} & \text{in } L^\infty(0,T;H^1(\Omega)), \\
\nabla\hat{\psi}_{M,t} & \longrightarrow & \nabla\hat{\psi}_t & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)^*).
\end{array} \tag{3.44}$$

Taking (3.42) and (3.43) into account, we see that $\bar{u} = \hat{u}$ and $\bar{\psi} = \hat{\psi}$. It follows from (3.44) that we may pass to the limit as $M \to \infty$ in (3.37). The convergence of the linear terms in (3.37) are standard. We take a closer look on the convergence of the nonlinear term

$$\begin{aligned}
\int_0^T\int_\Omega (\mu_\varepsilon(u)\nabla(w+\psi) - \mu_\varepsilon(u_M)\nabla(w_M+\psi_M)) \cdot \nabla\varphi \, dxdt \\
= \int_0^T\int_\Omega (\mu_\varepsilon(u) - \mu_\varepsilon(u_M))\nabla(w+\psi) \cdot \nabla\varphi \, dxdt \\
+ \int_0^T\int_\Omega \mu_\varepsilon(u_M)\nabla\left[(w-w_M) + (\psi-\psi_M)\right] \cdot \nabla\varphi \, dxdt.
\end{aligned} \tag{3.45}$$

Because of the Lipschitz continuity of μ_ε and the compactness results in (3.44) the first term on the right hand side converges to zero. The second term converges to zero again by taking into account the compactness results in (3.44).

We denote the solution of the regularized problem by $(u_\varepsilon, w_\varepsilon, \psi_\varepsilon)$. We can state the following energy estimate.

Lemma 4 (Energy estimate) *There exists an* ε_0 *(see* **Remark 6***) such that for all* $0 < \varepsilon \le \varepsilon_0$ *the following estimate holds with constants* C_7, C_8 *independent of* ε*:*

$$\frac{\gamma}{2}\max_{0\le t\le T}\int_\Omega |\nabla\psi_\varepsilon(t)|^2dx + \int_0^T\int_\Omega |\psi_\varepsilon|^2dxdt + \int_0^T\int_\Omega \mu_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2dxdt \le C_7, \tag{3.46}$$

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2dxdt \le C_8. \tag{3.47}$$

Proof. 1. The function $v_\varepsilon = f'_\varepsilon(u_\varepsilon) + w_\varepsilon + \psi_\varepsilon \in L^2(0,T;H^1(\Omega))$ is a valid testfunction in (3.4). Therefore we obtain

$$\int_0^t \langle \partial_t u_\varepsilon, f'_\varepsilon(u_\varepsilon) + w_\varepsilon + \psi_\varepsilon\rangle dt = -\int_0^t\int_\Omega \mu_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2dxdt \tag{3.48}$$

for all $t \in [0,T]$. To prove this we define *steklov averaged* functions

$$u_{\varepsilon h}(t,x) := \frac{1}{h}\int_{t-h}^t u_\varepsilon(\tau,x)d\tau, \tag{3.49}$$

where we set $u_\varepsilon(t,x) = u_0(x)$ when $t \le 0$. From [23] $u_{\varepsilon h}$ converge strongly to u_ε in $L^2(0,T;H^1(\Omega))$. Because of **(A2)**, **(B4)** and the continuity of f'_ε it is easily proved that

$$\begin{array}{lllll} w_{\varepsilon h} & \longrightarrow & w_\varepsilon & \text{strongly} & \text{in} \quad L^2(0,T;H^1(\Omega)), \\ f'_{\varepsilon h}(u_{\varepsilon h}) & \longrightarrow & f'_\varepsilon(u_\varepsilon) & \text{strongly} & \text{in} \quad L^2(0,T;H^1(\Omega)). \end{array} \tag{3.50}$$

We define $g_{\varepsilon h} := f'_{\varepsilon h}(u_{\varepsilon h}) + w_{\varepsilon h}$, and $v_{\varepsilon h} := g_{\varepsilon h} + \psi_{\varepsilon h}$. It follows from (3.54) that

$$\nabla\psi_{\varepsilon h} \longrightarrow \nabla\psi_\varepsilon \text{ strongly} \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{3.51}$$

Furthermore, we can show $\partial_t u_{\varepsilon h} \longrightarrow \partial_t u_\varepsilon$ strongly in $L^2(0,T;H^1(\Omega)^*)$. For any $\varphi \in L^2(0,T;H^1(\Omega))$ we have

$$\begin{aligned} |\langle \partial_t u_{\varepsilon h} - \partial_t u_\varepsilon, \varphi\rangle| &= \frac{1}{h}\left|\int_0^T \left\langle \int_{t-h}^t (\partial_t u_\varepsilon(\tau) - \partial_t u_\varepsilon(t))d\tau, \varphi \right\rangle dt\right| \\ &= \frac{1}{h}\left|\int_0^T \left\langle \int_{-h}^0 (\partial_t u_\varepsilon(t+s) - \partial_t u_\varepsilon(t))ds, \varphi \right\rangle dt\right| \\ &\le \frac{1}{h}\int_{-h}^0 \left|\int_0^T\int_\Omega (\mu_\varepsilon(u_\varepsilon(t+s))\nabla v_\varepsilon - \mu_\varepsilon(u_\varepsilon(t))\nabla v_\varepsilon)\,\nabla\varphi dxdt\right| ds \\ &\le \max_{-h\le s\le 0} \|(\mu_\varepsilon(u_\varepsilon(t+s))\nabla v_\varepsilon(t+s) - \mu_\varepsilon(u_\varepsilon(t))\nabla v_\varepsilon(t)\|_{L^2(Q_T)}\,\|\nabla\varphi\|_{L^2(Q_T)}. \end{aligned}$$

We have

$$\begin{aligned}
\max_{-h\le s\le 0}&\|\mu_\varepsilon(u_\varepsilon(t+s))\nabla v_\varepsilon(t+s)-\mu_\varepsilon(u_\varepsilon(t))\nabla v_\varepsilon(t)\|_{L^2(Q_T)}\\
&\le \max_{-h\le s\le 0}\|[\mu_\varepsilon(u_\varepsilon(t+s))-\mu_\varepsilon(u_\varepsilon(t))]\nabla v_\varepsilon(t+s)\|_{L^2(Q_T)}\\
&\quad + C\max_{-h\le s\le 0}\|g_\varepsilon(t+s)-g_\varepsilon(t)\|_{L^2(0,T;H^1(\Omega))}\\
&\quad + C\max_{-h\le s\le 0}\|\nabla\psi_\varepsilon(t+s)-\nabla\psi_\varepsilon(t)\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

The first part of the right hand side tends as $h\to 0$ pointwise to zero, because of the Lipschitz continuity of $u_\varepsilon \mapsto \mu_\varepsilon(u_\varepsilon)$ and the convergence

$$\max_{-h\le s\le 0}\|u_\varepsilon(t+s)-u_\varepsilon(t)\|_{L^2(Q_T)}\to 0 \quad \text{as} \quad h\to 0.$$

The second and the third part follow from (3.50) and (3.51). It follows that

$$\partial_t u_{\varepsilon h}\longrightarrow \partial_t u_\varepsilon \quad \text{strongly in } L^2(0,T;H^1(\Omega)^*).$$

Using $\partial_t u_{\varepsilon h}\in L^2(0,T;L^2(\Omega))$, we have for almost all $t\in[0,T]$

$$\begin{aligned}
\int_0^t\langle\partial_t u_{\varepsilon h}, f_\varepsilon'(u_{\varepsilon h})+w_{\varepsilon h}+\psi_{\varepsilon h}\rangle dt &= \int_0^t\int_\Omega \partial_t u_{\varepsilon h}(f_\varepsilon'(u_{\varepsilon h})+w_{\varepsilon h}+\psi_{\varepsilon h})dxdt\\
&= \partial_t\int_0^t\int_\Omega\left(f_\varepsilon(u_{\varepsilon h})+\frac{1}{2}u_{\varepsilon h}w_{\varepsilon h}+\frac{1}{2}|\nabla\psi_{\varepsilon h}|^2\right)dxdt\\
&+\int_0^t\int_\Omega|\psi_{\varepsilon h}|^2dxdt\\
&=\int_\Omega\left(f_\varepsilon(u_{\varepsilon h}(t))+\frac{1}{2}u_{\varepsilon h}(t)w_{\varepsilon h}(t)+\frac{1}{2}|\nabla\psi_{\varepsilon h}(t)|^2\right)dxdt\\
&+\int_\Omega\left(f_\varepsilon(u_0)+\frac{1}{2}u_0w_0+\frac{1}{2}|\nabla\psi_0|^2\right)dxdt+\int_0^t\int_\Omega|\psi_{\varepsilon h}|^2dxdt.
\end{aligned}$$

Passing to the limit ($h\searrow 0$) in this equation, where we apply the convergence properties of $u_{\varepsilon h}$ proved above, and using **Remark 6**, (3.48) gives for almost all t

$$\begin{aligned}
\int_\Omega\frac{1}{2}|\nabla\psi_{\varepsilon h}(t)|^2+\int_0^t\int_\Omega|\psi_{\varepsilon h}|^2dxdt+\int_0^t\int_\Omega\mu_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2dxdt&\\
\le F_{NL,\varepsilon}(u_0)+\int_\Omega\frac{1}{2}|\nabla\psi_0|^2dxdt\le C_7.&
\end{aligned}$$

The proof of (3.47) is similar to the proof in the discrete case (3.25). We get further a priori estimates for $\partial_t u_\varepsilon$ and $\nabla\partial_t\psi_\varepsilon$ in a similar way to the discrete case (3.40) and (3.41).

Lemma 5 *There exists an ε_0 (see* **Remark 6***) such that for all $0<\varepsilon\leq\varepsilon_0$ the following estimate holds with a constant C_8 independent of ε:*

$$\max_{0\leq t\leq T}\|\Delta\psi_\varepsilon\|^2_{L^2(\Omega)}\leq C_{11}. \tag{3.52}$$

Proof. 1. We again make use of (3.49). We apply the admissible testfunction $-\Delta\psi_{\varepsilon h}\in L^2(\Omega)$ in (3.5) and get

$$\gamma\int_0^t\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta\psi_{\varepsilon h}|^2dxdt+\int_0^t\int_\Omega|\nabla\psi_{\varepsilon h}|^2dxdt+\int_0^t\int_\Omega\partial_t u_{\varepsilon h}\Delta\psi_{\varepsilon h}dxdt=0,$$

for all $t\in[0,T]$.
2. We obtain by using $-u_{\varepsilon h}/\gamma$ as a testfunction

$$-\int_0^t\int_\Omega\partial_t\nabla\psi_{\varepsilon h}\cdot\nabla u_{\varepsilon h}dxdt-\frac{1}{\gamma}\int_0^t\int_\Omega\psi_{\varepsilon h}u_{\varepsilon h}dxdt+\frac{1}{\gamma}\int_0^t\frac{1}{2}\frac{d}{dt}\int_\Omega|u_{\varepsilon h}|^2dxdt=0$$

for all $t\in[0,T]$. We find after standard calculations

$$\begin{aligned}\frac{\gamma}{2}\|\Delta\psi_{\varepsilon h}(t)\|^2_{L^2(\Omega)}+\int_0^t\|\nabla\psi_{\varepsilon h}\|^2_{L^2(\Omega)}dt+\frac{1}{2\gamma}\|u_{\varepsilon h}(t)\|^2_{L^2(\Omega)}&=\frac{\gamma}{2}\|\Delta\psi_{\varepsilon h}(0)\|^2_{L^2(\Omega)}+\frac{1}{2\gamma}\|u_{\varepsilon h}(0)\|^2_{L^2(\Omega)}\\&+\frac{1}{\gamma}\int_0^t\int_\Omega\psi_{\varepsilon h}u_{\varepsilon h}dxdt-\int_\Omega\Delta\psi_{\varepsilon h}(t)u_{\varepsilon h}(t)dx+\int_\Omega\Delta\psi_{\varepsilon h}(0)u_{\varepsilon h}(0)dx\end{aligned}\tag{3.53}$$

for all $t\in[0,T]$.
3. Using Young's inequality

$$\int_\Omega\Delta\psi_{\varepsilon h}(t)u_{\varepsilon h}(t)dx\leq\frac{\gamma}{2}\|\Delta\psi_{\varepsilon h}(t)\|^2_{L^2(\Omega)}+\frac{1}{2\gamma}\|u_{\varepsilon h}(t)\|^2_{L^2(\Omega)},$$

$$\frac{1}{\gamma}\int_0^t\int_\Omega\psi_{\varepsilon h}u_{\varepsilon h}dxdt\leq\frac{1}{2\epsilon\gamma}\int_0^t\|\psi_{\varepsilon h}\|^2_{L^2(\Omega)}dt+\frac{\epsilon}{2\gamma}\int_0^t\|u_{\varepsilon h}\|^2_{L^2(\Omega)}dt,$$

and the Poincaré inequality in the form

$$\frac{1}{2\epsilon\gamma}\int_0^t\|\psi_{\varepsilon h}\|^2_{L^2(\Omega)}dt\leq\frac{c_p}{2\epsilon\gamma}\int_0^t\|\nabla\psi_{\varepsilon h}\|^2_{L^2(\Omega)}dt$$

for all $t \in [0,T]$, where c_p is the Poincaré constant, we find from (3.53)

$$\left(1 - \frac{c_p}{2\epsilon\gamma}\right) \int_0^T \|\nabla\psi_{\varepsilon h}\|^2_{L^2(\Omega)} dt \le \gamma \|\Delta\psi_{\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|u_{\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{\epsilon}{2\gamma}\int_0^T \|u_{\varepsilon h}\|^2_{L^2(\Omega)} dt.$$

We obtain choosing $\epsilon = c_p/\gamma$ and passing to the limit ($h \searrow 0$)

$$\int_0^T \|\nabla\psi_\varepsilon\|^2_{L^2(\Omega)} dt \le 2\gamma \|\Delta\psi_\varepsilon(0)\|^2_{L^2(\Omega)} + \frac{2}{\gamma}\|u_\varepsilon(0)\|^2_{L^2(\Omega)} + \frac{c_p}{\gamma^2}\int_0^T \|u_\varepsilon\|^2_{L^2(\Omega)} dt. \tag{3.54}$$

4. Furthermore applying Young's inequality in the form

$$\frac{2}{\gamma}\int_\Omega \Delta\psi_{\varepsilon h}(t) u_{\varepsilon h}(t) dx \le \frac{1}{2}\|\Delta\psi_{\varepsilon h}(t)\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\|u_{\varepsilon h}(t)\|^2_{L^2(\Omega)},$$

$$\frac{2}{\gamma^2}\int_0^T\int_\Omega \psi_{\varepsilon h} u_{\varepsilon h} dx dt \le \frac{1}{\gamma^2}\int_0^T \|\psi_{\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{1}{\gamma^2}\int_0^T \|u_{\varepsilon h}\|^2_{L^2(\Omega)} dt,$$

we get from (3.53) the estimate

$$\|\Delta\psi_{\varepsilon h}(t)\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\int_0^t \|\nabla\psi_{\varepsilon h}\|^2_{L^2(\Omega)} dt = 3\|\Delta\psi_{\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{8}{\gamma^2}\|u_{\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\int_0^t \|\psi_{\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{2}{\gamma^2}\int_0^t \|u_{\varepsilon h}\|^2_{L^2(\Omega)} dt.$$

By the Poincaré inequality in the form

$$\int_0^t \|u_{\varepsilon h}\|^2_{L^2(\Omega)} dt \le c_p \int_0^t \|\nabla u_{\varepsilon h}\|^2_{L^2(\Omega)} dt,$$

we get passing to the limit ($h \searrow 0$)

$$\max_{0\le t\le T} \|\Delta\psi_\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\int_0^T \|\nabla\psi_\varepsilon\|^2_{L^2(\Omega)} dt = 3\|\Delta\psi_\varepsilon(0)\|^2_{L^2(\Omega)} + \frac{8}{\gamma^2}\|u_\varepsilon(0)\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\int_0^T \|\psi_\varepsilon\|^2_{L^2(\Omega)} dt + \frac{2c_p}{\gamma^2}\int_0^T \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} dt.$$

By (3.46) and (3.47) we find (3.52) We find following standard compactness properties

$$\begin{array}{lllll} u_\varepsilon & \longrightarrow & u & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)), \\ \partial_t u_\varepsilon & \longrightarrow & \partial_t u & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)^*), \\ \psi_\varepsilon & \longrightarrow & \psi & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\ \nabla\psi_\varepsilon & \longrightarrow & \nabla\psi & \text{weakly-star} & \text{in } L^\infty(0,T;H^1(\Omega)), \\ \nabla\partial_t\psi_\varepsilon & \longrightarrow & \nabla\psi_t & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)^*). \end{array} \tag{3.55}$$

so that as $\varepsilon \longrightarrow 0$ we may pass to the limit in (3.4)-(3.6). The convergence of the linear terms in (3.4)-(3.6) are standard. We take a closer look on the convergence of the nonlinear term

$$\begin{aligned} \int_0^T\int_\Omega (\mu(u)\nabla(w+\psi) - \mu_\varepsilon(u_\varepsilon)\nabla(w_\varepsilon+\psi_\varepsilon))\cdot\nabla\varphi\,dxdt \\ = \int_0^T\int_\Omega (\mu(u)-\mu_\varepsilon(u_\varepsilon))\nabla(w+\psi)\cdot\nabla\varphi\,dxdt \\ + \int_0^T\int_\Omega \mu_\varepsilon(u_\varepsilon)\nabla\left[(w-w_\varepsilon)+(\psi-\psi_\varepsilon)\right]\cdot\nabla\varphi\,dxdt. \end{aligned} \tag{3.56}$$

Using the fact that for all $z \in \mathbb{R}$

$$|\mu(z)-\mu_\varepsilon(z)| \le \sup_{\substack{0\le z\le\varepsilon \\ 1-\varepsilon\le z\le 1}} |\mu(z)| \longrightarrow 0, \quad \text{as} \quad \varepsilon \longrightarrow 0,$$

it follows that $\mu_\varepsilon \longrightarrow \mu$ uniformly and the first term on the right hand side tends to zero. The convergence of the second term is a standard consequence of the compactness result (3.55).

Now we have shown that the problem (2.6), (2.7) and (3.6) has a solution. The next step is to overcome the truncation in (3.6) and to show that the truncation is effectless. The rest of the proof is formulated as a

Proposition 1 *Let (u,w,ψ) be solution of the problem (2.6)-(2.8) then*

$$0 \le u(t,x) \le 1, \quad for\ a.a.\ (t,x) \in Q_T. \tag{3.57}$$

Proof. 1. Using in (2.6) the admissible testfunction $u^\bullet := \min(u,0)$ we get

$$\frac{1}{2}\int_\Omega |u^\bullet(t)|^2 dx + \int_0^T\int_\Omega |\nabla u^\bullet|^2 dxdt + \int_0^T\int_\Omega \mu(u)\nabla(w+\psi)\cdot\nabla u^\bullet dxdt = 0.$$

2. Because of $\mu(u)\nabla u^{\bullet} = 0$ the last term vanishes and we get

$$0 = \frac{1}{2}\int_{\Omega} |u^{\bullet}(t)|^2 dx + \int_0^T \int_{\Omega} |\nabla u^{\bullet}|^2 dxdt \geq \frac{1}{2}\int_{\Omega} |u^{\bullet}(t)|^2 dx,$$

that means $u^{\bullet}(t,x) = 0,$ for a.a. $(t,x) \in Q_T$, hence $u(t,x) \geq 0,$ for a.a. $(t,x) \in Q_T$.

3. We use in (2.6) the admissible testfunction $u^{\diamond} := \min(1-u, 0)$ and find

$$\frac{1}{2}\int_{\Omega} |u^{\diamond}(t)|^2 dx + \int_0^T \int_{\Omega} |\nabla u^{\diamond}|^2 dxdt + \int_0^T \int_{\Omega} \mu(u)\nabla(w+\psi)\cdot\nabla u^{\diamond} dxdt = 0.$$

2. Because of $\mu(u)\nabla u^{\diamond} = 0$ the last term vanishes and we get

$$0 = \frac{1}{2}\int_{\Omega} |u^{\diamond}(t)|^2 dx + \int_0^T \int_{\Omega} |\nabla u^{\diamond}|^2 dxdt \geq \frac{1}{2}\int_{\Omega} |u^{\diamond}(t)|^2 dx,$$

that means $u^{\diamond}(t,x) = 0,$ for a.a. $(t,x) \in Q_T$, hence $1-u(t,x) \geq 0,$ for a.a. $(t,x) \in Q_T$.

3.2 Uniqueness

Proof. 1. For $i = 1, 2$, suppose that

$$\int_0^T \langle \partial_t u_i, \varphi \rangle \, dt + \int_0^T \int_{\Omega} \nabla u_i \cdot \nabla\varphi dxdt$$

$$+ \int_0^T \int_{\Omega} \mu(u_i)\nabla(w_i+\psi_i)\cdot\nabla\varphi dxdt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)) \tag{3.58}$$

$$\gamma \int_0^T \langle \nabla\partial_t\psi_i, \nabla\varphi \rangle \, dt + \int_0^T \int_{\Omega} \psi_i\varphi dxdt = \int_0^T \langle \partial_t u_i, \varphi \rangle \, dt, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{3.59}$$

$$w_i = P(1-2u_i) \text{ a.e. } (t,x) \in Q_T. \tag{3.60}$$

The difference $u := u_1 - u_2, w := w_1 - w_2, \psi := \psi_1 - \psi_2$ then satisfies

$$\begin{aligned}&\int_0^T \langle u_t, \varphi\rangle\, dt + \int_0^T \int_\Omega \nabla u \cdot \nabla\varphi dxdt \\ &+ \int_0^T \int_\Omega (\mu(u_1)\nabla(w_1+\psi_1) - \mu(u_2)\nabla(w_2+\psi_2)) \cdot \nabla\varphi dxdt = 0, \forall\varphi \in L^2(0,T;H^1(\Omega)),\end{aligned} \tag{3.61}$$

$$\gamma \int_0^T \langle \nabla\psi_t, \nabla\varphi\rangle\, dt + \int_0^T \int_\Omega \psi\varphi dxdt = \int_0^T \langle u_t, \varphi\rangle\, dt, \quad \forall\varphi \in L^2(0,T;H^1(\Omega)), \tag{3.62}$$

$$w = P(-2u) \text{ a.e. } (t,x) \in Q_T. \tag{3.63}$$

Testing (3.61) by u and ψ and (3.62) by ψ we find

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)} dt &= -\int_0^t \int_\Omega (\mu(u_1)\nabla(w_1+\psi_1) - \mu(u_2)\nabla(w_2+\psi_2)) \cdot \nabla u\, dxdt \\ &= -\int_0^t \int_\Omega (\mu(u_1) - \mu(u_2))\nabla(w_1+\psi_1) \cdot \nabla u\, dxdt \\ &\quad - \int_0^t \int_\Omega \mu(u_2)\nabla w \cdot \nabla u\, dx - \int_0^t \int_\Omega \mu(u_2)\nabla\psi \cdot \nabla u\, dxdt \\ &=: \int_0^t I_5 dt + \int_0^t I_6 dt + \int_0^t I_7 dt.\end{aligned}$$

and

$$\begin{aligned}\frac{\gamma}{2}\|\nabla\psi(t)\|^2_{L^2(\Omega)} + \int_0^t \|\psi\|^2_{L^2(\Omega)} dt + \int_0^t \int_\Omega \mu(u_2)|\nabla\psi|^2 dxdt \\ = -\int_0^t \int_\Omega (\mu(u_1) - \mu(u_2))\nabla(w_1+\psi_1) \cdot \nabla\psi\, dx \\ - \int_0^t \int_\Omega \nabla u \cdot \nabla\psi\, dx - \int_0^t \int_\Omega \mu(u_2)\nabla w \cdot \nabla\psi\, dx \\ =: \int_0^t I_8 dt + \int_0^t I_9 dt + \int_0^t I_{10} dt.\end{aligned}$$

We will estimate the $I_i; i = 1, ..., 6$ separately. Because of the Lipschitz continuity of μ we get by using Hölder's inequality

$$\begin{aligned} |I_5| &\leq C_{12} \int_\Omega |u|\,|\nabla w_1|\,|\nabla u|\,dx + C_{12} \int_\Omega |u|\,|\nabla \psi_1|\,|\nabla u|\,dx \\ &\leq C_{12} \|u\|_{L^2(\Omega)}\,\|\nabla w_1\|_{L^\infty(\Omega)}\,\|\nabla u\|_{L^2(\Omega)} + C_{12}\|u\|_{L^3(\Omega)}\,\|\nabla\psi_1\|_{L^6(\Omega)}\,\|\nabla u\|_{L^2(\Omega)}, \end{aligned} \tag{3.64}$$

where C_{12} is the Lipschitz constant. For the first term on the right hand side we get using **(A2)** and **(B4)**

$$\begin{aligned} C_{12}\|u\|_{L^2(\Omega)}\,\|\nabla w_1\|_{L^\infty(\Omega)}\,\|\nabla u\|_{L^2(\Omega)} &\leq C_{12}\|u\|_{L^2(\Omega)}\,\|w_1\|_{W^{1,\infty}(\Omega)}\,\|\nabla u\|_{L^2(\Omega)} \\ &\leq C_{13}\|u\|_{L^2(\Omega)}\,\|u_1\|_{L^\infty(\Omega)}\,\|\nabla u\|_{L^2(\Omega)} \\ &\leq C_{13}\|u\|_{L^2(\Omega)}\,\|\nabla u\|_2 \\ &\leq \frac{C_{13}}{2\epsilon_1}\|u\|^2_{L^2(\Omega)} + \frac{C_{13}\epsilon_1}{2}\|\nabla u\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have used for the last operation Young's inequality. The Gagliardo-Nirenberg inequality (A.14) for $\dim(\Omega) = 3$ and Sobolev embedding theorems (A.8) give for the second term in (3.64)

$$\begin{aligned} C_{12}\|u\|_{L^3(\Omega)}\,\|\nabla\psi_1\|_{L^6(\Omega)}\,\|\nabla u\|_2 &\leq C_{14}\|u\|^{1/2}_{L^2(\Omega)}\,\|\nabla\psi_1\|_{H^2(\Omega)}\,\|\nabla u\|^{3/2}_{L^2(\Omega)} \\ &\leq C_{15}\|u\|^{1/2}_{L^2(\Omega)}\,\|\nabla u\|^{3/2}_{L^2(\Omega)} \\ &\leq \frac{C_{15}}{4\epsilon_2}\|u\|^2_{L^2(\Omega)} + \frac{3C_{15}\epsilon_2}{4}\|\nabla u\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have used Young's inequality for the last row. Finally we find

$$|I_5| \leq \left(\frac{C_{13}}{2\epsilon_1} + \frac{C_{15}}{4\epsilon_2}\right)\|u\|^2_{L^2(\Omega)} + \left(\frac{C_{13}}{2}\epsilon_1 + \frac{3C_{15}\epsilon_2}{4}\right)\|\nabla u\|^2_{L^2(\Omega)}.$$

Young's inequality and (3.63) together with **(A2)** and **(B4)** give

$$\begin{aligned} |I_6| &\leq \|\nabla w\|^2_{L^2(\Omega)} + \frac{1}{4^2}\|\nabla u\|^2_{L^2(\Omega)} \\ &\leq \|w\|^2_{H^1(\Omega)} + \frac{1}{4^2}\|\nabla u\|^2_{L^2(\Omega)} \\ &\leq r_2^2\|u\|^2_{L^2(\Omega)} + \frac{1}{4^2}\|\nabla u\|^2_{L^2(\Omega)}, \end{aligned}$$

$$|I_7| \leq \|\nabla\psi\|^2_{L^2(\Omega)} + \frac{1}{8^2}\|\nabla u\|^2_{L^2(\Omega)}.$$

Furthermore we get

$$\begin{aligned} |I_8| &\le C_{16} \int_\Omega |u|\, |\nabla w_1|\, |\nabla \psi|\, dx + C_{16} \int_\Omega |u|\, |\nabla \psi_1|\, |\nabla \psi|\, dx \\ &\le C_{16} \|u\|_2 \|\nabla w_1\|_\infty \|\nabla \psi\|_2 + C_{16} \|u\|_3 \|\nabla \psi_1\|_6 \|\nabla \psi\|_2. \end{aligned}$$

where C_{16} is the Lipschitz constant. For the first term we get by using **(A2)**, **(B4)** and Young's inequality

$$\begin{aligned} C_{16} \|u\|_2 \|\nabla w_1\|_{L^\infty(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} &\le C_{17} \|u\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\ &\le \frac{C_{17}}{2} \|u\|^2_{L^2(\Omega)} + \frac{C_{17}}{2} \|\nabla \psi\|^2_{L^2(\Omega)}. \end{aligned}$$

The Gagliardo-Nirenberg inequality (A.14) for $\dim(\Omega) = 3$ and Sobolev embedding theorem (A.8) give for the second term

$$\begin{aligned} C_{16} \|u\|_{L^3(\Omega)} \|\nabla \psi_1\|_{L^6(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} &\le C_{18} \|u\|^{1/2}_{L^2(\Omega)} \|\nabla u\|^{1/2}_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\ &\le \frac{C_{18}}{2} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} + \frac{C_{18}}{2} \|\nabla \psi\|^2_{L^2(\Omega)} \\ &\le \frac{C_{18}}{4\epsilon_3} \|u\|^2_{L^2(\Omega)} + \frac{C_{18}\epsilon_3}{4} \|\nabla u\|^2_{L^2(\Omega)} + \frac{C_{18}}{2} \|\nabla \psi\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have applied Young's inequality for the last estimates. Thus, we get

$$|I_8| \le \left(\frac{C_{17}}{2} + \frac{C_{18}}{4\epsilon_3} \right) \|u\|^2_{L^2(\Omega)} + \frac{C_{18}\epsilon_3}{4} \|\nabla u\|^2_{L^2(\Omega)} + \left(\frac{C_{17}}{2} + \frac{C_{18}}{2} \right) \|\nabla \psi\|^2_{L^2(\Omega)}. \quad (3.65)$$

Using Hölder's inequality and **(A2)** and **(B4)** we get

$$\begin{aligned} |I_9| &\le 2 \|\nabla \psi\|^2_{L^2(\Omega)} + \frac{1}{8} \|\nabla u\|^2_{L^2(\Omega)}, \\ |I_{10}| &\le \frac{1}{8} \|\nabla w\|^2_{L^2(\Omega)} + \frac{1}{8} \|\nabla \psi\|^2_{L^2(\Omega)} \\ &\le \frac{r_2^2}{2} \|u\|^2_{L^2(\Omega)} + \frac{1}{8} \|\nabla \psi\|^2_{L^2(\Omega)}, \end{aligned}$$

Finally we conclude

$$\begin{aligned} &\frac{1}{2} \left[\|u(t)\|^2_{L^2(\Omega)} + \gamma \|\nabla \psi(t)\|^2_{L^2(\Omega)} \right] + \nu(\epsilon_1, \epsilon_2, \epsilon_3) \int_0^t \|\nabla u\|^2_{L^2(\Omega)} dt + \int_0^t \|\psi\|^2_{L^2(\Omega)} dt \\ &+ \|\psi\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \mu(u_2) |\nabla \psi|^2 dx dt \le C_{19}(\epsilon_1, \epsilon_2, \epsilon_3) \int_0^t \|u\|^2_{L^2(\Omega)} dt + C_{20}(\epsilon_1, \epsilon_2, \epsilon_3) \int_0^t \|\nabla \psi\|^2_{L^2(\Omega)} dt, \end{aligned}$$

where $\nu(\epsilon_1, \epsilon_2, \epsilon_3) := 1 - C_{13}\epsilon_1/2 - 3C_{15}\epsilon_2/4 - C_{18}\epsilon_3/4 - 13/4^3$. Choose $\epsilon_i, i = 1, 2, 3$, so that $\nu > 0$ Gronwall's Lemma (A.4) gives the uniqueness.