Asymptotic Models for Planetary Scale Atmospheric Motions

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Chapter 1

Introduction

Observations indicate the existence of a large number of low-frequency (periods longer than 10 days) atmospheric regimes with planetary spatial scales (of the order of the earth's radius $\sim$ 6300 km) that have an important influence on the variability of the atmosphere. This motivates us to study in this thesis the atmospheric dynamics on the planetary scale making use of multiple scale asymptotic analysis and of numerical simulations. In particular, we aim to identify the relevant physical mechanisms on these large scales and to construct simplified models for their theoretical description. These models have to incorporate in a systematic way the important interactions between the planetary scale flow and the synoptic eddies (periods of 2 to 6 days and spatial scales of 1000 km). Such planetary scale atmospheric models are of particular interest not only because they elucidate general features of the atmospheric dynamics, but also because they are potentially useful in the construction of reduced complexity models for long-term climate simulations.

Planetary and synoptic scales in the atmosphere

A considerable part of the atmospheric variability shows spatial structures on planetary scales (e.g. Hoskins and Pearce, 2001). We attribute to these structures atmospheric phenomena such as the quasi-stationary Rossby waves, teleconnection patterns and the polar/subtropical jet. Fig. 1.1 shows time-averaged 500 hPa geopotential height of the northern hemisphere for the winter season. The pattern of a typical steady wave is visible: two pronounced troughs over the eastern parts of North America and Asia and a third weaker trough over western Asia; the wavenumber 2-3 structure implies wavelengths of the order of 12000 - 8000 km for 50°N. Such stationary waves are the resonant response of free Rossby waves to thermal and orographic forcing from below; they are often referred to as quasi-stationary Rossby waves, since they are persistent over long periods of time. They have nearly an equivalent barotropic vertical structure and play an important role for the momentum, heat and water vapor transport in the atmosphere.

There is a high resemblance between the wavetrains generated in the horizontal propagation of stationary Rossby waves and the teleconnection patterns in the real atmosphere (Hoskins and Karoly, 1981). Such planetary scale patterns can be identified from correlation maps of the 500 hPa geopotential and surface pressure (Wallace and Gutzler, 1981) and they represent the

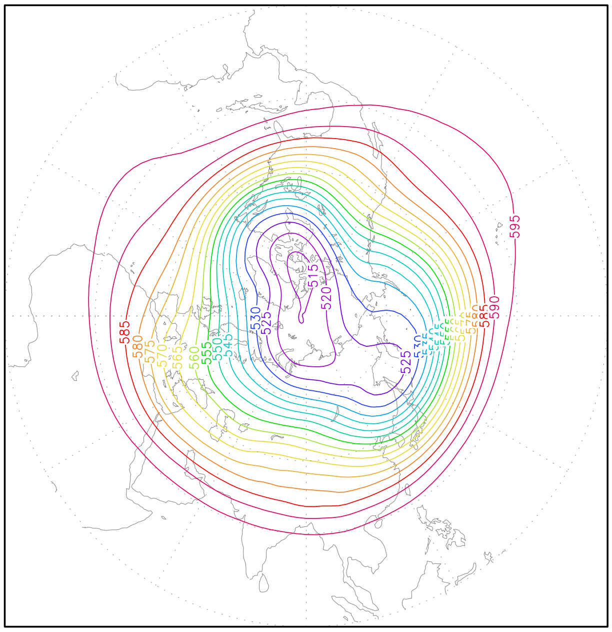

Figure 1.1: Time mean geopotential height of the 500 hPa surface for the northern hemisphere (DJF), units gpdm. Based upon ERA40 reanalysis data (Simmons and Gibson, 2000).

leading modes of the regional atmospheric variability, e.g., the Pacific-North-American pattern (PNA) or the North Atlantic Oscillation (NAO). On the other hand, a zonally symmetric structure characterizes the global modes of atmospheric variability described by the northern and southern annular modes (AM) (Thompson and Wallace, 2000) with a time scale of about 1-2 weeks. Other zonal phenomena with similar or longer time scales are the poleward propagation of zonal mean zonal wind anomalies (Riehl et al., 1959) and the zonal index oscillation (Rossby, 1939) describing the transitions between blocked and enhanced midlatitude westerly flow.

One of the pioneering works on the subject of stationary Rossby waves is from Charney and Eliassen (1949) who reproduced the steady anomalies of the geopotential field at a fixed latitude using the linearized equivalent barotropic vorticity equation forced by orography. The excitation of stationary disturbances by diabatic source terms representing the land-sea thermal contrast was studied by Smagorinsky (1953). Their vertical propagation was investigated in the seminal paper of Charney and Drazin (1961), where it was shown that for easterly and for strong westerly background flow the waves are trapped in the lower atmosphere, whereas for weak westerlies and for sufficiently large wavelengths they can propagate to the middle atmosphere. The stationary planetary waves play an important role in the dynamics of the stratosphere, where they can decelerate the polar night jet and even lead to the breakdown of the polar vortex (Matsuno, 1970, 1971; Holton, 1976). The horizontal propagation of Rossby waves on the sphere was studied by Hoskins et al. (1977) using a linearized barotropic model. In Grose and Hoskins (1979) the steady response to orography was investigated with a linearized spherical shallow water model. They associated stationary trough and ridges with Rossby wavetrains excited by the mountains. Their study was extended by Hoskins and Karoly (1981) by incorporating both thermal and orographic forcing in a linear baroclinic model.

Apart from the stationary Rossby waves, other phenomena relevant to the transports in the ex-

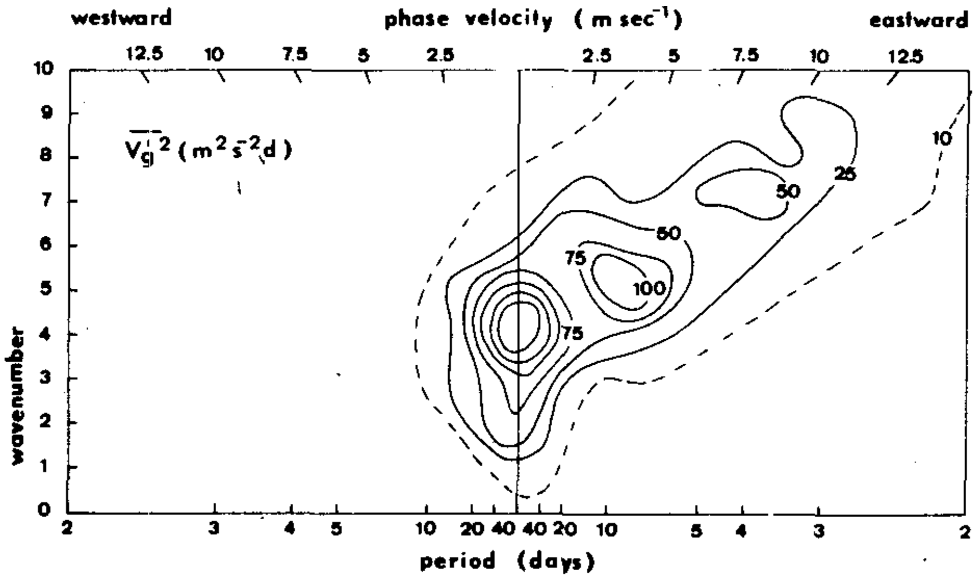

Figure 1.2: Power spectrum density of the meridional geostrophic wind at 500 hPa and $50°$N (Fraedrich and Böttger, 1978).

tratropical atmosphere are the synoptic eddies. They are generated by the baroclinic instability process (Charney, 1947; Eady, 1949) and are described by the quasi-geostrophic (QG) theory (Pedlosky, 1987). The synoptic spatial scales are characterized by the internal Rossby deformation radius (Holton, 1992), denoting the characteristic length which an internal gravity wave will travel during the earth's rotation time. This radius is around 1000 km for the atmosphere, typical time periods associated with the synoptic waves are 2-6 days.

The different planetary and synoptic scales are also evident in spectral analysis of tropospheric data: observations (e.g. Blackmon, 1976; Fraedrich and Böttger, 1978; Fraedrich and Kietzig, 1983) as well as simulations (e.g. Gall, 1976; Hayashi and Golder, 1977) show the presence of isolated peaks in the wavenumber-frequency domain. Fig. 1.2 displays three such peaks in the spectrum of the meridional geostrophic wind. There is a maximum associated with the quasi-stationary Rossby waves with zonal wavenumber k = 1-4 and with periods larger than 20 days. The other two maxima at k = 5-6, periods of 10 days and at k = 7-8, periods of 4 days result from the synoptic waves. These are eastward propagating long and short waves associated with different background stratifications (Fraedrich and Böttger, 1978). The overall picture of three maxima persists during the different seasons for the northern hemisphere. This indicates a separation between the planetary and the synoptic scales. However, the interactions between the two scales are of great relevance to the atmospheric dynamics as stressed in many studies (e.g. Hoskins et al., 1983).

Reduced atmospheric models

One approach in atmospheric modeling, applied in the construction of general circulation models (GCMs), is based on the idea of solving numerically the full hydro and thermodynamic equations using the finest possible resolution and parameterizing all the unresolved phenomena. This is a tremendous task and currently different components of GCMs are being developed in order to achieve more sophisticated representation of the processes in the real atmosphere. However, their enormous complexity and the restrictions of the modern computational facilities do not allow one to utilize GCMs for addressing all problems in the atmospheric dynamics. Thus, the need for simplified atmospheric models do arise.

First, our understanding of the atmosphere is to a great extent based on reduced models, e.g., in connection with the study of regime behavior (Charney and Devore, 1979; Marshall and Molteni, 1993) or ultra-low-frequency variability (James and James, 1989, 1992). Some models are even analytically tractable, e.g., the energy balanced models (North, 1975; Oerlemans and van den Dool, 1978). In climate modeling there is a special need for numerically attractive reduced models, e.g., for paleoclimate studies, where interactions on time scales of the order of millennia and more are involved, or for ensemble simulations. This has led to the development of the Earth System Models of Intermediate Complexity (EMICs; Claussen et al., 2001) and in the context of atmospheric modeling, more precisely, the development of the Statistical Dynamical Models (SDMs; Saltzman, 1978).

The concept of the SDMs is based on the assumption that equations, governing the large-scale, long term “climate” variables, can be derived by averaging the original primitive equations over the smaller (e.g., synoptic) scales. This averaging procedure is similar to the Reynolds averaging applied in the classical boundary layer theory, it naturally gives rise to the appearance in the new equations of unknown correlation terms (e.g., synoptic fluxes). These terms have to be closed, either by deriving evolution equations for them (higher order closures, e.g. Kurihara, 1970; Petoukhov et al., 2003) or parameterizing them (Saltzman and Vernekar, 1971; Yao and Stone, 1987). Here we have to state that the parameterization of the synoptic fluxes remains an important (not only for SDMs) topic of ongoing research. Part of the difficulties encountered are due to the non-negligible contribution from third order moments (Petoukhov et al., 2008), e.g., in the regions of synoptic eddy generation.

SDMs allowing zonal variations have a large range of applicability, e.g., global warming scenarios, paleoclimate and feedback studies (e.g., Ganopolski et al., 2001; Claussen et al., 2001; Petoukhov et al., 2005; Calov and Ganopolski, 2005). The equations for the large scale motion of such models (Petoukhov et al., 1998, 2000) are based on the planetary geostrophic equations (PGEs). The history of the PGEs goes back to the work of Burger (1958) who pointed out that on the planetary scale the vorticity remains quasi-stationary and the vorticity equation reduces to a balance between the horizontal divergence of the wind and the advection of planetary vorticity. The PGEs were proposed by Phillips (1963) as a reduced system of equations for the planetary scale motions (also referred to as geostrophic motions of type two). They consists of the geostrophic and hydrostatic balance, of a $3D$ divergence constraint (where the vertical velocity results from variations of the Coriolis parameter) and of a transport equation for the potential temperature.

The PGEs for a Boussinesq fluid are widely used for modeling the large-scale ocean circulation (Salmon, 1998). Some of the pioneering works on the subject were from Robinson and Stommel (1959) and Welander (1959), where the authors studied the steady version of the equations as a model of the ocean thermocline. In the former paper the authors could reproduce some features of the thermocline, e.g., the upwelling at the equator and the deepening in the west direction. In their study they included simple source terms representing temperature diffusion and surface wind stress. The PGEs can be solved numerically in a closed domain, (e.g., Samelson and Vallis (1997)) when Laplacian and biharmonic diffusion are added to the temperature equation. In Wiin-Nielsen (1961) it was shown that in the presence of a vertically sheared zonal wind the PGEs exhibit baroclinic instability. Further, it was demonstrated that the growth rates of the disturbances increase linearly with the zonal wave number leading to an ill posed mathematical problem; this could be overcome if a diffusive friction is included (e.g., Colin de Verdiere, 1986). In Mundt et al. (1997) one can find the results of numerical simulations with the shallow water formulation of the PGEs, it was shown that the numerical efficiency of the model was at least an order of magnitude larger than other balanced models.

One important feature of the PGEs is that they require a closure for the barotropic component of the flow, because the pressure cannot be determined through the solution of a Poisson equation (the invertibility principle in the QG theory). The barotropic component of the flow can be closed if Rayleigh friction or some prescribed surface wind stress are added to the equations. However, such an approach is applicable to the ocean but not to the atmosphere. Current closures for the atmosphere (Petoukhov et al., 1998, 2000) are diagnostic and based on the temperature (the only prognostic variable in the PGEs); it is assumed that they might be a cause for the limited atmospheric variability observed in some of the intermediate complexity models.

As already mentioned, the PGEs resolve only the planetary scale and the important fluxes due to the synoptic eddies have to be parameterized. On the other hand, the theory for the synoptic waves, the QG theory, is valid only on the synoptic scale and is unable to describe planetary scale flows with order one variations of the Coriolis parameter and horizontal variations of the background stratification. This motivates our interest in a theory that merges in a systematic manner the QG and PG model and captures the interactions between the synoptic and planetary scales.

Asymptotic regimes for the planetary scale and synoptic scales

In this thesis we study reduced models appropriate for the description of the planetary and synoptic dynamics in the atmosphere, in particular we focus on the feedbacks between the two scales. The character of the problem motivates the use of multiple scales asymptotic analysis and we utilize a method referred to as an unified asymptotic approach to meteorological modeling (Klein, 2000, 2004). It has been applied in the development of reduced models, e.g., for the tropical dynamics (Majda and Klein, 2003), deep mesoscale convection (Klein and Majda, 2006), Hadley type circulations (Dolaptchiev, 2006) and concentrated atmospheric vortices (Mikusky, 2007).

We consider three asymptotic regimes, accounting for the different types of phenomena on the planetary scale, and derive reduced model equations for each. The characteristic length and time scales of some of the regimes are presented in Fig. 1.3. The first one is the Planetary Regime (PR), there we consider the scales described by the PGEs: planetary horizontal scales (of the order of the earth's radius) and a corresponding advective time scale of 7 days. For reasons that will become clear in the next chapter, these characteristic length and time scales can be expressed as $\varepsilon^{-3}h_{sc}$ and $\varepsilon^{-3}h_{sc}/u_{ref}$, respectively, where we have used the small parameter $\varepsilon \sim \frac{1}{8} \ldots \frac{1}{6}$, the scale height $h_{sc} \approx 10$ km and a reference velocity $u_{ref} = 10$ ms^{-1} defining the time scale $h_{sc}/u_{ref} \approx 20$ min.

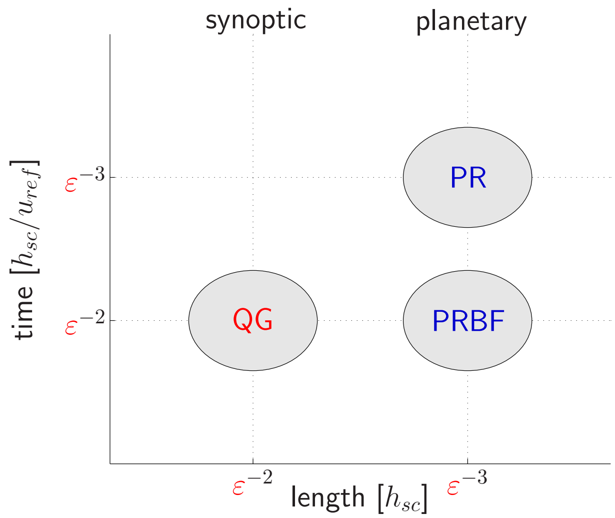

Figure 1.3: Scale map for the PR and PRBF, the validity range of the quasi-geostrophic (QG) theory is also shown, see text for explanations.

Whereas in the PR the horizontal variations of the potential temperature are consistent with the QG theory, an order of magnitude larger fluctuations are assumed in the Planetary Regime with a Background Flow (PRBF). These are motivated by the observed equator to pole temperature gradients and generate a strong zonal background flow. The larger velocities requires a faster time scale of 1 day ($\varepsilon^{-2}h_{sc}/u_{ref}$) for the description of the dynamics on the planetary length scale. It matches the time scale of the QG model for the synoptic motions with a characteristic length scale of about 1000 km ($\varepsilon^{-2}h_{sc}$).

The wavenumber-frequency spectrum in Fig. 1.2 provides no information about the meridional scale of the quasi-stationary Rossby waves at k = 1-4. On the other hand, in many theoretical models it is often assumed that their meridional extent is smaller than the planetary scale in order to guarantee that the advection by the geostrophic wind of the relative vorticity and of the planetary vorticity are of the same order. This has motivated us to consider the Anisotropic Planetary Regime (APR) (not shown in Fig. 1.3). It describes motions with zonal variations on a planetary scale but with meridional variations on the synoptic scale. In all three regimes we present, we resolve in addition the temporal and spatial scales of the QG model, aiming to investigate the planetary-synoptic interactions.

The outline of this thesis is as follows: in Chapter 2 we introduce the asymptotic approach and the rescaled coordinates resolving the planetary and the synoptic scales. In Chapter 3 we consider the PR, first a reduced planetary scale model and after that a two scale planetary-synoptic model is derived. In Chapters 4 and 5 we present the APR and the PRBF, respectively. In Chapter 6 we study the leading balances on the planetary and synoptic scales by performing numerical simulations with a primitive equations model; the results are interpreted with respect to the reduced model equations derived in the previous chapters. A summary of the thesis is presented in Chapter 7.

Chapter 2

Multiple Scales Asymptotic Approach

In Section 2.1 of this chapter we consider the asymptotic representation of the $3D$ compressible flow equation. We utilize the governing equations on the sphere, because we are interested in motions with horizontal scales of the order of the earth's radius. In Section 2.2 we introduce spatial and temporal coordinates resolving the planetary and synoptic scales. A solvability condition imposed in the asymptotics is discussed in Section 2.3 by using the example of the linear damped harmonic oscillator.

In order to derive simplified model equations for the atmospheric dynamics on the planetary and synoptic scales we use a unified asymptotic approach to meteorological modeling. It was introduced by Klein (2000, 2004) and is based on multi-scale perturbation methods. It provides a self-consistent mathematical description of a phenomenon capturing only the essential physics and is a useful theoretical tool for multi-scale interaction studies. Majda and Klein (2003) applied the approach successfully in the systematic development of some reduced equations for the tropical dynamics. These can be regarded as a promising theoretical model for explaining some aspects of the Madden-Julian Oscillation (Majda and Biello, 2004; Biello and Majda, 2005). The same method was used by Dolaptchiev (2006) for the description of large-scale convectively driven Hadley type circulations in the tropics. Klein and Majda (2006) extended the asymptotic approach to include moist processes. New reduced model equations describing deep mesoscale convection together with important interactions between different spatial scales were derived. In the work by Mikusky (2007) the method was applied to study concentrated atmospheric vortices (e.g. hurricanes); in particular, to describe the vortex core structure and the influence of the background shear flow on the vortex trajectory.

For an introduction to the unified asymptotic approach to meteorological modeling as well as some new aspects of it we refer the reader to a recent paper by Klein (2007). The multi-scale perturbation techniques are discussed in depth by Kevorkian and Cole (1981); Holmes (1995).

2.1 Asymptotic representation of the governing equations

We start from the governing equations in spherical coordinates for a compressible fluid on the rotating earth and nondimensionalize them. We use the following reference quantities: the thermodynamic pressure $p_{ref} = 10^5$ kg m^{-1} s^{-2}, the air density $\rho_{ref} = 1.25\,\text{kg/m}^3$, a characteristic flow velocity $u_{ref} = 10\,\mathrm{m/s}$, the scale height $h_{sc} = p_{ref}/g/\rho_{ref} \approx 10\,\mathrm{km}$ ($g \approx 10\,\mathrm{m\,s^{-2}}$ is the gravity acceleration) and a time scale $t_{ref} = h_{sc}/u_{ref} \approx 20\,\mathrm{min}$. After nondimensionalization the governing equations take the form

$$\frac{d}{dt}u - \frac{uv\tan\phi}{r} - \frac{uw}{r} + \frac{1}{Ro}(w\cos\phi - v\sin\phi) + \frac{1}{M^2}\frac{1}{r\rho\cos\phi}\frac{\partial p}{\partial\lambda} = S_u\,, \tag{2.1}$$

$$\frac{d}{dt}v + \frac{u^2\tan\phi}{r} + \frac{vw}{r} + \frac{1}{Ro}u\sin\phi + \frac{1}{M^2}\frac{1}{r\rho}\frac{\partial p}{\partial\phi} = S_v\,, \tag{2.2}$$

$$\frac{d}{dt}w - \frac{u^2}{r} - \frac{v^2}{r} - \frac{1}{Ro}u\cos\phi + \frac{1}{M^2}\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{Fr^2} = S_w\,, \tag{2.3}$$

$$\frac{d}{dt}\theta = S_\theta\,, \tag{2.4}$$

$$\frac{d}{dt}\rho + \frac{\rho}{r\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v cos\phi}{\partial\phi}\right) + \rho\frac{\partial w}{\partial r} + \frac{2w\rho}{r} = 0\,, \tag{2.5}$$

$$\rho\theta = p^{\frac{1}{\gamma}}\,, \tag{2.6}$$

where the coordinates λ, ϕ and r measure longitude, latitude and the distance from the center of the earth, the corresponding spherical unit vectors are $\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$. The non-dimensional variables $p, \rho, \theta, u, v,$ and w denote pressure, density, potential temperature and the velocity components in the direction of $\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$, respectively. $S_{u,v,w}$ and S_θ represent momentum and diabatic source terms and γ is the isentropic exponent. The operator d/dt is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r}\,. \tag{2.7}$$

The Mach, Froude and Rossby number $M, Fr,$ and Ro are defined as

$$M = \frac{u_{ref}}{\sqrt{p_{ref}/\rho_{ref}}}\,, \tag{2.8}$$

$$Fr = \frac{u_{ref}}{\sqrt{gh_{sc}}}\,, \tag{2.9}$$

$$Ro = \frac{u_{ref}}{2\Omega h_{sc}}\,, \tag{2.10}$$

with $\Omega \approx 7\;10^{-5}\,\mathrm{s^{-1}}$ denoting the earth's rotation frequency. We introduce a small parameter

$$\varepsilon = \left(\frac{a\Omega^2}{g}\right)^{\frac{1}{3}}\,, \tag{2.11}$$

where a is the earth's radius $\approx 6\;10^3\,\mathrm{km}$ and $\varepsilon \sim \frac{1}{8}\ldots\frac{1}{6}$, $\varepsilon \ll 1$. Next, the Mach, Froude and Rossby numbers are expressed in terms of ε in a carefully chosen limit. A detailed discussion of this step can be found in Klein (2000, 2004); Majda and Klein (2003), see also Keller and Ting (1951) who suggested to introduce ε as a expansion parameter. The distinguished limit reads

$$\sqrt{M} \sim \sqrt{Fr} \sim 1/Ro \sim \varepsilon \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{2.12}$$

It follows naturally for the radius of the earth: $a = \varepsilon^{-3} a^* h_{sc}$, where a^* is a constant of order unity. Since we are interested in motions in the atmosphere, we can introduce a new non-dimensional coordinate z, measuring the altitude from the ground

$$r = \varepsilon^{-3} a^* + z\,. \tag{2.13}$$

Finally, the governing equations take the form

$$\begin{aligned}
\frac{d}{dt}u - \varepsilon^3\left(\frac{uv\tan\phi}{R} - \frac{uw}{R}\right) + \varepsilon(w\cos\phi - v\sin\phi) &= -\frac{\varepsilon^{-1}}{R\rho\cos\phi}\frac{\partial p}{\partial\lambda} + S_u\,, && (2.14)\\
\frac{d}{dt}v + \varepsilon^3\left(\frac{u^2\tan\phi}{R} + \frac{vw}{R}\right) + \varepsilon u\sin\phi &= -\frac{\varepsilon^{-1}}{R\rho}\frac{\partial p}{\partial\phi} + S_v\,, && (2.15)\\
\frac{d}{dt}w - \varepsilon^3\left(\frac{u^2}{R} + \frac{v^2}{R}\right) - \varepsilon u\cos\phi &= -\frac{\varepsilon^{-4}}{\rho}\frac{\partial p}{\partial z} - \varepsilon^{-4} + S_w\,, && (2.16)\\
\frac{d}{dt}\theta &= S_\theta\,, && (2.17)\\
\frac{d}{dt}\rho + \frac{\varepsilon^3\rho}{R\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v cos\phi}{\partial\phi}\right) + \rho\frac{\partial w}{\partial z} + \frac{\varepsilon^3 2w\rho}{R} &= 0\,, && (2.18)\\
\rho\theta &= p^{\frac{1}{\gamma}}\,, && (2.19)
\end{aligned}$$

where $R = a^* + \varepsilon^3 z$ and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\varepsilon^3 u}{R\cos\phi}\frac{\partial}{\partial\lambda} + \frac{\varepsilon^3 v}{R}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}\,. \tag{2.20}$$

Next, we introduce rescaled coordinates in order to resolve planetary and synoptic motions.

2.2 Scaling of the coordinates

The length δs of a path increment on the surface of the earth can be expressed in terms of variations of longitude $\delta\lambda$ if the latitude is fixed at ϕ_0

$$\delta s = a\cos\phi_0\delta\lambda\,. \tag{2.21}$$

If we keep the longitude constant and vary the latitude, we have

$$\delta s = a\delta\phi \,. \tag{2.22}$$

We are interested in motions with planetary spatial scales, i.e., motions with a reference length of the order of the radius of the earth: $x_{plan} \sim a$. For planetary scale motions horizontal variations δs_{plan} divided by the reference length have to be order one: $\frac{\delta s_{plan}}{x_{plan}} = \mathcal{O}(1)$, as $\varepsilon \to 0$. Taking into account (2.21) and (2.22), we obtain the conditions

$$\cos\phi_0 \delta\lambda_P,\; \delta\phi_P = \mathcal{O}(1) \;, \tag{2.23}$$

where we denote longitudinal and latitudinal variations on the planetary scale with $\delta\lambda_P$ and $\delta\phi_P$, respectively. This constraint is satisfied if we take variations appropriate for the planetary scale motions: $\delta\lambda_P \sim \frac{\pi}{2} \ldots \pi \sim \mathcal{O}(1)$, $\delta\phi_P \sim \frac{\pi}{2} \sim \mathcal{O}(1)$ and if we assume that the motion is not in the vicinity of the poles: $\cos\phi_0 \sim \mathcal{O}(1)$. Thus, the nondimensional coordinates λ and ϕ resolve already motions on a planetary scale, they do not have to be rescaled and we will denote them with λ_P and ϕ_P.

Suppose, we want to resolve synoptic scale motions, then the reference length scale is given by $x_{syn} \sim \varepsilon^{-2} h_{sc} = \varepsilon a$. We denote synoptic scale longitudinal and latitudinal variations with $\delta\lambda_S$ and $\delta\phi_S$, respectively. Substituting for δs_{syn} (2.21) and (2.22) and requiring $\delta s_{syn}/x_{syn} \sim \mathcal{O}(1)$ to hold, we obtain the conditions:

$$\varepsilon^{-1}\cos\phi_0 \delta\lambda_S,\; \varepsilon^{-1}\delta\phi_S \sim \mathcal{O}(1) \;. \tag{2.24}$$

This is satisfied only if we set

$$\delta\lambda_S = \varepsilon\delta\lambda_P \text{ and } \delta\phi_S = \varepsilon\delta\phi_P \,, \tag{2.25}$$

here again $\delta\lambda_P \sim \frac{\pi}{2} \ldots \pi \sim \mathcal{O}(1)$ and $\delta\phi_P \sim \frac{\pi}{2} \sim \mathcal{O}(1)$. In order to resolve synoptic scale motions, we have to introduce new “stretched” coordinates $\lambda_S,\ \phi_S$

$$\lambda_S = \frac{1}{\varepsilon}\lambda_P \text{ and } \phi_S = \frac{1}{\varepsilon}\phi_P \,. \tag{2.26}$$

Next, we consider the time coordinate. An appropriate planetary advective time scale, based on the reference velocity u_{ref}, is given by

$$t_{plan} = \frac{x_{plan}}{u_{ref}} = \frac{\varepsilon^{-3} a^* h_{sc}}{h_{sc}/t_{ref}} = \varepsilon^{-3} a^* t_{ref} = \frac{a}{h_{sc}} t_{ref} \sim 7 \text{ days} \,. \tag{2.27}$$

A suitable time coordinate, resolving motions on the planetary time scale is

$$t_P = \frac{t'}{t_{plan}} = \varepsilon^3 t \,, \tag{2.28}$$

where t' stands for the dimensional time coordinate and t for the time coordinate nondimensionalized by t_{ref}. Similarly, the characteristic synoptic time scale reads

$$t_{syn} = \frac{x_{syn}}{u_{ref}} = \varepsilon^{-2} t_{ref} \sim 1 \text{ days}\,, \tag{2.29}$$

and the synoptic time coordinate is given by

$$t_S = \varepsilon^2 t\,. \tag{2.30}$$

We assume that each dependant variable from (2.14) - (2.19) can be represented in general as an asymptotic series in terms of ε

$$U(\lambda, \phi, z, t; \varepsilon) = \sum_i \varepsilon^i U^{(i)}(\lambda_P, \phi_P, \lambda_S, \phi_S, z, t_P, t_S)\,. \tag{2.31}$$

In some of the considered regimes we will omit the dependence on the planetary/synoptic spatial and temporal scales.

2.3 Sublinear growth condition

In order to guarantee a well defined asymptotic expansion (2.31), we have to require that $U^{(i)}$ grows slower than linearly in any of the coordinates, which is known as the sublinear growth condition. Suppose, X_S denotes one of the synoptic coordinates λ_S, ϕ_S, t_S and X_P the corresponding planetary coordinate λ_P, ϕ_P or t_P. Since we have $X_S = X_P/\varepsilon$, we can formulate the sublinear growth condition for the coordinate X_S as

$$\lim_{\varepsilon \to 0} \frac{U^{(i)}(\ldots, X_S)}{X_S + 1} = \lim_{\varepsilon \to 0} \frac{U^{(i)}(\ldots, \frac{X_P}{\varepsilon})}{\frac{X_P}{\varepsilon} + 1} = 0\,, \tag{2.32}$$

where all coordinates except X_S are held fixed with respect to ε in the limit process. An immediate consequence from the last constraint is the disappearing of averages over X_S of terms, which can be represented as derivatives with respect to X_S. In particular we have

$$\overline{\frac{\partial}{\partial X_S} U^{(i)}}^{X_S} = 0\,. \tag{2.33}$$

Here the averaging operator $\overline{()}^{X_S}$ is defined for the different synoptic coordinates as

$$\overline{U^{(i)}}^{t_S}(\ldots) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{2T} \int\limits_{\frac{t_P}{\varepsilon}-\frac{T}{\varepsilon}}^{\frac{t_P}{\varepsilon}+\frac{T}{\varepsilon}} U^{(i)}(\ldots, t_S)\,\mathrm{d}t_S\,, \tag{2.34}$$

$$\overline{U^{(i)}}^{\lambda_S}(\ldots) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{2\pi} \int\limits_{\frac{\lambda_P}{\varepsilon}}^{\frac{\lambda_P}{\varepsilon}+\frac{2\pi}{\varepsilon}} U^{(i)}(\ldots, \lambda_S)\,\mathrm{d}\lambda_S\,, \tag{2.35}$$

$$\overline{U^{(i)}}^{\phi_S}(\ldots) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\pi} \int\limits_{\frac{\phi_P}{\varepsilon}-\frac{\pi}{2\varepsilon}}^{\frac{\phi_P}{\varepsilon}+\frac{\pi}{2\varepsilon}} U^{(i)}(\ldots, \phi_S)\,\mathrm{d}\phi_S\,, \tag{2.36}$$

where T in (2.34) is a characteristic time averaging scale. Finally, we can define an averaging operator $\overline{()}^S$ over all synoptic scales

$$\overline{U^{(i)}}^S(\lambda_P, \phi_P, z, t_P) = \overline{\overline{\overline{U^{(i)}}^{\phi_S}}^{\lambda_S}}^{t_S}\,. \tag{2.37}$$

We note that in $\overline{()}^S$ the order for averaging over the different synoptic scales is arbitrary.

Example: the linear harmonic oscillator

Typically, when multiple scales are involved in a problem, the leading order asymptotic equations determine only the structure of the (leading order) solution on the “small” scales, whereas its “large” scale distribution appears in the next order equations together with higher order unknown variables. In such cases the application of the sublinear growth condition provides the uniqueness of the solution. It can be viewed as a solvability condition for the next order equation which determines the “large” scale structure of the leading order solution. We discuss here the sublinear growth condition for the classical example of a weakly damped harmonic oscillator (Holmes, 1995). In nondimensional form the problem is described by

$$\frac{\partial^2}{\partial t^2}y + \delta\frac{\partial}{\partial t}y + y = 0\,, \tag{2.38}$$

$$\frac{\partial}{\partial t}y(0) = 1, \quad y(0) = 0\,, \tag{2.39}$$

where t is the time coordinate, $\delta \ll 1$ the friction coefficient, both nondimensionalized using the frequency of the undamped oscillation, and $y(t)$ measures the displacement from some reference state. Intuitively, the problem involves two time scales: the first is fast and is given by the period of the undamped oscillation, the second is the much longer e-folding time of the weak damping. In order to resolve variations on these time scales, we introduce two time coordinates t_1, t_2. Expressed in terms of the original one, t, they read

$$t_1 = t\,, \tag{2.40}$$
$$t_2 = \delta t\,, \tag{2.41}$$

where no scaling is needed for t_1, since t is already scaled with the frequency of the undamped oscillation. We assume that the solution of (2.38), (2.39) can be represented as an asymptotic series in powers of δ

$$y(t) = \sum_i \delta^i y_i(t_1, t_2)\,, \tag{2.42}$$

where the different factors y_i depend on the new time coordinates. We transform the derivative operators in (2.38), (2.39) according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t_1} + \delta\frac{\partial}{\partial t_2}\,, \tag{2.43}$$
$$\frac{\partial^2}{\partial t^2} \to \frac{\partial^2}{\partial t_1^2} + \delta^2\frac{\partial^2}{\partial t_2^2} + 2\delta\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\,. \tag{2.44}$$

With the help of the results above, we substitute (2.42) in (2.38), (2.39) and obtain as a leading order equation system

$$\mathcal{O}(1):\quad \left(\frac{\partial^2}{\partial t_1^2} + 1\right) y_0 = 0\,, \tag{2.45}$$
$$y_0 = 0,\ \frac{\partial}{\partial t_1} y_0 = 1,\ \text{for } t_1, t_2 = 0 \tag{2.46}$$

The formal solutions reads

$$y_0 = a_0(t_2)\sin(t_1) + b_0(t_2)\cos(t_1)\,, \tag{2.47}$$
$$a_0(0) = 1, b_0(0) = 0\,. \tag{2.48}$$

Till now, only the evolution of y_0 on the fast time scale has been found, the constants $a_0(t_2)$ and $b_0(t_2)$ have to be determined from the next order asymptotic equation

$$\mathcal{O}(\delta): \quad \left(\frac{\partial^2}{\partial t_1^2}+1\right)y_1 = -2\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}y_0 - \frac{\partial}{\partial t_1}y_0\,, \tag{2.49}$$

$$y_1 = 0, \frac{\partial}{\partial t_1}y_1 = -\frac{\partial}{\partial t_2}y_0, \text{ for } t_1, t_2 = 0\,. \tag{2.50}$$

Substituting (2.47) in(2.49), we obtain

$$\mathcal{O}(\delta): \quad \left(\frac{\partial^2}{\partial t_1^2}+1\right)y_1 = \left(2\frac{\partial b_0}{\partial t_1}+b_0\right)\sin(t_1) - \left(2\frac{\partial a_0}{\partial t_1}+a_0\right)\cos(t_1)\,. \tag{2.51}$$

Since the linear operator on the left hand side of (2.51) is forced from the right with its eigenfrequency, we expect a resonant behavior of the solution. It can be easily verified that the solution will contain terms proportional to $t_1\sin(t_1)$, $t_1\cos(t_1)$. Such terms will grow unbounded in t_1 leading to a violation of the asymptotic expansion (2.42). They can be suppressed by requiring the terms in the brackets on the right hand side of (2.51) to vanish. This solvability condition and the boundary conditions (2.48) give us

$$a_0 = e^{-t_2/2}\,, \tag{2.52}$$

$$b_0 = 0\,. \tag{2.53}$$

Finally, the leading order solution of (2.38), (2.39) reads

$$y_0(t_1,t_2) = \sin(t_1)e^{-t_2/2}\,. \tag{2.54}$$

Chapter 3

The Planetary Regime

In Section 3.1 of this chapter we systematically derive reduced model equations for the Planetary Regime (PR). It describes atmospheric motions with planetary spatial scales and a temporal scale of the order of about one week, see Fig 1.3. We assume variations of the background potential temperature comparable in magnitude with those adopted in the classical quasi-geostrophic theory. At leading order the resulting equations include the planetary geostrophic balance. In order to apply these equations to the atmosphere, one has to prescribe a closure for the barotropic component of the flow. In Section 3.1.3 such closure is derived in a systematic way from the asymptotic analysis, it represents an evolution equation for the vertically averaged pressure. In Section 3.2 the planetary scale model is extended to a two scale model by incorporating the synoptic scales in it, different interaction mechanisms between the two scales are discussed.

The results from Section 3.1 have been published by Dolaptchiev and Klein (2008).

3.1 Single scale model

A priori assumptions for the background stratification

In this regime we assume that the deviations from a constant reference value of the potential temperature θ are small throughout the troposphere and are of the order ε^2. This was justified in Majda and Klein (2003), where typical values of the dry buoyancy-frequencies of the atmosphere were considered. In this case the expansion for the potential temperature takes the form

$$\theta = 1 + \varepsilon^2 \Theta^{(2)}(\lambda_P, \phi_P, z, t_P) + \varepsilon^3 \Theta^{(3)}(\lambda_P, \phi_P, z, t_P) + \mathcal{O}(\varepsilon^4)\,. \tag{3.1}$$

As pointed out in Klein and Majda (2006); Klein (2007), variations of the potential temperature of the order ε are associated with long term radiative balances. This is confirmed if one considers the large equator-to-pole surface temperature difference: $\sim 40 - 60\,$K (Peixoto and Oort, 1992). The regime associated with $\mathcal{O}(\varepsilon)$ potential temperature variations will be presented in Chapter 5.

Source terms

Before starting with the asymptotic analysis, we consider the source terms in the governing equations. On the planetary scale radiative effects have an important contribution to S_θ, a simple parameterization of these processes is the relaxation ansatz (e.g. Fraedrich et al., 1998)

$$S_\theta = \frac{\Theta_e - \Theta}{\tau} \, . \tag{3.2}$$

Here τ is the radiative relaxation time scale, Θ_e denotes the radiative equilibrium temperature of the atmosphere. A typical value for the radiative relaxation time scale is 20 days $\sim \varepsilon^{-3} t_{ref}$. Taking into account (3.1), we can estimate the magnitude of S_θ to be $\mathcal{O}(\varepsilon^5)$. This is consistent with the values in the literature for the radiative heating/cooling rates of about $1\,\mathrm{K}$ day^{-1}, e.g., Gill (2003); Holton (1992). In nondimensional form they give the same order for S_θ as mentioned here, see Dolaptchiev (2006). Thus, we obtain $S_\theta^{(i)} = 0$ for $i = 0, \ldots, 4$ and the first nontrivial term has the form

$$S_\theta^{(5)} = \frac{\Theta_e - \Theta^{(2)}}{\tau} \, . \tag{3.3}$$

The source terms in the momentum equation represent effects due to friction. We will show later on that the vertical velocities disappear up to $\mathcal{O}(\varepsilon^3)$, consequently we will set $S_w^{(i)} = 0$ up to this order. For the sinks of horizontal momentum we use the same representation as in Marshall and Molteni (1993)

$$S_u = -ku \, , \qquad S_v = -kv \, , \tag{3.4}$$

with a drag coefficient $k(\lambda, \phi, z)$ given through

$$k = \frac{1}{\tau_f} \left(1 + \alpha_1 LS(\lambda, \phi) + \alpha_2 H(\lambda, \phi, z) \right) \, . \tag{3.5}$$

The function LS describes variations of the drag over land and sea, the function H variations due to the topography. The constants α_1, α_2 are user defined weights between 0 and 1. Taking the proposed value in Marshall and Molteni (1993) of 3 days for the relaxation time τ_f, we estimate the magnitude of the momentum source terms S_u, S_v to be somewhere between $\mathcal{O}(\varepsilon^3)$ and $\mathcal{O}(\varepsilon^2)$. Since we expect that frictional effects in the free atmosphere affect the time evolution of the wind but not its geostrophic balance, we consider in the current analysis only $\mathcal{O}(\varepsilon^3)$ momentum sources (compare (3.28) and (3.29),(3.30)). Thus, the first nontrivial friction terms read

$$S_u^{(3)} = -ku^{(0)} \, , \qquad S_v^{(3)} = -kv^{(0)} \, . \tag{3.6}$$

In order to study the influence of friction on the evolution of the wind on the fast synoptic time scale, one should consider $\mathcal{O}(\varepsilon^2)$ dissipation terms. This estimate is appropriate for the asymptotic analysis in Section 3.2, where we resolve the synoptic length and time scales. Next, we proceed with the asymptotic derivation of the reduced equations.

3.1.1 Derivation of the Planetary Regime

Notation

From here on we drop the subscripts of the temporal and spatial variables, keeping in mind that they resolve motions with temporal scales of the order of about 7 days and spatial scales comparable with the radius of the earth

$$t_P, \lambda_P, \phi_P \to t, \lambda, \phi\,. \tag{3.7}$$

The superscript of the order one variable a^* will be dropped as well. The following notation for the operators is used

$$\nabla = \frac{\boldsymbol{e}_\lambda}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{\boldsymbol{e}_\phi}{a}\frac{\partial}{\partial\phi}\,, \tag{3.8}$$

$$\Delta = \frac{1}{a^2\cos^2\phi}\left(\frac{\partial^2}{\partial\lambda^2} + \cos\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right)\right)\,, \tag{3.9}$$

$$\boldsymbol{e}_r \cdot (\nabla \times \boldsymbol{u}) = \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial u\cos\phi}{\partial\phi}\right)\,, \tag{3.10}$$

$$\boldsymbol{u} = \boldsymbol{e}_\lambda u + \boldsymbol{e}_\phi v\,. \tag{3.11}$$

Key steps of the derivation

We substitute the ansatz (2.31) in the governing equations and collect terms of the same order in ε. From the vertical momentum balance follows that the atmosphere is hydrostatically balanced up to $p^{(4)}$

$$\frac{\partial}{\partial z}p^{(i)} = -\rho^{(i)}, \quad i = 0, \dots 4\,. \tag{3.12}$$

From the horizontal momentum balance (2.14) and (2.15) we obtain that $p^{(0)}$ and $p^{(1)}$ do not depend on the horizontal coordinates

$$\nabla p^{(i)} = 0, \quad i = 0, 1\,, \tag{3.13}$$

where for the expansion of the advection operator (2.20) we have used the Taylor series

$$\frac{1}{R} = \frac{1}{a + \varepsilon^3 z} = \frac{1}{a} - \frac{1}{a^2}\varepsilon^3 z + \mathcal{O}\left(\varepsilon^6\right)\,. \tag{3.14}$$

We will drop the time dependence in $p^{(0)}$ and $p^{(1)}$, since it is unphysical that the leading orders of the pressure change in time horizontally uniform on the considered scales (it is possible to derive this assumption starting from the thermodynamic equation rewritten as an evolution equation for the pressure). Expanding the equation of state (2.19) we have

$$\rho^{(0)} = {p^{(0)}}^{\frac{1}{\gamma}} , \tag{3.15}$$

$$\rho^{(1)} = {p^{(0)}}^{\frac{1}{\gamma}} \frac{p^{(1)}}{\gamma p^{(0)}} , \tag{3.16}$$

$$\rho^{(2)} + \rho^{(0)}\Theta^{(2)} = {p^{(0)}}^{\frac{1}{\gamma}} \left(\frac{p^{(2)}}{\gamma p^{(0)}} + \frac{(1-\gamma){p^{(1)}}^2}{2\gamma^2 {p^{(0)}}^2} \right) . \tag{3.17}$$

If the pressure $p^{(i)}$ is hydrostatically balanced, we have the following useful relationship

$$\underbrace{-\frac{\rho^{(i)}}{\rho^{(0)}}}_{\frac{1}{\rho^{(0)}}\frac{\partial p^{(i)}}{\partial z}} + \underbrace{\frac{p^{(i)}}{\gamma p^{(0)}}}_{p^{(i)}\frac{\partial}{\partial z}\frac{1}{\rho^{(0)}}} = \frac{\partial}{\partial z}\pi^{(i)} , \tag{3.18}$$

here we have introduced the variable

$$\pi^{(i)} = p^{(i)}/\rho^{(0)} . \tag{3.19}$$

We combine (3.15) and (3.12) and obtain for the pressure

$$p^{(0)}(z) = p_0 \left(1 - \frac{\gamma - 1}{\gamma} z \right)^{\frac{\gamma}{\gamma-1}} . \tag{3.20}$$

p_0 is an integration constant. In the Newtonian limit, i.e., $\gamma - 1 = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$ (for details see Klein and Majda (2006)), the leading order pressure and density reads: $p^{(0)} = \rho^{(0)} = \exp(-z)$, which are exactly the profiles for an isothermal atmosphere. Transforming (3.16) with the help of (3.18) and integrating over z we have $p^{(1)}(z) = p_1 p^{(0)}$, where p_1 is another constant of integration. Note that in the expansion of the pressure $p^{(0)}$ can now absorb the $p^{(1)}$ term. Consequently the series for the pressure takes the form

$$p(\lambda, \phi, z, t) = (p_0 + \varepsilon p_1) p^{(0)}(z) + \varepsilon^2 p^{(2)}(\lambda, \phi, z, t) + \mathcal{O}\left(\varepsilon^3\right) , \tag{3.21}$$

where without loss of generality the constant factor $p_0 + \varepsilon p_1$ can be set to 1 by an appropriate choice of nondimensionalization. From (3.17) we can represent the hydrostatic balance of $p^{(2)}$ with the help of $\Theta^{(2)}$

$$\frac{\partial}{\partial z}\pi^{(2)} = \Theta^{(2)}\,. \tag{3.22}$$

For the zero order continuity equation we obtain

$$\frac{\partial}{\partial z}\rho^{(0)}w^{(0)} = 0\,. \tag{3.23}$$

Integration gives for $z \to \infty$ and $\rho^{(0)} \to 0 : w^{(0)}(\infty) \to \infty$ which is not physical. So we require $w^{(0)} = 0$. Analogously it can be shown from the next two order equations that $w^{(1)} = w^{(2)} = 0$. The higher order equations are

$$\nabla \cdot \boldsymbol{u}^{(0)} + \frac{1}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)} = 0\,, \tag{3.24}$$

$$\frac{\partial}{\partial z}\rho^{(0)}w^{(4)} + \rho^{(0)}\nabla \cdot \boldsymbol{u}^{(1)} = 0\,, \tag{3.25}$$

$$\frac{\partial}{\partial t}\rho^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla\rho^{(2)} + \frac{\partial}{\partial z}\left(\rho^{(0)}w^{(5)} + \rho^{(2)}w^{(3)}\right) + \rho^{(0)}\nabla \cdot \boldsymbol{u}^{(2)} + \rho^{(2)}\nabla \cdot \boldsymbol{u}^{(0)} = 0\,. \tag{3.26}$$

The first two terms in the expansion of the velocity field are geostrophically balanced

$$\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\pi^{(2)}\,, \tag{3.27}$$

$$\boldsymbol{u}^{(1)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\pi^{(3)}\,. \tag{3.28}$$

The time evolution of $\boldsymbol{u}^{(0)}$ appears in the next order

$$\frac{\partial}{\partial t}u^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla u^{(0)} + w^{(3)}\frac{\partial}{\partial z}u^{(0)} - \frac{u^{(0)}v^{(0)}\tan\phi}{a} - fv^{(2)} = -\frac{1}{a\rho^{(0)}\cos\phi}\left(\frac{\partial}{\partial\lambda}p^{(4)} - \frac{\rho^{(2)}}{\rho^{(0)}}\frac{\partial}{\partial\lambda}p^{(2)}\right) + {S_u}^{(3)}\,, \tag{3.29}$$

$$\frac{\partial}{\partial t}v^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla v^{(0)} + w^{(3)}\frac{\partial}{\partial z}v^{(0)} + \frac{u^{(0)}u^{(0)}\tan\phi}{a} + fu^{(2)} = -\frac{1}{a\rho^{(0)}}\left(\frac{\partial}{\partial\phi}p^{(4)} - \frac{\rho^{(2)}}{\rho^{(0)}}\frac{\partial}{\partial\phi}p^{(2)}\right) + {S_v}^{(3)}\,. \tag{3.30}$$

From the expansion of the potential temperature equation we obtain

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}^{(0)} \cdot \nabla + w^{(3)}\frac{\partial}{\partial z}\right)\Theta^{(2)} = {S_\theta}^{(5)}\,. \tag{3.31}$$

From the equations of the asymptotic expansion we can derive now some practical relations.

The vorticity and the PV equation

We can combine (3.27) and (3.22) in a thermal wind equation

$$\frac{\partial}{\partial z}\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\Theta^{(2)} \,. \tag{3.32}$$

The leading order vorticity balance can be obtained by calculating the divergence of (3.27)

$$f\nabla\cdot\boldsymbol{u}^{(0)} = -\boldsymbol{u}^{(0)}\cdot\nabla f = -\frac{v^{(0)}\cos\phi}{a} \,. \tag{3.33}$$

This equation states that the generation of vorticity through stretching is balanced by the advection of planetary vorticity (Sverdrup balance). Making use of the continuity equation (3.24) we can also write it as

$$\frac{1}{\rho^{(0)}}\frac{\partial\rho^{(0)}w^{(3)}}{\partial r} = \frac{1}{f}\boldsymbol{u}^{(0)}\cdot\nabla f \,. \tag{3.34}$$

Applying $-\frac{1}{a}\frac{\partial}{\partial\phi}$ on (3.29) and $\frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}$ on (3.30), one can derive a vorticity equation

$$\begin{aligned}\frac{\partial}{\partial t}\zeta^{(0)} + \nabla\cdot\boldsymbol{u}^{(0)}\zeta^{(0)} + w^{(3)}\frac{\partial}{\partial z}\zeta^{(0)} + \boldsymbol{e}_r\cdot(\nabla w^{(3)}\times\frac{\partial}{\partial z}\boldsymbol{u}^{(0)}) + \nabla\cdot f\boldsymbol{u}^{(2)} = \\ \boldsymbol{e}_r\cdot\left(\frac{1}{{\rho^{(0)}}^2}\nabla\rho^{(2)}\times\nabla p^{(2)}\right) + \boldsymbol{e}_r\cdot\nabla\times\boldsymbol{S}^{(3)} \,.\end{aligned} \tag{3.35}$$

Here $\boldsymbol{S}^{(3)} = (S_u^{(3)}, {S_v}^{(3)})^T$ and the vorticity $\zeta^{(0)}$ is given through

$$\zeta^{(0)} = \boldsymbol{e}_r\cdot(\nabla\times\boldsymbol{u}^{(0)}) = \frac{1}{f}\Delta\pi^{(2)} + \frac{u^{(0)}\cot\phi}{a} \,. \tag{3.36}$$

The first term on the right hand side of the last equation represents vorticity due to the curvature of the isobars, in contrast to the QG vorticity here f is not constant. The second term represents a shear vorticity – even in the presence of a constant meridional pressure gradient, the geostrophic zonal wind has meridional variations because f varies. In the vorticity equation (3.35) nearly all terms from the general form are present and it is quite complex when compared with its QG counterpart. This is in accordance with the study of Burger (1958), who pointed out that for PG motions it is difficult to gain more precise information than the quasi-stationary character of the vorticity (3.33).

Equations (3.24), (3.31) and (3.32) can be combined in a conservation equation for the potential vorticity (the exact derivation is presented in Appendix A.2)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}^{(0)} \cdot \nabla + w^{(3)} \frac{\partial}{\partial z}\right) \frac{f}{\rho^{(0)}} \frac{\partial \Theta^{(2)}}{\partial z} = S_{pv}{}^{(5)} \,, \tag{3.37}$$

where $S_{pv}{}^{(5)} = \frac{f}{\rho^{(0)}} \frac{\partial S_\theta^{(5)}}{\partial z}$. This completes the derivation of the hierarchy of perturbation equations needed for the construction of a closed, leading-order system of planetary scale equations. The system of equations derived up to this point is not closed because of the (usual) appearance of a higher-order velocity – here $u^{(2)}$ – in the relative vorticity transport equation (3.35). The subsequent derivation in Section 3.1.3 of the evolution equation for the barotropic part of the pressure provides the desired closure as it allows us to eliminate this higher-order velocity in a way similar to that encountered in the classical derivation of QG theory. In the next section the planetary geostrophic equations (PGEs) are summarized and we briefly discuss the closure problem.

3.1.2 The PGEs for the atmosphere

Equations (3.27), (3.22), (3.24) and (3.31) represent the PGEs for the atmosphere (Phillips (1963), for applications to the ocean see Robinson and Stommel (1959); Welander (1959)). Here we recapitulate them

$$\boldsymbol{u}^{(0)} = \frac{1}{f} \boldsymbol{e}_r \times \nabla \pi^{(2)} \,, \tag{3.38}$$

$$\frac{\partial}{\partial z} \pi^{(2)} = \Theta^{(2)} \,, \tag{3.39}$$

$$\nabla \cdot \boldsymbol{u}^{(0)} = -\frac{1}{\rho^{(0)}} \frac{\partial}{\partial z} \rho^{(0)} w^{(3)} \,, \tag{3.40}$$

$$\frac{\partial}{\partial t} \Theta^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla \Theta^{(2)} + w^{(3)} \frac{\partial}{\partial z} \Theta^{(2)} = S_\theta{}^{(5)} \,. \tag{3.41}$$

As shown in the previous section, these equations can be combined in one transport equation for the PV variable $\frac{f}{\rho^{(0)}} \frac{\partial \Theta^{(2)}}{\partial z}$, see (3.37).

The energy of the system is only potential – the PV equation contains only the stretching vorticity term and the relative vorticity is absent due to the fact that the momentum equation is inertialess. Consequently, the pressure cannot be found through the solution of an elliptic equation as in the QG theory. Suppose $\pi^{(2)}$ is known, then one can find the horizontal wind from the geostrophic balance, assuming periodic boundary conditions in λ and ϕ, and the vertical velocity from the divergence constraint, applying vanishing $w^{(3)}$ at the bottom of the atmosphere. Once the velocities are known, the potential temperature can be calculated from the evolution equation for it. By integrating vertically the hydrostatic balance, one can determine the pressure. In doing so one needs a boundary condition for the pressure – it has to be specified at some level, e.g., at the ground. In general, the pressure depends on the motion and prescribing it at some level using a closure or parameterization that is not rooted directly in the governing equations is a considerable limitation of the model.

In the next section we systematically derive a closure condition for the PGEs within the present asymptotic framework. In analogy with the classical derivation of the PV transport equation in the QG theory, we eliminate higher-order unknown terms from the transport equation of relative vorticity (3.35). We obtain a new evolution equation for the vertically averaged second-order pressure $\overline{p^{(2)}}^z$ that may be interpreted again as a planetary barotropic PV transport equation. Knowing the distribution of $\Theta^{(2)}$ and $\overline{p^{(2)}}^z$, the surface pressure $p_0^{(2)}$ (note that $p^{(2)} = \pi^{(2)}$ at $z = 0$) can be easily found from the hydrostatic balance

$$\overline{p^{(2)}}^z = \int_0^1 \left\{ \rho^{(0)}(z') \int_0^{z'} \Theta^{(2)}(\lambda, \phi, z, t) \mathrm{d}z \right\} \mathrm{d}z' + p_0^{(2)} \int_0^1 \rho^{(0)}(z') \mathrm{d}z' \,. \tag{3.42}$$

Again using the hydrostatic balance and $p_0^{(2)}$ as a boundary condition, the pressure at any level can be reconstructed.

3.1.3 The evolution of the barotropic pressure

Assuming a constant Coriolis parameter f, Bresch et al. (2006) proposed a closure for the PGEs in the form of a barotropic vorticity equation. Such assumption helped the authors to study the existence of unique solutions, however, it is not realistic for the planetary scale dynamics (remark 3 in Bresch et al. (2006)). Even a β plane approximation for f is inappropriate, since the PGEs describe motions with order one variations of the Coriolis parameter (Pedlosky, 1987). In our analysis we take into account the full variations of the Coriolis parameter; we make use of the higher order vorticity equation (3.35) from the asymptotic expansion and derive a closure condition for (3.38) - (3.41). This closure represents an evolution equation for the barotropic component of the pressure.

We performed an asymptotic analysis (not shown here) for the dynamics within a layer above the troposphere with vertical variations of the potential temperature similar to the observed in the stratosphere: of the order ε. The analysis revealed vanishing vertical velocities $w^{(3)}$ at the tropopause. Consequently, we assume a rigid lid as a boundary condition at the top for the equations presented here, since they are valid within the troposphere. If we average (3.40) with respect to z and apply $w^{(3)} = 0$ at $z = 0, 1$, we obtain

$$\overline{\nabla \cdot \rho^{(0)} \boldsymbol{u}^{(0)}}^z = 0 \,, \tag{3.43}$$

where we have used the averaging operator $\overline{()}^z$, defined for a general function $f(\lambda, \phi, z, t)$ as

$$\overline{f}^z(\lambda, \phi, t) = \int_0^1 f(\lambda, \phi, z, t) \, \mathrm{d}z \,. \tag{3.44}$$

From (3.43) using (3.33) we can represent the horizontal divergence through the geostrophically balanced meridional component $v^{(0)}$ and we have

$$\frac{\partial}{\partial \lambda} \overline{p^{(2)}}^z = 0 \,. \tag{3.45}$$

Consequently, the pressure can be written in the form

$$p^{(2)} = \overline{p^{(2)}}^{z}(\phi, t) + p^{(2)\prime}(\lambda, \phi, z, t)\,, \qquad \overline{p^{(2)\prime}}^{z} = 0. \tag{3.46}$$

Next, we can multiply the vorticity equation (3.35) by $\rho^{(0)}$ and average it with respect to z and λ (note that $\overline{p^{(2)}}^{z,\lambda} = \overline{p^{(2)}}^{z}$)

$$\begin{aligned} &\frac{\partial}{\partial t}\overline{\rho^{(0)}\zeta^{(0)}}^{z,\lambda} + \overline{\nabla\cdot \boldsymbol{u}^{(0)}\rho^{(0)}\zeta^{(0)} + \rho^{(0)}w^{(3)}\frac{\partial}{\partial z}\zeta^{(0)}}^{z,\lambda} + \overline{\rho^{(0)}\boldsymbol{e}_r\cdot\left(\nabla w^{(3)}\times\frac{\partial}{\partial z}\boldsymbol{u}^{(0)}\right)}^{z,\lambda} \\ &+ \overline{\boldsymbol{e}_r\cdot\frac{1}{\rho^{(0)}}\nabla\frac{\partial}{\partial z}p^{(2)}\times\nabla p^{(2)}}^{z,\lambda} + \overline{\nabla\cdot\rho^{(0)}f\boldsymbol{u}^{(2)}}^{z,\lambda} = \overline{\boldsymbol{e}_r\cdot\nabla\times\rho^{(0)}\boldsymbol{S}^{(3)}}^{z,\lambda}\,. \end{aligned} \tag{3.47}$$

We have to eliminate terms containing $\boldsymbol{u}^{(2)}$ in order to have a closed equation. The fifth term on the left hand side can be written as

$$\begin{aligned} \overline{\nabla\cdot\rho^{(0)}f\boldsymbol{u}^{(2)}}^{z,\lambda} &= f\,\overline{\nabla\cdot\rho^{(0)}\boldsymbol{u}^{(2)}}^{z,\lambda} + \beta\overline{\rho^{(0)}v^{(2)}}^{z,\lambda} \\ &= -f\left(\frac{\partial}{\partial t}\overline{\rho^{(2)}}^{z,\lambda} + \overline{\nabla\cdot\boldsymbol{u}^{(0)}\rho^{(2)}}^{z,\lambda}\right) + \beta\overline{\rho^{(0)}v^{(2)}}^{z,\lambda}\,. \end{aligned} \tag{3.48}$$

Here we have used the continuity equation (3.26) and the notation $\beta = \frac{1}{a}\frac{\partial f}{\partial \phi}$. Using (3.22) and $\rho^{(0)} = \exp(-z)$ we can express the density $\rho^{(2)}$ in terms of pressure and potential temperature

$$\begin{aligned} \overline{\rho^{(2)}}^{z,\lambda} = -\overline{\frac{\partial}{\partial z}p^{(2)}}^{z,\lambda} = -\overline{\frac{\partial}{\partial z}\rho^{(0)}\pi^{(2)}}^{z,\lambda} &= -\overline{\rho^{(0)}\frac{\partial}{\partial z}\pi^{(2)}}^{z,\lambda} + \overline{\rho^{(0)}\pi^{(2)}}^{z,\lambda} \\ &= -\overline{\rho^{(0)}\Theta^{(2)}}^{z,\lambda} + \overline{p^{(2)}}^{z,\lambda}\,. \end{aligned} \tag{3.49}$$

If we average the potential temperature equation (3.41) over z and λ, the temporal evolution of $\rho^{(2)}$ can be written as

$$-\frac{\partial}{\partial t}\overline{\rho^{(2)}}^{z,\lambda} = \frac{\partial}{\partial t}\left(\overline{\rho^{(0)}\Theta^{(2)}}^{z,\lambda} - \overline{p^{(2)}}^{z,\lambda}\right) \tag{3.50}$$

$$= -\overline{\rho^{(0)}\boldsymbol{u}^{(0)}\cdot\nabla\Theta^{(2)} - \rho^{(0)}w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}}^{z,\lambda} + \overline{\rho^{(0)}S_\theta{}^{(5)}}^{z,\lambda} - \frac{\partial}{\partial t}\overline{p^{(2)}}^{z} \tag{3.51}$$

$$= -\overline{\nabla\cdot\rho^{(0)}\boldsymbol{u}^{(0)}\Theta^{(2)}}^{z,\lambda} - \underbrace{\overline{\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}\Theta^{(2)}}^{z,\lambda}}_{=0} + \overline{\rho^{(0)}S_\theta{}^{(5)}}^{z,\lambda} - \frac{\partial}{\partial t}\overline{p^{(2)}}^{z}\,. \tag{3.52}$$

Applying (3.49) and (3.52), (3.48) takes the form

$$\overline{\nabla \cdot \rho^{(0)} f \boldsymbol{u}^{(2)}}^{z,\lambda} = - f \left(\frac{\partial}{\partial t} \overline{p^{(2)}}^{z} + \underbrace{\overline{\nabla \cdot \boldsymbol{u}^{(0)} p^{(2)}}^{z,\lambda}}_{=0} - \overline{\rho^{(0)} S_\theta{}^{(5)}}^{z,\lambda} \right) + \beta \overline{\rho^{(0)} v^{(2)}}^{z,\lambda} . \tag{3.53}$$

The second term on the r.h.s. disappears if periodic boundary conditions in λ are assumed. We can express $\overline{\rho^{(0)} v^{(2)}}^{z,\lambda}$ in terms of known variables, if we use the momentum equation (3.29)

$$\begin{aligned} \overline{\rho^{(0)} v^{(2)}}^{z,\lambda} = \frac{1}{f} \Bigg(& \overline{\frac{\partial \rho^{(0)} u^{(0)}}{\partial t}}^{z,\lambda} - \overline{\rho^{(0)} S_u{}^{(3)}}^{z,\lambda} \\ & + \overline{\rho^{(0)} \left(\boldsymbol{u}^{(0)} \cdot \nabla u^{(0)} + w^{(3)} \frac{\partial}{\partial z} u^{(0)} - \frac{u^{(0)} v^{(0)} \tan\phi}{a} - \frac{\rho^{(2)}}{a \cos\phi {\rho^{(0)}}^2} \frac{\partial p^{(2)}}{\partial \lambda} \right)}^{z,\lambda} \Bigg) . \end{aligned} \tag{3.54}$$

Substituting the last two equations in (3.47), the vorticity equation takes finally the form

$$\frac{\partial}{\partial t} \left(\overline{\rho^{(0)} \zeta^{(0)}}^{z,\lambda} + \frac{\beta}{f} \overline{\rho^{(0)} u^{(0)}}^{z,\lambda} - f \overline{p^{(2)}}^{z} \right) + \overline{NV}^{z,\lambda} + \overline{NM}^{z,\lambda} = \overline{S_p}^{z,\lambda} . \tag{3.55}$$

Here we have used the notations

$$\begin{aligned} \overline{NV}^{z,\lambda} = & \overline{\nabla \cdot \boldsymbol{u}^{(0)} \rho^{(0)} \zeta^{(0)} + \rho^{(0)} w^{(3)} \frac{\partial}{\partial z} \zeta^{(0)} +} \\ & \overline{\rho^{(0)} \boldsymbol{e}_r \cdot \left(\nabla w^{(3)} \times \frac{\partial}{\partial z} \boldsymbol{u}^{(0)} \right) + \frac{\boldsymbol{e}_r}{\rho^{(0)}} \cdot \nabla \frac{\partial}{\partial z} p^{(2)} \times \nabla p^{(2)}}^{z,\lambda} , \end{aligned} \tag{3.56}$$

$$\begin{aligned} \overline{NM}^{z,\lambda} = & \frac{\beta}{f} \Bigg(\overline{\rho^{(0)} \boldsymbol{u}^{(0)} \cdot \nabla u^{(0)} + \rho^{(0)} w^{(3)} \frac{\partial}{\partial z} u^{(0)} -} \\ & \overline{\frac{\rho^{(0)} u^{(0)} v^{(0)} \tan\phi}{a} - \frac{\rho^{(2)}}{a \cos\phi \rho^{(0)}} \frac{\partial p^{(2)}}{\partial \lambda}}^{z,\lambda} \Bigg) , \end{aligned} \tag{3.57}$$

$$\overline{S_p}^{z,\lambda} = \overline{\boldsymbol{e}_r \cdot \nabla \times \rho^{(0)} \boldsymbol{S}^{(3)}}^{z,\lambda} + \frac{\beta}{f} \overline{\rho^{(0)} S_u{}^{(3)}}^{z,\lambda} - f \overline{\rho^{(0)} S_\theta{}^{(5)}}^{z,\lambda} . \tag{3.58}$$

Terms from the vorticity equation (3.47) are contained in $\overline{NV}^{z,\lambda}$: horizontal advection of relative vorticity by the leading order wind $\boldsymbol{u}^{(0)}$, divergence of this wind multiplied by the relative vorticity, vertical advection of vorticity, the twisting term and the solenoidal term. The terms in $\overline{NM}^{z,\lambda}$ and the second term in the brackets of (3.55) result from the elimination of the advection of planetary vorticity by the zonally and vertically averaged ageostrophic meridional velocity $\overline{v^{(2)}}^{z,\lambda}$ (see (3.29)). The last term in the brackets of (3.55) results from the density tendencies caused by the divergence of $\boldsymbol{u}^{(2)}$ (see (3.26)). We express the bracketed terms in (3.55) in terms of $\overline{p^{(2)}}^{z}$ using (3.36) and the geostrophic balance

$$\frac{\partial}{\partial t}\left(\frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\frac{\cos\phi}{f}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^{z} - \frac{\beta}{f^2 a}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^{z} - f\overline{p^{(2)}}^{z}\right) + \overline{NV}^{z,\lambda} + \overline{NM}^{z,\lambda} = \overline{S_p}^{z,\lambda}. \quad (3.59)$$

The terms $\overline{NV}^{z,\lambda}$ and $\overline{NM}^{z,\lambda}$ can be calculated if the distribution of $p^{(2)}$ is known, then (3.59) can be integrated in time and after inverting the Helmholtz operator acting on $\overline{p^{(2)}}^{z}$ the evolution of the barotropic component of the pressure is determined. The surface pressure $p_0^{(2)}$ can be calculated from (3.42), this provides the necessary boundary condition for (3.38)-(3.41).

As we have already mentioned, some EMICs solve the PGEs presented in Section 3.1.2 but use a diagnostic parameterization for $\overline{p^{(2)}}^{z}$ in order to close the system. The closure is based on a linear relationship between the pressure and the temperature (Petoukhov et al., 2000). In this way the model has only one prognostic equation – an advection equation for the temperature. This considerably reduces the computational time but may also be a cause of the limited atmospheric variability observed in simulations based on this model. The closure presented here is an additional evolution equation, which will be added to the PGEs. Since it has only one spatial dimension, because of the averaging in λ and z, equation (3.59) will not severely decrease the numerical efficiency of the model. Nevertheless, it will add an additional degree of freedom to the system which can improve the representation of the atmospheric variability in the model. As shown, the evolution equation for the barotropic pressure arises from the vorticity equation; it contains terms such as advection of planetary vorticity and of relative vorticity which are not present in the classical PG model. This gives the possibility to capture additional physical phenomena with the model, e.g., zonal planetary Rossby waves.

Some EMICs are vertically averaged models, other have a very crude vertical resolution, e.g., some universal linear structure for the temperature is assumed (Claussen et al., 2001). This motivates us to analyze the special case when the pressure distribution is represented as the product of two functions, one depending only on z and another on the horizontal and time coordinates. From the condition (3.45) we obtain that there are no variations in λ. As a consequence $v^{(0)}$ and $w^{(3)}$ disappear, the terms $\overline{NV}^{z,\lambda}$ and $\overline{NM}^{z,\lambda}$ are zero and the initial pressure distribution remains constant in time. We conclude that in the model presented one should consider at least two modes in order to have non trivial evolution of $\overline{p^{(2)}}^{z}$. Such an assumption is implicitly made in the CLIMBER EMIC (Petoukhov et al., 2000), taking into account the atmospheric lapse rate dependence on the surface temperature, e.g., Mokhov and Akperov (2006).

3.1.4 Discussion

Using an asymptotic approach, we derived reduced model equations valid for one particular regime of planetary scale atmospheric motions with temporal variations of the order of about one week. Such temporal and spatial scales characterize atmospheric phenomena like the quasi-stationary Rossby waves and teleconnection patterns. Here we summarize the equations

$$\boldsymbol{u}^{(0)} = \frac{1}{f}\boldsymbol{e}_r \times \nabla\pi^{(2)}\,, \quad (3.60)$$

$$\frac{\partial}{\partial z}\pi^{(2)} = \Theta^{(2)}\,, \quad (3.61)$$

$$\nabla\cdot\boldsymbol{u}^{(0)} = -\frac{1}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}\,, \quad (3.62)$$

$$\frac{\partial}{\partial t}\Theta^{(2)} + \boldsymbol{u}^{(0)}\cdot\nabla\Theta^{(2)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)} = S_\theta{}^{(5)}\,, \quad (3.63)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\frac{\cos\phi}{f}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^z - \frac{\beta}{f^2 a}\frac{\partial}{\partial\phi}\overline{p^{(2)}}^z - f\overline{p^{(2)}}^z\right) + \overline{NV}^{z,\lambda} + \overline{NM}^{z,\lambda} = \overline{S_p}^{z,\lambda}\,, \quad (3.64)$$

see (3.56), (3.57), (3.58) for the definition of $\overline{NV}^{z,\lambda}$, $\overline{NM}^{z,\lambda}$ and $\overline{S_p}^{z,\lambda}$. The above equations contain the PGEs and a planetary barotropic vorticity equation (3.64). The PGEs alone do not represent a closed system, since a boundary condition for the surface pressure, or equivalently for the barotropic pressure, is needed. We derived the evolution equation (3.64) which uniquely determines the barotropic component of the flow and provides the desired closure. Consistent with previous studies on planetary scale motions, it contains terms absent in the classical QG model: the advection of planetary vorticity by the ageostrophic wind, the solenoidal, the twisting and the vertical advection term (where the vertical velocity results from the variation of f). The new equation gives the possibility to capture additional physical phenomena, not included in the models based on the PGEs only. It suggests itself as a prognostic alternative to the temperature-based diagnostic closure adopted in reduced-complexity planetary models (e.g., CLIMBER Petoukhov et al. (2000)) and may provide for more realistic increased large-scale, long-time variability in future implementations. Wiin-Nielsen (1961) showed that the PGEs produce baroclinic instability in the presence of a sheared wind. The addition of (3.64) to the system should not affect this property, since the last equation governs the barotropic component of the flow and we regard it as a boundary condition for closing the PGEs. Nevertheless, it is important to emphasize that in our model there is an important coupling between the barotropic dynamics (3.64) and the temperature equation (3.63) through the surface pressure $p_0^{(2)}$ from (3.42). Equation (3.42) shows that changes in the barotropic pressure $\overline{p^{(2)}}^z$ will alter the $p_0^{(2)}$ distribution and thus the surface wind field, which will change the temperature through advection. In this way (3.64) will considerably modify the behavior of the model compared to other models based only on the classical PGEs.

Expressed in dimensional units the variations near the ground of $p^{(2)}$ from (3.64) are of the order ~ 20 hPa. Such fluctuations are comparable to those associated with meridional variations of the zonal mean surface pressure and with anomalies due to quasi-stationary Rossby waves (Peixoto and Oort, 1992). Since $\pi^{(2)}$ from (3.61) is defined as $p^{(2)}$ scaled with $\rho^{(0)}$ (3.19), the estimated fluctuations will increase with height in accordance with the equivalent barotropic structure of the quasi-stationary anomalies (Hoskins and Pearce, 2001).

Our analysis shows that the planetary distribution of the vertically averaged leading order pressure $\overline{p^{(2)}}^z$ is zonally symmetric. Such property possess some planetary oscillations with dynamical relevance to the atmosphere, e.g., the zonal index (Rossby, 1939) describing the transitions between blocked and enhanced midlatitude westerly flow. Another zonal phenomenon characterized by planetary scales is the poleward propagation of zonal mean zonal wind anomalies (Riehl et al., 1959) with period of about 60 days (Lee et al., 2007). The leading modes of variability in the extratropical circulation, also known as northern and southern annular modes (AM) (Thompson and Wallace, 2000), are also zonally symmetric; they are characterized by planetary time scales of about 1-2 weeks. The derived new reduced equations may help in understanding the structure of the AM better. In our model the zonal symmetry is a direct consequence from the averaged continuity equation. On the other hand, idealized experiments (Cash et al., 2002) have indicated that the zonally symmetric AM structure can be interpreted as the resulting distribution of many zonally localized events with a meridional structure similar to that of the AM.

In the derivation of the equation for the barotropic flow we used the boundary condition of vanishing vertical velocity at the bottom and at the top of the domain. This condition was motivated from the asymptotic analysis of the dynamics within a layer above the troposphere with vertical variations of the potential temperature similar to the observed in the stratosphere: of the order of ε. In this case we have shown that the vertical velocities $w^{(3)}$ vanish and we have assumed a rigid lid at the top of the troposphere which is also consistent with the QG theory. An open question here is how other boundary conditions, e.g., vanishing zonally averaged vertical mass flux, will modify the presented prognostic closure. Additional analysis is required in order to find the type of the energy conserved in the PR when the new evolution equation for the barotropic pressure is added. This equation was derived from the zonally and vertically averaged vorticity equation (3.47), which indicates that some zonally symmetric barotropic kinetic energy is conserved in addition to the potential energy (conserved in the PGEs).

We want to compare our approach for the derivation of reduced models with the one based on mode truncation. In the latter the governing equations, e.g., the PEs, are projected on suitable basis functions. One can choose basis functions motivated by the large-scale flow structures, e.g., the slow Hough harmonics (Kasahara, 1977; Tanaka, 2003) or some empirical orthogonal functions (EOFs) (Schubert, 1985; Achatz and Branstator, 1999; Achatz and Opsteegh, 2003). Such models predict the time evolution only of the leading functions and the effects from the unresolved modes are parameterized, e.g., through some linear regression. Instead of truncating the degrees of freedom of the large-scale solution by considering a small number of horizontal or vertical modes, here we filter the governing equations through the asymptotic technique so that they are valid only for the planetary scales. In this way phenomena not relevant for the planetary scale dynamics like barotropic acoustic waves or hydrostatic gravity waves (present in the PEs) are neglected, retaining the full 3D structure of the solution. In both approaches the question of the representation of the unresolved scales (here the synoptic eddies) remains open. They can be parameterized applying a linear regression fitting procedure (Tanaka, 1991; Achatz and Branstator, 1999) or a macroturbulent diffusion (Petoukhov et al., 2000). The unified asymptotic technique applied here gives us another tool for representing the synoptic scales and their interactions with the planetary scales. In the next section we capture these interactions in a systematic way. By using a two scale asymptotic expansion we derive coupled reduced equations governing both the planetary scale motion and the synoptic scale flow. In summary, we consider our approach as an alternative to the one based on mode truncation; it reveals new insights in the atmospheric dynamics and because of its systematic basis it has the potential to be utilized for studies of multiple scales phenomena.

3.2 Two scale model

A challenging problem in the atmospheric science is the question about the interactions between the synoptic scales and the planetary scale motions. In Section 2.2 we introduced the scaling of the coordinates for the synoptic scale motions. Assuming that the solution of the governing equations depends on these coordinates, one can rederive the classical QG equations on a sphere. The complete derivation is presented in Section 4.1.1. The unified asymptotic approach gives one the possibility of deriving model equations valid not only on the synoptic scale but on the planetary scale as well. Applying a two scale asymptotic expansion the PG theory and the QG theory can be merged in a systematic manner. In this section we present the corresponding model equations.

3.2.1 Derivation of the Planetary Regime with synoptic scale interactions

A priori assumptions for the background stratification

The a priori assumptions from Section 3.1 remain the same. In addition we require that the largest potential temperature variations on the synoptic scale (t_S, λ_S, ϕ_S) are ot the order ε^3. This is consistent with the classical QG theory, where a horizontally uniform Brunt-Väisälä frequency $\sim 2 \times 10^{-2}$ s^{-1} is assumed (Klein, 2000; Majda and Klein, 2003). In this case the expansion of the potential temperature takes the form

$$\theta = 1 + \varepsilon^2 \Theta^{(2)}(\lambda_P, \phi_P, z, t_P) + \varepsilon^3 \Theta^{(3)}(\lambda_P, \phi_P, \lambda_S, \phi_S, z, t_P, t_S) + \mathcal{O}(\varepsilon^4)\,. \tag{3.65}$$

Notation

We use the following notation

$$(\lambda_S, \phi_S), (\lambda_P, \phi_P) \to \boldsymbol{X}_S, \boldsymbol{X}_P \tag{3.66}$$

$$f = \sin\phi_P\,, \tag{3.67}$$

$$\beta = \frac{1}{a}\frac{\partial}{\partial \phi_p}\sin\phi_P\,, \tag{3.68}$$

$$\nabla_{S,P} = \frac{\boldsymbol{e}_\lambda}{a\cos\phi_P}\frac{\partial}{\partial \lambda_{S,P}} + \frac{\boldsymbol{e}_\phi}{a}\frac{\partial}{\partial \phi_{S,P}}\,, \tag{3.69}$$

$$\Delta_{S,P} = \frac{1}{a^2\cos^2\phi_P}\left(\frac{\partial^2}{\partial {\lambda_{S,P}}^2} + \cos\phi_P \frac{\partial}{\partial \phi_{S,P}}\left(\cos\phi_P \frac{\partial}{\partial \phi_{S,P}}\right)\right)\,, \tag{3.70}$$

$$\nabla_{S,P}\cdot\boldsymbol{u} = \frac{1}{a\cos\phi_P}\left(\frac{\partial u}{\partial \lambda_{S,P}} - \frac{\partial v\cos\phi_P}{\partial \phi_{S,P}}\right)\,, \tag{3.71}$$

$$\boldsymbol{e}_r\cdot(\nabla_{S,P}\times\boldsymbol{u}) = \frac{1}{a\cos\phi_P}\left(\frac{\partial v}{\partial \lambda_{S,P}} - \frac{\partial u\cos\phi_P}{\partial \phi_{S,P}}\right)\,, \tag{3.72}$$

$$\boldsymbol{u} = \boldsymbol{e}_\lambda u + \boldsymbol{e}_\phi v\,. \tag{3.73}$$

Note that we do not need to make the traditional β-plane approximation for the Coriolis parameter f, since its full variations are resolved by the planetary scale coordinate ϕ_P.

Asymptotic expansion

Here we present the leading non-trivial equations in the asymptotic expansion. The magnitudes of the source terms in (2.14) - (2.19) have been estimated at the beginning of Section 3.1 and here we omit them for simplicity.

Horizontal Momentum Balance

The leading order velocities are geostrophically balanced with respect to the pressure gradient on both synoptic and planetary scale

$$\mathcal{O}(\varepsilon^1): \qquad -\sin\phi_P v^{(0)} = \frac{1}{a\cos\phi_P}\left(-\frac{\partial}{\partial\lambda_S}\pi^{(3)} - \frac{\partial}{\partial\lambda_P}\pi^{(2)}\right), \tag{3.74}$$

$$\mathcal{O}(\varepsilon^1): \qquad \sin\phi_P u^{(0)} = \frac{1}{a}\left(-\frac{\partial}{\partial\phi_S}\pi^{(3)} - \frac{\partial}{\partial\phi_P}\pi^{(2)}\right) \tag{3.75}$$

As in the QG theory the evolution of the velocity field $\boldsymbol{u}^{(0)}$ on the synoptic time scale appears in the next order equation, here we have an additional planetary scale pressure gradient term

$$\begin{aligned}\mathcal{O}(\varepsilon^2): \qquad & \frac{\partial}{\partial t_S}u^{(0)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_S}u^{(0)} + \frac{v^{(0)}}{a}\frac{\partial}{\partial\phi_S}u^{(0)} - \sin\phi_P v^{(1)} \\ & = \frac{1}{a\cos\phi_P}\left(-\frac{\partial}{\partial\lambda_S}\pi^{(4)} - \frac{\partial}{\partial\lambda_P}\pi^{(3)}\right),\end{aligned} \tag{3.76}$$

$$\begin{aligned}\mathcal{O}(\varepsilon^2): \qquad & \frac{\partial}{\partial t_S}v^{(0)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_S}v^{(0)} + \frac{v^{(0)}}{a}\frac{\partial}{\partial\phi_S}v^{(0)} + \sin\phi_P u^{(1)} \\ & = \frac{1}{a}\left(-\frac{\partial}{\partial\phi_S}\pi^{(4)} - \frac{\partial}{\partial\phi_P}\pi^{(3)}\right).\end{aligned} \tag{3.77}$$

In the next order we have not only all terms from (3.29) and (3.30) but also terms such as synoptic scale advection by $\boldsymbol{u}^{(1)}$ and its time derivative with respect to t_S.

$$\begin{aligned}\mathcal{O}(\varepsilon^3): \qquad & \frac{\partial}{\partial t_S}\boldsymbol{u}^{(1)} + \frac{\partial}{\partial t_P}\boldsymbol{u}^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S\boldsymbol{u}^{(1)} + \boldsymbol{u}^{(1)}\cdot\nabla_S\boldsymbol{u}^{(0)} \\ & + \boldsymbol{u}^{(0)}\cdot\nabla_P\boldsymbol{u}^{(0)} + w^{(3)}\frac{\partial}{\partial z}\boldsymbol{u}^{(0)} + \sin\phi_P\boldsymbol{e}_r\times\boldsymbol{v}^{(2)} - \boldsymbol{e}_\lambda\frac{u^{(0)}v^{(0)}\tan\phi_P}{a} \\ & + \boldsymbol{e}_\phi\frac{u^{(0)}u^{(0)}\tan\phi}{a} = -\nabla_P\pi^{(4)} + \frac{\rho^{(2)}}{{\rho^{(0)}}^2}\nabla_P\pi^{(2)} - \nabla_S\pi^{(5)} + \frac{\rho^{(2)}}{{\rho^{(0)}}^2}\nabla_S\pi^{(3)}.\end{aligned} \tag{3.78}$$

Vertical momentum balance

The expansion of the vertical momentum equation shows that the atmosphere is in hydrostatic balance up to a very high order, the first non-trivial equations read

$$\mathcal{O}\big(\varepsilon^2\big): \qquad \Theta^{(2)} = \frac{\partial}{\partial z}\pi^{(2)}\,, \tag{3.79}$$

$$\mathcal{O}\big(\varepsilon^3\big): \qquad \Theta^{(3)} = \frac{\partial}{\partial z}\pi^{(3)}\,. \tag{3.80}$$

In accordance with the a priori assumption for the potential temperature variations and with the observations, we assume that the surface distribution of $\pi^{(2)}$ does not depend on the synoptic scales. If we allow for such dependence, the geostrophic balance will imply horizontal velocities of the order $\varepsilon^{-1}u_{ref}$ near the surface. Observations, however, show that the synoptic scale velocity fields are an order of magnitude weaker. We integrate (3.79) from 0 to z and we obtain

$$\pi^{(2)} = \pi^{(2)}(\lambda_P, \phi_P, t_P, z). \tag{3.81}$$

Applying the results (3.81), (3.74) and (3.75), we find that the synoptic scale divergence of $\boldsymbol{u}^{(0)}$ disappears, i.e.,

$$f\nabla_S \cdot \boldsymbol{u}^{(0)} = 0\,. \tag{3.82}$$

Continuity equation

The first three orders in the mass conservation expansion give $w^{(0)} = w^{(1)} = w^{(2)} = 0$. The $\mathcal{O}(\varepsilon^3)$ order equation reads

$$\mathcal{O}(\varepsilon^3): \qquad \nabla_P \cdot \rho^{(0)}\boldsymbol{u}^{(0)} + \nabla_S \cdot \rho^{(0)}\boldsymbol{u}^{(1)} + \frac{\partial}{\partial z}\rho^{(0)}w^{(3)} = 0\,. \tag{3.83}$$

Here the synoptic scale divergence of $\boldsymbol{u}^{(1)}$ (interpreted in the classical QG theory as the divergence due to the ageostrophic velocities) appears in the same order as the planetary scale divergence of the leading order field $\boldsymbol{u}^{(0)}$.

Potential temperature

From the expansion of the potential temperature equation we have

$$\begin{aligned}\mathcal{O}(\varepsilon^5): \qquad & \frac{\partial}{\partial t_S}\Theta^{(3)} + \frac{\partial}{\partial t_P}\Theta^{(2)} + \frac{u^{(0)}}{a\cos\phi_P}\left(\frac{\partial}{\partial \lambda_S}\Theta^{(3)} + \frac{\partial}{\partial \lambda_P}\Theta^{(2)}\right) \\ & + \frac{v^{(0)}}{a}\left(\frac{\partial}{\partial \phi_S}\Theta^{(3)} + \frac{\partial}{\partial \phi_P}\Theta^{(2)}\right) + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)} = 0\,.\end{aligned} \tag{3.84}$$

It is worth to compare again the result for the two scale model with the corresponding QG equation. In the latter theory $\Theta^{(2)}$ is interpreted as a constant background temperature distribution and all terms involving it, except the stratification term, are set to zero. Here we consider the variations on the planetary spatial and temporal scales of $\Theta^{(2)}$ and their influence on the synoptic scale dynamics of $\Theta^{(3)}$.

We have summarized the equations for the dynamics of the leading order non-trivial variables. These equations involve higher-order unknown variables, e.g., $\boldsymbol{u}^{(1)}$; in the next section they will be eliminated.

3.2.2 PV formulation

In this section we proceed with a derivation of a PV type equation in a way similar to that encountered in the classical QG theory.

Applying $-\frac{1}{a\cos\phi_P}\frac{\partial}{\partial\phi_S}\cos\phi_P$ to (3.76) and $\frac{1}{a\cos\phi_P}\frac{\partial}{\partial\lambda_S}$ to (3.77) we obtain

$$\begin{aligned}&\frac{\partial}{\partial t_S}\zeta^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S\zeta^{(0)} + f\nabla_S\cdot\boldsymbol{u}^{(1)}\\ &= \frac{1}{a^2\cos\phi_P}\frac{\partial}{\partial\phi_S}\frac{\partial}{\partial\lambda_P}\pi^{(3)} - \frac{1}{a^2\cos\phi_P}\frac{\partial}{\partial\lambda_S}\frac{\partial}{\partial\phi_P}\pi^{(3)}\,,\end{aligned} \tag{3.85}$$

where

$$\zeta^{(0)} = \boldsymbol{e}_r\cdot(\nabla_S\times\boldsymbol{u}^{(0)}) = \frac{1}{f}\Delta_S\pi^{(3)}\,. \tag{3.86}$$

With the help of (3.74) and (3.75) we can write

$$f\nabla_P\cdot\boldsymbol{u}^{(0)} = -\frac{1}{a^2\cos\phi_P}\frac{\partial}{\partial\phi_S}\frac{\partial}{\partial\lambda_P}\pi^{(3)} + \frac{1}{a^2\cos\phi_P}\frac{\partial}{\partial\lambda_S}\frac{\partial}{\partial\phi_P}\pi^{(3)} - \beta v^{(0)}\,. \tag{3.87}$$

Thus, the vorticity equation reduces to

$$\frac{\partial}{\partial t_S}\zeta^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S\zeta^{(0)} + f\nabla_P\cdot\boldsymbol{u}^{(0)} + f\nabla_S\cdot\boldsymbol{u}^{(1)} + \beta v^{(0)} = 0\,. \tag{3.88}$$

Using the continuity equation (3.83) the last equation can be expressed as

$$\frac{\partial}{\partial t_S}\zeta^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S\zeta^{(0)} + \beta v^{(0)} = \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}\,. \tag{3.89}$$

Eliminating the vertical velocity with the help of (3.84), we have

$$\begin{aligned} \frac{\partial}{\partial t_S}\zeta^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_S \zeta^{(0)} + \beta v^{(0)} = \\ -\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}}{\frac{\partial}{\partial z}\Theta^{(2)}}\left(\frac{\partial}{\partial t_S}\Theta^{(3)} + \frac{\partial}{\partial t_P}\Theta^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla_S \Theta^{(3)} + \boldsymbol{u}^{(0)} \cdot \nabla_P \Theta^{(2)}\right). \end{aligned} \tag{3.90}$$

In the equation above both the planetary and the synoptic scales are involved; we have reduced the unknown variables to two $\pi^{(2)}(\boldsymbol{X}_P, t_P, z)$ and $\pi^{(3)}(\boldsymbol{X}_S, \boldsymbol{X}_P, t_S, t_P, z)$, since $\boldsymbol{u}^{(0)}, \Theta^{(2)}, \Theta^{(3)}$ and $\zeta^{(0)}$ can be expressed in terms of them, see (3.75),(3.76), (3.79), (3.80) and (3.86). Next, we derive two separate equations for the unknowns, as usual in the multiple scales asymptotic techniques this is achieved by applying the sublinear growth condition.

Sublinear growth condition

The variable $\pi^{(3)}$ can be represented as

$$\pi^{(3)}(\boldsymbol{X}_S, \boldsymbol{X}_P, t_S, t_P, z) = \overline{\pi^{(3)}}^S(\boldsymbol{X}_P, t_P, z) + \pi_S^{(3)}(\boldsymbol{X}_S, \boldsymbol{X}_P, t_S, t_P, z) \tag{3.91}$$

$$= \pi_P^{(3)}(\boldsymbol{X}_P, t_P, z) + \pi_S^{(3)}(\boldsymbol{X}_S, \boldsymbol{X}_P, t_S, t_P, z)\,, \tag{3.92}$$

where the operator $\overline{()}^S$ was defined in (2.37) and we have

$$\overline{\pi_S^{(3)}}^S = 0\,. \tag{3.93}$$

Consequently, we can write

$$\boldsymbol{u}^{(0)} = \underbrace{\frac{1}{f}\boldsymbol{e}_r \times \nabla_S \pi_S^{(3)}}_{:= \boldsymbol{u}_S^{(0)}} + \underbrace{\frac{1}{f}\boldsymbol{e}_r \times \nabla_P \pi^{(2)}}_{:= \boldsymbol{u}_P^{(0)}}\,. \tag{3.94}$$

Note that $\boldsymbol{u}_S^{(0)}$ is a function of the synoptic and planetary scales but $\boldsymbol{u}_P^{(0)}$ of the planetary scales only and we have

$$\overline{\boldsymbol{u}_S^{(0)}}^S = 0\,. \tag{3.95}$$

Equation (3.90) can be rewritten, with the terms depending on the planetary scales only appearing on the right hand side, as

$$\frac{\partial}{\partial t_S}\left(\zeta^{(0)} + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\partial\pi^{(3)}/\partial z}{\partial\Theta^{(2)}/\partial z}\right)\right) + \left(\boldsymbol{u}_S^{(0)} + \boldsymbol{u}_P^{(0)}\right)\cdot\nabla_S\left(\zeta^{(0)} + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\partial\pi^{(3)}/\partial z}{\partial\Theta^{(2)}/\partial z}\right)\right)$$
$$+\beta v_S^{(0)} + \frac{f}{\rho^{(0)}}\boldsymbol{u}_S^{(0)}\cdot\frac{\partial}{\partial z}\frac{\nabla_P\rho^{(0)}\Theta^{(2)}}{\partial\Theta^{(2)}/\partial z} = -\beta v_P^{(0)} - \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{\partial\Theta^{(2)}\partial z}\left(\frac{\partial}{\partial t_P}\Theta^{(2)} + \boldsymbol{u}_P^{(0)}\cdot\nabla_P\Theta^{(2)}\right)\right). \tag{3.96}$$

Here we used the hydrostatic balance and the transformation relation

$$\frac{f}{\rho^{(0)}}\left(\frac{\partial}{\partial z}\boldsymbol{u}_S^{(0)}\right)\cdot\nabla_P\Theta^{(2)} + \frac{f}{\rho^{(0)}}\left(\frac{\partial}{\partial z}\boldsymbol{u}_P^{(0)}\right)\cdot\nabla_S\Theta^{(3)}$$
$$= \frac{f}{\rho^{(0)}}\left(\frac{\partial}{\partial z}\boldsymbol{u}_S^{(0)}\right)\cdot\nabla_S\Theta^{(3)} = \frac{f}{\rho^{(0)}}\left(\frac{\partial}{\partial z}\boldsymbol{u}_P^{(0)}\right)\cdot\nabla_P\Theta^{(2)} = 0\,. \tag{3.97}$$

With the definition

$$PV^{(3)} = \frac{1}{f}\Delta_S\pi^{(3)} + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}\partial\pi^{(3)}/\partial z}{\partial\Theta^{(2)}/\partial z}\right), \tag{3.98}$$

equation (3.96) can be written in the form

$$\frac{\partial}{\partial t_S}PV^{(3)} + \nabla_S\cdot\left((\boldsymbol{u}_S^{(0)} + \boldsymbol{u}_P^{(0)})PV^{(3)} + \frac{\beta\boldsymbol{e}_\lambda\pi^{(3)}}{f} - \frac{\boldsymbol{e}_z\pi^{(3)}}{\rho^{(0)}}\times\frac{\partial}{\partial z}\frac{\nabla_P\rho^{(0)}\Theta^{(2)}}{\partial\Theta^{(2)}/\partial z}\right)$$
$$= -\beta v_P^{(0)} - \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{\partial\Theta^{(2)}\partial z}\left(\frac{\partial}{\partial t_P}\Theta^{(2)} + \boldsymbol{u}_P^{(0)}\cdot\nabla_P\Theta^{(2)}\right)\right). \tag{3.99}$$

The left hand side of (3.99) vanishes after averaging the equation over the synoptic scales and applying the sublinear growth condition. Thus

$$\beta v_P^{(0)} + \frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{\partial\Theta^{(2)}/\partial z}\left(\frac{\partial}{\partial t_P}\Theta^{(2)} + \boldsymbol{u}_P^{(0)}\cdot\nabla_P\Theta^{(2)}\right)\right) = 0\,, \tag{3.100}$$

therefore we have from (3.99)

$$\frac{\partial}{\partial t_S}PV^{(3)} + \left(\boldsymbol{u}_S^{(0)} + \boldsymbol{u}_P^{(0)}\right)\cdot\nabla_S PV^{(3)} + \beta v_S^{(0)} + \frac{f}{\rho^{(0)}}\boldsymbol{u}_S^{(0)}\cdot\frac{\partial}{\partial z}\frac{\nabla_P\rho^{(0)}\Theta^{(2)}}{\partial\Theta^{(2)}/\partial z} = 0\,. \tag{3.101}$$

Equation (3.100) can further be simplified. We average (3.83) and (3.84) over the synoptic scales to obtain

$$\nabla_P \cdot \boldsymbol{u}_P^{(0)} + \frac{1}{\rho^{(0)}} \frac{\partial}{\partial z} \rho^{(0)} w_P^{(3)} = 0\,, \tag{3.102}$$

$$\frac{\partial}{\partial t_P} \Theta^{(2)} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)} + w_P^{(3)} \frac{\partial}{\partial z} \Theta^{(2)} = 0\,, \tag{3.103}$$

where $w_P^{(3)} = \overline{w^{(3)}}^S$. Thus, (3.100) can be written as

$$\begin{aligned}
&\beta v_P^{(0)} + \frac{f}{\rho^{(0)}} \left(\frac{\partial}{\partial z} \frac{\rho^{(0)}}{\partial \Theta^{(2)} / \partial z} \right) \left(\frac{\partial}{\partial t_P} \Theta^{(2)} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)} \right) \\
&+ \frac{f}{\rho^{(0)}} \frac{\rho^{(0)}}{\partial \Theta^{(2)} / \partial z} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t_P} \Theta^{(2)} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)} \right) \\
&= \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \rho^{(0)} w_P^{(3)} + \frac{f}{\rho^{(0)}} \left(\frac{\partial}{\partial z} \frac{\rho^{(0)}}{\partial \Theta^{(2)} / \partial z} \right) \left(-w_P \frac{\partial}{\partial z} \Theta^{(2)} \right) \\
&+ \frac{f}{\rho^{(0)}} \frac{\rho^{(0)}}{\partial \Theta^{(2)} \partial z} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t_P} \Theta^{(2)} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)} \right) \\
&= \frac{f}{\rho^{(0)}} \frac{\rho^{(0)}}{\partial \Theta^{(2)} / \partial z} \frac{\partial}{\partial z} \rho^{(0)} w_P^{(3)} + \frac{f}{\rho^{(0)}} \frac{\rho^{(0)}}{\partial \Theta^{(2)} \partial z} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t_P} \Theta^{(2)} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)} \right) = 0\,.
\end{aligned} \tag{3.104}$$

Applying similar steps as in the derivation of the potential vorticity equation (A.37), equation (3.104) can be written in a compact form as

$$\left(\frac{\partial}{\partial t_P} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P + w_P \frac{\partial}{\partial z} \right) \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \Theta^{(2)} = 0\,. \tag{3.105}$$

This completes the derivation of the two scale model for the Planetary Regime. In the next section we summarize the model equations and discuss them.

3.2.3 Discussion

Using a two scale asymptotic ansatz, we extended in a systematic way the region of validity of the planetary scale model from Section 3.1 to the synoptic spatial and temporal scales. The model presented relies on the assumption that the variations of the background potential temperature are comparable in magnitude with those adopted in the classical quasi-geostrophic theory. The model equations can be transformed into two advection equations (3.105), (3.101) for a PV type quantity, namely,

$$\left(\frac{\partial}{\partial t_P} + \boldsymbol{u}_P{}^{(0)} \cdot \nabla_P + w_P^{(3)} \frac{\partial}{\partial z}\right) PV^{(2)} = 0\,, \tag{3.106}$$

$$\left(\frac{\partial}{\partial t_S} + \left(\boldsymbol{u}_S^{(0)} + \underline{\boldsymbol{u}_P^{(0)}\right) \cdot \nabla_S\right) PV^{(3)}} + \beta v_S^{(0)} + \underline{\frac{f}{\rho^{(0)}} \boldsymbol{u}_S^{(0)} \cdot \frac{\partial}{\partial z} \frac{\nabla_P \rho^{(0)} \Theta^{(2)}}{\partial \Theta^{(2)}/\partial z}} = 0\,, \tag{3.107}$$

where the underlined terms, discussed below in details, describe planetary-synoptic interactions and we have

$$PV^{(2)} = \frac{f}{\rho^{(0)}} \frac{\partial \Theta^{(2)}}{\partial z}\,, \tag{3.108}$$

$$PV^{(3)} = \frac{1}{f} \Delta_S \pi^{(3)} + \frac{f}{\rho^{(0)}} \frac{\partial}{\partial z} \left(\frac{\rho^{(0)} \partial \pi^{(3)}/\partial z}{\partial \Theta^{(2)}/\partial z} \right)\,. \tag{3.109}$$

Equation (3.106) describes the planetary scale dynamics and (3.107) – the synoptic scale dynamics. If we leave the planetary scales dependence of the variables out, equation (3.106) reduces trivially and the underlined terms in (3.107) vanish. In this case (3.107) is the classical PV equation from the QG theory. On the other hand, if we assume that the variables do not depend on the synoptic scales, only (3.106) remains and we have the planetary scale model derived in Section 3.1.

In the general case, when both synoptic and planetary scales are included, equations (3.106) and (3.107) with appropriate boundary conditions provide the planetary scale structure of $\Theta^{(2)}$ ($\pi^{(2)}$) and the synoptic scale structure of $\pi^{(3)}$. The derivative $\frac{\partial \Theta^{(2)}}{\partial z}$ in (3.107),(3.109) can be interpreted as the background stratification. But whereas in the classical QG model a horizontally uniform stratification is assumed, here it is governed by the evolution equation (3.106). Further difference to the QG theory is that we do not utilize a β-plane approximation in the derivation of the synoptic scale model (3.107). In the last model variation of the Coriolis parameter f (as well as β) on a planetary length scale are allowed. As we will discuss in Chapter 7, further investigation is required for the case when f tends to zero. In this limit the model should be matched in a systematic way to the planetary equatorial synoptic scale model of Majda and Klein (2003).

As usual in the asymptotic analysis, the planetary scale structure of $\pi^{(3)}$ and its evolution on the slow time scale t_P appear in the equations one order higher than (3.106), (3.107). These higher order equations involve unknowns such as the fast time variations of $\pi^{(4)}$ and its gradients on the small spatial scale, see (3.76), (3.77) and (3.78). We postpone the discussion of the planetary scale dynamics of $\pi^{(3)}$ to the next chapter, where motions with planetary zonal variations are considered.

The two underlined terms in (3.107) describe interactions between the planetary and the synoptic scales, or more precisely the influence of the planetary scale variations of the background temperature (pressure) distribution on the synoptic pressure field. The first term can be interpreted as the advection of synoptic scale PV by the planetary scale velocity field, the second as the advection of PV resulting from the planetary scale gradient of $\Theta^{(2)}$ by the synoptic scale velocities. It is important to note that in (3.106) there is no feedback from the synoptic scale to the planetary scale distribution of $\Theta^{(2)}$; we discuss later an interaction mechanism through the boundary condition. From the perspective of the classical wave-mean-flow interaction theory, one might expect that the divergence over the $\boldsymbol{X}_S$ spatial scale of the synoptic fluxes will change the planetary background state. The reason for the absence of interactions of this kind is sublinear growth condition. Because of the vanishing synoptic scale divergence of $\boldsymbol{u}^{(0)}$ (3.82), the synoptic scale advection by $\boldsymbol{u}^{(0)}$ of any quantity can be represented as the synoptic scale divergence of a flux. Averages over the synoptic scales of such terms should vanish because of the sublinear growth condition; consequently there is no net influence from the synoptic scale on the planetary scale variable $\Theta^{(2)}$.

As pointed out in Section 3.1.2 the PGEs are closed up to a boundary condition for the barotropic component of the flow, e.g., for the vertically averaged pressure $\overline{p^{(2)}}^z$. Considering planetary scales only, in Section 3.1.3 we derived a closure condition in the form of planetary barotropic vorticity equation (3.55). Note, that in the last equation the vorticity is defined as $\zeta^{(0)} = \boldsymbol{e}_r \cdot \nabla_P \times \boldsymbol{u}^{(0)}$ and should not be confused with the vorticity from (3.86). If we include the synoptic scales in the analysis, terms of the form $\overline{\boldsymbol{u}^{(1)} \cdot \nabla_S \rho^{(0)} \zeta^{(0)}}^{z,\lambda}$ will appear in (3.55), with $\zeta^{(0)}$ build with ∇_P. After averaging over the synoptic scales, such terms will give nonzero contribution, since the synoptic scale divergence of $\boldsymbol{u}^{(1)}$ does not vanish. These terms represent a feedback from the synoptic scale to the planetary scale dynamics. We should note that without applying a solvability condition these terms are not closed at this stage (see also the discussion at the end of Section 4.2.2).

The closure problem motivated us to study a case, where the departures from the geostrophic wind are an order of magnitude smaller, by simply setting $\boldsymbol{u}^{(1)}$ to zero in the two scale model. In this case we can apply similar manipulations as in Section 3.2.2 and we obtain the same equations as (3.106) and (3.107). Moreover, the unknown terms involving $\boldsymbol{u}^{(1)}$ vanish in the evolution equation for the barotropic pressure and it takes the same form as the synoptic scale averaged (3.55). An interesting finding is that some of the nonlinear terms in the evolution equation (3.55) generate new forms of synoptic-planetary interactions. In the two scale case the term $\overline{\nabla_P \cdot \boldsymbol{u}^{(0)} \rho^{(0)} \zeta^{(0)}}^{z,\lambda}$ has a contribution from $\overline{\nabla_P \cdot \boldsymbol{u}_S^{(0)} \rho^{(0)} (\boldsymbol{e}_r \cdot \nabla_P \times \boldsymbol{u}_S^{(0)})}^{z,\lambda}$. The latter term can be interpreted as planetary divergence of a flux depending on the synoptic scales and it does not vanish after applying the averaging procedure. The discussion above shows that the two scale version of (3.55) will provide the closure condition for the planetary scale dynamics in (3.106) and a new mechanism for a feedback from the synoptic scale dynamics. We have studied other possible types of closed interaction terms, which can affect the slow time evolution of $\boldsymbol{u}^{(0)}$ ($\pi^{(2)}$). It can be easily shown that terms of the form $\nabla_S \times (\boldsymbol{u}^{(0)} \cdot \nabla_P \boldsymbol{u}^{(0)})$ and $\nabla_P \times (\boldsymbol{u}^{(0)} \cdot \nabla_S \boldsymbol{u}^{(0)})$ will vanish after synoptic scale averaging. On the other hand, nonlinear terms from (3.55) involving $w^{(3)}$ and $\boldsymbol{u}^{(0)}$ components will give nonzero contributions.

Equations (3.106) and (3.107) can be regarded as the anelastic analogon of Pedlosky's two scale model for the large-scale oceanic circulation (Pedlosky, 1984). As in the atmosphere, also in the ocean there are planetary scale phenomena important for the heat and momentum transport, e.g., the gyre scale circulation and the thermocline. On the other hand, we have in the ocean some considerable differences: an incompressible flow with much smaller characteristic velocities $\approx$ 10 cm/s and horizontal scales of the eddies generated by baroclinic instability (synoptic eddies) about 50 km. In his study Pedlosky (1984) applied an asymptotic expansion in two small parameters: one is the Rossby number and the other is the ratio between the synoptic and the planetary length scale. For the derivation of his model he considered the case when the ratio between the two small parameters is of the order one. Expressing in terms of ε Pedlosky's expansion parameters for our model, it can be shown that their ratio is again one, which means that we have considered the same distinguished limit. The analysis of Pedlosky starts from the incompressible equations on a plane, here we study the compressible ones on a sphere. Nevertheless, both model equations have the same structure and are identical if we set $\rho^{(0)}$ in (3.106), (3.107) to one and neglect the effects due to the spherical geometry. A fundamental difference is the absence of a counterpart to the barotropic vorticity equation (3.55) in the Pedlosky's model. In the ocean the barotropic component of the planetary scale flow is determined either by prescribing the surface wind or by including some additional friction in the leading order momentum equation. This is not applicable to the atmosphere, since the surface winds should be a part of the solution and the frictional effects are much smaller than in the ocean.

We have already mentioned that in (3.106) there is no feedback from the synoptic scale dynamics to the planetary scale flow, whereas there is an interaction in the reverse direction. The same is also true for the model of Pedlosky (1984), but in the last paper some terms are discussed, which might provide the missing feedback under a particular distinguished limit. Our analysis showed that such terms will vanish (regardless of the distinguished limit) after averaging over the synoptic scales (they can be brought in a form similar to the term $\nabla_S \times (\boldsymbol{u}^{(0)} \cdot \nabla_P \boldsymbol{u}^{(0)})$ which we already discussed above). The last finding stresses the importance of the evolution equation for the barotropic flow for the complete representation of the interactions between the synoptic and planetary scales.

Chapter 4

Anisotropic Planetary Regime

The existence of anisotropic quasi-stationary Rossby waves motivated us to consider in the Anisotropic Planetary Regime (APR) motions with zonal variations on a planetary scale but with meridional extent restricted to the synoptic scale. We resolve the synoptic zonal and the planetary/synoptic temporal coordinates as well. The same magnitude of the potential temperature variations as in the PR is assumed. As leading order reduced system the QG model is derived (Section 4.1.1), it determines the dynamics on the synoptic scale, see Fig. 1.3. For the evolution on the planetary time scale of the leading order solution we consider the next order system of equations in the asymptotic expansion (Section 4.1.2). We discuss this system in Section 4.2.1 for the case of small meridional velocities and in Section 4.2.2 for the case of a plane geometry. All results are summarized and discussed in Section 4.3.

4.1 Derivation of the Anisotropic Planetary Regime

Coordinates scaling

We introduced in Section 2.2 the coordinates (λ_P, ϕ_P, t_P) and (λ_S, ϕ_S, t_S) resolving planetary and synoptic spatial and temporal scales. In this chapter we will use the same coordinates, but since we consider motions with meridional variations confined to the synoptic scale only, we set $\phi_P = const$. Thus, ϕ_P can be interpreted as the latitude in the classical QG theory around which one expands the Coriolis parameter f, typically $\phi_P = 45\,^\circ$.

A priori assumptions

In accordance with the discussions in the previous chapter, see Sections 3.2.1, we allow potential temperature variations on the planetary zonal scale of the order ε^2. We assume that the fluctuations on the synoptic spatial and temporal scales are one order of magnitude smaller, thus the expansion for the potential temperature takes the form

$$\theta = 1 + \varepsilon^2 \Theta^{(2)}(\lambda_P, z, t_P) + \varepsilon^3 \Theta^{(3)}(\lambda_P, \lambda_S, \phi_S, z, t_P, t_S) + \mathcal{O}\big(\varepsilon^4\big)\ . \tag{4.1}$$

Preliminary expansions

We will make use of the following Taylor series expansions in the derivation

$$\sin(\phi_P + \varepsilon\phi_G) = \sin\phi_P + \varepsilon\phi_S\cos\phi_P + \mathcal{O}\big(\varepsilon^2\big)\ , \tag{4.2}$$

$$\begin{aligned} \frac{1}{\cos(\phi_P+\varepsilon\phi_S)} &= \frac{1}{\cos\phi_P - \varepsilon\phi_S\sin\phi_P + \mathcal{O}(\varepsilon^2)} \\ &= \frac{1}{\cos\phi_P}\left(1+\varepsilon\phi_S\tan\phi_P + \mathcal{O}\big(\varepsilon^2\big)\right)\ , \end{aligned} \tag{4.3}$$

In the APR the divergence from the continuity equation (2.18) can be expanded as

$$\begin{aligned} &\frac{\varepsilon^2}{a\cos(\phi_P+\varepsilon\phi_S)}\left(\frac{\partial u}{\partial\lambda_S} + \frac{\partial v\cos(\phi_P+\varepsilon\phi_S)}{\partial\phi_S}\right) + \frac{\varepsilon^3}{a\cos(\phi_P+\varepsilon\phi_S)}\frac{\partial u}{\partial\lambda_P} = \\ &\frac{\varepsilon^2}{a\cos\phi_P}\left(\frac{\partial u}{\partial\lambda_S} + \frac{\partial v\cos\phi_P}{\partial\phi_S}\right) + \varepsilon^3\frac{\phi_S\tan\phi_P}{a\cos\phi_P}\left(\frac{\partial u}{\partial\lambda_S} + \frac{\partial v\cos\phi_P}{\partial\phi_S}\right) - \\ &\frac{\varepsilon^3}{a\cos\phi_P}\frac{\partial}{\partial\phi_S}v\phi_S\sin\phi_P + \varepsilon^4\frac{\phi_S^2}{2a}\left(\frac{1}{\cos\phi_P} + \frac{2\sin^2\phi_P}{\cos^3\phi_P}\right)\left(\frac{\partial u}{\partial\lambda_S} - \frac{\partial v\cos\phi_P}{\partial\phi_S}\right) \\ &-\varepsilon^4\frac{\phi_S\tan\phi_P}{a\cos\phi_P}\frac{\partial}{\partial\phi_S}v\phi_S\sin\phi_P - \varepsilon^4\frac{1}{a\cos\phi_P}\frac{\partial}{\partial\phi_S}v\phi_S^2\frac{\cos\phi_P}{2} \\ &+\frac{\varepsilon^3}{a\cos\phi_P}\frac{\partial u}{\partial\lambda_P} + \varepsilon^4\frac{\phi_S\tan\phi_P}{a\cos\phi_P}\frac{\partial u}{\partial\lambda_P} + \mathcal{O}\big(\varepsilon^5\big)\ . \end{aligned} \tag{4.4}$$

Notation

Taking into account a constant planetary scale variable ϕ_P, we introduce the following notation for the regime considered here

$$f_0 = \sin\phi_P\,, \beta = \frac{1}{a}\cos\phi_P\,, \tag{4.5}$$

$$\nabla_P = \frac{\boldsymbol{e}_\lambda}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\,, \tag{4.6}$$

$$\nabla_S = \frac{\boldsymbol{e}_\lambda}{a\cos\phi_P}\frac{\partial}{\partial\lambda_S} + \frac{\boldsymbol{e}_\phi}{a}\frac{\partial}{\partial\phi_S}\,, \tag{4.7}$$

$$\nabla_S\cdot\boldsymbol{u} = \frac{1}{a\cos\phi_P}\left(\frac{\partial u}{\partial\lambda_S} + \frac{\partial v\cos\phi_P}{\partial\phi_S}\right)\,, \tag{4.8}$$

$$\nabla_P\cdot\boldsymbol{u} = \frac{1}{a\cos\phi_P}\frac{\partial u}{\partial\lambda_P}\,, \tag{4.9}$$

$$\Delta_S = \frac{1}{a^2\cos^2\phi_P}\left(\frac{\partial^2}{\partial{\lambda_S}^2} + \cos\phi_P\frac{\partial}{\partial\phi_S}\left(\cos\phi_P\frac{\partial}{\partial\phi_S}\right)\right)\,, \tag{4.10}$$

$$\boldsymbol{e}_r\cdot(\nabla_S\times\boldsymbol{u}) = \frac{1}{a\cos\phi_P}\left(\frac{\partial v}{\partial\lambda_S} - \frac{\partial u\cos\phi_P}{\partial\phi_S}\right)\,. \tag{4.11}$$

Asymptotic expansion

We represent all dependent variables in (2.14) - (2.19) as an asymptotic series (2.31) and collect terms with the same order in ε. Again we omit for simplicity the source terms. We estimated their magnitudes at the beginning of Chapter 3 and the terms can be added to the equations without changing the derivation. Here we summarize the results.

Horizontal Momentum Balance

Making use of (4.2) and (4.3), the first two orders from the u and v components of the momentum equation read

$$\mathcal{O}(\varepsilon):\qquad -f_0 v^{(0)} = -\frac{1}{a\cos\phi_P}\left(\frac{\partial}{\partial\lambda_S}\pi^{(3)} + \frac{\partial}{\partial\lambda_P}\pi^{(2)}\right)\,, \tag{4.12}$$

$$\mathcal{O}(\varepsilon):\qquad f_0 u^{(0)} = -\frac{1}{a}\frac{\partial}{\partial\phi_S}\pi^{(3)} \tag{4.13}$$

$$\begin{aligned}\mathcal{O}(\varepsilon^2):\qquad & \frac{\partial}{\partial t_S}u^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S u^{(0)} - f_0 v^{(1)} - \phi_S\cos\phi_P v^{(0)} \\ & = \frac{1}{a\cos\phi_P}\left(-\frac{\partial}{\partial\lambda_S}\pi^{(4)} - \frac{\partial}{\partial\lambda_P}\pi^{(3)}\right) - \frac{\phi_S\tan\phi_P}{a\cos\phi_P}\left(\frac{\partial}{\partial\lambda_S}\pi^{(3)} + \frac{\partial}{\partial\lambda_P}\pi^{(2)}\right)\,,\end{aligned} \tag{4.14}$$

$$\mathcal{O}(\varepsilon^2):\qquad \frac{\partial}{\partial t_S}v^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S v^{(0)} + f_0 u^{(1)} + \phi_S\cos\phi_P u^{(0)} = -\frac{1}{a}\frac{\partial}{\partial\phi_S}\pi^{(4)}\,. \tag{4.15}$$

In the equations above one can identify all terms from the momentum equations in the quasi-geostrophic approximation. Additionally, we have terms involving derivatives with respect to the planetary zonal coordinate λ_P and the $\phi_S\tan\phi_P$ terms result from the inclusion of spherical geometry. For reasons which will become clear later, we go one order further in the asymptotic expansion. The next order corrections to the QG momentum equations are

$$
\begin{aligned}
\mathcal{O}(\varepsilon^3): \quad & \frac{\partial}{\partial t_S}u^{(1)} + \frac{\partial}{\partial t_P}u^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_S u^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_S u^{(0)} + \frac{\phi_S \tan\phi_P}{a\cos\phi_P}u^{(0)}\frac{\partial}{\partial \lambda_S}u^{(0)} \\
&+\boldsymbol{u}^{(0)} \cdot \nabla_P u^{(0)} + w^{(3)}\frac{\partial}{\partial z}u^{(0)} - f_0 v^{(2)} - \phi_S \cos\phi_P v^{(1)} + \frac{\phi_S^2}{2}\sin\phi_P v^{(0)} - \frac{u^{(0)}v^{(0)}\tan\phi_P}{a} \\
&= -\frac{1}{a\cos\phi_P}\left(\frac{\partial}{\partial\lambda_S}\pi^{(5)} + \frac{\partial}{\partial\lambda_P}\pi^{(4)}\right) - \frac{\phi_S\tan\phi_P}{a\cos\phi_P}\left(\frac{\partial}{\partial\lambda_S}\pi^{(4)} + \frac{\partial}{\partial\lambda_P}\pi^{(3)}\right) \\
&\quad -\frac{\phi_S^2}{2a}\left(\frac{1}{\cos\phi_P} + \frac{2\tan^2\phi_P}{\cos\phi_P}\right)\left(\frac{\partial}{\partial\lambda_S}\pi^{(3)} + \frac{\partial}{\partial\lambda_P}\pi^{(2)}\right) \\
&\quad +\frac{\rho^{(2)}}{a\cos\phi_P\rho^{(0)}}\left(\frac{\partial}{\partial\lambda_S}\pi^{(3)} + \frac{\partial}{\partial\lambda_P}\pi^{(2)}\right),
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
\mathcal{O}(\varepsilon^3): \quad & \frac{\partial}{\partial t_S}v^{(1)} + \frac{\partial}{\partial t_P}v^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_P v^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_S v^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_S v^{(0)} + \\
& \phi_S\frac{\tan\phi_P}{a\cos\phi_P}u^{(0)}\frac{\partial}{\partial\lambda_S}v^{(0)} + w^{(3)}\frac{\partial}{\partial z}v^{(0)} + f_0 u^{(0)} + \phi_S\cos\phi_P u^{(1)} \\
& -\frac{\phi_S^2}{2}\sin\phi_P u^{(0)} + \frac{u^{(0)}u^{(0)}\tan\phi_P}{a} = -\frac{1}{a}\frac{\partial}{\partial\phi_S}\pi^{(5)} + \frac{\rho^{(2)}}{a\rho^{(0)}}\frac{\partial}{\partial\phi_S}\pi^{(3)}.
\end{aligned}
\tag{4.17}
$$

In accordance to the two scale model from Section 3.2, we have in (4.16), (4.17) the evolution on the planetary time scale of the leading order velocity field and the synoptic evolution of the next order corrections of that field. Additional terms due to the expansion of the trigonometric functions in $\varepsilon\phi_S$ appear.

Vertical momentum balance

The expansion of the vertical momentum equation remains the same as in the previous chapter and we obtain hydrostatic balance up to $p^{(4)}$

$$
\frac{\partial}{\partial z}p^{(i)} = -\rho^{(i)}, \quad i = 0, \dots 4\,. \tag{4.18}
$$

Making use of the ideal gas law (see Section 3.1.1), we can rewrite the first nontrivial equations as

$$
\mathcal{O}\big(\varepsilon^2\big): \quad \Theta^{(2)} = \frac{\partial}{\partial z}\pi^{(2)}, \tag{4.19}
$$

$$
\mathcal{O}\big(\varepsilon^3\big): \quad \Theta^{(3)} = \frac{\partial}{\partial z}\pi^{(3)}, \tag{4.20}
$$

$$
\mathcal{O}\big(\varepsilon^4\big): \quad \Theta^{(4)} = \frac{\partial}{\partial z}\pi^{(4)} + \frac{\Theta^{(2)}}{\rho^{(0)}}\frac{\partial}{\partial z}p^{(2)} + \frac{p^{(2)}}{2{p^{(0)}}^2}\frac{1}{\gamma}\left(\frac{1}{\gamma} - 1\right). \tag{4.21}
$$

Continuity equation

Applying the arguments from Sections 3.1, 3.2 it can be shown that the vertical velocities disappear up to $w^{(2)}$. Making use of (4.4), we obtain from the continuity equation

$$\mathcal{O}(\varepsilon^3): \quad \nabla_P \cdot \rho^{(0)}\boldsymbol{u}^{(0)} + \nabla_S \cdot \rho^{(0)}\boldsymbol{u}^{(1)} - \frac{\tan\phi_P \rho^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(0)}\phi_S + \frac{\partial}{\partial z}\rho^{(0)}w^{(3)} = 0\,, \tag{4.22}$$

$$\begin{aligned}\mathcal{O}(\varepsilon^4): \quad & \nabla_P \cdot \rho^{(0)}\boldsymbol{u}^{(1)} + \phi_S \tan\phi_P \nabla_P \cdot \rho^{(0)}\boldsymbol{u}^{(0)} + \nabla_S \cdot \rho^{(0)}\boldsymbol{u}^{(2)} + \\ & \phi_S \tan\phi_P \nabla_S \cdot \rho^{(0)}\boldsymbol{u}^{(1)} - \frac{\tan\phi_P \rho^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(1)}\phi_S - \frac{\phi_S \tan^2\phi_P \rho^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(0)}\phi_S \\ & - \frac{\rho^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(0)}\frac{\phi_S^2}{2} + \frac{\partial}{\partial z}\rho^{(0)}w^{(4)} = 0\,.\end{aligned} \tag{4.23}$$

Potential temperature

At leading order we have for $\Theta^{(3)}$ the potential temperature equation from the QG theory. As in Section 3.2 we have additional terms describing planetary scale variations of the background $\Theta^{(2)}$

$$\mathcal{O}(\varepsilon^5): \quad \frac{\partial}{\partial t_S}\Theta^{(3)} + \frac{\partial}{\partial t_P}\Theta^{(2)} + \boldsymbol{u}^{(0)} \cdot \nabla_S \Theta^{(3)} + \boldsymbol{u}^{(0)} \cdot \nabla_P \Theta^{(2)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)} = 0\,. \tag{4.24}$$

Again we proceed in the derivation one order beyond the expansion for the QG theory

$$\begin{aligned}\mathcal{O}(\varepsilon^6): \quad & \left(\frac{\partial}{\partial t_S} + \boldsymbol{u}^{(0)} \cdot \nabla_S\right)\Theta^{(4)} + \left(\frac{\partial}{\partial t_P} + \boldsymbol{u}^{(0)} \cdot \nabla_P\right)\Theta^{(3)} + \boldsymbol{u}^{(1)} \cdot \nabla_S \Theta^{(3)} \\ & \boldsymbol{u}^{(1)} \cdot \nabla_P \Theta^{(2)} + \frac{\phi_S \tan\phi_P u^{(0)}}{a\cos\phi_P}\left(\frac{\partial}{\partial \lambda_S}\Theta^{(3)} + \frac{\partial}{\partial \lambda_P}\Theta^{(2)}\right) + w^{(3)}\frac{\partial}{\partial z}\Theta^{(3)} \\ & + w^{(4)}\frac{\partial}{\partial z}\Theta^{(2)} = 0\,.\end{aligned} \tag{4.25}$$

Up until now we have summarized the equations resulting from the asymptotic expansion, next we derive reduced equations for the leading order solution.

4.1.1 Leading order solution: QG model

The next manipulations follow closely the derivation of the two scale model in Section 3.2.2. Taking $-\frac{\partial}{\partial \phi_S}$ of (4.14) and $\frac{\partial}{\partial \lambda_S}$ of (4.15), using (4.22), (4.12), (4.13) we derive a vorticity equation of the form

$$\frac{\partial}{\partial t_S}\zeta^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_S \zeta^{(0)} + \beta v^{(0)} = \frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}\,, \tag{4.26}$$

where

$$\zeta^{(0)} = \boldsymbol{e}_r \cdot (\nabla_S \times \boldsymbol{u}^{(0)}) = \frac{1}{f_0} \Delta_S \pi^{(3)} \,. \tag{4.27}$$

We eliminate in (4.26) the unknown vertical velocity $w^{(3)}$ using (4.24) and rewrite the result in such a way that all terms on the right hand side depend on the planetary scale λ_P only, whereas the terms on the left depend on the synoptic scales as well (compare with (3.99))

$$\begin{aligned}\frac{\partial}{\partial t_S} PV^{(3)} + \nabla_S \cdot \left((\boldsymbol{u}_S^{(0)} + \boldsymbol{u}_P^{(0)}) PV^{(3)} + \frac{\beta \boldsymbol{e}_\lambda \pi^{(3)}}{f_0} - \frac{\boldsymbol{e}_z \pi^{(3)}}{\rho^{(0)}} \times \frac{\partial}{\partial z} \frac{\nabla_P \rho^{(0)} \Theta^{(2)}}{\partial \Theta^{(2)} / \partial z} \right) \\ = -\beta v_P^{(0)} - \frac{f_0}{\rho^{(0)}} \frac{\partial}{\partial z} \left(\frac{\rho^{(0)}}{\partial \Theta^{(2)} \partial z} \left(\frac{\partial}{\partial t_P} \Theta^{(2)} + \boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)} \right) \right) .\end{aligned} \tag{4.28}$$

Here $\boldsymbol{u}_S^{(0)}, \boldsymbol{u}_P^{(0)}$ and $PV^{(3)}$ are defined as (compare with (3.94), (3.98))

$$\boldsymbol{u}^{(0)} = \underbrace{\frac{1}{f_0} \boldsymbol{e}_r \times \nabla_S \pi^{(3)}}_{:= \boldsymbol{u}_S^{(0)}} + \underbrace{\frac{1}{f_0} \boldsymbol{e}_r \times \nabla_P \pi^{(2)}}_{:= \boldsymbol{u}_P^{(0)}} \,, \tag{4.29}$$

$$PV^{(3)} = \frac{1}{f_0} \Delta_S \pi^{(3)} + \frac{f_0}{\rho^{(0)}} \frac{\partial}{\partial z} \left(\frac{\rho^{(0)} \partial \pi^{(3)} / \partial z}{\partial \Theta^{(2)} / \partial z} \right) . \tag{4.30}$$

We average (4.28) over the synoptic scales and apply the sublinear growth condition. The left hand side vanishes and we obtain the solvability condition that the right hand side is equal to zero. We note that in contrast to the regime presented in Chapter 3, here we have vanishing planetary scale divergence of $\boldsymbol{u}_P^{(0)}$

$$\nabla_P \cdot \boldsymbol{u}_P^{(0)} = 0 \,. \tag{4.31}$$

By averaging the continuity equation (4.22) over the synoptic scales we obtain $\overline{w^{(3)}}^S = 0$. The averaged potential temperature equation (4.24) reads

$$\frac{\partial}{\partial t_P} \Theta^{(2)} + \underbrace{\boldsymbol{u}_P^{(0)} \cdot \nabla_P \Theta^{(2)}}_{=0} = 0 \,. \tag{4.32}$$

We obtain $\Theta^{(2)} = \Theta^{(2)}(\lambda_P, z)$. The requirement that the right hand side of (4.28) vanishes, reduces to

$$\beta v_P^{(0)} = 0\,. \tag{4.33}$$

It follows $\Theta^{(2)} = \Theta^{(2)}(z)$. Thus, the interaction terms on the left hand side of (4.28) involving gradients in λ_P of $\Theta^{(2)}$ or $\pi^{(2)}$ disappear and we obtain the classical QG model on a sphere for the synoptic scale dynamics

$$\frac{\partial}{\partial t_S} PV^{(3)} + \boldsymbol{u}_S^{(0)} \cdot \nabla_S PV^{(3)} + \beta v_S^{(0)} = 0\,. \tag{4.34}$$

4.1.2 Next order equations, general case

Here we summarize the next order equations system derived from the asymptotic expansion. We have shown in the previous section that $\Theta^{(2)}$ is function of z only, consequently some terms in (4.25) can be set to zero. From (4.16) and (4.17) we obtain the general form of the next order vorticity equation

$$\begin{aligned}
&\frac{\partial}{\partial t_S}\zeta^{(1)} + \boldsymbol{u}^{(1)} \cdot \nabla_S \zeta^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_S \zeta^{(1)} + \zeta^{(0)} \nabla_S \cdot \boldsymbol{u}^{(1)} + w^{(3)} \frac{\partial}{\partial z}\zeta^{(0)} + \boldsymbol{e}_r \cdot \left(\nabla w^{(3)} \times \frac{\partial}{\partial z}\boldsymbol{u}^{(0)}\right) \\
&-\frac{1}{a}\frac{\partial}{\partial \phi_S}\frac{\phi_S \tan\phi_P}{a\cos\phi_P} u^{(0)} \frac{\partial}{\partial \lambda_S} u^{(0)} + \frac{1}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}\frac{\phi_S \tan\phi_P}{a\cos\phi_P} u^{(0)} \frac{\partial}{\partial \lambda_S} v^{(0)} + \nabla_S \cdot \phi_S \cos\phi_P \boldsymbol{u}^{(1)} + \\
&\quad f_0 \nabla_S \cdot \boldsymbol{u}^{(2)} - \nabla_S \cdot \frac{\phi_S^2}{2}\sin\phi_P \boldsymbol{u}^{(0)} + \frac{1}{a}\frac{\partial}{\partial \phi}\frac{u^{(0)} v^{(0)} \tan\phi_P}{a} + \frac{1}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}\frac{u^{(0)} u^{(0)} \tan\phi_P}{a} + \\
&\quad \frac{\partial}{\partial t_P}\zeta^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_P \zeta^{(0)} + \frac{1}{a\cos\phi_P}\frac{\partial u^{(0)}}{\partial \lambda_S}\frac{1}{a\cos\phi_P}\frac{\partial v^{(0)}}{\partial \lambda_P} - \frac{1}{a}\frac{\partial u^{(0)}}{\partial \phi_S}\frac{1}{a\cos\phi_P}\frac{\partial u^{(0)}}{\partial \lambda_P} \\
&\quad = \frac{1}{a^2\cos\phi_P}\frac{\partial}{\partial \lambda_P}\frac{\partial}{\partial \phi_S}\pi^{(4)} + \frac{1}{a}\frac{\partial}{\partial \phi_S}\frac{\phi_S \tan\phi_P}{a\cos\phi_P}\left(\frac{\partial}{\partial \lambda_S}\pi^{(4)} + \frac{\partial}{\partial \lambda_P}\pi^{(3)}\right) \\
&\quad + \frac{1}{a}\frac{\partial}{\partial \phi_S}\left(\frac{\phi_S^2}{2}\left(\frac{1}{\cos\phi} + \frac{2\tan^2\phi}{\cos\phi}\right)\frac{\partial}{\partial \lambda_S}\pi^{(3)}\right)\,,
\end{aligned} \tag{4.35}$$

where

$$\zeta^{(1)} = \boldsymbol{e}_r \cdot \left(\nabla_S \times \boldsymbol{u}^{(1)}\right)\,. \tag{4.36}$$

Making use of the continuity equation (4.23), (4.35) can be written as

$$\frac{\partial}{\partial t_S}\zeta^{(1)} + \boldsymbol{u}^{(1)}\cdot\nabla_S\zeta^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S\zeta^{(1)} + \zeta^{(0)}\nabla_S\cdot\boldsymbol{u}^{(1)} + w^{(3)}\frac{\partial}{\partial z}\zeta^{(0)} + \boldsymbol{e}_r\cdot(\nabla w^{(3)}\times\frac{\partial}{\partial z}\boldsymbol{u}^{(0)})$$
$$+\frac{\phi_S\tan\phi_P}{a\cos\phi_P}\left(u^{(0)}\frac{\partial}{\partial\lambda_S}\zeta^{(0)} + \zeta^{(0)}\frac{\partial}{\partial\lambda_S}u^{(0)}\right) - \frac{\tan\phi_P}{a\cos\phi_P}u^{(0)}\frac{\partial}{\partial\lambda_S}u^{(0)} + \frac{\phi_S\tan\phi_P f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)}$$
$$-\frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(4)} - \frac{\phi_S\cos\phi_P}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}w^{(3)} + \beta v^{(1)} - \frac{2f_0\phi_S}{a}v^{(0)} + \frac{1}{a}\frac{\partial}{\partial\phi_S}\frac{u^{(0)}v^{(0)}\tan\phi_P}{a}$$
$$+\frac{1}{a\cos\phi_P}\frac{\partial}{\partial\lambda_S}\frac{u^{(0)}u^{(0)}\tan\phi_P}{a} + \frac{\partial}{\partial t_P}\zeta^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_P\zeta^{(0)} + \frac{1}{a\cos\phi_P}\frac{\partial u^{(0)}}{\partial\lambda_S}\frac{1}{a\cos\phi_P}\frac{\partial v^{(0)}}{\partial\lambda_P}$$
$$-\frac{1}{a}\frac{\partial u^{(0)}}{\partial\phi_S}\frac{1}{a\cos\phi_P}\frac{\partial u^{(0)}}{\partial\lambda_P} - \phi_S\tan\phi_P f_0\nabla_P\cdot\boldsymbol{u}^{(0)} + \frac{\partial}{\partial t_S}\frac{1}{a\cos\phi_P}\frac{\partial v^{(0)}}{\partial\lambda_P}$$
$$+\frac{1}{a^2\cos^2\phi_P}\frac{\partial}{\partial\lambda_P}\left(u^{(0)}\frac{\partial}{\partial\lambda_S}v^{(0)}\right) + \frac{1}{a^2\cos\phi_P}\frac{\partial}{\partial\lambda_P}\left(v^{(0)}\frac{\partial}{\partial\phi_S}v^{(0)}\right)$$
$$= -\frac{\tan\phi_P}{a}\left(\frac{\partial}{\partial t_S}u^{(0)} + \boldsymbol{u}^{(0)}\cdot\nabla_S u^{(0)}\right)$$
$$+\phi_S\tan\phi_P\left(\frac{\partial}{\partial t_S} + \boldsymbol{u}^{(0)}\cdot\nabla_S\right)\left(-\frac{1}{a\cos\phi_P}\frac{\partial u^{(0)}\cos\phi_P}{\partial\phi_S}\right). \tag{4.37}$$

The vertical velocity $w^{(4)}$ in (4.37) can be eliminated with the help of (4.25). In this way an equation combining the synoptic evolution of $\zeta^{(1)}$, $\frac{\partial}{\partial z}\Theta^{(4)}$ and the planetary evolution of $\zeta^{(0)}$, $\frac{\partial}{\partial z}\Theta^{(3)}$ can be derived. This will be demonstrated in Section 4.2.2 for a plane geometry.

4.2 Special cases

We proceed with a discussion of two special cases of the APR: in the first case we assume that the leading order meridional velocities are $\mathcal{O}(\varepsilon)$, in the second we consider a plane geometry.

4.2.1 Regime $(\lambda_P, \lambda_S, \phi_S, t_P, t_S), v^{(0)} = 0$

In this section we restrict the analysis to the case when the leading order meridional velocities vanish. Since $v^{(0)}$ is geostrophically balanced we obtain

$$\frac{\partial}{\partial\lambda_S}\pi^{(3)} = 0 \rightarrow \pi^{(3)}(\lambda_P, \phi_S, t_P, t_S, z)\,. \tag{4.38}$$

Now, averaging (4.14), (4.22) and (4.24) over λ_S and applying the sublinear growth condition we have

$$\frac{\partial}{\partial t_S}\overline{u^{(0)}}^{\lambda_S} - f_0\overline{v^{(1)}}^{\lambda_S} = -\frac{1}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\overline{\pi^{(3)}}^{\lambda_S}\,, \tag{4.39}$$

$$\frac{1}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\rho^{(0)}\overline{u^{(0)}}^{\lambda_S} + \frac{\rho^{(0)}}{a}\frac{\partial}{\partial\phi_S}\overline{v^{(1)}}^{\lambda_S} + \frac{\partial}{\partial z}\rho^{(0)}\overline{w^{(3)}}^{\lambda_S} = 0\,, \tag{4.40}$$

$$\frac{\partial}{\partial t_S}\overline{\Theta^{(3)}}^{\lambda_S} + \overline{w^{(3)}}^{\lambda_S}\frac{\partial}{\partial z}\Theta^{(2)} = 0\,. \tag{4.41}$$

The equations above can be combined into one

$$\frac{\partial}{\partial t_S}\left(\overline{\zeta_x^{(0)}}^{\lambda_S} + \frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}\overline{\Theta^{(3)}}^{\lambda_S}}{\partial\Theta^{(2)}/\partial z}\right) = 0\,, \tag{4.42}$$

where we have used the definition

$$\zeta_x^{(i)} = -\frac{1}{a\cos\phi_P}\frac{\partial u^{(i)}\cos\phi_P}{\partial\phi_S}\,, \quad i = 0, 1, \ldots\,. \tag{4.43}$$

From (4.42) it follows that $\pi^{(3)}$ does not depend on t_S. Together with (4.39), (4.40) and (4.41) this gives

$$\pi^{(3)} = \pi^{(3)}(\lambda_P, \phi_S, t_P, z)\,, \tag{4.44}$$

$$\overline{w^{(3)}}^{\lambda_S} = 0\,, \tag{4.45}$$

$$\overline{v^{(1)}}^{\lambda_S} = \overline{v^{(1)}}^{\lambda_S}(\lambda_P, \phi_S, t_P, z)\,. \tag{4.46}$$

We average the continuity equation (4.23) over λ_S and make use of (4.40), we obtain

$$\frac{\rho^{(0)}}{a\cos\phi_P}\left(\frac{\partial}{\partial\lambda_P}\overline{u^{(1)}}^{\lambda_S} + \frac{\partial}{\partial\phi_S}\cos\phi_P\overline{v^{(2)}}^{\lambda_S}\right) - \frac{\tan\phi_P\rho^{(0)}}{a}\frac{\partial}{\partial\phi_S}\overline{v^{(1)}}^{\lambda_S}\phi_S + \frac{\partial}{\partial z}\rho^{(0)}\overline{w^{(4)}}^{\lambda_S} = 0\,. \tag{4.47}$$

By applying $-\frac{\partial}{\partial\phi_S}$ and $\frac{\partial}{\partial\lambda_P}$ to the averaged in λ_S (4.16) and (4.15) and combining the result with (4.47), we obtain a vorticity equation of the form

$$\begin{aligned}\frac{\partial}{\partial t_S}\overline{\zeta_x^{(1)}}^{\lambda_S} + \frac{\partial}{\partial t_P}\zeta_x^{(0)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\zeta_x^{(0)} + \frac{\overline{v^{(1)}}^{\lambda_S}}{a}\frac{\partial}{\partial\phi_S}\zeta_x^{(0)} \\ +\beta\overline{v^{(1)}}^{\lambda_S} = \frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\rho^{(0)}\overline{w^{(4)}}^{\lambda_S}\,.\end{aligned} \tag{4.48}$$

The averaged potential temperature equation (4.25) reads

$$\frac{\partial}{\partial t_S}\overline{\Theta^{(4)}}^{\lambda_S} + \frac{\partial}{\partial t_P}\Theta^{(3)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}\Theta^{(3)} + \frac{\overline{v^{(1)}}^{\lambda_S}}{a}\frac{\partial}{\partial \phi_S}\Theta^{(3)} + \overline{w^{(4)}}^{\lambda_S}\frac{\partial}{\partial z}\Theta^{(2)} = 0\,. \tag{4.49}$$

From the results above we can derive a PV equation

$$\frac{\partial}{\partial t_S}\overline{PV^{(4)}}^{\lambda_S} + \frac{\partial}{\partial t_P}PV^{(3)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}PV^{(3)} + \frac{\overline{v^{(1)}}^{\lambda_S}}{a}\frac{\partial}{\partial \phi_S}PV^{(3)} + \beta\overline{v^{(1)}}^{\lambda_S} = 0\,, \tag{4.50}$$

where

$$PV^{(i)} = \left(\zeta_x^{(i-3)} + \frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}\Theta^{(i)}}{\partial\Theta^{(2)}/\partial z}\right)\,. \tag{4.51}$$

Since $\overline{v^{(1)}}^{\lambda_S}, \pi^{(3)}$ and $\Theta^{(3)}$ are functions only of λ_P, ϕ_S, t_P and z, it can be shown from (4.50), applying the sublinear growth condition it t_S, that

$$\frac{\partial}{\partial t_S}\overline{PV^{(4)}}^{\lambda_S} = 0\,, \tag{4.52}$$

$$\frac{\partial}{\partial t_P}PV^{(3)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}PV^{(3)} + \frac{\overline{v^{(1)}}^{\lambda_S}}{a}\frac{\partial}{\partial \phi_S}PV^{(3)} + \beta\overline{v^{(1)}}^{\lambda_S} = 0\,. \tag{4.53}$$

Using the averaged versions of (4.15) and (4.21), we can write (4.52) in terms of $\pi^{(4)}$ as

$$\frac{\partial}{\partial t_S}\left(\frac{1}{f_0 a^2}\frac{\partial^2}{\partial \phi_S^2}\overline{\pi^{(4)}}^{\lambda_S} + \frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}\partial\overline{\pi^{(4)}}^{\lambda_S}/\partial z}{\partial\Theta^{(2)}/\partial z}\right) = 0\,. \tag{4.54}$$

It follows that $\overline{\pi^{(4)}}^{\lambda_S}$ does not depend on t_S. On the other hand (4.53) gives us the evolution on planetary time scale of the (λ_P, ϕ_S) spatial structure of $\pi^{(3)}$. Knowing this distribution, $\Theta^{(3)}, u^{(0)}$ and $\overline{v^{(1)}}^{\lambda_S}$ can be calculated.

Next, we discuss some dispersion properties of the equations.

Barotropic Case

We introduce the Mercator coordinates X_P, y_S (Holton, 1992), given through

$$\partial X_P = a \cos\phi_P \partial\lambda_P \,, \tag{4.55}$$

$$\partial y_S = a \partial\phi_S \,. \tag{4.56}$$

We consider the barotropic version of the t_S averaged vorticity equation (4.48), in the new coordinates it reads

$$\frac{\partial}{\partial t_P}\zeta_x^{(0)} + u^{(0)}\frac{\partial}{\partial X_P}\zeta_x^{(0)} + \overline{v^{(1)}}^{\lambda_S}\frac{\partial}{\partial y_S}\zeta_x^{(0)} + \beta\overline{v^{(1)}}^{\lambda_S} = 0\,. \tag{4.57}$$

We look for solutions of the form

$$\pi^{(3)} = \Pi(kX_P - \omega t_P)e^{ily_S}\,. \tag{4.58}$$

Substituting the ansatz in the vorticity equation, one can show that the nonlinear terms drop out

$$\overline{v^{(1)}}^{\lambda_S}\frac{\partial}{\partial y_S}\zeta_x + u^{(0)}\frac{\partial}{\partial X_P}\zeta_x = 0\,. \tag{4.59}$$

We end with an equation for the amplitudes $\Pi(kX_P - \omega t_P)$

$$-l^2\frac{\partial}{\partial t_P}\Pi + \beta\frac{\partial}{\partial X_P}\Pi = 0\,. \tag{4.60}$$

From here we obtain the phase speed $c = \frac{\omega}{k}$ of the anisotropic barotropic Rossby waves

$$c = -\frac{\beta}{l^2}\,, \quad \omega = -\frac{\beta k}{l^2}\,. \tag{4.61}$$

This dispersion relation can be considered as the $k \to 0$ limit of the classical Rossby wave dispersion relation $c = -\beta/(k^2 + l^2)$. If we linearize the barotropic vorticity equation (4.57) about a zonal mean state $\overline{u}$ and look for stationary solutions of the form $\pi^{(3)} = \Pi(X_P)e^{ily_S}$, we obtain

$$\overline{u} - \frac{\beta}{l^2} = 0\,. \tag{4.62}$$

For a given background zonal flow $\overline{u}$, the last result can be interpreted as a constraint on the meridional wavenumber l for which the Rossby waves became stationary. Eq. (4.62) can be obtained by letting $k \to 0$ in the corresponding classical result; which shows some consistency between the considered regime and the QG theory. Nevertheless, it is important to state that both results in (4.61), (4.62) were derived under much weaker assumptions when comparing them with the QG theory. In the QG theory the wavy ansatz is applied for the zonal structure of the solution as well. In our regime Π can be arbitrary for (4.62) to be valid and has to satisfy only $\Pi = \Pi(kX_P - \omega t_P)$ for (4.61). As a consequence the derived dispersion relations can be applied to a larger class of problems.

4.2.2 Plane geometry, Regime (X, x, y, t_P, t_S)

Here we consider the special case when we have a plane geometry, we convert the spherical coordinates to Cartesian ones. For simplicity we set $\rho^{(0)}$ and $\frac{\partial}{\partial z}\Theta^{(2)}$ to one.

Notation

$$\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi, \boldsymbol{e}_r \to \boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z \tag{4.63}$$

$$\lambda_P, \lambda_S, \phi_S \to X, x, y \tag{4.64}$$

$$\frac{1}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}, \frac{1}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}, \frac{1}{a}\frac{\partial}{\partial \phi_S} \to \frac{\partial}{\partial X}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \tag{4.65}$$

$$\rho^{(0)}, \frac{\partial}{\partial z}\Theta^{(2)} = 1 \tag{4.66}$$

$$\frac{d}{dt_{S,P}} = \left(\frac{\partial}{\partial t_{S,P}} + \boldsymbol{u}^{(0)} \cdot \nabla_{S,P}\right) \tag{4.67}$$

$$\tilde{\nabla}_S = \nabla_S + \boldsymbol{e}_z \frac{\partial}{\partial z}\,, \tag{4.68}$$

$$\tilde{\Delta}_S = \frac{1}{f_0}\Delta_S + f_0 \frac{\partial^2}{\partial z^2}\,. \tag{4.69}$$

For the sake of completeness we write the leading order solution from Section 4.1.1

$$\frac{d}{dt_S} q^{(3)} = 0\,. \tag{4.70}$$

where

$$q^{(3)} = \zeta^{(0)} + f_0 \frac{\partial}{\partial z}\Theta^{(3)} + \beta y = \tilde{\Delta}_S \pi^{(3)} + \beta y\,. \tag{4.71}$$

Next, we rewrite (4.25) and (4.37) in the new Cartesian coordinates, the metric terms and all terms resulting from the expansion of trigonometric functions, except for the expansion of the Coriolis parameter, vanish. Combining the two equations into one PV equation gives

$$\begin{aligned}\frac{d}{dt_S}\, q^{(4)} + \boldsymbol{u}^{(1)} \cdot \nabla_S q^{(3)} + w^{(3)} \frac{\partial}{\partial z} q^{(3)} + f_0 \frac{\partial w^{(3)}}{\partial z}\frac{\partial \Theta^{(3)}}{\partial z} - \zeta_a^{(0)} \frac{\partial}{\partial z} w^{(3)} + f_0 \frac{\partial \boldsymbol{u}^{(1)}}{\partial z} \cdot \nabla_S \Theta^{(3)} \\ + f_0 \frac{\partial \boldsymbol{u}^{(0)}}{\partial z} \cdot \nabla_S \Theta^{(4)} + \boldsymbol{e}_z \cdot \nabla_S w^{(3)} \times \frac{\partial}{\partial z}\boldsymbol{u}^{(0)} = -\frac{d}{dt_P} q^{(3)} - \frac{d}{dt_S}\frac{\partial}{\partial X} v^{(0)}\,,\end{aligned} \tag{4.72}$$

where we have used the definitions

$$\zeta_a^{(0)} = \zeta^{(0)} + \beta y\,, \tag{4.73}$$

$$q^{(4)} = \zeta^{(1)} + f_0 \frac{\partial}{\partial z}\Theta^{(4)} - \frac{f_0}{2} y^2\,. \tag{4.74}$$

Using (4.24) and (4.26), we combine the fourth and fifth term on the left hand side of (4.72) as

$$\begin{aligned} f_0 \frac{\partial w^{(3)}}{\partial z}\frac{\partial \Theta^{(3)}}{\partial z} - \zeta_a^{(0)} \frac{\partial}{\partial z} w^{(3)} &= \frac{\partial \Theta^{(3)}}{\partial z}\frac{d}{dt_S}\zeta_a^{(0)} + \zeta_a^{(0)} \frac{d}{dt_S}\frac{\partial \Theta^{(3)}}{\partial z} \\ = \frac{d}{dt_S}\left(\zeta_a^{(0)} \frac{\partial \Theta^{(3)}}{\partial z}\right) &= \frac{d}{dt_S}\left(\left(\frac{1}{f_0}\Delta_S \pi^{(3)} + \beta y\right)\frac{\partial^2 \pi^{(3)}}{\partial z^2}\right)\end{aligned} \tag{4.75}$$

We express the last three terms on the left hand side of (4.72) in terms of $\pi^{(3)}$ and $\boldsymbol{u}^{(0)}$, too. Applying (4.14), (4.15) and (4.24) we obtain

$$\begin{aligned} f_0 \frac{\partial \boldsymbol{u}^{(1)}}{\partial z} \cdot \nabla_S \Theta^{(3)} + f_0 \frac{\partial \boldsymbol{u}^{(0)}}{\partial z} \cdot \nabla_S \Theta^{(4)} &= -\frac{d}{dt_S}\frac{1}{2f_0}\left(\nabla_S \frac{\partial}{\partial z}\pi^{(3)}\right)^2 \\ -\frac{\partial}{\partial x}\left(\frac{\partial \pi^{(3)}}{\partial z}\right)\frac{\partial \boldsymbol{u}^{(0)}}{\partial z} \cdot \nabla_S \frac{\partial \pi^{(3)}}{\partial x} &- \frac{\partial}{\partial y}\left(\frac{\partial \pi^{(3)}}{\partial z}\right)\frac{\partial \boldsymbol{u}^{(0)}}{\partial z} \cdot \nabla_S \frac{\partial \pi^{(3)}}{\partial y}\,,\end{aligned} \tag{4.76}$$

$$\begin{aligned} \boldsymbol{e}_z \cdot \nabla_S w^{(3)} \times \frac{\partial}{\partial z}\boldsymbol{u}^{(0)} &= -\frac{d}{dt_S}\frac{1}{2f_0}\left(\nabla_S \frac{\partial}{\partial z}\pi^{(3)}\right)^2 \\ -\frac{\partial}{\partial x}\left(\frac{\partial \pi^{(3)}}{\partial z}\right)\frac{\partial \boldsymbol{u}^{(0)}}{\partial x} \cdot \nabla_S \frac{\partial \pi^{(3)}}{\partial z} &- \frac{\partial}{\partial y}\left(\frac{\partial \pi^{(3)}}{\partial z}\right)\frac{\partial \boldsymbol{u}^{(0)}}{\partial y} \cdot \nabla_S \frac{\partial \pi^{(3)}}{\partial z}\,.\end{aligned} \tag{4.77}$$

Further, it can be easily shown that the last two terms on the right hand side of (4.76) and of (4.77) will cancel when summed. With the help of the results above, (4.72) takes the form

$$\frac{d}{dt_S}\, q^{(4)} + \boldsymbol{u}^{(1)} \cdot \nabla_S q^{(3)} = S_{qg} - \frac{d}{dt_P} q^{(3)} - \frac{d}{dt_S}\frac{\partial}{\partial X} v^{(0)}\,, \tag{4.78}$$

where we have used the definition

$$S_{qg} = -w^{(3)}\frac{\partial}{\partial z} q^{(3)} - \frac{d}{dt_S}\left(\zeta_a^{(0)}\frac{\partial \Theta^{(3)}}{\partial z}\right) + \frac{1}{f_0}\frac{d}{dt_S}\left(\nabla_S \frac{\partial}{\partial z}\pi^{(3)}\right)^2 . \tag{4.79}$$

Since we have the identity

$$\frac{1}{f_0}\frac{d}{dt_S}\left(\nabla_S \frac{\partial}{\partial z}\pi^{(3)}\right)^2 = \frac{1}{f_0}\frac{d}{dt_S}\left(\frac{\partial v^{(0)}}{\partial z}\frac{\partial \Theta^{(3)}}{\partial x} - \frac{\partial v^{(0)}}{\partial y}\frac{\partial \Theta^{(3)}}{\partial y}\right), \tag{4.80}$$

the last two terms in the definition of S_{qg} can be written as the advection of a PV type quantity

$$\frac{d}{dt_S}\left(\zeta_a^{(0)}\frac{\partial \Theta^{(3)}}{\partial z}\right) + \frac{1}{f_0}\frac{d}{dt_S}\left(\nabla_S \frac{\partial}{\partial z}\pi^{(3)}\right)^2 = \frac{d}{dt_S}\left(\left(\tilde{\nabla}_S \times \boldsymbol{u}^{(0)} + \boldsymbol{e}_z \beta y\right) \cdot \tilde{\nabla}_S \Theta^{(3)}\right) . \tag{4.81}$$

We briefly discuss the different terms in (4.78). The variable $q^{(3)}$ can be interpreted as the leading order synoptic scale PV, it is advected by the vertical motion and by the ageostrophic velocity field $\boldsymbol{u}^{(1)}$. The second term on the right hand side of (4.78) represents the planetary evolution ($\frac{d}{dt_P}$) of $q^{(3)}$. The variables $q^{(4)}$ and $(\tilde{\nabla}_S \times \boldsymbol{u}^{(0)} + \boldsymbol{e}_z \beta y) \cdot \tilde{\nabla}_S \Theta^{(3)}$ can be considered as the next order corrections to $q^{(3)}$; in (4.78) the dynamics of these variables evolves only on the synoptic scale ($\frac{d}{dt_S}$). Finally, the third term on the right hand side of (4.78) can be interpreted as the synoptic evolution of the planetary scale vorticity ($\frac{\partial}{\partial X} v^{(0)}$). Thus, equation (4.78) describes the coupling between the synoptic scale dynamics of the next order PV corrections,the planetary scale dynamics of the leading order PV and the synoptic evolution of the planetary scale vorticity. The equation however is not closed, since the synoptic structure of $q^{(4)}$ and the planetary structure of $q^{(3)}$ are unknown. The closure can be achieved by applying a solvability condition, i.e., the sublinear growth condition, we will return to this issue later on.

In a way similar to the QG theory we want to construct the synoptic distributions of the variables $\boldsymbol{u}^{(1)}, w^{(3)}$ and $\Theta^{(4)}$ from $q^{(4)}$. Making use of the three-dimensional Helmholtz decomposition introduced in Muraki et al. (1999) (see eq.(24) there), we can express all higher order variables in terms of a gradient potential $\Phi^{(4)}$ and a curl potential with the components $F^{(4)}$ and $G^{(4)}$

$$u^{(1)} = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi^{(4)} - f_0\frac{\partial}{\partial z}F^{(4)}\,, \tag{4.82}$$

$$v^{(1)} = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi^{(4)} + \frac{1}{f_0}\frac{\partial}{\partial X}\pi^{(3)} - f_0\frac{\partial}{\partial z}G^{(4)}\,, \tag{4.83}$$

$$\Theta^{(4)} = \frac{\partial}{\partial z}\Phi^{(4)} + \frac{\partial}{\partial x}G^{(4)} - \frac{\partial}{\partial y}F^{(4)}\,. \tag{4.84}$$

In order to be consistent with the asymptotic results, we introduced in the decomposition of $v^{(1)}$ the additional term $\frac{1}{f_0}\frac{\partial \pi^{(3)}}{\partial X}$. Consider the case when all variables (the potentials $\Phi^{(4)}, F^{(4)}, G^{(4)}$ too) do not depend on the synoptic scales x, y, t_S. Then it can easily be shown that $\boldsymbol{u}^{(0)}, w^{(3)} = 0$ and we obtain that $v^{(1)}$ is geostrophically balanced with respect to $\frac{1}{f_0}\frac{\partial \pi^{(3)}}{\partial X}$, which is guaranteed by the ansatz (4.83).

Differentiating the Cartesian versions of the momentum equations (4.16), (4.17) with respect to z and the Cartesian version of the temperature equation (4.24) with respect to x or y, one can derive elliptic equations for the potentials $F^{(4)}$ and $G^{(4)}$

$$\tilde{\Delta}_S F^{(4)} = -\frac{1}{f_0^2}\frac{\partial}{\partial x}\left(\frac{d}{dt_S}\Theta^{(3)}\right) - \frac{\beta y}{f_0}\frac{\partial}{\partial y}\Theta^{(3)} + \frac{1}{f_0}\frac{\partial}{\partial z}\left(\frac{d}{dt_S}v^{(0)}\right)\,, \tag{4.85}$$

$$\tilde{\Delta}_S G^{(4)} = -\frac{1}{f_0^2}\frac{\partial}{\partial y}\left(\frac{d}{dt_S}\Theta^{(3)}\right) + \frac{\beta y}{f_0}\frac{\partial}{\partial x}\Theta^{(3)} - \frac{1}{f_0}\frac{\partial}{\partial z}\left(\frac{d}{dt_S}u^{(0)}\right)\,. \tag{4.86}$$

We identify the right hand sides of the equations above as the components of the $\boldsymbol{Q}$ vector (Holton, 1992). Rewriting the continuity equation (4.22) in Cartesian coordinates and making use of (4.82) and (4.83), we derive a diagnostic relation for $w^{(3)}$

$$w^{(3)} = f_0\left(\frac{\partial}{\partial x}F^{(4)} + \frac{\partial}{\partial y}G^{(4)}\right)\,. \tag{4.87}$$

After applying $\tilde{\Delta}_S$ to the last equation, we will obtain the well known Omega equation (Holton, 1992). With the help of the Helmholtz decomposition we express $q^{(4)}$ in terms of the new potential $\Phi^{(4)}$

$$q^{(4)} = \tilde{\Delta}_S\Phi^{(4)} + \frac{1}{f_0}\frac{\partial}{\partial X}\frac{\partial}{\partial x}\pi^{(3)} - \frac{f_0}{2}y^2\,. \tag{4.88}$$

The conservation of $q^{(3)}$ (4.70) gives the synoptic scale structure of $\pi^{(3)}$. Ones we have $\pi^{(3)}$, we can evaluate $\boldsymbol{u}^{(0)}$ and $\Theta^{(3)}$ and hence the right hand side of (4.85) and (4.86). Inverting the Laplacians we can determine the synoptic structure of $F^{(4)}$ and $G^{(4)}$. From (4.87) we can find $w^{(3)}$ and all the fields needed to calculate S_{qg} in (4.78). We assume for a moment that we know the planetary scale structure of $\pi^{(3)}$ and its evolution on the t_P time scale. Provided with appropriate initial conditions, we can simply integrate (4.78) forward in time, determine $q^{(4)}$ and after inverting (4.88) find the new value of $\Phi^{(4)}$. Since we can determine the fields $F^{(4)}$ and $G^{(4)}$ by integrating (4.70) in time and solving (4.85) and (4.86), we can evaluate the velocity and temperature corrections $\boldsymbol{u}^{(1)}$ and $\Theta^{(4)}$ at the next time step. Repeating this procedure iteratively, we can solve (4.70) and (4.78).

The planetary scale structure of $\pi^{(3)}$ and its evolution on the t_P time scale have to be determined from a solvability condition applied to the right hand side of (4.78). The left hand side of (4.78) can be considered as a linear operator acting on $\Phi^{(4)}$. The sublinear growth condition states that we have to suppress terms on the right hand side which will lead to an unbounded growth of the solution $\Phi^{(4)}$. In other words we have to remove the terms exciting the linear operator with its eigenfrequency and thus leading to a resonant behavior. The solvability condition should lead to separation of (4.78) into two equations, one for the synoptic scale dynamics of $q^{(4)}$ and one for the planetary scale dynamics of $q^{(3)}$, compare with the condition for the two scale model in Section 3.2. Unfortunately, the author has been unsuccessful to perform this splitting in the presented regime, except for the special case of a meridionally averaged model. Applying the usual procedure by averaging over x_S and t_S, meridional fluxes of the form $\frac{\partial}{\partial y}\overline{v^{(0)}q^{(4)}}^{x,t_S}$ will remain not closed in the equation for $q^{(3)}$. Such fluxes can be parameterized from observational data.

4.3 Discussion

In this chapter we considered motions with synoptic meridional extent and with zonal and temporal variations on the synoptic and on the planetary scales. We assumed that the background potential temperature distribution does not depend on the synoptic scale and its fluctuations are at most $\mathcal{O}(\varepsilon^2)$ to be consistent with the QG scaling.

We compare the leading order reduced equations for the APR and for the two scale PR from Chapter 3.2. After applying a solvability condition, we have shown that in the present regime the background potential temperature distribution remains constant, whereas in the PR it is governed by the PGEs. Further, the synoptic dynamics in the APR is completely described by the classical QG theory and all planetary interaction terms from the two scale PR vanish.

In Section 4.2.1 we discussed the APR when $\mathcal{O}(\varepsilon)$ meridional velocities are assumed. We have shown that in this case the restoring β force enters the barotropic vorticity equation (4.57) and affects the slow time evolution of the synoptic vorticity. The dispersion properties of (4.57) indicate that Rossby waves are allowed as solutions, these waves are consistent with the classical Rossby waves in the long wavelength limit. Despite the fact that we resolved two zonal scales in Section 4.2.1, we obtain as a result that the leading order pressure does not depend on λ_S. Equation (4.53) which describes the evolution of that pressure on the slow planetary time scale, remains the same if we redo the analysis (not shown here) without the coordinate λ_S. This means that under the assumption of smaller meridional velocities, there is no net influence from the synoptic zonal scale on the planetary dynamics.

If spherical effects are neglected, eq. (4.53) takes the same form as the model of Dickinson (1968a) for stratospheric disturbances with planetary zonal scales. In this model the time and meridional coordinates are rescaled in such a way as to guarantee that the horizontal advection of relative vorticity balances the planetary vorticity advection. This implies zonal velocities of the order of 50 m/s and a Oboukhov meridional scale, if the value of the stratospheric static stability is used for the scaling. If instead of it the value for the troposphere is substituted, the rescaled length and time scales match exactly those from Section 4.2.1, the zonal velocities become 10 m/s and the meridional are an order of magnitude smaller.

The assumption of small meridional velocities was dropped in Section 4.2.2. In the case of a plane geometry we derived a PV type equation (4.78) for the next order dynamics (as already mentioned, in the general case we obtain the QG theory as leading order solution). This equation describes a coupling between the planetary evolution of the leading order PV field, the synoptic evolution of the planetary scale vorticity field and the synoptic dynamics of higher order PV corrections.

If we leave the planetary scale dependence in (4.78) out, we obtain the $\mathcal{O}(\varepsilon)$ corrections to the QG model, also known as the QG^{+1} model (Muraki et al., 1999). Similarly to the QG theory, QG^{+1} describes balanced flows and the dynamics is completely determined from the advection of the PV type variable $q^{(4)}$. A discussion of analytical solutions of the QG^{+1} model for the finite-amplitude Eady edge wave can be found in Muraki et al. (1999). In the last paper the authors also address the issue of boundary conditions and solvability. The region of validity of QG^{+1} equations is restricted only to the synoptic scale, however, these equations contain higher order effects not present in the QG model. Rotunno et al. (2000) demonstrated that these effects explain the mesoscale structure of the synoptic eddies and features of the frontogenesis processes. Studying numerically unstable baroclinic waves, they showed that the QG^{+1} equations account for the asymmetries between cyclones and anticyclones or between the cold and warm fronts. The QG^{+1} model can be also viewed as a tool for observational analysis, since its mathematical structure is relatively simple and it incorporates some well known diagnostic relations, e.g., the omega equation.

There is one essential difference in the derivation of the QG^{+1} model presented here and the one in Muraki et al. (1999) – the asymptotic expansions start from different equations. The latter QG^{+1} model is derived by QG rescaling of the primitive equations (PE) and of the Ertel's PV equation, assuming a hydrostatic Boussinesq fluid on a f plane. A priori it is not clear that the rescaled PV equation will be the same as the one derived from the rescaled PE, it is an implicit assumption that both equations are consistent. In the derivation presented here we do not make use of the additional PV equation. Moreover, we can show that from the leading two systems of asymptotic equations, one can derive the same PV equations as those from the asymptotic expansion of the PV equation in Muraki et al. (1999).

Another difference in our derivation is that we do not have to assume that the fluid is in hydrostatic balance, this can be shown as a leading order result. Of course the hydrostatic balance comes automatically from the vertical momentum equation when applying a QG scaling, but a rescaled general Ertel's PV differs from the PV in Muraki et al. (1999) derived under the hydrostatic approximation. In the definition of the latter PV there are no contributions from the vertical winds. It turns out that such terms can be neglected when $\mathcal{O}(\varepsilon)$ corrections to the QG are considered, however they will become important for the construction of higher order corrections. Finally, equations (4.37) and (4.25) can be regarded as a generalization of the QG^{+1} model to spherical geometry. The three-dimensional Helmholtz decomposition of the variables can be applied in the same way as in the case of plane geometry.

Chapter 5

Planetary Regime with Background Flow

In the Planetary Regime with Background Flow (PRBF) we consider systematically larger variations of the background potential temperature, namely $\mathcal{O}(\varepsilon)$. A justification for this can be found in Section 5.1. The stronger temperature gradients imply zonal velocities of the order of the jets, because of them the planetary scale dynamics in this regime evolves on the fast (synoptic) time scale, see Fig 1.3. Using an asymptotic ansatz resolving both the planetary and synoptic scales we derive in Section 5.2 a hierarchy of reduced equations. We show in Section 5.3 that the leading order equations determine the vertical structure of the solution. Its temporal and spatial structure enters the next order equations. These equations are presented in Section 5.4 and we consider them under the Boussinesq approximation (Section 5.4.1). The chapter ends with a summary and a discussion of the results.

5.1 Coordinates scaling and a priori assumptions

In this chapter we use the two scale asymptotic ansatz from Section 3.2, resolving the planetary and the synoptic spatial and temporal variations.

A priori assumptions

Observations of the potential temperature distribution (Gill, 2003; Peixoto and Oort, 1992) reveal the following key features: i) large equator to pole and surface to tropopause temperature differences $\delta\Theta \sim 40-60\,\mathrm{K}$ evolving on a seasonal time scale, the nondimensional order of these temperature differences is $\delta\Theta/\Theta_{ref} \sim 1/6 \sim \mathcal{O}(\varepsilon)$ ($\Theta_{ref} = 300\,\mathrm{K}$); ii) an order of magnitude smaller zonal variations on the planetary scale; iii) $\mathcal{O}(\varepsilon^3)$ synoptic scale variations, see the a priori assumptions from Sections 3.1, 3.2.1; iv) quasi linear vertical structure (Petoukhov et al., 2000; Mokhov and Akperov, 2006). The properties i)-iv) motivate the following asymptotic expansion for the potential temperature

$$\theta = 1 + \varepsilon\Theta^{(1)}(\phi_P, z) + \varepsilon^2\Theta^{(2)}(\lambda_P, \phi_P, z, t_P, t_S) + \varepsilon^3\Theta^{(3)}(\lambda_P, \phi_P, \lambda_S, \phi_S, z, t_P, t_S) + \mathcal{O}(\varepsilon^4)\,. \tag{5.1}$$

Here $\Theta^{(1)}$ represents a prescribed (since it evolves on a much longer time scale than the one considered here) background potential temperature profile of the form

$$\Theta^{(1)}(\phi_P, z) = \Theta_h(\phi_P) + Nz\,, \tag{5.2}$$

where the meridional structure $\Theta_h(\phi_P)$ and the constant lapse rate N are given. As we will show later, $\mathcal{O}(\varepsilon)$ temperature variations induce through the hydrostatic and geostrophic balance strong zonal winds of the order of $\varepsilon^{-1}u_{ref}$. Such winds are comparable in magnitude with the atmospheric jets. We use for the horizontal wind the expansion

$$\boldsymbol{u} = \varepsilon^{-1}\boldsymbol{u}^{(-1)}(\lambda_P, \phi_P, z, t_P, t_S) + \boldsymbol{u}^{(0)} + \varepsilon\boldsymbol{u}^{(1)} + \mathcal{O}\left(\varepsilon^3\right)\,, \tag{5.3}$$

where $\boldsymbol{u}^{(i)}$ for $i \geq 0$ depends on both the planetary and synoptic scales. Since $\varepsilon^{-1}u_{ref}$ planetary scale surface winds are not observed and the meridional variations of the zonally averaged surface pressure are small (Peixoto and Oort, 1992, p. 146): $\delta p \sim 5 - 10\,\text{hPa} \sim \mathcal{O}(\varepsilon^2 - \varepsilon^3)$, we set

$$\pi^{(1)}(\phi_P, z = 0) = 0\,, \tag{5.4}$$

as a lower boundary condition for the model.

5.2 Derivation of the Planetary Regime with Background Flow

Notation

We use the following notation

$$
\begin{aligned}
(\lambda_S, \phi_S), (\lambda_P, \phi_P) &\to \boldsymbol{X}_S, \boldsymbol{X}_P && (5.5)\\
f &= \sin\phi_P\,, && (5.6)\\
\beta &= \frac{1}{a}\frac{\partial}{\partial\phi_P}\sin\phi_P\,, && (5.7)\\
\nabla_{S,P} &= \frac{\boldsymbol{e}_\lambda}{a\cos\phi_P}\frac{\partial}{\partial\lambda_{S,P}} + \frac{\boldsymbol{e}_\phi}{a}\frac{\partial}{\partial\phi_{S,P}}\,, && (5.8)\\
\Delta_{S,P} &= \frac{1}{a^2\cos^2\phi_P}\left(\frac{\partial^2}{\partial{\lambda_{S,P}}^2} + \cos\phi_P\frac{\partial}{\partial\phi_{S,P}}\left(\cos\phi_P\frac{\partial}{\partial\phi_{S,P}}\right)\right)\,, && (5.9)\\
\nabla_{S,P}\cdot\boldsymbol{u} &= \frac{1}{a\cos\phi_P}\left(\frac{\partial u}{\partial\lambda_{S,P}} - \frac{\partial v\cos\phi_P}{\partial\phi_{S,P}}\right)\,, && (5.10)\\
\boldsymbol{e}_r\cdot(\nabla_{S,P}\times\boldsymbol{u}) &= \frac{1}{a\cos\phi_P}\left(\frac{\partial v}{\partial\lambda_{S,P}} - \frac{\partial u\cos\phi_P}{\partial\phi_{S,P}}\right)\,, && (5.11)\\
\boldsymbol{u} &= \boldsymbol{e}_\lambda u + \boldsymbol{e}_\phi v\,. && (5.12)
\end{aligned}
$$

Asymptotic expansion

We summarize the leading order equations resulting from the asymptotic expansion, again we omit for simplicity the momentum and diabatic source terms.

Horizontal momentum balance

At leading order we obtain a geostrophically balanced zonal velocity $u^{(-1)}$ and vanishing $v^{(-1)}$ component

$$\mathcal{O}(1): \qquad u^{(-1)}(\phi_P, z) = -\frac{1}{fa}\frac{\partial}{\partial \phi_P}\pi^{(1)}(\phi_P, z)\,, \tag{5.13}$$

$$\mathcal{O}(1): \qquad v^{(-1)} = 0\,. \tag{5.14}$$

The $\boldsymbol{u}^{(0)}$ components of the velocity are no longer in geostrophic balance, since terms like the synoptic scale advection by $u^{(-1)}$ and the metric terms appear at the same order as the Coriolis force and the pressure gradient

$$\mathcal{O}(\varepsilon^1): \qquad \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}u^{(0)} - \sin\phi_P v^{(0)} = \frac{1}{a\cos\phi_P}\left(-\frac{\partial}{\partial \lambda_S}\pi^{(3)} - \frac{\partial}{\partial \lambda_P}\pi^{(2)}\right)\,, \tag{5.15}$$

$$\begin{aligned}\mathcal{O}(\varepsilon^1): \qquad & \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}v^{(0)} + \frac{u^{(-1)}u^{(-1)}\tan\phi_P}{a} + \sin\phi_P u^{(0)} \\ & = \frac{1}{a}\left(-\frac{\partial}{\partial \phi_S}\pi^{(3)} - \frac{\partial}{\partial \phi_P}\pi^{(2)} + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial \phi_P}\pi^{(1)}\right)\,.\end{aligned} \tag{5.16}$$

The $\mathcal{O}(\varepsilon^2)$ momentum equations read

$$\begin{aligned}\mathcal{O}(\varepsilon^2): \qquad & \frac{\partial}{\partial t_S}u^{(0)} + \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}u^{(1)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}u^{(0)} + \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}u^{(0)} \\ & + \frac{v^{(0)}}{a}\frac{\partial}{\partial \phi_S}u^{(0)} + \frac{v^{(0)}}{a}\frac{\partial}{\partial \phi_P}u^{(-1)} + w^{(3)}\frac{\partial}{\partial z}u^{(-1)} - \frac{u^{(-1)}v^{(0)}\tan\phi_P}{a} \\ & - \sin\phi_P v^{(1)} = \frac{1}{a\cos\phi_P}\left(-\frac{\partial}{\partial \lambda_S}\pi^{(4)} + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial \lambda_S}\pi^{(3)} - \frac{\partial}{\partial \lambda_P}\pi^{(3)} + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial \lambda_P}\pi^{(2)}\right)\,,\end{aligned} \tag{5.17}$$

$$\begin{aligned}\mathcal{O}(\varepsilon^2): \qquad & \frac{\partial}{\partial t_S}v^{(0)} + \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}v^{(1)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}v^{(0)} + \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}v^{(0)} \\ & + \frac{v^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(0)} + \frac{2u^{(-1)}u^{(0)}\tan\phi_P}{a} + \sin\phi_P u^{(1)} = \frac{1}{a}\left(-\frac{\partial}{\partial \phi_S}\pi^{(4)}\right. \\ & \left. + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial \phi_S}\pi^{(3)} - \frac{\partial}{\partial \phi_P}\pi^{(3)} + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial \phi_P}\pi^{(2)} + \left(-\frac{{\rho^{(1)}}^2}{{\rho^{(0)}}^2} + \frac{\rho^{(2)}}{\rho^{(0)}}\right)\frac{\partial}{\partial \phi_P}\pi^{(1)}\right)\,.\end{aligned} \tag{5.18}$$

In the last two equations we can identify all terms in (3.76) and (3.77). In addition we have the $u^{(-1)}$ advection terms and the planetary gradient of $\pi^{(1)}$.

Vertical momentum balance

Pressure and density variations up to the order of ε^3 are hydrostatically balanced.

$$\frac{\partial}{\partial z}p^{(i)} = -\rho^{(i)}, \quad i = 0, \dots 3\,. \tag{5.19}$$

In the next order equation the hydrostatic balance of $p^{(4)}$ and $\rho^{(4)}$ is disturbed by the Coriolis force resulting from the horizontal component of the earth's rotation vector.

$$\mathcal{O}(1): \qquad \rho^{(0)}\boldsymbol{u}^{(-1)}\cos\phi_P = \frac{\partial}{\partial z}p^{(4)} + \rho^{(4)}\,. \tag{5.20}$$

Making use of the ideal gas law (see Section 3.1.1, but now $\Theta^{(1)} \neq 0$), we obtain from the first nontrivial equations in the vertical momentum balance

$$\rho^{(0)} = {p^{(0)}}^{1/\gamma} \Rightarrow \rho^{(0)}(z), p^{(0)}(z)\,, \tag{5.21}$$

$$\Theta^{(1)}(\phi_P, z) = \frac{\partial}{\partial z}\pi^{(1)} \Rightarrow \pi^{(1)}(\phi_P, z), \rho^{(1)}(\phi_P, z)\,, \tag{5.22}$$

$$\begin{aligned}\Theta^{(2)} &= \frac{\partial}{\partial z}\pi^{(2)} - \frac{\rho^{(1)}\Theta^{(1)}}{\rho^{(0)}} + \frac{(1-\gamma)z^2{\rho^{(0)}}^2}{2\gamma^2{p^{(0)}}^2}{\Theta^{(1)}}^2\,,\\ &\Rightarrow \pi^{(2)}(\phi_P, \lambda_P, z, t_P, t_S)\end{aligned} \tag{5.23}$$

Potential temperature

The leading order potential temperature equation reduces to

$$\mathcal{O}(\varepsilon^3): \qquad w^{(2)}\frac{\partial}{\partial z}\Theta^{(1)} = 0 \Rightarrow w^{(2)} = 0\,. \tag{5.24}$$

The next order equation takes the form

$$\mathcal{O}(\varepsilon^4): \qquad \frac{u^{(-1)}}{a\cos\phi_P}\left(\frac{\partial}{\partial\lambda_S}\Theta^{(3)} + \frac{\partial}{\partial\lambda_P}\Theta^{(2)}\right) + \frac{v^{(0)}}{a}\frac{\partial}{\partial\phi_P}\Theta^{(1)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(1)} = 0\,. \tag{5.25}$$

The time evolution of $\Theta^{(2)}$ and $\Theta^{(3)}$ on the planetary and synoptic scale, respectively, appears in the next order equation

$$
\begin{aligned}
\mathcal{O}(\varepsilon^5): \qquad & \frac{\partial}{\partial t_S}\Theta^{(3)} + \frac{\partial}{\partial t_P}\Theta^{(2)} + \frac{u^{(-1)}}{a\cos\phi_P}\left(\frac{\partial}{\partial \lambda_S}\Theta^{(4)} + \frac{\partial}{\partial \lambda_P}\Theta^{(3)}\right) \\
& + \frac{u^{(0)}}{a\cos\phi_P}\left(\frac{\partial}{\partial \lambda_S}\Theta^{(3)} + \frac{\partial}{\partial \lambda_P}\Theta^{(2)}\right) + \frac{v^{(0)}}{a}\left(\frac{\partial}{\partial \phi_S}\Theta^{(3)} + \frac{\partial}{\partial \phi_P}\Theta^{(2)}\right) \\
& + \frac{v^{(1)}}{a}\frac{\partial}{\partial \phi_P}\Theta^{(1)} + w^{(4)}\frac{\partial}{\partial z}\Theta^{(1)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)} = 0\,. \qquad (5.26)
\end{aligned}
$$

Higher order unknown variables $\Theta^{(4)}$, $v^{(1)}$ and $w^{(4)}$ appear in this potential temperature equation unlike (3.84).

Continuity equation

Making use of the fact that $u^{(-1)}$ does not depend on the synoptic scales, we obtain from the leading order continuity equation

$$
\mathcal{O}(\varepsilon^1): \qquad \underbrace{\frac{\rho^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}u^{(-1)} + \frac{\rho^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(-1)}}_{=0} + \frac{\partial}{\partial z}\rho^{(0)}w^{(1)} = 0 \qquad \Rightarrow w^{(1)} = 0\,. \qquad (5.27)
$$

Although not geostrophically balanced, the $\boldsymbol{u}^{(0)}$ field is divergence free on the synoptic scale

$$
\mathcal{O}(\varepsilon^2): \qquad \frac{\rho^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_S}u^{(0)} + \frac{\rho^{(0)}}{a}\frac{\partial}{\partial \phi_S}v^{(0)} = 0\,. \qquad (5.28)
$$

The continuity equation imposes a constraint on the synoptic scale structure of $\pi^{(3)}$. Together with (5.15) and (5.16) it gives

$$
\frac{\partial}{\partial \lambda_S}\boldsymbol{e}_r \cdot \left(\nabla_S \times \boldsymbol{u}^{(0)}\right) = \frac{\partial}{\partial \lambda_S}\Delta_S\frac{\pi^{(3)}}{f} = 0\,. \qquad (5.29)
$$

The next order continuity equation reads

$$
\mathcal{O}(\varepsilon^3): \qquad \nabla_P \cdot \rho^{(0)}\boldsymbol{u}^{(0)} + \nabla_S \cdot \rho^{(0)}\boldsymbol{u}^{(1)} + \frac{\partial}{\partial z}\rho^{(0)}w^{(3)} = 0\,. \qquad (5.30)
$$

5.2.1 Averaging over the synoptic scales $\lambda_S,\ \phi_S$ and t_S

Next, we regroup the results from the asymptotic analysis in different equation systems, each containing a momentum, mass and potential temperature balance. In order to see the net effect from the synoptic scale on the planetary scale, we average the equations over $\boldsymbol{X}_S$ and t_S and apply the sublinear growth condition.

We have the prescribed background state

$$u^{(-1)}(\phi_P, z) = -\frac{1}{af}\frac{\partial}{\partial \phi_P}\pi^{(1)}(\phi_P, z)\,, \tag{5.31}$$

$$\Theta^{(1)}(\phi_P, z) = \frac{\partial}{\partial z}\pi^{(1)} \rightarrow \pi^{(1)}(\phi_P, z), \rho^{(1)}(\phi_P, z)\,, \tag{5.32}$$

(5.33)

The first nontrivial system of equations reads

$$\sin\phi_P \overline{v^{(0)}}^S = \frac{1}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}\pi^{(2)}\,, \tag{5.34}$$

$$\frac{u^{(-1)}u^{(-1)}\tan\phi_P}{a} + \sin\phi_P \overline{u^{(0)}}^S = \frac{1}{a}\left(-\frac{\partial}{\partial \phi_P}\pi^{(2)} + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial \phi_P}\pi^{(1)}\right) \tag{5.35}$$

$$\Theta^{(2)} = \frac{\partial}{\partial z}\pi^{(2)} - \frac{\rho^{(1)}\Theta^{(1)}}{\rho^{(0)}} + \frac{(1-\gamma)z^2{\rho^{(0)}}^2}{2\gamma^2{p^{(0)}}^2}{\Theta^{(1)}}^2\,, \tag{5.36}$$

$$\frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial \lambda_P}\Theta^{(2)} + \frac{\overline{v^{(0)}}^S}{a}\frac{\partial}{\partial \phi_P}\Theta^{(1)} + \overline{w^{(3)}}^S\frac{\partial}{\partial z}\Theta^{(1)} = 0\,, \tag{5.37}$$

$$\nabla_P \cdot \rho^{(0)}\overline{\boldsymbol{u}^{(0)}}^S + \frac{\partial}{\partial z}\rho^{(0)}\overline{w^{(3)}}^S = 0\,, \tag{5.38}$$

where the operator $\overline{()}^S$ was defined in (2.37). It is shown in the next Section that the last system of equations determines the vertical structure of $\pi^{(2)}$. The next order equations read

$$\frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\overline{u^{(0)}}^S + \frac{\overline{v^{(0)}}^S}{a}\frac{\partial}{\partial\phi_P}u^{(-1)} + \overline{w^{(3)}}^S\frac{\partial}{\partial z}u^{(-1)} - \frac{u^{(-1)}\overline{v^{(0)}}^S\tan\phi_P}{a}$$
$$-\sin\phi_P\overline{v^{(1)}}^S = \frac{1}{a\cos\phi_P}\left(-\frac{\partial}{\partial\lambda_P}\overline{\pi^{(3)}}^S + \frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial\lambda_P}\pi^{(2)}\right), \tag{5.39}$$

$$\frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\overline{v^{(0)}}^S + \frac{2u^{(-1)}\overline{u^{(0)}}^S\tan\phi_P}{a} + \sin\phi_P\overline{u^{(1)}}^S = -\frac{1}{a}\frac{\partial}{\partial\phi_P}\overline{\pi^{(3)}}^S$$
$$+\frac{1}{a}\left[\frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial\phi_P}\pi^{(2)} + \left(-\frac{{\rho^{(1)}}^2}{{\rho^{(0)}}^2} + \frac{\rho^{(2)}}{\rho^{(0)}}\right)\frac{\partial}{\partial\phi_P}\pi^{(1)}\right], \tag{5.40}$$

$$\frac{\partial}{\partial t_P}\Theta^{(2)} + \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\overline{\Theta^{(3)}}^S + \frac{\overline{u^{(0)}}^S}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\Theta^{(2)} + \frac{\overline{v^{(0)}}^S}{a}\frac{\partial}{\partial\phi_P}\Theta^{(2)}$$
$$+\frac{\overline{v^{(1)}}^S}{a}\frac{\partial}{\partial\phi_P}\Theta^{(1)} + \overline{w^{(4)}}^S\frac{\partial}{\partial z}\Theta^{(1)} + \overline{w^{(3)}}^S\frac{\partial}{\partial z}\Theta^{(2)} = 0, \tag{5.41}$$

$$\overline{\Theta^{(3)}}^S = \frac{\partial}{\partial z}\overline{\pi^{(3)}}^S - \frac{\rho^{(1)}\Theta^{(2)}}{\rho^{(0)}} - \frac{\rho^{(2)}\Theta^{(1)}}{\rho^{(0)}} - \frac{\rho^{(1)}\Theta^{(1)}}{\rho^{(0)}}$$
$$+\frac{(1-\gamma)p^{(1)}p^{(2)}}{\gamma^2{p^{(0)}}^2} + \frac{(1-\gamma)(1-2\gamma){p^{(1)}}^3}{6\gamma^3{p^{(0)}}^3}, \tag{5.42}$$

$$\frac{\partial}{\partial z}\left(\rho^{(0)}\overline{w^{(4)}}^S + \rho^{(1)}\overline{w^{(3)}}^S\right) + \nabla_P\cdot\left(\rho^{(0)}\overline{\boldsymbol{u}^{(1)}}^S + \rho^{(1)}\overline{\boldsymbol{u}^{(0)}}^S\right)$$
$$+\frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\overline{\rho^{(2)}}^S = 0. \tag{5.43}$$

In the above system there is no feedback from the synoptic scale to the planetary scale dynamics, since the averages over the synoptic scale advection terms in (5.17), (5.18) and (5.26) vanish because of (5.28) and of the sublinear growth condition. Following the discussion from Section 3.2.3, we expect that some planetary-synoptic interaction terms will appear in the higher order averaged equations; for example in the form of $\overline{\boldsymbol{u}^{(1)}\cdot\nabla_S(\boldsymbol{e}_r\cdot\nabla_S\times\boldsymbol{u}^{(0)})}^{z,\lambda}$ (the synoptic advection of synoptic scale relative vorticity $\boldsymbol{e}_r\cdot\nabla_S\times\boldsymbol{u}^{(0)}$ by higher order velocity corrections) or $\overline{\nabla_P\cdot\boldsymbol{u}^{(0)}(\boldsymbol{e}_r\cdot\nabla_P\times\boldsymbol{u}^{(0)})}^{z,\lambda}$ (the planetary divergence of the flux of planetary scale relative vorticity $\boldsymbol{e}_r\cdot\nabla_P\times\boldsymbol{u}^{(0)}$).

5.3 Vertical structure $\pi^{(2)}$

We proceed with the discussion of the leading order equations (5.34)-(5.38). We show that the system (5.34)-(5.38) can be written in the same form as the stationary, linearized (about a zonally symmetric flow) potential vorticity equation for the PR (3.106). Further, we present some analytical solutions for the vertical structure of $\pi^{(2)}$ for the cases where first the background density $\rho^{(0)}$ and second the background zonal flow are set to constant.

Case: $\rho^{(0)} = const$

Differentiating (5.37) with respect to z and expressing all unknown variables in terms of $\pi^{(2)}$, we can combine (5.34) – (5.38) in one equation

$$\frac{\rho^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\left(\frac{\partial^2}{\partial z^2}\pi^{(2)} - \underbrace{\frac{\beta N}{fFz}}_{:=\alpha^2/z}\pi^{(2)}\right) = 0\,, \tag{5.44}$$

where N is defined in (5.2) and $F = \frac{1}{a}\frac{\partial}{\partial\phi_P}\Theta^{(1)}$. The formal solution for $\pi^{(2)}$ reads

$$\pi^{(2)}(\lambda_P, \phi_P, z, t_P) = \int_0^{\lambda_P} \pi_h(\lambda, \phi_P, z, t_P)\mathrm{d}\lambda + \pi_0(\phi_P, z, t_P)\,. \tag{5.45}$$

Here π_h satisfies $(\frac{\partial^2}{\partial z^2} + \frac{\alpha^2}{z})\pi_h = 0$ and is given by

$$\pi_h(\boldsymbol{X}_P, z, t_P) = \sqrt{z}J_1(2\alpha\sqrt{z})C_1^*(\boldsymbol{X}_P, t_P) + \sqrt{z}Y_1(2\alpha\sqrt{z})C_2^*(\boldsymbol{X}_P, t_P)\,, \tag{5.46}$$

where J_1, Y_1 are the Bessel functions of the first and second kind (Abramowitz and Stegun, 1964), C_1^*, C_2^* and π_0 are integration constants and we have $\alpha = \alpha(\phi_P)$.

Thus, $\pi^{(2)}$ has the form

$$\pi^{(2)}(\boldsymbol{X}_P, z, t_P) = f_1(\phi_P, z)C_1(\boldsymbol{X}_P, t_P) + f_2(\phi_P, z)C_2(\boldsymbol{X}_P, t_P) + \pi_0(\phi_P, z, t_P)\,, \tag{5.47}$$

where $f_1 = \sqrt{z}J_1(2\alpha\sqrt{z})$, $f_2 = \sqrt{z}Y_1(2\alpha\sqrt{z})$, $C_1 = \int_0^{\lambda_P} C_1^*\mathrm{d}\lambda$ and $C_2 = \int_0^{\lambda_P} C_2^*\mathrm{d}\lambda$. We expect the functions C_1, C_2 and π_0 to be determined from the next order asymptotic equations by applying the sublinear growth condition, see Section 5.4 and Appendix A.3. The profiles of f_1 and f_2 are displayed in Fig. 5.1. For large z these functions behave like

$$f_1 \sim z^{1/4}\cos(2\alpha\sqrt{z} - \frac{3}{4}\pi) + \mathcal{O}\left(|z|^{-1}\right)\,, \tag{5.48}$$

$$f_2 \sim z^{1/4}\sin(2\alpha\sqrt{z} - \frac{3}{4}\pi) + \mathcal{O}\left(|z|^{-1}\right)\,, \tag{5.49}$$

(Abramowitz and Stegun, 1964) implying that the amplitude of $\pi^{(2)}$ oscillates between $+\infty$ and $-\infty$ for $z \to \infty$. Such an unbounded growth is limited if we set the background zonal wind $u^{(-1)} = const$ above some height z_t (denoting the height of tropopause)

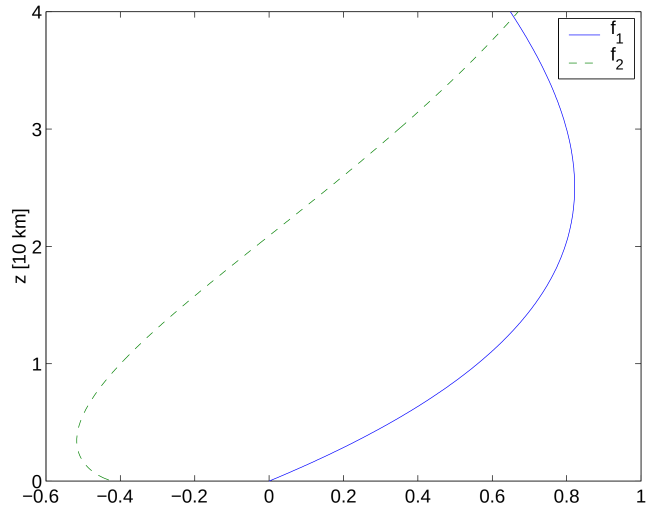

Figure 5.1: The vertical structure of f_1, f_2. The functions are computed with N and Θ_h in (5.44) equal to one and for $\phi_P = 60°$N.

$$u^{(-1)} = \begin{cases} -\frac{1}{f}Fz, & z \leq z_t \\ \\ -\frac{1}{f}Fz_t, & z \geq z_t\,. \end{cases} \tag{5.50}$$

For the layer above the troposphere we can set $z = z_t$ in (5.44) and the solution takes the form

$$\pi^{(2)} = {C_1}^t(\boldsymbol{X}_P, t_P)\cos(\alpha_t z) + {C_2}^t(\boldsymbol{X}_P, t_P)\sin(\alpha_t z) + \pi_0^t(\phi_P, z, t_P)\,, \tag{5.51}$$

where $\alpha_t^2 = \alpha^2/z_t$ and ${C_1}^t$, ${C_2}^t$,π_0^t are integration constants. Further investigation is required here to determine a matching condition for the solutions (5.47) and (5.51) at $z = z_t$.

Case $\rho^{(0)} = e^{-z}, u^{(-1)} = const$

Next, we consider the vertical structure equation for $\pi^{(2)}$ assuming $u^{(-1)} = const$ but relaxing the condition of constant density $\rho^{(0)}$. We set the background density to a much more realistic profile: $\rho^{(0)} = e^{-z}$, which was derived analytically in Section 3.1.1, see (3.20) and the discussion thereafter. Under these assumptions, (5.34) – (5.38) can be written as a single equation for $\pi^{(2)}$

$$\frac{\partial}{\partial\lambda_P}\frac{\partial}{\partial z}\rho^{(0)}\frac{\partial}{\partial z}\pi^{(2)} + \frac{\beta}{f^2 u^{(-1)}}\frac{\partial\Theta^{(1)}}{\partial z}\frac{\partial}{\partial\lambda_P}\rho^{(0)}\pi^{(2)} = 0\,. \tag{5.52}$$

Substituting the ansatz

$$\pi^{(2)} = A(z) e^{z/2} e^{ik\lambda_P} V(\phi_P)\,, \tag{5.53}$$

we obtain an equation for the amplitude $A(z)$

$$\frac{\partial^2}{\partial z^2} A(z) + \left(\frac{\beta}{f^2 u^{(-1)}} \frac{\partial \Theta^{(1)}}{\partial z} - \frac{1}{4} \right) A(z) = 0\,. \tag{5.54}$$

We want to compare this result with the QG theory. Linearizing the PV equation (4.70) about a constant zonal mean flow U and using the ansatz $\pi^{(3)} = A(z) e^{z/2} e^{i(kx+ly)}$, we obtain the vertical structure equation for stationary Rossby waves in the QG theory

$$\frac{\partial^2}{\partial z^2} A(z) + \left(\frac{\beta}{f_0^2 U} \frac{\partial \Theta^{(2)}}{\partial z} - k^2 - l^2 - \frac{1}{4} \right) A(z) = 0\,. \tag{5.55}$$

Depending on the sign of the expression in the bracket in (5.55), the solution $A(z)$ will either be a vertically propagating or evanescent wave. We note that for the large wavelength limit $k, l \to 0$, (5.55) takes the same form as (5.54).

5.4 Vertical structure $\pi^{(3)}$, horizontal structure $\pi^{(2)}$

Usually in the asymptotic analysis the leading order system of equations determines only the spatial structure of the solution, its temporal evolution is found by applying the sublinear growth condition in the next order equations. In the case considered here, we obtain from the leading order equations (5.34) – (5.38) the vertical structure of $\pi^{(2)}$, but the time evolution together with the horizontal structure are determined from the next order equations.

Motivated by the discussion in the previous section, one can represent (5.39) – (5.43) as the linear operator from (5.44) but now acting on $\pi^{(3)}$ with the right hand side depending on $\pi^{(2)}$ only. Introducing the functions $F_x^{(1)}, F_y^{(1)}, G^{(1)}, T^{(1)}$ and $K^{(1)}$, which do not depend on $\Theta^{(3)}, \pi^{(3)}, w^{(4)}$ or $u^{(1)}$, we can write (5.39) – (5.43) as

$$-f v^{(1)} + \frac{1}{a \cos\phi_P} \frac{\partial}{\partial \lambda_P} \pi^{(3)} = F_x^{(1)}\,, \tag{5.56}$$

$$f u^{(1)} + \frac{1}{a} \frac{\partial}{\partial \phi_P} \pi^{(3)} = F_y^{(1)}\,, \tag{5.57}$$

$$\Theta^{(3)} - \frac{\partial}{\partial z} \pi^{(3)} = G^{(1)}\,, \tag{5.58}$$

$$u^{(-1)} \frac{1}{a \cos\phi_P} \frac{\partial}{\partial \lambda_P} \Theta^{(3)} + v^{(1)} \frac{1}{a} \frac{\partial}{\partial \phi_P} \Theta^{(1)} + w^{(4)} \frac{\partial}{\partial z} \Theta^{(1)} = T^{(1)}\,, \tag{5.59}$$

$$\frac{\partial}{\partial z} \rho^{(0)} w^{(4)} + \nabla_P \cdot \rho^{(0)} \boldsymbol{u}^{(1)} = K^{(1)}\,, \tag{5.60}$$

where we have dropped the bars denoting synoptic scale averages. From (5.39) and (5.40) we can derive the vorticity equation

$$\begin{aligned}&\nabla_P \cdot f\boldsymbol{u}^{(1)} + \boldsymbol{u}^{(-1)} \cdot \nabla_P \zeta^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_P \zeta^{(-1)} + \zeta^{(-1)} \nabla_P \cdot \boldsymbol{u}^{(0)} + w^{(3)} \frac{\partial}{\partial z} \zeta^{(-1)} \\ &+\boldsymbol{e}_r \cdot (\nabla_P w^{(3)} \times \frac{\partial}{\partial z} \boldsymbol{u}^{(-1)}) = \frac{\boldsymbol{e}_r}{{\rho^{(0)}}^2} \cdot \left(\nabla_P \rho^{(1)} \times \nabla p^{(2)} + \nabla_P \rho^{(2)} \times \nabla_P p^{(1)} \right) ,\end{aligned} \tag{5.61}$$

where

$$\zeta^{(-1)} = \boldsymbol{e}_r \cdot (\nabla_P \times \boldsymbol{u}^{(-1)}) = \frac{1}{f} \Delta_P \pi^{(1)} + \frac{u^{(-1)} \cot \phi_P}{a} , \tag{5.62}$$

$$\begin{aligned}\zeta^{(0)} = \boldsymbol{e}_r \cdot (\nabla_P \times \boldsymbol{u}^{(0)}) &= \frac{1}{f} \Delta_P \pi^{(2)} - \frac{\rho^{(1)}}{\rho^{(0)} f} \Delta_P \pi^{(1)} + \frac{u^{(0)} \cot \phi_P}{a} - \\ &\frac{\rho^{(1)}}{\rho^{(0)}} \frac{u^{(-1)} \cot \phi_P}{a} + \frac{u^{(-1)}}{a \rho^{(0)}} \frac{\partial}{\partial \phi_P} \rho^{(1)} + \frac{1}{a^2 \cos \phi} \frac{\partial}{\partial \phi_P} {u^{(-1)}}^2 .\end{aligned} \tag{5.63}$$

Comparing the last equation with the definition of the planetary scale vorticity $\zeta^{(0)}$ in the PR (3.36), we note the additional terms due to the $\rho^{(1)}, \pi^{(1)}$ variations. Introducing $V^{(1)}$ as an abbreviation, (5.61) takes the form

$$\nabla_P \cdot f\boldsymbol{u}^{(1)} = V^{(1)} \tag{5.64}$$

Case: $\rho^{(0)}(z) = e^{-z}$

Making use of the fact that $\rho^{(0)}(z) = e^{-z}$, we combine (5.56) – (5.60) to

$$\frac{1}{a \cos \phi_P} \frac{\partial}{\partial \lambda_P} \left\{ -Fz \frac{\partial}{\partial z} \left(\rho^{(0)} \frac{\partial}{\partial z} \pi^{(3)} \right) + \left(\frac{\beta N}{f} - F \right) \rho^{(0)} \pi^{(3)} \right\} = Q' , \tag{5.65}$$

where

$$\begin{aligned}Q' = f \frac{\partial}{\partial z} \rho^{(0)} T^{(1)} + N \rho^{(0)} V^{(1)} - N f K^{(1)} + \frac{N \rho^{(0)} \beta}{f} F_x^{(1)} + \\ F \frac{\partial}{\partial z} \rho^{(0)} \left(\frac{z}{a \cos \phi_P} \frac{\partial}{\partial \lambda_P} G^{(1)} + F_x^{(1)} \right) .\end{aligned} \tag{5.66}$$

Substituting the expressions for $T^{(1)}, V^{(1)}, K^{(1)}, F_x^{(1)}$ and $G^{(1)}$, Q' can be written as

$$\begin{aligned}
Q' = -f\frac{\partial}{\partial z}\Bigg\{\rho^{(0)}\left(\frac{\partial}{\partial t_P}\Theta^{(2)} + \frac{u^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\Theta^{(2)} + \frac{v^{(0)}}{a}\frac{\partial}{\partial\phi_P}\Theta^{(2)} + w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)\Bigg\} \\
-N\rho^{(0)}\Bigg(\boldsymbol{u}^{(-1)}\cdot\nabla_P\zeta^{(0)} + \nabla_P\cdot\boldsymbol{u}^{(0)}\zeta^{(-1)} + w^{(3)}\frac{\partial}{\partial z}\zeta^{(-1)} + \boldsymbol{e}_r\cdot\nabla_P w^{(3)}\times\frac{\partial}{\partial z}\boldsymbol{u}^{(-1)} \\
-\frac{\boldsymbol{e}_r}{{\rho^{(0)}}^2}\cdot\left(\nabla_P\rho^{(1)}\times\nabla_P p^{(2)} + \nabla_P\rho^{(2)}\times\nabla_P p^{(1)}\right)\Bigg) + fN\Bigg(\frac{\partial}{\partial z}\rho^{(1)}w^{(3)} + \nabla_P\cdot\rho^{(1)}\boldsymbol{u}^{(0)} \\
+\frac{u^{(-1)}}{a\cos\pi_P}\frac{\partial}{\partial\lambda_P}\rho^{(2)}\Bigg) - \frac{\beta\rho^{(0)}N}{f}\Bigg(\frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}u^{(0)} + \frac{v^{(0)}}{a}\frac{\partial}{\partial\phi_P}u^{(-1)} + w^{(3)}\frac{\partial}{\partial r}u^{(-1)} \\
-\frac{u^{(-1)}v^{(0)}\tan\phi_P}{a} - \frac{1}{a\cos\phi_P}\frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial\lambda_P}\pi^{(2)}\Bigg) - F\frac{\partial}{\partial z}\Bigg\{\rho^{(0)}\Bigg[\frac{z}{a\cos\phi}\frac{\partial}{\partial\lambda_P}\Bigg(\frac{\rho^{(1)}\Theta^{(2)}}{\rho^{(0)}} + \frac{\rho^{(2)}\Theta^{(1)}}{\rho^{(0)}} \\
-\frac{(1-\gamma)p^{(1)}p^{(2)}}{\gamma^2{p^{(0)}}^2}\Bigg) + \frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}u^{(0)} + \frac{v^{(0)}}{a}\frac{\partial}{\partial\phi_P}u^{(-1)} + w^{(3)}\frac{\partial}{\partial z}u^{(-1)} \\
-\frac{u^{(-1)}v^{(0)}\tan\phi_P}{a} - \frac{1}{a\cos\phi_P}\frac{\rho^{(1)}}{\rho^{(0)}}\frac{\partial}{\partial\lambda_P}\pi^{(2)}\Bigg]\Bigg\}.
\end{aligned} \tag{5.67}$$

Case: $\rho^{(0)} = const$

When we assume a constant density $\rho^{(0)}$, the operator on the left hand side of (5.65) takes the same form as the one in (5.44) and we have

$$\frac{\rho^{(0)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_P}\left(\frac{\partial^2}{\partial z^2} - \frac{\beta N}{fFz}\right)\pi^{(3)} = -\frac{1}{Fz}Q'\,. \tag{5.68}$$

In the next section we analyze this equation under some additional assumptions.

5.4.1 Boussinesq fluid

Since we are interested in the general properties of (5.68), we restrict the analysis to a Boussinesq fluid in order to make the discussion easier. We assume a constant background density state $\rho^{(0)} = const$ and set all higher order density fluctuations $\rho^{(i)}$, $i \geq 1$ to zero, except those in the vertical momentum equation. In this case Q' simplifies to

$$
\begin{aligned}
Q' = -\rho^{(0)} N \Bigg\{ & \left(\frac{\partial}{\partial t_P} + \boldsymbol{u}^{(0)} \cdot \nabla_P + w^{(3)} \frac{\partial}{\partial z} \right) \frac{f}{N} \frac{\partial^2}{\partial z^2} \pi^{(2)} + \boldsymbol{u}^{(-1)} \cdot \nabla_P \zeta^{(0)} + \nabla_P \cdot \boldsymbol{u}^{(0)} \zeta^{(-1)} \\
& + w^{(3)} \frac{\partial}{\partial z} \zeta^{(-1)} + \boldsymbol{e}_r \cdot \left(\nabla_P w^{(3)} \times \frac{\partial}{\partial z} \boldsymbol{u}^{(-1)} \right) - \frac{\tan\phi_P}{aN} \frac{\partial u^{(-1)}}{\partial z} \frac{2u^{(-1)}}{a\cos\phi_P} \frac{\partial}{\partial \lambda_P} \frac{\partial \pi^{(2)}}{\partial z} \\
& + \frac{\beta}{f} \left(\frac{u^{(-1)}}{a\cos\phi_P} \frac{\partial}{\partial \lambda_P} u^{(0)} + \frac{v^{(0)}}{a} \frac{\partial}{\partial \phi_P} u^{(-1)} + w^{(3)} \frac{\partial}{\partial z} u^{(-1)} - \frac{u^{(-1)} v^{(0)} \tan\phi_P}{a} \right) \\
& + \frac{F}{N} \frac{\partial}{\partial z} \left(\frac{u^{(-1)}}{a\cos\phi_P} \frac{\partial}{\partial \lambda_P} u^{(0)} + \frac{v^{(0)}}{a} \frac{\partial}{\partial \phi_P} u^{(-1)} + w^{(3)} \frac{\partial}{\partial z} u^{(-1)} - \frac{u^{(-1)} v^{(0)} \tan\phi_P}{a} \right) \Bigg\} .
\end{aligned}
\tag{5.69}
$$

We have the following relations

$$
\boldsymbol{u}^{(0)} = \frac{1}{f} \boldsymbol{e}_z \times \left(\nabla_P \pi^{(2)} + \frac{{u^{(-1)}}^2 \tan\phi_P}{a} \boldsymbol{e}_\phi \right), \tag{5.70}
$$

$$
\Theta^{(2)} = \frac{\partial}{\partial z} \pi^{(2)} , \tag{5.71}
$$

$$
\zeta^{(-1)} = \frac{1}{f} \Delta_P \pi^{(1)} + \frac{u^{(-1)} \cot\phi_P}{a} , \tag{5.72}
$$

$$
\zeta^{(0)} = \frac{1}{f} \Delta_P \pi^{(2)} + \frac{u^{(0)} \cot\phi_P}{a} + \frac{1}{a^2 \cos\phi_P} \frac{\partial}{\partial \phi_P} {u^{(-1)}}^2 , \tag{5.73}
$$

$$
0 = \nabla_P \cdot \boldsymbol{u}^{(0)} + \frac{\partial}{\partial z} w^{(3)} , \tag{5.74}
$$

$$
\begin{aligned}
v^{(1)} = & \frac{1}{af\cos\phi_P} \frac{\partial}{\partial \lambda_P} \pi^{(3)} + \frac{u^{(-1)}}{a\cos\phi_P} \frac{\partial}{\partial \lambda_P} u^{(0)} + \frac{v^{(0)}}{a} \frac{\partial}{\partial \phi_P} u^{(-1)} \\
& + w^{(3)} \frac{\partial}{\partial z} u^{(-1)} - \frac{u^{(-1)} v^{(0)} \tan\phi_P}{a} .
\end{aligned}
\tag{5.75}
$$

The terms that appear in (5.69) include: advection of the relative vorticity $\zeta^{(0)}$ by the $u^{(-1)}$ field, vertical vorticity advection, twisting term, advection of planetary vorticity by the ageostrophic component of $v^{(1)}$ (third line) and the product of the vertical derivative of this component with the meridional background temperature gradient F (last line). From Section 3.1.2 we know that in the case when $\Theta^{(1)}$ is set to zero, the left hand side of (5.68) vanishes and Q' contains only the terms involving $\frac{f}{N}\frac{\partial^2}{\partial z^2}\pi^{(2)}$.

One can view on (5.68) as an equation for the vertical structure of $\pi^{(3)}$. One has to ensure that the linear operator acting on $\pi^{(3)}$ is not excited by its eigenfrequency, which will lead to an unbounded growth of $\pi^{(3)}$ (secular terms). We expect that such solvability condition for the right hand side of (5.68) may provide an equation for $\zeta^{(0)}$ which will uniquely determine $\pi^{(2)}$.

It remains an open question how to suppress the secular terms in (5.68). To our knowledge there is no general approach applicable to practical problems. In Appendix A.3 we give an example how the secular terms in (5.68) can be removed under some additional assumptions. We also note that if we average zonally (5.68) and apply periodic boundary conditions, the linear operator vanishes and we obtain a closed model.

5.5 Discussion

In this chapter we systematically extended the two scale PR allowing $\mathcal{O}(\varepsilon)$ variations of the background potential temperature. Such temperature fluctuations are motivated by the observed equator to pole and surface to tropopause temperature gradients. Since they evolve on a much slower – seasonal time scale, we have prescribed the background temperature distribution in the present model. The background state of the model is characterized through $\mathcal{O}(\varepsilon^{-1})$ zonal winds, linearly increasing with height and in thermal wind balance. The leading order asymptotic equations (5.34)-(5.38) can be combined into one PV equation which has the same form as the steady, linearized (about a zonal mean flow) PV equation (3.106) for the PR. In the case of a constant background density this PV equation is (5.44) and in the case of a constant background zonal flow – (5.52). At this asymptotic order the PV contains only the vorticity stretching term and the energy of the system is only potential. As in the two scale PR, at this asymptotic order there is no net influence from the synoptic scales on the leading order pressure correction $\pi^{(2)}$.

We discussed in Section 5.3 that the leading order system (5.34)-(5.38) can be interpreted as a constraint for the vertical structure of $\pi^{(2)}$. Analytical solutions have been given in the case of constant background density: (5.47), (5.51). It was found that the PV transport equation (5.44) represents the steady version of the planetary wave model of Welander (1961) if the vertical variations of the background stratification are neglected there. We could show (not presented here) that if we allow z variations in the lapse rate N from (5.2), both models are equivalent (if no time variations are considered). Here we have to mention that the Welander (1961) model was derived by combining the linearized PEs into a wave equation and expanding its coefficients in small Rossby and Richardson numbers under the assumption of order one zonal wave numbers.

In the case of a constant background zonal flow, eq. (5.54) for the vertical structure of $\pi^{(2)}$ is the long wave length limit of the corresponding equation in the QG theory. This demonstrates a consistency between the two models. Such consistency is not a priori guaranteed since the QG theory is derived under assumptions which cannot be applied to the planetary scale (e.g., small variations of the Coriolis parameter, constant background stratification). This has as a consequence that identical terms in both models describe different physical mechanisms. Whereas in the QG the vertical derivative of $w^{(3)}$ balances the divergence of the ageostrophic wind components, in the present regime (in the PR as well) it balances the advection of planetary vorticity.

The second order equation system in the PRBF is given through (5.39)-(5.43) and it can be regarded as a constraint on the vertical structure of the pressure correction term $\pi^{(3)}$, see (5.65). The terms on the right hand side of the last equation represent a coupling with the dynamics of the $\pi^{(2)}$ field, e.g., the nonlinear advection of $\Theta^{(2)}$ and the linearized advection of the relative vorticity $\zeta^{(0)}$. In the case of a Boussinesq fluid, (5.65) is modified to (5.68) (with the right hand side given in (5.69)). We discussed the problem with the solvability condition for (5.68), we expect that such additional condition determines the full horizontal structure of $\pi^{(2)}$. Our expectation was supported by the derivation in Appendix A.3 of a solvability condition for (5.68) under some additional assumptions. The presented analysis in the appendix can be extended straightforward to the general case considered here.

In the literature on the vertical propagation of planetary waves (e.g. Charney and Drazin, 1961; Dickinson, 1968a,b; Matsuno, 1970; Tung and Lindzen, 1979) it is often assumed that the advection by the geostrophic wind of relative vorticity and of planetary vorticity are of the same order. In (5.69) the planetary vorticity is advected by the next order velocity corrections $v^{(1)}$, which are not in geostrophic balance. The reason for the difference to the above mentioned models in the literature, is that (although not stated often explicitly) these models apply only to anisotropic flows, similar to those discussed in Chapter 4. These flows a characterized by sub-planetary meridional scales and "small" variations of the Coriolis parameter, whereas in the PRBF we consider planetary meridional scales and $\mathcal{O}(1)$ variations of f.

Recently, Klein (2007) considered a slightly modified distinguished limit in the multiple scales asymptotic approach. The new limit implies higher reference velocity $\sim$ 25 m/s, whereas here a value of 10 m/s was assumed. We have studied if our results are sensitive with respect to the new modification. It was shown by Klein (2007) applying the new approach that the variations of the background potential temperature in the QG theory are of the same order as the observed equator to pole and surface to tropopause temperature variations, namely $\mathcal{O}(\varepsilon)$. In the present approach the variations of background temperature in the QG model are only $\mathcal{O}(\varepsilon^2)$, because of this we studied the PRBF. Applying the modified approach, all regimes considered in the last three chapters were rederived. The new reduced model equations involve everywhere an order of magnitude larger temperature and pressure fluctuations (the variables $\Theta^{(i)}, \pi^{(i)}$ change to $\Theta^{(i-1)}, \pi^{(i-1)}$ in the equations). This is feasible if one considers that larger geostrophically balanced horizontal velocities require larger pressure and temperature variations. Thus, in the modified approach the planetary scale evolution of $\mathcal{O}(\varepsilon)$ potential temperature variations is described by the PR from Chapter 3, rather than by the PRBF. In order to derive the model equations for the PRBF with the new approach, one has to assume $\mathcal{O}(1)$ background temperature variations. However, temperature variations exceeding $\mathcal{O}(\varepsilon)$ are not observed in the troposphere. The discussion above gives us a hint that the PR is of greater relevance for the real atmosphere than the PRBF. This is supported also by the numerical simulations with a PEs model which are presented in the next chapter. Nevertheless, the study of the PRBF helped us to understand some general properties of the PR such as the vertical structure of the solution. This was possible because the leading order asymptotic equations in the PRBF and the steady, linearized PGEs from the PR have similar form. Finally, we want to mention another advantage of the modified distinguished limit introduced by Klein (2007). It allows one to distinguish between the Oboukhov scale (or external Rossby deformation radius) and the planetary scale. The Oboukhov scale is defined as the ratio between the fast barotropic wave speed and the earth's rotation frequency. Applying the new limit, this scale can be expressed in terms of ε as $\varepsilon^{-5/2}h_{sc}$. In the approach used here, on the other hand, the Oboukhov scale is of the same order as the planetary scale, namely, $\varepsilon^{-3}h_{sc}$. The modified approach gives one the possibility to study phenomena characterized by the Oboukhov scale such as atmospheric blockings. An example of asymptotic models for blockings is given in Appendix A.5.

Chapter 6

Balances on the Planetary and Synoptic Scales in Numerical Experiments

In the previous chapters we presented three asymptotic regimes valid for planetary and synoptic scales and various background stratifications. In this chapter we address the question how close the reduced models are able to describe the atmospheric flow. For that purpose we perform simulations with a model based on the primitive equations (PEs). Since the PEs are derived from the full compressible flow equations by assuming only hydrostatic balance and a small aspect ratio of the vertical to horizontal length scale, these equations are much more comprehensive than the asymptotic models and apply to a wider range of scales. From the simulations with the PEs model we study the balances in the vorticity transport on the planetary and synoptic scale. After comparing the results with the reduced asymptotic equations, we find that whereas the PR and APR capture main features of the large-scale atmospheric dynamics, the PRBF fails. In the next section we introduce briefly the PEs model, the experiment setup and the method we use for calculating the balances. The results from the simulations are interpreted in Section 6.2 with respect to the PR and in Section 6.3 with respect to the PRBF and APR.

6.1 Model description and methodology

6.1.1 The model

For the numerical experiments we use the simplified global circulation model Portable University Model of the Atmosphere (PUMA; Fraedrich et al., 1998). The model solves the primitive equations on a sphere for a dry ideal gas applying the spectral transform method. A semi implicit time scheme (Hoskins and Simmons, 1975) and a finite difference vertical scheme (Simmons and Burridge, 1981) are implemented in the model; the vertical levels are equally spaced σ levels. All diabatic and dissipation effects are linearly parameterized through Newtonian cooling and Raylaigh friction, respectively (Held and Suarez, 1994). This reduced complexity model represents the dynamical core of an atmospheric general circulation model (AGCM) and it is widely used for idealized experiments, e.g., for studying low-frequency variability, storm track dynamics (Franzke et al., 2000, 2001; Franzke, 2002) and the atmospheric entropy production (Kleidon et al., 2003). We continue with a summary of the model equations, for a complete description of the model we refer the reader to Fraedrich et al. (1998); Lunkei et al. (2005); Liakka (2006).

The PEs implemented in PUMA are in the form

Vorticity equation

$$\frac{\partial}{\partial t}\zeta_a = \frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}F_v - \frac{\partial}{\partial \mu}F_u - \frac{\zeta}{\tau_F} - K(-1)^h\nabla^{2h}\zeta\,, \tag{6.1}$$

Divergence equation

$$\frac{\partial}{\partial t}D = \frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}F_u + \frac{\partial}{\partial \mu}F_v - \nabla^2\left(\frac{U^2+V^2}{2(1-\mu^2)} + \Phi + T_0\ln p_S\right) - \frac{D}{\tau_F} - K(-1)^h\nabla^{2h}D\,, \tag{6.2}$$

Temperature equation

$$\frac{\partial}{\partial t}T' = -\frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}UT' - \frac{\partial}{\partial \mu}VT' + DT' - \dot{\sigma}\frac{\partial}{\partial \sigma}T + \kappa\frac{T\omega}{p} + \frac{T_R - T}{\tau_R} - K(-1)^h\nabla^{2h}T'\,, \tag{6.3}$$

Continuity equation

$$\frac{\partial}{\partial t}\ln p_S = -\frac{U}{1-\mu^2}\frac{\partial}{\partial \lambda}\ln p_S - V\frac{\partial}{\partial \mu}\ln p_S - D - \frac{\partial}{\partial \sigma}\dot{\sigma}\,, \tag{6.4}$$

Hydrostatic balance

$$\frac{\partial}{\partial \ln \sigma}\Phi = -T\,, \tag{6.5}$$

where

$$F_u = V\zeta_a - \dot{\sigma}\frac{\partial}{\partial \sigma}U - T'\frac{\partial}{\partial \lambda}\ln p_S\,, \tag{6.6}$$

$$F_v = -U\zeta_a - \dot{\sigma}\frac{\partial}{\partial \sigma}V - T'(1-\mu^2)\frac{\partial}{\partial \mu}\ln p_S\,. \tag{6.7}$$

All variables have been nondimensionalized using Ω, a, p_{ref} and g (for the definitions see Section 2.1) and the reference temperature $a^2\Omega^2/R$ (R ideal gas constant). In the model equations t, λ and $\mu = \sin(\phi)$ denote time, zonal and meridional coordinate, respectively, where ϕ measures the latitude. The vertical coordinate $\sigma = p/p_s$ is a pressure coordinate p scaled with the surface pressure p_s. The absolute vorticity ζ_a is the sum of the relative vorticity ζ and the planetary vorticity f. The variables D, T and Φ denote divergence, temperature and geopotential, respectively. T' measures the departure of the temperature from a constant reference profile T_0; $\dot{\sigma}$ and w are vertical velocities defined by the σ and p coordinate, respectively. Expressed in terms of the zonal and meridional velocities u, v, the variables U and V read: $U = \cos\phi u$, $V = \cos\phi v$. Further we have the restoration temperature T_R, the diabatic and friction relaxation time scales τ_R and τ_f, the adiabatic coefficient κ and the hyperdiffusion coefficient K. With the help of a stream function ψ and a velocity potential χ the vorticity and the divergence are expressed as

$$\zeta = \Delta\psi\,, \qquad D = \Delta\chi\,, \tag{6.8}$$

and the horizontal velocities are given through

$$U = -\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu} + \frac{\partial\chi}{\partial\lambda}\,, \tag{6.9}$$

$$V = \frac{\partial\psi}{\partial\lambda} + \left(1-\mu^2\right)\frac{\partial\chi}{\partial\mu}\,. \tag{6.10}$$

6.1.2 The methodology

As already mentioned, the equations in PUMA are solved using the spectral transform method (Bourke, 1988), where all nonlinear products are calculated on the grid but are then transformed spectrally for the computation of the λ, μ derivatives. For this purpose each prognostic variable (denoted here with Q) is expressed in terms of a truncated series of spherical harmonics

$$Q(t,\lambda,\mu,\sigma) = \sum_{m}^{N}\sum_{n=m}^{N} Q_n^m(t,\sigma)P_n^m(\mu)e^{im\lambda}\,, \tag{6.11}$$

where Q_n^m denotes the spectral coefficients, m the zonal wavenumber and $P_n^m(\mu)$ the associated Legendre polynomials. N gives the number of the considered modes, since a triangular truncation is applied, the model resolution is denoted with TN.

Using PUMA output we compare the magnitude of the different terms in the vorticity equation (6.1). For this purpose the tendency and the dissipation terms are given directly by the model, but the nonlinear terms have to be calculated from the output. Here we give an example of how this is done by considering the $\frac{\partial V\zeta_a}{\partial\mu}$ term. Applying the product rule we have

$$\frac{\partial}{\partial\mu}V\zeta_a = \zeta\frac{\partial V}{\partial\mu} + f\frac{\partial V}{\partial\mu} + V\frac{\partial\zeta}{\partial\mu} + V\frac{\partial f}{\partial\mu}\,. \tag{6.12}$$

Whereas in PUMA the whole product of $V\zeta_a$ is differentiated with respect to μ, here we are interested in the contributions from the different terms on the right hand side of (6.12). The asymptotic analysis from the previous three chapters showed that at leading order the wind is geostrophically balanced, this has as a consequence that on the synoptic scale $f\frac{\partial V}{\partial \mu}$ is much larger than all other terms (see the discussion in Section 6.2). However, we found that the approximation error in the computation of $f\frac{\partial V}{\partial \mu}$ can be comparable in magnitude with some of the terms on the right hand side of (6.12), if in the computation the same spectral resolution as the one of the original V field is used. This makes a comparison of the terms difficult even for high resolutions with $N = 85$. The large approximation error can easily be understood by substituting in the definition of V (6.10) the spectral representation (6.11) for χ.

$$V = (1-\mu^2)\frac{\partial}{\partial \mu}\sum_{m=0}^{N}\sum_{n=m}^{N}\chi_n^m P_n^m(\mu)e^{im\lambda} + \frac{\partial \psi}{\partial \lambda}\,. \tag{6.13}$$

Making use of this equation and of the recursive relation for the associated Legendre polynomials

$$(1-\mu^2)\frac{\partial P_n^m}{\partial \mu} = \epsilon_{n+1}^m P_{n+1}^m + \epsilon_{n-1}^m P_{n-1}^m\,, \tag{6.14}$$

where $\epsilon_{n-1}^m, \epsilon_{n+1}^m$ are some constants dependant only on m and n (Abramowitz and Stegun, 1964), it can be shown that V requires $N+1$ spectral coefficients more than those needed for the representations of χ, ψ

$$V = \sum_{m=0}^{N}\sum_{n=m}^{N+1} V_n^m P_n^m(\mu)e^{im\lambda}\,. \tag{6.15}$$

These additional spectral coefficients are omitted by each transformation from the physical space into the spectral one leading to large approximation errors. The problem could be overcome by doubling the number of spectral modes in the transformation. We used T21 model output but a T42 resolution for the computation of the nonlinear terms. Finally, only modes corresponding to a T21 resolution were used in the analysis. Applying this method the relative approximation error was at most $\mathcal{O}(10^{-6})$, which is reasonable if one considers that PUMA variables are single precision.

6.1.3 Model setup

We performed simulations with an aquaplanet or a realistic orography as lower boundary condition. The model was run at a T21 resolution, with 10 vertical σ-levels and with a time step of 30 min. For the analysis an output with 1 day time increment was used, the first 360 days were ignored due to spin up effects. We used the default value of 70 K for the equator to pole temperature difference in the restoration temperature profile and the seasonal cycle in the model was switched off. The initial condition was an atmosphere at rest with a small amplitude perturbation of the surface pressure.

PUMA is able to produce all key features of the atmospheric circulation reasonably well for a simplified atmospheric model. At midlatitudes in the lower and middle troposphere a pronounced wavenumber 6-7 structure with a period of ca. 7 days is visible over the most time of the simulations. This wave implies a characteristic length scale of ~ 2000 km for the individual synoptic eddies, its time period is overestimated compared with the real atmosphere where the maximum of the synoptic activity lies around 4 days (Fig. 1.2). In the simulation with orography we studied time mean fields of the 500 hPa geopotential height. The model reproduces the trough over Eastern Asia, but it shifts the trough over Canada to Greenland. In the experiment the weak trough over Western Asia is absent but a weak minimum over the Aleutian islands is visible. In the real atmosphere the depression over these islands is confined to the lower troposphere only. These discrepancies can be due to absence of land-sea thermal forcing in the model.

6.2 The PR in simulations

In this section we analyze the magnitudes of the different terms in the PUMA vorticity equation (6.1) and compare the leading order balances with the two scale PR model. We consider the vorticity formulation of the PR momentum equations, the first two orders vorticity equations (see (3.82), (3.88)) are

$$f \nabla_S \cdot \boldsymbol{u}^{(0)} = 0\,, \tag{6.16}$$

$$\frac{\partial}{\partial t_S}\zeta^{(0)} + \boldsymbol{u}^{(0)} \cdot \nabla_S \zeta^{(0)} + f \nabla_P \cdot \boldsymbol{u}^{(0)} + f \nabla_S \cdot \boldsymbol{u}^{(1)} + \beta v^{(0)} = 0\,. \tag{6.17}$$

Next, we present the results for the balances in the PUMA vorticity transport on the synoptic and planetary scales.

6.2.1 Synoptic scale dynamics

All terms in the PUMA vorticity equation (6.1) are listed in Table 6.1, Fig. 6.1 and 6.2 display the zonal and temporal variations of some of them. Overall, depending on the amplitude of the fluctuations three groups of terms can be identified. The first includes V_7 and V_8 denoting the horizontal divergence multiplied with the Coriolis parameter f. The second group includes the vorticity tendency V_1; the zonal and meridional vorticity advection V_2, V_3; the horizontal divergence multiplied with vorticity V_4, V_5 and the planetary vorticity advection term V_6. The third group of terms contains corrections from the σ coordinate transformation V_9, V_{10}; the vertical vorticity advection and soledoinal term combined as V_{11}, V_{12} and the dissipation term V_{13}. Group one contains terms with the largest variations. The terms in the second group have typically smaller amplitudes by a factor of 4 to 10 as compared to group one with an exception of V_2. In Fig. 6.1(a) and 6.2(a) we see that V_2 fluctuations are much larger than those of the other terms in the group and are sometimes even comparable with V_7 and V_8 (but rarely exceed them). This can be understood if one takes into account that V_2 describes the zonal advection of the vorticity ζ and that the results presented in the figures correspond to geographical locations nearly coinciding with the position of the jets. At other locations, outside the maximum of zonal wind, V_2 was comparable with the terms in the second group (not shown here). This is why we have attributed V_2 to the second group. The fluctuations of the different terms from group one and two discussed so far are mainly due to the synoptic eddies and are characterized by the synoptic spatial and temporal scales: see the wavenumber 6 structure in Fig. 6.1(a) or the 7 day oscillation in Fig. 6.2(a). From Fig. 6.1(c) and 6.2(b) we conclude that the variations of the terms in the third group are an order of magnitude smaller than group two. Overall, the separation between the three groups of terms remained pronounced at all vertical levels between 30°N(S) and 80°N(S) and for experiments with or without orography.

We studied balances between the terms in the different groups. One would expect from Fig. 6.1(a) and 6.2(a) that V_7 and V_8 nearly balance. This is confirmed by Fig. 6.3(a) where a typical time series of $V_7 + V_8$ is plotted. The balance between V_7 and V_8 implies that the leading order contribution to the PUMA wind comes from a component that is divergence-free on the synoptic scale. This result is in accordance with the leading order asymptotic balance (6.16) and we conclude that this divergent-free component corresponds to $\boldsymbol{u}^{(0)}$ in the asymptotic analysis. This analysis states further that if we consider only the terms in the leading order asymptotic balance the error due to omitting all other terms in the vorticity equation is not larger than the next order correction terms, namely, at most $\mathcal{O}(\varepsilon)$. Fig. 6.3(a) confirms this result too, one can see that around day 102 the sum $V_7 + V_8$ (equal to the contribution from all other terms) is comparable with the terms in the second group and around day 111 it is of the order of the terms in group three.

Fig. 6.3(b) shows the time evolution of V_{qg}, where V_{qg} is the sum of the terms from group one and two together with the term V_{13}. Looking at the time between day 100 and 110, one can say that the error we make in the vorticity transport by taking the V_{qg} terms is an order of magnitude smaller than the one if we take only the leading order terms V_7 and V_8. Interestingly the effects due to friction cannot be neglected here, as the $V_{qg} - V_{13}$ curve shows. If we substitute in the V_{qg} according to the QG approximation the geostrophic and ageostrophic wind, we obtain all terms in the classical QG vorticity equation with friction. Each term in V_{qg} have a counterpart in (6.17); the friction term V_{13} is an exception because no frictional effects have been considered in the two scale model. We showed that the wind in the simulations is to a first approximation described by a divergence-free (on the synoptic scale) wind. This wind amounts for the largest variations of the terms in group two. Thus, we can say that at leading order these terms can be approximated by substituting everywhere the divergence-free wind in their definitions. In this case V_6 corresponds to $\beta v^{(0)}$. Since the spatial and temporal variations in Fig. 6.1(a),(b) and 6.2(a) are on the synoptic scale, V_1, V_2, V_3, V_4 and V_5 can be associated with $\frac{\partial}{\partial t_S}\zeta^{(0)}, \boldsymbol{u}^{(0)} \cdot \nabla_S \zeta^{(0)}$ and $\zeta^{(0)} \nabla_S \cdot \boldsymbol{u}^{(0)}$. The term $\zeta^{(0)} \nabla_S \cdot \boldsymbol{u}^{(0)}$ is absent in (6.17) since $\boldsymbol{u}^{(0)}$ is divergence free (6.16). From the perspective of the asymptotics the residual between V_7 and V_8 is (to a first approximation) due to the synoptic divergence of the first order wind corrections: $f \nabla_S \cdot \boldsymbol{u}^{(1)}$ and, as we will show in Section 6.2.2, due to the planetary divergence of the leading order wind: $f \nabla_P \cdot \boldsymbol{u}^{(0)}$. Comparing this result with the QG theory, we interpret $f \nabla_S \cdot \boldsymbol{u}^{(1)}$ as the divergence due to the ageostrophic wind components, however, the term $f \nabla_P \cdot \boldsymbol{u}^{(0)}$ does not appear in the

$V_1 = \frac{\partial}{\partial t}\zeta$	$V_9 = \frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}\left(T'(1-\mu^2)\frac{\partial}{\partial \mu}\ln p_S\right)$
$V_2 = U\frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}\zeta$	$V_{10} = -\frac{\partial}{\partial \mu}\left(T'\frac{\partial}{\partial \lambda}\ln p_S\right)$
$V_3 = V\frac{\partial}{\partial \mu}\zeta$	$V_{11} = \frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}\left(\dot{\sigma}\frac{\partial}{\partial \sigma}V\right)$
$V_4 = \zeta\frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}U$	$V_{12} = -\frac{\partial}{\partial \mu}\left(\dot{\sigma}\frac{\partial}{\partial \sigma}U\right)$
$V_5 = \zeta\frac{\partial}{\partial \mu}V$	$V_{13} = \frac{\zeta}{\tau_F} + K(-1)^h\nabla^{2h}\zeta$
$V_6 = V\frac{\partial}{\partial \mu}f$	$V_{qg} = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8 + V_{13}$
$V_7 = f\frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}U$	$V_{dv} = V_2 + V_3 + V_4 + V_5$
$V_8 = f\frac{\partial}{\partial \mu}V$	$V_{df} = V_6 + V_7 + V_8$

Table 6.1: Notation used for the different terms in the vorticity equation (6.1).

classical QG model.

As mentioned earlier in this section, the variations considered up until now are mostly on the synoptic scales. In order to see fluctuations on different spatial and temporal scales, we performed a wavenumber-frequency analysis of the different terms. Some of the results for the experiment with orography are presented in Fig. 6.4. For the calculation of the frequency spectra we have multiplied the data with a Bartlett window (Press et al., 2002). In all spectra of terms from group one and two the maximum at k = 6, 7 and around 7 days associated with the synoptic waves is clearly evident. Its magnitude is at least an order of magnitude larger for V_7 and V_8. In the spectra of the terms from group three (not shown) no synoptic peak can be identified, the spectral density there is overall an order of magnitude smaller than the one corresponding to group two. In the spectrum of V_6 a second maximum of activity at k = 2 and periods larger than 40 days is visible. This peak results from the quasi-stationary Rossby waves, its magnitude is comparable to the one of the synoptic peak and it appears only in the simulation with orography. In general, the spectral properties discussed above are robust with regard to the length of the time series and are observed for different vertical levels and latitudes.

6.2.2 Planetary scale dynamics

In order to study the net effect from the synoptic scales on the planetary scale motions, we average (6.17) over the synoptic scales and obtain

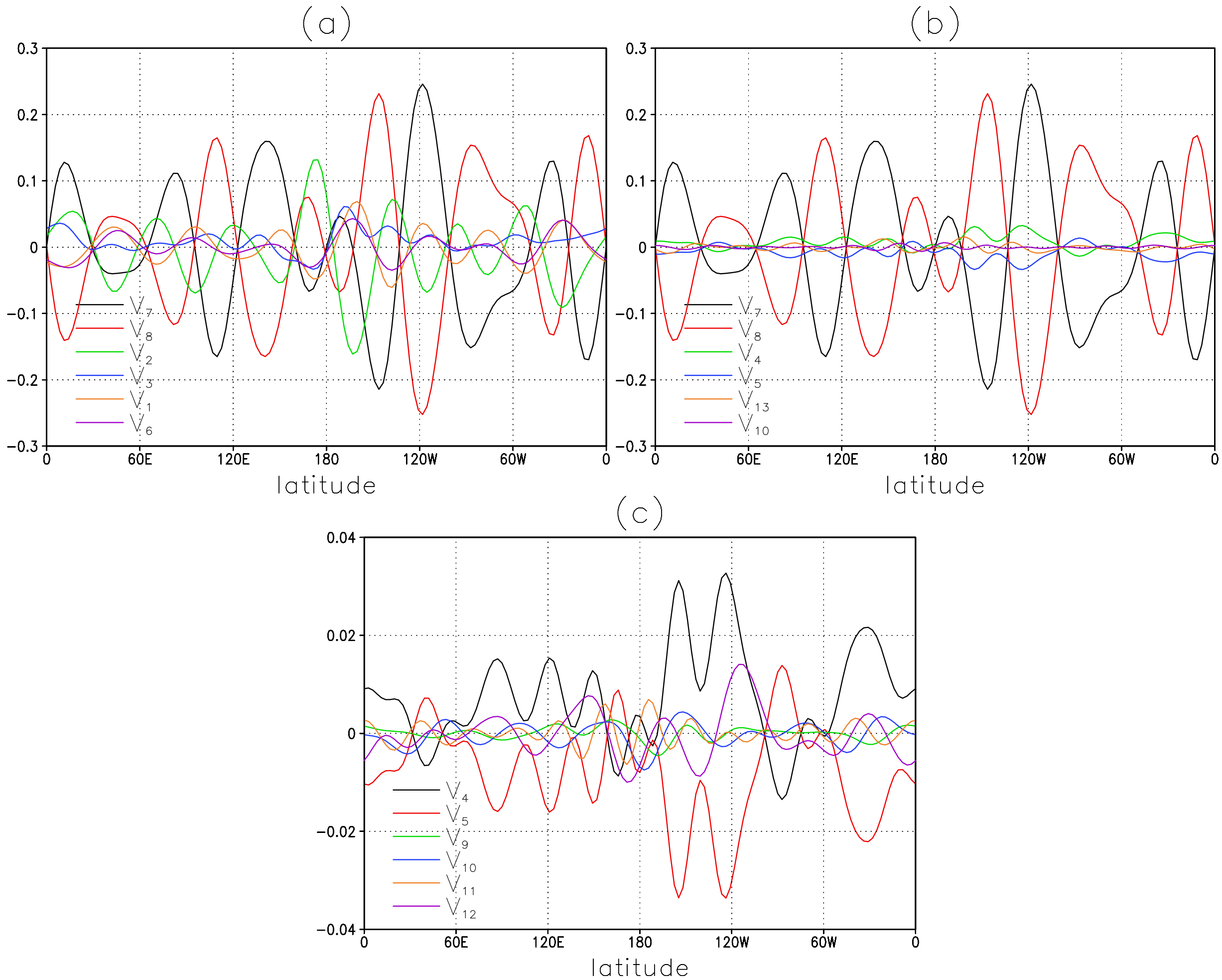

Figure 6.1: Zonal variations of terms from Table 6.1. Shown is the distribution at ca. 50°N and at 500 hPa for day 100 of the aquaplanet simulation; all terms are nondimensionalized using Ω^2.

$$\overline{f\nabla_P \cdot \boldsymbol{u}^{(0)}}^S + \beta\overline{v^{(0)}}^S = 0\,. \tag{6.18}$$

Here we applied the sublinear growth condition, which amongst others requires

$$\overline{\nabla_S \cdot \boldsymbol{u}^{(0)}\zeta^{(0)}}^S = 0\,. \tag{6.19}$$

Equations (6.18) and (6.19) motivated us to study the terms V_{df} and V_{dv} representing the divergence of the f- and of the ζ-flux (see Table. 6.1). The averaging over the synoptic spatial scales can be performed in the spectral model by simply omitting all modes higher than some cut-off mode (if scales smaller than the synoptic are neglected). Suppose we have a function dependent on λ only and represented as a Fourier series with N zonal modes. We assume that there is a spectral gap at the zonal wavenumber k = 4; all modes with $k \leq 3$ are attributed to the planetary scale and those with $k \geq 5$ to the synoptic one. Then the average of the function over the

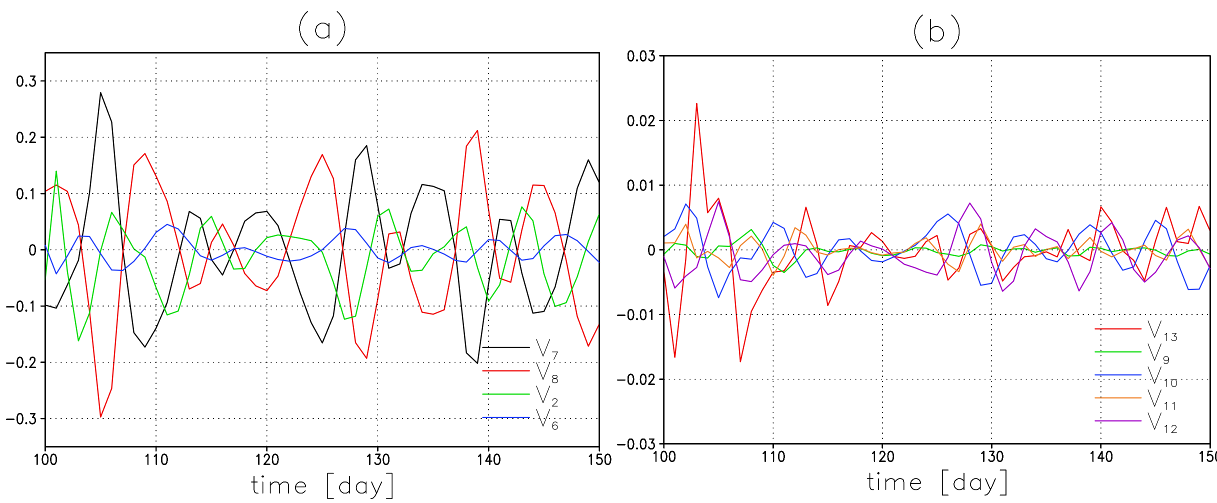

Figure 6.2: Time variations of terms from Table 6.1. Shown is the distribution at ca. 100°E, 50°N and at 500 hPa from the aquaplanet simulation; all terms are nondimensionalized using Ω^2.

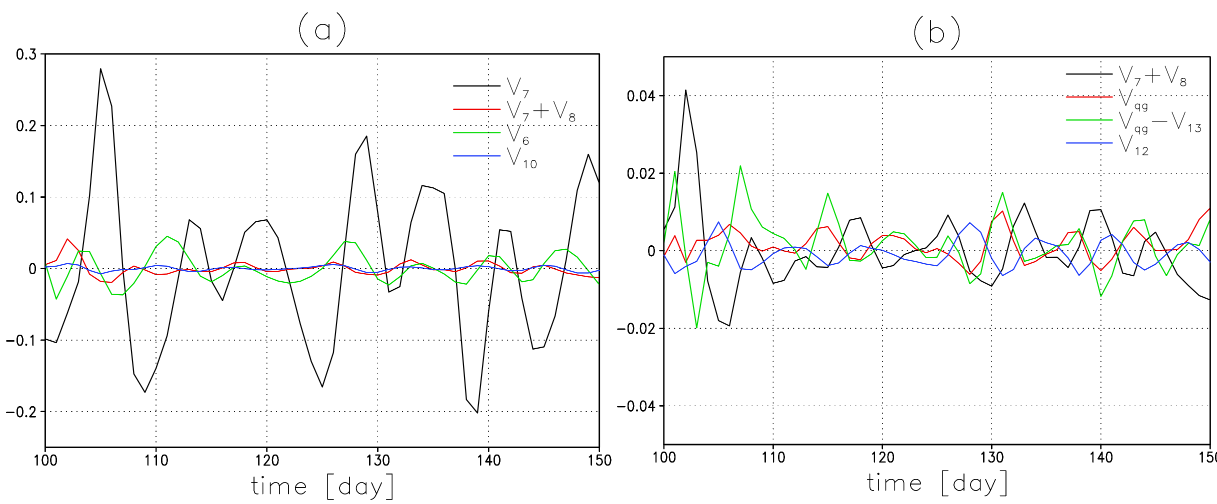

Figure 6.3: The same as in Fig. 6.2 but for different terms

synoptic scale is simply the truncated series at k = 3, since all higher modes will vanish when integrating over them. The same argument can be applied when averaging a series of spherical harmonics, but instead of some cut-off zonal wavenumber, there is a cut-off truncation number N and all modes with $n > N$ are ignored. The synoptic time averaging can be performed by simply filtering the fields in time.

From (6.18) and (6.19) we expect that V_{dv} and V_{df} vanish on the planetary scale. Since the spherical harmonics form an orthogonal set, we consider separate modes from V_{dv} and V_{df} and inspect for which total wavenumbers n their amplitudes vanish. For that reason we transformed the data into spherical harmonics and analyzed the spectral coefficients weighted with the corresponding Legendre polynomials P_n^m. The difference $n - m$ defines the so-called meridional wavenumber and gives the zero-crossings of the polynomial. Looking at the structure of the polynomials in Fig. A.1, one can say that modes with small n and a low number of nodes are characterized by planetary meridional scales, whereas those with large n have smaller merid-

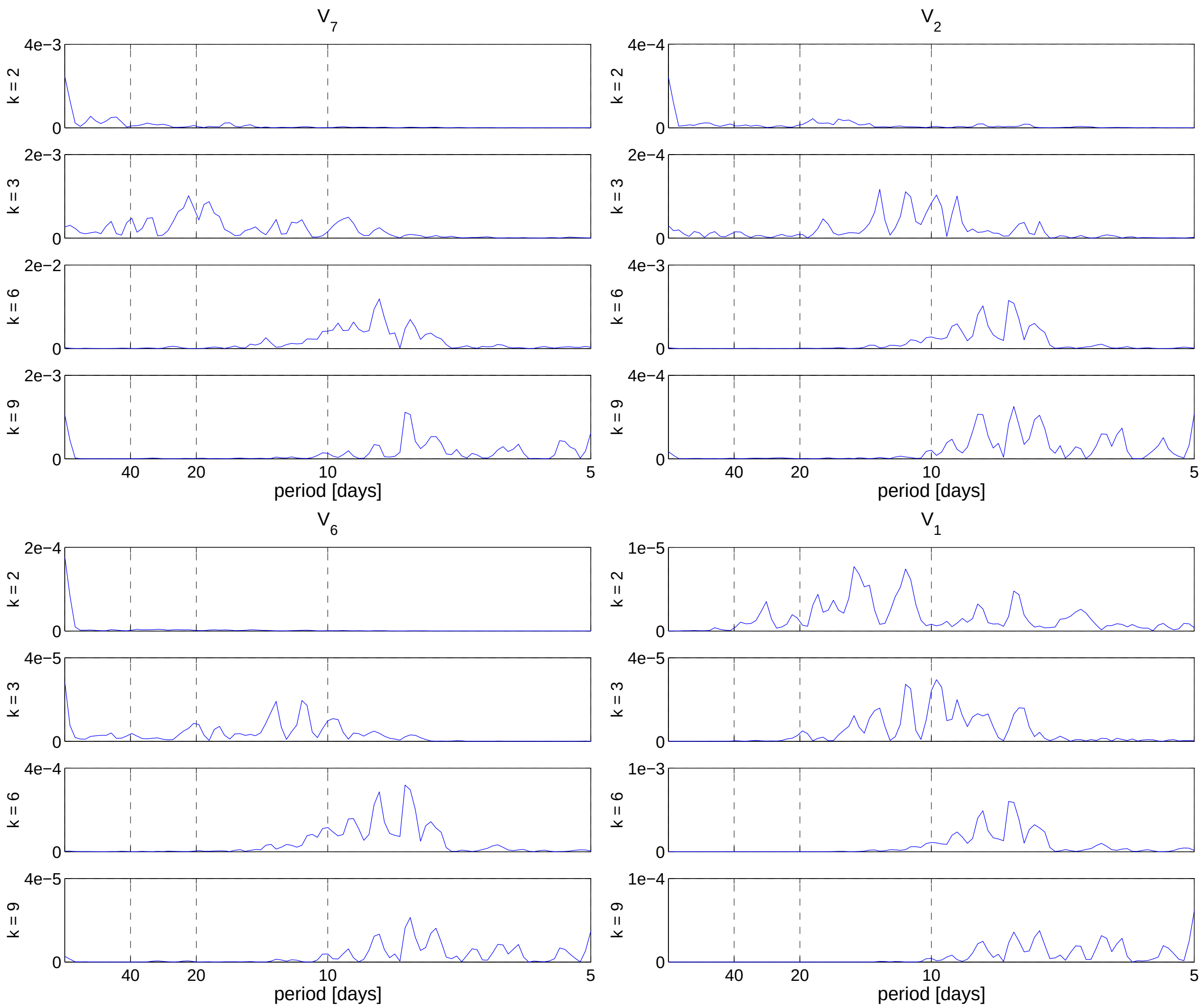

Figure 6.4: Frequency spectra of terms from Table 6.1 for different zonal wavenumbers k. Shown are spectra at ca. 50°N and 500 hPa from an 360 day experiment with orography, see text for details.

ional scales. By weighting the Legendre polynomial with the corresponding spectral coefficient its contribution to the amplitude of the field is evaluated.

We compared the amplitudes of the weighted spectral coefficients for V_{dv} and V_{df} with the amplitudes associated with individual terms in their definitions. The corresponding time series for different coefficients are displayed in Fig. 6.5 and 6.6. Indeed, for Legendre polynomials with $n \leq 2$ the fluctuations of V_{dv} are very small [see Fig. 6.5(a),(b)], we refer to these modes as planetary modes. It is important to note that the projections of the different terms V_2, V_3, V_4 and V_5 on the planetary modes show "large" variations, but when combined in V_{dv} they nearly balance. Another point we want to stress is that no spatial or temporal filtering was applied in the computation of V_2, V_3, V_4 and V_5. The inspection of the profiles of ζ, U and V shows that the leading order spatial variations are on the synoptic scale (and not on the planetary), consequently at leading order only the synoptic gradients are involved in V_{dv}, such gradients vanish when averaging over the synoptic spatial scales (a synoptic time averaging is not necessary for that). If modes with $n > 2$ are considered, the term V_{dv} is comparable or even larger

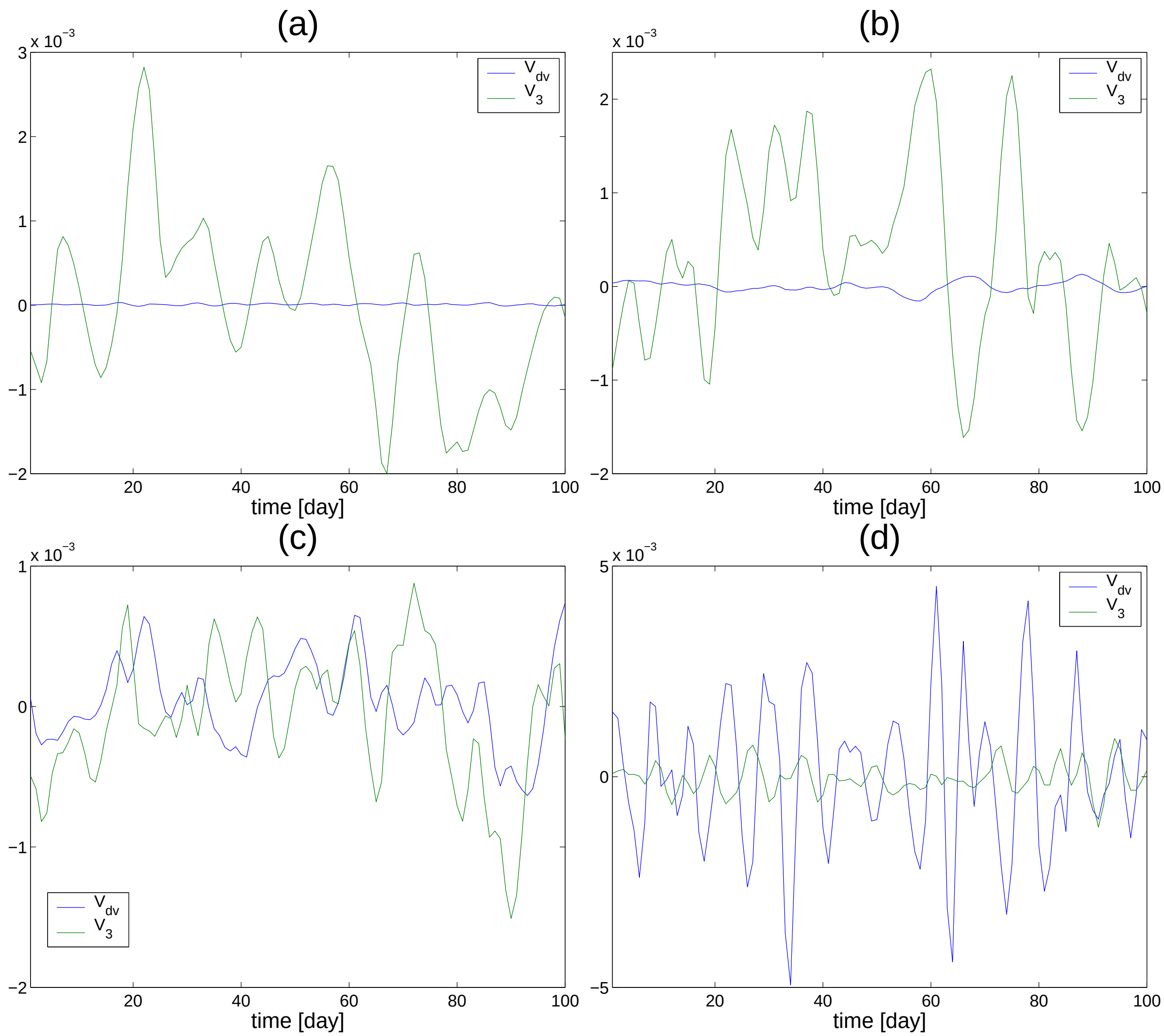

Figure 6.5: Time series of the spectral coefficients (real parts) for V_{dv} and V_3 weighted with the value at 50°N of the Legendre polynomial: (a) P_1^1, (b) P_2^2, (c) P_6^3 and (d) P_{15}^7. Shown are the results at 500 hPa for the experiment with orography, for notation see Table 6.1. Similar plots are obtained for the imaginary parts of the spectral coefficients and for different latitudes.

than V_3 [Fig. 6.5(c), (d)]. This corresponds to the synoptic case already discussed, which is characterized through a balance between the terms in V_{qg}.

Further, we found that for planetary modes V_6 is of the same order as V_7, V_8 and the three terms approximately balance [Fig. 6.6(a),(b)]. For such modes the variations of V_{df} are smaller than these of the individual terms V_6, V_7, V_8 and are mainly on the synoptic time scale. If these synoptic variations are filtered out (which is required for the synoptic scale averaging) we will obtain for V_{df} approximately some time constant value. The fact that V_6, V_7 and V_8 balance only up to some constant, becomes clear if we take into account that in the asymptotic averaging we divide the integral by an interval growing as $\frac{1}{\varepsilon}$ for $\varepsilon \to 0$, see (2.34)-(2.36). No such weighting was applied in the numerical calculation of V_{df}. It was found that for planetary modes the synoptic fluctuations in V_{df} and its time averaged mean are removed, if we add the terms V_9

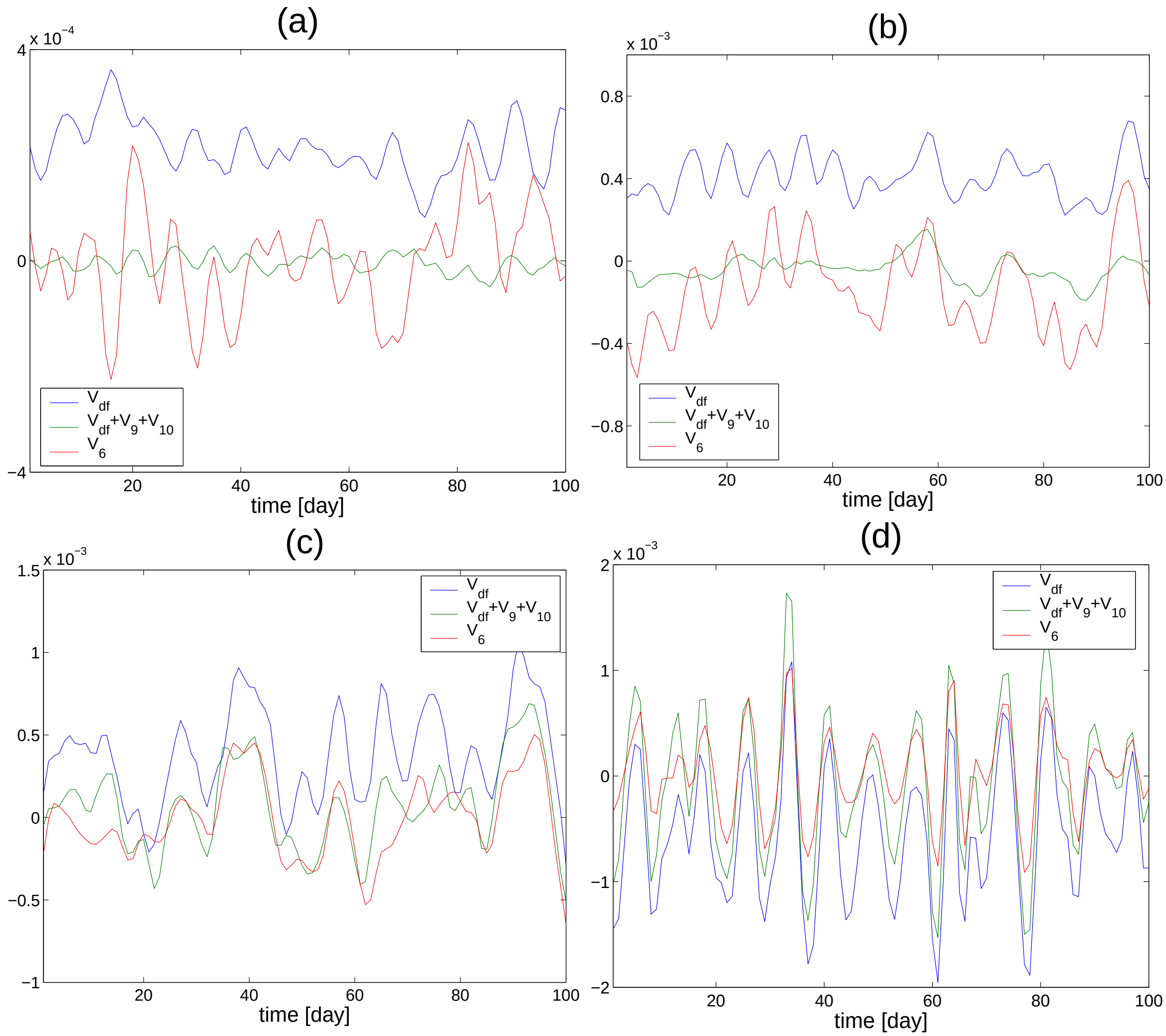

Figure 6.6: The same notation as in Fig. 6.5 but for V_{df}, $(V_{df} + V_9 + V_{10})$ and V_6.

and V_{10} [Fig. 6.6(a),(b)]. The Legendre polynomial P_2^0 was here an exception, one can show analytically that for it the relation $V_6 + V_7 + V_8/2 = 0$ is satisfied exactly. The results for all other modes indicates, that on the planetary scale ($n \leq 2$) we have a balance in the form of (6.18). For modes with $n > 2$ we observed again the synoptic balance [Fig. 6.6(c),(d)].

6.3 The PRBF and the APR in simulations

We compare the observations from the numerical simulations with the other two asymptotic regimes. In the APR we obtain that $\boldsymbol{u}^{(0)}$ is divergence-free (see (4.12), (4.13) and (4.33)), which is consistent with the balance between V_7 and V_8 as already discussed. Further, in the APR we obtain as leading order model the QG model (4.34). This is also supported by the numerical experiments, since we find that for anisotropic modes with planetary zonal scale but with a synoptic meridional extent (e.g., $P_6^3(\mu)$) we have in the vorticity equation a balance

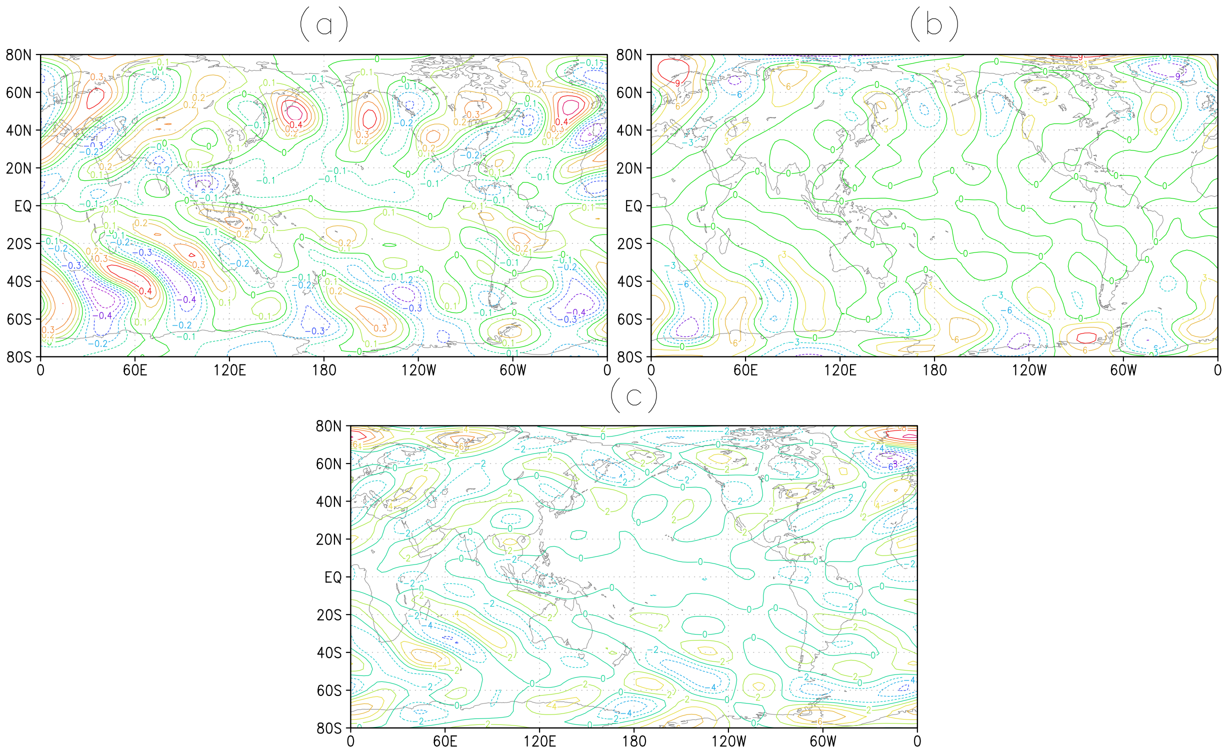

Figure 6.7: Contour lines of (a) ζ, (b) $\frac{1}{1-\mu^2}\frac{\partial\zeta}{\partial\lambda}$ and (c) $\frac{\partial\zeta}{\partial\mu}$ at 500 hPa for day 30; all terms are nondimensionalized using Ω.

between the terms in V_{qg}.

No evidence was found for the vorticity constraint (5.29) in the PRBF. This constraint implies that the zonal gradient of the vorticity is at least an order of magnitude smaller than the meridional one. No such anisotropy can be found in the data, see Fig. 6.7. Without the constraint the leading order nontrivial vorticity balance in the PRBF reads (see (5.15), (5.16))

$$\frac{u^{(-1)}}{a\cos\phi_P}\frac{\partial}{\partial\lambda_S}\zeta^{(0)} + f\nabla_S\cdot\boldsymbol{u}^{(0)} = 0\,. \tag{6.20}$$

Fig. 6.1(a) and 6.2(a) show that in the regions of the jets the zonal advection term V_2 is comparable to the divergence terms V_7 and V_8, nevertheless, no balance between the three terms was observed. V_7+V_8 is balanced on the synoptic scale by the complete sum $V_1+V_2+V_3+V_6+V_{13}$, rather than by V_2 alone. The simulations reveal also that the assumption of a constant zonal background flow $u^{(-1)}$ is not satisfied. The inspection of the eddy momentum fluxes and of the zonal mean wind indicates that the synoptic eddies transform eddy kinetic energy to kinetic energy of the mean flow, in this way zonal jets result. However, such mechanism is not captured in the present asymptotic setup of the PRBF, since we assumed that $u^{(-1)}$ does not depend on the synoptic dynamics of the $\boldsymbol{u}^{(0)}$ field. The observations discussed in this chapter show that atmospheric motions with isotropic planetary horizontal scales are governed by the Planetary Regime rather than by the Planetary Regime with Background Flow.

Chapter 7

Conclusion

Aiming to improve our understanding of the atmospheric dynamics on the planetary and synoptic scale, we presented in this thesis an approach based on asymptotic analysis and numerical experiments. We applied a multiple scales asymptotic method (Klein, 2000, 2004, 2007) and systematically derived reduced model equations describing three different planetary scale regimes and accounting for the planetary-synoptic interactions. Additionally, we performed numerical simulations with a much more comprehensive primitive equations (PEs) model and studied the balances on the planetary and synoptic scale between different terms in the vorticity transport. The combination between multiple scales asymptotic analysis and model simulations turned out to be advantageous. On the one hand, the numerical experiments helped us to identify the relevance of the asymptotic regimes for the atmosphere and on the other hand the asymptotic analysis was useful for the interpretation of different aspects of the planetary scale dynamics in the numerical simulations.

We summarize the asymptotic regimes in this thesis: the Planetary Regime (PR), the Anisotropic Planetary Regime (APR) and the Planetary Regime with Background Flow (PRBF). The PR is characterized by planetary horizontal scales and by a corresponding advective time scale of about one week. We assume variations of the background potential temperature comparable in magnitude with those adopted in the classical quasi-geostrophic theory. At leading order the resulting equations include the planetary geostrophic equations (PGEs). In order to apply these equations to the atmosphere, one has to prescribe a closure for the vertically averaged (barotropic) pressure. We presented an evolution equation for this component of the pressure, which was derived in a systematic way from the asymptotic analysis. Relative to the prognostic closures adopted in existing reduced-complexity planetary models, this new dynamical closure may provide for more realistic increased large scale, long term variability in future implementations. Using a two scale asymptotic ansatz, we extended the region of validity of the PR to the synoptic spatial and temporal scales. The derived two scale model includes in addition to the equations governing the single scale PR a modified quasi-geostrophic potential vorticity equation, describing the dynamics on the synoptic scale. Without the evolution equation for the barotropic pressure, the two scale model in the PR can be regarded as the anelastic analogon of the model of Pedlosky (1984) for the large scale oceanic circulation. This model accounts only for a feedback from the planetary scale dynamics to the synoptic scale but not for the reverse interaction. We discussed in this thesis different terms describing such reverse interactions, e.g., the planetary scale divergence of the vorticity flux with vorticity resulting from the planetary scale curl of the wind. We found that the evolution equation for the barotropic pressure provides a feedback from the synoptic scale to the planetary scale.

The PRBF describes motions with isotropic planetary horizontal scales too, but unlike the PR we consider in this regime systematically larger variations of the background potential temperature. Such temperature variations are of the order of the observed equator to pole temperature difference and we have further assumed that they do not evolve on the 7 days time scale. The numerical results presented in this thesis showed that planetary atmospheric motions are governed by the PR rather than by the PRBF. A theoretical justification was found when we performed the analysis for the PR and PRBF with the slightly modified distinguished limit recently introduced by Klein (2007). Under the new limit the variations of the background potential temperature in the QG theory are of the same order as the equator to pole temperature difference. Variations of such order are then described by the PR alone. Larger background temperature variations, required for the derivation of the PRBF under the new limit, are not observed in the troposphere, but they may become relevant for motions in the upper atmosphere or on other planets. Nevertheless, the study of the PRBF was valuable for understanding the PR, since the leading order model in the PRBF and the steady, linearized version of the PGEs from the PR have similar form. We showed that this leading order model represents the long wavelength limit of the vertical structure equation in the QG theory and found some analytical solutions of it.

Motions with planetary modulation in zonal direction but with a meridional extent confined to the synoptic scale are investigated in the APR. This regime is motivated by the large body of theoretical studies on the quasi-stationary planetary waves (e.g. Charney and Drazin, 1961; Dickinson, 1968a; Matsuno, 1970; Tung and Lindzen, 1979) where it is assumed that the advection of the relative vorticity and of the planetary vorticity by the geostrophic wind are of the same magnitude. In accordance with these studies, we assume in the APR the same magnitude of the background potential temperature variations as in the PR. We resolve the planetary and the synoptic time scales and, in addition to the planetary zonal and meridional synoptic scales, the synoptic zonal scale too. As a leading order model we obtained a condition for a horizontally uniform background and the QG model. This model determines the evolution of the leading order synoptic potential vorticity on the synoptic time scale. The next order model equations represent a coupling between the planetary evolution of the leading order synoptic PV field, the synoptic evolution of the planetary scale vorticity field and the synoptic dynamics of higher order PV corrections. In the case of small meridional velocities we showed that the dynamics evolves only on the planetary time scale and derived a closed transport equation for the leading order synoptic PV. Further, we demonstrated that this equation allows anisotropic Rossby waves and that it does not contain a feedback from the synoptic zonal scale. In the case when the planetary scales are left out, the APR equations are a generalization of the QG^{+1} model of Muraki et al. (1999) to spherical geometry. The QG^{+1} model accounts for the leading order corrections to the QG theory and our general equations for the APR show that such corrections influence the planetary scale dynamics.

In order to explore the validity of the derived asymptotic models, we studied the balances in the vorticity transport utilizing a PEs model. As expected, the synoptic spatial and temporal variations of the different terms in the vorticity equation are explained by the QG model. We obtain as leading order balance the divergence-free condition for the horizontal wind, the next order balance is between all terms entering the QG vorticity equation. However, this picture changes drastically when variations on planetary scales are considered. By projecting the terms on different spectral modes, it was shown that the horizontal divergence of the wind is of the same order as the advection of planetary vorticity f and they approximately balance, if modes with a total wavenumber less than or equal to two are considered. Further, we observed that the divergence of the horizontal vorticity flux vanishes for such modes. We demonstrated that both results are consistent with the two scale model for the PR. The synoptic scale averaged vorticity equation of this model reduces to a balance between the advection of planetary vorticity and the planetary divergence of the leading order geostrophically balanced wind. Such planetary divergence term is absent in the classical QG model where the divergence is only due to the higher order ageostrophic wind components. By considering anisotropic modes with planetary zonal scale but with a synoptic meridional extent, we found in the numerical experiments as leading order balance the QG balance and confirmed the results from the APR. On the other hand, no evidence was found in the simulations for the predicted by the PRBF weaker zonal vorticity gradients and we concluded that this asymptotic regime is not applicable for the real atmosphere.

The comparison between the numerical experiments and the asymptotic models can be extended in the present framework by considering the thermodynamic equation or higher order balances between terms on the planetary scale. The asymptotic analysis revealed that some higher order terms involve corrections to the leading order wind. These corrections can be calculated from the model output by considering only the divergent part of the wind. One can apply a time filtering to the data too, in order to distinguish for example between the vorticity tendency on the planetary and on the synoptic time scale. Of course the best way to prove the validity of the asymptotic models is to solve the equations numerically and see if they reproduce the planetary scale atmospheric flow. The two scale PR is of particular interest here since it can be used as a global model: it accounts for planetary-synoptic interactions and allows order one variations of the Coriolis parameter f. One important question is how the model behaves in the tropics where f tends to zero. This would mean that the geostrophically balanced leading order wind has a singularity at the equator. However, the asymptotic analysis of Majda and Klein (2003) showed that the background temperature field in the tropics is horizontally uniform (also known as the weak temperature gradient approximation). This condition on the temperature implies a vanishing leading order pressure gradient which in the case of the single scale PR compensates the growth due to f. In the case of the two scale PR further analysis is required, this model should be matched in a systematic way to the intraseasonal planetary equatorial synoptic scale model of Majda and Klein (2003).

The multiple scales asymptotic approach in this thesis can be easily applied for the derivation of reduced models for the ocean dynamics. In the ocean the scale separation between the planetary and synoptic scales is much more pronounced than in the atmosphere. Whereas the characteristic length for the planetary scale flow in the ocean remains the earth's radius, the synoptic eddies have a length scale only of the order of 50 km. Because of this, we expect asymptotic regimes for the synoptic and planetary scales in the ocean to be much more pronounced in observation and simulation data. We briefly discuss which modifications are required in the asymptotic approach when applying it for ocean studies. First, the governing equations change: the continuity equation reduces to an incompressibility condition; in the absence of salinity the density can be expressed as a function only of the temperature (equation of state) and the thermodynamic equation takes the form of a transport equation for the density. Since the pressure is hydrostatically balanced to a high accuracy, the Mach number in the equations becomes of the order of the Froude number. For comparison, we have in the atmosphere $M \sim Fr$ too, because we insert in the definition of Fr (and M) the scale height h_{sc}, denoting the e-folding length of the hydrostatically balanced pressure. Taking into account characteristic values for the ocean depth $\approx$ 4 km and for the horizontal velocity $\approx$ 10 cm/s, the Rossby and Froude numbers for the ocean can be expressed in terms of the universal parameter ε as: $Ro \sim \varepsilon$ and $Fr \sim \varepsilon^4$. The rescaled coordinates $\boldsymbol{X}_S = \varepsilon \boldsymbol{x}'$ and $\boldsymbol{X}_P = \varepsilon^3 \boldsymbol{x}'$ resolve the synoptic and planetary length scales, respectively. Since the characteristic vertical length scale of the planetary and synoptic motions in the ocean is about 1 km, we introduce a new vertical coordinate $z = \frac{1}{\varepsilon} z'$. With the above mentioned distinguished limit and rescaled coordinates we rederived the PG and the QG equations for the ocean as well as the two scale model of Pedlosky (1984). In order to study the feedback from the synoptic scale to the planetary scale, higher order asymptotic equations have to be considered as we have done this for the atmosphere. However, we stop the discussion here, since it goes beyond the scope of the thesis.

We derived in this thesis simplified models by reducing the full hydro- and thermodynamic equations on the basis of asymptotic analysis. There are other more empirical approaches for the construction of low order models, which are based on the fitting of mathematical models to observation or simulation data. Such models proved to be a useful tool for understanding the low-frequency variability of the planetary scale flow. In the recent works by Majda et al. (2006) and Franzke et al. (2008) hidden Markov models (HMMs) were utilized successfully for determining metastable regime behavior of planetary waves. Horenko et al. (2008b) presented a method which simultaneously combines the metastability analysis of the HMM with dimension reduction and provides a reduced model in the form of multidimensional stochastic differential equations. This method employs the concept of local principal component analysis (Horenko et al., 2006b) in combination with the fitting of stochastic models for the dynamics within the different metastable states (Horenko et al., 2006a). We demonstrated (Horenko et al., 2008b) the performance of the technique by analyzing surface temperature data for Europe. In comparison with standard multidimensional autoregressive methods (such as the seasonal autoregressive moving average model), the new method is much less computationally expensive. Further, it provides additional insight into the dynamics of the system in the form of a Markov jump process describing the transitions between the hidden metastable states and in the form of correlation patterns characterizing the leading modes of variability within each metastable state. Horenko et al. (2008a) extended the method in order to study time series with gaps showing some memory in the underlying process. Using the idea of extended space representation (Horenko, 2008) such processes can be casted into the Markovian framework. We applied the new method (Horenko et al., 2008a) for analyzing 500 hPa geopotential height fields (daily mean values from the ERA 40 data set for a period of 44 winters) and identified two metastable states characterizing a weakening of the zonal flow. We found that the time evolution of the most blocking events in the considered atmospheric region is described by the hidden probability paths for these two states.

Another approach for the construction of reduced models, often combined with some empirical method, is based on a truncation of the degrees of freedom of the large-scale solution by considering a small number of horizontal or vertical modes (e.g. Kasahara, 1977; Schubert, 1985; Achatz and Opsteegh, 2003). The unresolved scales are then parameterized, e.g., applying a linear regression fitting procedure (Tanaka, 1991; Achatz and Branstator, 1999). In our approach, we filter the governing equations through the asymptotic technique, the new equations are valid for some particular scales but the full 3D structure of the solution is retained. The unresolved scales, e.g., the synoptic scales in the single scale PR, can be represented using the same methods as in the mode truncation approach. Recently a stochastic mode reduction strategy referred to as MTV (Majda et al., 2003; Franzke et al., 2005; Franzke and Majda, 2006) was introduced. An additional asymptotic analysis may be advantageous for the MTV procedure. As we shown applying two scale asymptotic expansions, our method depicts the important interaction terms accounting for the feedback from the smaller scales (the synoptic scales) to the planetary scale flow. One may use the stochastic mode reduction strategy for closing only such terms, which may considerably reduce the computation time for the MTV procedure. The unified multiple scales asymptotic technique itself may be viewed as a strategy for constructing models for the smaller scales. For the two scale PR we derived such reduced model, it describes the synoptic scale motion and is coupled to the planetary scale flow. Thus, the asymptotic method gives the possibility systematically to build a hole hierarchy of coupled reduced models covering a width range of scales: from the meso up to the planetary scale. Such models will be a useful tool for studying multiple scales phenomena in the atmosphere.

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Appendix A

A.1 Spherical coordinates

In this appendix we introduce the spherical coordinate system and show how some operators, e.g., the nabla operator, are transformed into the new coordinate system.

Using the spherical coordinates λ, ϕ and r and Cartesian unit vectors $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$, we can represent an arbitrary vector $\boldsymbol{r}$ as

$$\boldsymbol{r} = r\cos\lambda\cos\phi\boldsymbol{e}_x + r\sin\lambda\cos\phi\boldsymbol{e}_y + r\sin\phi\boldsymbol{e}_z\,, \tag{A.1}$$

where λ denotes longitude, ϕ latitude and r is the distance from the center of the earth. The spherical coordinates can be expressed in terms of the Cartesian coordinates $x, y,$ and z as

$$r = \sqrt{x^2+y^2+z^2}\,, \tag{A.2}$$

$$\phi = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\,, \tag{A.3}$$

$$\lambda = \arctan\left(\frac{y}{x}\right)\,. \tag{A.4}$$

Using the last equations, we obtain the following useful relations

$$\frac{\partial r}{\partial x} = \cos\phi\cos\lambda\,, \frac{\partial r}{\partial y} = \cos\phi\sin\lambda\,, \frac{\partial r}{\partial z} = \sin\phi\,, \tag{A.5}$$

$$\frac{\partial \lambda}{\partial x} = -\frac{\sin\lambda}{r\cos\phi}\,, \frac{\partial \lambda}{\partial y} = \frac{\cos\lambda}{r\cos\phi}\,, \frac{\partial \lambda}{\partial z} = 0\,, \tag{A.6}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi\cos\lambda}{r}\,, \frac{\partial \phi}{\partial y} = -\frac{\sin\phi\sin\lambda}{r}\,, \frac{\partial \phi}{\partial z} = \frac{\cos\phi}{r}\,. \tag{A.7}$$

In the new coordinate system (λ, ϕ, r) we have as unit vectors $\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$. Applying the definition of the unit vector $\boldsymbol{e}_i$ belonging to the coordinate i

$$\boldsymbol{e}_i = \frac{1}{|\partial \boldsymbol{r}/\partial i|}\frac{\partial \boldsymbol{r}}{\partial i}\,, \tag{A.8}$$

we can express $\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$ in terms of $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$

$$\boldsymbol{e}_\lambda = -\sin\lambda \boldsymbol{e}_x + \cos\lambda \boldsymbol{e}_y\,, \tag{A.9}$$
$$\boldsymbol{e}_\phi = -\cos\lambda\sin\phi \boldsymbol{e}_x - \sin\lambda\sin\phi \boldsymbol{e}_y + \cos\phi \boldsymbol{e}_z\,, \tag{A.10}$$
$$\boldsymbol{e}_r = \cos\lambda\cos\phi \boldsymbol{e}_x + \sin\lambda\cos\phi \boldsymbol{e}_y + \sin\phi \boldsymbol{e}_z\,. \tag{A.11}$$

We can solve the above equations for $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$

$$\boldsymbol{e}_x = \cos\lambda\cos\phi \boldsymbol{e}_r + \sin\lambda \boldsymbol{e}_\lambda - \cos\lambda\sin\phi \boldsymbol{e}_\phi\,, \tag{A.12}$$
$$\boldsymbol{e}_y = \sin\lambda\cos\phi \boldsymbol{e}_r + \cos\lambda \boldsymbol{e}_\lambda - \sin\phi\sin\lambda \boldsymbol{e}_\phi\,, \tag{A.13}$$
$$\boldsymbol{e}_z = \sin\phi \boldsymbol{e}_r + \cos\phi \boldsymbol{e}_\phi\,. \tag{A.14}$$

The Cartesian Nabla operator can be written formally as

$$\begin{aligned}\nabla &= \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z} \\ &= \boldsymbol{e}_x\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \lambda}{\partial x}\frac{\partial}{\partial \lambda} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\right) + \boldsymbol{e}_y\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \lambda}{\partial y}\frac{\partial}{\partial \lambda} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right) \\ &+ \boldsymbol{e}_z\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \lambda}{\partial z}\frac{\partial}{\partial \lambda} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}\right)\,.\end{aligned} \tag{A.15}$$

Making use of (A.2), (A.3), (A.4) (in order to compute the partial derivatives of λ, ϕ, r with respect to x, y, z) and of (A.12), (A.13), (A.14), the Nabla operator takes in spherical coordinates the form

$$\nabla = \frac{\boldsymbol{e}_\lambda}{r\cos\phi}\frac{\partial}{\partial \lambda} + \frac{\boldsymbol{e}_\phi}{r}\frac{\partial}{\partial \phi} + \boldsymbol{e}_r\frac{\partial}{\partial r}\,. \tag{A.16}$$

In the spherical coordinate system some of the unit vectors $\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$ depend on the coordinates λ and ϕ.

$$\frac{\partial}{\partial\phi}\boldsymbol{e}_\phi = -\cos\lambda\cos\phi\boldsymbol{e}_x - \sin\lambda\cos\phi\boldsymbol{e}_y - \sin\phi\boldsymbol{e}_z = -\boldsymbol{e}_r\,, \tag{A.17}$$

$$\frac{\partial}{\partial\phi}\boldsymbol{e}_\lambda = 0\,, \tag{A.18}$$

$$\frac{\partial}{\partial\phi}\boldsymbol{e}_r = \boldsymbol{e}_\phi\,, \tag{A.19}$$

$$\frac{\partial}{\partial\lambda}\boldsymbol{e}_r = \cos\phi\boldsymbol{e}_\lambda\,, \tag{A.20}$$

$$\frac{\partial}{\partial\lambda}\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_\lambda\,, \tag{A.21}$$

$$\frac{\partial}{\partial\lambda}\boldsymbol{e}_\lambda = -\cos\phi\boldsymbol{e}_r + \sin\phi\boldsymbol{e}_\phi\,, \tag{A.22}$$

$$\frac{\partial}{\partial r}\boldsymbol{e}_\lambda = \frac{\partial}{\partial r}\boldsymbol{e}_\phi = \frac{\partial}{\partial r}\boldsymbol{e}_r = 0\,. \tag{A.23}$$

Note that

$$\frac{\partial\boldsymbol{e}_{(x,y,z)}}{\partial(\lambda,\phi,r)} = 0\,. \tag{A.24}$$

Thus, we can easily derive the following useful relationship for the substantial (material) derivative of the unit vectors in spherical coordinates

$$\frac{d}{dt}\boldsymbol{e}_\lambda = \frac{u\tan\phi}{r}\boldsymbol{e}_\phi - \frac{u}{r}\boldsymbol{e}_r\,, \tag{A.25}$$

$$\frac{d}{dt}\boldsymbol{e}_\phi = -\frac{u\tan\phi}{r}\boldsymbol{e}_\lambda - \frac{v}{r}\boldsymbol{e}_r\,, \tag{A.26}$$

$$\frac{d}{dt}\boldsymbol{e}_r = \frac{u}{r}\boldsymbol{e}_\lambda + \frac{v}{r}\boldsymbol{e}_\phi\,. \tag{A.27}$$

Here u, v and w are components of the velocity vector in the direction of $\boldsymbol{e}_\lambda, \boldsymbol{e}_\phi$ and $\boldsymbol{e}_r$, respectively. They are defined as

$$u = \frac{d}{dt}\lambda,\; v = \frac{d}{dt}\phi,\; w = \frac{d}{dt}r\,. \tag{A.28}$$

Using (A.25), (A.26), (A.27), it can be easily shown for $\boldsymbol{u} = u\boldsymbol{e}_\lambda + v\boldsymbol{e}_\phi$ that

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial v cos\phi}{\partial\phi}\right)\,, \tag{A.29}$$

and

$$\boldsymbol{e}_r \cdot (\nabla \times \boldsymbol{u}) = \frac{1}{r\cos\phi}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial u cos\phi}{\partial \phi}\right) . \tag{A.30}$$

For the $3D$ divergence we obtain:

$$\nabla \cdot (\boldsymbol{u} + w\boldsymbol{e}_r) = \frac{1}{r\cos\phi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial v cos\phi}{\partial \phi}\right) + \frac{1}{r^2}\frac{\partial r^2 w}{\partial r} . \tag{A.31}$$

Applying the horizontal Laplacian to a scalar ζ, we have

$$\Delta_h \zeta = \frac{1}{r^2 cos^2\phi}\left(\frac{\partial^2}{\partial \lambda^2}\zeta + \cos\phi\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial}{\partial \phi}\zeta\right)\right) . \tag{A.32}$$

Spherical harmonics

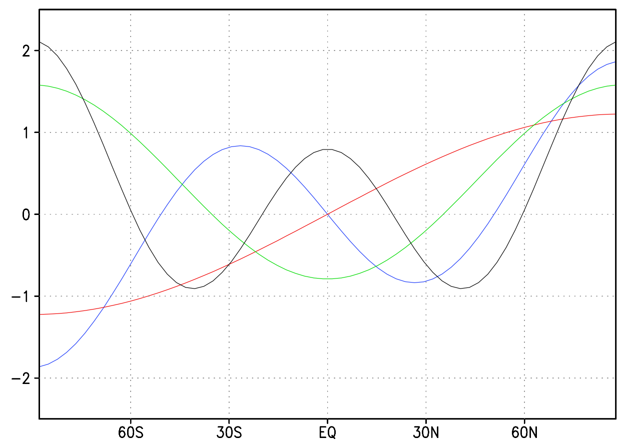

Figure A.1: The leading spherical harmonics $Y_n^m(\lambda, \mu) = P_n^m(\mu)e^{im\lambda}$: red Y_1^0, green Y_2^0, blue Y_3^0 and black Y_4^0. For the notation see (6.11) and the text below.

A.2 PV equation

We differentiate (3.31) w.r.t. z and multiply it by $f/\rho^{(0)}$

$$\frac{\partial}{\partial t}\left(\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\right)+f\boldsymbol{u}^{(0)}\cdot\nabla\left(\frac{1}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\right)+\frac{f}{\rho^{(0)}}\underbrace{\frac{\partial\boldsymbol{u}^{(0)}}{\partial z}\cdot\nabla\Theta^{(2)}}_{=0\,(3.32)}$$
$$+\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)=\frac{f}{\rho}\frac{\partial}{\partial z}S_\theta^{(5)}\,. \tag{A.33}$$

For the last term we can write also

$$\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)=w^{(3)}\frac{\partial}{\partial z}\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}+\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial\rho^{(0)}w^{(3)}}{\partial z}\,. \tag{A.34}$$

From (3.24) and (3.33) we obtain

$$\frac{1}{\rho^{(0)}}\frac{\partial\rho^{(0)}w^{(3)}}{\partial z}=\frac{1}{f}\boldsymbol{u}^{(0)}\cdot\nabla f\,. \tag{A.35}$$

Using the last two equations, the fourth term on the l.h.s. of (A.33) takes the form

$$\frac{f}{\rho^{(0)}}\frac{\partial}{\partial z}\left(w^{(3)}\frac{\partial}{\partial z}\Theta^{(2)}\right)=w^{(3)}\frac{\partial}{\partial z}\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}+\frac{1}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}\boldsymbol{u}^{(0)}\cdot\nabla f\,. \tag{A.36}$$

So (A.33) can be finally written in the form

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}^{(0)}\cdot\nabla+w^{(3)}\frac{\partial}{\partial z}\right)\frac{f}{\rho^{(0)}}\frac{\partial\Theta^{(2)}}{\partial z}=S_{pv}{}^{(5)}\,, \tag{A.37}$$

where we have defined $S_{pv}{}^{(5)}=f/\rho^{(0)}\partial S_\theta{}^{(5)}/\partial z$.

A.3 Solvability condition for the PRBF

In this appendix we derive a solvability condition for (5.68) under the assumption of positive F. In this case $\pi^{(2)}$ takes the same form as in (5.47) but with f_1 and f_2 defined as

$$f_1 = \alpha\sqrt{z}I_1(2\alpha\sqrt{z})\,, \tag{A.38}$$

$$f_2 = 2\alpha\sqrt{z}K_1(2\alpha\sqrt{z})\,. \tag{A.39}$$

Here I_1,K_1 are the modified Bessel functions of the first and second kind, respectively. Further, we make the additional assumption that our solution consists only of the f_1 component and we set C_2 and π_0 from (5.47) to zero. All terms in Q' from (5.69) which can be represented as some z-independent factor multiplied by zf_1 will lead to a resonance, since Q' is divided by z and f_1 is a eigenfunction to the linear operator in (5.68). In order to suppress these terms, we have to set their coefficients to zero. This will provide an equation for the horizontal structure of $C_1(\boldsymbol{X}_P, t_P)$.

We proceed with the identification of the terms in Q' parallel to zf_1. For this purpose, we make some preliminary calculations. We have some useful relations for the modified Bessel function $I_n(x)$

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\Gamma(n+k+1)}\,, \tag{A.40}$$

$$\frac{d}{dx}I_n(x) = I_{n-1}(x) - \frac{n}{x}I_n(x)\,, \tag{A.41}$$

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x}I_n(x)\,. \tag{A.42}$$

With the help of these relations we can calculated the following derivatives of f_1

$$\frac{\partial}{\partial z}f_1 = \alpha^2 I_0(2\alpha\sqrt{z})\,, \tag{A.43}$$

$$\frac{\partial^2}{\partial z^2}f_1 = \frac{\alpha^3}{\sqrt{z}}I_{-1}(2\alpha\sqrt{z}) = \frac{\alpha^2}{z}f_1\,, \tag{A.44}$$

$$\frac{\partial^3}{\partial z^3}f_1 = -\frac{\alpha^2}{z^2}f_1 + \frac{\alpha^3}{\sqrt{z}}I_0\,, \tag{A.45}$$

$$\frac{\partial}{\partial \phi}f_1 = 2\alpha\alpha' z I_0(2\alpha\sqrt{z})\,, \tag{A.46}$$

$$\frac{\partial}{\partial \phi}\frac{\partial}{\partial z}f_1 = 2\alpha\alpha'\left(I_0 + f_1\right)\,, \tag{A.47}$$

$$\frac{\partial^2}{\partial \phi^2}f_1 = 2zI_0(\alpha'^2 + \alpha\alpha'') + 4\alpha'^2 z f_1\,, \tag{A.48}$$

$$\int_0^z f_1(\phi, z')\mathrm{d}z' = zI_2(2\alpha\sqrt{z})\,, \tag{A.49}$$

$$\frac{\partial}{\partial\phi}\int_0^z f_1(\phi, z')\mathrm{d}z' = \frac{2z\alpha'}{\alpha}\left(f_1 - I_2\right)\,, \tag{A.50}$$

$$\frac{\partial}{\partial\phi}\frac{\partial^2}{\partial z^2}f_1 = \frac{2\alpha\alpha'}{z}f_1 + 2\alpha^3\alpha' I_0\,, \tag{A.51}$$

where we have dropped all subscripts of the planetary variables. All linear in $\pi^{(2)}$ terms from Q' can be represented as linear combinations of the terms above. From there it is easy to see which will give contributions parallel to zf_1. In order to identify the contributions from the nonlinear terms in Q', we calculate the products

$$f_1\frac{\partial^2}{\partial z^2}f_1 = \frac{\alpha^4}{\Gamma(2)}f_1 + \frac{\alpha^6}{\Gamma(3)}zf_1 + \frac{\alpha^8}{2!\Gamma(4)}z^2 f_1 + \ldots\,, \tag{A.52}$$

$$\frac{\partial}{\partial\phi}f_1\frac{\partial^2}{\partial z^2}f_1 = 2\alpha^3\alpha'\left\{\frac{1}{\Gamma(1)}f_1 + \frac{\alpha^2}{\Gamma(2)}zf_1 + \frac{\alpha^4}{2!\Gamma(3)}z^2 f_1 + \ldots\right\}\,, \tag{A.53}$$

$$zI_2\frac{\partial^3}{\partial z^3}f_1 = \alpha^3\sqrt{z}I_0 I_2 - \frac{\alpha^4}{\Gamma(3)}f_1 - \frac{\alpha^6}{\Gamma(4)}zf_1 - \frac{\alpha^8}{\Gamma(5)}z^2 f_1 + \ldots\,. \tag{A.54}$$

We consider the zonal advection of $\frac{f}{N}\frac{\partial^2}{\partial z^2}\pi^{(2)}$, it can be written in terms of f_1 as

$$\frac{u^{(0)}}{a\cos\phi}\frac{\partial}{\partial\lambda}\frac{f}{N}\frac{\partial^2}{\partial z^2}\pi^{(2)} = \frac{u^{(0)}}{a\cos\phi}\frac{f}{N}\frac{\partial}{\partial\lambda}C_1\frac{\partial^2}{\partial z^2}f_1 \tag{A.55}$$

$$= \frac{f}{a\cos\phi N}\left\{-\frac{1}{fa}\frac{\partial}{\partial\phi}\pi^{(2)} - \frac{u^{(-1)2}}{a\cos\phi}\right\}\frac{\partial}{\partial\lambda}C_1\frac{\partial^2}{\partial z^2}f_1 \tag{A.56}$$

$$= -\frac{1}{a^2\cos\phi N}\left\{f_1\frac{\partial}{\partial\phi}C_1 + C_1\frac{\partial}{\partial\phi}f_1 + u^{(-1)2}\tan\phi\right\}\frac{\partial}{\partial\lambda}C_1\frac{\partial^2}{\partial z^2}f_1 \tag{A.57}$$

Making use of (A.52) and (A.53), we collect all terms in the equation above which are multiplied with zf_1, these are

$$-\frac{1}{a^2 N\cos\phi}\left\{\frac{\alpha^6}{\Gamma(3)}\frac{\partial}{\partial\lambda}C_1\frac{\partial}{\partial\phi}C_1 + \frac{2\alpha^5\alpha'}{\Gamma(2)}C_1\frac{\partial}{\partial\lambda}C_1\right\}zf_1 - \frac{\tan\phi}{a^2 N\cos\phi}\frac{F^4\alpha^4}{\beta^2 N^2}\frac{\partial}{\partial\lambda}C_1 zf_1\,. \tag{A.58}$$

When substituted in the right hand side of (5.68), the terms will lead to an unbounded growth of $\pi^{(3)}$. Therefore their coefficients have to be set to zero. We proceed analogous with the remaining terms in Q' and we can derive an equation for $C_1(\lambda, \phi, t)$

$$\left\{a_1\frac{\partial^2}{\partial\lambda^2}+a_2\frac{\partial^2}{\partial\phi^2}+a_3\frac{\partial}{\partial\phi}+a_4\right\}\frac{\partial}{\partial\lambda}C_1+a_5\frac{\partial}{\partial\phi}C_1\frac{\partial}{\partial\lambda}C_1+a_6C_1\frac{\partial}{\partial\lambda}C_1=0 \tag{A.59}$$

Here α_i are known functions of N and F and are given through

$$\alpha_1=-\frac{F^2\alpha}{\beta N}\frac{1}{fa^3\cos^3\phi} \tag{A.60}$$

$$\alpha_2=-\frac{F^2\alpha}{\beta N}\frac{1}{a^3f\cos\phi} \tag{A.61}$$

$$\alpha_3=\frac{F^2\alpha}{\beta N}\left(\frac{1}{a^3\cos^2\phi}+\frac{1}{f^2a^2}\right) \tag{A.62}$$

$$\begin{aligned}\alpha_4=&-4\alpha'^2\frac{F^2\alpha}{\beta N}+\frac{1}{Nfa^2\cos\phi}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\alpha F^2\right)\\&-\frac{\beta}{Nf^2a\cos\phi}\left(\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\alpha F^2\right)\frac{\beta}{f}\frac{F^2\alpha}{Nf\beta a^2\cos\phi}+\frac{2\alpha'F^2}{Na^2f^2\cos\phi}+\frac{\beta}{f}\frac{\alpha F^2}{\beta N}\frac{1}{a^2\cos\phi}\\&-\frac{\beta}{f}\frac{1}{Nfa^2\cos\phi}\frac{\partial}{\partial\phi}\frac{F^2\alpha}{\beta}+\frac{F^2\alpha}{\beta N}\frac{2F\alpha\alpha'}{Nfa^2\cos\phi}\end{aligned} \tag{A.63}$$

$$\alpha_5=\frac{\alpha^6}{a^2N\cos\phi\Gamma(3)} \tag{A.64}$$

$$\alpha_6=\frac{\beta\alpha^6}{fNa^2\cos\phi\Gamma(3)}\frac{2\alpha^5\alpha'}{a^2N\cos\phi}\left(\frac{1}{\Gamma(3)}+\frac{1}{\Gamma(2)}\right)-\frac{\beta\alpha^6}{Nfa\cos\phi\Gamma(4)} \tag{A.65}$$

A.4 The Ekman layer and the planetary scale

In this appendix we study the effect of the strong zonal winds in the PRBF on the planetary boundary layer. As shown by Klein et al. (2005) the classical Ekman layer theory is derived by utilizing a two scale asymptotic expansion: horizontal and vertical scales of the order of 200 m and a 20 s time scale are resolved in addition to the synoptic spatial and temporal scales. With this expansion the vertical turbulent fluxes are obtained by averaging over the small-scale horizontal variables. Here we simplify the analysis and resolve in addition to the planetary scales only the 200 m vertical scale, we will make use of the analysis by Klein et al. (2005) and associate the vertical advection terms in expansion of the momentum equation with the turbulent momentum fluxes. The rescaled vertical coordinate for the Ekman layer reads

$$z_E = \frac{z}{\varepsilon^2} \,. \tag{A.66}$$

In the PRBF assume in the free atmosphere a constant background zonal wind $u^{(-1)}$. In the present regime we allow arbitrary directions of this background wind, deviations from the zonal direction may result from the strong frictional effects in the boundary layer. Thus, the expansion for the horizontal wind takes the form

$$\boldsymbol{u} = \varepsilon^{-1}\boldsymbol{u}^{(-1)}(\lambda_P, \phi_P, z_E, t_p) + \boldsymbol{u}^{(0)}(\lambda_P, \phi_P, z_E, t_p) + \mathcal{O}(\varepsilon) \ . \tag{A.67}$$

From the leading order horizontal and vertical momentum balance we obtain $p^{(0)} = const$, which combined with the expansion of the equation of state gives $\rho^{(0)} = const$. The $\mathcal{O}(\varepsilon^{-5})$ vertical momentum balance implies that the pressure $p^{(1)}$ shows no vertical variations in the Ekman layer

$$\mathcal{O}\big(\varepsilon^{-5}\big): \qquad \frac{\partial}{\partial z_E} p^{(1)} = 0 \,. \tag{A.68}$$

From the horizontal momentum equation we have

$$\mathcal{O}(1): \qquad f\boldsymbol{e}_r \times \boldsymbol{u}^{(-1)} + \underbrace{w^{(3)}\frac{\partial}{\partial z_E}\boldsymbol{u}^{(-1)}}_{-K\frac{\partial^2 \boldsymbol{u}^{(-1)}}{\partial z_E^2}} = -\frac{1}{\rho^{(0)}}\nabla_P p^{(1)} \,. \tag{A.69}$$

As we mentioned at the beginning, we associate the vertical momentum advection with the vertical transport due to turbulent eddies and use a simple gradient flux ansatz in order to parameterize it. Further, we assume as boundary conditions: i) no surface wind ii) the wind at the upper boundary of the Ekman layer matches the geostrophically balanced wind in the free atmosphere iii) the pressure in the Ekman layer matches the value of the pressure from the free atmosphere. The last condition implies that the pressure depends only on ϕ, since we have zonal wind in the free atmosphere. Summarized the boundary conditions read

$$\boldsymbol{u}^{(-1)}(z_E = 0) = 0\,, \tag{A.70}$$
$$v^{(-1)}(z_E \to \infty) = 0\,, \tag{A.71}$$
$$u^{(-1)}(z_E \to \infty) = u_g^{(-1)}\,, \tag{A.72}$$
$$p^{(1)} = p_g^{(1)}(\phi)\,, \tag{A.73}$$

where we denoted with the subscript g the variables from the free atmosphere. Making use of the last boundary condition, the momentum equation (A.69) can be written as

$$-K\frac{\partial^2 u}{\partial z^2} - fv = 0\,, \tag{A.74}$$
$$-K\frac{\partial^2 v}{\partial z^2} + fu = -\frac{1}{a\rho}\frac{\partial p}{\partial \phi}\,, \tag{A.75}$$
$$\tag{A.76}$$

where we dropped the indices of $\rho^{(0)}, \boldsymbol{u}^{(-1)}$ and $p^{(1)}$ and the subscripts of the independent variables. From the meridional component of the momentum equation we obtain

$$u = \frac{1}{f}\left(K\frac{\partial^2 v}{\partial z^2} - \frac{1}{a\rho}\frac{\partial p}{\partial \phi}\right). \tag{A.77}$$

Substituting the last result in the zonal component, we obtain a fourth order ODE for v

$$\frac{\partial^4 v}{\partial z^4} + \underbrace{\frac{f^2}{K^2}}_{=\alpha^2} v = 0\,. \tag{A.78}$$

This equation is solved by

$$v = \sum_{k=1}^{4} c_k(\lambda, \phi, t) e^{\beta_k z}\,, \tag{A.79}$$

where

$$\beta_{1,2} = \pm\sqrt{\frac{i \sin\phi}{K}}\,, \tag{A.80}$$
$$\beta_{3,4} = \pm\sqrt{\frac{-i \sin\phi}{K}}\,. \tag{A.81}$$

Making use of the relations

$$\sqrt{i} = \exp(\frac{i}{2}(\frac{\pi}{2} + 2\pi n)) = \sqrt{\frac{1}{2}}\left((-1)^n + i(-1)^n\right) , n = 0, 1, \dots , \tag{A.82}$$

$$\sqrt{-1i} = \exp(\frac{3i}{2}(\frac{\pi}{2} + 2\pi n)) = \sqrt{\frac{1}{2}}\left((-1)^{n+1} + i(-1)^n\right) , n = 0, 1, \dots , \tag{A.83}$$

we obtain for the meridional velocity

$$v = c_1 e^{\sqrt{\frac{\alpha}{2}}(1+i)z} + c_2 e^{-\sqrt{\frac{\alpha}{2}}(1+i)z} + c_3 e^{-\sqrt{\frac{\alpha}{2}}(1-i)z} + c_4 e^{\sqrt{\frac{\alpha}{2}}(1-i)z} . \tag{A.84}$$

We set $c_1 = c_4 = 0$ to prevent an unbounded growth of the solution for $z \to \infty$. In this case the conditions $v(z \to \infty) = 0$ and $u(z \to \infty) = u_g$ are satisfied. From the condition $v(z = 0) = 0$ we have

$$c_2 + c_3 = 0 . \tag{A.85}$$

Substituting the condition $u(z = 0) = 0$ in the meridional momentum balance, we have

$$\frac{\partial^2 v}{\partial z^2} = -\frac{1}{a\rho K}\frac{\partial p_g}{\partial \phi} = -\frac{f}{K}u_g \tag{A.86}$$

where we have used the definition of the geostrophic wind. This gives the constraint for the constant

$$c_3 = -\frac{-iu_g}{2} \tag{A.87}$$

and we obtain for the meridional velocity the classical result

$$v = u_g \sin(\sqrt{\frac{\alpha}{2}}z)e^{-\sqrt{\frac{\alpha}{2}}z} . \tag{A.88}$$

A.5 Blockings

In this section we derive two asymptotic regimes for length and time scales characterizing blocking situations in the atmosphere. In the first regime we consider isotropic horizontal length scales of the order of the external Rossby deformation radius (Oboukhov scale), this regime corresponds to a typical blocking anticyclone. The second regime describes anisotropic blockings like the Omega blocking with meridional scale of the order of the external Rossby deformation radius but with a planetary zonal extent. In both regimes we resolve the corresponding advective time scales. In order to distinguish between the external Rossby deformation radius and the planetary scale we apply the recently introduced modified asymptotic approach of Klein (2007). In this new approach the distinguished limit for the Mach, Froude and Rossby numbers reads (compare with (2.12))

$$M^{\frac{2}{3}} \sim Fr^{\frac{2}{3}} \sim 1/Ro \sim \varepsilon \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{A.89}$$

For simplicity we consider the case of plane geometry.

Regime 1

The rescaled spacial and temporal scales for the first asymptotic regime read

$$x_O = \varepsilon^{\frac{5}{2}} x\,, \tag{A.90}$$
$$y_O = \varepsilon^{\frac{5}{2}} y\,, \tag{A.91}$$
$$t_O = \varepsilon^{\frac{5}{2}} t\,, \tag{A.92}$$
$$z = z\,. \tag{A.93}$$

We can expand the Coriolis parameter f around some constant latitude ϕ_o as

$$f = \sin(\phi) = \sin(\phi_0 + \frac{y}{a}) = \sin(\phi_0 + \sqrt{\varepsilon}\frac{y_O}{a}) = f_0 + \sqrt{\varepsilon}\beta y_O + \mathcal{O}(\varepsilon)\ . \tag{A.94}$$

The expansion for the potential temperature and the wind takes the form

$$\theta = 1 + \varepsilon\Theta^{(\frac{2}{2})}(z) + \varepsilon^{\frac{3}{2}}\Theta^{(\frac{3}{2})}(x_O, y_O, t_O, z) + \mathcal{O}(\varepsilon^{\frac{4}{2}})\,, \tag{A.95}$$
$$\boldsymbol{u} = \boldsymbol{u}^{(0)}(x_O, y_O, t_O, z) + \varepsilon^{\frac{1}{2}}\boldsymbol{u}^{(\frac{1}{2})}(x_O, y_O, t_O, z) + \mathcal{O}(\varepsilon^{\frac{2}{2}})\,. \tag{A.96}$$

Next, we summarize the equations resulting from the asymptotic expansion of the governing equations

Horizontal Momentum Balance

$$\mathcal{O}(\varepsilon^{i-\frac{1}{2}}): \qquad \nabla_O p^{(i)} = 0\,, \qquad i = 0, \frac{1}{2}, 1\,, \tag{A.97}$$

$$\mathcal{O}(\varepsilon^{1}): \qquad f_0 \boldsymbol{e}_r \times \boldsymbol{u}^{(0)} = -\frac{1}{\rho^{(0)}} \nabla_O p^{(\frac{3}{2})}\,, \tag{A.98}$$

$$\mathcal{O}(\varepsilon^{\frac{3}{2}}): \qquad f_0 \boldsymbol{e}_r \times \boldsymbol{u}^{(\frac{1}{2})} + \beta y_O \boldsymbol{e}_r \times \boldsymbol{u}^{(0)} = -\left(\frac{1}{\rho}\nabla_O p\right)^{(\frac{4}{2})}\,. \tag{A.99}$$

Vertical momentum balance

$$\frac{\partial}{\partial z} p^{(i)} = -\rho^{(i)}, \quad i = 0, \ldots \frac{8}{2}\,. \tag{A.100}$$

Continuity equation

$$\mathcal{O}(i): \qquad w^{(i)} = 0,\,, \qquad i = 0, \ldots, \frac{4}{2}\,, \tag{A.101}$$

$$\mathcal{O}(\varepsilon^{\frac{5}{2}}): \qquad \nabla_O \cdot \rho^{(0)} \boldsymbol{u}^{(0)} = 0\,, \tag{A.102}$$

$$\mathcal{O}(\varepsilon^{\frac{6}{2}}): \qquad \nabla_O \cdot \rho^{(0)} \boldsymbol{u}^{(\frac{1}{2})} + \frac{\partial}{\partial z} \rho^{(0)} w^{(\frac{6}{2})} = 0\,. \tag{A.103}$$

Potential temperature

$$\mathcal{O}(\varepsilon^{\frac{7}{2}}): \qquad w^{(\frac{5}{2})} \frac{\partial}{\partial z} \Theta^{(\frac{2}{2})} = 0\,, \tag{A.104}$$

$$\mathcal{O}(\varepsilon^{\frac{8}{2}}): \qquad \frac{\partial}{\partial t_O} \Theta^{(\frac{3}{2})} + \boldsymbol{u}^{(0)} \cdot \nabla_O \Theta^{(\frac{3}{2})} + w^{(\frac{6}{2})} \frac{\partial}{\partial z} \Theta^{(\frac{2}{2})} = 0\,. \tag{A.105}$$

From the momentum equation (A.99) we obtain

$$\nabla_O \cdot \boldsymbol{u}^{(\frac{1}{2})} = -\frac{\beta}{f_0} \boldsymbol{u}^{(0)} \tag{A.106}$$

Combining the last result with (A.103) and (A.105), we derive the following PV equation

$$\left(\frac{\partial}{\partial t_O} + \boldsymbol{u}^{(0)} \cdot \nabla_O\right) \frac{f_0}{\rho^{(0)}} \frac{\partial}{\partial z} \frac{\rho^{(0)} \frac{\partial}{\partial z} \pi^{(\frac{3}{2})}}{\frac{\partial \Theta^{(1)}}{\partial z}} + \beta v^{(0)} = 0\,. \tag{A.107}$$

Regime 2

The rescaled spacial and temporal scales for the second asymptotic regime read

$$X_P = \varepsilon^{\frac{6}{2}} x \,, \tag{A.108}$$
$$y_O = \varepsilon^{\frac{5}{2}} y \,, \tag{A.109}$$
$$t_O = \varepsilon^{\frac{5}{2}} t \,, \tag{A.110}$$
$$t_P = \varepsilon^{\frac{6}{2}} t \,, \tag{A.111}$$
$$z = z \,. \tag{A.112}$$

We use the following expansion for the potential temperature and the horizontal wind

$$\theta = 1 + \varepsilon \Theta^{(\frac{2}{2})}(X_P, z) + \varepsilon^{\frac{3}{2}} \Theta^{(\frac{3}{2})}(X_P, y_O, t_{O,P}, z) + \mathcal{O}(\varepsilon^{\frac{4}{2}}) \,, \tag{A.113}$$
$$\boldsymbol{u} = \boldsymbol{u}^{(0)}(X_P, y_O, t_{O,P}, z) + \varepsilon^{\frac{1}{2}} \boldsymbol{u}^{(\frac{1}{2})}(X_P, y_O, t_{O,P}, z) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \,. \tag{A.114}$$

We introduce the notation

$$\nabla_O = \boldsymbol{e}_y \frac{\partial}{\partial y_O} \,, \tag{A.115}$$
$$\nabla_P = \boldsymbol{e}_x \frac{\partial}{\partial X_P} \tag{A.116}$$

We summarize the results from the asymptotic expansion of the governing equations

Horizontal Momentum Balance

$$\mathcal{O}(\varepsilon^1): \qquad f_0 \boldsymbol{e}_r \times \boldsymbol{u}^{(0)} = -\frac{1}{\rho^{(0)}} \nabla_O p^{(\frac{3}{2})} - \frac{1}{\rho^{(0)}} \nabla_P p^{(\frac{2}{2})} \,, \tag{A.117}$$
$$\mathcal{O}(\varepsilon^{\frac{3}{2}}): \qquad f_0 \boldsymbol{e}_r \times \boldsymbol{u}^{(\frac{1}{2})} + \beta y_O \boldsymbol{e}_r \times \boldsymbol{u}^{(0)} = -\left(\frac{1}{\rho} \nabla_O p\right)^{(\frac{4}{2})} - \left(\frac{1}{\rho} \nabla_P p\right)^{(\frac{3}{2})} \,. \tag{A.118}$$

From the last equations we have

$$\nabla_O \cdot \boldsymbol{u}^{(0)} = \frac{\partial}{\partial y_O} v^{(0)} = 0 \,, \tag{A.119}$$
$$f_0 \nabla_O \cdot \boldsymbol{u}^{(\frac{1}{2})} = \frac{\partial}{\partial y_O} \frac{\partial}{\partial X_P} \pi^{(\frac{3}{2})} - \beta v^{(0)} \,. \tag{A.120}$$

Vertical momentum balance

$$\frac{\partial}{\partial z}p^{(i)} = -\rho^{(i)}, \quad i = 0, \ldots, \frac{8}{2}\,. \tag{A.121}$$

Continuity equation

$$\mathcal{O}(i): \qquad w^{(i)} = 0, \qquad i = 0, \ldots, \frac{4}{2}\,, \tag{A.122}$$

$$\mathcal{O}(\varepsilon^{\frac{5}{2}}): \qquad \nabla_O \cdot \rho^{(0)}\boldsymbol{u}^{(0)} = 0\,, \tag{A.123}$$

$$\mathcal{O}(\varepsilon^{\frac{6}{2}}): \qquad \nabla_O \cdot \rho^{(0)}\boldsymbol{u}^{(\frac{1}{2})} + \nabla_P \cdot \rho^{(0)}\boldsymbol{u}^{(0)} + \frac{\partial}{\partial z}\rho^{(0)}w^{(\frac{6}{2})} = 0\,. \tag{A.124}$$

Potential temperature

$$\mathcal{O}(\varepsilon^{\frac{8}{2}}): \qquad \frac{\partial}{\partial t_O}\Theta^{(\frac{3}{2})} + \boldsymbol{u}^{(0)} \cdot \nabla_O \Theta^{(\frac{3}{2})} + \frac{\partial}{\partial t_P}\Theta^{(\frac{2}{2})} + \boldsymbol{u}^{(0)} \cdot \nabla_P \Theta^{(\frac{2}{2})} + w^{(\frac{6}{2})}\frac{\partial}{\partial z}\Theta^{(\frac{2}{2})} = 0\,. \tag{A.125}$$

The leading order PV equation for this asymptotic regime takes the form

$$\begin{aligned}\left(\frac{\partial}{\partial t_O} + v^{(0)}\frac{\partial}{\partial y_O}\right)\frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}\frac{\partial}{\partial z}\pi^{(\frac{3}{2})}}{\frac{\partial\Theta^{(1)}}{\partial z}} + \beta v^{(0)} + u^{(0)}\frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}\frac{\partial}{\partial X_P}\frac{\partial}{\partial z}\pi^{(1)}}{\frac{\partial\Theta^{(1)}}{\partial z}} \\ + \frac{f_0}{\rho^{(0)}}\frac{\partial}{\partial z}\frac{\rho^{(0)}\frac{\partial}{\partial t_P}\frac{\partial}{\partial z}\pi^{(1)}}{\frac{\partial\Theta^{(1)}}{\partial z}} = 0\,.\end{aligned} \tag{A.126}$$

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