The C_1 conjecture for the moduli space of stable vector bundles with fixed determinant on a smooth projective curve

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Summary

The C_1 conjecture states that every separably rationally connected variety over a C_1 field has a rational point. The conjecture has been proven in several cases by the works of Esnault, Graber, Harris, Starr, de Jong and Colliot-Thélène. The conjecture is still open in the case when the C_1 field is the fraction field K of a Henselian discrete valuation ring R of mixed characteristic with algebraically closed residue field k. In this thesis we prove the conjecture in this setting for a special case. Fix integers r, d coprime with $r \geq 2$. Let X_K be a smooth, projective, geometrically connected curve of genus $g \geq 2$ defined over K and $\mathcal{L}_K$ a fixed invertible sheaf on X_K of degree d. The moduli space of geometrically stable locally free sheaves of rank r and determinant $\mathcal{L}_K$ on the curve X_K is a separably rationally connected variety. In this thesis we prove the C_1 conjecture for this variety under the assumption that the curve X_K has a semistable model $X_R \to \operatorname{Spec}(R)$ with the special fibre X_k, a generalised tree-like curve whose singular components do not normalise to a rational curve. In order to show the existence of a K-rational point, we prove the existence of a geometrically stable locally free sheaf of rank r and determinant $\mathcal{L}_K$ on the curve X_K under our assumptions.

By modifying the classical proof by Le Potier, we first prove the existence of a semistable locally free sheaf of fixed rank and determinant on a smooth curve of genus $g \geq 1$, defined over an algebraically closed field of arbitrary characteristic. Then using the theory of generalised parabolic sheaves we prove the same result on an irreducible nodal curve defined over an algebraically closed field of arbitrary characteristic. Using these results and stability conditions given by Teixidor i Bigas, we prove the existence of a semistable locally free sheaf with fixed rank and determinant on the special fibre X_k. Then using Grothendieck algebraisation and Artin approximation, we lift this semistable locally free sheaf of fixed rank and determinant to the model X_R. Finally using standard arguments, we conclude that the pull back of this sheaf to the generic fibre X_K gives a geometrically stable locally free sheaf of required rank and determinant on X_K.

Zusammenfassung

Die C_1-Vermutung besagt, dass jede separabel rational zusammenhängende Varietät über einem C_1-Körper einen rationalen Punkt besitzt. Die Vermutung wurde in mehreren Fällen in den Arbeiten von Esnault, Graber, Harris, Starr, de Jong und Colliot-Thélène bewiesen. Sie ist jedoch noch immer offen im Fall, dass der C_1-Körper der Quotientenkörper K eines Henselschen diskreten Bewertungsringes R von gemischter Charakteristik mit algebraisch abgeschlossenem Restklassenkörper k ist. In dieser Arbeit beweisen wir die Vermutung in dieser Situation für einen Spezialfall. Wir fixieren koprime ganze Zahlen r, d mit $r \geq 2$. Sei X_K eine glatte, projektive, geometrisch verbunden, Kurve über K vom Geschlecht $g \geq 2$ und sei $\mathcal{L}_K$ eine fixe invertierbare Garbe auf X_K vom Grad d. Der Modulraum geometrisch stabiler lokal freier Garben vom Rang r und mit Determinante $\mathcal{L}_K$ auf der Kurve X_K ist eine separabel rational zusammenhängende Varietät. In dieser Arbeit beweisen wir die C_1-Vermutung für diese Varietät unter der Voraussetzung, dass die Kurve X_K ein semistabiles Modell $X_R \to \mathrm{Spec}(R)$ besitzt, in dem die spezielle Faser eine generalisierte baumartige Kurve ist, deren singuläre Komponenten eine nicht-rationale Normalisierung haben. Um die Existenz eines K-rationalen Punktes zu beweisen, zeigen wir die Existenz einer geometrisch stabilen lokal freien Garbe vom Rang r und mit Determinante $\mathcal{L}_K$ auf der Kurve X_K unter unseren Voraussetzungen.

Indem wir einen klasischen Beweis von Le Potier modifizieren, beweisen wir zuerst die Existenz einer semistabilen lokal freien Garbe mit festgelegtem Rang und Determinante auf einer glatten Kurve vom Geschlecht $g \geq 1$ über einem algebraisch abgeschlossenen Körper von beliebiger Charakteristik. Dann benutzen wir die Theorie der generalisierten parabolischen Garben, um dasselbe Resultat für eine irreduzible nodale Kurve, die über einem algebraisch abgeschlossenen Körper beliebiger Charakteristik definiert ist, zu beweisen. Mit Hilfe dieser Resultate und von Stabilitätsbedingungen von Teixidor i Bigas beweisen wir die Existenz einer semistabilen lokal freien Garbe von festem Rang und fester Determinante auf der speziellen Faser X_k. Dann benutzen wir Grothendieck-Algebraisierung und Artin-Approximation, um diese semistabile lokal freie Garbe auf das Modell X_R zu heben. Schließlich benutzen wir Standardargumente, um festzustellen, dass die Zurückziehung dieser Garbe auf die generische Faser eine geometrisch stabile lokal freie Garbe von benötigtem Rang und Determinante ergibt.

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Contents

Notations

Table 1: Notations

R	a Henselian discrete valuation ring.
$\mathfrak{m}$	the maximal ideal of R.
k	residue field of R, assumed to be algebraically closed.
K	fraction field of R.
$\overline{K}$	algebraic closure of K.
R_n	$R/\mathfrak{m}^{n+1}$ for $n \geq 1$.
Y_n	the spectrum of the ring R_n.
$\hat{R}$	completion of the discrete valuation ring R.
$\overline{K}$	algebraic closure of K.
X_K	smooth, projective, geometrically connected curve of genus $g \geq 2$, defined over K.
X_R	semistable model of X_K over R with special fibre a generalised tree-like curve whose singular components do not normalise to a rational curve.
$\mathcal{L}_K$	a fixed invertible sheaf on X_K.
$\mathcal{L}_R$	a lift of $\mathcal{L}_K$ to X_R.
$\mathrm{Quot}^{r,d}_{X/S/\mathcal{H}}$	see Notation A.2.3.
$\mu(\mathcal{E})$	slope of a sheaf $\mathcal{E}$, see Definition A.1.3.
$\mu_{\mathrm{sesh}}(\mathcal{E})$	Seshadri slope of a sheaf $\mathcal{E}$, see Definition A.1.1.
$M^s_{X_K}(r,d)$	see Definition 2.3.4.
$M^s_{X_K,\mathcal{L}_K}(r,d)$	see Proposition 2.3.6.
T^0	internal nodes of a generalised tree-like curve T, see Notation A.4.3.
T^b	boundary nodes of a generalised tree-like curve T, see Notation A.4.3.
$B(i)$	see Lemma A.4.2.
$G(i)$	see Lemma A.4.2.
$\nu(i)$	see Lemma A.4.2 and Notation A.4.3.

To all my teachers ...

Chapter 1

Introduction

A field L is said to be C_1 if any hypersurface in $\mathbf{P}^n_L$ of degree $d \leq n$ has a rational point. The C_1 conjecture states that every separably rationally connected variety over a C_1 field has a rational point. The conjecture has already been proven in the case when the C_1 field is a finite field (see [Esn03]). For the case when the C_1 field is a function field of a curve defined over an algebraically closed field of characteristic zero (resp. arbitrary characteristic), it was proven by Graber, Harris and Starr in [GHS03] (resp. by de Jong and Starr in [JS03]). The conjecture has also been understood for the quotient field of an equal characteristic Henselian discrete valuation ring with algebraically closed residue field using [CT10]. However, little is known in the case of mixed characteristic. In this thesis we prove the conjecture for a specific rationally connected variety defined over the quotient field of a Henselian discrete valuation ring of mixed characteristic with algebraically closed residue field.

Notation 1.0.1. Let R be a Henselian discrete valuation ring with fraction field denoted K, of characteristic 0 and algebraically closed residue field denoted k, of characteristic $p > 0$. Fix integers r, d coprime with $r \geq 2$. Let X_K be a smooth, projective, geometrically connected curve of genus $g \geq 2$ defined over K and $\mathcal{L}_K$ a fixed invertible sheaf on X_K of degree d.

Consider the moduli space, denoted $M^s_{X_K,\mathcal{L}_K}(r,d)$ of geometrically stable locally free sheaves of rank r and determinant $\mathcal{L}_K$ on X_K. This is a Fano variety (see Proposition 2.3.7) and as K is of characteristic 0, it is rationally connected and also separably rationally connected (see Remark 2.3.8). By the statement of the C_1 conjecture, this variety has a K-rational point. In order to prove the existence of this rational point, it suffices to prove the existence of a geometrically stable locally free sheaf with the required rank and determinant on the curve X_K.

The moduli of (semi)stable locally free sheaves of fixed rank and degree over a curve, have been studied for decades and there is a plethora of results on the subject. However, for most of these results the curve is defined over an algebraically closed field. In fact when the field is non-algebraically closed, there may not even exist invertible sheaves of certain degrees over a smooth, projective curve (see for example [BB08]). To the best of our knowledge, there is no result proving the existence of a stable locally free sheaf of rank ≥ 2 and fixed determinant on a smooth, projective curve of genus $g \geq 2$, defined over a non-algebraically closed field.

We now discuss the possible approaches one can take and survey existing techniques. Let $X_R \to \mathrm{Spec}(R)$ be a semistable model for the curve X_K and denote by $X_k := X_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ the special fibre. It is natural to ask whether $M^s_{X_K,\mathcal{L}_K}(r,d)$ specializes to a moduli space say $M^s_{X_k,\mathcal{L}_k}(r,d)$, as X_K specializes to the semistable curve X_k and $\mathcal{L}_K$ specializes to an invertible sheaf $\mathcal{L}_k := \mathcal{L}_R \otimes_R k$. One could then use the fact that k is algebraically closed to prove the non-emptiness of $M^s_{X_k,\mathcal{L}_k}(r,d)$ and check if the k-point can be lifted to an R-point. However, this approach only works in the case when X_k is smooth. This is because to define a moduli space (i.e. a scheme corepresenting a moduli functor) of stable, pure sheaves with determinant an invertible sheaf, one needs to define the determinant morphism

$$\det : M^s_{X_k}(r,d) \to \mathrm{Pic}^d(X_k)$$

where $\mathrm{Pic}^d(X_k)$ is the Picard group of invertible sheaves on X_k of degree d and $M^s_{X_k}(r,d)$ is the moduli space of stable, pure sheaves of rank r and degree d. Note that for stable pure sheaves which are not locally free, one defines the determinant by taking a locally free resolution of the sheaf (see [HL97, Proposition 2.1.10]), which is finite when the underlying curve is smooth. However, if the underlying curve is nodal, the locally free resolution need not be of finite length, so one cannot use this definition for the determinant. As a result, there is no good definition of a moduli functor for stable locally free sheaves of arbitrary rank and fixed determinant (where determinant is an invertible sheaf) on an irreducible nodal curve. Some results using the theory of generalised parabolic bundles have been proven but only for the case when the rank is 2 and the degree is coprime to the rank (see [Sun03, Theorem 2]).

Another approach one may consider is to study the degeneration of the moduli space of semistable torsion free sheaves of rank r and degree d on the generic fibre to the corresponding moduli space on the special fibre and use further arguments for the determinant. One of the first results concerning the degenerations of moduli spaces over families of curves is due to Gieseker [Gie84]. Denote by X_0 an irreducible, projective curve with one node $\{\delta\}$ defined over the complex numbers.

Let $U(r,d)$ (resp. $U(r,d)^0$) denote the moduli space of stable torsion free (resp. locally free) sheaves of rank r and degree d on X_0. In the case of rank 2 and degree 1, Gieseker constructed a compactification denoted $G(2,1)$ for $U(2,1)^0$. The points of $G(2,1)\backslash U(2,1)^0$ consist of locally free sheaves $\mathcal{E}$, on curves say X which are semi-stably equivalent to X_0 (i.e., there exists a morphism $\pi : X \to X_0$ such that π is an isomorphism over $X_0\backslash\{\delta\}$ and $\pi^{-1}(\delta)$ is a chain of projective lines). One of the interesting features of Gieseker's compactification is that when $(r,d) = 1$, it has good specialization properties i.e. if a smooth, projective curve X specializes to X_0, then the moduli space of stable locally free sheaves of rank r and degree d on X specializes to $G(r,d)$. In [NS99], Nagaraj and Seshadri generalised Gieseker's construction for higher ranks. For the case when X_0 is reducible, the construction was generalised by Xia in [Xia95] for rank 2 and by Schmitt in [Sch04] for higher ranks. Unfortunately, none of these results can be used in our setting because in all of these results the curve X_0 is defined over the field of complex numbers. The discrete valuation ring in all of these constructions is of equal characteristic and is in fact a complex algebra. This assumption is used crucially for obtaining the specialization property mentioned above (see for example [NS99, Proposition 8]). One expects that these results could be used with some modifications for when the curve X_K is defined over a C_1 field K of *same* characteristic as k. However, in that case the moduli spaces $M^s_{X_K,\mathcal{L}_K}(r,d)$ already has a rational point by [GHS03], [JS03] and [CT10] as mentioned in the first paragraph. Moreover by [HN75, Proposition 1.2.1] that rational point corresponds to a semistable locally free sheaf on X_K of rank r and determinant $\mathcal{L}_K$.

Note that the moduli space $M^s_{X_K,\mathcal{L}_K}(r,d)$ is non-empty (see Corollary 2.4.8) and of finite type. Since $M^s_{X_K,\mathcal{L}_K}(r,d)$ is a coarse moduli space, a $\overline{K}$ point corresponds to a stable locally free sheaf $\mathcal{F}_L$ with determinant $\mathcal{L}_L$ on X_L, where L is a finite extension of K. Denote by R_L the integral closure of L in $\overline{K}$ and since k is algebraically closed, the residue field of R_L is also k. Then one could consider the degeneration of the stable locally free sheaf $\mathcal{F}_L$ to the special fibre X_k. However $\mathcal{F}_L$ need not degenerate to a locally sheaf on X_k unless $X_{R_L} \to \mathrm{Spec}(R_L)$ is regular (see [Oss14, Proposition 4.1]). As regularity is not preserved under base change this approach only works if we assume $X_R \to \mathrm{Spec}(R)$ to be a smooth model. However using a semistable locally free sheaf with fixed determinant on the special fibre (not necessarily obtained as a degeneration) one can still obtain a semistable locally free sheaf on the generic fibre. This is the key idea in the approach we take.

We now discuss our approach in detail and the main results of this thesis. The results in this thesis are proven under the following assumption:

Assumptions 1.0.2. Assume that there exists a semistable model $X_R \to \mathrm{Spec}(R)$ for the curve X_K with the special fibre X_k being a generalised tree-like curve (see

Definition 2.4.12), whose irreducible singular components do not normalise to a rational curve.

Denote by $Y_1, \dots Y_N$ the irreducible components of X_k. In the first step, we prove the existence of semistable locally free sheaves $\mathcal{E}_i$ of fixed rank and determinant on the irreducible components Y_i. The proof of existence of a semistable locally free sheaf of fixed rank and degree on a smooth curve of genus $g \geq 1$ defined over the field of complex numbers is given in [LP97, Theorem 8.6.1]. In Theorem 2.4.6, we modify this proof by replacing the steps which fail in positive characteristic. We then prove the following:

Theorem 1.0.3 (see Theorem 2.4.6 and Corollary 2.4.8)**.** *There exists a semistable locally free sheaf of fixed rank and determinant on a smooth, projective curve Y of genus $g \geq 1$ defined over an algebraically closed field.*

However, a locally free sheaf is semistable on a rational component of X_k only if its degree is a multiple of its rank. Therefore to produce semistable locally free sheaves on the non-singular, rational components of the curve X_k, we prove the following lemma.

Lemma 1.0.4 (see Lemma 3.2.4)**.** *Let r be a fixed integer. Denote by S the set of indices of the rational components of X_k. There exists an invertible sheaf $\mathcal{L}_R$ on X_R such that $\mathcal{L}_R \otimes \mathcal{O}_{X_K} = \mathcal{L}_K$ and for all $s_i \in S$, $\deg(\mathcal{L}_R \otimes \mathcal{O}_{Y_{s_i}})$ is a multiple of r.*

For any component Y of X_k which is irreducible, nodal with normalisation $\tilde{Y}$ a smooth curve of genus $g \geq 1$, we use the theory of generalised parabolic bundles and Theorem 1.0.3 to produce a semistable locally free sheaf with fixed rank and degree. We then use arguments related to the Picard group of the normalisation $\tilde{Y}$ and that of Y to prove that we can also get the required determinant. Namely, we prove the following:

Theorem 1.0.5 (Theorem 3.1.9)**.** *Let Y be an irreducible, nodal curve with normalisation $\tilde{Y}$ a smooth curve of genus $g \geq 1$. Denote by $\mathcal{Q}$ an invertible sheaf on Y of degree d. There exists a semistable locally free sheaf on Y of rank r and degree d with determinant $\mathcal{Q}$.*

Then we obtain a locally free sheaf, say $\mathcal{E}_k$ of fixed rank and determinant on the curve X_k by gluing the semistable locally free sheaves $\mathcal{E}_i$ on the components Y_i. However, $\mathcal{E}_k$ need not be semistable. In [Big91, Step 2], we see a sufficient criterion for a locally free sheaf to be Seshadri semistable (for any choice of polarisation), over a tree-like curve given the restriction of the sheaf is semistable on each of the irreducible components of the curve. In Theorem A.4.11, we check that the criterion given holds also for *generalised* tree-like curves which include rational components (see Remark A.4.1). In the proof of the following theorem we show the existence of a line bundle $\mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$, $a_i \in \mathbf{Z}$ such that $\mathcal{F}_k := \mathcal{E}_k \otimes_k \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$ satisfies the criterion given in [Big91, Step 2]. The key observation is that this line bundle comes from irreducible components of the curve X_k. As a consequence of this, $\det(\mathcal{F}_k) \simeq \mathcal{L}_R \otimes_R k$ where $\mathcal{L}_R$ is an invertible sheaf on X_R such that $\mathcal{L}_R \otimes_R K \simeq \mathcal{L}_K$.

Theorem 1.0.6 (see Theorem 3.2.6 and Lemma 3.2.9)**.** There exists a Gieseker semistable locally free sheaf $\mathcal{F}_k$ of rank r on X_k with $\det(\mathcal{F}_k) \simeq \mathcal{L}_R \otimes \mathcal{O}_{X_k}$ on X_k, where $\mathcal{L}_R$ is an invertible sheaf on X_R such that $\mathcal{L}_R \otimes \mathcal{O}_{X_K} \cong \mathcal{L}_K$.

In Chapter 4, using Grothendieck formal function theorem in Proposition 4.1.15 and Artin approximation in Proposition 4.2.6 we lift this Gieseker semistable locally free sheaf on X_k to a Gieseker semistable locally free sheaf on the model X_R. Finally, using the fact that Gieseker geometric stability is an open condition we prove the following theorem under our assumptions.

Theorem 1.0.7 (see Theorem 4.3.1)**.** Keep Notations 1.0.1 and Assumptions 1.0.2. Then, there exists a geometrically stable locally free sheaf $\mathcal{F}_K$ on X_K of rank r and determinant $\mathcal{L}_K$.

As a consequence, the moduli space $M^s_{X_K,\mathcal{L}_K}(r,d)$ has a K-rational point:

Theorem 1.0.8 (see Theorem 4.3.2)**.** Keep Notations 1.0.1 and Assumptions 1.0.2. Denote by $M^s_{X_K,\mathcal{L}_K}(r,d)$ the moduli space of geometrically stable locally free sheaves over X_K of rank r and determinant $\mathcal{L}_K$. Then $M^s_{X_K,\mathcal{L}_K}(r,d)$ has a K-rational point.

Therefore the C_1 conjecture is true for the variety $M^s_{X_K,\mathcal{L}_K}(r,d)$.

The appendices cover all the preliminary definitions and lemmas which are specific to this thesis and not necessarily covered in a standard algebraic geometry textbook. Also we use certain known results under different hypothesis and therefore we reprove them in the appendices. We define the notations necessary for each section at the beginning of the section. However, to prevent the reader from getting lost, we also include a list containing the most commonly used notations.

Chapter 2

Preliminaries

In this chapter we introduce some necessary preliminaries and state the question we will be answering in this thesis (see Question 2.4.14). We use definitions and notations from Appendices A.1 and A.2.

2.1 A brief overview of C_1 fields

In this section we recall the definition of a C_1 field and state some examples.

Definition 2.1.1. A field K is called a C_1 field if for any integer $d > 0$, every homogeneous form over K of degree d in $n > d$ variables has a non-trivial solution.

Example 2.1.2. We state without proof some examples of C_1 fields:

1. An algebraically closed field is trivially C_1.
2. Finite fields are C_1 (see [Che35]).
3. The function field of an irreducible curve defined over an algebraically closed field is C_1(see [Tse33]).

In this thesis, we will be mainly interested in the following example of a C_1 field.

Theorem 2.1.3 ([Lan52, Theorem 14])**.** Let R be a Henselian discrete valuation ring of characteristic 0 with residue field denoted k, of characteristic p and fraction field denoted K. If k is algebraically closed, then K is C_1.

2.2 Rationally connected varieties and the C_1 conjecture

In this section we recall the basic definitions and facts related to rationally connected varieties that we require. We recall the C_1 conjecture and state known results.

Notation 2.2.1. Let K be a field and $\overline{K}$ its algebraic closure.

Definition 2.2.2. Let Y be a variety over $\overline{K}$. We have the following definitions:

1. The variety Y over $\overline{K}$ is called *unirational* if there exists a dominant, rational map from $\mathbf{P}^n_{\overline{K}} \dashrightarrow Y$ for some integer $n > 0$.
2. A variety Y of dimension n is called *uniruled* if there exists a variety Z over $\overline{K}$ of dimension $n-1$ and a dominant rational map $\mathbf{P}^1_{\overline{K}} \times Z \dashrightarrow Y_{\overline{K}}$.
3. A variety Y is called *Fano* if the anticanonical divisor of Y denoted $\omega_Y^\vee$ is ample.

Lemma 2.2.3. Let Y be a smooth unirational variety of dimension n over an algebraically closed field $\overline{K}$ of characteristic 0. Then the canonical divisor ω_Y is not numerically effective.

Proof. Since Y is a unirational variety and the field $\overline{K}$ has characteristic 0, it is uniruled. By [Deb01, Corollary 4.11], there is a free rational curve $f : \mathbf{P}^1 \to Y$ which implies by [Deb01, Remark 4.6] that $H^1(\mathbf{P}^1, f^*T_Y \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$, where T_Y denotes the tangent bundle of Y. Since the variety Y is smooth, f^*T_Y is a rank $n = \dim(Y)$ locally free sheaf on $\mathbf{P}^1$. By Grothendieck's theorem, f^*T_Y decomposes as a sum of invertible sheaves

$$f^*T_Y \simeq \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_n)$$

where we can assume $a_1 \geq \cdots \geq a_n$. Since f is nonconstant, f^*T_Y contains $T_{\mathbf{P}^1} \simeq \mathcal{O}_{\mathbf{P}^1}(2)$ and $a_1 \geq 2$. Therefore, $\omega_Y \cdot f_*\mathcal{O}_{\mathbf{P}^1} = -\sum_{i=1}^{n} a_i \leq -2$. Hence, ω_Y is not numerically effective. □

Lemma 2.2.4. Let Y be a variety defined over K and $Y_{\overline{K}} := Y \times_K \operatorname{Spec} \overline{K}$, its base change to the algebraic closure. If the variety $Y_{\overline{K}}$ is Fano, then so is Y.

Proof. Let $f : Y \to \operatorname{Spec} K$, $g : \operatorname{Spec} \overline{K} \to \operatorname{Spec} K$ the natural morphisms and $f' : Y_{\overline{K}} \to \operatorname{Spec} \overline{K}$, the base change by g. By [Har77, Proposition II.8.10], we have

$\omega_{Y_{\overline{K}}} \simeq g'^{*}\omega_Y$, where $g' : Y_{\overline{K}} \to Y$ is the natural morphism. Note that $(g'^{*}\omega_Y)^{\vee} \simeq g'^{*}\omega_Y^{\vee}$.

Since the variety $Y_{\overline{K}}$ is Fano, the anticanonical divisor $\omega^{\vee}_{Y_{\overline{K}}}$ is ample. By [Gro65, Corollary 2.7.2], this implies $\omega_Y^{\vee}$ is also ample. Hence, the variety Y is Fano. □

Definition 2.2.5. Over a general field K, a variety Y over K is *rationally connected* if there exists a K-scheme T of finite type and a morphism

$$F : T \times \mathbf{P}^1_K \to Y$$

(which one can think of as a family of rational curves on Y parametrised by T) such that the induced morphism

$$\begin{aligned} T \times \mathbf{P}^1 \times \mathbf{P}^1 &\longrightarrow Y \times Y \\ (t, u, u') &\longmapsto (F(t,u), F(t,u')) \end{aligned}$$

is dominant.

Theorem 2.2.6 ([Kol13, Corollary V.2.15])**.** A Fano variety over an algebraically closed field of characteristic 0 is rationally connected.

Definition 2.2.7. A variety Y over an algebraically closed field $\overline{K}$ is *separably rationally connected* if there exists a morphism $f : \mathbf{P}^1 \to Y$ such that $f^{*}(T_Y)$ is ample.

Remark 2.2.8. Note that over an algebraically closed field $\overline{K}$ of characteristic 0, rationally connected is equivalent to separably rationally connected (see [Kol13, Proposition IV.3.3.1]).

Conjecture 2.2.9. The C_1 conjecture (Lang-Manin-Kollár): A smooth, proper, separably rationally connected variety over a C_1 field always has a rational point.

Known results: The conjecture has already been proven for various C_1 fields. We state without proof the known cases.

1. Esnault [Esn03] proved the conjecture for finite fields.

2. Graber, Harris and Starr [GHS03] settled the conjecture in the case of the function field of an irreducible curve defined over an algebraically closed field of characteristic 0.

3. The previous result was generalised by de Jong and Starr [JS03] to the case when the underlying field is of arbitrary characteristic.

4. Using [CT10] the conjecture has also been understood in the case of the quotient field of an equal characteristic Henselian discrete valuation ring with algebraically closed residue field.

Remark 2.2.10. The conjecture remains open in the case when the C_1 field is the fraction field of a maximal unramified discrete valuation ring with algebraically closed residue field of mixed characteristic. The goal of this thesis is to verify the conjecture in this case for a certain separably rationally connected variety. We describe this variety in the next section.

2.3 The moduli space of stable locally free sheaves with fixed determinant

In this section we introduce the moduli space of (semi)stable locally free sheaves with fixed determinant on a smooth, projective curve defined over a C_1 field. We also recall the proof that this variety is Fano.

The basic definitions and results necessary for this section are covered in Appendix A.2.

Notation 2.3.1. Let R be a Henselian discrete valuation ring with maximal ideal $\mathfrak{m}$, fraction field K of characteristic 0 and algebraically closed residue field k of characteristic $p > 0$. Let X_K be a smooth, projective, geometrically connected curve of genus $g \geq 2$ defined over K. Fix integers r, d coprime with $r \geq 2$. Denote by $\mathcal{L}_K$ an invertible sheaf of degree d on X_K. Let $\overline{K}$ denote the algebraic closure of K and $X_{\overline{K}} := X_K \times_K \operatorname{Spec} \overline{K}$ the base change of X_K. Denote by $\mathcal{L}_{\overline{K}} := \mathcal{L}_K \otimes_K \overline{K}$ the pull back of the invertible sheaf $\mathcal{L}_K$ to $\overline{K}$.

Definition 2.3.2. Let Sch$/K$ denote the category of schemes of finite type over K and Sch$^\circ/K$ its opposite category. Denote by $X_T := X_K \times_{\operatorname{Spec}(K)} T$ and let r, d be as in Notation 2.3.1. We define a functor $\mathcal{M}_{X_K}(r, d)$ as follows:

$$\mathcal{M}_{X_K}(r, d) : \mathrm{Sch}^\circ/K \to \mathrm{Sets}$$

such that for a K-scheme T,

$$\mathcal{M}_{X_K}(r,d)(T) := \left\{ \begin{array}{c} S\text{-equivalence classes of locally free sheaves } \mathcal{F} \text{ on} \\ X_T \text{ such that for every geometric point} t \in T, \mathcal{F}_t \text{ is a} \\ \text{slope semistable sheaf of rank } r \text{ and degree } d \text{ on } X_t. \end{array} \right\} / \sim$$

where $\mathcal{F} \sim \mathcal{F}'$ if and only if there exists an invertible sheaf $\mathcal{L}$ on T, such that $\mathcal{F} \simeq \mathcal{F}' \otimes \pi_T^* \mathcal{L}$ where $\pi_T : X_T \to T$ is the second projection map.

We denote by $\mathcal{M}^s_{X_K}(r,d)$ the subfunctor for stable sheaves.

Remark 2.3.3. We observe the following:

1. Since X_K is a curve over a field, fixing the rank and the degree of a locally free sheaf is the same as fixing its Hilbert polynomial. Therefore on a smooth curve Gieseker semistability coincides with slope semistability (see Lemma A.1.4).
2. By assumption, the rank r and degree d of the locally free sheaves in Definition 2.3.2 are coprime and X_K is integral. Then by Lemma A.1.5 the semistable locally free sheaves of rank r and degree d are in fact stable. Hence $\mathcal{M}_{X_K}(r,d)$ and $\mathcal{M}^s_{X_K}(r,d)$ coincide.

Definition 2.3.4. By Remark 2.3.3 and Theorem A.2.9, the functor $\mathcal{M}_{X_K}(r,d)$ is universally corepresented by a projective K-scheme. We denote this scheme by $M^s_{X_K}(r,d)$.

Now we define the functor of stable locally free sheaves with fixed determinant on the curve X_K.

Definition 2.3.5. Let $\mathcal{L}_K$ be as in Notation 2.3.1. Denote by $X_T := X_K \times_{\mathrm{Spec}(K)} T$ and let r,d be as in Notation 2.3.1. We define a functor $\mathcal{M}_{X_K,\mathcal{L}_K}(r,d)$ as follows:

$$\mathcal{M}_{X_K,\mathcal{L}_K}(r,d) : \mathrm{Sch}^\circ/K \to \mathrm{Sets}$$

such that for a K-scheme T,

$$\mathcal{M}_{X_K,\mathcal{L}_K}(r,d)(T) := \left\{ \begin{array}{c} S\text{-equivalence classes of locally free sheaves } \mathcal{F} \text{ on } X_T \\ \text{such that for every geometric point } t \in T, \mathcal{F}_t \text{ is a slope} \\ \text{semistable sheaf of rank } r \text{ and degree } d \text{ on } X_t \text{ and for} \\ \text{some invertible sheaf } \mathcal{Q} \text{ on } T, \det(\mathcal{F}) \simeq \pi_{X_K}^* \mathcal{L}_K \otimes \pi_T^* \mathcal{Q} \end{array} \right\} / \sim$$

where $\pi_{X_K} : X_T \to X_K$, $\pi_T : X_T \to T$ are the first and second projections respectively and $\mathcal{F} \sim \mathcal{F}'$ if and only if there exists an invertible sheaf $\mathcal{L}$ on T such that $\mathcal{F} \simeq \mathcal{F}' \otimes \pi_T^* \mathcal{L}$.

We denote by $\mathcal{M}^s_{X_K,\mathcal{L}_K}(r,d)$ the subfunctor for the stable sheaves. By Remark 2.3.3(2), $\mathcal{M}^s_{X_K,\mathcal{L}_K}(r,d)$ coincides with $\mathcal{M}_{X_K,\mathcal{L}_K}(r,d)$.

Recall the Picard functor $\mathcal{P}ic_{X_K}$ and the natural transformation $\mathcal{M}^s_{X_K}(r,d) \to \mathcal{P}ic_{X_K}$ which is defined by taking the determinant of the locally free sheaves. This induces the *determinant* morphism $\det : M^s_K(r,d) \to \operatorname{Pic}(X_K)$.

Proposition 2.3.6. The functor $\mathcal{M}^s_{X_K,\mathcal{L}_K}(r,d)$ is corepresented by a K-scheme of finite type. We denote this scheme by $M^s_{X_K,\mathcal{L}_K}(r,d)$. Furthermore, $M^s_{X_K,\mathcal{L}_K}(r,d) \simeq \det^{-1}(\mathcal{L}_K)$.

Proof. We know from the proof of Theorem A.2.9 that there exists a subset of the Quot scheme, denoted $\mathcal{R}^s$, such that $M^s_{X_K}(r,d)$ is a categorical quotient of this subset by the action of a certain general linear group. Denote by $\alpha : \mathcal{R}^s \to M^s_{X_K}(r,d)$ this quotient.

By composing the morphism det with α we obtain, a morphism

$$\det_{\mathcal{R}} : \mathcal{R}^s \to M^s_{X_K}(r,d) \to \operatorname{Pic}(X_K).$$

Let $\mathcal{R}^s_{\mathcal{L}_K} := \det_{\mathcal{R}}^{-1}(\mathcal{L}_K)$ denote the fibre of the map $\det_{\mathcal{R}}$ at the point corresponding to $\mathcal{L}_K$ and $N_{K,\mathcal{L}_K} := \det^{-1}(\mathcal{L}_K)$. It is easy to see that $\mathcal{M}^s_{X_K,\mathcal{L}_K}(r,d)$ is corepresented by a K-scheme $M^s_{X_K,\mathcal{L}_K}(r,d)$ which is the categorical quotient of $\mathcal{R}^s_{\mathcal{L}_K}$ by the same general linear group.

Furthermore, by the universal property of categorical quotients, there exists an unique morphism $\phi : M^s_{X_K,\mathcal{L}_K}(r,d) \to N_{K,\mathcal{L}_K}$. Since the characteristic of the field K is 0, [HL97, Theorem 4.2.10] implies ϕ is an isomorphism. This completes the proof. □

Let us recall the proof of $M^s_{X_K,\mathcal{L}_K}(r,d)$ being a Fano variety.

Proposition 2.3.7. The moduli space $M^s_{X_K,\mathcal{L}_K}(r,d)$ is Fano.

Proof. By [DN89, Theorem *B*], the smooth, projective variety $M^s_{\overline{K},\mathcal{L}_{\overline{K}}}(r,d)$ has Picard group isomorphic to $\mathbb{Z}$. Therefore, we can assume that it is generated by an ample invertible sheaf, say $\mathcal{L}'$. Moreover by [Ses, pp 53], $M^s_{X_{\overline{K}},\mathcal{L}_{\overline{K}}}(r,d)$ is a unirational variety. Hence by Lemma 2.2.3 the canonical divisor ω^s_0 of $M^s_{X_{\overline{K}},\mathcal{L}_{\overline{K}}}(r,d)$ is not numerically effective. Therefore it cannot be ample or trivial. Then by [Har77, Proposition II7.5], $\omega^\vee_0$ is ample. Therefore, the variety $M^s_{X_{\overline{K}},\mathcal{L}_{\overline{K}}}(r,d)$ is Fano and by Lemma 2.2.4, so is $M^s_{X_K,\mathcal{L}_K}(r,d)$. □

Remark 2.3.8. By Theorem 2.2.6 and Proposition 2.3.7, $M^s_{X_K,\mathcal{L}_K}(r,d)$ is rationally connected. Moreover, since the field K is of characteristic 0, [Kol13, Proposition IV.3.3.1] it is also separably rationally connected. Therefore if the C_1 conjecture is true, the variety $M^s_{X_K,\mathcal{L}_K}(r,d)$ has a K-rational point. In this thesis we prove that this is indeed the case under certain assumptions.

2.4 An example of the C_1 conjecture in mixed characteristic

In this section we prove that there always exists a semistable locally free sheaf of fixed rank and determinant on a smooth, projective curve defined over an algebraically closed field of arbitrary characteristic. Using this we prove that the variety $M^s_{K,\mathcal{L}_K}(r,d)$ is non-empty. We then state the question we will be answering in this thesis.

Remark 2.4.1. By [LP97, Proposition 8.6.1], we know that that there always exists a semistable locally free sheaf of degree d and rank r on a smooth, projective curve defined over the field of complex numbers. The proof as it is given holds true for any algebraically closed field of characteristic 0. However the proof fails when the curve is defined over an algebraically closed field of characteristic p. In particular, the use of Bertini's theorem does not hold in characteristic $p > 0$. To circumvent this problem we replace the Bertini argument by Lemma 2.4.4 proven below. Hence we give a proof for the existence of a semistable locally free sheaf of degree d and rank r on a smooth curve defined over an algebraically closed field of arbitrary characteristic.

It should be noted that there are other proofs for the existence of (semi)stable locally free sheaves of fixed rank and degree over algebraically closed fields. We give the proof that works in any characteristic since we will be using this result several times in this thesis and over fields of different characteristics.

Notation 2.4.2. Let F be an algebraically closed field and X_F a smooth, projective curve on F of genus at least 1. Given any F-variety S, denote by $X_S := X_F \times_{\mathrm{Spec}(F)} S$. For any triple $(S, X_S, \mathcal{F})$, where S is a F-scheme and $\mathcal{F}$ a coherent sheaf on X_S, denote by $\mathrm{Quot}^{r,d}_{X_S/S/\mathcal{F}}$ the relative Quot scheme parametrizing coherent quotients of $\mathcal{F}$ of degree d and rank r. Denote by $\pi : \mathrm{Quot}^{r,d}_{X_S/S/\mathcal{F}} \to S$ the natural morphism. In the case $S = \mathrm{Spec}(F)$ we will simply denote the Quot scheme by $\mathrm{Quot}^{r,d}_{X_F/\mathcal{F}}$.

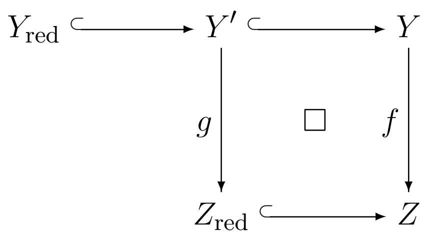
We need the following lemma to replace a step in the original proof of [LP97, Proposition 8.6.1] which does not hold in characteristic $p > 0$.

Definition 2.4.3. We say that a scheme X is *densely reduced* if there exists a open dense subset U of X such that for all $u \in U$, the local ring $\mathcal{O}_{X,u}$ does not contain any nilpotent element.

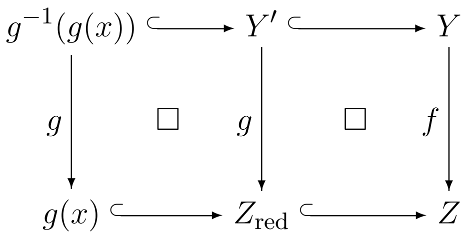
Lemma 2.4.4. Let $f : Y \to Z$ be a dominant morphism of F-schemes, locally of finite type. Suppose that the geometric generic fiber of the induced morphism $h : Y_{\mathrm{red}} \to Z_{\mathrm{red}}$ is densely reduced. Then, for every irreducible component Y' of Y mapping dominantly onto Z and a general $x \in Y'$ closed point (by general we mean outside finitely many proper closed subsets),

$$\dim T_x Y - \dim T_x f^{-1}(f(x)) \geq \dim_x Y - \dim_x f^{-1}(f(x)).$$

Proof. Denote by g the pull-back of f by the morphism $Z_{\mathrm{red}} \to Z$ i.e., $g : Y' := Y \times_Z Z_{\mathrm{red}} \to Z_{\mathrm{red}}$ is the pull-back of f. Consider the following diagram:



As f is assumed to be locally of finite type between schemes of finite type over a field, closed points map to closed points. But closed points are reduced, hence for x closed, x lies in Y_{red} (see [Har77, Ex. II.2.3]). Then by the commutativity of the above diagram, $g(x)$ and $f(x)$ are the same point on Z. Moreover we have the following diagram:



As $g(x) = f(x)$, by the universal property of pullback, we can check $f^{-1}(f(x)) \cong g^{-1}(g(x))$ (scheme-theoretic isomorphism). Hence

$$\dim T_x Y - \dim T_x f^{-1}(f(x)) \geq \dim T_x Y' - \dim T_x g^{-1}(g(x)). \tag{2.1}$$

Denote by $h : Y_{\text{red}} = Y'_{\text{red}} \hookrightarrow Y' \xrightarrow{g} Z_{\text{red}}$. Hence, for any $x \in Y_{\text{red}}$, the natural morphisms $T_x h^{-1}(h(x)) \to T_x g^{-1}(g(x))$ and $T_x Y_{\text{red}} \to T_x Y'$ are injective. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_x h^{-1}(h(x)) & \longrightarrow & T_x Y_{\text{red}} & \xrightarrow{dh_x} & T_{g(x)} Z_{\text{red}} \\
 & & \big\downarrow & \circlearrowleft & \big\downarrow & \circlearrowleft & \big\downarrow \text{Id} \\
0 & \longrightarrow & T_x g^{-1}(g(x)) & \longrightarrow & T_x Y' & \xrightarrow{dg_x} & T_{g(x)} Z_{\text{red}}
\end{array}$$

Therefore, $\dim T_x Y' - \dim T_x g^{-1}(g(x)) \geq \dim T_x Y_{\text{red}} - \dim T_x h^{-1}(h(x))$. As the geometric generic fiber of h is densely reduced, [Gro66, Corollary 12.1.17] implies for every irreducible component Y' of Y mapping dominantly onto Z and a general, closed point $x \in Y'$, $h^{-1}(h(x))$ is densely reduced. Hence

$$\dim_x f^{-1}(f(x)) = \dim_x h^{-1}(h(x)) = \dim T_x h^{-1}(h(x)).$$

Combining with (2.1) we get,

$$\dim_x Y - \dim_x f^{-1}(f(x)) = \dim T_x Y_{\text{red}} - \dim T_x h^{-1}(h(x)) \leq$$
$$\leq \dim T_x Y - \dim T_x f^{-1}(f(x)).$$

This completes the proof of the lemma. □

We will use the following proposition to satisfy the hypothesis of Lemma 2.4.4. This will play a vital role in the proof of Theorem 2.4.6 below.

Proposition 2.4.5. Let A be a regular F-algebra and $f : X \to \text{Spec}(A)$ a dominant morphism of finite type between F-schemes. Suppose that X is reduced and f has non-reduced geometric generic fiber. Then, there exists a normal affine ring A' with the same Krull dimension as A and a dominant morphism from $\text{Spec}(A')$ to $\text{Spec}(A)$ such that the geometric generic fiber of the composition

$$(X \times_A \text{Spec}(A'))_{\text{red}} \to X \times_A \text{Spec}(A') \to \text{Spec}(A')$$

is densely reduced.

Proof. As reducedness is a local property, it suffices to prove the statement for the restriction of f to an affine open dense subscheme $\mathrm{Spec}(B)$ such that $f|_{\mathrm{Spec}(B)}$ is also dominant. Denote by $g: A \to B$ the morphism induced by $f|_{\mathrm{Spec}(B)}$, $L' = \mathrm{Frac}(A)$ and $\overline{L'}$ an algebraic closure of L'. As a finite type morphism is stable under base-change, $\overline{L'} \to \overline{L'} \otimes_A B$ is of finite type. Hence, $\overline{L'} \otimes_A B$ is a noetherian ring. This implies that the nilradical $N \subset \overline{L'} \otimes_A B$ is generated by finitely many elements, say $x_1, ..., x_m$.

Let L be a finite field extension of L' over which $x_1, ..., x_m$ are defined i.e., there exists $y_1, ..., y_m \in B \otimes_A L$ such that under the induced morphism $B \otimes_A L \to B \otimes_A \overline{L'}$, y_i maps to x_i. As A is regular, it is integrally closed in L'. Denote by A' the integral closure of A in L. Then, the Krull dimension of A is the same as A'. Denote by

$$g_{A'}: \mathrm{Spec}(B) \times_A \mathrm{Spec}(A') \to \mathrm{Spec}(A'),$$

the base change of g by the dominant morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. Denote by N' the nilradical of the ring $B \otimes_A A'$. The geometric generic fiber of the morphism

$$g_{A'_{\mathrm{red}}}: \mathrm{Spec}(B \otimes_A A')_{\mathrm{red}} \to \mathrm{Spec}(B \otimes_A A') \to \mathrm{Spec}(A')$$

is isomorphic to $\mathrm{Spec}((B \otimes_A A')/N' \otimes_{A'} \overline{L'})$. It now remains to prove that $\mathrm{Spec}((B \otimes_A A')/N' \otimes_{A'} \overline{L'})$ is reduced i.e., $(B \otimes_A A')/N' \otimes_{A'} \overline{L'}$ does not contain any nilpotent element.

Consider the short exact sequence,

$$0 \to N' \to B \otimes_A A' \to (B \otimes_A A')/N' \to 0$$

and tensor it by $- \otimes_{A'} L$. As $(B \otimes_A A') \otimes_{A'} L \cong B \otimes_A L$, $N' \otimes_{A'} L$ is a nilpotent ideal of $B \otimes_A L$ and $(B \otimes_A A')/N' \otimes_{A'} L$ is reduced, we have $(B \otimes_A A')/N' \otimes_{A'} L \cong (B \otimes_A L)/(y_1, ..., y_m)$ (uniqueness of reduced scheme structure). Consider now the short exact sequence:

$$0 \to (y_1, ..., y_m) \to B \otimes_A L \to (B \otimes_A L)/(y_1, ..., y_m) \to 0.$$

Tensoring this by $- \otimes_L \overline{L'}$ and observing $(y_1, ..., y_m) \otimes_L \overline{L'} \cong (x_1, ..., x_m)$, we have $(B \otimes_A L)/(y_1, ..., y_m) \otimes_L \overline{L'} \cong (B \otimes_A \overline{L'})/(x_1, ..., x_m)$. But $(B \otimes_A \overline{L'})/(x_1, ..., x_m)$ is reduced. To summarise, we have

$$(B \otimes_A A')/N' \otimes_{A'} \overline{L'} \cong ((B \otimes_A A')/N' \otimes_{A'} L) \otimes_L \overline{L'} \cong (B \otimes_A L)/(y_1, ..., y_m) \otimes_L \overline{L'}$$

which is isomorphic to $(B \otimes_A \overline{L'})/(x_1, ..., x_m)$ is reduced. This completes the proof of the proposition. $\square$

Now we recall the following theorem from [LP97, Proposition 8.6.1] with some modifications so that it holds in arbitrary characteristic.

Theorem 2.4.6. For any pair of integers r, d with $r > 1$, there exists a semistable locally free sheaf of rank r and degree d on X_F.

Proof. Take $\mathcal{Q}$ an invertible sheaf of degree d on X_F. Then $\mathcal{E} := \mathcal{Q} \oplus \mathcal{O}_{X_F}^{\oplus r-1}$ is a locally free sheaf of rank r and degree d. By Serre's vanishing theorem, there exists $m >> 0$ such that $h^1(\mathcal{E}(m)) = 0$ for $m \gg 0$ and $\mathcal{E}(m)$ is generated by global sections. Take such m and $\mathcal{E}' := \mathcal{E}(m)$. Denote by $d' := \deg(\mathcal{E}')$. Let $\mathcal{H} := H^0(\mathcal{E}') \otimes \mathcal{O}_{X_F}$. If $\mathcal{E}'$ is semistable then so is $\mathcal{E}$ as twisting by a invertible sheaf does not change semistability. Then we are done.

Suppose $\mathcal{E}'$ is not semistable. Denote by $S_0 := \mathrm{Spec}(F)$. Since $\mathcal{E}'$ is locally free, there exists an affine open neighbourhood $S \subset \mathrm{Quot}^{r,d'}_{X_F/S_0/\mathcal{H}}$ such that every element of S corresponds to a locally free sheaf. Denote by $\mathbb{F}$ the restriction of the universal quotient of $\mathrm{Quot}^{r,d'}_{X_F/S_0/\mathcal{H}}$ to $X_F \times S$. Furthermore, by upper-semicontinuity theorem, we can assume that for any $s \in S$, the corresponding element $[\mathcal{H} \twoheadrightarrow \mathbb{F}(s)]$ satisfies: $h^1(\mathbb{F}(s)) = 0$. By [Har10, Theorem 7.1], infinitesimal deformation of locally free sheaves along families of curves is unobstructed (use Grothendieck vanishing theorem). Hence, S is non-singular. It suffices to prove that a general element of S is semistable. Indeed, for such an element $[\mathcal{H} \twoheadrightarrow \mathbb{F}(s)]$, $\mathbb{F}(s)(-m)$ is semistable of degree d and rank r.

Let $s \in S$ be a closed point and the corresponding quotient of $\mathcal{H}$ sits in the following short exact sequence:

$$0 \to \mathcal{K}(s) \to \mathcal{H} \to \mathbb{F}(s) \to 0.$$

Let $\mathbb{F}(s) \twoheadrightarrow \mathcal{G}$ be a coherent quotient of $\mathbb{F}(s)$. Denote by j the composition $\mathcal{H} \to \mathbb{F}(s) \to \mathcal{G}$. Since $\deg(\mathcal{H}) = 0$ (degree of trivial sheaves is zero) and degree is additive, $-\deg(\ker j) = \deg(\mathcal{G})$. Moreover, trivial bundles are semistable, hence $\deg(\ker j) \leq 0$. Therefore, $\deg(\mathcal{G}) \geq 0$. To summarize, every coherent quotient of $\mathbb{F}(s)$ has degree non-negative.

Note that, there are finitely many choices of r'', d'' satisfying $0 < r'' < r, 0 \leq d''$ and $d''/r'' < d'/r$. Since for any $s \in S$, the degree of any coherent quotient of $\mathbb{F}(s)$ is non-negative, as seen above, it suffices to prove that for any such pair r'', d'', the image of the natural morphism $\pi^{r'',d''} : \mathrm{Quot}^{r'',d''}_{X_F \times S/S/\mathbb{F}} \to S$ is not the whole of S. Indeed, for any such r'', d'' the image of $\pi^{r'',d''}$ consists of *all* points $s \in S$ such that there exists a coherent quotient $\mathbb{F}(s) \twoheadrightarrow \mathcal{G}$ such that $\mathrm{rk}(\mathcal{G}) = r''$ and $\deg(\mathcal{G}) = d''$ in particular, $\mathbb{F}(s)$ is not semistable. Conversely, for any closed point $s \in S$ corresponding to a non-semistable sheaf $\mathbb{F}(s)$ there exists such a pair r'', d''

such that $s \in \operatorname{Im} \pi^{r'',d''}$. Since there are finitely many choices of such r'', d'', we have our claim.

Suppose $\pi^{r'',d''}$ is dominant of finite type. Applying Proposition 2.4.5 to the morphism

$$\left(\operatorname{Quot}^{r'',d''}_{X_F \times S/S/\mathbb{F}}\right)_{\text{red}} \to \operatorname{Quot}^{r'',d''}_{X_F \times S/S/\mathbb{F}} \xrightarrow{\pi^{r'',d''}} S,$$

there exists a normal affine scheme $\overline{S}$ of the same dimension as S and a dominant morphism $\overline{S} \to S$ of finite type such that the fiber product

$$\begin{array}{ccc} \operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}} & \longrightarrow & \operatorname{Quot}^{r'',d''}_{X_F \times S/S/\mathbb{F}} \\ {\scriptstyle \overline{\pi}^{r'',d''}} \downarrow & \square & \downarrow {\scriptstyle \pi^{r'',d''}} \\ \overline{S} & \longrightarrow & S \end{array}$$

satisfies the property: the geometric generic fiber of the composition

$$\left(\operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}}\right)_{\text{red}} \to \operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}} \xrightarrow{\overline{\pi}^{r'',d''}} \overline{S}$$

is densely reduced. Using Lemma 2.4.4 we conclude that for any irreducible component of $\operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}}$ mapping dominantly onto $\overline{S}$ and a general closed point x on it,

$$\dim \overline{S} = \dim_x \operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}} - \dim_x \left(\overline{\pi}^{r'',d''}\right)^{-1} (\overline{\pi}^{r'',d''}(x)) \leq$$

$$\leq \dim T_x \operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}} - \dim T_x \left(\overline{\pi}^{r'',d''}\right)^{-1} (\overline{\pi}^{r'',d''}(x)).$$

Recall by [Ser06, Corollary 4.4.5] for such a general closed point $x \in \operatorname{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}}$, corresponding to a short exact sequence of the form,

$$0 \to \mathcal{E}(x) \to \mathbb{F}(\overline{\pi}^{r'',d''}(x)) \to \mathcal{G}(x) \to 0,$$

$T_x \left(\overline{\pi}^{r'',d''}\right)^{-1} \left(\overline{\pi}^{r'',d''}(x)\right) \cong \operatorname{Hom}(\mathcal{E}(x), \mathcal{G}(x))$. Denote by $s = \overline{\pi}^{r'',d''}(x)$. As

$$h^1(\mathbb{F}(s)) = 0,\ h^1(\mathcal{H}om(\mathcal{H}, \mathbb{F}(s))) = \oplus h^1(\mathbb{F}(s)) = 0.$$

By the Grothendieck spectral sequence, this implies $\operatorname{Ext}^1(\mathcal{H}, \mathbb{F}(s)) = 0$ (recall $\mathcal{E}xt^1(\mathcal{H}, \mathbb{F}(s)) = 0$ as $\mathcal{H}$ is locally free). By Lemma A.2.4, we conclude the Kodaira-Spencer map κ is surjective. Therefore, ω is surjective by Lemma A.2.6. Using the short exact sequence (A.1) we finally have,

$$\dim \overline{S} \leq \dim T_x \mathrm{Quot}^{r'',d''}_{X_F \times \overline{S}/\overline{S}/\mathbb{F}} - \dim T_x \left(\overline{\pi}^{r'',d''}\right)^{-1}(s) =$$

$$= \dim T_s \overline{S} - \dim \mathrm{Ext}^1(\mathcal{E}(x), \mathcal{G}(x)).$$

Note that, $\mathrm{rk}(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x)) = \mathrm{rk}(\mathcal{E}(\mathrm{x})).\mathrm{rk}(\mathcal{G}(\mathrm{x}))$ and

$$\deg(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x))) = \deg(\mathcal{E}(x)^{\vee} \otimes \mathcal{G}(x)) = \mathrm{rk}(\mathcal{G}(x))\deg(\mathcal{E}(x)^{\vee}) +$$

$$+ \mathrm{rk}(\mathcal{E}(x))\deg(\mathcal{G}(x)) = \mathrm{rk}(\mathcal{E}(x))\deg(\mathcal{G}(x)) - \mathrm{rk}(\mathcal{G}(x))\deg(\mathcal{E}(x)).$$

Since $\mathrm{rk}(\mathcal{G}(x)) = r'', \deg(\mathcal{G}(x)) = d'', \mathrm{rk}(\mathbb{F}(x)) = r$, $\deg \mathbb{F}(x) = d'$ and rank and degree of vector bundles are additive, we have $\mathrm{rk}(\mathcal{E}(x)) = r - r''$ and $\deg(\mathcal{E}(x)) = d' - d''$.

The Riemann-Roch theorem for a vector bundle E on X_F states,

$$\chi(E) = \deg(E) + \mathrm{rk}(E)(1 - g(X_F)).$$

Hence,

$$\chi(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x)) = (r - r'')d'' - r''(d' - d'') + r''(r - r'')(1 - g(X_F)) =$$

$$= r''(r - r'')\left(\frac{d''}{r''} - \frac{d' - d''}{r - r''} + 1 - g(X_F)\right).$$

As $\mathcal{E}(x)$ is locally free, $\mathcal{E}xt^1(\mathcal{E}(x), \mathcal{G}(x)) = 0$. Applying Grothendieck Spectral sequence once again, we have $\mathrm{Ext}^1(\mathcal{E}(x), \mathcal{G}(x)) = H^1(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x)))$. We then have

$$\dim \mathrm{Ext}^1(\mathcal{E}(x), \mathcal{G}(x)) = h^1(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x))) = -\chi(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x))) +$$

$$+ h^0(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x))) \geq -\chi(\mathcal{H}\mathrm{om}(\mathcal{E}(x), \mathcal{G}(x))) =$$

$$= r''(r - r'')\left(\frac{d' - d''}{r - r''} - \frac{d''}{r''} + g(X_F) - 1\right) = (d'r'' - rd'') + r''(r - r'')(g(X_F) - 1) > 0$$

as $g(X_F) \geq 1$ and $d''/r'' < d'/r$. Hence, $\dim \overline{S} < \dim T_s \overline{S} = \dim \overline{S}$ where the last equality follows from the fact that $\overline{S}$ is normal and s is general, closed. This contradicts the assumption, $\pi^{r'',d''}$ is dominant. This completes the proof of the theorem. □

Proposition 2.4.7. Replace in Definition 2.3.2, K by F and X_K by X_F. The determinant morphism $\det : M^s_{X_F}(r,d) \to \mathrm{Pic}^d(X_F)$ is surjective.

Proof. By Theorem 2.4.6 there exists a semistable locally free sheaf say $\mathcal{E}$ on X_F with rank r and degree d. Let ρ denote the morphism

$$\mathrm{Pic}^0(X_F) \to M_{X_F}(r,d) \xrightarrow{\det} \mathrm{Pic}^d(X_F), \quad \mathcal{L} \mapsto \mathcal{E} \otimes \mathcal{L} \mapsto \det(\mathcal{E} \otimes \mathcal{L}).$$

Since the det map is closed, so is ρ. Note that the kernel of ρ is the r-th torsion subgroup of $\mathrm{Pic}^0(X_K)$ i.e

$$\ker \rho = \mathrm{Pic}^0(X_F)[r] := \{\mathcal{L} \in \mathrm{Pic}^0(X_F) | \mathcal{L}^r = \mathcal{O}_{X_F}\}.$$

This is because

$$\det(\mathcal{E} \otimes \mathcal{L}) = \det(\mathcal{E} \otimes \mathcal{L}') \Leftrightarrow \mathcal{L}^{\otimes r} \otimes (\mathcal{L}'^{-1})^{\otimes r} = \mathcal{O}_{X_F} \Leftrightarrow \mathcal{L} \otimes \mathcal{L}'^{-1} \in \mathrm{Pic}^0(X_F)[r].$$

This induces a map:

$$\rho' : \frac{\mathrm{Pic}^0(X_F)}{\mathrm{Pic}^0(X_F)[r]} \hookrightarrow \mathrm{Pic}^d(X_F).$$

Since

$$\dim\left(\frac{\mathrm{Pic}^0(X_F)}{\mathrm{Pic}^0(X_F)[r]}\right) = \dim(\mathrm{Pic}^0(X_F)) = \dim(\mathrm{Pic}^d(X_F)),$$

$\mathrm{Im}(\rho')$ is dense in $\mathrm{Pic}^d(X_F)$. Hence ρ is dominant in $\mathrm{Pic}^d(X_F)$. Since ρ is closed, this implies that ρ is surjective. Therefore the morphism

$$\det : M^s_{X_F}(r,d) \to \mathrm{Pic}^d(X_F)$$

is surjective. □

Corollary 2.4.8. Let F be an algebraically closed field (of arbitrary characteristic), X_F a smooth, projective curve of genus $g \geq 1$ over F and r,d a pair of integers with $r > 1$. Let $\mathcal{L}_F$ be a fixed invertible sheaf of degree d on X_F. There exists a semistable locally free sheaf on X_F of rank r and determinant $\mathcal{L}_F$.

Proof. By Theorem 2.4.6, there exists a semistable locally free sheaf of rank r and degree d on X_F. Hence $M^s_{X_F}(r,d)$ is non-empty. By Proposition 2.4.7 $M^s_{X_F}(r,d) \to \mathrm{Pic}^d(X_F)$ is surjective. Therefore there exists at least one semistable locally free sheaf of rank r and degree d with determinant $\mathcal{L}_F$. □

Remark 2.4.9. Replace F by $\overline{K}$ and $\mathcal{L}_F$ by $\mathcal{L}_{\overline{K}}$, a fixed invertible sheaf of degree d, in Corollary 2.4.8. Then by Corollary 2.4.8 the variety $M_{X_K,\mathcal{L}_K}(r,d)$ has a $\overline{K}$-point. By Remark 2.3.3, $M^s_{X_K,\mathcal{L}_K}(r,d)$ has a $\overline{K}$-point In this thesis we prove that under certain assumptions, it also has a K-point.

In order to state these assumptions we need the following definitions.

Definition 2.4.10. Let $S := \mathrm{Spec}(R)$.

1. A *fibred surface* over S is an integral projective S-scheme $\mathcal{C} \twoheadrightarrow S$ of dimension 2.
2. Let C be a smooth, projective connected curve over K. A *model* of C over S is a fibred surface $\mathcal{C} \xrightarrow{f} S$ together with an isomorphism $\mathcal{C}_\eta \simeq C$, where $\mathcal{C}_\eta$ is the generic fibre of f. The model is said to be *smooth* if the morphism f is smooth. It is said to be *regular* if $\mathcal{C}$ is regular .
3. Given a smooth curve C over a discretely valued field K, a *stable* (resp. *semistable*) *model* of C over R is a flat, proper morphism $\mathcal{C} \to S$ with a specified isomorphism $\mathcal{C}_\eta \simeq C$ and the special fiber $\mathcal{C}_s$ is a curve which is reduced, connected, has only nodal singularites, all of whose irreducible components which are rational meet the other components in at least 3 points (resp. 2 points).

The following theorem gives the existence of a stable model for certain curves.

Theorem 2.4.11 ([DM69, Theorem 2.4.11])**.** Let C be a smooth, geometrically connected curve over K, of genus $g \geq 2$. Then there exists a finite field extension $L|K$ such that the curve C_L has a stable model.

Definition 2.4.12. Given a curve C with irreducible components $C_1,\dots,C_n$ we can associate a *dual graph* to it as follows: every irreducible component C_i of C is a vertex weighted by its genus and two vertices are linked by an edge if and only if the corresponding irreducible components share a nodal singularity.

A curve is called a *generalised tree-like curve* if after ignoring the singularities of the individual components, the dual graph associated to the curve does not have any loops.

Assumptions 2.4.13. Throughout this thesis, we assume that there exists a semistable model $X_R \to \mathrm{Spec}(R)$ of X_K with special fibre $X_k := X_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ a generalised tree-like curve whose singular components do not normalise to a rational curve.

Now we state the question we aim to answer in this thesis.

Question 2.4.14. Given Assumptions 2.4.13, does there exist a K-point of the moduli space $M^s_{K,\mathcal{L}_K}(r,d)$?

We answer this in §4.3 (see Theorem 4.3.2).

Chapter 3

Existence of semistable locally free sheaves with fixed determinant

Keep Notations 2.3.1 and Assumptions 2.4.13. This chapter uses definitions and notations from Appendices A.1, A.3 and A.4.

By Corollary 2.4.8, we know that there always exists a Gieseker (and slope) semistable locally free sheaf of rank ≥ 2 and fixed determinant on a smooth curve of genus $g \geq 1$, defined over an algebraically closed field of arbitrary characteristic. In this chapter we prove the same result for the semistable generalised tree-like curve X_k whose irreducible components do not normalise to a rational curve.

We do this by first proving the existence of a slope semistable locally free sheaf of fixed rank and determinant on an irreducible nodal curve defined over an algebraically closed field of arbitrary characteristic, which does not normalise to a rational curve (see Theorem 3.1.9). For this we use Corollary 2.4.8 and the theory of generalised parabolic bundles. It is possible that our semistable, generalised tree-like curve X_k contains non-singular rational components. In order to obtain semistable locally free sheaves on these components, we prove the existence of an invertible sheaf $\mathcal{L}_R$ on X_R such that $\mathcal{L}_k := \mathcal{L}_R \otimes_R k$ has degree a multiple of r on any irreducible, non-singular rational component of X_k and $\mathcal{L}_R \otimes K \simeq \mathcal{L}_K$ (see Lemma 3.2.2 and Lemma 3.2.4). Then using Corollary 2.4.8, Theorem 3.1.9 and Lemma 3.2.4 we prove the existence of a locally free sheaf of rank r and determinant $\mathcal{L}_k$ on the whole curve X_k such that its restriction to each of the irreducible components is semistable. However, this sheaf need not be Seshadri semistable (see Definition A.1.3) with respect to any polarisation. In Appendix A.4 we recall results on Seshadri semistability for locally free sheaves on generalised tree-like curves. We use these results in Theorem 3.2.6, to prove the existence of a Seshadri semistable locally free sheaf $\mathcal{F}_k$ of rank r and determinant $\mathcal{L}_k$ on the curve X_k, for any choice of polarisation. Finally we show that there exists a choice of polarisation for which this sheaf is also Gieseker semistable (see Lemma 3.2.9).

3.1 Existence of a semistable locally free sheaf with fixed determinant on a nodal curve

We recall preliminary definitions and results for this section in Appendix A.3. Here we directly apply them to our situation.

Notation 3.1.1. Let Y be an irreducible nodal curve defined over an algebraically closed field of arbitrary characteristic. Denote by $\pi : \tilde{Y} \to Y$ the normalisation map and assume that $\tilde{Y}$ has genus $g \geq 1$. Let $\mathcal{Q}$ be an invertible sheaf on Y of degree d. Denote by J the set of singular points of Y and let γ be the number of singular points. For all $1 \leq i \leq \gamma$, let p_i, q_i be the two points in $\tilde{Y}$ lying over the double point $x_i \in Y$ and let $D_i := p_i + q_i$ be an effective divisor on $\tilde{Y}$. Let $\mathcal{E}$ be a semistable locally free sheaf on $\tilde{Y}$ of rank r and determinant $\pi^* Q$, the existence of which we have proven in Corollary 2.4.8.

Denote by $\mathcal{E}|_{D_i} := H^0(\mathcal{E} \otimes \mathcal{O}_{D_i}) \otimes \mathcal{O}_{D_i}$. For $x_i \in J$, let

$$\mathcal{E}(p_i) := \mathcal{E}_{p_i} \otimes k(p_i), \quad \mathcal{E}(q_i) := \mathcal{E}_{q_i} \otimes k(q_i),$$

where $k(p_i)$ and $k(q_i)$ are the residue fields at the points p_i and q_i respectively. Fix a set of basis elements $\{e_j\}_{j=1}^r$ and $\{f_j\}_{j=1}^r$ of $\mathcal{E}_{p_i}$ and $\mathcal{E}_{q_i}$, respectively. By abuse of notation, we will again denote by e_j and f_j their image in $\mathcal{E}(p_i)$ and $\mathcal{E}(q_i)$, respectively.

In this section we prove the existence of a semistable locally free sheaf with fixed rank and determinant on an irreducible nodal curve. Our main tool is the theory of generalised parabolic bundles given in [Bho92].

Using the existence of a semistable locally free sheaf with fixed determinant proven in Corollary 2.4.8, we define a generalised parabolic bundle on the normalisation $\tilde{Y}$ (see Definition 3.1.3). We then use the theory of generalised parabolic bundles to prove the existence of a locally free sheaf of fixed rank and degree on the curve Y (see Lemma 3.1.5 and Lemma 3.1.6). Moreover we use parabolic semistability to prove the slope semistability of this locally free sheaf (see Lemma 3.1.7 and Proposition 3.1.8). Finally in Theorem 3.1.9 we prove that there exists a locally free sheaf of fixed rank and determinant on the nodal curve Y.

Remark 3.1.2. Assume that there exists a semistable torsion free sheaf with fixed determinant say $\mathcal{Q}$, on an irreducible nodal curve of genus $g \geq 2$ defined over the complex numbers. Then given this assumption, in [Sun03, Lemma 1.7] X. Sun proves the existence of a semistable locally free sheaf with determinant $\mathcal{Q}$ on the irreducible nodal curve. We do not have this assumption.

Instead, we obtain a semistable locally free sheaf with determinant $\mathcal{Q}$ on an irreducible nodal curve (defined over an algebraically closed field of *arbitrary* characteristic) using a semistable locally free sheaf with determinant $\pi^*\mathcal{Q}$ on its normalisation, the existence of which we have proven in Corollary 2.4.8.

Definition 3.1.3. Recall the definition of a generalised parabolic bundle (see Definition A.3.3). We define a generalised parabolic structure σ of $\mathcal{E}$ over the divisors D_i as follows. Denote by $F_1^i(\mathcal{E})$ the k-vector space generated by $e_j \oplus f_j$ for all $i = 1, \dots \gamma$. We assign to each singular point x_i, $1 \leq i \leq \gamma$:

1. a flag of vector subspaces $\Lambda^i : F_0^i(\mathcal{E}) = \mathcal{E}|_{D_i} \supset F_1^i(\mathcal{E}) \supset F_2^i(\mathcal{E}) = 0$.
2. weights $\underline{\alpha}^i = (0, 1)$.

We define a generalised parabolic locally free sheaf $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ where $\underline{\Lambda} = (\Lambda^1, \dots, \Lambda^\gamma)$ and $\underline{\alpha} = (\underline{\alpha}^1, \dots, \underline{\alpha^\gamma})$.

Definition 3.1.4. We associate to the generalised parabolic bundle $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$, the torsion free sheaf $\phi(\mathcal{E})$ of rank r and degree d on the nodal curve Y as the kernel of the composition:

$$\pi_*(\mathcal{E}) \to \bigoplus_i^\gamma \pi_*(\mathcal{E}) \otimes k(x_i) \to \bigoplus_i^\gamma \frac{\pi_*(\mathcal{E}) \otimes k(x_i)}{F_1^i(\mathcal{E})} \to 0 \tag{3.1}$$

Our choice of $F_1^i(\mathcal{E})$ in Definition 3.1.3 gives the following.

Lemma 3.1.5. *The torsionfree sheaf $\phi(\mathcal{E})$ on the curve Y is locally free of rank r and degree d.*

Proof. Recall, the projection morphisms pr_j^i as in Proposition A.3.8. By definition

$$\mathrm{pr}_1^i : F_1^i(\mathcal{E}) \xrightarrow{\simeq} k(p_i)^{\oplus r} \quad \text{and} \quad \mathrm{pr}_2^i : F_1^i(\mathcal{E}) \xrightarrow{\simeq} k(q_i)^{\oplus r}.$$

Therefore by Proposition A.3.8, the torsionfree sheaf $\phi(\mathcal{E})$ is locally free.

We now show that $\phi(\mathcal{E})$ has the same rank and degree as $\mathcal{E}$. Since Y is irreducible and π is birational, $\mathrm{rk}(\pi_*(\mathcal{E})) = \mathrm{rk}(\mathcal{E})$. Furthermore, the rank of $\frac{\pi_*(\mathcal{E})\otimes k(x_i)}{F_1^i(\mathcal{E})} = 0$ since it is supported on points. Then using the additivity of rank, we have

$$\mathrm{rk}(\phi(\mathcal{E})) = \mathrm{rk}(\pi_*(\mathcal{E})) = \mathrm{rk}(\mathcal{E}).$$

Moreover $\chi(\pi_*(\mathcal{E})) = \chi(\phi(\mathcal{E})) + \sum_{i=1}^{\gamma} \dim(F_1^i(\mathcal{E}))$. Since pushforwards preserve Euler characteristic, using Riemann-Roch for locally free sheaves, we have

$$r(1-(\rho_a(Y)-\gamma)) + \deg(\mathcal{E}) = r(1-\rho_a(Y)) + \deg(\phi(\mathcal{E})) + r\gamma.$$

where ρ_a denotes the genus. Therefore $\deg(\phi(\mathcal{E})) = \deg(\mathcal{E})$. As $\mathcal{E}$ has degree d, so does $\phi(\mathcal{E})$. □

Using this we can prove the following.

Lemma 3.1.6. Let $\mathcal{E}$ and $\phi(\mathcal{E})$ be as above. Then

$$\pi^*\phi(\mathcal{E}) \cong \mathcal{E}.$$

Proof. Consider the pull-back of the morphism $\phi(\mathcal{E}) \to \pi_*\mathcal{E}$ under the normalisation map π and the natural map $\pi^*\pi_*\mathcal{E} \to \mathcal{E}$. Denote by τ the composition

$$\pi^*\phi(\mathcal{E}) \to \pi^*\pi_*(\mathcal{E}) \to \mathcal{E}.$$

Denote by $\mathcal{K}$ the kernel of τ. Note that the localization of τ at x is an isomorphism for all $x \notin \pi^{-1}(J)$. Hence $\mathcal{K}$ is supported at finitely many points and is therefore a torsion sheaf. However, $\pi^*\phi(\mathcal{E})$ is locally free since by Lemma 3.1.5, $\phi(\mathcal{E})$ is locally free and the pull back of a locally free sheaf is locally free. Hence $\pi^*\phi(\mathcal{E})$ cannot contain a non-zero torsion sheaf implying $\mathcal{K} = 0$. Therefore τ is injective.

Since τ is an isomorphism for all $x \notin \pi^{-1}(J)$, $\text{coker}(\tau)$ is a skyscraper sheaf with support in $\pi^{-1}(J)$. But degree of a non-trivial skyscraper sheaf is strictly positive. Since degree is additive, by the short exact sequence

$$0 \to \pi^*\phi(\mathcal{E}) \xrightarrow{\tau} \mathcal{E} \to \text{coker}(\tau) \to 0$$

$\deg(\pi^*(\phi(\mathcal{E})) \leq \deg(\mathcal{E})$ with strict inequality if τ is not surjective. Since π is the normalisation map, $\deg(\pi^*\phi(\mathcal{E})) = \deg(\phi(\mathcal{E}))$. By Lemma 3.1.5, $\deg(\phi(\mathcal{E})) = \deg(\mathcal{E})$. Hence, τ must be surjective.

Therefore τ is an isomorphism and hence $\pi^*\phi(\mathcal{E}) \cong \mathcal{E}$ as required. □

The following lemma follows easily from definitions.

Lemma 3.1.7. Recall the definition of parabolic semistability for generalised parabolic bundles (see Definition A.3.6). The generalised parabolic bundle $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ defined in Definition 3.1.3 is parabolic semistable.

Proof. By definition the generalised parabolic bundle $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ is parabolic semistable if for any sub-bundle $\mathcal{K} \subset \mathcal{E}$ with the induced parabolic structure

$$\mathrm{par}\mu(\mathcal{K}) \leq \mathrm{par}\mu(\mathcal{E}).$$

Since we take weights $\underline{\alpha}^i = (0,1)$ for all $1 \leq i \leq \gamma$, we have

$$\mathrm{par}\mu(\mathcal{E}) = \frac{\deg(\mathcal{E}) + \gamma(\mathrm{rk}(\mathcal{E}))}{\mathrm{rk}(\mathcal{E})}.$$

Note that $\dim\left(\frac{F_1^i(\mathcal{K})}{F_2^i(\mathcal{K})}\right) = \dim(F_1^i(\mathcal{K}))$ because $F_2^i(\mathcal{E}) = 0$, where

$$F_j^i(\mathcal{K}_1) = F_j^i(\mathcal{E}) \cap (\pi_* \mathcal{K}_1 \otimes k(x_i)) \text{ for } j = 1, 2.$$

Moreover

$$\dim(F_1^i(\mathcal{K})) = \dim(F_1^i(\mathcal{E}) \cap H^0(\mathcal{K} \otimes \mathcal{O}_{D_i})) \leq \mathrm{rk}(\mathcal{K}).$$

Therefore

$$\mathrm{par}\mu(\mathcal{K}) \leq \frac{\deg(\mathcal{K}) + \gamma(\mathrm{rk}(\mathcal{K}))}{\mathrm{rk}(\mathcal{K})}.$$

Since $\mathcal{E}$ is a semistable locally free sheaf we have

$$\mathrm{par}\mu(\mathcal{K}) \leq \frac{\deg(\mathcal{K}) + \gamma(\mathrm{rk}(\mathcal{K}))}{\mathrm{rk}(\mathcal{K})} \leq \frac{\deg(\mathcal{E}) + \gamma(\mathrm{rk}(\mathcal{E}))}{\mathrm{rk}(\mathcal{E})} = \mathrm{par}\mu(\mathcal{E})$$

and therefore $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ is a semistable generalised parabolic bundle. □

The following proposition is proven in [Bho92]. We include it here for the sake of completion with small elaborations.

Proposition 3.1.8 ([Bho92, Proposition 4.2])**.** The torsion free sheaf $\phi(\mathcal{E})$ is (semi)stable if and only if the generalised parabolic bundle $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ is (semi)stable.

Proof. Recall, $\pi^*\phi(\mathcal{E}) \cong \mathcal{E}$. Take $\mathcal{K} \subset \phi(\mathcal{E})$ a coherent subsheaf. Denote by $\mathcal{K}_1 :=$ Image$(\pi^*\mathcal{K} \to \pi^*\phi(\mathcal{E}) \cong \mathcal{E})$. Composing $\pi_*\pi^*\mathcal{K} \to \pi_*\mathcal{K}_1$ with the natural morphism $\mathcal{K} \to \pi_*\pi^*\mathcal{K}$, one defines $\rho : \mathcal{K} \to \pi_*\pi^*\mathcal{K} \to \pi_*\mathcal{K}_1$. Since π is an isomorphism when restricted to $U := \tilde{\mathrm{Y}} \setminus \pi^{-1}(J)$, for all $u \in U$,

$$(\pi^*\mathcal{K})_u \xrightarrow{\sim} \mathcal{K}_{1_u} \text{ and } \mathcal{K}_u \xrightarrow{\sim} (\pi_*\pi^*\mathcal{K})_u.$$

Hence, $\ker(\rho)$ is supported on finitely many points of Y which implies it is a torsion sheaf. But $\mathcal{K}$ is a torsion-free subsheaf, hence does not contain any non-trivial torsion sheaf. Therefore, ρ is injective.

Using the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \xrightarrow{\ \rho\ } & \pi_*\mathcal{K}_1 & & & & \\
 & & \cap\downarrow & \circlearrowleft & \cap\downarrow & & & & \\
0 & \longrightarrow & \phi(\mathcal{E}) & \longrightarrow & \pi_*\mathcal{E} & \longrightarrow & \bigoplus_{i=1}^{\gamma}\left(\frac{\pi_*\mathcal{E}\otimes k(x_i)}{F_1^i(\mathcal{E})}\right) & \longrightarrow & 0
\end{array}$$

we observe,

$$\operatorname{coker}(\rho) = \oplus_{x_i\in J}(\pi_*\mathcal{K}_1\otimes k(x_i)/F_1^i(\mathcal{K}_1)),$$

where $F_1^i(\mathcal{K}_1) = F_1^i(\mathcal{E})\cap(\pi_*\mathcal{K}_1\otimes k(x_i))$. By arguments as in proof of Lemma 3.1.5, we have

$$\deg(\pi_*\mathcal{K}_1) = \deg(\mathcal{K}_1)+\gamma\operatorname{rk}(\mathcal{K}_1) \text{ and } \operatorname{rk}(\pi_*\mathcal{K}_1) = \operatorname{rk}(\mathcal{K}_1).$$

Since $\mathcal{K}_1$ is locally free, $\dim\pi_*\mathcal{K}_1\otimes k(x_i) = 2\operatorname{rk}(\mathcal{K}_1)$ for all $x_i \in J$. Hence,

$$\deg\operatorname{coker}(\rho) = 2\gamma(\operatorname{rk}(\mathcal{K}_1)) - \sum_{i=1}^{\gamma}\dim(F_1^i(\mathcal{K}_1)).$$

As degree and rank are additive, we have:

$$\deg(\mathcal{K}) = \deg(\pi_*\mathcal{K}_1) - \deg\operatorname{coker}(\rho)$$

$$= \deg(\mathcal{K}_1)+\gamma(\operatorname{rk}(\mathcal{K}_1)) - 2\gamma(\operatorname{rk}(\mathcal{K}_1)) + \sum_{i=1}^{\gamma}\dim(F_1^i(\mathcal{K}_1)) \text{ and } \operatorname{rk}(\mathcal{K}) = \operatorname{rk}(\mathcal{K}_1).$$

By Lemma 3.1.7 $(\mathcal{E},\underline{\Lambda},\underline{\alpha})$ is a semistable generalised parabolic sheaf, therefore

$$\frac{\deg(\mathcal{K}_1)+\sum_{i=1}^{\gamma}\dim F_1^i(\mathcal{K}_1)}{\operatorname{rk}(\mathcal{K}_1)} = \mu\ \operatorname{par}(\mathcal{K}_1) \leq \mu\ \operatorname{par}(\mathcal{E}) = \frac{\deg(\mathcal{E})+\gamma\dim F_1^1(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}.$$

Since $\mathcal{E}$ is locally free of rank r, $\dim F_1^1(\mathcal{E}) = r$, hence we have the following inequality:

$$\frac{\deg(\mathcal{K})}{\mathrm{rk}(\mathcal{K})} = \frac{\deg(\mathcal{K}_1) + \sum_{i=1}^{\gamma} \dim F_1^i(\mathcal{K}_1) - \gamma \mathrm{rk}(\mathcal{K}_1)}{\mathrm{rk}(\mathcal{K}_1)} \leq \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} = \frac{\deg(\phi(\mathcal{E}))}{\mathrm{rk}(\phi(\mathcal{E}))}.$$

Hence, $\phi(\mathcal{E})$ is semistable. □

We now prove the main result of this section.

Theorem 3.1.9. Let Y be an irreducible nodal curve defined over an algebraically closed field of arbitrary characteristic with normalisation $\tilde{Y}$ a smooth curve of genus $g \geq 1$. Denote by $\mathcal{Q}$ an invertible sheaf on Y of degree d. There exists a semistable locally free sheaf on Y of rank r and determinant $\mathcal{Q}$.

Proof. By Corollary 2.4.8, there exists a semistable locally free sheaf $\mathcal{E}$ of rank r and determinant $\pi^*\mathcal{Q}$ on the normalisation $\tilde{Y}$. Let $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ be the generalised parabolic locally free sheaf as in Definition 3.1.3. Denote by $\phi(\mathcal{E})$ the corresponding torsion free sheaf as in Definition 3.1.4. By Lemma 3.1.5, this is a locally free sheaf on Y of rank r and degree d. The generalised parabolic bundle $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ is semistable by Lemma 3.1.7. Then by Proposition 3.1.8, $\phi(\mathcal{E})$ is semistable.

By assumption $\det(\mathcal{E}) = \pi^*\mathcal{Q}$. For any $x \in Y$, denote by $\tilde{\mathcal{O}}_{Y,x}$ the integral closure of $\mathcal{O}_{Y,x}$. For any $x_i \in J$, $\tilde{\mathcal{O}}^*_{Y,x_i}/\mathcal{O}^*_{Y,x_i} \cong k^*$. By [Har77, Ex. II.6.9], we have a short exact sequence

$$0 \to \oplus_{x_i \in J} k^* \to \mathrm{Pic}(Y) \xrightarrow{\pi^*} \mathrm{Pic}(\tilde{Y}) \to 0. \tag{3.2}$$

Since pull-back commutes with tensor product,

$$\pi^*(\mathcal{Q} \otimes \det(\phi(\mathcal{E}))^{-1}) = \pi^*(\mathcal{Q}) \otimes \pi^*(\det(\phi(\mathcal{E}))^{-1}).$$

By Lemma 3.1.6, $\pi^*\phi(\mathcal{E}) \cong \mathcal{E}$. Hence,

$$\pi^*(\mathcal{Q}) \otimes \pi^*(\det(\phi(\mathcal{E}))^{-1}) = \pi^*(\mathcal{Q}) \otimes (\det(\mathcal{E}))^{-1} = \mathcal{O}_{\tilde{Y}}.$$

Therefore, the invertible sheaf $\mathcal{R} := \mathcal{Q} \otimes_{\mathcal{O}_Y} (\det(\phi(\mathcal{E}))^{-1}$ is in the kernel of π^*. As k is algebraically closed, the morphism

$$[r] : \oplus_{x_i \in Q} k^* \to \oplus_{x_i \in Q} k^*; \quad [r](a_1, a_2, \dots a_n) = (a_1^r, a_2^r, \dots a_n^r)$$

is surjective. Therefore there exists an invertible sheaf, say $\mathcal{R}'$ on Y such that $\mathcal{R} \cong \mathcal{R}'^{\otimes r}$.

Let $\mathcal{G} := \phi(\mathcal{E}) \otimes \mathcal{R}'$. It is easy to see that $\mathcal{G}$ is locally free and as twisting with an invertible sheaf does not change local freeness or the rank, $\text{rank}(\mathcal{G}) = r$. By Lemma A.1.6, stability is also preserved under twisting with a invertible sheaf. By earlier arguments, $\phi(\mathcal{E})$ is semistable, therefore so is $\mathcal{G}$. Moreover,

$$\det(\mathcal{G}) \cong \det(\phi(\mathcal{E}) \otimes \mathcal{R}') \cong \det(\phi(\mathcal{E})) \otimes \mathcal{R}'^{\otimes r} \cong \det(\phi(\mathcal{E})) \otimes \mathcal{R} \cong \mathcal{Q}.$$

The degree of a locally free sheaf is the same as that of its determinant, hence $\deg(\mathcal{G}) = d$. This proves the theorem. □

3.2 Existence of a semistable locally free sheaf with fixed determinant on X_k

Notation 3.2.1. Keep Notations 2.3.1 and Assumption 2.4.13. Denote by $Y_1, \ldots, Y_N$ the irreducible components of the semistable, generalised tree-like curve X_k. Recall Lemma A.4.2 and Notations A.4.3. For $1 \leq i \leq N$, denote by $Y_{\nu(i)}$ the unique component in $B(i)$ which intersects Y_i. Let S denote the set of indices s_i such that the irreducible component Y_{s_i} of X_k is a rational curve and let $|S| = m$. We recall the basic definitions and results needed for this section in Appendix A.4.

In this section we prove the existence of a Gieseker semistable locally free sheaf $\mathcal{F}_k$ on X_k such that $\det(\mathcal{F}_k) \simeq \mathcal{L}_R \otimes_R k$ where $\mathcal{L}_R$ is an invertible sheaf on X_R and $\mathcal{L}_R \otimes_R K \simeq \mathcal{L}_K$.

We do this as follows. We first prove the existence of an invertible sheaf, say $\mathcal{L}'_R$ on X_R such that $\mathcal{L}'_R \otimes \mathcal{O}_{X_K} = \mathcal{L}_K$ and for all $s_i \in S$, $\deg(\mathcal{L}'_R \otimes \mathcal{O}_{Y_{s_i}})$ is a multiple of r (see Lemma 3.2.4). Using this we obtain semistable locally free sheaves of rank r and fixed determinant on the rational components of X_k. Combining this result with Corollary 2.4.8 and Theorem 3.1.9, we show that there exists a semistable locally free sheaf $\mathcal{E}_i$ of rank r and degree d on each component Y_i with determinant $\mathcal{L}_k|_{Y_i}$. We then glue the sheaves $\mathcal{E}_i$ to obtain a locally free sheaf $\mathcal{E}$ of rank r and determinant $\mathcal{L}_k$ on the entire curve X_k which is slope semistable on each of the irreducible components of X_k. However, this sheaf need not be Seshadri semistable on the curve X_k. For this we prove the existence of a line bundle $\mathcal{O}_{X_k}\left(\sum_{i=1}^{N} a_i Y_i\right)$, $a_i \in \mathbf{Z}$ such that $\mathcal{F}_k := \mathcal{E} \otimes_k \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$ satisfies the stability criterion given in Definition A.4.6 which is a sufficient criterion for a sheaf to be Seshadri semistable for any choice of polarisation (see Theorem A.4.11). Note that the determinant of $\mathcal{F}_k$ is still the restriction of a lift of $\mathcal{L}_K$ to X_R. Finally in Lemma 3.2.9, we show that there exists a choice of polarisation such that $\mathcal{F}_k$ is also Gieseker semistable.

By the following lemma we know that for a locally free sheaf to be semistable on a rational component of X_k, its degree must be a multiple of its rank.

Lemma 3.2.2. The only semistable locally free sheaves of rank r on $\mathbf{P}^1$ are of the form $\oplus_{i=1}^{r}\mathcal{O}_{\mathbf{P}^1}(d)$ for some $d \in \mathbf{Z}$.

Proof. Let $\mathcal{E}$ be a locally free sheaf on $\mathbf{P}^1$ of rank r. By Grothendieck's theorem any locally free sheaf on $\mathbf{P}^1$ is of the form

$$\mathcal{E} \simeq \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_r).$$

Note that the degree of $\mathcal{E}$ is $\Sigma_{i=1}^{r} a_i$. By definition $\mathcal{E}$ is μ semistable if for every subsheaf $\mathcal{F} \subset \mathcal{E}$,

$$\frac{\deg(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} \leq \frac{\sum_{i=1}^{r} a_i}{r}$$

Let $b = \max\{a_i\}$ and consider the invertible sheaf $\mathcal{F} := \mathcal{O}_{\mathbf{P}^1}(b)$ of $\mathcal{E}$.

Then

$$\mu(\mathcal{F}) = b \geq \frac{\sum_{i=1}^{r} a_i}{r} = \mu(\mathcal{E})$$

with equality if and only if for all $1 \leq i \leq r$, $a_i = b$. This proves the lemma. □

Therefore a locally free sheaf on a irreducible, non-singular rational component of X_k is semi-stable only if its degree is a multiple of its rank. Our goal is to obtain a semistable locally free sheaf say $\mathcal{F}_k$, on X_k of rank r with $\det(\mathcal{F}_k) \simeq \mathcal{L}_R \otimes_R k$ where $\mathcal{L}_R$ is an invertible sheaf on X_R such that $\mathcal{L}_R \otimes_R K \simeq \mathcal{L}_K$. Therefore both the rank and the degree of $\mathcal{F}_k$ are fixed.

Note that there may be lifts of $\mathcal{L}_K$ on X_R such that the restriction of the sheaf on the irreducible, non-singular rational components of X_k does not have degree a multiple of r. We now prove that for any r, *there does exist* atleast one invertible sheaf on X_R which fulfills this property. Moreover, using sequence (3.3) we observe that this invertible sheaf has restriction $\mathcal{L}_K$ on the generic fibre X_K.

Recall the following exact sequence:

$$\oplus_{i\in I}\mathbb{Z}[Y_i] \to \mathrm{Pic}(X_R) \xrightarrow{\psi} \mathrm{Pic}(X_K) \to 0 \tag{3.3}$$

where ψ is induced by the immersion $X_K \to X_R$. Since the morphism ψ need not be injective, the lift of $\mathcal{L}_K$ to X_R need not be unique. However ψ is surjective because given any divisor D on X_K, its closure $\overline{D}$ in X_R defines a Cartier divisor on X_R which restricts to D in X_K.

We use this observation in proving Lemma 3.2.4 below.

Notation 3.2.3. Recall, the notations of $G(i)$ and $B(i)$ as given in Lemma A.4.2.

Lemma 3.2.4. Let r be a fixed integer. There exists an invertible sheaf $\mathcal{L}_R$ on X_R such that $\mathcal{L}_R \otimes \mathcal{O}_{X_K} = \mathcal{L}_K$ and for all $s_i \in S$, $\deg(\mathcal{L}_R \otimes \mathcal{O}_{Y_{s_i}})$ is a multiple of r.

Proof. Consider any lift $\mathcal{L}'_R$ of $\mathcal{L}_K$ on X_R. By the sequence (3.3), twisting $\mathcal{L}'_R$ by $\mathcal{O}_{X_R}(\sum_{i=1}^{N} a_i Y_i)$ does not change its restriction to X_K. Therefore, we need to find integers a_i such that $\mathcal{L}'_R \otimes \mathcal{O}_{X_R}(\sum_{i=1}^{N} a_i Y_i) \otimes \mathcal{O}_{Y_{s_j}}$ is a multiple of r for every $s_j \in S$. We find such a_i by decreasing induction on s_j for $j = 1, ..., m$ for $m = |S|$.

Using the ordering of the irreducible components of X_k, we numbering $s_1, s_2, .., s_m$ on the indices of S such that $s_1 < s_2 < ... < s_m$. As X_k is a semistable curve, every rational component must intersect at least two other irreducible components of X_k i.e., $Y_{s_j}.\overline{(X_k \backslash Y_{s_j})} \geq 2$ for all $j = 1, ..., m$. By Lemma A.4.2, $Y_{s_j}.B(s_j) = 1$. As $X_k \backslash Y_{s_j} = B(s_j) \cup (G(s_j) \backslash Y_{s_j})$ for each $1 \leq j \leq m$, this implies $Y_{s_j}.(\overline{G(s_j) \backslash Y_{s_j}}) \geq 1$. In particular, $G(j)$ contains at least one curve other than Y_{s_j}. For each j, choose an index $s'_j \neq s_j$ such that the corresponding curve $Y_{s'_j}$ is contained in $G(s_j)$ and intersects Y_{s_j}. Note that by Lemma A.4.2, $s'_j < s_j$ and $Y_{s'_j}$ does not intersect any curve in $B(s_j)$ for all $j = 1, ..., m$.

Base Case: $s_j = s_m$. Suppose that $\deg(\mathcal{L}_k|_{Y_{s_m}})$ is not a multiple of r. As X_k is a tree-like curve $Y_{s'_m}.Y_{s_m} = 1$. Then there exists (by the Euclidean algorithm) an integer $a_{s'_m}$ such that

$$\deg(\mathcal{L}_k \otimes \mathcal{O}_{X_k}(a_{s'_m} Y_{s'_m}) \otimes \mathcal{O}_{Y_{s_m}}) = \deg(\mathcal{L}_k \otimes \mathcal{O}_{Y_{s_m}}) + a_{s'_m}$$

is a multiple of r.

Inductive hypothesis: Assume for some $t < m$, we have integers $a_1, a_2, ..., a_N$ such that

$$\deg\left(\mathcal{L}_k \otimes \mathcal{O}_{X_k}\left(\sum_{i=1}^{N} a_i Y_i\right) \otimes \mathcal{O}_{Y_{s_j}}\right) \qquad t < j \leq m$$

is a multiple of r.

Inductive step:$(j = t)$ As $Y_{s'_t}.Y_{s_t} = 1$, similarly as above, there exists b such that

$$\deg\left(\mathcal{L}_k \otimes \mathcal{O}_{X_k}\left(bY_{s'_t} + \sum_{i=1}^{N} a_i Y_i\right) \otimes \mathcal{O}_{Y_{s_t}}\right)$$

is a multiple of r. As $Y_{s'_t}.B(s_t) = 0$ and $Y_{s_j} \in B(s_t)$ for all $j > t$, $Y_{s'_t}.Y_{s_j} = 0$ for all $j > t$. Hence,

$$\deg\left(\mathcal{L}_k \otimes \mathcal{O}_{X_k}\left(bY_{s'_t} + \sum_{i=1}^{N} a_i Y_i\right) \otimes \mathcal{O}_{Y_{s_j}}\right) = \deg\left(\mathcal{L}_k \otimes \mathcal{O}_{X_k}\left(\sum_{i=1}^{N} a_i Y_i\right) \otimes \mathcal{O}_{Y_{s_j}}\right)$$

is a multiple of r, for all $j > t$. Finally, reassign the values for a_i as follows: Keep a_i unchanged if $i \neq s'_t$ and replace $a_{s'_t}$ by $a_{s'_t} + b$. Therefore, we get integers $a_1, ..., a_N$, such that

$$\deg\left(\mathcal{L}_k \otimes \mathcal{O}_{X_k}\left(\sum_{i=1}^{N} a_i Y_i\right) \otimes \mathcal{O}_{Y_{s_j}}\right) \qquad t \leq j \leq m$$

is a multiple of r. This gives us the induction step and the proof of the lemma. □

The following lemma tells us that twisting a locally free sheaf on the whole curve with a divisor coming from its components does not change the Euler characteristic of the sheaf.

Lemma 3.2.5. Let Z be a connected subcurve of X_k and $\mathcal{E}$ be a locally free sheaf on X_k. Denote by $Z_1, ..., Z_t$ the irreducible components of Z. Fix integers $a_1, ..., a_t$ such that $a_i = 0$ if Y_i intersect $\overline{X_k \backslash Z}$. Denote by

$$\mathcal{L}_0 := \mathcal{O}_{X_R}\left(\sum_{i=1}^{t} a_i Z_i\right) \otimes \mathcal{O}_Z.$$

Then, $\chi(\mathcal{E} \otimes \mathcal{O}_Z) = \chi(\mathcal{E} \otimes \mathcal{L}_0 \otimes \mathcal{O}_Z)$. In particular,

$$\sum_{i=1}^{t} \chi(\mathcal{E} \otimes \mathcal{O}_{Z_i}) = \sum_{i=1}^{t} \chi(\mathcal{E} \otimes \mathcal{L}_0 \otimes \mathcal{O}_{Z_i}).$$

Proof. First observe that it suffices to prove the statement in the case $\mathcal{L}_0 := \mathcal{O}_{X_R}(Z_j) \otimes \mathcal{O}_Z$ where Z_j does not intersect $\overline{X_k \backslash Z}$. Then the argument can be completed by recursion. Denote by $\mathcal{E}_i := \mathcal{E} \otimes \mathcal{O}_{Z_i}$. Let $r := \mathrm{rk}(\mathcal{E})$. Using Lemma A.4.4, one obtains the short exact sequences

$$0 \to \mathcal{E} \otimes \mathcal{O}_Z \to \bigoplus_{i=1}^{t} \mathcal{E}_i \to \bigoplus_{P \in Z^0} \mathcal{O}_P^r \to 0,$$

$$\text{and} \quad 0 \to \mathcal{E} \otimes \mathcal{L}_0 \otimes \mathcal{O}_Z \to \bigoplus_{i=1}^{t} \mathcal{E}_i \otimes \mathcal{L}_0 \to \bigoplus_{P \in Z^0} \mathcal{O}_P^r \to 0.$$

Using the above short exact sequences we notice,

$$\chi(\mathcal{E} \otimes \mathcal{O}_Z) - \chi(\mathcal{E} \otimes \mathcal{L}_0 \otimes \mathcal{O}_Z) = \sum_{i=1}^{t} (\chi(\mathcal{E}_i) - \chi(\mathcal{E}_i \otimes \mathcal{L}_0)). \tag{3.4}$$

Our goal is to show that the right hand side is 0. Since tensor product by an invertible sheaf does not change the rank of a locally free sheaf, $\mathrm{rk}(\mathcal{E}_i) = \mathrm{rk}(\mathcal{E}_i \otimes \mathcal{L}_0)$, for all i. Now,

$$\deg(\mathcal{E}_j \otimes \mathcal{O}_Z(Z_j)) = \deg(\mathcal{E}_j) + r(Z_j^2) = \deg(\mathcal{E}_j) - r \sum_{i=1, Y_i \neq Z_j}^{N} Y_i.Z_j$$

which is equal to $\deg(\mathcal{E}_j) - r \sum_{i=1, i \neq j} Z_i Z_j$ because $Y_i.Z_j = 0$ for Y_i not in Z. Also,

$$\deg(\mathcal{E}_i \otimes \mathcal{O}_Z(Z_j)) = \deg(\mathcal{E}_i) + r Z_i.Z_j \text{ for } i \neq j.$$

Hence, $\sum_{i=1}^{t} \deg(\mathcal{E}_i \otimes \mathcal{O}_Z(Z_j)) = \sum_{i=1}^{t} \deg(\mathcal{E}_i)$. Therefore, $\sum_{i=1}^{t} \chi(\mathcal{E}_i) = \sum_{i=1}^{t} \chi(\mathcal{E}_i \otimes \mathcal{O}_Z(Z_j))$, which implies the lemma. □

Now we prove the main result of this chapter.

Theorem 3.2.6. For any polarisation $\lambda := (\lambda_1, \lambda_2, ..., \lambda_N)$, there exists a Seshadri semistable locally free sheaf $\mathcal{F}_k$ of rank r on X_k with $\det(\mathcal{F}_k) \simeq \mathcal{L}_R \otimes_R k$ on X_k, where $\mathcal{L}_R$ is an invertible sheaf on X_R such that $\mathcal{L}_R \otimes_R K \cong \mathcal{L}_K$.

Proof. By assumption X_k is a semistable generalised tree-like curve having at worst nodal singularities. Hence the irreducible components are either smooth curves of genus $g \geq 0$ or irreducible nodal curves of genus $g \geq 2$ with normalisation a smooth curve of genus $g \geq 1$. By Lemma 3.2.4, there exists a lift of $\mathcal{L}_K$, say $\mathcal{L}'_R$

such that $\deg(\mathcal{L}'_R \otimes Y_{s_i})$ is a multiple of r, for Y_{s_i} any rational component of X_k, $s_i \in S$. Denote by $\mathcal{L}'_k := \mathcal{L}'_R \otimes_R k$ and by $\mathcal{L}'_i := \mathcal{L}'_k|_{Y_i}$. In the case Y_i is rational (resp. smooth of genus $g \geq 1$, resp. irreducible nodal with normalisation of genus $g \geq 1$) there exists a slope semistable locally free sheaf $\mathcal{E}_i$ with determinant $\mathcal{L}'_i$ on the component Y_i by Lemma 3.2.2 (resp. Corollary 2.4.8, resp. Theorem 3.1.9). Define $\mathcal{E}$ to be the locally free sheaf on X_k obtained by glueing $\mathcal{E}_i$ for $1 \leq i \leq n$, on the intersection points. Then $\det(\mathcal{E}) = \mathcal{L}'_k$. If $\mathcal{E}$ is Seshadri semistable, then we are done. Suppose not.

We now prove by recursion the existence of integers $a_1, a_2, \ldots a_N$ such that $\mathcal{E} \otimes_k \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$ is λ-semistable i.e for all $1 \leq i \leq N$, $\mathcal{E} \otimes \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$, satisfies the following inequality:

$$(\sum_{Y_j \in G(i)} \lambda_j)\chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)) + r(|G(i)|-1) \leq \sum_{Y_j \in G(i)} \chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i) \otimes \mathcal{O}_{Y_j}) \leq$$

$$\leq (\sum_{Y_j \in G(i)} \lambda_j)\chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)) + r|G(i)|. \quad (*)$$

By Theorem A.4.11 if a locally free sheaf is λ-semistable, then it is Seshadri semistable, so it suffices to obtain a λ-semistable sheaf.

Note that as a consequence of the ordering given in Lemma A.4.2, any locally free sheaf $\mathcal{E}$ is λ-semistable on Y_N. Indeed, for $i = N$, we have $\sum_{i=1}^{N} \lambda_i = 1$, $G(N) = X_k$ and $|G(N)| = N$. Hence the inequality $(*)$ can be written as,

$$(\sum_{Y_j \in X_k} \lambda_j)\chi(\mathcal{E}) + r(N-1) \leq \sum_{Y_j \in X_k} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_j}) \leq$$

$$\leq (\sum_{Y_j \in X_k} \lambda_j)\chi(\mathcal{E}) + rN.$$

By Lemma A.4.4 we have

$$\sum_{i=1}^{N} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_i}) = \chi(\mathcal{E}) + \sum_{i=1}^{N-1} \chi(\mathcal{E} \otimes \mathcal{O}_{P_i}).$$

where P_i are the nodes in X_k connecting two irreducible components. Since there are $n-1$ such nodes and $\dim(\mathcal{E} \otimes \mathcal{O}_{P_i}) = r$ for all i, we have

$$(\sum_{Y_j \in X_k} \lambda_j)\chi(\mathcal{E}) + r(N-1) \leq \chi(\mathcal{E}) + rN - r \leq$$

$$\leq (\sum_{Y_j \in X_k} \lambda_j)\chi(\mathcal{E}) + rN.$$

Therefore $\mathcal{E}$ is λ-semistable on Y_N.

Assume that for some $n_0 \leq N$, $\mathcal{E}$ is λ-semistable on $Y_{n_0+1}, \ldots Y_N$. Therefore for $i = n_0$, we have

$$\sum_{Y_j \in G(n_0)} \deg(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0}) \otimes \mathcal{O}_{Y_j}) = \sum_{Y_j \in G(n_0)} \left(\deg(\mathcal{E} \otimes \mathcal{O}_{Y_j}) + r a_{n_0} Y_{n_0}.Y_j\right).$$

Note that for any i, $Y_i.Y_{\nu(i)} = 1$ and $\nu(i) > i$. By [Liu02, Proposition 9.1.21], $Y_i^2 = -Y_i.Y_{\nu(i)} - \sum_{Y_j \in G(i) \setminus Y_i} Y_j.Y_i$. Therefore for $i = n_0$ we have

$$\sum_{Y_j \in G(n_0)} \left(\deg(\mathcal{E} \otimes \mathcal{O}_{Y_j}) + r a_{n_0} Y_{n_0}.Y_j\right) = \left(\sum_{Y_j \in G(n_0)} \deg(\mathcal{E} \otimes \mathcal{O}_{Y_j})\right) - r a_{n_0}.$$

Since the rank of $\mathcal{E}$ does not change after twisting with a invertible sheaf, the Euler characteristic depends only on the degree. Furthermore for any i, the difference between the upper-bound and the lower bound in the inequality $*$ is equal to r. Then by the Euclidean algorithm, there must exist an integer a_{n_0} such that

$$(\sum_{Y_j \in G(n_0)} \lambda_j)\chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})) + r(|G(n_0)| - 1) \leq$$

$$\leq \sum_{Y_j \in G(n_0)} \chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0}) \otimes \mathcal{O}_{Y_j}) \leq (\sum_{Y_j \in G(n_0)} \lambda_j)\chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})) + r|G(n_0)|.$$

Hence, $\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})$ is λ-semistable at Y_{n_0}.

Note that $\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})$ is also λ-semistable on Y_t for all $n_0 < t < n$. This is because Y_{n_0} does not intersect any curve in $\overline{B(n_0) \setminus Y_{\nu(n_0)}}$, neither does any curve in $G(n_0)$ (the only curve in $G(n_0)$ that intersects $B(n_0)$ is Y_{n_0}). By Lemma 3.2.5, this implies

$$\sum_{Y_j \in G(n_0) \cup Y_{\nu(n_0)}} \chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0}) \otimes \mathcal{O}_{Y_j}) = \sum_{Y_j \in G(n_0) \cup Y_{\nu(n_0)}} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_j}).$$

Furthermore, for any curve $Y_j \in \overline{B(n_0) \backslash Y_{\nu(n_0)}}$,

$$\chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0}) \otimes \mathcal{O}_{Y_j}) = \chi(\mathcal{E} \otimes \mathcal{O}_{Y_j}).$$

Note that $G(n_0) \cup Y_{\nu(n_0)}$ is connected. Hence, for any $t > n_0$, either $G(n_0) \cup Y_{\nu(n_0)}$ is entirely contained in $G(t)$ or in $B(t)$. Therefore for any $t > n_0$,

$$\sum_{Y_j \in G(t)} \chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0}) \otimes \mathcal{O}_{Y_j}) = \sum_{Y_j \in G(t)} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_j}).$$

By hypothesis, $\mathcal{E}$ is λ-semistable at Y_t for all $t > n_0$. Furthermore by Lemma 3.2.5, we have

$$\chi(\mathcal{E}) = \chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})).$$

Since the sum of the Euler characteristics of $\mathcal{E}$ when restricted to curves in $G(t)$ is the same as that of $\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})$, this implies $\mathcal{E} \otimes \mathcal{O}_{X_k}(a_{n_0} Y_{n_0})$ is also λ-semistable for all $t > n_0$.

By recursion we can find integers a_i for all $1 \le i < N$ such that $\mathcal{E} \otimes \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$ is λ-semistable on Y_i. Therefore the locally free sheaf $\mathcal{F}_k := \mathcal{E} \otimes \mathcal{O}_{X_k}(\sum_{i=1}^{N} a_i Y_i)$ is λ-semistable. By Theorem A.4.11 it is also Seshadri semistable. Note that

$$\det(\mathcal{F}_k) \simeq \left(\mathcal{L}'_R \otimes \left(\mathcal{O}_{X_R}(\sum_{i=1}^{N} a_i Y_i) \right)^r \right) \otimes \mathcal{O}_k.$$

Denote by $\mathcal{L}_R := \mathcal{L}'_R \otimes \left(\mathcal{O}_{X_R}(\sum_{i=1}^{N} a_i Y_i) \right)^r$. Then by the exact sequence (3.3)

$$\mathcal{L}_R \otimes_R K \simeq \mathcal{L}'_R \otimes_R K \simeq \mathcal{L}_K.$$

This proves the theorem. □

Remark 3.2.7. Theorem 3.2.6 proves the existence of a Seshadri semistable (with respect to any polarisation) locally free sheaf of given rank and determinant, on a semistable generalised tree-like curve (defined over an algebraically closed field of arbitrary characteristic) whose singular components do not normalise to a rational curve, provided that the degree of the restriction of the determinant to any non-singular, irreducible rational component of the curve is a multiple of the rank.

Remark 3.2.8. In Theorem 3.2.6, we have proven that for *any* choice of polarisation, there exists a Seshadri semistable locally free sheaf $\mathcal{F}_k$ on X_k. We now prove that there exists a choice of polarisation for which $\mathcal{F}_k$ is also Gieseker semistable. We use this fact in the next chapter.

Lemma 3.2.9. The Seshadri semistable sheaf $\mathcal{F}_k$ on X_k obtained in Theorem 3.2.6 is also Gieseker semistable.

Proof. Recall the notations from Definition A.1.1. To prove that the sheaf $\mathcal{F}_k$ is Gieseker semistable, we need to show that for any subsheaf $\mathcal{G} \subset \mathcal{F}_k$,

$$\frac{\chi(\mathcal{G} \otimes \mathcal{O}_{X_k}(t))}{\alpha_d(\mathcal{G})} \leq \frac{\chi(\mathcal{F}_k \otimes \mathcal{O}_{X_k}(t))}{\alpha_d(\mathcal{F}_k)}.$$

Since X_k is a curve, the dimension of support of any sheaf on X_k is at most 1. Therefore $d = 1$. By Lemma A.4.5 we have

$$\chi(\mathcal{G} \otimes X_k(t)) = \sum_{i=1}^{N} \chi(\mathcal{G}|_{Y_i} \otimes X_k(t)) - \sum_{P \in X_k^0} \dim(\mathcal{G}|_P \otimes X_k(t)).$$

Using this and the fact that dimension of $\mathcal{G}|_{P_i} \otimes \mathcal{O}_{X_k}(t)$ is zero since P_i are points, we get

$$\alpha_1(\mathcal{G}) = \alpha_1(\mathcal{G}_1) + \alpha_1(\mathcal{G}_2) + \cdots + \alpha_1(\mathcal{G}_N) \qquad (*)$$

where $\mathcal{G}_i := \mathcal{G} \otimes Y_i$ and α_1 is the leading coefficient of the Hilbert polynomial. Since $\mathcal{F}_k$ is locally free of rank r and Seshadri semistable we have

$$\frac{\chi(\mathcal{G})}{\Sigma \lambda_i l_i} \leq \frac{\chi(\mathcal{F}_k)}{\Sigma \lambda_i r}.$$

where l_i denotes the rank of $\mathcal{G}_i$. Let $\lambda_i = \frac{\alpha_1(\mathcal{O}_{Y_i})}{\alpha_1(\mathcal{O}_{X_k})}$. Then

$$\Sigma \lambda_i l_i = \frac{1}{\alpha_1(\mathcal{O}_{X_k})} (\alpha_1(\mathcal{O}_{Y_1}) l_1 + \alpha_1(\mathcal{O}_{Y_2}) l_2 + \cdots + \alpha_1(\mathcal{O}_{Y_m}) l_m).$$

By definition $l_i = \frac{\alpha_1(\mathcal{G}_i)}{\alpha_1(\mathcal{O}_{Y_i})}$. Substituting for l_i we get

$$\Sigma\lambda_i l_i = \frac{1}{\alpha_1(\mathcal{O}_{X_k})}(\alpha_1(\mathcal{O}_{Y_1})\frac{\alpha_1(\mathcal{G}_1)}{\alpha_1(\mathcal{O}_{Y_1})} + \alpha_1(\mathcal{O}_{Y_2})\frac{\alpha_1(\mathcal{G}_2)}{\alpha_1(\mathcal{O}_{Y_2})} + \cdots + \alpha_1(\mathcal{O}_{Y_N})\frac{\alpha_1(\mathcal{G}_N)}{\alpha_1(\mathcal{O}_{Y_N})}).$$

Using the inequality $(*)$, we have

$$\Sigma\lambda_i l_i = \frac{1}{\alpha_1(\mathcal{O}_{X_k})}(\alpha_1(\mathcal{G}_1) + \alpha_1(\mathcal{G}_2) + \cdots + \alpha_1(\mathcal{G}_N)) = \frac{\alpha_1(\mathcal{G})}{\alpha_1(\mathcal{O}_{X_k})}.$$

Hence,

$$\frac{\chi(\mathcal{G})}{\alpha_1(\mathcal{G})/\alpha_1(\mathcal{O}_{X_k})} = \frac{\chi(\mathcal{G})}{\Sigma\lambda_i l_i} \leq \frac{\chi(\mathcal{F}_k)}{r} = \frac{\chi(\mathcal{F}_k)}{\alpha_1(\mathcal{F}_k)/\alpha_1(\mathcal{O}_{X_k})}.$$

As $\chi(\mathcal{G} \otimes \mathcal{O}_{X_k}(t))$ (resp. $\chi(\mathcal{F}_k \otimes \mathcal{O}_{X_k}(t))$ are linear polynomials with leading coefficient $\alpha_1(\mathcal{G})$ (resp. $\alpha_1(\mathcal{F}_k)$) and constant term $\chi(\mathcal{G})$ (resp. $\chi(\mathcal{F}_k)$), we have the inequality:

$$\frac{\chi(\mathcal{G} \otimes \mathcal{O}_{X_k}(t))}{\alpha_1(\mathcal{G})} \leq \frac{\chi(\mathcal{F}_k \otimes \mathcal{O}_{X_k}(t))}{\alpha_1(\mathcal{F}_k)}$$

This proves the lemma. □

Chapter 4

Verifying the C_1 conjecture for a particular example

This chapter uses definitions and notations from Appendices A.1 and A.2.

Notation 4.0.1. Keep Notations 2.3.1 and Assumption 2.4.13. Denote by $\hat{R}$ the completion of the discrete valuation ring R and by $X_{\hat{R}} := X_R \times_R \mathrm{Spec}(\hat{R})$. Denote by $\mathcal{L}_R$ a lift of $\mathcal{L}_K$ to X_R such that the degree of the restriction of $\mathcal{L}_R$ to the rational components of X_k is a multiple of r. By Lemma 3.2.4, this is always possible. Denote by $\mathcal{L}_{\hat{R}} := \mathcal{L}_R \otimes_R \hat{R}$ and by $\mathcal{L}_k := \mathcal{L}_R \otimes_R k$.

The goal of this chapter is to answer Question 2.4.14. We answer this in §4.3 in the affirmative (see Theorem 4.3.2). We show the existence of a K-rational point of the moduli space $M^s_{X_K,\mathcal{L}_K}(r,d)$ by proving the existence of a geometrically stable locally free sheaf of rank r and determinant $\mathcal{L}_K$ on the curve X_K. Using results from Chapter 3 and Grothendieck formal function theorem, we first prove the existence of a locally free sheaf with determinant $\mathcal{L}_{\hat{R}}$ on $X_{\hat{R}}$ (see Proposition 4.1.15). Note that for this we require the underlying ring to be complete (see Proposition 4.1.3 and Theorem 4.1.14), therefore we use $\hat{R}$ instead of R. Then using Artin approximation and properties of semistability of sheaves, we obtain a geometrically semistable locally free sheaf $\mathcal{F}_R$ with determinant $\mathcal{L}_R$ on the model X_R (see Proposition 4.2.6). Finally in Theorem 4.3.1 we obtain a geometrically stable locally free sheaf on the curve X_K with determinant $\mathcal{L}_K$ as required.

Remark 4.0.2. Throughout this chapter semistability always refers to Gieseker semistability. We proved in Lemma 3.2.9 that there exists a polarisation such that the Seshadri semistable locally free sheaf $\mathcal{F}_k$ (with respect to this polarisation) obtained in Theorem 3.2.6 is also Gieseker semistable. We use the property of it being Gieseker semistable in Proposition 4.2.6 and Theorem 4.3.1. As we know from Lemma A.1.4 Gieseker semistability is the same as slope semistability on the smooth curve X_K.

4.1 Lifting locally free sheaves with fixed determinant

Notation 4.1.1. Keep notations 3.1.1 and 4.0.1. Denote by $\mathcal{F}_k$ the semistable locally free sheaf on X_k obtained in Theorem 3.2.6. For $n \geq 1$, let $R_n := R/\mathfrak{m}^{n+1}$. Denote by Y_n, the spectrum of the ring R_n and by $\hat{R}$ the projective limit of the rings R_n. Let $X_n := X_{\hat{R}} \times_{\hat{R}} \operatorname{Spec}(R_n)$. Since $X_{\hat{R}}$ is flat over $\operatorname{Spec}(\hat{R})$, the scheme X_n is flat over $\operatorname{Spec}(R_n)$ and $X_k \simeq X_n \times_{R_n} \operatorname{Spec}(k)$. Therefore X_n is a deformation of X_k over R_n. Let $\mathcal{F}_n$ be a locally free sheaf on the curve X_n with determinant $\mathcal{L}_n := \mathcal{L}_{\hat{R}} \otimes_{\hat{R}} R_n$.

In this section using Grothendieck's formal function theorem we show how to lift the Gieseker semistable locally free sheaf $\mathcal{F}_k$ on X_k with determinant $\mathcal{L}_k$, obtained in Theorem 3.2.6, to a locally free sheaf $\mathcal{F}_{\hat{R}}$ on $X_{\hat{R}}$ with determinant $\mathcal{L}_{\hat{R}}$.

Recall the following general definitions.

Definition 4.1.2. We define a ringed space called the *formal spectrum* of R, denoted $\mathcal{Y} := (\operatorname{Spf}(R), \mathcal{O}_{\mathcal{Y}})$ as follows: The topological space $\operatorname{Spf}(R)$ consists of the closed point $\operatorname{Spec}(k)$, with the discrete topology and the sheaf of rings $\mathcal{O}_{\mathcal{Y}}$ is $\hat{R}$.

Proposition 4.1.3 ([Har10, Proposition 21.1])**.** Let $(\hat{R}, \mathfrak{m})$ be a complete local ring with residue field k, and for each n, the schemes X_n flat and of finite type over Y_n and maps $X_n \to X_{n+1}$ inducing isomorphisms $X_n \simeq X_{n+1} \times_{Y_{n+1}} Y_n$. Then there is a noetherian formal scheme $\mathcal{X}$, flat over $\mathcal{Y}$, the formal spectrum of $\hat{R}$, such that for each n, $X_n \simeq \mathcal{X} \times_{\hat{R}} Y_n$.

Definition 4.1.4. Let $\mathcal{F}_n$ (resp. $\mathcal{F}_k$) be a coherent sheaf on X_n (resp. X_k). We define an *extension* $\mathcal{F}_{n+1}$ *of* $\mathcal{F}_n$ (resp. $\mathcal{F}_k$) over R_{n+1} to be a coherent sheaf $\mathcal{F}_{n+1}$ on X_{n+1} flat over R_{n+1}, together with a map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ (resp. $\mathcal{F}_n \to \mathcal{F}_k$) inducing an isomorphism $\mathcal{F}_{n+1} \otimes_{R_{n+1}} R_n \simeq \mathcal{F}_n$ (resp. $\mathcal{F}_n \otimes_{R_n} k \simeq \mathcal{F}_k$).

The following theorem tells us when is it possible to extend a sheaf on X_n to a sheaf on X_{n+1}. Note that the following theorem is general i.e. it applies to any locally free sheaf $\mathcal{F}_k$ on a curve X_k, one does note require our assumptions.

Theorem 4.1.5 ([Har10, Theorem 7.3])**.** Let $\mathcal{F}_k$ denote a locally free sheaf on X_k and suppose that $\mathcal{F}_n$ is a locally free sheaf on X_n such that $\mathcal{F}_k \simeq \mathcal{F}_n \otimes_{R_n} k$. We have the following:

1. If $\mathcal{F}_{n+1}$ is an extension of $\mathcal{F}_n$ on X_{n+1}, then the group $\operatorname{Aut}(\mathcal{F}_{n+1}/\mathcal{F}_n)$ of automorphisms of $\mathcal{F}_{n+1}$ inducing the identity automorphism on $\mathcal{F}_n$ is isomorphic to $H^0(\mathcal{H}om_{X_k}(\mathcal{F}_k, \mathcal{F}_k) \otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$.

2. Given $\mathcal{F}_n$ on X_n, there is an obstruction in $H^2(\mathcal{H}\mathrm{om}_{X_k}(\mathcal{F}_k,\mathcal{F}_k)\otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$ whose vanishing is a necessary and sufficient condition for the existence of an extension $\mathcal{F}_{n+1}$ of $\mathcal{F}_n$ over X_{n+1}.

3. If an extension $\mathcal{F}_{n+1}$ of $\mathcal{F}_n$ over X_{n+1} exists, then the set of all such is a torsor under the action of $H^1(\mathcal{H}\mathrm{om}_{X_k}(\mathcal{F}_k,\mathcal{F}_k)\otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$.

Now, we show that the extension of a locally free sheaf if it exists is again locally free.

Proposition 4.1.6 ([Har10, Exercise 7.1])**.** Assume that we can extend the sheaf $\mathcal{F}_n$ to $\mathcal{F}_{n+1}$ over X_{n+1}. If $\mathcal{F}_n$ is locally free on X_n, then $\mathcal{F}_{n+1}$ is a locally free sheaf on X_{n+1}.

To prove this we first recall a lemma from commutative algebra.

Lemma 4.1.7. Let $(R,\mathfrak{m})$ be a local ring with nilpotent maximal ideal $\mathfrak{m}$. Let M be a flat R-module. If A is a set and $x_\alpha \in M$, $\alpha \in A$ is a collection of elements of M, then the following are equivalent:

1. $\{\overline{x}_\alpha\}_{\alpha\in A}$ forms a basis for the vector space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, and

2. $\{x_\alpha\}_{\alpha\in A}$ forms a basis for M over R.

where $\overline{x}_\alpha$ is the image of x_α under the quotient map $M \to M/(\mathfrak{m}M)$.

Proof of Lemma. The implication (2) $\Rightarrow$ (1) is immediate. We will prove the other implication by using induction on n to show that $\{x_\alpha\}_{\alpha\in A}$ forms a basis for $M/\mathfrak{m}^nM$ over $R/\mathfrak{m}^n$. The case $n=1$ holds by assumption (1). Assume the statement holds for some $n\geq 1$. By Nakayama's Lemma the elements x_α generate M, in particular $M/\mathfrak{m}^{n+1}M$. The exact sequence

$$0\to \mathfrak{m}^n/\mathfrak{m}^{n+1}\to R/\mathfrak{m}^{n+1}\to R/\mathfrak{m}^n\to 0$$

gives on tensoring with M the exact sequence

$$0\to \mathfrak{m}^nM/\mathfrak{m}^{n+1}M\to M/\mathfrak{m}^{n+1}M\to M/\mathfrak{m}^nM\to 0.$$

Here we are using that M is flat. Moreover, we have $\mathfrak{m}^nM/\mathfrak{m}^{n+1}M = M/\mathfrak{m}M\otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ by flatness of M again. Now suppose that $\sum\overline{f}_\alpha\overline{x}_\alpha = 0$ in $M/\mathfrak{m}^{n+1}M$. Then by induction hypothesis $f_\alpha\in\mathfrak{m}^n$ for each α. By the short exact sequence above we then conclude that $\sum\overline{f}_\alpha\otimes\overline{x}_\alpha$ is zero in $\mathfrak{m}^n/\mathfrak{m}^{n+1}\otimes_{R/\mathfrak{m}} M/\mathfrak{m}M$. Since $\overline{x}_\alpha$ forms a basis we conclude that each of the congruence classes $\overline{f}_\alpha\in\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is zero. □

Proof of Proposition. We have that $X_k \to X_{n+1}$ is a closed immersion, giving the following exact sequence

$$0 \to \ker f \to \mathcal{O}_{X_{n+1}} \xrightarrow{f} \mathcal{O}_{X_k} \to 0.$$

Let us localise at a point $x \in X_{n+1}$. Since R_{n+1} is a local Artin ring , by [Mat80, Proposition 3.G], $\mathcal{O}_{X_{n+1},x}$ is a free R_{n+1}-module. Let $\{f_\alpha\}_{\alpha \in I}$ with $f_\alpha \in \mathcal{O}_{X_{n+1},x}$ be the basis of this module. Denote by $\overline{f}_\alpha$ the image of f_α under the natural morphism $\mathcal{O}_{X_{n+1},x} \to \mathcal{O}_{X_{n+1},x}/\mathfrak{m}\mathcal{O}_{X_{n+1},x}$ where $\mathfrak{m}$ is the maximal ideal of R. By Lemma 4.1.7, $\{\overline{f}_\alpha\}_{\alpha \in I}$ is the k-basis of $\mathcal{O}_{X_{n+1},x} \otimes_{R_{n+1}} k \simeq \mathcal{O}_{X_{n+1},x}/\mathfrak{m}\mathcal{O}_{X_{n+1},x}$.

By assumption, $\mathcal{F}_{n+1,x} \otimes_{\mathcal{O}_{X_{n+1},x}} (\mathcal{O}_{X_{n+1},x} \otimes_{R_{n+1}} k)$ is a free $\mathcal{O}_{X_k,x}$ module with basis say, $x_1, \ldots x_r$. Hence, $\{\overline{f}_\alpha x_1, \ldots, \overline{f}_\alpha x_r\}_{\alpha \in I}$ is an k-basis of $\mathcal{F}_{n+1,x} \otimes_{\mathcal{O}_{X_{n+1},x}} (\mathcal{O}_{X_{n+1},x} \otimes_{R_{n+1}} k)$.

Let $\tilde{x}_i$ be a lift of x_i under the natural morphism from $\mathcal{F}_{n+1,x}$ to $\mathcal{F}_{n+1,x} \otimes_{\mathcal{O}_{X_{n+1},x}} (\mathcal{O}_{X_{n+1},x} \otimes_{R_{n+1}} k)$. Using Lemma 4.1.7, we have that $\{f_\alpha \tilde{x}_1, \ldots, f_\alpha \tilde{x}_r\}_{\alpha \in I}$ is a R-basis of $\mathcal{F}_{n+1,x}$. Therefore, $\{\tilde{x}_i\}_{i=1}^r$ is a $\mathcal{O}_{X_{n+1},x}$-basis of $\mathcal{F}_{n+1,x}$. Hence $\mathcal{F}_{n+1}$ is a locally free $\mathcal{O}_{X_{n+1}}$ module as required. □

Notation 4.1.8. Let $\mathcal{F}_n$ be a locally free sheaf on the curve X_n with determinant $\mathcal{L}_n := \mathcal{L}_{\hat{R}} \otimes_{\hat{R}} R_n$.

Now we discuss how the extensions of $\mathcal{F}_n$ relate to those of $\mathcal{L}_n$.

Definition 4.1.9. We have the following definitions:

1. The *trace map* $\text{tr} : M(r, \mathcal{O}_{X_{n+1}}) \to \mathcal{O}_{X_{n+1}}$ is given by taking the trace of the matrices.

2. We define the map

$$\begin{aligned} \pi^n \text{tr} : \pi^n M(r, \mathcal{O}_{X_{n+1}}) &\to \pi^n \mathcal{O}_{X_{n+1}}. \\ \pi^n A &\mapsto \pi^n \text{tr}(A) \end{aligned}$$

3. We define the map

$$\begin{aligned} 1 + \pi^n \text{tr} : 1 + \pi^n M(r, \mathcal{O}_{X_{n+1}}) &\to 1 + \pi^n \mathcal{O}_{X_{n+1}}. \\ 1 + \pi^n A &\mapsto 1 + \pi^n \text{tr}(A) \end{aligned}$$

Then we have the natural short exact sequence:

$$1 \to 1+\ker(\pi^n \mathrm{tr}) \to 1+\pi^n M(r,\mathcal{O}_{X_{n+1}}) \xrightarrow{1+\pi^n \mathrm{tr}} 1+\pi^n \mathcal{O}_{X_{n+1}} \to 1. \tag{4.1}$$

Lemma 4.1.10. Consider the natural surjective morphism

$$\mathrm{SL}(r,\mathcal{O}_{X_{n+1}}) \xrightarrow{\alpha} \mathrm{SL}(r,\mathcal{O}_{X_n}).$$

Then $1+\ker(\pi^n \mathrm{tr}) = \ker(\alpha)$.

Proof. Note that the morphism α is a group homomorphism because it is induced by the ring homomorphism $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$. Let $N := (a_{ij})$ be a matrix in $\mathrm{SL}(r,\mathcal{O}_{X_{n+1}})$ with image the identity matrix in $\mathrm{SL}(r,\mathrm{O}_{X_n})$. Then $a_{ij} = \pi^n b_{ij}$ for $i \neq j$ with $b_{ij} \in \mathcal{O}_{X_{n+1}}$ and $a_{ii} = 1+\pi^n b_{ii}$. Since $\pi^{n+1} = 0$ in $\mathcal{O}_{X_{n+1}}$ and $(a_{ij}) \in \mathrm{SL}(r,\mathcal{O}_{X_{n+1}})$,

$$1 = \det(a_{ij}) = 1+\pi^n \sum(b_{ii}).$$

Hence $\pi^n \sum(b_{ii})$ must be 0.

Since $\mathrm{tr}(N-\mathrm{Id}) = \pi^n \sum_i b_{ii}$, this implies $N \in 1+\ker(\pi^n \mathrm{tr})$. Hence $\ker(\alpha) \subseteq 1+\ker(\pi^n \mathrm{tr})$. The reverse inclusion $1+\ker(\pi^n \mathrm{tr}) \subseteq \ker(\alpha)$ is direct. □

This gives us the following short exact sequence

$$1 \to 1+\ker(\pi^n \mathrm{tr}) \to \mathrm{SL}(r,\mathcal{O}_{X_{n+1}}) \to \mathrm{SL}(r,\mathcal{O}_{X_n}) \to 1. \tag{4.2}$$

Definition 4.1.11. Recall the determinant map $\det : \mathrm{GL}(r,\mathcal{O}_{X_{n+1}}) \to \mathcal{O}^\times_{X_{n+1}}$ given by taking the determinant of the matrices. Since $\mathrm{SL}(r,\mathcal{O}_{X_{n+1}})$ are the matrices with determinant 1, we have the following exact sequence

$$1 \to \mathrm{SL}(r,\mathcal{O}_{X_{n+1}}) \to \mathrm{GL}(r,\mathcal{O}_{X_{n+1}}) \xrightarrow{\det} \mathcal{O}^\times_{X_{n+1}} \to 1. \tag{4.3}$$

Using sequences (4.1), (4.2), (4.3) we obtain the following diagram

$$\begin{array}{ccccccccc}
1 & \hookrightarrow & 1+\ker(\pi^n\mathrm{tr}) & \longrightarrow & \mathrm{SL}(r,\mathcal{O}_{X_{n+1}}) & \twoheadrightarrow & \mathrm{SL}(r,\mathcal{O}_{X_n}) & \longrightarrow & 1 \\
 & & \big\downarrow & \circlearrowleft & \big\downarrow & \circlearrowleft & \big\downarrow & & \\
1 & \hookrightarrow & 1+\pi^n M(r,\mathcal{O}_{X_{n+1}}) & \longrightarrow & \mathrm{GL}(r,\mathcal{O}_{X_{n+1}}) & \twoheadrightarrow & \mathrm{GL}(r,\mathcal{O}_{X_n}) & \longrightarrow & 1. \\
 & & {\scriptstyle 1+\pi^n\mathrm{tr}}\big\downarrow & \circlearrowleft & {\scriptstyle \det}\big\downarrow & \circlearrowleft & {\scriptstyle \det}\big\downarrow & & \\
1 & \hookrightarrow & 1+\pi^n\mathcal{O}_{X_{n+1}} & \longrightarrow & \mathcal{O}^\times_{X_{n+1}} & \longrightarrow & \mathcal{O}^\times_{X_n} & \longrightarrow & 1
\end{array}$$

Note that the short exact sequence (4.1) splits i.e. there exists a section ϕ to the trace map,

$$\phi : \mathcal{O}_{X_{n+1}}(U) \to M(r,\mathcal{O}_{X_{n+1}})(U), \quad \lambda \mapsto \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all $U \subseteq X_{n+1}$. Since the short exact sequence (4.1) is split exact,

$$H^1(1+\ker(\pi^n\mathrm{tr})) \to H^1(1+\pi^n M(r,\mathcal{O}_{X_{n+1}}))$$

is injective. It follows directly from definition, $1+\ker(\pi^n\mathrm{tr})$ is a sheaf of abelian groups. Using Grothendieck vanishing, this implies $H^2(1+\ker(\pi^n\mathrm{tr})) = 0$. Therefore, we have the following short exact sequence:

$$1 \to H^1(1+\ker(\pi^n\mathrm{tr})) \to H^1(1+\pi^n M(r,\mathcal{O}_{X_{n+1}})) \to H^1(1+\pi^n\mathcal{O}_{X_{n+1}}) \to 1.$$

Similarly, the short exact sequence (4.3) splits i.e there exists a section ψ to the determinant map given by

$$\psi : \mathcal{O}^\times_{X_{n+1}}(U) \to \mathrm{GL}(r,\mathcal{O}_{X_{n+1}})(U), \quad \lambda \mapsto \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

for all for all $U \subseteq X_{n+1}$.

The following diagram summarizes all this.

$$\begin{array}{ccccc}
H^1(1+\ker(\pi^n \mathrm{tr})) & \longrightarrow & H^1(\mathrm{SL}(r,\mathcal{O}_{X_{n+1}})) & \longrightarrow & H^1(\mathrm{SL}(r,\mathcal{O}_{X_n})) \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\
H^1(1+\pi^n M(r,\mathcal{O}_{X_{n+1}})) & \longrightarrow & H^1(\mathrm{GL}(r,\mathcal{O}_{X_{n+1}})) & \twoheadrightarrow & H^1(\mathrm{GL}(r,\mathcal{O}_{X_n})) \\
\downarrow{\scriptstyle 1+\pi^n \mathrm{tr}} & \circlearrowleft & \downarrow{\scriptstyle \det} & \circlearrowleft & \downarrow{\scriptstyle \det} \\
H^1(1+\pi^n \mathcal{O}_{X_{n+1}}) & \xrightarrow{\psi_1} & H^1(\mathcal{O}^\times_{X_{n+1}}) & \xrightarrow{\psi_2} & H^1(\mathcal{O}^\times_{X_n})
\end{array} \tag{4.4}$$

Here the north-east square is a diagram of pointed sets, all the other groups are abelian.

We can now prove the following:

Theorem 4.1.12. Suppose there exists a locally free sheaf $\mathcal{F}_n$ on X_n with determinant $\mathcal{L}_n := \mathcal{L}_{\hat{R}} \otimes_{\hat{R}} R_n$ which is an extension of $\mathcal{F}_k$. Then there exists an extension $\mathcal{F}_{n+1}$ of the locally free sheaf $\mathcal{F}_n$ such that $\det(\mathcal{F}_{n+1}) \simeq \mathcal{L}_{n+1}$ where $\mathcal{L}_{n+1} := \mathcal{L}_{\hat{R}} \otimes_{\hat{R}} R_{n+1}$.

Proof. Since X_k is a curve, by Grothendieck's vanishing theorem,

$$H^2(\mathcal{H}om_{X_k}(\mathcal{F}_k,\mathcal{F}_k) \otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}) = 0$$

for all $n \geq 0$. Hence by Theorem 4.1.5, there is no obstruction to extending $\mathcal{F}_n$ to a coherent sheaf say, $\mathcal{F}'_{n+1}$ over X_{n+1}. Furthermore, by Proposition 4.1.6, the sheaf $\mathcal{F}'_{n+1}$ is in fact a locally free sheaf on X_{n+1}. Let $\mathcal{L}'_{n+1} := \det(\mathcal{F}'_{n+1})$. If $\mathcal{L}'_{n+1} \simeq \mathcal{L}_{n+1}$, then we are done. Suppose not, we now show how we can modify the extension $\mathcal{F}'_{n+1}$ so that its determinant bundle is in fact isomorphic to $\mathcal{L}_{n+1}$.

By Theorem 4.1.5 the set of extensions of $\mathcal{L}_n$ on X_{n+1} is a torsor under the action of $H^1(\mathcal{O}^\times_{X_{n+1}})$. Hence there exists $\gamma \in H^1(1+\pi^n \mathcal{O}_{X_{n+1}})$ such that $[\mathcal{L}_{n+1}] = \gamma \bullet [\mathcal{L}'_{n+1}]$, where $\bullet$ indicates the torsor action of $H^1(\mathcal{O}^\times_{X_{n+1}})$. Since the morphism $1+\pi^n \mathrm{tr}$ is surjective, there exists a preimage of γ, say $\Gamma \in H^1(1+\pi^n M(r,\mathcal{O}_{X_{n+1}}))$. Denote by $\mathcal{F}_{n+1} := \Gamma \circ \mathcal{F}'_{n+1}$ in $H^1(\mathrm{GL}(r,\mathcal{O}_{X_{n+1}}))$ where $\circ$ indicates the torsor action in $H^1(\mathrm{GL}(r,\mathcal{O}_{X_{n+1}}))$. Since the torsor action is compatible with taking determinant, the commutativity of the lower left square of diagram (4.4) implies $\det(\mathcal{F}_{n+1}) = [\mathcal{L}_{n+1}]$. □

The following theorem tells us that there exists a locally free sheaf on the formal scheme $\mathcal{X}$.

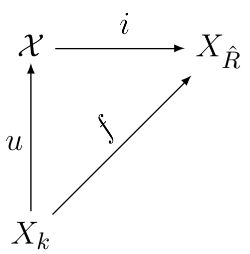
Proposition 4.1.13 ([Har77, Proposition 9.6])**.** Given a collection of locally free sheaves $\mathcal{F}_n$ on X_n over R_n and maps $\mathcal{F}_n \simeq \mathcal{F}_{n+1} \otimes_{R_{n+1}} R_n$, we have $\hat{\mathcal{F}} = \varprojlim \mathcal{F}_n$ is a locally free sheaf on $\mathcal{X}$ with $\det(\hat{\mathcal{F}}) = \varprojlim \mathcal{L}_n$ and for the natural map $u_n : X_n \to \mathcal{X}$, $u_n^* \hat{F} \simeq F_n$.

However, we would like to obtain a locally free sheaf on $X_{\hat{R}}$ and not just on $\mathcal{X}$. For this, we use the following theorem.

Theorem 4.1.14 ([FGI+05, Theorem 8.4.2])**.** Let $X_{\hat{R}}$ be a noetherian fibered surface, separated and of finite type over $\mathrm{Spec}(\hat{R})$, and let $\mathcal{X}$ be as in Proposition 4.1.3. Then the functor $F \to \hat{F}$ from the category of coherent sheaves on X whose support is proper over $\mathrm{Spec}(R)$ to the category of coherent sheaves on $\mathcal{X}$ whose support is proper over $\mathcal{Y}$ is an equivalence, where $\mathcal{Y}$ is as in Definition 4.1.2.

Proposition 4.1.15. There exists a locally free sheaf $\mathcal{F}_{\hat{R}}$ on $X_{\hat{R}}$ such that $\mathcal{F}_k \simeq \mathcal{F}_{\hat{R}} \otimes_{\hat{R}} k$ with determinant $\mathcal{L}_{\hat{R}}$.

Proof. By Proposition 4.1.13, we obtain a locally free sheaf $\hat{\mathcal{F}}$ on the formal scheme $\mathcal{X}$. Then by Theorem 4.1.14, there exists a locally free sheaf $\mathcal{F}_{\hat{R}}$ on $X_{\hat{R}}$ such that for the flat morphism $i : \mathcal{X} \to X_{\hat{R}}$, $\hat{\mathcal{F}}$ is isomorphic to $i^*(\mathcal{F}_{\hat{R}})$. By the commutativity of the following diagram



and Proposition 4.1.13, we obtain $f^*(\mathcal{F}_{\hat{R}}) = (i \circ u)^* \mathcal{F}_{\hat{R}}$ is isomorphic to $u^* \hat{\mathcal{F}} = \mathcal{F}_k$. □

4.2 Artin approximation for the moduli functor

Keep Notations 4.0.1.

In this section we obtain a Gieseker semistable locally free sheaf $\mathcal{F}_R$ of rank r on X_R with the property that determinant $\det(\mathcal{F}_R) \otimes_R K \simeq \mathcal{L}_K$. We do this using the locally free sheaf $\mathcal{F}_{\hat{R}}$ on $X_{\hat{R}}$ obtained in Proposition 4.1.15 and Artin approximation.

We first recall the relevant definitions and results on Artin approximation we require. For a full treatment the reader is referred to [Art69].

Definition 4.2.1. Let A be a noetherian ring. A functor $\mathfrak{F} : A-\text{algebras} \to \text{Sets}$ is said to be *locally of finite presentation* if for every filtered inductive system of A-algebras, $\{B_i\}$, the canonical map

$$\varinjlim \mathfrak{F}(B_i) \to \mathfrak{F}(\varinjlim B_i)$$

is bijective.

We refer to the following result as *Artin approximation* .

Theorem 4.2.2 ([Art69, Theorem 1.12])**.** Let R be a field or an excellent discrete valuation ring and A the henselization of an R-algebra of finite type at a prime ideal. Denote by $\mathfrak{m}$ a proper ideal of A, $\hat{A}$ the $\mathfrak{m}$-adic completion of A and $A_n := A \otimes_R R/\mathfrak{m}^n$. Let $\mathfrak{F}$ be a functor which is locally of finite presentation. Then given any $\bar{\xi} \in \mathfrak{F}(\hat{A})$, there exists $\xi \in \mathfrak{F}(A)$ such that under the natural restriction morphisms:

$$r_n : \mathfrak{F}(\hat{A}) \to \mathfrak{F}(A_n) \text{ and } r'_n : \mathfrak{F}(A) \to \mathfrak{F}(A_n),$$

$r_n(\bar{\xi}) = r'_n(\xi)$ for all $n \in \mathbb{N}$.

Remark 4.2.3. Recall that a discrete valuation ring R is *excellent* if the field extension $\hat{K}/K$ is separable where K denotes the quotient field of R and $\hat{K}$ its completion. This condition is trivially satisfied if K has characteristic 0 or R is complete. Hence Theorem 4.2.2 can be applied to the ring R in Notation 4.0.1.

The following is a consequence of the Quot functor being locally of finite presentation.

Lemma 4.2.4. Let $f : X_R \to \operatorname{Spec}(R)$ be a flat, projective morphism and $\mathcal{H}$ a free sheaf on X_R of the form $\oplus_{i=1}^{N} \mathcal{O}_{X_R}$ for some N. Recall the Quot-functor $\mathcal{Q}uot^P_{X_R/\operatorname{Spec}(R)/\mathcal{H}}$ (see Definition A.2.1) for a fixed Hilbert polynomial P. Given a projective system $\{Z_i\}_{i \in I}$ of affine schemes, the natural morphism

$$\rho : \varinjlim_{i \in I} \mathcal{Q}uot^P_{X_R/\operatorname{Spec}(R)/\mathcal{H}}(Z_i) \to \mathcal{Q}uot^P_{X_R/\operatorname{Spec}(R)/\mathcal{H}}(\varprojlim_{i \in I} Z_i)$$

is bijective.

Proof. By [Ser06, Proposition 4.4.1], the Quot-functor $\mathcal{Q}uot^P_{X_R/\mathrm{Spec}(R)/\mathcal{H}}$ is represented by a projective Spec(R)-scheme. In particular, the natural morphism

$$\phi : \mathrm{Quot}^P_{X_R/\mathrm{Spec}(R)/\mathcal{H}} \to \mathrm{Spec}(R)$$

is of finite type. Since Spec(R) is locally noetherian, the morphism ϕ is locally of finite presentation. Then by [Gro66, Proposition 8.14.2] the Quot-functor $\mathcal{Q}uot^P_{X_R/\mathrm{Spec}(R)/\mathcal{H}}$ is locally of finite presentation. Hence the lemma follows. □

We use the following lemma to prove Proposition 4.2.6.

Lemma 4.2.5. Consider the fiber product:

$$\begin{array}{ccc} X_{\hat{R}} & \xrightarrow{j_1} & X_R \\ \downarrow{\scriptstyle \hat{f}} & \square & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(\hat{R}) & \xrightarrow{j_0} & \mathrm{Spec}(R) \end{array}$$

where j_0 is the natural morphism. Denote by $i_n : X_n \hookrightarrow X_R$ the natural closed immersion.

If $\mathcal{F}_R$ is an invertible sheaf on X_R satisfying $i_n^*\mathcal{F}_R \cong \mathcal{O}_{X_n}$ for all $n \geq 1$, then $\mathcal{F}_R \cong \mathcal{O}_{X_R}$.

Proof. By Theorem 4.2.2 the natural morphism

$$i : \mathrm{Pic}(X_R) \to \varprojlim \mathrm{Pic}(X_n)$$

is injective. Since $i(\mathcal{F}_R) = i(\mathcal{O}_{X_R})$, we have $\mathcal{F}_R \cong \mathcal{O}_{X_R}$. □

Now we apply Artin approximation to our situation to obtain a Gieseker semistable locally free sheaf with determinant $\mathcal{L}_R$ on the model X_R.

Proposition 4.2.6. Let $\mathcal{F}_{\hat{R}}$ be the locally free sheaf on $X_{\hat{R}}$ with determinant $\mathcal{L}_{\hat{R}} := j_1^*\mathcal{L}_R$ obtained using Proposition 4.1.15. Then, there exists a geometrically stable locally free sheaf $\mathcal{F}_R$ on X_R with determinant $\mathcal{L}_R$.

Proof. By Lemma 4.2.4, the Quot functor is locally of finite presentation. Using Artin's approximation theorem [Art69, Theorem 1.12], we conclude that there exists a coherent sheaf $\mathcal{F}_R$ on X_R such that

$$i_n^* \mathcal{F}_R \cong {i'_n}^* \mathcal{F}_{\hat{R}}, \quad \forall \; n \geq 1. \tag{4.5}$$

where $i'_n : X_n \to X_{\hat{R}}$ is the morphism induced by the natural morphism $\hat{R} \to R/m^n$. By Theorem 3.1.9, $\mathcal{F}_k \simeq \mathcal{F}_{\hat{R}} \otimes_{\hat{R}} k$. Note that locally freeness and geometric semistability are open properties by Lemma A.1.8 and Lemma A.1.10 respectively. Therefore $\mathcal{F}_R$ is locally free and geometrically stable.

Let $\mathcal{L} := \det(\mathcal{F}_R)$. Using the fact that determinant commutes with pull-back and the isomorphism (4.5)

$$i_n^* \mathcal{L} \cong \det i_n^* \mathcal{F}_R \cong \det {i'_n}^* \mathcal{F}_{\hat{R}}.$$

By assumption $\det(\mathcal{F}_{\hat{R}}) \cong j_1^* \mathcal{L}_R$. Hence $\det {i'_n}^* \mathcal{F}_{\hat{R}} \cong i'^*_n \circ j_1^* \mathcal{L}_R$. By the universal property of inverse limits $j_1 \circ i'_n = i_n$, hence $i'^*_n \circ j_1^* \mathcal{L}_R \cong i_n^* \mathcal{L}_R$. Therefore for all $n \geq 1$,

$$i_n^* (\mathcal{L} \otimes_{\mathcal{O}_{X_R}} \mathcal{L}_R^{\vee}) \cong \mathcal{O}_{X_n}.$$

Hence by Lemma 4.2.5, $\mathcal{L} \cong \mathcal{L}_R$ i.e., $\det(\mathcal{F}_R) \simeq \mathcal{L}_R$. This proves the proposition. □

4.3 Rational points of the moduli space of stable locally free sheaves with fixed determinant

Keep Notations 4.0.1. In this section we answer Question 2.4.14.

Theorem 4.3.1. Let R be a Henselian discrete valuation ring with maximal ideal $\mathfrak{m}$, fraction field K of characteristic 0 and algebraically closed residue field k of characteristic $p > 0$. Let X_K be a smooth, projective, geometrically connected curve of genus $g \geq 2$ defined over K. Fix integers r, d coprime with $r \geq 2$. Let $\mathcal{L}_K$ be an invertible sheaf of degree d on X_K. Assume that there exists a semistable model $X_R \to \mathrm{Spec}(R)$ of X_K with special fibre $X_k := X_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ a generalised tree-like curve whose singular components do not normalise to a rational curve.

Then there exists a geometrically stable locally free sheaf say $\mathcal{F}_K$ on X_K of rank r and determinant $\mathcal{L}_K$.

Proof. Let $\mathcal{F}_R$ be as in Proposition 4.2.6. Denote by $\mathcal{F}_K := \mathcal{F}_R \otimes_R K$ its pullback to the generic fibre. Since $\mathcal{F}_R$ is locally free, so is $\mathcal{F}_K$ because the pull-back of a locally free sheaf is locally free. Since $\mathcal{L}_R$ was chosen to be a lift of $\mathcal{L}_K$, determinant of $\det(\mathcal{F}_K)$ is $\mathcal{L}_K$.

The locally free sheaf $\mathcal{F}_R|_{X_k} \cong \mathcal{F}_k$ and by Lemma 3.2.9 the sheaf $\mathcal{F}_k$ is Gieseker semistable. As k is algebraically closed, it is Gieseker geometrically semistable.

By Lemma A.1.10, Gieseker geometric stability is an open property. Since any open set in Spec(R) contains the generic point Spec(K), $\mathcal{F}_K$ is also (Gieseker) geometrically semistable. Since X_K is a smooth curve, by Lemma A.1.4 $\mathcal{F}_K$ is also slope semistable. Moreover, the rank and degree of $\mathcal{F}_k$ are coprime. Therefore by Lemma A.1.5, the sheaf $\mathcal{F}_K$ is stable. This proves the theorem. □

Theorem 4.3.2. Let R be a Henselian discrete valuation ring with maximal ideal $\mathfrak{m}$, fraction field K of characteristic 0 and algebraically closed residue field k of characteristic $p > 0$. Let X_K be a smooth, projective, geometrically connected curve of genus $g \geq 2$ defined over K. Fix integers r, d coprime with $r \geq 2$. Let $\mathcal{L}_K$ be an invertible sheaf of degree d on X_K. Denote by $M^s_{X_K,\mathcal{L}_K}(r,d)$ the moduli space of geometrically stable locally free sheaves on X_K of rank r and determinant $\mathcal{L}_K$. Assume that there exists a semistable model $X_R \to \mathrm{Spec}(R)$ of X_K with special fibre $X_k := X_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ a generalised tree-like curve whose singular components do not normalise to a rational curve.

Then $M^s_{X_K,\mathcal{L}_K}(r,d)$ has a K-rational point.

Proof. By Theorem 4.3.1 there exists a geometrically stable locally free sheaf $\mathcal{F}_K$ on X_K of rank r, degree d and determinant $\mathcal{L}_K$. Then there exists a K-rational point of $M^s_{X_K,\mathcal{L}_K}(r,d)$ corresponding to $\mathcal{F}_K$. □

Appendix A

Generalities on moduli spaces and locally free sheaves

We now recall some of the basic definitions and facts used in the text which are specific to the subject of this thesis.

Remark A.0.1. In order to make the main body of the text easy to follow, we will often state definitions and results in the form that we need them. It should be noted however that many of the definitions and results hold more generally than stated here.

A.1 Stability

Keep Remark A.0.1. In this section we briefly recall the different definitions of stability we use in the thesis. The references for this section are [HL97] and [Ses].

Definition A.1.1. Recall the following:

1. Let X be a smooth, projective curve over an algebraically closed field k and $\mathcal{E}$ a coherent sheaf on X. The *slope* of $\mathcal{E}$ is defined as

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

2. Let X be a projective scheme and $\mathcal{E}$ a coherent sheaf on X.

   a The *Hilbert polynomial* denoted $P(\mathcal{E})$ is defined as

$$P(\mathcal{E})(t) := \chi(X, \mathcal{E} \otimes \mathcal{O}_X(t)) = \sum_{i=0}^{d} \alpha_i(\mathcal{E}) \frac{t^i}{i!} \text{ for } t >> 0,$$

   and $\mathcal{O}_X(t) = \mathcal{O}_X(1)^{\otimes t}$.

b The *reduced Hilbert polynomial* is defined as $P_{red}(\mathcal{E}) := \dfrac{P(\mathcal{E})}{\alpha_d(\mathcal{E})}$.

Remark A.1.2. In the thesis, we often use the property of the Euler characteristic being additive, i.e. for a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

we have $\chi(\mathcal{E}) = \chi(\mathcal{E}') + \chi(\mathcal{E}'')$.

Definition A.1.3. We recall the different types of stability.

1. Let X be a smooth, projective curve. A coherent sheaf $\mathcal{E}$ of dimension d is called *slope (semi)stable* if for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F})(\leq) < \mu(\mathcal{E})$.

2. Let X be a reducible, projective curve, say $X = \cup_{1 \leq i \leq n} Y_i$. Then in [Ses], C.S Seshadri generalised slope semistability as follows. Let $\mathcal{E}$ be a semistable torsion free sheaf on X with rank r_i on each component Y_i. Then

$$\mu_{\text{sesh}}(\mathcal{E}) := \frac{\chi(\mathcal{E})}{\Sigma \lambda_i r_i}$$

   where λ_i are rational numbers with $0 < \lambda_i < 1$ and $\Sigma \lambda_i = 1$. The tuple $(\lambda_1, \ldots, \lambda_n)$ is called a *polarisation* . We call a sheaf *Seshadri-(semi)stable* with respect to the polarisation $(\lambda_1, \ldots \lambda_n)$ if for every subsheaf

$$\mu_{\text{sesh}}(\mathcal{F})(\leq) < \mu_{\text{sesh}}(\mathcal{E}).$$

   By definition Seshadri semistability depends on the choice of the polarisation. For ease of notation, we do not always specify this but it should be clear from the context.

3. Let X be a projective curve. A coherent sheaf $\mathcal{E}$ is called *Gieseker semi(stable)* if for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$, $P_{red}(\mathcal{F}) \leq (<) P_{red}(\mathcal{E})$ i.e. the coefficients of $P_{red}(\mathcal{F})$ are smaller (strictly) than the coefficients of $P_{red}(\mathcal{E})$.

Lemma A.1.4. Let X be a smooth, projective curve over a field k and $\mathcal{E}$ a coherent sheaf on X. Then $\mathcal{E}$ is slope semistable if and only if $\mathcal{E}$ is Gieseker semistable.

Proof. Let $\mathcal{E}$ be Gieseker (semi)stable i.e. $p_{red}(\mathcal{F})(\leq) < p_{red}(\mathcal{E})$ for any proper subsheaf $\mathcal{F}$ of $\mathcal{E}$. Then by Riemann Roch we have

$$t + \frac{deg(\mathcal{F})}{r(\mathcal{F})} - \frac{deg(\omega_X)}{2}(\leq) < t + \frac{deg(\mathcal{E})}{r(\mathcal{E})} - \frac{deg(\omega_X)}{2}.$$

where t is a variable. This is the same as slope (semi)stability. □

Now we recall some well-known results on semistability.

Lemma A.1.5. Let X be an integral scheme and $\mathcal{E}$ a coherent sheaf of dimension $d = \dim(X)$ such that $(rk(\mathcal{E}), d(\mathcal{E})) = 1$. If $\mathcal{E}$ is μ-semistable, then $\mathcal{E}$ is μ-stable.

Proof. If $\mathcal{E}$ is not μ-stable, then there exits a subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F})$ and $\deg(\mathcal{F}).\mathrm{rk}(\mathcal{E}) = \deg(\mathcal{E}).\mathrm{rk}(\mathcal{F})$. But this contradicts the assumption $(rk(\mathcal{E}), d(\mathcal{E})) = 1$. □

Lemma A.1.6. Let X be an irreducible, projective curve, $\mathcal{E}$ a semistable locally free sheaf. Then, for any invertible sheaf $\mathcal{L}$ on X, $\mathcal{E} \otimes \mathcal{L}$ is a semistable locally free sheaf on X.

Proof. We prove this by contradiction. Suppose $\mathcal{E} \otimes \mathcal{L}$ is not semistable. Then, there exists a coherent subsheaf $\mathcal{F} \subset \mathcal{E} \otimes \mathcal{L}$ such that

$$\mu(\mathcal{F}) > \mu(\mathcal{E} \otimes \mathcal{L}) = \mu(\mathcal{E}) + \deg(\mathcal{L}) \Rightarrow \mu(\mathcal{F}) - \deg(\mathcal{L}) = \mu(\mathcal{F} \otimes \mathcal{L}^{-1}) > \mu(\mathcal{E}).$$

But this contradicts the semi-stability of $\mathcal{E}$, hence proves the lemma. □

Definition A.1.7. Let P be a property of coherent sheaves on noetherian schemes, P is said to be an *open property* , if for any projective morphism $f : X \to S$ of noetherian schemes and any flat family $\mathcal{F}$ of sheaves on the fibres of f, the set of points $s \in S$ such that $\mathcal{F}_s$ has P is an open subset in S. The family of sheaves $\mathcal{F}$ is said to have property P if for all $s \in S$, the sheaf $\mathcal{F}_s$ has P.

Lemma A.1.8 ([HL97, Proposition 2.1.8])**.** Let $X \to S$ be a morphism of finite type of noetherian schemes. Denote by $\mathcal{F}_s$ the pullback of $\mathcal{F}$ to the fibre $X_s := X \times_S \mathrm{Spec}(k(s))$. Let $\mathcal{F}$ be a flat family of coherent sheaves. Then the set

$$\{s \in S |\ \mathcal{F}_s \text{ is locally free at } s\}$$

is an open subset of S.

Definition A.1.9. Suppose k is not algebraically closed and X a projective scheme over k. A coherent sheaf $\mathcal{E}$ on X is called *geometrically stable* if $\mathcal{E} \otimes_k \operatorname{Spec} \overline{k}$ is stable, where $\overline{k}$ denotes the algebraic closure of k.

The following result is used quite often.

Proposition A.1.10 ([HL97, Proposition 2.3.1])**.** Gieseker semistability and Gieseker geometric stability are open properties in flat families.

A.2 The Quot and Moduli functor

We recall basic definitions and results about the Quot functor that we use in the thesis.

Definition A.2.1. Let $f : X \to S$ be a projective morphism of algebraic schemes and $\mathcal{O}_X(1)$ an f-ample invertible sheaf on X. Let $\mathcal{H}$ be a coherent sheaf on X, flat over S, $\mathcal{O}_{X\times S}$ module and $P \in \mathbf{Q}[z]$ a polynomial. We define the Quot functor denoted

$$\mathcal{Q}(\mathcal{H}, P) := \mathcal{Q}uot^P_{X/S/\mathcal{H}} : (\mathrm{Sch}/S) \to (\mathrm{Sets}),$$

as follows: for $T \in (\mathrm{Sch}/S)$, $\mathcal{Q}(\mathcal{H}, P)(T)$ is the set of T-flat coherent sheaves $\mathcal{H}_T := \mathcal{O}_T \otimes \mathcal{H} \twoheadrightarrow F$ with Hilbert polynomial P. If $g : T' \to T$ is an S-morphism,

$$\mathcal{Q}(\mathcal{H}, P)(g) : \mathcal{Q}(\mathcal{H}, P)(T) \to \mathcal{Q}(\mathcal{H}, P)(T'), \quad \mathcal{H}_T \to \mathcal{F} \mapsto \mathcal{H}_{T'} \to g_X^*\mathcal{F}.$$

Note that when X is a curve over a field k, the Hilbert polynomial P is linear, i.e. the leading coefficient gives us the rank and degree. Then we write $\mathcal{Q}^{r,d}(\mathcal{H})$ instead of $\mathcal{Q}(\mathcal{H}, P)$.

We have the following:

Theorem A.2.2 ([Ser06, Theorem 4.4.1,Proposition 4.4.3])**.** Let $f : X \to S$ be a projective morphism of algebraic schemes. The functor $\mathcal{Q}(\mathcal{H}, P)$ defined in Definition A.2.1 is represented by a projective S-scheme $\pi : \mathrm{Quot}^P_{X/S/\mathcal{H}} \to S$, called the *associated Quot-scheme* .

Moreover, it has the following base change property. For a morphism $T \to S$, the Quot scheme $\mathrm{Quot}^P_{X\times_S T/T/\mathcal{H}_T} = T \times_S \mathrm{Quot}^P_{X/S/\mathcal{H}}$, where $\mathcal{H}_T$ is the pullback of $\mathcal{H}$ under the induced morphism $X \times_S T \to X$.

Notation A.2.3. If X is a family of curves, the Hilbert polynomial of any S-flat coherent sheaf on X is uniquely determined by its rank and degree. In this case, denote by $\pi : \mathrm{Quot}^{r,d}_{X/S/\mathcal{H}} \to S$ the associated Quot-scheme when P is the Hilbert polynomial of a rank r, degree d, S-flat coherent sheaf on X. If S is the spectrum of a field then we simply denote by $\mathrm{Quot}^{r,d}_{X/\mathcal{H}}$ the associated Quot-scheme.

Lemma A.2.4 ([LP97, Lemma 8.6.6])**.** Let r_0, d_0, r, d be integers with $r_0 > 0, r > 0$ and S an open subset of $\mathrm{Quot}^{r,d}_{X_{\overline{K}}/\mathcal{H}}$ where $\mathcal{H} = H \otimes \mathcal{O}_{X_{\overline{K}}}$ for some k-vector space H. Denote by $\mathbb{F}_S$ the universal quotient over S. Let $s \in S$ be a closed point, $\mathcal{H} \twoheadrightarrow \mathbb{F}(s)$ the corresponding coherent quotient and $\mathcal{K}(s)$ the kernel of the morphism. Then, the *Kodaira-Spencer infinitesimal deformation map of* $\mathbb{F}$ at the point s is the morphism

$$\kappa : \mathrm{Hom}_X(\mathcal{K}(s), \mathbb{F}(s)) \to \mathrm{Ext}^1_X(\mathbb{F}(s), \mathbb{F}(s))$$

arising from applying $\mathrm{Hom}_X(-, \mathbb{F}(s))$ to the short exact sequence

$$0 \to \mathcal{K}(s) \to \mathcal{H} \to \mathbb{F}(s) \to 0.$$

Notation A.2.5. Notations as in Lemma A.2.4. Let $x \in \mathrm{Quot}^{r_0,d_0}_{X_S/S/\mathbb{F}}$ be a closed point such that $\pi_0(x) = s$ for $\pi_0 : \mathrm{Quot}^{r_0,d_0}_{X_S/S/\mathcal{F}} \to S$ the natural morphism. Denote by

$$0 \to \mathcal{E}'(s) \to \mathbb{F}(s) \to \mathcal{E}(s) \to 0$$

the corresponding short exact sequence. Applying the functors $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathbb{F}(s))$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}'(s), -)$, respectively, we get the following morphisms

$$i : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{F}(s), \mathbb{F}(s)) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}'(s), \mathbb{F}(s)),$$

$$j : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}'(s), \mathbb{F}(s)) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}'(s), \mathcal{E}(s)).$$

Define the morphism $\omega_x : T_s S \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}(s), \mathcal{E}(s))$ to be the composition $j \circ i \circ \kappa$.

Lemma A.2.6. Notations as in Lemma A.2.4. Suppose the Kodaira-Spencer map κ is surjective. If $\mathcal{E}(s)$ is locally free then the morphism ω_x is surjective.

Proof. By construction, the cokernel of i (resp. j) is isomorphic to $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}(s), \mathbb{F}(s))$ (resp. $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}'(s), \mathcal{E}'(s))$). As $\mathcal{E}'(s)$ is a subsheaf of a locally free sheaf on a non-singular curve, it is locally free, hence $\mathcal{E}xt^i(\mathcal{E}'(s), \mathcal{E}'(s)) = 0$ for $i \geq 1$. Moreover, since $\mathcal{E}(s)$ is locally free, $\mathcal{E}xt^i(\mathcal{E}(s), \mathbb{F}(s)) = 0$ for $i \geq 1$. Therefore, by the Grothendieck spectral sequence, $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}(s), \mathbb{F}(s))$ (resp. $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}'(s), \mathcal{E}'(s))$) are isomorphic to $H^2(\mathcal{H}\mathrm{om}(\mathcal{E}(s), \mathbb{F}(s)))$ (resp. $H^2(\mathcal{H}\mathrm{om}(\mathcal{E}'(s), \mathcal{E}'(s)))$) which vanish by the Grothendieck vanishing theorem. This implies the lemma. □

Recall, the following results in deformation theory.

Proposition A.2.7 ([Ser06, Proposition 4.4.4])**.** We have a short exact sequence of the form

$$0 \to \mathrm{Hom}(\mathcal{E}'(s), \mathcal{E}(s)) \to T_x\mathrm{Quot}^{r_0,d_0}_{X_S/S/\mathbb{F}} \xrightarrow{T_x\pi_0} T_sS \xrightarrow{\omega_x} \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(\mathcal{E}'(s), \mathcal{E}(s)) \quad \text{(A.1)}$$

We will now recall the basic definitions and results about moduli functors.

Definition A.2.8. Let Y be a scheme of finite type over a universally Japanese ring R, $f : X \to Y$ a projective morphism of R-schemes of finite type with geometrically connected fibers and $\mathcal{O}_X(1)$ an f-very ample invertible sheaf. Let T be a Y-scheme and P a fixed Hilbert polynomial. We have the following definitions:-

1. A *family of pure Gieseker semistable sheaves on the fibres of* $X_T := X \times_Y T \to T$ is a T-flat coherent $\mathcal{O}_{X_T}$ module $\mathcal{F}$ such that for every geometric point t of T,the restriction of $\mathcal{F}$ to the fibre X_t is pure and Gieseker semistable.
2. Let $\mathcal{F}$, $\mathcal{F}'$ be two families of pure Gieseker semistable sheaves on the fibres of X_T. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent*, denoted $\mathcal{F} \sim \mathcal{F}'$ if and only if there exist filtrations $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}$ and $0 = \mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \cdots \subset \mathcal{F}'_m = \mathcal{F}'$ by coherent $\mathcal{O}_{X_T}$ modules such that $\oplus_{i=1}^m \mathcal{F}_i/\mathcal{F}_{i-1}$ is a family of pure Gieseker semistable sheaves on the fibres of X_T and there exists an invertible sheaf $\mathcal{L}$ on T such that $\oplus_{i=1}^m \mathcal{F}'_i/\mathcal{F}'_{i-1} \simeq (\oplus_{i=1}^m \mathcal{F}_i/\mathcal{F}_{i-1}) \otimes_{\mathcal{O}_T} \mathcal{L}$.
3. We define the moduli functor $\mathcal{M}_{X/Y}(P) : (\mathrm{Sch}/Y)^\circ \to (\mathrm{Sets})$ from the category of locally noetherian schemes over Y to the category of sets by

$$\mathcal{M}_{X/Y}(P)(T) := \left\{ \begin{array}{l} \sim \text{ equivalence classes of families of pure Gieseker} \\ \text{semistable sheaves on the fibres of } T \times_Y X \to T, \\ \text{which have Hilbert polynomial } P. \end{array} \right\}$$

We denote the open subfunctor for stable sheaves by $\mathcal{M}^s_{X/Y}(P)$.

The following theorem tells us that this functor is uniformly corepresented.

Theorem A.2.9 ([Lan04, Theorem 0.2])**.** Let R be a universally Japanese ring and Y a scheme of finite type over R. Let $f : X \to Y$ be a projective morphism of R-schemes of finite type with geometrically connected fibers and $\mathcal{O}_X(1)$ an f-ample invertible sheaf. For a fixed polynomial P there exists a projective Y-scheme $M_{X/Y}(P)$ of finite type over Y and a natural transformation of functors

$$\theta : \mathcal{M}_{X/Y}(P) \to \mathrm{Hom}_Y(-, M_{X/Y}(P)),$$

which uniformly corepresents the functor $\mathcal{M}_{X/Y}(P)$. For every geometric point $y \in Y$, the induced map $\theta(y)$ is a bijection. Moreover, there is an open scheme $M^s_{X/Y}(P) \subset M_{X/Y}(P)$ which universally corepresents the subfunctor of families of geometrically Gieseker stable sheaves.

A.3 Generalised parabolic bundles

In this section we recall the basic definitions and results on generalised parabolic bundles that we need. We refer the reader to Bhosle [Bho92] for a full treatment of generalised parabolic bundles.

Notation A.3.1. Let Y_k be an irreducible nodal curve defined over an algebraically closed field k of characteristic $p > 0$. Denote by $\pi : \tilde{Y}_k \to Y_k$ the normalisation map and assume that $\tilde{Y}_k$ has genus $g \geq 1$. Denote by J the set of singular points of Y_k and γ the number of singular points. For all $1 \leq i \leq \gamma$, let p_i, q_i be the two points in $\tilde{Y}_k$ lying over the double point $x_i \in Y$. Let $D_i := p_i + q_i$ be an effective divisor on $\tilde{Y}_k$.

Definition A.3.2. A *generalised parabolic structure* on a locally free sheaf $\mathcal{E}$ over an effective divisor D of $\tilde{Y}_k$ consists of

1. a flag Λ of vector subspaces of $H^0(\mathcal{E} \otimes \mathcal{O}_D)$ given by

$$\Lambda : F_0(\mathcal{E}) = H^0(\mathcal{E} \otimes \mathcal{O}_D) \supset F_1(\mathcal{E}) \supset \cdots \supset F_r(\mathcal{E}) = 0$$

2. real numbers $\alpha_1, \ldots \alpha_r$ with $0 \leq \alpha_1 < \cdots < \alpha_r < 1$ called *weights* associated to the flag.

Definition A.3.3. A *generalised parabolic bundle* is a locally free sheaf $\mathcal{E}$ together with parabolic structures over finitely many disjoint divisors, say $\{D_i\}_{i=1,\ldots,\gamma}$. We denote it by a triple $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ where $\underline{\Lambda} := (\Lambda^1, \ldots, \Lambda^\gamma)$ and $\underline{\alpha} := (\underline{\alpha}^1, \ldots, \underline{\alpha}^\gamma)$ with

$$\Lambda^i : F^i_0(\mathcal{E}) = H^0(\mathcal{E} \otimes \mathcal{O}_{D_i}) \supset F^i_1(\mathcal{E}) \supset \cdots \supset F^i_r(\mathcal{E}) = 0$$

being the flag on the divisor D_i and $\underline{\alpha}^i := (\alpha^i_1, \ldots, \alpha^i_r)$ the weights associated to it.

Definition A.3.4. A locally free sub-sheaf $\mathcal{K}$ of $\mathcal{E}$ gets a natural structure of a generalised parabolic bundle. Denote by $F^i_l(\mathcal{K}) = F^i_l(\mathcal{E}) \cap H^0(\mathcal{K} \otimes \mathcal{O}_{D_i})$. The *induced flag* is given by

$$\Lambda^i(\mathcal{K}) : F^i_0(\mathcal{K}) = H^0(\mathcal{K} \otimes \mathcal{O}_{D_i}) \supset F^i_1(\mathcal{K}) \supset \cdots \supset F^i_r(\mathcal{K}) = 0$$

We associate to the vector space $F^i_l(\mathcal{K})$ the weight $\beta^i_l := \alpha^i_l$. We define a *subbundle of a generalised parabolic locally free sheaf* $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ as a locally free subsheaf $\mathcal{K}$ with the induced parabolic structure $(\underline{\Lambda}(\mathcal{K}), \underline{\beta})$ where $\underline{\beta} = (\underline{\beta}^1, \ldots \underline{\beta}^\gamma)$, $\underline{\beta}^i := (\beta^i_1, ..., \beta^i_r)$.

Definition A.3.5. Let $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ be a generalised parabolic locally free sheaf with generalised parabolic structures $(\Lambda^i, \underline{\alpha}^i)$ over the divisors $\{D_i\}_{i=1,\ldots\gamma}$.

1. Denote by $m^i_l := \dim F^i_{l-1}(\mathcal{E}) / \dim F^i_l(\mathcal{E})$ for $l = 1, \ldots r$. We define the *weight* of $\mathcal{E}$ *over a divisor* D_i as $wt_{D_i}(\mathcal{E}) = \sum_{l=1}^{r} m^i_l \alpha^i_l$. The *weight* of $\mathcal{E}$, denoted $\mathrm{wt}(\mathcal{E})$, is defined as $\sum_i wt_{D_i}(\mathcal{E})$.
2. The *parabolic degree* of $\mathcal{E}$ is defined as $\mathrm{pardeg}(\mathcal{E}) = \deg(\mathcal{E}) + \mathrm{wt}(\mathcal{E})$.
3. The *parabolic slope* of $\mathcal{E}$, denoted $\mathrm{par}\mu(\mathcal{E})$, is defined as $\frac{\mathrm{pardeg}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}$.

Definition A.3.6. A generalised parabolic bundle $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ is *parabolic semistable* (respectively *parabolic stable*) if for every proper subbundle $(\mathcal{K})$ of $\mathcal{E}$, one has $\mathrm{par}\mu(\mathcal{K}) \leq \mathrm{par}\mu(\mathcal{E})$ (respectively $< \mathrm{par}\mu(\mathcal{E})$).

Remark A.3.7. Consider the normalisation map, $\pi : \mathcal{O}_{\tilde{Y}_k} \to \mathcal{O}_{Y_k}$ and a double point $x \in Y_k$. For $\{p, q\} = \pi^{-1}(x)$, there is a natural identification $\mathcal{O}_{Y_k,x} \cong k[[S,T]]/(ST), \mathcal{O}_{\tilde{X}_k,p} = k[[S]]$ and $\mathcal{O}_{\tilde{Y}_k,q} = k[[T]]$, under which the morphism

$$i^\#_x : \mathcal{O}_{Y_k,x} \to \mathcal{O}_{\widetilde{Y}_k,p} \oplus \mathcal{O}_{\widetilde{Y}_k,q}$$

can be identified with the natural morphism

$$\begin{aligned} k[[S,T]]/(ST) &\to k[[S]] \oplus k[[T]] \\ c + Sf(S) + Tg(T) &\mapsto (c + Sf(S), c + Tg(T)). \end{aligned}$$

where $c \in k$ and $f(S), g(T) \in k[[S,T]]$. The cokernel of $i_x^{\#}$ is given by the following short exact sequence,

$$0 \to k[[S,T]]/(ST) \xrightarrow{i_x^{\#}} k[[S]] \oplus k[[T]] \xrightarrow{\nu} k \to 0$$

where $\nu(f,g) = c_0(f) - c_0(g)$ for $c_0(f), c_0(g)$ the constant terms of f, g, respectively. In otherwords, the natural restriction morphism $\rho : \mathcal{O}_{\tilde{Y}_k,p} \oplus \mathcal{O}_{\tilde{Y}_k,q} \to k(p) \oplus k(q)$ satisfies the following condition: for $\Delta \subset k(p) \oplus k(q)$ the diagonal, $\rho^{-1}(\Delta) \cong \mathcal{O}_{Y,x}$.

Proposition A.3.8. Let $(\mathcal{E}, \underline{\Lambda}, \underline{\alpha})$ be a generalised parabolic bundle. For $x_i \in J$, denote by

$$\mathcal{E}(p_i) := \mathcal{E}_{p_i} \otimes k(p_i), \mathcal{E}(q_i) := \mathcal{E}_{q_i} \otimes k(q_i),$$

where $k(p_i)$ and $k(q_i)$ are the residue fields at the points p_i and q_i respectively. Fix a set of basis elements $\{e_j\}_{j=1}^r$ and $\{f_j\}_{j=1}^r$ of $\mathcal{E}_{p_i}$ and $\mathcal{E}_{q_i}$, respectively. By abuse of notation, we will again denote by e_j and f_j their image in $\mathcal{E}(p_i)$ and $\mathcal{E}(q_i)$, respectively. Let

$$\mathrm{pr}_1^i : F_1^i(\mathcal{E}) \to \mathcal{E}(p_i) \oplus \mathcal{E}(q_i) \to \mathcal{E}(p_i),\ \mathrm{pr}_2^i : F_1^i(\mathcal{E}) \to \mathcal{E}(p_i) \oplus \mathcal{E}(q_i) \to \mathcal{E}(q_i)$$

Suppose that for all singular points $x_i \in Y_k$, pr_1^i and pr_2^i are isomorphisms. Then, the kernel of the composition

$$\pi_*\mathcal{E} \to \bigoplus_{i=1}^{\gamma} \pi_*\mathcal{E} \otimes k(x_i) \cong \bigoplus_{i=1}^{\gamma} (\mathcal{E}(p_i) \oplus \mathcal{E}(q_i))/F_1^i(\mathcal{E}) \otimes \mathcal{O}_{x_i} \to 0$$

denoted $\phi(\mathcal{E})$ is a locally-free sheaf.

Proof. Denote by U the open set $Y_k \backslash J$. Since $\phi(\mathcal{E})$ is defined using the short exact sequence

$$0 \to \phi(\mathcal{E}) \to \pi_*\mathcal{E} \to \bigoplus_{i=1}^{\gamma} (\mathcal{E}(p_i) \oplus \mathcal{E}(q_i))/F_1^i(\mathcal{E}) \otimes \mathcal{O}_{x_i} \to 0 \tag{A.2}$$

and π is an isomorphism of $\pi^{-1}(U)$, we have $\phi(\mathcal{E})_x \cong \mathcal{E}_x$ for all $x \in U$. Therefore, it just remains to prove for all $x_i \in J$, $\phi(\mathcal{E})_{x_i} \cong \mathcal{O}_{Y_k,x_i}^{\oplus r}$.

Denote by $\sigma_i := \mathrm{pr}_2^i \circ (\mathrm{pr}_1^i)^{-1} : \mathcal{E}(p_i) \to \mathcal{E}(q_i)$. Since $\mathrm{pr}_1^i, \mathrm{pr}_2^i$ are isomorphisms, so is σ_i. Let $\{B_{jh}\}$ be the matrix with $B_{jh} \in k(q_i)$ satisfying $\sigma_i(e_j) = \sum_{h=1}^{r} B_{jh} f_h$. Since σ_i is an isomorphism, (B_{jh}) is invertible. Denote by (A_{jh}) its inverse. This means for each l, $\sum_{j=1}^{r} B_{lj} \left(\sum_{k=1}^{r} A_{jk} f_k \right) = f_l$. Denote by $\mathrm{Id} : \mathcal{E}(p_i) \to \mathcal{E}(q_i)$ the morphism sending e_j to f_j for all $j = 1,...,r$ and $\Gamma_{\sigma_i}, \Gamma_{\text{Id}}$ the graphs of the corresponding morphisms. Then, the linear transformation

$$\psi : \mathcal{E}(p_i) \oplus \mathcal{E}(q_i) \to \mathcal{E}(p_i) \oplus \mathcal{E}(q_i) \text{ defined by } e_l \oplus f_j \mapsto e_l \oplus \sum_{k=1}^{r} A_{jk} f_k$$

is an isomorphism and satisfies $\psi(\Gamma_{\sigma_i}) = \Gamma_{\text{Id}}$.

Denote by $(\widetilde{A_{jh}})$ the lift of the matrix (A_{jh}). Since $\det(A_{jh})$ is a unit in $k(q_i)$, $\det(\widetilde{A_{jh}})$ is also a unit in $\mathcal{O}_{\tilde{Y},q_i}$ (as it is a lift of $\det(A_{jh})$). Hence, the morphism

$$\psi' : \mathcal{E}_{p_i} \oplus \mathcal{E}_{q_i} \to \mathcal{E}_{p_i} \oplus \mathcal{E}_{q_i} \text{ defined by } e_l \oplus f_j \mapsto e_l \oplus \sum_{h=1}^{r} \widetilde{A_{jh}} f_h$$

is an automorphism.

Denote by $\rho : \mathcal{E}_{p_i} \oplus \mathcal{E}_{q_i} \to \mathcal{E}(p_i) \oplus \mathcal{E}(q_i)$ the natural restriction morphism. It follows from definition that $\psi \circ \rho = \rho \circ \psi'$. By Remark A.3.7, $\rho^{-1}(\Gamma_{\text{Id}}) \cong \mathcal{O}^{\oplus r}_{Y_k,x_i}$. Hence,

$$\mathcal{O}^{\oplus r}_{Y_k,x_i} \cong \rho^{-1}(\Gamma_{\text{Id}}) \cong (\psi')^{-1} \circ \rho^{-1}(\Gamma_{\text{Id}}) = \rho^{-1} \circ \psi^{-1}(\Gamma_{\text{Id}}) = \rho^{-1}(\Gamma_{\sigma_i}) = \phi(\mathcal{E})_{x_i},$$

where the last equality follows from the short exact sequence (A.2). Hence, $\phi(\mathcal{E})$ is locally free. □

A.4 Semi-stability results for locally free sheaves on tree-like curves.

In this appendix we review some semistability conditions for locally free sheaves on tree-like curves given in [Big91]. To make the application of these results in the main text easier to follow, we have changed the notation.

Remark A.4.1. Let X_k be a generalised tree-like curve with N components Y_i. By a generalised tree-like curve we mean that after ignoring the singularities of the individual components, the dual graph associated to X_k does not have any loops. We assume that the singularities are at worst nodal. Note that the assumption that the singular components do not normalise to a rational curve is not necessary in this appendix.

It should be noted that in [Big91], the irreducible components are assumed to be smooth of genus $g \geq 1$. However, for the results we use from [Big91], this assumption is not necessary. We check this here by recalling the proofs.

Lemma A.4.2 ([Big91, Lemma 1]). Let $X_k = Y_1 \cup Y_2 \cdots \cup Y_N$ be a tree-like curve. There is an ordering of the components of X_k such that for every $i \leq N-1$, there exists at most one connected component of $\overline{X_k \backslash Y_i}$ which contains curves with indices greater than i. Denote by $B(i)$ this connected component and by $G(i) := \overline{X_k \backslash B(i)}$. Furthermore, $G(i)$ is connected and intersects $B(i)$ at exactly 1 point.

Proof. Observe that the statement of the lemma is equivalent to proving that there exists an ordering on the the irreducible components of X_k such that $Y_i \cup Y_{i+1} \cup ... \cup Y_N$ is connected. Ofcourse if for all i, $Y_{i+1} \cup ... \cup Y_N$ is connected then there exists an unique connected component of $X_k \backslash Y_i$ containing it, hence proves the lemma. Conversely, if the conclusion of the lemma is true but there exists i such that $Y_{i+1} \cup ... \cup Y_N$ is not connected then there exists a subcurve $Z \subset B(i)$ satisfying $Z \cup Y_{i+1} \cup ... \cup Y_N$ is connected and each of the irreducible component of Z has index strictly less than i. Take any such $Y_j \subset Z$, $j < i$. Then, $B(j)$ is connected and contains $Y_{i+1} \cup ... \cup Y_N$ but not the entire Z. Hence, there are two distinct paths connecting Y_{i+1} to Y_N. This implies X_k contains a loop which contradicts the assumption of X_k being tree-like. Therefore, for all i, $Y_{i+1} \cup ... \cup Y_N$ must be connected.

We prove the equivalent condition by decreasing induction. Denote by Y_N an irreducible component of X_k which intersects $\overline{X_k \backslash Y_N}$ at exactly one point. The existence of such a component is guaranteed since the curve is tree-like. Suppose for some n_0, all $i > n_0$ satisfies the equivalent condition. Denote by $Y := Y_{n_0+1} \cup Y_{n_0+2} \cup ... \cup Y_N$. As X_k is connected, there exists a curve in $\overline{X_k \backslash Y}$ which intersects Y. Denote by X_{n_0} any such curve. Note that, $X_{n_0} \cup Y$ is connected. This proves the induction step and hence the lemma. □

Notation A.4.3. Observe that the above Lemma A.4.2 implies that for any $1 \leq i \leq N$, Y_i intersects exactly one irreducible curve, say $Y_{\nu(i)}$ in $B(i)$ and $\nu(i) > i$. Hence, the curve $Y_i \cup Y_{\nu(i)}$ is connected and for any $j < \nu(i), j \neq i$, $Y_i \cup Y_{\nu(i)}$ is contained in $B(j)$. We fix the following notations.

1. Fix from now on an ordering on X_k as mentioned in Lemma A.4.2. For a given irreducible component Y_i of X_k, denote by $\nu(i)$ the index such that the corresponding curve $Y_{\nu(i)}$ has the property that $Y_{\nu(i)} \in B(i)$ and intersects Y_i at exactly one point.

2. Denote by X_k^0 the set of *internal nodes* i.e., all points of intersection of any two irreducible components of X_k. For Y a subcurve of X_k denote by Y^0 the set of *internal nodes* of Y i.e., the intersection points of any two curves in Y. Denote by Y^b the nodes on Y which are on the *boundary of* Y i.e., internal nodes of X_k which lie on an irreducible component of Y but is not an internal node of Y.

3. For Y a connected curve, denote by $|Y|$ the number of irreducible components of Y.

4. Let $\mathcal{E}$ be a locally free sheaf on X_k, $\mathcal{F}$ a subsheaf of $\mathcal{E}$, denote by $\mathcal{F}|_{Y_i}$ the image of the natural morphism $\mathcal{F}\otimes\mathcal{O}_{Y_i}\to\mathcal{E}\otimes\mathcal{O}_{Y_i}$. For any point $P\in X_k$, again denote by $\mathcal{F}|_P$ the image of the natural morphism $\mathcal{F}\otimes\mathcal{O}_P\to\mathcal{E}\otimes\mathcal{O}_P$.

We now observe how such a sheaf restricts to the different components of X_k.

Lemma A.4.4. Let $\mathcal{F}$ be a non-zero coherent sheaf on X_k. Then, there exists a natural short exact sequence:

$$0\to\mathcal{F}\to\bigoplus_{i=1}^{N}(\mathcal{F}\otimes\mathcal{O}_{Y_i})\to\bigoplus_{P\in X_k^0}(\mathcal{F}\otimes\mathcal{O}_P)\to 0.$$

Proof. Recall the natural short exact sequence

$$0\to\mathcal{O}_{X_k}\xrightarrow{\phi}\mathcal{O}_{Y_1}\oplus\ldots\oplus\mathcal{O}_{Y_N}\xrightarrow{\psi}\bigoplus_{P\in X_k^0}\mathcal{O}_P\to 0$$

where ϕ is induced by the restriction map to each irreducible component and ψ is defined in the following way: ψ is non-zero only at the points $P\in X_k^0$. For any $P\in X_k^0$, there exists exactly two curves Y_i,Y_j such that $P=Y_i\cap Y_j$. Define the map ψ at the point P, denoted $\psi_p:\mathcal{O}_{Y_i,P}\oplus\mathcal{O}_{Y_j,P}\to\mathcal{O}_P$ by $(f,g)\mapsto f-g$.

Since $\mathcal{F}$ is a non-zero coherent sheaf, the induced morphism $\psi:\mathcal{F}\to\bigoplus_{i=1}^{N}\mathcal{F}\otimes\mathcal{O}_{Y_i}$ is injective. Indeed, since ψ is an isomorphism away from X_k^0, it suffices to prove the morphism is injective at the points $P\in X_k^0$. Fix one such P. Denote by M the kernel of the morphism $\phi_P:\mathcal{F}_P\to\bigoplus_{i=1}^{N}(\mathcal{F}\otimes\mathcal{O}_{Y_i})_P$. As $M\otimes\mathcal{O}_{Y_i,P}=0$ for all i, $M\otimes\mathcal{O}_P=0$. Denote by m_P the maximal ideal of $\mathcal{O}_{X_k,P}$. Tensoring the following short exact sequence by M,

$$0\to m_P\to\mathcal{O}_{X_k,P}\to\mathcal{O}_P\to 0,$$

we therefore get $M.m_P\cong M$. Since M is a finitely generated $\mathcal{O}_{X_k,P}$-module, the Nakayama lemma implies $M=0$. Hence, ϕ is injective. Finally, by the right-exactness of tensor product we get the short exact sequence,

$$0\to\mathcal{F}\to\bigoplus_{i=1}^{N}(\mathcal{F}\otimes\mathcal{O}_{Y_i})\to\bigoplus_{P\in X_k^0}(\mathcal{F}\otimes\mathcal{O}_P)\to 0.$$

This completes the proof of the lemma. □

Using this we have the following.

Corollary A.4.5. Let $\mathcal{E}$ be a locally free sheaf and $\mathcal{F}$ a coherent subsheaf of $\mathcal{E}$. Then,

$$0 \to \mathcal{F} \to \bigoplus_{i=1}^{N} \mathcal{F}|_{Y_i} \to \bigoplus_{P \in X_k^0} \mathcal{F}|_P \to 0.$$

In particular, $\chi(\mathcal{F}) = \sum_{i=1}^{N} \chi(\mathcal{F}|_{Y_i}) - \sum_{P \in X_k^0} \dim(\mathcal{F}|_P)$.

Proof. Consider the short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

for some coherent sheaf $\mathcal{G}$. Using Lemma A.4.4, we have the following diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \xrightarrow{\phi_{10}} & \bigoplus_{i=1}^{N} \mathcal{F} \otimes \mathcal{O}_{Y_i} & \xrightarrow{\phi_9} & \bigoplus_{P \in X_k^0} \mathcal{F} \otimes \mathcal{O}_P & \longrightarrow & 0 \\
 & & \big\downarrow & \circlearrowleft & \big\downarrow \phi_1 & \circlearrowleft & \big\downarrow \phi_2 & & \\
0 & \longrightarrow & \mathcal{E} & \xrightarrow{\phi_4} & \bigoplus_{i=1}^{N} \mathcal{E} \otimes \mathcal{O}_{Y_i} & \xrightarrow{\phi_3} & \bigoplus_{P \in X_k^0} \mathcal{E} \otimes \mathcal{O}_P & \longrightarrow & 0 \\
 & & \big\downarrow & \circlearrowleft & \big\downarrow & \circlearrowleft & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \bigoplus_{i=1}^{N} \mathcal{G} \otimes \mathcal{O}_{Y_i} & \longrightarrow & \bigoplus_{P \in X_k^0} \mathcal{G} \otimes \mathcal{O}_P & \longrightarrow & 0
\end{array}$$

where the vertical columns are exact and the vertical maps in the last row are surjective because tensor product is right exact. By the Snake lemma, this means $\ker \phi_1 \cong \ker \phi_2$. Hence, $\phi_1(\ker(\phi_2 \circ \phi_9)) = \phi_1(\ker \phi_9)$. Therefore, $\phi_1(\ker(\phi_3 \circ \phi_1)) = \phi_1(\ker(\phi_2 \circ \phi_9)) = \phi_1(\ker \phi_9) = \phi_1(\operatorname{Im} \phi_{10})$. Hence, we have the short exact sequence:

$$0 \to \mathcal{F} \xrightarrow{\phi_1 \circ \phi_{10}} \operatorname{Im} \phi_1 \xrightarrow{\phi_3} \operatorname{Im} \phi_2 \to 0$$

As $\operatorname{Im} \phi_1 = \bigoplus_{i=1}^{N} \mathcal{F}|_{Y_i}$ and $\operatorname{Im} \phi_2 = \bigoplus_{P \in X_k^0} \mathcal{F}|_P$ by definition, we have the short exact sequence in the corollary. Since Euler characteristic is additive, we have later equality. This completes the proof of the corollary. □

For tree-like curves we have the following semistability defined by Teixidor (see [Big91, Inequality 1]).

Definition A.4.6. Let $\mathcal{E}$ be a locally free sheaf of rank r on X_k such that $\mathcal{E} \otimes \mathcal{O}_{Y_i}$ is semi-stable for each $i = 1, ..., N$. Given a polarisation, $\lambda := (\lambda_1, \lambda_2, ..., \lambda_N)$, we say that $\mathcal{E}$ is λ-*semistable* if for each $i \leq N$, the following inequality is satisfied:

$$(\sum_{Y_j \in G(i)} \lambda_j)\chi(\mathcal{E}) + r(|G(i)| - 1) \leq \sum_{Y_j \in G(i)} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_j}) \leq (\sum_{Y_j \in G(i)} \lambda_j)\chi(\mathcal{E}) + r|G(i)|.$$

This condition can be simplified for subcurves as follows.

Lemma A.4.7 ([Big91, Lemma 2])**.** Let $Y = Y_{a_1} \cup Y_{a_2} ... \cup Y_{a_t}$ be a connected subcurve of X_k. Let $P_1, ..., P_\alpha$ be the points of intersection of Y and the closure of $X_k \backslash Y$. Suppose $\mathcal{E}$ is λ-semistable. Then, the following relations are satisfied:

$$(\lambda_{a_1} + ... + \lambda_{a_t})\chi(\mathcal{E}) + r(t-1) \leq \chi(\mathcal{E} \otimes \mathcal{O}_{Y_{a_1}}) + ... + \chi(\mathcal{E} \otimes \mathcal{O}_{Y_{a_t}}) \leq$$
$$\leq (\lambda_{a_1} + ... + \lambda_{a_t})\chi(\mathcal{E}) + r(t - 1 + \alpha).$$

Proof. For each point P_l where $1 \leq l \leq \alpha$, denote by Y_{u_l} (resp. Y_{m_l}) the irreducible component of Y (resp. $\overline{X_k \backslash Y}$) which contains P_l. Note that there are two possibilities. Either $u_l > m_l$ for all l or there exists atleast one l such that $u_l < m_l$. Since $\mathcal{E}$ is λ-semistable, we have for all $1 \leq i \leq t$,

$$(\sum_{Y_j \in G(i)} \lambda_j)\chi(\mathcal{E}) + r(|G(i)| - 1) \leq \sum_{Y_j \in G(i)} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_j}) \leq$$
$$\leq (\sum_{Y_j \in G(i)} \lambda_j)\chi(\mathcal{E}) + r|G(i)|. \quad \star$$

Case 1: $u_l > m_l$ for all l. Then, $G(m_l)$ is the connected component in $\overline{X_k \backslash Y}$ which contains Y_{m_l}. Hence, $\bigcup_{l=1}^{\alpha} G(m_l) = X_k \backslash Y$. Denote by $t_l := |G(m_l)| - 1$. Then,

$$\sum (t_l + 1) = r - |Y| = r - t.$$

As $\mathcal{E}$ is λ-semistable, we have for each l,

$$(\sum_{Y_i \in G(m_l)} \lambda_i)\chi(\mathcal{E}) + rt_l \leq \sum_{Y_i \in G(m_l)} \chi(\mathcal{E} \otimes \mathcal{O}_{Y_i}) \leq (\sum_{Y_i \in G(m_l)} \lambda_i)\chi(\mathcal{E}) + r(t_l + 1).$$

Adding these inequalities, using $\sum_{i=1}^{N}\lambda_i = 1$ and $\chi(\mathcal{E}) = \sum_{i=1}^{N}\chi(\mathcal{E}\otimes\mathcal{O}_{Y_i})$, we get

$$(\lambda_{a_1}+...+\lambda_{a_t})\chi(\mathcal{E})+r(t-1) < \chi(\mathcal{E}\otimes\mathcal{O}_{Y_{a_1)}})+...+\chi(\mathcal{E}\otimes\mathcal{O}_{Y_{a_t}}) <$$

$$< (\lambda_{a_1}+...+\lambda_{a_t})\chi(\mathcal{E})+r(t-1+\alpha).$$

This proves the lemma in this case.

Case 2 Assume $m_l > u_l$ for atleast one l and without loss of generality assume $l = 1$. By definition of $\nu(u_l)$ it follows that $m_1 = \nu(u_1)$. Then, $G(u_1)$ is the union of Y_{u_1} and the connected components of $\overline{X_k\backslash Y_{u_1}}$ which does not contain $Y_{\nu(u_1)}$. Since Y is connected and X_k is a tree-like curve, Y intersects $B(u_1)$ uniquely at P_1. Therefore, Y is contained in $G(u_1)$. Furthermore, Y_{m_l} intersects Y for all $l \geq 2$. So $G(m_l)$ is contained in $G(u_1)$ for all $l \geq 2$. Hence $Y\cup Y_{m_2}\cup Y_{m_3}\cup...\cup Y_{m_t}$ are contained in $G(u_1)$. Therefore, $m_l < u_1$ for all $l \geq 2$, which means $G(m_l)$ is the connected component in $\overline{X_k\backslash Y}$ containing Y_{m_l}. Hence the inequalities (*) corresponding to the component Y_{u_1} can be written as

$$(\sum_{Y_i\in Y}\lambda_i)\chi(\mathcal{E})+\sum_{l=2}^{\alpha}(\sum_{Y_i\in G(m_l)}\lambda_i)\chi(\mathcal{E})+r(|Y|-1)+r\sum_{l=2}^{\alpha}(t_l+1) <$$

$$< (\sum_{Y_i\in Y}\chi(\mathcal{E}\otimes\mathcal{O}_{Y_i}))+\sum_{l=2}^{t}\sum_{Y_i\in G(m_l)}\chi(\mathcal{E}\otimes\mathcal{O}_{Y_i}) < (\sum_{Y_i\in Y}\lambda_i)\chi(\mathcal{E})+$$

$$+\sum_{l=2}^{\alpha}(\sum_{Y_i\in G(m_l)}\lambda_i)\chi(\mathcal{E})+r|Y|+r\sum_{l=2}^{\alpha}(t_l+1)$$

where $t_l = |G(m_l)|-1$ and

$$(\sum_{Y_i\in G(m_l)}\lambda_i)\chi(\mathcal{E})+rt_l \leq \sum_{Y_i\in G(m_l)}\chi(\mathcal{E}\otimes\mathcal{O}_{Y_i}) \leq (\sum_{Y_i\in G(m_l)}\lambda_i)\chi(\mathcal{E})+r(t_l+1)$$

for $l \geq 2$. Together this implies

$$(\lambda_{a_1}+...+\lambda_{a_t})\chi(\mathcal{E})+r(t-1) < \chi(\mathcal{E}\otimes\mathcal{O}_{Y_{a_1}})+...+\chi(\mathcal{E}\otimes\mathcal{O}_{Y_{a_t}}) <$$

$$< (\lambda_{a_1}+...+\lambda_{a_t})\chi(\mathcal{E})+r(t-1+\alpha).$$

This completes the proof of the lemma. □

Definition A.4.8. Let $\mathcal{F}$ be a locally free sheaf on X_k which is λ-semistable and let $\mathcal{F}\subset\mathcal{F}$ be a subsheaf. For Y a connected curve, denote by $|Y|$ the number of irreducible components of Y. We define $D_Y(\mathcal{F})$ as follows: For $|Y| > 1$, define

$$D_Y(\mathcal{F}) := \min_{j \in Y^0} \dim(\mathcal{F}|_j) - \sum_{Y \subsetneq T} D_T(\mathcal{F})$$

where T varies over connected subcurves of X_k containing Y. For $|Y| = 1$,

$$D_Y(\mathcal{F}) = \mathrm{rk}(\mathcal{F}|_Y) - \sum_{Y \subsetneq T} D_Y(\mathcal{F})$$

where T varies over connected subcurves of X_k containing Y.

Lemma A.4.9 ([Big91, Lemma 3])**.** Let Y be a proper connected subcurve of X_k and apply Definition A.4.8. Denote by $D_Y(\mathcal{F})^0 := \min_{j \in Y^0} \dim(\mathcal{F}_j)$ and by $D_Y^b(\mathcal{F}) := \max_{j \in Y^b} \dim(\mathcal{F}_j)$.

1. If $D_Y^b(\mathcal{F}) \geq D_Y^0(\mathcal{F})$ then $D_Y(\mathcal{F}) = 0$.
2. If $D_Y^b(\mathcal{F}) \leq D_Y^0(\mathcal{F})$ then $D_Y(\mathcal{F}) = D_Y^0(\mathcal{F}) - D_Y^b(\mathcal{F})$.

Proof. We use descending induction on the number of irreducible components of the curve Y, i.e. $|Y|$. Base Case: The case $|Y| = N - 1$ follows directly from definition.

Inductive hypothesis: assume that for some $0 < t \leq N - 1$, the results hold true for all connected subcurves Y with $|Y| > t$.

Inductive step: Take a connected subcurve Y such that $|Y| = t$. We will prove the lemma in this case. Let j_0 be a point in Y^b for which $\dim(\mathcal{F}|_{j_0}) = D_Y^b(\mathcal{F})$. Denote by Y' the irreducible component of X_k containing j_0 not contained in Y and $W := Y \cup Y'$.

For the first part assume that $D_Y^b(\mathcal{F}) \geq D_Y^0(\mathcal{F})$. By construction, j_0 is an internal node of the curve W. Hence, $D_W^0(\mathcal{F}) = D_Y^0(\mathcal{F})$ i.e.,

$$D_W(\mathcal{F}) = D_Y^0(\mathcal{F}) - \sum_{W \subsetneq T} D_T(F) \text{ and } D_Y(\mathcal{F}) = D_Y^0(\mathcal{F}) - \sum_{Y \subsetneq T} D_T(\mathcal{F}).$$

Therefore,

$$D_Y(\mathcal{F}) = -\sum_{\substack{Y \subsetneq T \\ W \not\subseteq T}} D_T(\mathcal{F}).$$

For any such T, $D^0_T(\mathcal{F}) \le D^0_Y(\mathcal{F}) \le D^b_Y(\mathcal{F}) \le D^b_T(\mathcal{F})$ where the last inequality follows from the fact that $j_0 \in T^b$. By induction hypothesis, then $D_T(\mathcal{F}) = 0$. This proves the lemma in this case.

Assume next $D^b_Y(\mathcal{F}) \le D^0_Y(\mathcal{F})$. Then, $D^0_W(\mathcal{F}) = D^b_Y(\mathcal{F})$ i.e.,

$$D_W(\mathcal{F}) = D^b_Y(\mathcal{F}) - \sum_{W \subsetneq T} D_T(\mathcal{F}) \text{ and } D_Y(\mathcal{F}) = D^0_Y(\mathcal{F}) - \sum_{Y \subsetneq T} D_T(\mathcal{F}).$$

Therefore,

$$D_Y(\mathcal{F}) = D^0_Y(\mathcal{F}) - D^b_Y(\mathcal{F}) - \sum_{\substack{Y \subsetneq T \\ W \not\subseteq T}} D_T(\mathcal{F}).$$

For any such T, $j_0 \in T^b$. By definition, there exist at least one $j \in Y^b \cap T^0$ such that $\dim(\mathcal{F}|_j) \le \dim(\mathcal{F}|_{j_0}) = D^b_Y(\mathcal{F})$. Hence, $D^0_T(\mathcal{F}) \le D^b_T(\mathcal{F})$. By induction hypothesis, then $D_T(\mathcal{F}) = 0$. This concludes the proof of the lemma. □

Corollary A.4.10 ([Big91, Corollary 4])**.** For any connected subcurve Y of X_k, the corresponding $D_Y(\mathcal{F}) \ge 0$ for any subsheaf $\mathcal{F} \subset \mathcal{E}$.

The following theorem tells us that λ semistability is a sufficient criterion for a locally free sheaf to be μ_{sesh} semistable.

Theorem A.4.11 ([Big91])**.** Let $\mathcal{E}$ be a locally free sheaf on X_k which is λ-semistable. Then $\mathcal{E}$ is Seshadri semistable.

Proof. Let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{E}$. Denote by $s_i := \text{rk}(\mathcal{F} \otimes \mathcal{O}_{Y_i})$. By definition to prove that $\mathcal{E}$ is μ_{sesh} semistable we need to show

$$\frac{\chi(\mathcal{F})}{\lambda_1 s_1 + \ldots + \lambda_N s_N} \le \frac{\chi(\mathcal{E})}{r}.$$

By Corollary A.4.5, we have

$$\frac{\chi(\mathcal{F})}{\lambda_1 s_1 + \ldots + \lambda_N s_N} = \frac{\sum_{i=1}^N \chi(\mathcal{F}|_{Y_i}) - \sum_{P \in X^0_k} \dim \mathcal{F}|_P}{\lambda_1 s_1 + \ldots + \lambda_N s_N} =$$

$$= \frac{\sum_{i=1}^N s_i \chi(\mathcal{F}|_{Y_i})/s_i - \sum_{P \in X^0_k} \dim \mathcal{F}|_P}{\lambda_1 s_1 + \ldots + \lambda_N s_N} =$$

$$= \frac{\sum_{i=1}^{N} \left(\sum_{Y} D_Y(\mathcal{F}) \right) \chi(\mathcal{F}|_{Y_i})/s_i - \sum_{P \in X_k^0} \left(\sum_{Y, P \in Y^0} D_Y(\mathcal{F}) \right)}{\lambda_1 s_1 + \ldots + \lambda_N s_N} =$$

$$= \frac{\sum_{Y} D_Y(\mathcal{F}) \left(\sum_{Y_i \in Y} \chi(\mathcal{F}|_{Y_i})/s_i - \sum_{P \in Y^0} 1 \right)}{\lambda_1 s_1 + \ldots + \lambda_N s_N}$$

where Y ranges over all connected curves in X_k.

Denote by $P_1^{(i)}, \ldots, P_{l_i}^{(i)}$ the set of all points where Y_i intersects other curves in Y and by $Q_1^{(i)}, \ldots, Q_{m_i}^{(i)}$ the set of all points where Y_i intersects other curves in $\overline{X_k \backslash Y}$. One can check that the last identity still holds if we replace every $\chi(\mathcal{F}|_{Y_i})$ by

$$s_Y^i := \chi(\mathcal{F}|_{Y_i}) + \sum_j \left(\sum_{T, P_j^{(i)} \in T^b} D_T(\mathcal{F}) \right) - \sum_j \left(\sum_{T, Q_j^{(i)} \in T^0} D_T(\mathcal{F}) \right),$$

where T always contains the curve Y_i. If $P_j^{(i)}$ is an interior node of Y but a boundary node of T, then $D_T(\mathcal{F})$ appears positive sign in s_Y^i but with a negative sign in s_T^i. By symmetry nothing changes. Hence, there is no overall change in the expression after the above mentioned substitution.

Note that by adding and subtracting the same expression, we can write s_Y^i as

$$s_Y^i = \chi(\mathcal{F}|_{Y_i}) + \sum_j \left(\sum_{T, P_j^{(i)} \in T^b} D_T(\mathcal{F}) \right)$$

$$- \sum_j \left(\sum_{T, Q_j^{(i)} \in T^0} D_T(\mathcal{F}) \right) + \sum_j \left(\sum_{T, P_j^{(i)} \in T^0} D_T(\mathcal{F}) \right) - \sum_j \left(\sum_{T, P_j^{(i)} \in T^0} D_T(\mathcal{F}) \right)$$

By definition, for any point $P_j^{(i)}$ and $Q_j^{(i)}$,

$$\sum_{T, Y_i \subset T} D_T(\mathcal{F}) = s_i, \quad \sum_{T, P_j^{(i)} \in T^0} D_T(\mathcal{F}) = \dim(\mathcal{F}|_{P_j^{(i)}}) \text{ and } \sum_{T, Q_j^{(i)} \in T^0} D_T(\mathcal{F}) = \dim(\mathcal{F}|_{Q_j^{(i)}}).$$

Therefore,

$$s_Y^i = \chi(\mathcal{F}|_{Y_i}) - \sum_{P_j^{(i)}} \dim \mathcal{F}|_{P_j^{(i)}} - \sum_{Q_j^{(i)}} \dim \mathcal{F}|_{Q_j^{(i)}} + \sum_{P_j^{(i)}} s_i =$$

$$= \chi(\mathcal{F}|_{Y_i}(-P_1^{(i)} - \dots - P_{l_i}^{(i)} - Q_1^{(i)} - \dots - Q_{s_i}^{(i)})) + l_i s_i.$$

As $\mathcal{E}|_{Y_i}$ is semistable, so is $\mathcal{E}|_{Y_i}(-P_1^{(i)} - \dots - P_{l_i}^{(i)} - Q_1^{(i)} - \dots - Q_{m_i}^{(i)})$. Using the fact that the rank of $\mathcal{E}|_{Y_i} = r$,

$$s_Y^i \leq \frac{s_i \chi(\mathcal{E}|_{Y_i}(-P_1^{(i)} - \dots - P_{l_i}^{(i)} - Q_1^{(i)} - \dots - Q_{m_i}^{(i)}))}{r} + l_i s_i$$
$$= \frac{s_i}{r}(\chi(\mathcal{E}|_{Y_i}) - r(m_i + l_i)) + l_i s_i.$$

Therefore, the coefficient of $D_Y(\mathcal{F})$ is equal to

$$\sum_{Y_i \in Y} s_Y^i / s_i - \sum_{P \in Y^0} 1 \leq \sum_{Y_i \in Y} (\chi(\mathcal{E}|_{Y_i})/r - m_i) - \sum_{P \in Y^0} 1.$$

Using Lemma A.4.7 this is bounded above by $(\sum_{Y_i \in Y} \lambda_i)\chi(\mathcal{E}))/r$ (the number of internal nodes of Y equals the number of curves in Y minus one, as Y is connected tree-like curve).

Furthermore, the denominator of the expression can be written as follows:

$$\sum_{i=1}^{N} \lambda_i s_i = \sum_{i=1}^{N} \lambda_i \left(\sum_{Y_i \in Y} D_Y(\mathcal{F}) \right) = \sum_{Y} D_Y(\mathcal{F}) \left(\sum_{Y_i \in Y} \lambda_i \right).$$

As $D_Y(\mathcal{F}) \geq 0$ for all Y (see Corollary A.4.10), we have

$$\frac{\chi(\mathcal{F})}{\lambda_1 s_1 + \dots + \lambda_N s_N} \leq \frac{(\chi(\mathcal{E})/r)\left(\sum_{Y} D_Y(\mathcal{F})\left(\sum_{Y_i \in Y} \lambda_i\right)\right)}{\sum_{Y} D_Y(\mathcal{F})\left(\sum_{Y_i \in Y} \lambda_i\right)} = \frac{\chi(\mathcal{E})}{r}.$$

This completes the proof of the theorem. □

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I, Inder Kaur, declare that this thesis titled, "The C_1 conjecture for the moduli space of stable vector bundles with fixed determinant on a smooth projective curve" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
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