Chapter 2

Statement of the problems and assumptions

Let be $\Omega \subset \mathbb{R}^3$ an open, bounded domain with boundary $\Gamma = \partial\Omega$ and ν the outer unit normal on Γ. In the sequel, $|\Omega|$ denotes the Lesbegue measure of Ω. We denote by $L^p(\Omega), W^{k,p}(\Omega)$ for $1 \leq p \leq \infty$ the Lesbegue spaces and Sobolev spaces of functions on Ω with the usual norms $\|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{W^{k,p}(\Omega)}$, and we write $H^k(\Omega) = W^{k,2}(\Omega)$ (see [13]). For a Banach space X we denote its dual by X^*, the dual pairing between $f \in X^*$, $g \in X$ will be denoted by $\langle f, g\rangle$. If X is a Banach space with the norm $\|\cdot\|_X$, we denote for $T > 0$ by $L^p(0,T;X)$ $(1 \leq p \leq \infty)$ the Banach space of all (equivalence classes of) Bochner measurable functions $u : (0,T) \longrightarrow X$ such that $\|u(\cdot)\|_X \in L^p(0,T)$. We set $R^1_+ = (0,\infty)$ and, as already mentioned, $Q_T = (0,T) \times \Omega$, $\Gamma_T = (0,T) \times \Gamma$. "Generic" positive constants are denoted by C and for $u \in L^1(\Omega)$ we put

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx.$$

Now we are going to formulate the nonlocal viscous Cahn-Hillard equation (1.13) with complemented initial and boundary values. So the initial-boundary value problem we want to discuss takes the form:

$$u_t - \nabla \cdot \overbrace{(\nabla u + \mu\nabla(w+\psi))}^{=\mu\nabla v} = 0 \quad \text{in } Q_T, \tag{2.1}$$

$$-\gamma\Delta\psi_t + \psi = u_t, \quad w = P(1-2u) \quad \text{in } Q_T, \tag{2.2}$$

$$\mu\nu\cdot\nabla v = \mu\nu\cdot\nabla w = \nu\cdot\nabla\psi = 0 \quad \text{on } \Gamma_T, \tag{2.3}$$

$$\nu\cdot\nabla\psi_0 = 0, u(0,x) = u_0(x), \psi(0,x) = \psi_0(x) \quad x \in \Omega. \tag{2.4}$$

Consider the system (2.1)-(2.4). We make the following general assumptions.
(A1) $f(u) = u\log u + (1-u)\log(1-u)$.
(A2) the **potential operator** P defined by

$$\rho \mapsto P\rho = \int_\Omega \mathcal{K}(|x-y|)\rho(y)dy$$

satisfies

$$\|P\rho\|_Y \le r_p \|\rho\|_{L^p}, \qquad 1 \le p \le \infty,$$

where the kernel $\mathcal{K} \in (\mathbb{R}^1_+ \mapsto \mathbb{R}^1)$ is such that

$$\int_\Omega \int_\Omega |\mathcal{K}(|x-y|)| dx dy = m_0 < \infty, \qquad \sup_{x\in\Omega} \int_\Omega |K(|x-y|)| dy = m_1 < \infty.$$

(A3) the **mobility** μ has the form

$$\mu(u) = \frac{1}{f''(u)} = u(1-u). \tag{2.5}$$

(A4) $0 \le u_0(x) \le 1, x \in \Omega, 0 < \bar{u}_0 < 1.$

The next assumptions concern different regularity assumptions on the data.

(B1) $\Omega \in C^{0,1}$	or	**(B1')** $\Omega \in C^4$,
(B2) $u_0 \in L^\infty(\Omega)$	or	**(B2')** $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$,
(B3) $\psi_0 \in H^2(\Omega)$	or	**(B3')** $\psi_0 \in H^3(\Omega)$,
(B4) $Y := H^{1,p}(\Omega)$	or	**(B4')** $Y := H^{2,p}(\Omega)$.

Remark 1 The kernel $\mathcal{K}$ is chosen to be symmetric. Consequently the potential operator P is symmetric, too.

Remark 2 Examples for kernels $\mathcal{K}$ satisfying **(A2)** are Newton potentials

$$\mathcal{K}(|x|) = \begin{cases} \kappa_n |x|^{2-n} & n \ne 2; \\ -\kappa_2 \log |x| & n = 2; \end{cases}$$

and Gauss functions $\mathcal{K}(s) = c \exp(-s^2/\lambda)$ and usual mollifiers like

$$\mathcal{K}(|x|) = \begin{cases} C \exp\left(-\frac{h^2}{h^2-|x|^2}\right) & \text{if } |x| < h, \\ 0 & \text{if } |x| \ge h \end{cases}$$

where h characterizes the range of interaction.

Remark 3 A concentration-dependent mobility appeared in the original derivation of the Cahn-Hillard equation (see [7]), and a natural and thermodynamically reasonable choice is of the form (2.5) and were considered in [11].

Due to different regularity assumptions on the initial data we formulate two different Theorems, which will be proven separately in the next two chapters.

Theorem 1 *Suppose that the assumptions* **(A1)** *to* **(A4)** *and* **(B1)** *to* **(B4)** *hold. Then there exists a unique triple of functions* (u, w, ψ) *such that*

1. $u \in C(0,T;L^\infty) \cap L^2(0,T;H^1(\Omega)), \quad 0 \leq u(t,x) \leq 1$ *for a.a.* $(t,x) \in Q_T$,
2. $u_t \in L^2(0,T;H^1(\Omega)^*)$,
3. $w \in C(0,T;H^{1,\infty}(\Omega))$,
4. $\psi \in L^2(0,T;L^2(\Omega))$,
5. $\nabla\psi \in L^\infty(0,T;H^1(\Omega))$,
6. $\nabla\psi_t \in L^2(0,T;H^1(\Omega)^*)$,

which satify equations (2.1)-(2.4) in the following sense:

$$\int_0^T \langle u_t, \varphi \rangle \, dt + \int_0^T \int_\Omega (\nabla u + \mu \nabla (w + \psi)) \nabla \varphi dx dt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{2.6}$$

$$\gamma \int_0^T \langle \nabla\psi_t, \nabla\varphi \rangle \, dt + \int_0^T \int_\Omega \psi\varphi dx dt = \int_0^T \langle u_t, \varphi \rangle \, dt, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{2.7}$$

$$w = P(1-2u) \text{ a.e. } (t,x) \in Q_T. \tag{2.8}$$

Theorem 2 *Suppose that the assumptions* **(A1)** *to* **(A4)** *and* **(B1')** *to* **(B4')** *hold. Then there exists a unique triple of functions* (u, w, ψ) *such that*

1. $u \in C(0,T;L^\infty) \cap L^2(0,T;H^2(\Omega)), \quad 0 \leq u(t,x) \leq 1$ *for a.a.* $(t,x) \in Q_T$,
2. $u_t \in L^2(0,T;L^2(\Omega))$,
3. $w \in C(0,T;H^{2,\infty}(\Omega))$,
4. $\psi \in L^2(0,T;L^2(\Omega))$,
5. $\nabla\psi \in L^\infty(0,T;H^2(\Omega))$,
6. $\nabla\psi_t \in L^2(0,T;L^2(\Omega))$,

which satifies equations (2.1)-(2.4) in the following sense:

$$\int_0^T \int_\Omega u_t \varphi dx dt + \int_0^T \int_\Omega \left(\nabla u + \mu \nabla (w + \psi)\right) \nabla \varphi dx dt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{2.9}$$

$$\gamma \int_0^T \int_\Omega \nabla \psi_t \cdot \nabla \varphi dt + \int_0^T \int_\Omega \psi \varphi dx dt = \int_0^T \int_\Omega u_t \varphi dx dt, , \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{2.10}$$

$$w = P(1-2u) \ a.e. \ (t,x) \in Q_T. \tag{2.11}$$

Remark 4 Note that the testfunction $\varphi = 1$ gives

$$\begin{aligned} &\int_\Omega u(t,x) dx = \int_\Omega u_0(x) dx = u_\alpha |\Omega|, \\ &\int_0^T \int_\Omega \psi(t,x) dx dt = 0. \end{aligned} \tag{2.12}$$