Kapitel 5

Hilfsmittel

In diesem Kapitel werden die Hilfsmittel für die im nächsten Kapitel folgende gemischte Integraldarstellung bereitgestellt. Im Folgenden stellen die eckigen Klammern [.] die Größte-Ganze Funktion auf $\mathbb{R}$ dar.

5.1 Allgemeine Hilfsmittel

In der Clifford Algebra gilt nicht die Produktregel der Differenziation:

Es gilt $\partial(x\overline{x}) = \partial|x|^2 = 2x$, aber für $m > 1$ gilt

$$\begin{aligned}(\partial x)\overline{x} + x(\partial\overline{x}) &= (1-m)\overline{x} + x(m+1) \\ &= \overline{x} + x + m(x-\overline{x}) \\ &= 2x_0 + 2m\sum_{k=1}^{m} x_k e_k \\ &\neq 2x.\end{aligned}$$

Die Produktregel ist also nicht anwendbar, da die Kommutativität fehlt.
Man kann jedoch immer gemischte Potenzen von x und $\overline{x}$ durch Ausnutzung von $x\overline{x} = |x|^2$ verändern.

Es gilt (für $x \in \mathbb{R}^{m+1}$): $|x|^2 = \sum\limits_{k=0}^{m} x_k^2$, und deshalb

$$\begin{aligned}
\partial(x|x|^2) &= \sum_{k=0}^{m} e_k \frac{\partial}{\partial x_k}\left(\sum_{l=0}^{m} x_l e_l \sum_{\mu=0}^{m} x_\mu^2\right) \\
&= \sum_{k=0}^{m}\sum_{l=0}^{m} e_k e_l \frac{\partial x_l}{\partial x_k}|x|^2 + \sum_{k=0}^{m}\sum_{l=0}^{m}\sum_{\mu=0}^{m} e_k e_l x_l \frac{\partial x_\mu^2}{\partial x_k} \\
&= \sum_{k=0}^{m} e_k^2 |x|^2 + \sum_{k=0}^{m} e_k \sum_{l=0}^{m} e_l x_l 2x_k \\
&= (1-m)|x|^2 + x2x \\
&= (\partial x)|x|^2 + x(\partial|x|^2)
\end{aligned}$$

bzw. ($\alpha, \beta \in \mathbb{N}$):

$$\begin{aligned}
\partial(x^\alpha|x|^\beta) &= \sum_{k=0}^{m} e_k \frac{\partial}{\partial x_k}\left\{\left(\sum_{l=0}^{m} x_l e_l\right)^\alpha \left(\sum_{\mu=0}^{m} x_\mu^2\right)^{\frac{\beta}{2}}\right\} \\
&= |x|^\beta(\partial x^\alpha) + \beta x|x|^{\beta-2}x^\alpha \\
&= (\partial x^\alpha)|x|^\beta + x^\alpha(\partial|x|^\beta)
\end{aligned}$$

(mit $\overline{x}$, bzw. Δ analog.)

Satz 5.1.1 *Es gilt*

$$\begin{aligned}
\sum_{\mu=1}^{l} (l-\mu+1)\,\overline{x}^{\,l-\mu} x^{\mu-1} &= \sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]} (l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)} \\
&\quad + \sum_{\mu=\left[\frac{1}{2}(l+3)\right]}^{l} (l-\mu+1)\, x^{2\mu-l-1}|x|^{2(l-\mu)}.
\end{aligned} \tag{5.1}$$

Beweis :

$$\begin{aligned}
&\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]} (l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)} + \sum_{\mu=\left[\frac{1}{2}(l+3)\right]}^{l} (l-\mu+1)\, x^{2\mu-l-1}|x|^{2(l-\mu)} \\
= &\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]} (l-\mu+1)\overline{x}^{l-2\mu+1+\mu-1}x^{\mu-1} + \sum_{\mu=\left[\frac{1}{2}(l+3)\right]}^{l} (l-\mu+1)x^{2\mu-l-1+l-\mu}\overline{x}^{l-\mu} \\
= &\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]} (l-\mu+1)\overline{x}^{l-\mu}x^{\mu-1} + \sum_{\left[\frac{1}{2}(l+3)\right]}^{l} (l-\mu+1)x^{\mu-1}\overline{x}^{l-\mu} \\
= &\sum_{\mu=1}^{l} (l-\mu+1)x^{\mu-1}\overline{x}^{l-\mu},
\end{aligned}$$

da $\left[\frac{1}{2}(l+1)\right]+1=\left[\frac{1}{2}(l+3)\right]$.

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Satz 5.1.2 *(Umformung der Differenziationsregeln)*

Für $2 \le k$ *gilt*

$$\partial\overline{x}^k = (m+2k-1)\overline{x}^{k-1}+(m-1)x\left\{\sum_{\nu=1}^{\left[\frac{k}{2}\right]}\overline{x}^{k-2\nu}|x|^{2(\nu-1)}+\sum_{\nu=\left[\frac{1}{2}(k+2)\right]}^{k-1}x^{2\nu-k}|x|^{2(k-1-\nu)}\right\}$$

Für $1 \le k$ *gilt*

$$\partial x^k = (1-m)\left\{\sum_{\nu=1}^{\left[\frac{1}{2}(k+1)\right]}\overline{x}^{k-2\nu+1}|x|^{2(\nu-1)}+\sum_{\nu=\left[\frac{1}{2}(k+3)\right]}^{k}x^{2\nu-k-1}|x|^{2(k-\nu)}\right\}.$$

Beweis :

$$\partial\overline{x}^k=(m+2k-1)\overline{x}^{k-1}+(m-1)\sum_{\nu=0}^{k-2}\overline{x}^{k-2-\nu}x^{\nu+1}=(m+2k-1)\overline{x}^{k-1}+(m-1)x\sum_{\nu=0}^{k-2}\overline{x}^{k-2-\nu}x^{\nu}$$

1.

$$\begin{aligned}\sum_{\nu=0}^{k-2}\overline{x}^{k-2-\nu}x^{\nu} &= \sum_{\nu=1}^{k-1}\overline{x}^{k-2-(\nu-1)}x^{\nu-1}\\ &= \sum_{\nu=1}^{k-1}\overline{x}^{k-1-\nu}x^{\nu-1}\\ &\overset{(*)\binom{l=k-1}{\mu=\nu}}{=} \sum_{\nu=1}^{\left[\frac{k}{2}\right]}\overline{x}^{k-2\nu}|x|^{2(\nu-1)}+\sum_{\nu=\left[\frac{1}{2}(k+2)\right]}^{k-1}x^{2\nu-(k-1)-1}|x|^{2(k-1-\nu)}\\ &= \sum_{\nu=1}^{\left[\frac{k}{2}\right]}\overline{x}^{k-2\nu}|x|^{2(\nu-1)}+\sum_{\nu=[\frac{1}{2}(k+2)]}^{k-1}x^{2\nu-k}|x|^{2(k-1-\nu)}\end{aligned}$$

2.

$$\begin{aligned}\frac{\partial x^k}{1-m} &= \sum_{\nu=0}^{k-1}\overline{x}^{k-\nu-1}x^{\nu}=\sum_{\nu=1}^{k}\overline{x}^{k-(\nu-1)-1}x^{\nu-1}=\sum_{\nu=1}^{k}\overline{x}^{k-\nu}x^{\nu-1}\\ &\overset{(*)\binom{l=k}{\mu=\nu}}{=} \sum_{\nu=1}^{\left[\frac{1}{2}(k+1)\right]}\overline{x}^{k-2\nu+1}|x|^{2(\nu-1)}+\sum_{\nu=\left[\frac{1}{2}(k+3)\right]}^{k}x^{2\nu-k-1}|x|^{2(k-\nu)}\end{aligned}$$

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Satz 5.1.3 *Für $l, n \in \mathbb{N}$, $2n < l+1$ und $1 \leq \mu \leq l-2n+1$ gilt*

$$\sum_{\nu=1}^{\mu-1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=l-2n-\mu+2}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1} - \binom{l-n-\mu+1}{n}\binom{n+\mu-1}{n}.$$

Beweis:

Die Aussage gilt offenbar für $\mu = 1$.

1. *Induktionsanfang ($\mu = 2$): Es gilt*

$$\sum_{\nu=1}^{1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=l-2n}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \binom{l-n+1}{n+1} - \binom{l-n+2-l+2n}{n+1}\binom{n-1+l-2n}{n}$$
$$-\binom{l-n+2-l+2n-1}{n+1}\binom{n-1+l-2n+1}{n}$$
$$= \binom{l-n+1}{n+1} - (n+2)\binom{l-n-1}{n} - \binom{l-n}{n}$$
$$= \binom{l-n}{n} + \binom{l-n}{n+1} - (n+2)\binom{l-n-1}{n} - \binom{l-n}{n}$$
$$= \binom{l-n-1}{n+1} + \binom{l-n-1}{n} - (n+2)\binom{l-n-1}{n}$$
$$= \binom{l-n-1}{n+1} - (n+1)\binom{l-n-1}{n}$$
$$= \binom{l-n-1}{n+1}\binom{n+1}{n+1} - \binom{l-n-1}{n}\binom{n+1}{n}.$$

2. *Induktionsvoraussetzung: Die Aussage gilt für $2n < l+1$ und $1 \leq \mu \leq l-2n+1$.*

3. *Induktionsschritt: Es ist*

$$\sum_{\nu=1}^{\mu} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=l-2n-\mu+1}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1} - \binom{l-n-\mu}{n}\binom{n+\mu}{n}$$

zu zeigen. Es gilt

$$
\begin{aligned}
&\sum_{\nu=1}^{\mu-1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\\
&-\sum_{\nu=l-2n-\mu+2}^{l-2n+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}\\
&-\binom{l-n+2-l+2n+\mu-1}{n+1}\binom{n-1+l-2n-\mu+1}{n}\\
\overset{Iv.}{=}\;&\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}-\binom{n+\mu+1}{n+1}\binom{l-n-\mu}{n}\\
&+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}-\binom{l-n-\mu+1}{n}\binom{n+\mu-1}{n}\\
=\;&\left[\binom{l-n-\mu+1}{n+1}+\binom{l-n-\mu+1}{n}\right]\binom{n+\mu-1}{n}\\
&-\binom{l-n-\mu+1}{n}\binom{n+\mu-1}{n}+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}\\
&-\binom{n+\mu+1}{n+1}\binom{l-n-\mu}{n}\\
=\;&\binom{l-n-\mu+1}{n+1}\left[\binom{n+\mu-1}{n}+\binom{n+\mu-1}{n+1}\right]\\
&-\binom{n+\mu+1}{n+1}\binom{l-n-\mu}{n}\\
=\;&\left[\binom{l-n-\mu}{n+1}+\binom{l-n-\mu}{n}\right]\binom{n+\mu}{n+1}\\
&-\binom{l-n-\mu}{n}\left[\binom{n+\mu}{n+1}+\binom{n+\mu}{n}\right]\\
=\;&\binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}-\binom{l-n-\mu}{n}\binom{n+\mu}{n}.
\end{aligned}
$$

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Satz 5.1.4 *Es gilt für $2n < l+1$ und $1 \leq \mu \leq l+1-2n$*

$$\sum_{\nu=1}^{l-2n-\mu+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=\mu+1}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \binom{l-n-\mu+1}{n+1}\binom{n+\mu}{n+1}.$$

Beweis:

1. *Induktionsanfang: $\mu = 1$: Es gilt*

$$\sum_{\nu=1}^{1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=l-2n+1}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \binom{l-n+1}{n+1} - \binom{l-n}{n}$$
$$= \binom{l-n}{n+1}.$$

2. *Induktionsschritt: Es ist*

$$\sum_{\nu=1}^{l-2n-\mu+2} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=\mu}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n+1}$$

zu zeigen.
Es gilt

$$\sum_{\nu=1}^{l-2n-\mu+2} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=\mu}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$= \sum_{\nu=1}^{l-2n-\mu+3} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} - \sum_{\nu=\mu-1}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$
$$-\binom{n+\mu-1}{n+1}\binom{l-n-\mu+2}{n} + \binom{l-n-\mu+3}{n+1}\binom{n+\mu-2}{n}$$
$$\overset{\text{I.V.}}{=} \binom{l-n-\mu+3}{n+1}\binom{n+\mu-2}{n+1} - \binom{n+\mu-1}{n+1}\binom{l-n-\mu+2}{n}$$

$$+\binom{l-n-\mu+3}{n+1}\binom{n+\mu-2}{n}$$
$$= \binom{l-n-\mu+3}{n+1}\binom{n+\mu-1}{n+1} - \binom{l-n-\mu+2}{n}\binom{n+\mu-1}{n+1}$$
$$= \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n+1}.$$

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5.2 Stammfunktionen

Um die im nächsten Kapitel folgende gemischte Integraldarstellung zu finden, wird eine allgemeine Stammfunktion der Funktion

$$F_0 := \frac{\overline{x}|x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)},$$

bzgl. ∂ benötigt. Weiterhin seien mit den eckigen Klammern $[\alpha]$ die Größte-Ganze Funktion (von α) gemeint.

Definition 5.2.1 *Sei für $1 \leq k$ und $1 \leq m$*

$$F_0 := \frac{\overline{x}|x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)}.$$

Satz 5.2.1 *Dann hat die l-te Stammfunktion F_l von F_0 bzgl. ∂ die Form*

$$F_l(x) := \frac{\overline{x}^{l+1}|x|^{2k-m-1}}{2^{k+l}(k+l)! \prod_{j=1}^{k}(2j-m-1)} + \sum_{n=0}^{\left[\frac{1}{2}(l-1)\right]} \frac{\sum_{\mu=1}^{l-2n}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\overline{x}^{l-\mu-2n}x^{\mu-1}|x|^{2(k+n+1)-m-1}}{2^{k+l}(k+l)! \prod_{j=n+2}^{k}(2j-m-1)\prod_{j=1}^{n+1}\{2(k+j)-m-1\}}.$$

Beweis: (durch Induktion)

Benuzt man die Umformung nach (5.1) für F_l, so ergibt sich

$$
\begin{aligned}
F_l(x) &= \frac{\overline{x}^{\,l+1}|x|^{2k-m-1}}{2^{k+l}(k+l)!\prod\limits_{j=1}^{k}(2j-m-1)} \qquad (2)\\
&+\frac{|x|^{2(k+1)-m-1}}{2^{k+l}(k+l)!\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\{\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]}(l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)}\right.\\
&\qquad\left.+\sum_{\mu=\left[\frac{1}{2}(l+3)\right]}^{l}(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\right\}\\
&+\sum_{n=1}^{\left[\frac{1}{2}(l-1)\right]}\left\langle\frac{|x|^{2(k+n+1)-m-1}}{2^{k+l}(k+l)!\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}}\right.\\
&\quad\times\left\{\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+1)\right]}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\overline{x}^{\,l-2\mu-2n+1}|x|^{2(\mu-1)}\right.\\
&\qquad\left.\left.+\sum_{\mu=\left[\frac{1}{2}(l-2n+3)\right]}^{l-2n}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}\right\}\right\rangle.
\end{aligned}
$$

Induktionsanfang:

Es gilt

$$
F_1(x) := \frac{\overline{x}^2\,|x|^{2k-m-1}}{2^{k+1}\,(k+1)!\prod\limits_{j=1}^{k}(2j-m-1)}+\frac{|x|^{2(k+1)-m-1}}{2^{k+1}\,(k+1)!\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}
$$

und

$$
\begin{aligned}
2^{k+1}(k+1)!\prod_{j=2}^{k}(2j-m-1)\partial F_1(x) &= \frac{|x|^{2k-m-1}}{1-m}\{(m+3)\overline{x}+(m-1)x\}\\
&+\frac{\overline{x}^2}{1-m}x(2k-m-1)|x|^{2(k-1)-m-1}\\
&+\frac{\{2(k+1)-m-1\}x}{\{2(k+1)-m-1\}}|x|^{2k-m-1}\\
&= \frac{|x|^{2k-m-1}}{1-m}\{\overline{x}(m+3+2k-m-1)+x(m-1+1-m)\}\\
&= \frac{2(k+1)\,\overline{x}\,|x|^{2k-m-1}}{1-m}.
\end{aligned}
$$

Also ist $F_1(x)$ bezüglich ∂ die Stammfunktion von $F_0 = \frac{1}{\prod_{j=1}^{k}(2j-m-1)}\,\overline{x}\,|x|^{2k-m-1}$.

Nun zum Induktionsschritt:
Es gilt

$$\begin{aligned}
F_{l+1}(x) \quad = \quad & \frac{\overline{x}^{\,l+2}|x|^{2k-m-1}}{2^{k+l+1}(k+l+1)!\prod_{j=1}^{k}(2j-m-1)} \\
& + \frac{|x|^{2(k+1)-m-1}}{2^{k+l+1}(k+l+1)!\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \\
& \qquad \times \left\{ \sum_{\mu=1}^{\left[\frac{1}{2}(l+2)\right]} (l-\mu+2)\overline{x}^{\,l-2\mu+2}|x|^{2(\mu-1)} + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1} (l-\mu+2)x^{2\mu-l-2}|x|^{2(l-\mu+1)} \right\} \\
& + \sum_{n=1}^{\left[\frac{l}{2}\right]} \left\langle \frac{|x|^{2(k+n+1)-m-1}}{2^{k+l+1}(k+l+1)!\prod_{j=n+2}^{k}(2j-m-1)\prod_{j=1}^{n+1}\{2(k+j)-m-1\}} \right. \\
& \qquad \times \left\{ \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\overline{x}^{\,l-2\mu-2n+2}|x|^{2(\mu-1)} \right. \\
& \qquad \left.\left. + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} x^{2(n+\mu)-l-2}|x|^{2(l-2n-\mu+1)} \right\} \right\rangle,
\end{aligned}$$

also gilt mit $l - 2\left[\frac{l}{2}\right] = l \bmod 2$

$$\begin{aligned}
& 2^{k+l+1}\,(k+l+1)!\,\partial F_{l+1}(x) \\
= \quad & \frac{|x|^{2k-m-1}}{\prod_{j=1}^{k}(2j-m-1)} \left\langle \{m+2(l+2)-1\}\overline{x}^{l+1} + (m-1)x \left\{ \sum_{\nu=1}^{\left[\frac{1}{2}(l+2)\right]} \overline{x}^{l+2-2\nu}|x|^{2(\nu-1)} \right.\right. \\
& \qquad\qquad \left.\left. + \sum_{\nu=\left[\frac{1}{2}(l+4)\right]}^{l+1} x^{2\nu-(l+2)}|x|^{2(l+2-1-\nu)} \right\} \right\rangle \\
& + \frac{|x|^{2(k-1)-m-1}x(2k-m-1)}{\prod_{j=1}^{k}(2j-m-1)}\overline{x}^{l+2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|x|^{2(k+1)-m-1}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \\
&\quad\times\Bigg\langle \sum_{\mu=1}^{\left[\frac{1}{2}(l+2)\right]}(l-\mu+2)\left\{|x|^{2(\mu-1)}(\partial\overline{x}^{l+2(1-\mu)})+\overline{x}^{l+2(1-\mu)}(2\mu-2)x|x|^{2(\mu-2)}\right\} \\
&\qquad+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1}(l-\mu+2)\left\{|x|^{2(l-\mu+1)}(\partial x^{2(\mu-1)-l})+x^{2(\mu-1)-l}2(l-\mu+1)x|x|^{2(l-\mu)}\right\}\Bigg\rangle \\
&+\frac{|x|^{2k-m-1}x\{2(k+1)-m-1\}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l+2)\right]}(l-\mu+2)\overline{x}^{\,l+2(1-\mu)}|x|^{2(\mu-1)}+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1}(l-\mu+2)x^{2(\mu-1)-l}|x|^{2(l-\mu+1)}\Bigg\rangle \\
&+\sum_{n=1}^{\left[\frac{l}{2}\right]}\Bigg(\frac{|x|^{2(k+n+1)-m-1}}{\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}} \\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\Big\{|x|^{2(\mu-1)}(\partial\overline{x}^{l+2(1-\mu-n)}) \\
&\qquad\qquad+\overline{x}^{l+2(1-\mu-n)}\{2(\mu-1)\}x|x|^{2(\mu-2)}\Big\} \\
&\qquad\quad+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\Big\{|x|^{2(l-2n-\mu+1)}(\partial x^{2(n+\mu-1)-l}) \\
&\qquad\quad+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}\binom{l-n-\mu+1}{n}\binom{n+\mu-1}{n}x^{2(n+\mu-1)-l}|x|^{2(l-2n-\mu+1)}\Bigg\rangle \\
&\quad+\frac{|x|^{2(k+n)-m-1}\,x\,\{2(k+n+1)-m-1\}}{\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}} \\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\overline{x}^{l+2(1-\mu-n)}|x|^{2(\mu-1)} \\
&\qquad\quad+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}x^{2(n+\mu-1)-l}|x|^{2(l-2n-\mu+1)}\Bigg\rangle\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{|x|^{2k-m-1}}{\prod\limits_{j=1}^{k}(2j-m-1)}\overline{x}^{l+1}\underbrace{(m+2l+4-1+2k-m-1)}_{2(k+l+1)} \\
& + \frac{|x|^{2k-m-1}x\{2(k+1)-m-1\}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\Bigg\langle -\sum_{\nu=1}^{\left[\frac{1}{2}(l+2)\right]}\overline{x}^{l+2(1-\nu)}|x|^{2(\nu-1)} \\
& \qquad - \sum_{\nu=\left[\frac{1}{2}(l+4)\right]}^{l+1} x^{2(\nu-1)-l}|x|^{2(l-\nu+1)} + \sum_{\nu=1}^{\left[\frac{1}{2}(l+2)\right]}(l-\nu+2)\overline{x}^{l+2(1-\nu)}|x|^{2(\nu-1)} \\
& \qquad + \sum_{\nu=\left[\frac{1}{2}(l+4)\right]}^{l+1}(l-\nu+2)x^{2(\nu-1)-l}|x|^{2(l-\nu+1)}\Bigg\rangle \\
& + \frac{|x|^{2(k+1)-m-1}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \\
& \quad \times\Bigg(\sum_{\mu=1}^{\left[\frac{1}{2}(l+2)\right]}(l-\mu+2)\Bigg\langle\Bigg(|x|^{2(\mu-1)}\{m+2(l+2-2\mu)-1\}\overline{x}^{l-2\mu+1} \\
& \qquad +(m-1)|x|^{2(\mu-1)}x\Bigg\{\sum_{\nu=1}^{\left[\frac{1}{2}(l-2\mu+2)\right]}\overline{x}^{l-2\mu+2-2\nu}|x|^{2(\nu-1)} \\
& \qquad + \sum_{\nu=\left[\frac{1}{2}(l-2\mu+4)\right]}^{l-2\mu+1}x^{2\nu-l+2\mu-2}|x|^{2(l-2\mu+2-1-\nu)}\Bigg\}\Bigg) \\
& \qquad +2(\mu-1)\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Bigg\rangle \\
& \quad + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1}(l-\mu+2)\Bigg\langle|x|^{2(l-\mu+1)}(1-m)\Bigg\{\sum_{\nu=1}^{\left[\frac{1}{2}(2\mu-1-l)\right]}\overline{x}^{2\mu-l-2-2\nu+1}|x|^{2(\nu-1)} \\
& \qquad + \sum_{\nu=\left[\frac{1}{2}(2\mu+1-l)\right]}^{2\mu-l-2}x^{2\nu-2\mu+2+l-1}|x|^{2(2\mu-l-2-\nu)}\Bigg\} \\
& \qquad +2(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\Bigg\rangle\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\left[\frac{l}{2}\right]}\Bigg(\frac{|x|^{2(k+n+1)-m-1}}{\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}}\\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\\
&\qquad\times\Bigg\{\Bigg(|x|^{2(\mu-1)}\Big\{m+2(l+2-2\mu-2n)-1\Big\}\overline{x}^{l+1-2\mu-2n}\\
&\qquad\quad+|x|^{2(\mu-1)}(m-1)x\Bigg\langle\sum_{\nu=1}^{\left[\frac{1}{2}(l+2-2\mu-2n)\right]}\overline{x}^{l+2-2\mu-2n-2\nu}|x|^{2(\nu-1)}\\
&\qquad\quad+\sum_{\nu=\left[\frac{1}{2}(l+4-2\mu-2n)\right]}^{l+1-2\mu-2n}x^{2\nu-l-2+2\mu+2n}|x|^{2(l+1-2\mu-2n-\nu)}\Bigg\rangle\Bigg)\operatorname{sgn}\{l+2(1-\mu-n)\}\\
&\qquad\quad+2(\mu-1)\operatorname{sgn}\{2(\mu-1)\}\,\overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)}\Bigg\}\\
&\qquad+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\\
&\qquad\quad\times\Bigg\{|x|^{2(l-2n-\mu+1)}(1-m)\Bigg\langle\sum_{\nu=1}^{\left[\frac{1}{2}\{2(n+\mu)-1-l\}\right]}\overline{x}^{2(n+\mu-1)-l-2\nu+1}|x|^{2(\nu-1)}\\
&\qquad\quad+\sum_{\nu=\left[\frac{1}{2}(2(n+\mu)+1-l)\right]}^{2(n+\mu-1)-l}x^{2\nu-1-2(n+\mu-1)+l}|x|^{2\{2(n+\mu-1)-l-\nu\}}\Bigg\rangle\operatorname{sgn}\{2(n+\mu-1)-l\}\\
&\qquad\quad+2(l-2n-\mu+1)\operatorname{sgn}\{2(l-2n-\mu+1)\}\,x^{2(n+\mu)-1-l}|x|^{2(l-2n-\mu)}\Bigg\}\Bigg\rangle\\
&\quad+\frac{|x|^{2(k+n+1)-m-1}\{2(k+n+1)-m-1\}}{\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}}\\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)}\\
&\qquad\quad+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}\Bigg\rangle\Bigg)\\
&+\sum_{n=0}^{\left[\frac{1}{2}(l-2)\right]}\frac{|x|^{2(k+n+1)-m-1}\,x\,(2n+4-m-1)}{\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}}\\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]}\binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}\overline{x}^{l-2(\mu+n)}|x|^{2(\mu-1)}\\
&\qquad\quad+\sum_{\mu=\left[\frac{1}{2}(l-2n+2)\right]}^{l-2n-1}\binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}x^{2(n+\mu)-l}|x|^{2(l-2n-2-\mu+1)}\Bigg\rangle
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{2(k+l+1)\overline{x}^{l+1}|x|^{2k-m-1}}{\prod\limits_{j=1}^{k}(2j-m-1)} \\
& + \frac{|x|^{2(k+1)-m-1}\,\{2(k+1)-m-1\}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \Bigg\langle \sum_{\nu=1}^{\left[\frac{1}{2}(l+2)\right]} (l-\nu+1)\overline{x}^{l-2\nu+1}|x|^{2(\nu-1)} \\
& \qquad + \sum_{\nu=\left[\frac{1}{2}(l+4)\right]}^{l} (l-\nu+1)x^{2\nu-l-1}|x|^{2(l-\nu)} \Bigg\rangle \\
& + \frac{|x|^{2(k+1)-m-1}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \\
& \quad \times \Bigg(\sum_{\mu=1}^{\left[\frac{1}{2}(l+2)\right]} \Bigg\langle \overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Big\{(l-\mu+2)(m+2l+3-4\mu)+(l-\mu+2)\,2(\mu-1)\Big\} \\
& \qquad + (m-1)(l-\mu+2) \sum_{\nu=1}^{\left[\frac{1}{2}(l-2\mu+2)\right]} \overline{x}^{l-2\mu-2\nu+1}|x|^{2(\mu+\nu-1)} \\
& \qquad + (m-1)(l-\mu+2) \sum_{\nu=\left[\frac{1}{2}(l-2\mu+4)\right]}^{l-2\mu+1} x^{2(\mu+\nu)-l-1}|x|^{2(l-\mu-\nu)} \Bigg\rangle \\
& \quad + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1} \Bigg\langle (l-\mu+2)\,2(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)} \\
& \qquad + (1-m)(l-\mu+2) \sum_{\nu=1}^{\left[\frac{1}{2}(2\mu-1-l)\right]} \overline{x}^{2(\mu-\nu)-l-1}|x|^{2(l-\mu+\nu)} \\
& \qquad + (1-m)(l-\mu+2) \sum_{\nu=\left[\frac{1}{2}(2\mu+1-l)\right]}^{2\mu-l-2} x^{2(\nu-\mu)+l+1}|x|^{2(\mu-\nu-1)} \Bigg\rangle \Bigg) \\
& + \frac{|x|^{2(k+1)-m-1}(3-m)}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \Bigg\langle \sum_{\mu=1}^{\left[\frac{l}{2}\right]} (l-\mu)\,\mu\,\overline{x}^{l-2\mu-1}|x|^{2\mu} \\
& \qquad + \sum_{\mu=\left[\frac{1}{2}(l+2)\right]}^{l-1} (l-\mu)\,\mu\,x^{2\mu-l+1}|x|^{2(l-\mu-1)} \Bigg\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\left[\frac{1}{2}(l-2)\right]}\Bigg(\frac{|x|^{2(k+n+1)-m-1}}{\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}}\\
&\qquad\times\Bigg\langle\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]}\Bigg\{\overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)}\Bigg\langle\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\\
&\qquad\qquad\times\Big\{(m+2l+3-4\mu-4n)\operatorname{sgn}\{l+2(1-\mu-n)\}+\{2(\mu-1)\}\Big\}\\
&\qquad\qquad\qquad+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\{2(k+n+1)-m-1\}\Bigg\rangle\\
&\qquad\qquad\qquad+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\operatorname{sgn}\{l+2(1-\mu-n)\}\,(m-1)\\
&\qquad\qquad\qquad\qquad\times\sum_{\nu=1}^{\left[\frac{1}{2}(l+2-2\mu-2n)\right]}\overline{x}^{l+1-2(\mu+n+\nu)}|x|^{2(\mu+\nu-1)}\\
&\qquad\qquad\qquad+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\operatorname{sgn}\{l+2(1-\mu-n)\}\,(m-1)\\
&\qquad\qquad\qquad\qquad\times\sum_{\nu=\left[\frac{1}{2}(l+4-2\mu-2n)\right]}^{l+1-2\mu-2n}x^{2(\mu+\nu+n)-l-1}|x|^{2(l-\mu-2n-\nu)}\Bigg\}\\
&\qquad+\sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]}\Bigg\{\binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}(2n+4-m-1)\overline{x}^{l-2(\mu+n)-1}|x|^{2\mu}\Bigg\}\\
&\qquad+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}\Bigg\{x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}\\
&\qquad\quad\times\Bigg\langle\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\{2(k+n+1)-m-1\}\\
&\qquad\qquad+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,2(l-2n-\mu+1)\operatorname{sgn}\{2(l-2n-\mu+1)\}\Bigg\rangle\\
&\qquad\qquad+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\operatorname{sgn}\{2(n+\mu-1)-l\}\,(1-m)\\
&\qquad\qquad\qquad\times\sum_{\nu=1}^{\left[\frac{1}{2}(2(n+\mu)-l-1)\right]}\overline{x}^{2(n+\mu-\nu)-l-1}|x|^{2(l-2n+\nu-\mu)}\\
&\qquad\qquad+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\operatorname{sgn}\{2(n+\mu-1)-l\}\,(1-m)\\
&\qquad\qquad\qquad\times\sum_{\nu=\left[\frac{1}{2}(2(n+\mu)-l+1)\right]}^{2(n+\mu-1)-l}x^{2(\nu-n-\mu)+1+l}|x|^{2(\mu-\nu-1)}\Bigg\}\\
&\qquad+\sum_{\mu=\left[\frac{1}{2}(l-2n+2)\right]}^{l-2n-1}x^{2(n+\mu)-l+1}|x|^{2(l-2n-\mu-1)}\binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}(2n+4-m-1)
\end{aligned}
$$

$$
+\quad \frac{|x|^{2\left(k+\left[\frac{l}{2}\right]+1\right)-m-1}}{\prod\limits_{j=\left[\frac{l}{2}\right]+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\left[\frac{l}{2}\right]+1}\{2(k+j)-m-1\}}
$$

$$
\times\Bigg\langle \sum_{\mu=1}^{\left[\frac{1}{2}(l-2\left[\frac{l}{2}\right]+2)\right]=1} \Bigg\{ \overline{x}^{l+1-2\mu-2\left[\frac{l}{2}\right]}|x|^{2(\mu-1)} \Bigg\langle \binom{\left[\frac{1}{2}(l+1)\right]-\mu+2}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}
$$

$$
\times\Big\{(m+2l+3-4\mu-4\left[\frac{l}{2}\right])\ \mathrm{sgn}\{l+2(1-\mu-\left[\frac{l}{2}\right])\}+\{2(\mu-1)\}\Big\}
$$

$$
+\binom{\left[\frac{1}{2}(l+1)\right]-\mu+1}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\{2(k+\left[\frac{l}{2}\right]+1)-m-1\}\Bigg\rangle
$$

$$
+\binom{\left[\frac{1}{2}(l+1)\right]-\mu+2}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\ \mathrm{sgn}\{l+2(1-\mu-\left[\frac{l}{2}\right])\}\ (m-1)
$$

$$
\times \sum_{\nu=1}^{\left[\frac{1}{2}(l+2-2\mu-2\left[\frac{l}{2}\right])\right]=(1-\mu)=0} \overline{x}^{l+1-2\mu-2\left[\frac{l}{2}\right]-2\nu}|x|^{2(\mu+\nu-1)}
$$

$$
+\binom{\left[\frac{1}{2}(l+1)\right]-\mu+2}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\ \mathrm{sgn}\{l+2(1-\mu-\left[\frac{l}{2}\right])\}\ (m-1)
$$

$$
\times \sum_{\nu=2-\mu=1}^{l+1-2\mu-2\left[\frac{l}{2}\right]\le 0} x^{2\nu-l-1+2\mu+2\left[\frac{l}{2}\right]}|x|^{2(l-\mu-2\left[\frac{l}{2}\right]-\nu)}\Bigg\}
$$

$$
+\sum_{\mu=2}^{1+(l \bmod 2)} \Bigg\{ x^{2\left[\frac{l}{2}\right]+2\mu-l-1}|x|^{2(l-2\left[\frac{l}{2}\right]-\mu)}
$$

$$
\times\Bigg\langle \binom{\left[\frac{1}{2}(l+1)\right]-\mu+1}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\{2(k+\left[\frac{l}{2}\right]+1)-m-1\}
$$

$$
+\binom{\left[\frac{1}{2}(l+1)\right]-\mu+2}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\ 2(l-2\left[\frac{l}{2}\right]-\mu+1)\Bigg\rangle
$$

$$
+\binom{\left[\frac{1}{2}(l+1)\right]-\mu+2}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\ \mathrm{sgn}\{2(\left[\frac{l}{2}\right]+\mu-1)-l\}
$$

$$
\times(1-m)\sum_{\nu=1}^{\left[\frac{1}{2}(2\left[\frac{l}{2}\right]+2\mu-1-l)\right]=1} \overline{x}^{2(\mu-\nu)-1-(l \bmod 2)}|x|^{2(l-2\left[\frac{l}{2}\right]-\mu+\nu)}
$$

$$
+\binom{\left[\frac{1}{2}(l+1)\right]-\mu+2}{\left[\frac{l}{2}\right]+1}\binom{\left[\frac{l}{2}\right]+\mu-1}{\left[\frac{l}{2}\right]}\ \mathrm{sgn}\{2(\left[\frac{l}{2}\right]+\mu-1)-l\}
$$

$$
\times(1-m)\sum_{\nu=2}^{2\left[\frac{l}{2}\right]+2\mu-2-l=2\mu-2-(l \bmod 2)} x^{2(\nu-\mu)+1+(l \bmod 2)}|x|^{2(\mu-\nu-1)}\Bigg\}\Bigg\rangle
$$

Seien

$$\Upsilon_1 := \frac{2(k+l+1)\overline{x}^{l+1}|x|^{2k-m-1}}{\prod\limits_{j=1}^{k}(2j-m-1)},$$

$$\begin{aligned}
\Upsilon_2 \ := \ & \frac{|x|^{2(k+1)-m-1}\,\{2(k+1)-m-1\}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\langle \sum_{\nu=1}^{\left[\frac{1}{2}(l+2)\right]}(l-\nu+1)\overline{x}^{l-2\nu+1}|x|^{2(\nu-1)}\right. \\
& \qquad \left. + \sum_{\nu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\nu+1)x^{2\nu-l-1}|x|^{2(l-\nu)}\right\rangle \\
& + \frac{|x|^{2(k+1)-m-1}}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}} \\
& \quad \times\left(\sum_{\mu=1}^{\left[\frac{1}{2}(l+2)\right]}\left\langle \overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Big\{(l-\mu+2)(m+2l+3-4\mu)+(l-\mu+2)\,2(\mu-1)\Big\}\right.\right. \\
& \qquad +(m-1)(l-\mu+2)\sum_{\nu=1}^{\left[\frac{1}{2}(l-2\mu+2)\right]}\overline{x}^{l-2\mu-2\nu+1}|x|^{2(\mu+\nu-1)} \\
& \qquad \left. +(m-1)(l-\mu+2)\sum_{\nu=\left[\frac{1}{2}(l-2\mu+4)\right]}^{l-2\mu+1}x^{2(\mu+\nu)-l-1}|x|^{2(l-\mu-\nu)}\right\rangle \\
& \quad + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1}\left\langle (l-\mu+2)\,2(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\right. \\
& \qquad +(1-m)(l-\mu+2)\sum_{\nu=1}^{\left[\frac{1}{2}(2\mu-1-l)\right]}\overline{x}^{2(\mu-\nu)-l-1}|x|^{2(l-\mu+\nu)} \\
& \qquad \left.\left. +(1-m)(l-\mu+2)\sum_{\nu=\left[\frac{1}{2}(2\mu+1-l)\right]}^{2\mu-l-2}x^{2(\nu-\mu)+l+1}|x|^{2(\mu-\nu-1)}\right\rangle\right) \\
& + \frac{|x|^{2(k+1)-m-1}(3-m)}{\prod\limits_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\langle \sum_{\mu=1}^{\left[\frac{l}{2}\right]}(l-\mu)\,\mu\,\overline{x}^{l-2\mu-1}|x|^{2\mu}\right. \\
& \qquad \left. + \sum_{\mu=\left[\frac{1}{2}(l+2)\right]}^{l-1}(l-\mu)\,\mu\,x^{2\mu-l+1}|x|^{2(l-\mu-1)}\right\rangle,
\end{aligned}$$

$$
\begin{aligned}
\Upsilon_3 \;:=\; & \sum_{n=1}^{\left[\frac{1}{2}(l-2)\right]} \Bigg(\frac{|x|^{2(k+n+1)-m-1}}{\prod_{j=n+2}^{k}(2j-m-1)\prod_{j=1}^{n+1}\{2(k+j)-m-1\}} \\
& \times \Bigg\langle \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n+2)\right]} \Bigg\{ \overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)} \Bigg\langle \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
& \times \Big\{(m+2l+3-4\mu-4n)\,\mathrm{sgn}\{l+2(1-\mu-n)\}+\{2(\mu-1)\}\Big\} \\
& + \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\{2(k+n+1)-m-1\}\Bigg\rangle \\
& + \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,\mathrm{sgn}\{l+2(1-\mu-n)\}\,(m-1) \\
& \times \sum_{\nu=1}^{\left[\frac{1}{2}(l+2-2\mu-2n)\right]} \overline{x}^{l+1-2(\mu+n+\nu)}|x|^{2(\mu+\nu-1)} \\
& + \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,\mathrm{sgn}\{l+2(1-\mu-n)\}\,(m-1) \\
& \times \sum_{\nu=\left[\frac{1}{2}(l+4-2\mu-2n)\right]}^{l+1-2\mu-2n} x^{2(\mu+\nu+n)-l-1}|x|^{2(l-\mu-2n-\nu)} \Bigg\} \\
& + \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]} \Bigg\{ \binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}(2n+4-m-1)\overline{x}^{l-2(\mu+n)-1}|x|^{2\mu} \Bigg\} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \Bigg\{ x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)} \\
& \times \Bigg\langle \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\{2(k+n+1)-m-1\} \\
& + \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,2(l-2n-\mu+1)\,\mathrm{sgn}\{2(l-2n-\mu+1)\}\Bigg\rangle \\
& + \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,\mathrm{sgn}\{2(n+\mu-1)-l\}\,(1-m) \\
& \times \sum_{\nu=1}^{\left[\frac{1}{2}(2(n+\mu)-l-1)\right]} \overline{x}^{2(n+\mu-\nu)-l-1}|x|^{2(l-2n+\nu-\mu)} \\
& + \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,\mathrm{sgn}\{2(n+\mu-1)-l\}\,(1-m) \\
& \times \sum_{\nu=\left[\frac{1}{2}(2(n+\mu)-l+1)\right]}^{2(n+\mu-1)-l} x^{2(\nu-n-\mu)+1+l}|x|^{2(\mu-\nu-1)} \Bigg\} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+2)\right]}^{l-2n-1} x^{2(n+\mu)-l+1}|x|^{2(l-2n-\mu-1)} \binom{l-n-\mu}{n+1}\binom{n+\mu}{n+1}(2n+4-m-1)
\end{aligned}
$$

und

$$
\begin{aligned}
\Upsilon_4 \quad := \quad & \frac{|x|^{2\left(k+\left[\frac{l}{2}\right]+1\right)-m-1}}{\prod\limits_{j=\left[\frac{l}{2}\right]+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\left[\frac{l}{2}\right]+1}\{2(k+j)-m-1\}} \\
& \times\Bigg\langle \sum_{\mu=1}^{\left[\frac{1}{2}(l-2\left[\frac{l}{2}\right]+2)\right]=1} \Bigg\{ \overline{x}^{l+1-2\mu-2\left[\frac{l}{2}\right]}|x|^{2(\mu-1)} \Bigg\langle \begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+2 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \\
& \quad \times\Big\{(m+2l+3-4\mu-4\left[\frac{l}{2}\right])\operatorname{sgn}\{l+2(1-\mu-\left[\frac{l}{2}\right])\}+\{2(\mu-1)\}\Big\} \\
& \qquad +\begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+1 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \{2(k+\left[\frac{l}{2}\right]+1)-m-1\}\Bigg\rangle \\
& \qquad +\begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+2 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \operatorname{sgn}\{l+2(1-\mu-\left[\frac{l}{2}\right])\}\,(m-1) \\
& \qquad\qquad \times \sum_{\nu=1}^{\left[\frac{1}{2}(l+2-2\mu-2\left[\frac{l}{2}\right])\right]=(1-\mu)=0} \overline{x}^{l+1-2\mu-2\left[\frac{l}{2}\right]-2\nu}|x|^{2(\mu+\nu-1)} \\
& \qquad +\begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+2 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \operatorname{sgn}\{l+2(1-\mu-\left[\frac{l}{2}\right])\}\,(m-1) \\
& \qquad\qquad \times \sum_{\nu=2-\mu=1}^{l+1-2\mu-2\left[\frac{l}{2}\right]\leq 0} x^{2\nu-l-1+2\mu+2\left[\frac{l}{2}\right]}|x|^{2(l-\mu-2\left[\frac{l}{2}\right]-\nu)}\Bigg\} \\
& \quad + \sum_{\mu=2}^{1+(l \bmod 2)} \Bigg\{ x^{2\left[\frac{l}{2}\right]+2\mu-l-1}|x|^{2(l-2\left[\frac{l}{2}\right]-\mu)} \\
& \qquad \times\Bigg\langle \begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+1 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \{2(k+\left[\frac{l}{2}\right]+1)-m-1\} \\
& \qquad\quad +\begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+2 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} 2(l-2\left[\frac{l}{2}\right]-\mu+1)\Bigg\rangle \\
& \qquad\quad +\begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+2 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \operatorname{sgn}\{2(\left[\frac{l}{2}\right]+\mu-1)-l\} \\
& \qquad\qquad \times(1-m) \sum_{\nu=1}^{\left[\frac{1}{2}(2\left[\frac{l}{2}\right]+2\mu-1-l)\right]=1} \overline{x}^{2(\mu-\nu)-1-(l \bmod 2)}|x|^{2(l-2\left[\frac{l}{2}\right]-\mu+\nu)} \\
& \qquad\quad +\begin{pmatrix} \left[\frac{1}{2}(l+1)\right]-\mu+2 \\ \left[\frac{l}{2}\right]+1 \end{pmatrix} \begin{pmatrix} \left[\frac{l}{2}\right]+\mu-1 \\ \left[\frac{l}{2}\right] \end{pmatrix} \operatorname{sgn}\{2(\left[\frac{l}{2}\right]+\mu-1)-l\} \\
& \qquad\qquad \times(1-m) \sum_{\nu=2}^{2\left[\frac{l}{2}\right]+2\mu-2-l=2\mu-2-(l \bmod 2)} x^{2(\nu-\mu)+1+(l \bmod 2)}|x|^{2(\mu-\nu-1)}\Bigg\}\Bigg\rangle,
\end{aligned}
$$

Dann gilt

$$2^{k+l+1}\,(k+l+1)!\,\partial F_{l+1}(x) \;=\; \Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4.$$

Nun haben wir die Formel in 4 Teile zerlegt, die einzeln betrachtet werden:
1. Υ_1 *ist der erste Bruch mit* $|x|^{2k-m-1}$*, er ist bereits in der gewünschten Form.*
2. Nun wird Υ_2 *betrachtet:*
Mit

$$0=l+2(1-\mu)\Leftrightarrow \mu=\frac{l+2}{2}=\begin{cases}\frac{2\alpha+2}{2}=\alpha+1=[\alpha+1]=\left[\frac{1}{2}(l+2)\right] & \text{, falls } l \text{ gerade}\\ \frac{2\alpha-1+2}{2}=\frac{2\alpha+1}{2}=\alpha+\frac{1}{2}\notin\mathbb{N}_0 & \text{, falls } l \text{ ungerade,}\end{cases}$$

und $0=2(l-\mu+1)\Leftrightarrow \mu=l+1$ *gilt*

$$\begin{aligned}
\Upsilon_2 \;=\; & \frac{|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\Bigg(\overline{x}^{l-1}(l+1)(m+2l-1)+l\{2(k+1)-m-1\}\overline{x}^{l-1}\\
& +\sum_{\mu=2}^{\left[\frac{l}{2}\right]}\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Big\langle(l-\mu+2)\{m+2l+3-4\mu+2\mu-2\}\\
& \qquad +(l-\mu+1)\{2(k+1)-m-1\}\Big\rangle\\
& +\sum_{\mu=2}^{\left[\frac{l}{2}\right]}(l-\mu+1)(\mu-1)(3-m)\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\\
& +\sum_{\mu=1}^{\left[\frac{l}{2}\right]}(m-1)(l-\mu+2)\sum_{\nu=1}^{\left[\frac{1}{2}(l-2\mu+2)\right]}\overline{x}^{l-2\mu-2\nu+1}|x|^{2(\mu+\nu-1)}\\
& +(l \bmod 2)\,\overline{x}^{l-2\left[\frac{1}{2}(l+2)\right]+1}|x|^{2\left\{\left[\frac{1}{2}(l+2)\right]-1\right\}}\left\{l-\left[\frac{1}{2}(l+2)\right]+2\right\}\left\{m+2l+3-4\left[\frac{1}{2}(l+2)\right]\right\}\\
& +\{2(k+1)-m-1\}\left\{l-\left[\frac{1}{2}(l+2)\right]+1\right\}\overline{x}^{l-2\left[\frac{1}{2}(l+2)\right]+1}|x|^{2\left\{\left[\frac{1}{2}(l+2)\right]-1\right\}}\\
& +\overline{x}^{l-2\left[\frac{1}{2}(l+2)\right]+1}|x|^{2\left\{\left[\frac{1}{2}(l+2)\right]-1\right\}}\left\{l-\left[\frac{1}{2}(l+2)\right]+2\right\}\left\langle 2\left\{\left[\frac{1}{2}(l+2)\right]-1\right\}\right\rangle\\
& +(l \bmod 2)\,(m-1)\left\{l-\left[\frac{1}{2}(l+2)\right]+2\right\}\\
& \qquad\times\sum_{\nu=1}^{\left[\frac{1}{2}\left\{l-2\left[\frac{1}{2}(l+2)\right]+2\right\}\right]=\left[\frac{1}{2}(l \bmod 2)\right]=0}\overline{x}^{l-2\left[\frac{1}{2}(l+2)\right]-2\nu+1}|x|^{2\left\{\left[\frac{1}{2}(l+2)\right]+\nu-1\right\}}\\
& +\left[\frac{1}{2}(l+1)\right]\left[\frac{l}{2}\right](3-m)\overline{x}^{l-2\left[\frac{l}{2}\right]-1}|x|^{2\left[\frac{l}{2}\right]}\\
& +\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1}(1-m)(l-\mu+2)\sum_{\nu=1}^{\left[\frac{1}{2}(2\mu-l-1)\right]}\overline{x}^{2(\mu-\nu)-l-1}|x|^{2(l-\mu+\nu)}
\end{aligned}$$

$$+\sum_{\mu=1}^{\left[\frac{l}{2}\right]}(m-1)(l-\mu+2)\sum_{\nu=\left[\frac{1}{2}(l-2\mu+4)\right]}^{l-2\mu+1}x^{2(\mu+\nu)-l-1}|x|^{2(l-\mu-\nu)}$$

$$+(l \bmod 2)\,(m-1)\left\{l-\left[\frac{1}{2}(l+2)\right]+2\right\}\sum_{\nu=\left[\frac{1}{2}\left\{l-2\left[\frac{1}{2}(l+2)\right]+4\right\}\right]}^{l-2\left[\frac{1}{2}(l+2)\right]+1}x^{2\left[\frac{1}{2}(l+2)\right]+2\nu-l-1}|x|^{2\left\{l-\left[\frac{1}{2}(l+2)\right]-\nu\right\}}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+1)\{2(k+1)-m-1\}x^{2\mu-l-1}|x|^{2(l-\mu)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+2)\,2(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l+1}(1-m)(l-\mu+2)\sum_{\nu=\left[\frac{1}{2}(2\mu-l+1)\right]}^{2\mu-l-2}x^{l+1-2(\mu-\nu)}|x|^{2(\mu-\nu-1)}$$

$$\left.+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+1)(\mu-1)(3-m)x^{2\mu-l-1}|x|^{2(l-\mu)}\right).$$

Folgende Terme werden in einfache Summen umgewandelt:

1. *Sei*

$$S_1 := \sum_{\mu=1}^{\left[\frac{l}{2}\right]}(l-\mu+2)\sum_{\nu=1}^{\left[\frac{1}{2}(l-2\mu+2)\right]}\overline{x}^{l-2\mu-2\nu+1}|x|^{2(\mu+\nu-1)}.$$

Sei $\lambda := \mu+\nu$. *Für* $\nu = 1$ *gilt* $\lambda = \mu+1$, *für* $\nu = \left[\frac{l}{2}\right]-\mu+1$ *gilt* $\lambda = \mu+\left[\frac{l}{2}\right]-\mu+1$, *also gilt*

$$S_1 = \sum_{\mu=1}^{\left[\frac{l}{2}\right]}(l-\mu+2)\sum_{\lambda=\mu+1}^{\left[\frac{l}{2}\right]+1}\overline{x}^{l-2\lambda+1}|x|^{2(\lambda-1)}.$$

Es gilt $2 = 1+1 \leq \mu+1 \leq \lambda \leq \left[\frac{l}{2}\right]+1$ *und* $1 \leq \mu \leq \lambda-1$. *Also gilt*

$$\begin{aligned}
S_1 &= \sum_{\lambda=2}^{\left[\frac{l}{2}\right]+1}\overline{x}^{l-2\lambda+1}|x|^{2(\lambda-1)}\sum_{\mu=1}^{\lambda-1}(l-\mu+2)\\
&= \sum_{\lambda=2}^{\left[\frac{l}{2}\right]+1}\overline{x}^{l-2\lambda+1}|x|^{2(\lambda-1)}\left\{-\frac{\lambda(\lambda-1)}{2}+(\lambda-1)(l+2)\right\}\\
&= \sum_{\lambda=2}^{\left[\frac{l}{2}\right]+1}\overline{x}^{l-2\lambda+1}|x|^{2(\lambda-1)}(\lambda-1)(l-\frac{\lambda}{2}+2).
\end{aligned}$$

2. Es gilt

$$\left[\frac{1}{2}(-l-1)\right] = \left\{\begin{array}{ll} \frac{-l-2}{2} = -\frac{l+2}{2} = -\left[\frac{1}{2}(l+2)\right] = -\left[\frac{1}{2}(l+3)\right] & \text{, falls } l \text{ gerade} \\ \frac{-l-1}{2} = -\frac{l+1}{2} = -\left[\frac{1}{2}(l+1)\right] & \text{, falls } l \text{ ungerade.} \end{array}\right\}$$
$$= -\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right].$$

Sei

$$S_2 := \sum_{\mu=\left[\frac{l}{2}\right]+2}^{l+1} (l-\mu+2) \sum_{\nu=1}^{\mu-\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]} \overline{x}^{2(\mu-\nu)-l-1}|x|^{2\{l-(\mu-\nu)\}}.$$

Sei $\lambda := \mu - \nu$. *Für* $\nu = 1$ *gilt* $\lambda = \mu - 1$.
Für $\nu = \mu - \left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]$ *gilt*
$\lambda = \mu - \mu + \left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right] = \left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]$.
Also gilt

$$S_2 = \sum_{\mu=\left[\frac{l}{2}\right]+2}^{l+1} (l-\mu+2) \sum_{\lambda=\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]}^{\mu-1} \overline{x}^{2\lambda-l-1}|x|^{2(l-\lambda)}.$$

Mit

$$\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right] \leq \lambda \leq \mu - 1 \leq l+1-1 = l,$$

$$\lambda + 1 \leq \mu \leq l+1$$

und

$$l+1-\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]$$
$$= \left\{\begin{array}{ll} l+1-\left[\frac{1}{2}(l+1+2)\right] = l+1-\left[\frac{1}{2}(l+3)\right] & \text{, falls } l \text{ gerade} \\ l+1-\left[\frac{1}{2}\{l+1+2(1-1)\}\right] = l+1-\left[\frac{1}{2}(l+1)\right] & \text{, falls } l \text{ ungerade} \end{array}\right\}$$
$$= \left\{\begin{array}{ll} l+1-\frac{l+2}{2} = \frac{1}{2}l = \left[\frac{l}{2}\right] = \left[\frac{1}{2}(l+1)\right] & \text{, falls } l \text{ gerade} \\ l+1-\frac{l+1}{2} = \frac{l+1}{2} = \left[\frac{1}{2}(l+1)\right] & \text{, falls } l \text{ ungerade} \end{array}\right\}$$
$$= \left[\frac{1}{2}(l+1)\right],$$

folgt

$$S_2 = \sum_{\lambda=\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]}^{l} \overline{x}^{2\lambda-l-1}|x|^{2(l-\lambda)} \sum_{\mu=\lambda+1}^{l+1} l-\mu+2$$
$$= \sum_{\lambda=\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]}^{l} \overline{x}^{2\lambda-l-1}|x|^{2(l-\lambda)} \sum_{\mu=1}^{l-\lambda+1} l-\mu-\lambda+2$$

$$
\begin{aligned}
&= \sum_{\lambda=\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]}^{l} \overline{x}^{2\lambda-l-1}|x|^{2(l-\lambda)}\Big\{-\frac{(l-\lambda+1)(l-\lambda+2)}{2}+(l-\lambda+1)(l-\lambda+2)\Big\}\\
&= \sum_{\lambda=\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]}^{l} \frac{1}{2}(l-\lambda+1)(l-\lambda+2)\overline{x}^{2\lambda-l-1}|x|^{2(l-\lambda)}\\
&= \sum_{-\lambda=\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]}^{l} \frac{1}{2}(l+\lambda+1)(l+\lambda+2)\overline{x}^{-2\lambda-l-1}|x|^{2(l+\lambda)}\\
&= \sum_{\lambda=-l}^{-\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]} \frac{1}{2}(l+\lambda+1)(l+\lambda+2)\overline{x}^{-2\lambda-l-1}|x|^{2(l+\lambda)}\\
&= \sum_{\lambda=-l+l+1}^{-\left[\frac{1}{2}(l+1+2\{1-(l \bmod 2)\})\right]+l+1} \frac{1}{2}(l+\lambda-l-1+1)(l+\lambda-l-1+2)\overline{x}^{\,-2(\lambda-l-1)-l-1}|x|^{2(l+\lambda-l-1)}\\
&= \sum_{\lambda=1}^{\left[\frac{1}{2}(l+1)\right]} \frac{\lambda}{2}(\lambda+1)\overline{x}^{l-2\lambda+1}|x|^{2(\lambda-1)}
\end{aligned}
$$

3. Sei

$$S_3 := \sum_{\mu=1}^{\left[\frac{l}{2}\right]} (l-\mu+2) \sum_{\nu=\left[\frac{l}{2}\right]-\mu+2}^{l-2\mu+1} x^{2(\mu+\nu)-l-1}|x|^{2\{l-(\mu+\nu)\}}.$$

Sei $\lambda := \mu+\nu$.
Für $\nu = \left[\frac{l}{2}\right]-\mu+2$ *gilt dann* $\lambda = \mu+\left[\frac{l}{2}\right]-\mu+2 = \left[\frac{l}{2}\right]+2$ *und für* $\nu = l-2\mu+1$ *gilt* $\lambda = \mu+l-2\mu+1 = l-\mu+1$.

Also gilt

$$S_3 = \sum_{\mu=1}^{\left[\frac{l}{2}\right]} (l-\mu+2) \sum_{\lambda=\left[\frac{l}{2}\right]+2}^{l-\mu+1} x^{2\lambda-l-1}|x|^{2(l-\lambda)}.$$

Es gelten $\left[\frac{l}{2}\right]+2 \le \lambda \le l-\mu+1 \le l$ *und* $1 \le \mu \le l-\lambda+1$.

Also gilt

$$\begin{aligned} S_3 &= \sum_{\lambda=\left[\frac{l}{2}\right]+2}^{l} x^{2\lambda-l-1}|x|^{2(l-\lambda)} \sum_{\mu=1}^{l-\lambda+1} l-\mu+2 \\ &= \sum_{\lambda=\left[\frac{l}{2}\right]+2}^{l} x^{2\lambda-l-1}|x|^{2(l-\lambda)}\left\{-\frac{(l-\lambda+1)(l-\lambda+2)}{2}+(l-\lambda+1)(l+2)\right\} \\ &= \sum_{\lambda=\left[\frac{l}{2}\right]+2}^{l} x^{2\lambda-l-1}|x|^{2(l-\lambda)}\left\{(l-\lambda+1)\,\frac{2l+4-l+\lambda-2}{2}\right\} \\ &= \sum_{\lambda=\left[\frac{l}{2}\right]+2}^{l} \frac{1}{2}(l-\lambda+1)(l+\lambda+2)x^{2\lambda-l-1}|x|^{2(l-\lambda)} \end{aligned}$$

4. *Sei*

$$S_4 := \sum_{\mu=\left[\frac{l}{2}\right]+2}^{l+1} (l-\mu+2) \sum_{\nu=\mu+\left[\frac{1}{2}(1-l)\right]}^{2\mu-l-2} x^{l+1-2(\mu-\nu)}|x|^{2(\mu-\nu-1)}.$$

Sei weiterhin $\lambda := \mu-\nu$.
Dann gilt für $\nu = \mu+\left[\frac{1}{2}(1-l)\right]$*:* $\quad \lambda = \mu-\mu-\left[\frac{1}{2}(1-l)\right] = -\left[\frac{1}{2}(1-l)\right]$ *und für* $\nu = 2\mu-l-2$ *gilt* $\lambda = \mu-2\mu+l+2 = l-\mu+2$.
Also gilt

$$S_4 = \sum_{\mu=\left[\frac{l}{2}\right]+2}^{l+1} (l-\mu+2) \sum_{\lambda=l-\mu+2}^{-\left[\frac{1}{2}(1-l)\right]} x^{l-2\lambda+1}|x|^{2(\lambda-1)}$$

und mit $1 = l-l-1+2 \le l-\mu+2 \le \lambda \le -\left[\frac{1}{2}(1-l)\right]$ *und* $l-\lambda+2 \le \mu \le l+1$ *gilt*

$$\begin{aligned} S_4 &= \sum_{\lambda=1}^{-\left[\frac{1}{2}(1-l)\right]} x^{l-2\lambda+1}|x|^{2(\lambda-1)} \sum_{\mu=l-\lambda+2}^{l+1} l-\mu+2 \\ &= \sum_{\lambda=1}^{-\left[\frac{1}{2}(1-l)\right]} x^{l-2\lambda+1}|x|^{2(\lambda-1)} \sum_{\mu=1}^{l+1-l+\lambda-1=\lambda} l+2-\mu-l+\lambda-1 \\ &= \sum_{\lambda=1}^{-\left[\frac{1}{2}(1-l)\right]} x^{l-2\lambda+1}|x|^{2(\lambda-1)} \sum_{\mu=1}^{\lambda} \lambda-\mu+1 \\ &= \sum_{\lambda=1}^{-\left[\frac{1}{2}(1-l)\right]} x^{l-2\lambda+1}|x|^{2(\lambda-1)}\left\{-\frac{\lambda(\lambda+1)}{2}+\lambda(\lambda+1)\right\} \\ &= \sum_{\lambda=1}^{-\left[\frac{1}{2}(1-l)\right]} \frac{\lambda}{2}(\lambda+1)x^{l-2\lambda+1}|x|^{2(\lambda-1)}. \end{aligned}$$

Mit

$$
\begin{aligned}
l+1+\left[\frac{1}{2}(1-l)\right] &= l+2+\left[\frac{1}{2}(-1-l)\right] = l+2-\left[\frac{1}{2}(l+1+2\{(l \bmod 2)\})\right] \\
&= \left\{\begin{array}{ll} \left[\frac{1}{2}(l+2)\right] & \text{, falls } l \text{ gerade} \\ \left[\frac{1}{2}(l+4)\right] & \text{, falls } l \text{ ungerade} \end{array}\right\} = \left[\frac{1}{2}\left\{l+2+2(l \bmod 2)\right\}\right]
\end{aligned}
$$

folgt

$$
\begin{aligned}
S_4 &= \sum_{-\mu=1}^{-\left[\frac{1}{2}(1-l)\right]} -\frac{\mu}{2}(1-\mu)x^{l+2\mu+1}|x|^{2(-1-\mu)} \\
&= \sum_{\mu=\left[\frac{1}{2}(1-l)\right]}^{-1} -\frac{\mu}{2}(1-\mu)x^{l+2\mu+1}|x|^{-2(\mu+1)} \\
&= \sum_{\mu=l+1+\left[\frac{1}{2}(1-l)\right]}^{-1+l+1} -\frac{\mu-l-1}{2}(1-\mu+l+1)x^{l+2(\mu-l-1)+1}|x|^{-2(1+\mu-l-1)} \\
&= \sum_{\mu=\left[\frac{1}{2}\left\{l+2+2(l \bmod 2)\right\}\right]}^{l} \frac{l-\mu+1}{2}(l-\mu+2)x^{2\mu-l-1}|x|^{2(l-\mu)} \\
&= \sum_{\mu=\left[\frac{1}{2}\left\{l+2+2(l \bmod 2)\right\}\right]}^{l} \frac{1}{2}(l-\mu+1)(l-\mu+2)x^{2\mu-l-1}|x|^{2(l-\mu)}.
\end{aligned}
$$

Also gilt

$$
\begin{aligned}
\Upsilon_2 \;=\; & \frac{|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\Bigg(\overline{x}^{l-1}(l+1)(m+2l-1)+l\{2(k+1)-m-1\}\overline{x}^{l-1} \\
&+\sum_{\mu=2}^{\left[\frac{l}{2}\right]}\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Big\langle(l-\mu+2)\big\{m+2(l-\mu)+1\big\}+(l-\mu+1)\{2(k+1)-m-1\}\Big\rangle \\
&+\sum_{\mu=2}^{\left[\frac{l}{2}\right]}(l-\mu+1)(\mu-1)(3-m)\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)} \\
&+(m-1)\sum_{\mu=2}^{\left[\frac{l}{2}\right]+1}\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}(\mu-1)(l-\frac{\mu}{2}+2) \\
&+(l \bmod 2)\,\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}\left[\frac{1}{2}(l+3)\right]\{m-1+2(l \bmod 2)\} \\
&+\{2(k+1)-m-1\}\left[\frac{1}{2}(l+1)\right]\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]} \\
&+\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}\left[\frac{1}{2}(l+3)\right]\,2\left[\frac{l}{2}\right]+\left[\frac{1}{2}(l+1)\right]\left[\frac{l}{2}\right](3-m)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}
\end{aligned}
$$

$$+(1-m)\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]}\frac{\mu}{2}(\mu+1)\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}$$

$$+(m-1)\sum_{\mu=\left[\frac{l}{2}\right]+2}^{l}\frac{1}{2}(l-\mu+1)(l+\mu+2)x^{2\mu-l-1}|x|^{2(l-\mu)}$$

$$+(l \bmod 2)\,(m-1)\left[\frac{1}{2}(l+3)\right]\sum_{\mu=1}^{(l \bmod 2)-1\leq 0}x^{2\left[\frac{l}{2}\right]+2\mu-l+1}|x|^{2\left\{\left[\frac{1}{2}(l+1)\right]-\mu-1\right\}}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+1)\{2(k+1)-m-1\}x^{2\mu-l-1}|x|^{2(l-\mu)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+2)\,2(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}$$

$$+(1-m)\sum_{\mu=\left[\frac{1}{2}\left\{l+2+2(l \bmod 2)\right\}\right]}^{l}\frac{1}{2}(l-\mu+1)(l-\mu+2)x^{2\mu-l-1}|x|^{2(l-\mu)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+1)(\mu-1)(3-m)x^{2\mu-l-1}|x|^{2(l-\mu)}\Bigg)$$

$$=\frac{|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}$$

$$\times\Bigg(\overline{x}^{l-1}\Big\langle lm+2l^2-l+m+2l-1+2kl+2l-lm-l+1-m\Big\rangle$$

$$+\sum_{\mu=2}^{\left[\frac{l}{2}\right]}\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Big\langle lm+2l^2-2l\mu+l-\mu m-2l\mu+2\mu^2-\mu+2m+4l-4\mu+2$$

$$+2kl+2l-lm-l-2k\mu-2\mu+\mu m+\mu+2k+2-m-1$$

$$+(l\mu-l-\mu^2+\mu+\mu-1)(3-m)$$

$$+(m-1)(l\mu-\frac{1}{2}\mu^2+2\mu-l+\frac{\mu}{2}-2)+(1-m)(\frac{1}{2}\mu^2+\frac{\mu}{2})\Big\rangle$$

$$+\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}\Big\langle(m-1)(\left[\frac{1}{2}(l+2)\right]-1)\left\{l-\frac{1}{2}\left[\frac{1}{2}(l+2)\right]+2\right\}$$

$$+(l \bmod 2)\left[\frac{1}{2}(l+3)\right](m+1)+\{2(k+1)-m-1\}\left[\frac{1}{2}(l+1)\right]+\left[\frac{1}{2}(l+3)\right]\,2\left[\frac{l}{2}\right]$$

$$+\left[\frac{1}{2}(l+1)\right]\left[\frac{l}{2}\right](3-m)+(1-m)\,\frac{1}{2}\left[\frac{1}{2}(l+1)\right]\left[\frac{1}{2}(l+3)\right]$$

$$+(1-m)\{1-(l \bmod 2)\}\,\frac{1}{2}\left(l-\left[\frac{1}{2}(l+2)\right]+1\right)\left(l-\left[\frac{1}{2}(l+2)\right]+2\right)\Big\rangle$$

$$
\begin{aligned}
& + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l} x^{2\mu-l-1}|x|^{2(l-\mu)}\Big\langle \frac{1}{2}(l^2+l\mu+2l-l\mu-\mu^2-2\mu+l+\mu+2)(m-1) \\
& \qquad +2kl+2l-lm-l-2k\mu-2\mu+\mu m+\mu+2k+2-m \\
& \qquad -1+2l^2-2l\mu+2l-2l\mu+2\mu^2-2\mu+4l-4\mu+4 \\
& \qquad +(1-m)\frac{1}{2}(l^2-l\mu+2l-l\mu+\mu^2-2\mu+l-\mu+2) \\
& \qquad +(l\mu-l-\mu^2+\mu+\mu-1)(3-m)\Big\rangle\Bigg) \\
= & \frac{|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\Bigg(2l(k+l+1)\overline{x}^{l-1} \\
& + \sum_{\mu=2}^{\left[\frac{l}{2}\right]} \overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Big\langle 2l^2-4l\mu+6l+2\mu^2-6\mu+m+3+2kl-2k\mu+2k+3l\mu-3l \\
& \qquad -3\mu^2+6\mu-3-ml\mu+ml+m\mu^2-2m\mu+m+ml\mu-\frac{1}{2}m\mu^2 \\
& \qquad +2m\mu-lm+\frac{1}{2}m\mu-2m-l\mu+\frac{1}{2}\mu^2-2\mu+l-\frac{1}{2}\mu+2+\frac{1}{2}\mu^2 \\
& \qquad +\frac{1}{2}\mu-\frac{1}{2}m\mu^2-\frac{1}{2}m\mu\Big\rangle \\
& +\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}\Big\langle (m-1)\left(\left[\frac{1}{2}(l+2)\right]-1\right)\left\{l-\frac{1}{2}\left[\frac{1}{2}(l+2)\right]+2\right\} \\
& +(l \bmod 2)\left[\frac{1}{2}(l+3)\right](m+1)+\{2(k+1)-m-1\}\left[\frac{1}{2}(l+1)\right] \\
& +\left[\frac{1}{2}(l+3)\right]2\left[\frac{l}{2}\right]+\left[\frac{1}{2}(l+1)\right]\left[\frac{l}{2}\right](3-m)+(1-m)\frac{1}{2}\left[\frac{1}{2}(l+1)\right]\left[\frac{1}{2}(l+3)\right]\Big\rangle \\
& + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l} x^{2\mu-l-1}|x|^{2(l-\mu)}\Big\langle \frac{1}{2}ml^2-\frac{1}{2}l^2+\frac{1}{2}ml\mu-\frac{1}{2}l\mu+lm-l-\frac{1}{2}ml\mu+\frac{1}{2}l\mu-\frac{1}{2}m\mu^2 \\
& \qquad +\frac{1}{2}\mu^2-m\mu+\mu+\frac{1}{2}ml-\frac{1}{2}l+\frac{1}{2}m\mu-\frac{1}{2}\mu+m-1+2kl \\
& \qquad +7l-lm-2k\mu-7\mu+\mu m+2k+5-m+2l^2-4l\mu+2\mu^2 \\
& \qquad +\frac{1}{2}l^2-\frac{1}{2}l\mu+l-\frac{1}{2}l\mu+\frac{1}{2}\mu^2-\mu+\frac{1}{2}l-\frac{1}{2}\mu+1-\frac{1}{2}ml^2 \\
& \qquad +\frac{1}{2}ml\mu-ml+\frac{1}{2}ml\mu-\frac{1}{2}m\mu^2+m\mu-\frac{1}{2}ml+\frac{1}{2}m\mu-m \\
& \qquad +3l\mu-3l-3\mu^2+6\mu-3-ml\mu+ml+m\mu^2-2m\mu+m\Big\rangle\Bigg).
\end{aligned}
$$

Seien

$$
\begin{aligned}
\Upsilon_{2,1} \;:=\; & 2l^2 - 4l\mu + 6l + 2\mu^2 - 6\mu + m + 3 + 2kl - 2k\mu + 2k + 3l\mu - 3l - 3\mu^2 + 6\mu - 3 - ml\mu \\
& + ml + m\mu^2 - 2m\mu + m + ml\mu - \frac{1}{2}m\mu^2 + 2m\mu - lm + \frac{1}{2}m\mu - 2m - l\mu + \frac{1}{2}\mu^2 - 2\mu \\
& + l - \frac{1}{2}\mu + 2 + \frac{1}{2}\mu^2 + \frac{1}{2}\mu - \frac{1}{2}m\mu^2 - \frac{1}{2}m\mu,
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{2,2} \;:=\; & \overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}\Big\langle (m-1)(\left[\frac{1}{2}(l+2)\right] - 1)\left\{l - \frac{1}{2}\left[\frac{1}{2}(l+2)\right] + 2\right\} \\
& + (l \bmod 2)\left[\frac{1}{2}(l+3)\right](m+1) + \{2(k+1) - m - 1\}\left[\frac{1}{2}(l+1)\right] \\
& + \left[\frac{1}{2}(l+3)\right] 2\left[\frac{l}{2}\right] + \left[\frac{1}{2}(l+1)\right]\left[\frac{l}{2}\right](3-m) + (1-m)\frac{1}{2}\left[\frac{1}{2}(l+1)\right]\left[\frac{1}{2}(l+3)\right]\Big\rangle
\end{aligned}
$$

und

$$
\begin{aligned}
\Upsilon_{2,3} \;:=\; & \frac{1}{2}ml^2 - \frac{1}{2}l^2 + \frac{1}{2}ml\mu - \frac{1}{2}l\mu + lm - l - \frac{1}{2}ml\mu + \frac{1}{2}l\mu - \frac{1}{2}m\mu^2 + \frac{1}{2}\mu^2 - m\mu + \mu + \frac{1}{2}ml - \frac{1}{2}l \\
& + \frac{1}{2}m\mu - \frac{1}{2}\mu + m - 1 + 2kl + 7l - lm - 2k\mu - 7\mu + \mu m + 2k + 5 - m + 2l^2 - 4l\mu + 2\mu^2 \\
& + \frac{1}{2}l^2 - \frac{1}{2}l\mu + l - \frac{1}{2}l\mu + \frac{1}{2}\mu^2 - \mu + \frac{1}{2}l - \frac{1}{2}\mu + 1 - \frac{1}{2}ml^2 + \frac{1}{2}ml\mu - ml + \frac{1}{2}ml\mu - \frac{1}{2}m\mu^2 \\
& + m\mu - \frac{1}{2}ml + \frac{1}{2}m\mu - m + 3l\mu - 3l - 3\mu^2 + 6\mu - 3 - ml\mu + ml + m\mu^2 - 2m\mu + m.
\end{aligned}
$$

Dann gilt

$$
\begin{aligned}
\Upsilon_2 \;=\; & \frac{|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\Bigg(2l(k+l+1)\overline{x}^{l-1} + \sum_{\mu=2}^{\left[\frac{l}{2}\right]}\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}\Upsilon_{2,1} \\
& + \Upsilon_{2,2} + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l} x^{2\mu-l-1}|x|^{2(l-\mu)}\Upsilon_{2,3}\Bigg).
\end{aligned}
$$

Es gelten

$$
\begin{aligned}
\Upsilon_{2,1} \;=\; & l^2(2) + l\mu(-4+3-1) + l(6-3+1) + \mu^2(2-3+\frac{1}{2}+\frac{1}{2}) + \mu(-6+6-2-\frac{1}{2}+\frac{1}{2}) \\
& + m(1+1-2) + (3-3+2) + kl(2) + k\mu(-2) + k(2) + ml\mu(-1+1) + ml(1-1) \\
& + m\mu^2(1-\frac{1}{2}-\frac{1}{2}) + m\mu(-2+2+\frac{1}{2}-\frac{1}{2}) \\
=\; & 2l^2 - 2l\mu + 4l - 2\mu + 2 + 2kl - 2k\mu + 2k \\
=\; & 2kl - 2k\mu + 2k + 2l^2 - 2l\mu + 2l + 2l - 2\mu + 2 \\
=\; & (l-\mu+1)\,2(k+l+1)
\end{aligned}
$$

und

$$\begin{aligned}
\Upsilon_{2,3} &= ml^2(\frac{1}{2}-\frac{1}{2}) + l^2(-\frac{1}{2}+2+\frac{1}{2}) + ml\mu(\frac{1}{2}-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1) + l\mu(-\frac{1}{2}+\frac{1}{2}-4-\frac{1}{2}-\frac{1}{2}+3)\\
&\quad +lm(1+\frac{1}{2}-1-1-\frac{1}{2}+1) + l(-1-\frac{1}{2}+7+1+\frac{1}{2}-3) + m\mu^2(-\frac{1}{2}-\frac{1}{2}+1)\\
&\quad +\mu^2(\frac{1}{2}+2+\frac{1}{2}-3) + m\mu(-1+\frac{1}{2}+1+1+\frac{1}{2}-2) + \mu(1-\frac{1}{2}-7-1-\frac{1}{2}+6)\\
&\quad +m(1-1-1+1) + (-1+5+1-3) + k\mu(-2) + kl(2) + k(2)\\
&= 2l^2 - 2l\mu + 4l - 2\mu + 2 - 2k\mu + 2kl + 2k\\
&= (l-\mu+1)\,2(k+l+1).
\end{aligned}$$

Fallunterscheidung:

1. *Sei l ungerade. Dann gilt mit $\left[\frac{l}{2}\right] = \frac{l-1}{2}$ und $\left[\frac{1}{2}(l+1)\right] = \frac{l+1}{2}$*

$$\begin{aligned}
\Upsilon_{2,2} &= (m-1)\underbrace{\left\{\frac{l+1}{2}-1\right\}}_{\frac{l-1}{2}}\left\{l-\frac{1}{2}\,\frac{l+1}{2}+2\right\} + \frac{l+3}{2}(m+1) + (2k+1-m)\frac{l+1}{2}\\
&\quad +\frac{l+3}{2}\,2\frac{l-1}{2} + \frac{l+1}{2}\,\frac{l-1}{2}(3-m) + (1-m)\,\frac{1}{2}\frac{l+1}{2}\,\frac{l+3}{2}\\
&= \left(\frac{1}{2}ml-\frac{1}{2}m-\frac{1}{2}l+\frac{1}{2}\right)\underbrace{\left(l-\frac{1}{4}l-\frac{1}{4}+2\right)}_{\frac{3}{4}l+\frac{7}{4}} + \frac{1}{2}ml+\frac{1}{2}l+\frac{3}{2}m+\frac{3}{2}+kl+k+\frac{1}{2}l\\
&\quad +\frac{1}{2}-\frac{1}{2}ml-\frac{1}{2}m+\frac{1}{2}l^2-\frac{1}{2}l+\frac{3}{2}l-\frac{3}{2}+\left(\frac{1}{4}l^2-\frac{1}{4}l+\frac{1}{4}l-\frac{1}{4}\right)(3-m)\\
&\quad +\frac{1}{2}(1-m)\left(\frac{1}{4}l^2+\underbrace{\frac{3}{4}l+\frac{1}{4}l}_{l}+\frac{3}{4}\right)\\
&= \frac{3}{8}ml^2+\frac{7}{8}ml-\frac{3}{8}ml-\frac{7}{8}m-\frac{3}{8}l^2-\frac{7}{8}l+\frac{3}{8}l+\frac{7}{8}+2l+m+\frac{1}{2}+kl+k+\frac{1}{2}l^2\\
&\quad +\frac{3}{4}l^2-\frac{1}{4}ml^2-\frac{3}{4}+\frac{1}{4}m+\frac{1}{8}l^2+\frac{1}{2}l+\frac{3}{8}-\frac{1}{8}ml^2-\frac{1}{2}ml-\frac{3}{8}m\\
&= ml^2\left(\frac{3}{8}-\frac{1}{4}-\frac{1}{8}\right)+ml\left(\frac{7}{8}-\frac{3}{8}-\frac{1}{2}\right)+m\left(-\frac{7}{8}+1+\frac{1}{4}-\frac{3}{8}\right)\\
&\quad +l^2\left(-\frac{3}{8}+\frac{1}{2}+\frac{3}{4}+\frac{1}{8}\right)+l\left(-\frac{7}{8}+\frac{3}{8}+2+\frac{1}{2}\right)+\left(\frac{7}{8}+\frac{1}{2}-\frac{3}{4}+\frac{3}{8}\right)+kl+k\\
&= l^2+2l+1+kl+k\\
&= 2l^2-l^2-l+4l-l-1+2+2kl-kl-k+2k\\
&= 2l^2-2l\frac{l+1}{2}+4l-2\frac{l+1}{2}+2+2kl-2k\frac{l+1}{2}+2k
\end{aligned}$$

$$= 2l^2 - 2l\left[\frac{1}{2}(l+2)\right] + 4l - 2\left[\frac{1}{2}(l+2)\right] + 2 + 2kl - 2k\left[\frac{1}{2}(l+2)\right] + 2k$$

$$= \sum_{\mu=\left[\frac{1}{2}(l+2)\right]}^{\left[\frac{1}{2}(l+2)\right]} 2l^2 - 2l\mu + 4l - 2\mu + 2 - 2k\mu + 2kl + 2k$$

$$= \sum_{\mu=\left[\frac{1}{2}(l+2)\right]}^{\left[\frac{1}{2}(l+2)\right]} (l-\mu+1)\, 2(k+l+1).$$

Da

$2l(k+l+1)\overline{x}^{l-1} = \sum_{\mu=1}^{1}(l-\mu+1)\overline{x}^{l-2\mu+1}|x|^{2(\mu-1)},$

$\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]} = |x|^{l-1} = \sum_{\mu=\left[\frac{1}{2}(l+2)\right]}^{\left[\frac{1}{2}(l+2)\right]} \overline{x}^{l-2\mu+1}|x|^{2(\mu-1)}$

und $\left[\frac{1}{2}(l+2)\right] = \frac{l+1}{2}$ *gilt, wird* Υ_2 *zu*

$$\frac{2(k+l+1)|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\{\sum_{\mu=1}^{\frac{l+1}{2}}(l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)} + \sum_{\mu=\left[\frac{1}{2}(l+4)\right]}^{l}(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\right\}$$

$$= \frac{2(k+l+1)|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\{\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]}(l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)} + \sum_{\mu=\left[\frac{1}{2}(l+3)\right]}^{l}(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\right\}.$$

In diesem Fall ist Υ_2 *in der gewünschten Form.*

2. Sei l gerade. Dann gilt mit $\left[\frac{l}{2}\right]=\frac{l}{2}$ und $\left[\frac{1}{2}(l+1)\right]=\frac{l}{2}$

$$\begin{aligned}
\Upsilon_{2,2} &= (m-1)\frac{l}{2}\underbrace{\left\{l-\frac{l+2}{4}+2\right\}}_{\frac{3}{4}l+\frac{3}{2}}+(2k+1-m)\,\frac{l}{2}+\frac{l+2}{2}\,2\,\frac{l}{2}+\frac{1}{4}l^2(3-m)\\
&\quad +(1-m)\,\frac{1}{2}\,\frac{l}{2}\,\frac{l+2}{2}\\
&= \frac{3}{8}ml^2+\frac{3}{4}ml-\frac{3}{8}l^2-\frac{3}{4}l+kl+\frac{1}{2}l-\frac{1}{2}ml+\frac{1}{2}l^2+l+\frac{3}{4}l^2-\frac{1}{4}ml^2+\frac{1}{8}l^2+\frac{1}{4}l\\
&\quad -\frac{1}{8}ml^2-\frac{1}{4}ml\\
&= ml^2\left(\frac{3}{8}-\frac{1}{4}-\frac{1}{8}\right)+ml\left(\frac{3}{4}-\frac{1}{2}-\frac{1}{4}\right)+l^2\left(-\frac{3}{8}+\frac{1}{2}+\frac{3}{4}+\frac{1}{8}\right)\\
&\quad +l\left(-\frac{3}{4}+\frac{1}{2}+1+\frac{1}{4}\right)+kl\\
&= l^2+l+kl\\
&= 2l^2-l^2-2l+4l-l-2+2-kl-2k+2kl+2k\\
&= 2l^2-2l\frac{l+2}{2}+4l-2\frac{l+2}{2}+2-2k\frac{l+2}{2}+2kl+2k\\
&= \sum_{\mu=\frac{l+2}{2}}^{\frac{l+2}{2}} 2l^2-2l\mu+4l-2\mu+2-2k\mu+2kl+2k\\
&= \sum_{\mu=\frac{l+2}{2}}^{\frac{l+2}{2}} (l-\mu+1)\,2(k+l+1).
\end{aligned}$$

Da $\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{l}{2}\right]}=\overline{x}^{-1}|x|^l=x|x|^{l-2}=\sum_{\mu=\frac{l+2}{2}}^{\frac{l+2}{2}} x^{2\mu-l-1}|x|^{2(l-\mu)}$ *, und* $\left[\frac{l}{2}\right]=\left[\frac{1}{2}(l+1)\right]$ *gilt, wird* Υ_2 *in diesem Fall zu*

$$\begin{aligned}
&\frac{2(k+l+1)|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\{\sum_{\mu=1}^{\left[\frac{l}{2}\right]}(l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)}\right.\\
&\qquad\left.+\sum_{\mu=\left[\frac{1}{2}(l+2)\right]}^{l}(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\right\}\\
=&\frac{2(k+l+1)|x|^{2(k+1)-m-1}}{\prod_{j=2}^{k}(2j-m-1)\{2(k+1)-m-1\}}\left\{\sum_{\mu=1}^{\left[\frac{1}{2}(l+1)\right]}(l-\mu+1)\overline{x}^{\,l-2\mu+1}|x|^{2(\mu-1)}\right.\\
&\qquad\left.+\sum_{\mu=\left[\frac{1}{2}(l+3)\right]}^{l}(l-\mu+1)x^{2\mu-l-1}|x|^{2(l-\mu)}\right\}.
\end{aligned}$$

In diesem Fall ist Υ_2 *in der gewünschten Form.*

3. Nun wird Υ_3 betrachtet:
Es gilt

$$
\begin{aligned}
\Upsilon_3 \;=\; & \sum_{n=1}^{\left[\frac{1}{2}(l-2)\right]} \Bigg(\frac{|x|^{2(k+n+1)-m-1}}{\prod_{j=n+2}^{k}(2j-m-1)\prod_{j=1}^{n+1}\{2(k+j)-m-1\}} \\
& \Bigg\langle \overline{x}^{l-1-2n}\left\{ \binom{l-n+1}{n+1}(m+2l-1-4n) + \binom{l-n}{n+1}\{2(k+n+1)-m-1\} \right\} \\
& + \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
& \qquad \times (m+2l+3-4\mu-4n+2\mu-2) \\
& + \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)} \binom{l+1-n-\mu}{n+1}\binom{n+\mu-1}{n}\{2(k+n+1)-m-1\} \\
& + \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}(2n+3-m)\overline{x}^{l-2(\mu+n)+1}|x|^{2(\mu-1)} \\
& + \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}(m-1) \sum_{\nu=1}^{\left[\frac{l}{2}\right]+1-\mu-n} \overline{x}^{l+1-2(\mu+n+\nu)}|x|^{2(\mu+\nu-1)} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}(1-m) \\
& \qquad \times \sum_{\nu=1}^{n+\mu+\left[\frac{1}{2}(-l-1)\right]} \overline{x}^{2(n+\mu-\nu)-l-1}|x|^{2(l-2n+\nu-\mu)} \\
& +(l \bmod 2)\overline{x}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}} \binom{\left[\frac{1}{2}(l+3)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n} \\
& \qquad \times \left\{ m+2l+3-4\left[\frac{1}{2}(l-2n+2)\right]-4n \right\} \\
& +\overline{x}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}} \binom{\left[\frac{1}{2}(l+3)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n} 2\left\{ \left[\frac{1}{2}(l-2n+2)\right]-1 \right\} \\
& +\{2(k+n+1)-m-1\} \binom{\left[\frac{1}{2}(l+1)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n} \overline{x}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}} \\
& +(l \bmod 2)(m-1)\binom{\left[\frac{1}{2}(l+3)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n} \\
& \qquad \times \sum_{\nu=1}^{\left[\frac{1}{2}\left\{l+2-2\left[\frac{1}{2}(l-2n+2)\right]-2n\right\}\right]=0} \overline{x}^{l+1-2\left(\left[\frac{1}{2}(l-2n+2)\right]+n+\nu\right)}|x|^{2\left(\left[\frac{1}{2}(l-2n+2)\right]-1\right)}
\end{aligned}
$$

$$+\binom{\left[\frac{1}{2}(l+1)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n+1}(2n+3-m)\overline{x}^{l-2\left(\left[\frac{1}{2}(l-2n)\right]+n\right)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]}$$

$$+\sum_{\mu=1}^{\left[\frac{l}{2}\right]-n}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}(m-1)\sum_{\nu=\left[\frac{1}{2}(l+4-2\mu-2n)\right]}^{l+1-2\mu-2n}x^{2(\mu+\nu+n)-l-1}|x|^{2(l-\mu-2n-\nu)}$$

$$+(l \bmod 2)(m-1)\binom{\left[\frac{1}{2}(l+3)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n}$$

$$\times\sum_{\nu=\left[\frac{1}{2}\left\{l+4-2\left[\frac{1}{2}(l-2n+2)\right]-2n\right\}\right]=1}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n=(l \bmod 2)-1\leq 0}x^{2\left(\left[\frac{1}{2}(l-2n+2)\right]+\nu+n\right)-l-1}|x|^{2\left(l-\left[\frac{1}{2}(l-2n+2)\right]-2n-\nu\right)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n}\binom{l+1-n-\mu}{n+1}\binom{n+\mu-1}{n}\{2(k+n+1)-m-1\}x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}$$

$$+0\binom{l-n}{n}\{2(k+n+1)-m-1\}x^{2(n+l-2n+1)-l-1}|x|^{2(l-2n-l+2n-1)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}2(l-2n-\mu+1)x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}(1-m)\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\sum_{\nu=\left[\frac{1}{2}\{2(n+\mu)-l+1\}\right]}^{2(n+\mu-1)-l}x^{2(\nu-n-\mu)+l+1}|x|^{2(\mu-\nu-1)}$$

$$+\sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}(2n+3-m)x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}.$$

1. *Sei*

$$S_5 := \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\sum_{\nu=1}^{\left[\frac{1}{2}\{l+2(1-\mu-n)\}\right]}\overline{x}^{l+1-2(n+\mu+\nu)}|x|^{2(\mu+\nu-1)}.$$

Sei $\lambda := \mu+\nu$. *Für* $\nu = 1$ *gilt* $\lambda = \mu+1$*; für* $\nu = \left[\frac{1}{2}\{l+2(1-n)\}\right]-\mu$ *gilt* $\lambda = \mu + \left[\frac{1}{2}\{l+2(1-n)\}\right]-\mu = \left[\frac{1}{2}\{l+2(1-n)\}\right]$, *also*

$$S_5 = \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\sum_{\lambda=\mu+1}^{\left[\frac{1}{2}\{l+2(1-n)\}\right]}\overline{x}^{l+1-2(n+\lambda)}|x|^{2(\lambda-1)}.$$

Es gilt $2=1+1\leq\mu+1\leq\lambda\leq\left[\frac{1}{2}\{l+2(1-n)\}\right]$ *und* $1\leq\mu\leq\lambda-1$, *also*

$$S_5 = \sum_{\lambda=2}^{\left[\frac{1}{2}\{l+2(1-n)\}\right]}\overline{x}^{l+1-2(n+\lambda)}|x|^{2(\lambda-1)}\sum_{\mu=1}^{\lambda-1}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}.$$

2. Sei

$$S_6 := \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \sum_{\nu=1}^{n+\mu+\left[\frac{1}{2}(-l-1)\right]} \overline{x}^{2(n+\mu-\nu)-l-1}|x|^{2(l-2n-\mu+\nu)}.$$

Sei $\lambda := \mu-\nu$. *Dann gilt für* $\nu = 1$, $\lambda = \mu - 1$ *und mit* $\left[\frac{1}{2}(-l-1)\right] = -\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]$ *gilt für* $\nu = n+\mu+\left[\frac{1}{2}(-l-1)\right]$, $\lambda = \mu - n - \mu - \left[\frac{1}{2}(-l-1)\right] = \left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right] - n$.
Also gilt

$$S_6 = \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \sum_{\lambda=\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n}^{\mu-1} \overline{x}^{2(n+\lambda)-l-1}|x|^{2(l-2n-\lambda)}.$$

Mit $\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right] - n \le \lambda \le \mu - 1 \le l-2n$, $\lambda+1 \le \mu \le l-2n+1$ *und* $l+1-\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right] = \left[\frac{1}{2}(l+1)\right]$ *folgt*

$$\begin{aligned}
S_6 &= \sum_{\lambda=\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n}^{l-2n} \overline{x}^{2(n+\lambda)-l-1}|x|^{2(l-2n-\lambda)} \sum_{\mu=\lambda+1}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n}^{l-2n} \overline{x}^{2(n+\lambda)-l-1}|x|^{2(l-2n-\lambda)} \sum_{\mu=0}^{l-2n-\lambda} \binom{l-n-\mu-\lambda+1}{n+1}\binom{n+\mu+\lambda}{n} \\
&= \sum_{-\lambda=\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n}^{l-2n} \overline{x}^{2(n-\lambda)-l-1}|x|^{2(l-2n+\lambda)} \sum_{\mu=1-\lambda}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=2n-l}^{n-\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]} \overline{x}^{2(n-\lambda)-l-1}|x|^{2(l-2n+\lambda)} \sum_{\mu=1-\lambda}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=2n-l-2n+l+1}^{1+l-2n+n-\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]} \overline{x}^{2(n-2n+l+1-\lambda)-l-1}|x|^{2(l-2n+\lambda+2n-l-1)} \\
&\qquad \times \sum_{\mu=1-\lambda-2n+l+1=l-\lambda-2n+2}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=1}^{\left[\frac{1}{2}(l+1)\right]-n} \overline{x}^{l-2(n+\lambda)+1}|x|^{2(\lambda-1)} \\
&\qquad \times \sum_{\mu=l-\lambda-2n+2-l+\lambda+2n-1}^{l-2n+1-l+\lambda+2n-1} \binom{l-n+2-\mu-l+\lambda+2n-1}{n+1}\binom{n+\mu-1+l-\lambda-2n+1}{n} \\
&= \sum_{\lambda=1}^{\left[\frac{1}{2}(l+1)\right]-n} \overline{x}^{l-2(n+\lambda)+1}|x|^{2(\lambda-1)} \sum_{\mu=1}^{\lambda} \binom{n+1-\mu+\lambda}{n+1}\binom{l-n+\mu-\lambda}{n}.
\end{aligned}$$

3. Sei

$$S_7 := \sum_{\mu=1}^{\left[\frac{l}{2}\right]-n} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \sum_{\nu=\left[\frac{1}{2}(l+4-2\mu-2n)\right]}^{l+1-2\mu-2n} x^{2(\mu+\nu+n)-l-1}|x|^{2(l-\mu-2n-\nu)}.$$

Sei $\lambda := \mu+\nu$*, dann gilt für* $\nu = \left[\frac{1}{2}(l+4-2\mu-2n)\right]$, $\lambda = \mu+\left[\frac{1}{2}(l+4-2\mu-2n)\right] = \left[\frac{l}{2}\right]-n+2$ *und für* $\nu = l+1-2(\mu+n)$, $\lambda = \mu+l+1-2\mu-2n = l-2n-\mu+1$*, also*

$$S_7 = \sum_{\mu=1}^{\left[\frac{l}{2}\right]-n} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \sum_{\lambda=\left[\frac{l}{2}\right]-n+2}^{l-2n-\mu+1} x^{2(\lambda+n)-l-1}|x|^{2(l-\lambda-2n)}.$$

Es gilt $\left[\frac{l}{2}\right]-n+2 \le \lambda \le l-2n-\mu+1 \le l-2n$ *und* $1 \le \mu \le l-2n+1-\lambda$*, also gilt*

$$S_7 = \sum_{\lambda=\left[\frac{l}{2}\right]-n+2}^{l-2n} x^{2(\lambda+n)-l-1}|x|^{2(l-\lambda-2n)} \sum_{\mu=1}^{l-2n-\lambda+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}.$$

4. Sei

$$S_8 := \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \sum_{\nu=\left[\frac{1}{2}\{2(n+\mu)-l+1\}\right]}^{2(n+\mu-1)-l} x^{2(\nu-\mu-n)+l+1}|x|^{2(\mu-\nu-1)}.$$

Sei $\lambda := \mu-\nu$*. Dann gilt für* $\nu = n+\mu+\left[\frac{1}{2}(1-l)\right]$,
$\lambda = \mu-n-\mu-\left[\frac{1}{2}(1-l)\right] = -n-\left\{\left[\frac{1}{2}(-l-1)\right]+1\right\} = -n-1+\left[\frac{1}{2}\left\langle l+1+2\{1-(l \bmod 2)\}\right\rangle\right]$
und für $\nu = 2(n+\mu-1)-l$, $\lambda = \mu-2n-2\mu+2+l = l-2n-\mu+2$*, also gilt*

$$S_8 = \sum_{\mu=\left[\frac{l}{2}\right]-n+2}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \sum_{\lambda=l-2n-\mu+2}^{\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n-1} x^{l+1-2(\lambda+n)}|x|^{2(\lambda-1)}.$$

Es gilt $1 = l-2n+2-l+2n-1 \le l-2n-\mu+2 \le \lambda \le \left[\frac{1}{2}\left\langle l+1+2\{1-(l \bmod 2)\}\right\rangle\right]-n-1$

und $l-2n-\lambda+2 \le \mu \le l-2n+1$, *also*

$$
\begin{aligned}
S_8 &= \sum_{\lambda=1}^{\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n-1} x^{l+1-2(\lambda+n)}|x|^{2(\lambda-1)} \sum_{\mu=l-2n-\lambda+2}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{-\lambda=1}^{\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-n-1} x^{l+1+2(\lambda-n)}|x|^{-2(\lambda+1)} \sum_{\mu=l-2n+\lambda+2}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=n+1-\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]}^{-1} x^{l+1+2(\lambda-n)}|x|^{-2(\lambda+1)} \sum_{\mu=l-2n+\lambda+2}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=n+1-\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]-2n+l+1}^{-1-2n+l+1} x^{l+1+2(\lambda+2n-l-1-n)}|x|^{-2(\lambda+2n-l-1+1)} \\
&\qquad \times \sum_{\mu=l-2n+2+2n-l-1+\lambda}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=l+2-n-\left[\frac{1}{2}\langle l+1+2\{1-(l \bmod 2)\}\rangle\right]=l+2-n-\{1-(l \bmod 2)\}-\left[\frac{1}{2}(l+1)\right]}^{l-2n} x^{2(n+\lambda)-l-1}|x|^{2(l-2n-\lambda)} \\
&\qquad \times \sum_{\mu=1+\lambda}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
&= \sum_{\lambda=\left[\frac{l}{2}\right]+2-n-\{1-(l \bmod 2)\}=\left[\frac{1}{2}(l-2n+4)\right]-\{1-(l \bmod 2)\}}^{l-2n} x^{2(n+\lambda)-l-1}|x|^{2(l-2n-\lambda)} \\
&\qquad \times \sum_{\mu=1+\lambda}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}.
\end{aligned}
$$

Für gerade l gilt

$$
\begin{aligned}
S_8 &= \sum_{\mu=\left[\frac{1}{2}(l-2n+2)\right]}^{l-2n} x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)} \sum_{\nu=\mu+1}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
&= \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n} x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)} \sum_{\nu=\mu+1}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
&\quad +\overline{x}^{-1}|x|^{l-2n} \sum_{\nu=\frac{l}{2}-n+2}^{l-2n+1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}.
\end{aligned}
$$

Seien

$$
\begin{aligned}
\Upsilon_{3,1} &:= \binom{l-n+1}{n+1}(m+2l-1-4n) + \binom{l-n}{n+1}\{2(k+n+1)-m-1\} \\
&\quad +(1-m)\sum_{\mu=1}^{1} \binom{n+1-\mu+1}{n+1}\binom{l-n+\mu-1}{n},
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{3,2} \quad := \quad & \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n} \\
& \qquad \times(m+2l+3-4\mu-4n+2\mu-2) \\
& + \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)} \binom{l+1-n-\mu}{n+1}\binom{n+\mu-1}{n} \{2(k+n+1)-m-1\} \\
& + \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}(2n+3-m)\overline{x}^{l-2(\mu+n)+1}|x|^{2(\mu-1)} \\
& + \sum_{\mu=1}^{\left[\frac{1}{2}(l-2n)\right]} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}(m-1) \sum_{\nu=1}^{\left[\frac{l}{2}\right]+1-\mu-n} \overline{x}^{l+1-2(\mu+n+\nu)}|x|^{2(\mu+\nu-1)} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1} \binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}(1-m) \\
& \qquad \times \sum_{\nu=1}^{n+\mu+\left[\frac{1}{2}(-l-1)\right]} \overline{x}^{2(n+\mu-\nu)-l-1}|x|^{2(l-2n+\nu-\mu)} \\
& -(1-m)\sum_{\mu=1}^{1} \binom{n+1-\mu+1}{n+1}\binom{l-n+\mu-1}{n} \\
& -(m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}\{l+2(1-n)\}\right]-1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
& +(l \bmod 2)(m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}(l-2n+1)\right]} \binom{n+1-\nu+\mu}{n+1}\binom{l-n-\nu+\mu}{n},
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{3,3} \quad := \quad & (l \bmod 2)\overline{x}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}} \binom{\left[\frac{1}{2}(l+3)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n} \\
& \qquad \times\left\{m+2l+3-4\left[\frac{1}{2}(l-2n+2)\right]-4n\right\} \\
& +\overline{x}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}} \binom{\left[\frac{1}{2}(l+3)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n} 2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\} \\
& +\{2(k+n+1)-m-1\}\binom{\left[\frac{1}{2}(l+1)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n}\overline{x}^{l+1-2\left[\frac{1}{2}(l-2n+2)\right]-2n}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}} \\
& +\binom{\left[\frac{1}{2}(l+1)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n+1}(2n+3-m)\overline{x}^{l-2\left(\left[\frac{1}{2}(l-2n)\right]+n\right)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \\
& +(m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}\{l+2(1-n)\}\right]-1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}
\end{aligned}
$$

$$-(l \bmod 2)(m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]}\sum_{\nu=1}^{\left[\frac{1}{2}(l-2n+1)\right]}\begin{pmatrix} n+1-\nu+\mu \\ n+1 \end{pmatrix}\begin{pmatrix} l-n-\nu+\mu \\ n \end{pmatrix}$$
$$+\{1-(l \bmod 2)\}\,(1-m)\,x|x|^{l-2n-2}\sum_{\nu=\frac{l}{2}-n+2}^{l-2n+1}\begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu-1 \\ n \end{pmatrix}$$

und

$$\begin{aligned}\Upsilon_{3,4} \;:=\; & \sum_{\mu=1}^{\left[\frac{l}{2}\right]-n}\begin{pmatrix} l-n-\mu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\mu-1 \\ n \end{pmatrix}(m-1)\sum_{\nu=\left[\frac{1}{2}(l+4-2\mu-2n)\right]}^{l+1-2\mu-2n} x^{2(\mu+\nu+n)-l-1}|x|^{2(l-\mu-2n-\nu)} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n}\begin{pmatrix} l+1-n-\mu \\ n+1 \end{pmatrix}\begin{pmatrix} n+\mu-1 \\ n \end{pmatrix}\{2(k+n+1)-m-1\}x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n}\begin{pmatrix} l-n-\mu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\mu-1 \\ n \end{pmatrix}2(l-2n-\mu+1)x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n+1}(1-m)\begin{pmatrix} l-n-\mu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\mu-1 \\ n \end{pmatrix}\sum_{\nu=\left[\frac{1}{2}\{2(n+\mu)-l+1\}\right]}^{2(n+\mu-1)-l} x^{2(\nu-n-\mu)+l+1}|x|^{2(\mu-\nu-1)} \\
& + \sum_{\mu=\left[\frac{1}{2}(l-2n+4)\right]}^{l-2n}\begin{pmatrix} l-n-\mu+1 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\mu-1 \\ n+1 \end{pmatrix}(2n+3-m)x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)} \\
& - \{1-(l \bmod 2)\}\,(1-m)\,x|x|^{l-2n-2}\sum_{\nu=\frac{l}{2}-n+2}^{l-2n+1}\begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu-1 \\ n \end{pmatrix}.\end{aligned}$$

Dann gilt

$$\Upsilon_3 = \Upsilon_{3,1} + \Upsilon_{3,2} + \Upsilon_{3,3} + \Upsilon_{3,4}.$$

Für $\Upsilon_{3,1}$ *gilt*

$$\begin{aligned} & \begin{pmatrix} l-n+1 \\ n+1 \end{pmatrix}(m+2l-1-4n) + \begin{pmatrix} l-n \\ n+1 \end{pmatrix}(2k+2n+1-m) \\
& + \begin{pmatrix} n+1 \\ n+1 \end{pmatrix}\begin{pmatrix} l-n \\ n \end{pmatrix}(1-m) \\
= \; & \frac{(l-n+1)!}{(n+1)!(l-2n)!}(m+2l-1-4n) + \frac{(l-2n)!}{(n+1)!(l-2n-1)!}(2k+2n+1-m) \\
& + \frac{(l-n)!}{n!(l-2n)!}(1-m) \\
= \; & \frac{(l-n)!}{n!(l-2n-1)!}\left\{\frac{(l-n+1)(m+2l-1-4n)}{(n+1)(l-2n)} + \frac{2k+2n+1-m}{n+1} + \frac{1-m}{l-2n}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(l-n)!}{n!(l-2n-1)!}\frac{1}{(n+1)(l-2n)}(lm+2l^2-l-4ln-nm-2ln+n+4n^2+m \\
&\qquad \times +2l-1-4n+2kl+2ln+l-lm-4kn-4n^2-2n+2nm+n-mn+1-m) \\
&= \begin{pmatrix} l-n \\ n+1 \end{pmatrix}\frac{1}{l-2n}\left\{lm(1-1)+l^2(2)+l(-1+2+1)+ln(-4-2+2)+nm(-1+2-1)\right. \\
&\qquad \left. +n(1-4-2+1)+n^2(4-4)+m(1-1)+2kl-4kn+(-1+1)\right\} \\
&= \begin{pmatrix} l-n \\ n+1 \end{pmatrix}\frac{1}{l-2n}\left\{2kl-4kn+2l^2-4ln+2l-4n\right\} \\
&= \begin{pmatrix} l-n \\ n+1 \end{pmatrix}\frac{2(k+l+1)(l-2n)}{l-2n} \\
&= \begin{pmatrix} l-n-1+1 \\ n+1 \end{pmatrix}\begin{pmatrix} n+1-1 \\ n \end{pmatrix}2(k+l+1).
\end{aligned}
$$

Also hat $\Upsilon_{3,1}$ die richtige Form.

Seien

$$
S_5' := (m-1)\sum_{\nu=1}^{\mu-1}\begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu-1 \\ n \end{pmatrix} \qquad \left(\mu \in \left\{2,\dots\left[\frac{1}{2}(l-2n+2)\right]\right\}\right)
$$

und

$$
\begin{aligned}
S_6' &:= (1-m)\sum_{\nu=1}^{\mu}\begin{pmatrix} n+1-\nu+\mu \\ n+1 \end{pmatrix}\begin{pmatrix} l-n+\nu-\mu \\ n \end{pmatrix} \\
&= (1-m)\sum_{\nu=1+l-2n-\mu+1}^{\mu+l-2n-\mu+1}\begin{pmatrix} n+1+\mu-\nu+l-2n-\mu+1 \\ n+1 \end{pmatrix}\begin{pmatrix} l-n-\mu+\nu-l+2n+\mu-1 \\ n \end{pmatrix} \\
&= -(m-1)\sum_{\nu=l-2n-\mu+2}^{l-2n+1}\begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu-1 \\ n \end{pmatrix} \\
&\left(\nu \in \left\{l-2n+1-\left[\frac{1}{2}(l-2n+2)\right]=\left[\frac{1}{2}(l+1)\right]-n=\left[\frac{1}{2}(l-2n+1)\right],\dots,l-2n+1\right\}\right).
\end{aligned} \tag{5.2}
$$

Dann gilt mit Satz 5.1.3

$$
\begin{aligned}
S_5'+S_6' &= (m-1)\left\{\sum_{\nu=1}^{\mu-1}\begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu-1 \\ n \end{pmatrix}\right. \\
&\qquad \left. -\sum_{\nu=l-2n-\mu+2}^{l-2n+1}\begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu-1 \\ n \end{pmatrix}\right\} \\
&= (m-1)\left\{\sum_{\nu=0}^{\mu-2}\begin{pmatrix} l-n-\nu+1 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu \\ n \end{pmatrix}\right. \\
&\qquad \left. -\sum_{\nu=l-2n-\mu+1}^{l-2n}\begin{pmatrix} l-n-\nu+1 \\ n+1 \end{pmatrix}\begin{pmatrix} n+\nu \\ n \end{pmatrix}\right\}
\end{aligned}
$$

$$= \quad (m-1)\left\{\sum_{\nu=0}^{l-2n}\binom{l-n-\nu+1}{n+1}\binom{n+\nu}{n}-\sum_{\nu=\mu-1}^{l-2n}\binom{l-n-\nu+1}{n+1}\binom{n+\nu}{n}\right.$$
$$\left.-\sum_{\nu=l-2n-\mu+1}^{l-2n}\binom{l-n-\nu+1}{n+1}\binom{n+\nu}{n}\right\}.$$

Damit sind die Koeffizienten von $\sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]}\overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)}$ *in* $\Upsilon_{3,2}$

$$S_5'+S_6'+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}(m+2l+1-2\mu-4n)$$
$$+\binom{l+1-n-\mu}{n+1}\binom{n+\mu-1}{n}(2k+2n+1-m)$$
$$+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}(2n+3-m)$$
$$= \quad S_5'+S_6'+\frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-2)!}\,\left\{\frac{(m+2l+1-2\mu-4n)(l-n-\mu+2)}{(l-2n-\mu+1)(\mu-1)}\right.$$
$$\left.+\frac{2k+2n+1-m}{\mu-1}+\frac{2n+3-m}{n+1}\right\}$$
$$= \quad S_5'+S_6'+\frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-2)!}\,\frac{1}{(l-2n-\mu+1)(\mu-1)(n+1)}$$
$$\times\Big\{(n+1)(lm-mn-\mu m+2m+2l^2-2ln-2l\mu+4l+l-n-\mu+2-2l\mu+2\mu n+2\mu^2$$
$$-4\mu-4ln+4n^2+4\mu n-8n)$$
$$+(l-2n-\mu+1)(2kn+2k+2n^2+2n+n+1-mn-m)$$
$$+(l-2n-\mu+1)(2n\mu-2n+3\mu-3-\mu m+m)\Big\}$$
$$= \quad S_5'+S_6'+\frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-2)!}\,\frac{1}{(l-2n-\mu+1)(\mu-1)(n+1)}$$
$$\times\Big\{(n+1)(lm-mn-\mu m+2m+2l^2-6ln-4l\mu+5l-9n-5\mu+2+6\mu n+2\mu^2+4n^2)$$
$$+(l-2n-\mu+1)(2kn+2k+2n^2+n-mn+2n\mu+3\mu-2-\mu m)\Big\}$$
$$= \quad S_5'+S_6'+\frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-2)!}\,\frac{1}{(l-2n-\mu+1)(\mu-1)(n+1)}$$
$$\times\Big\{lmn-mn^2-\mu mn+2mn+2l^2n-6ln^2-4l\mu n+5ln-9n^2-5\mu n+2n+6\mu n^2+2\mu^2n$$
$$+4n^3+lm-mn-\mu m+2m+2l^2-6ln-4l\mu+5l-9n-5\mu+2+6\mu n+2\mu^2+4n^2$$
$$+2knl+2kl+2n^2l+ln-mnl+2n\mu l+3\mu l-2l-\mu ml-4kn^2-4kn-4n^3-2n^2$$
$$+2mn^2-4\mu n^2-6\mu n+4n+2\mu mn-2kn\mu-2k\mu-2\mu n^2-\mu n+\mu mn-2n\mu^2-3\mu^2$$
$$+2\mu+\mu^2m+2kn+2k+2n^2+n-mn+2n\mu+3\mu-2-\mu m\Big\}$$

$$
\begin{aligned}
= \quad & S_5' + S_6' + \frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-2)!}\,\frac{1}{(l-2n-\mu+1)(\mu-1)(n+1)} \\
& \times\Big\{ lmn(1-1) + mn^2(-1+2) + \mu mn(-1+2+1) + mn(2-1-1) + l^2n(2) + ln^2(-6+2) \\
& \quad + l\mu n(-4+2) + ln(5-6+1) + n^2(-9+4-2+2) + \mu n(-5+6-6-1+2) \\
& \quad + n(2-9+4+1) + \mu n^2(6-4-2) + \mu^2 n(2-2) + n^3(4-4) + lm(1) + \mu m(-1-1) \\
& \quad + m(2) + l\mu(-4+3) + l(5-2) + l^2(2) + \mu(-5+2+3) + (2-2) + 2knl + 2kl + \mu ml(-1) \\
& \quad + kn^2(-4) + kn(-4+2) + kn\mu(-2) + k\mu(-2) + \mu^2(2-3) + \mu^2 m(1) + 2k\Big\} \\
= \quad & S_5' + S_6' + \frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-2)!}\,\frac{1}{(l-2n-\mu+1)(\mu-1)(n+1)} \\
& \times\Big\{ mn^2 + 2\mu mn + 2l^2n - 4ln^2 - 2l\mu n - 5n^2 - 4\mu n - 2n + lm - 2\mu m + 2m - l\mu + 3l \\
& \quad + 2l^2 + 2knl + 2kl - \mu ml - 4kn^2 - 2kn - 2kn\mu - 2k\mu - \mu^2 + \mu^2 m + 2k\Big\}
\end{aligned}
$$

Für $\Upsilon_{3,2}$ sind noch die letzten Summanden von $(m-1)\,S_5$ und $(1-m)\,S_6$ zu beachten:
Es gilt

$$
\begin{aligned}
\Upsilon_{3,2} \quad = \quad & \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l+1-2\mu-2n}|x|^{2(\mu-1)} \Bigg\langle \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n} \frac{1}{(l-2n-\mu+1)(n+1)} \\
& \times\Big\{ n^2m + 2nm\mu + 2l^2n - 4l^2n - 2ln\mu - 5n^2 - 4n\mu - 2n + lm - 2m\mu \\
& \quad + 2m - l\mu + 3l + 2l^2 + 2kln + 2kl - lm\mu - 4kn^2 - 2kn - 2kn\mu - 2k\mu\Big\} \\
& + (m-1)\sum_{\nu=1}^{\mu-1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
& + (1-m)\sum_{\nu=1}^{\mu} \binom{n+1-\nu+\mu}{n+1}\binom{l-n+\nu-\mu}{n}\Bigg\rangle \\
& + (m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}\{l+2(1-n)\}\right]-1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
& - (l \bmod 2)(m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}(l-2n+1)\right]} \binom{n+1-\nu+\mu}{n+1}\binom{l-n-\nu+\mu}{n} \\
& - (m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}\{l+2(1-n)\}\right]-1} \binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
& + (l \bmod 2)(m-1)\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]} \sum_{\nu=1}^{\left[\frac{1}{2}(l-2n+1)\right]} \binom{n+1-\nu+\mu}{n+1}\binom{l-n-\nu+\mu}{n}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Bigg\langle \begin{pmatrix} l-n-\mu+1 \\ n+1 \end{pmatrix} \begin{pmatrix} n+\mu-1 \\ n \end{pmatrix} \frac{1}{(l-2n-\mu+1)(n+1)} \\
& \times \Big\{ n^2m + 2nm\mu + 2l^2n - 4l^2n - 2ln\mu - 5n^2 - 4n\mu - 2n + lm - 2m\mu \\
& +2m - l\mu + 3l + 2l^2 + 2kln + 2kl - lm\mu - 4kn^2 - 2kn - 2kn\mu - 2k\mu \Big\} \\
& +(m-1) \sum_{\nu=1}^{\mu-1} \begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix} \begin{pmatrix} n+\nu-1 \\ n \end{pmatrix} \\
& -(m-1) \sum_{\nu=l-2n-\mu+2}^{l-2n+1} \begin{pmatrix} l-n-\nu+2 \\ n+1 \end{pmatrix} \begin{pmatrix} n+\nu-1 \\ n \end{pmatrix} \Bigg\rangle \\
= & \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Bigg(\frac{(l-n-\mu+1)!}{n!(l-2n-\mu)} \frac{(m+\mu-1)!}{n!(\mu-2)!} \Bigg\langle \frac{1}{(l-2n-\mu+1)(n+1)^2(\mu-1)} \\
& \times \Big\{ n^2m + 2nm\mu + 2l^2n - 4l^2n - 2ln\mu - 5n^2 - 4n\mu - 2n + lm - 2m\mu \\
& +2m - l\mu + 3l + 2l^2 + 2kln + 2kl - lm\mu - 4kn^2 - 2kn - 2kn\mu - 2k\mu \Big\} \\
& \underbrace{+\frac{m-1}{(n+1)^2} - \frac{m-1}{(l-2n-\mu+1)(\mu-1)}}_{=\frac{m\mu-\mu-m+1}{(\mu-1)(n+1^2)} - \frac{nm-n+m-1}{(n+1)(l-2n-\mu+1)(\mu-1)}} \Bigg\rangle \Bigg) \\
= & \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Bigg(\frac{(l-n-\mu+1)!}{n!(l-2n-\mu)} \frac{(m+\mu-1)!}{n!(\mu-2)!} \Bigg\langle \frac{1}{(l-2n-\mu+1)(n+1)^2(\mu-1)} \\
& \times \Big\{ n^2m + 2nm\mu + 2l^2n - 4l^2n - 2ln\mu - 5n^2 - 4n\mu - 2n + lm - 2m\mu + 2m \\
& -l\mu + 3l + 2l^2 + 2kln + 2kl - lm\mu - 4kn^2 - 2kn - 2kn\mu - 2k\mu + lm\mu - l\mu \\
& -lm + l - 2nm\mu + 2n\mu + 2nm - 2n - m\mu^2 + \mu^2 + m\mu - \mu + m\mu - \mu - m \\
& +1 - n^2m + n^2 - nm + n - nm + n - m + 1 \Big\} \Bigg\rangle \Bigg) \\
= & \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Bigg(\frac{(l-n-\mu+1)!}{n!(l-2n-\mu)} \frac{(m+\mu-1)!}{n!(\mu-2)!} \Bigg\langle \frac{1}{(l-2n-\mu+1)(n+1)^2(\mu-1)} \\
& \times \Big\{ n^2m(1-1) + nm\mu(2-2) + l^2n(2) + ln^2(-4) + ln\mu(-2) + n^2(-5+1) \\
& +n\mu(-4+2) + n(-2-2+2) + lm(1-1) + m\mu(-2+1+1) + m(2-1-1) \\
& +l\mu(-1-1) + l(3+1) + l^2(2) + kln(2) + kl(2) + lm\mu(-1+1) + kn^2(-4) \\
& +kn(-2) + kn\mu(-2) + k\mu(-2) + \mu^2(-1+1) + m\mu^2(1-1) + k(2) + (1+1) \\
& +nm(2-1-1) + \mu(1-1) \Big\} \Bigg\rangle \Bigg)
\end{aligned}
$$

$$= \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Big\langle \binom{l-n-\mu+1}{n+1} \binom{n+\mu-1}{n} \frac{1}{(l-2n-\mu+1)(n+1)}$$
$$\times (2l^2n - 4ln^2 - 2ln\mu - 4n^2 - 2n\mu - 2n - 2l\mu + 4l + 2l^2 + 2kln + 2kl - 4kn^2$$
$$-2kn - 2kn\mu - 2k\mu + 2k + 2 - 2\mu) \Big\rangle$$

$$= \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Big\langle \binom{l-n-\mu+1}{n+1} \binom{n+\mu-1}{n} \frac{1}{(l-2n-\mu+1)(n+1)}$$
$$\times 2(kln + l^2n + ln + kl + l^2 + l - 2kn^2 - 2ln^2 - 2n^2 - 2kn - 2ln - 2n$$
$$-kn\mu - ln\mu - n\mu - k\mu - l\mu - \mu + kn + ln + n + k + l + 1) \Big\rangle$$

$$= \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Big\langle \binom{l-n-\mu+1}{n+1} \binom{n+\mu-1}{n} \frac{1}{(l-2n-\mu+1)(n+1)}$$
$$\times 2(l-2n-\mu+1)(kn + ln + n + k + l + 1) \Big\rangle$$

$$= \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Big\langle \binom{l-n-\mu+1}{n+1} \binom{n+\mu-1}{n} \frac{1}{(l-2n-\mu+1)(n+1)}$$
$$\times (l-2n-\mu+1)(n+1)\, 2(k+l+1) \Big\rangle$$

$$= \sum_{\mu=2}^{\left[\frac{1}{2}(l-2n)\right]} \overline{x}^{l-\mu-2n} x^{\mu-1} \Big\langle \binom{l-n-\mu+1}{n+1} \binom{n+\mu-1}{n} 2(k+l+1) \Big\rangle.$$

Damit hat $\Upsilon_{3,2}$ *die gewünschte Form.*
Nun wird $\Upsilon_{3,3}$ *betrachtet:*

1. Sei l *ungerade.*
Es gilt $\mu = \left[\frac{1}{2}(l-2n+1)\right] = \left[\frac{1}{2}(l-2n+2)\right]$.
Mit (5.2), Satz 5.1.3,

$$\sum_{\nu=1}^{\mu-1} \binom{l-n-\nu+2}{n+1} \binom{n+\nu-1}{n} - \sum_{\nu=l-2n-\mu+2}^{l-2n+1} \binom{l-n-\nu+2}{n+1} \binom{n+\nu-1}{n}$$
$$= \binom{l-n-\mu+1}{n+1} \binom{n+\mu-1}{n+1} - \binom{l-n-\mu+1}{n} \binom{n+\mu-1}{n}$$

$$
\begin{aligned}
&= \binom{l-n-\left[\frac{l}{2}\right]+n-1+1}{n+1}\binom{n-1+\left[\frac{l}{2}\right]-n+1}{n+1} \\
&\quad -\binom{l-n+1-\left[\frac{l}{2}\right]+n-1}{n}\binom{n-1+\left[\frac{l}{2}\right]-n+1}{n} \\
&= \binom{\left[\frac{1}{2}(l+1)\right]}{n+1}\binom{\left[\frac{l}{2}\right]}{n+1}-\binom{\left[\frac{1}{2}(l+1)\right]}{n}\binom{\left[\frac{l}{2}\right]}{n},
\end{aligned}
$$

$$
\overline{x}^{l+1-2n-2\left[\frac{1}{2}(l-2n+2)\right]}|x|^{2\left\{\left[\frac{1}{2}(l-2n+2)\right]-1\right\}}=\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]},
$$

und $\overline{x}^{(l \bmod 2)-1}|x|^{2\left[\frac{1}{2}(l-2n)\right]}=|x|^{l-2n-1}=\overline{x}^{\frac{l}{2}-n-\frac{1}{2}}x^{\frac{l}{2}-n-\frac{1}{2}}$ *gilt*

$$
\begin{aligned}
\frac{\Upsilon_{3,3}}{|x|^{l-2n-1}} &= (m-1)\binom{\frac{l+1}{2}}{n+1}\binom{\frac{l-1}{2}}{n+1}-(m-1)\binom{\frac{l+1}{2}}{n}\binom{\frac{l-1}{2}}{n} \\
&\quad +\binom{\frac{l+3}{2}}{n+1}\binom{\frac{l-1}{2}}{n}\{m-1+2(l \bmod 2)\}+\binom{\frac{l+3}{2}}{n+1}\binom{\frac{l-1}{2}}{n}(l-2n-1) \\
&\quad +(2k+2n+1-m)\binom{\frac{l+1}{2}}{n+1}\binom{\frac{l-1}{2}}{n}+\binom{\frac{l+1}{2}}{n+1}\binom{\frac{l-1}{2}}{n+1}(2n+3-m) \\
&= \frac{(\frac{l+1}{2})!}{n!(\frac{l+1}{2}-n-1)!}\,\frac{(\frac{l-1}{2})!}{n!(\frac{l-1}{2}-n-1)!}\left\{\frac{m-1}{(n+1)^2}-\frac{m-1}{(\frac{l+1}{2}-n)(\frac{l-1}{2}-n)}\right. \\
&\qquad +\frac{(m+1)\frac{l+3}{2}}{(n+1)(\frac{l+1}{2}-n)(\frac{l-1}{2}-n)}+\frac{(l-2n-1)\frac{l+3}{2}}{(n+1)(\frac{l+1}{2}-n)(\frac{l-1}{2}-n)} \\
&\qquad \left.+\frac{2k+2n+1-m}{(n+1)(\frac{l-1}{2}-n)}+\frac{2n+3-m}{(n+1)^2}\right\} \\
&= \frac{(\frac{l+1}{2})!}{n!(\frac{l+1}{2}-n-1)!}\,\frac{(\frac{l-1}{2})!}{n!(\frac{l-1}{2}-n-1)!}\left\{\frac{ln+l+n+1-2n^2-2n}{(n+1)^2(\frac{l+1}{2}-n)}\right. \\
&\qquad \left.+\frac{-2n+\frac{3}{2}+\frac{1}{2}l^2+2l+kl+k-2kn-2n^2}{(n+1)(\frac{l+1}{2}-n)(\frac{l-1}{2}-n)}\right\} \\
&= \frac{(\frac{l+1}{2})!}{n!(\frac{l+1}{2}-n-1)!}\,\frac{(\frac{l-1}{2})!}{n!(\frac{l-1}{2}-n-1)!} \\
&\quad \times\left\{\frac{\frac{1}{2}l^2n+\frac{1}{2}l^2+\frac{l}{2}-ln^2-\frac{1}{2}ln-\frac{1}{2}ln-\frac{l}{2}-\frac{1}{2}+n^2+\frac{1}{2}n-ln^2-ln-n+2n^3+n^2}{(n+1)^2(\frac{l+1}{2}-n)(\frac{l-1}{2}-n)}\right. \\
&\qquad -2n^2+\frac{3}{2}n+\frac{1}{2}k^2n+2ln+kln+kn-2kn^2-2n^3-2n+\frac{3}{2}+\frac{1}{2}l^2+2l \\
&\qquad \left.+kl+k-2kn-2n^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\frac{l+1}{2}!}{n!(\frac{l-1}{2}-n)!}\,\frac{(\frac{l-1}{2}!)}{n!(\frac{l-1}{2}-n-1)}\,\frac{1}{(n+1)^2(\frac{l-1}{2}-n)(\frac{l+1}{2}-n)}\\
&\quad\times\left\{kln-kn-2kn^2+kl+k+l^2n-2ln^2+l^2+2l-n-2n^2+1\right\}\\
&= \frac{\frac{l+1}{2}!}{n!(\frac{l-1}{2}-n)!}\,\frac{(\frac{l-1}{2}!)}{n!(\frac{l-1}{2}-n-1)}\,\frac{1}{(n+1)^2(\frac{l-1}{2}-n)(\frac{l+1}{2}-n)}\\
&\quad\times\left\{kln-kn-2kn^2+kl+k+l^2n-2ln^2+l^2+2l-n-2n^2+1\right\}\\
&= \frac{\frac{l+1}{2}!}{n!(\frac{l-1}{2}-n)!}\,\frac{(\frac{l-1}{2}!)}{n!(\frac{l-1}{2}-n-1)}\,\frac{1}{(n+1)^2(\frac{l-1}{2}-n)(\frac{l+1}{2}-n)}\\
&\quad\times\left\{kln+kn-2kn^2+kl+k-2kn+l^2n+ln-2ln^2+l^2+l-2ln+ln+n\right.\\
&\qquad\qquad\left.-2n^2+l+1-2n\right\}\\
&= \frac{\frac{l+1}{2}!}{n!(\frac{l-1}{2}-n)!}\,\frac{(\frac{l-1}{2}!)}{n!(\frac{l-1}{2}-n-1)}\,\frac{1}{(n+1)^2(\frac{l-1}{2}-n)(\frac{l+1}{2}-n)}\\
&\quad\times\left\{(2kn+2k+2ln+2l+2n+2)\left(\frac{l+1}{2}-n\right)\right\}\\
&= \frac{\frac{l+1}{2}!}{n!(\frac{l-1}{2}-n)!}\,\frac{(\frac{l-1}{2}!)}{n!(\frac{l-1}{2}-n-1)}\,\frac{2(k+l+1)(n+1)(\frac{l+1}{2}-n)}{(n+1)(\frac{l-1}{2}-n)(n+1)(\frac{l+1}{2}-n)}\\
&= \binom{\frac{l+1}{2}}{n+1}\binom{\frac{l-1}{2}}{n}2(k+l+1)\\
&= \binom{l-1+1-\frac{l+1}{2}+n}{n+1}\binom{n+\frac{l+1}{2}-n-1}{n}2(k+l+1)\\
&\overset{\mu=\frac{l+1}{2}-n}{=} \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}2(k+l+1).
\end{aligned}
$$

In diesem Fall hat $\Upsilon_{3,3}$ also die gewwünschte Form.

2. Sei l gerade. Also gilt mit $\mu=\left[\frac{1}{2}(l-2n+2)\right]=\frac{l}{2}-n+1$ und Satz 5.1.3

$$
\begin{aligned}
\frac{\Upsilon_{3,3}}{x|x|^{l-2n-2}} &= (m-1)\sum_{\nu=1}^{\frac{l}{2}-n}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}\\
&\quad-(m-1)\sum_{\nu=\frac{l}{2}-n+2}^{l-2n+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}+\binom{\frac{l+2}{2}}{n+1}\binom{\frac{l}{2}}{n}(l-2n)\\
&\quad+(2k+2n+1-m)\binom{\frac{l}{2}}{n+1}\binom{\frac{l}{2}}{n}+\binom{\frac{l}{2}}{n+1}\binom{\frac{l}{2}}{n+1}(2n+3-m)\\
&= (m-1)\sum_{\nu=1}^{\frac{l}{2}-n}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}\\
&\quad-(m-1)\sum_{\nu=\frac{l}{2}-n+1}^{l-2n+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}
\end{aligned}
$$

$$+(m-1)\binom{l-n+2-\frac{l}{2}+n-1}{n+1}\binom{n-1+\frac{l}{2}-n+1}{n}$$

$$+\frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\left\{\frac{(l-2n)\frac{l+2}{2}}{(\frac{l}{2}-n)^2}+\frac{2k+2n+1-m}{\frac{l}{2}-n}+\frac{2n+3-m}{n+1}\right\}$$

$$= (m-1)\sum_{\nu=1}^{\frac{l}{2}-n+1-1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}$$

$$-(m-1)\sum_{\nu=l-2n+2-\frac{l}{2}+n-1}^{l-2n+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}+(m-1)\binom{\frac{l}{2}+1}{n+1}\binom{\frac{l}{2}}{n}$$

$$+\frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\left\{\frac{\frac{1}{2}l^2+l-ln-2n}{(\frac{l}{2}-n)^2}\right.$$

$$\left.+\frac{kl+ln+\frac{l}{2}-\frac{1}{2}lm-2kn-2n^2-n+nm}{(\frac{l}{2}-n)^2}+\frac{ln+\frac{3}{2}l-\frac{1}{2}lm-2n^2-3n+nm}{(n+1)(\frac{l}{2}-n)}\right\}$$

$$= (m-1)\left\{\binom{\frac{l}{2}}{n+1}^2-\binom{\frac{l}{2}}{n}^2+\binom{\frac{l}{2}+1}{n+1}\binom{\frac{l}{2}}{n}\right\}$$

$$+\frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\left\{\frac{\frac{1}{2}l^2+\frac{3}{2}l-3n+kl-\frac{1}{2}lm-2kn-2n^2+nm}{(\frac{l}{2}-n)^2}\right.$$

$$\left.+\frac{\frac{1}{2}l^2n+\frac{3}{4}l^2-\frac{1}{4}l^2m-ln^2-\frac{3}{2}ln+\frac{1}{2}lnm-ln^2-\frac{3}{2}ln+\frac{1}{2}lnm+2n^3+3n^2-n^2m}{(n+1)(\frac{l}{2}-n)^2}\right\}$$

$$= \left(\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\right)^2\left\{\frac{m-1}{(n+1)^2}-\frac{m-1}{(\frac{l}{2}-n)^2}+\frac{(m-1)\frac{l+2}{2}}{(n+1)(\frac{l}{2}-n)^2}\right.$$

$$+\frac{\frac{1}{2}l^2+\frac{3}{2}l-3n+kl-\frac{1}{2}lm-2kn-2n^2+nm}{(n+1)(\frac{l}{2}-n)^2}$$

$$\left.+\frac{\frac{1}{2}l^2n+\frac{3}{4}l^2-\frac{1}{4}l^2m-ln^2-\frac{3}{2}ln+\frac{1}{2}lnm-ln^2-\frac{3}{2}ln+\frac{1}{2}lnm+2n^3+3n^2-n^2m}{(n+1)^2(\frac{l}{2}-n)^2}\right\}$$

$$= \left(\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\right)^2\left\{\frac{\frac{1}{2}lm-\frac{l}{2}-nm+n}{(n+1)^2(\frac{l}{2}-n)}+\frac{-nm+n-m+1+\frac{1}{2}lm+m-\frac{l}{2}-1}{(n+1)(\frac{l}{2}-n)^2}\right.$$

$$+\frac{\frac{1}{2}l^2n+\frac{3}{2}ln-3n^2+kln-\frac{1}{2}lnm-2kn^2-2n^3+n^2m+\frac{1}{2}l^2+\frac{3}{2}l-3n+kl}{(n+1)^2(\frac{l}{2}-n)^2}$$

$$+\frac{-\frac{1}{2}lm-2kn-2n^3+nm}{(n+1)^2(\frac{l}{2}-n)^2}$$

$$\left.+\frac{\frac{1}{2}l^2n+\frac{3}{4}l^2-\frac{1}{4}l^2m-ln^2-\frac{3}{2}ln+\frac{1}{2}lnm-ln^2-\frac{3}{2}ln+\frac{1}{2}lnm+2n^3+3n^2-n^2m}{(n+1)^2(\frac{l}{2}-n)^2}\right\}$$

$$= \left(\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\right)^2\frac{1}{(n+1)^2(\frac{l}{2}-n)^2}\left\{\frac{1}{4}l^2m-\frac{1}{4}l^2-\frac{1}{2}lnm+\frac{1}{2}ln-\frac{1}{2}lnm+\frac{1}{2}ln+n^2m-n^2\right.$$

$$-n^2m-n^2-nm+\frac{1}{2}lnm+nm-\frac{1}{2}ln-nm+n-m+\frac{1}{2}lm+m-\frac{l}{2}+l^2n-\frac{3}{2}ln$$

$$
\begin{aligned}
& -2n^2+kln+\frac{1}{2}lnm-2kn^2+\frac{5}{4}l^2+\frac{3}{2}l-3n+kl-\frac{1}{2}lm-2kn+nm-\frac{1}{4}l^2m-2ln^2\Big\} \\
= & \left(\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\right)^2\left\{l^2-ln-2n+l-2n^2+l^2n+kln-2kn^2+kl-2kn-2ln^2\right\} \\
= & \frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!} \\
& \times\frac{kln-2kn^2+kl-2kn+l^2n-2ln^2+l^2-ln-2n^2+l-2n}{(n+1)(\frac{l}{2}-n)^2} \\
= & \frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!} \\
& \times\frac{kln-2kn^2+kl-2kn+l^2n-2ln^2+l^2-2ln+ln-2n^2+l-2n}{(n+1)(\frac{l}{2}-n)^2} \\
= & \frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\,\frac{(2kn+2k+2ln+2l+2n+2)(\frac{l}{2}-n)}{(n+1)(\frac{l}{2}-n)^2} \\
= & \frac{(\frac{l}{2})!}{(n+1)!(\frac{l}{2}-n-1)!}\,\frac{(\frac{l}{2})!}{n!(\frac{l}{2}-n-1)!}\,\frac{2(k+l+1)}{\frac{l}{2}-n}\,\frac{(n+1)(\frac{l}{2}-n)}{(n+1)(\frac{l}{2}-n)} \\
= & \binom{\frac{l}{2}}{n+1}\binom{\frac{l}{2}}{n}2(k+l+1) \\
= & \binom{l-n-\frac{l}{2}+n-1+1}{n+1}\binom{n-1+\frac{l}{2}-n+1}{n}2(k+l+1).
\end{aligned}
$$

Also hat auch in diesem Fall $\Upsilon_{3,3}$ die gewünschte Form.

Für $\Upsilon_{3,4}$ gilt

$$
\begin{aligned}
\Upsilon_{3,4} = {} & (m-1)\sum_{\mu=[\frac{l}{2}]-n+2}^{l-2n}x^{2(\mu+n)-l-1}|x|^{2(l-\mu-2n)}\sum_{\nu=1}^{l-2n-\mu+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
& -(m-1)\sum_{\mu=[\frac{l}{2}]-n+2\underbrace{-\{1-(l \bmod 2)\}}_{\text{in }\Upsilon_{3,3}}}^{l-2n}x^{2(\mu+n)-l-1}|x|^{2(l-2n-\mu)} \\
& \qquad\times\sum_{\nu=\mu+1}^{l-2n+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n} \\
& +\sum_{\mu=[\frac{l}{2}]-n+2}^{l-2n}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\{2k+2n+1-m\}x^{2(\mu+n)-l-1}|x|^{2(l-2n-\mu)} \\
& +\sum_{\mu=[\frac{l}{2}]-n+2}^{l-2n}\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}\,2(l-2n-\mu+1)x^{2(n+\mu)-l-1}|x|^{2(l-2n-\mu)}
\end{aligned}
$$

$$+\sum_{\mu=\left[\frac{l}{2}\right]-n+2}^{l-2n}\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}(2n+3-m)x^{2(\mu+n)-l-1}|x|^{2(l-2n-\mu)}.$$

Also gilt

$$\begin{aligned}
&\frac{\Upsilon_{3,3}}{\sum\limits_{\mu=\left[\frac{l}{2}\right]-n+2}^{l-2n} x^{2(n+\mu)-l-1}|x|^{2(l-\mu-2n)}}\\
=\;& (m-1)\left\{\sum_{\nu=1}^{l-2n-\mu+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}\right.\\
&\left.-\sum_{\nu=\mu+1}^{l-2n+1}\binom{l-n-\nu+2}{n+1}\binom{n+\nu-1}{n}\right\}\\
&+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}(2k+2n+1-m)\\
&+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}2(l-2n-\mu+1)\\
&+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}(2n+3-m)\\
\overset{\text{Satz 5.1.4}}{=}\;& (m-1)\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\frac{n+\mu}{n+1}\\
&+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}(2k+2n+1-m)\\
&+\binom{l-n-\mu+2}{n+1}\binom{n+\mu-1}{n}2(l-2n-\mu+1)\\
&+\binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n+1}(2n+3-m)\\
=\;& \frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-1)!}\left\{\frac{(m-1)(n+\mu)}{n+1}+2k+2n+1-m\right.\\
&\left.+\frac{(l-n-\mu+2)\,2(l-2n-\mu+1)}{l-2n-\mu+1}+\frac{(2n+3-m)(\mu-1)}{n+1}\right\}\\
=\;& \frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-1)!}\,\frac{1}{n+1}\Big(nm+m\mu-n-\mu+2kn+2n^2+n-nm\\
&+2k+2n+1-m+2n\mu-2n+3\mu+m+2ln-2n^2-2n\mu+4n+2l-2n-2\mu+4\Big)\\
=\;& \frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-1)!}\,\frac{1}{n+1}(2kn+2n+2k+2+2ln+2l)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(l-n-\mu+1)!}{(n+1)!(l-2n-\mu)!}\,\frac{(n+\mu-1)!}{n!(\mu-1)!}\,(2k+2l+2)\,\frac{n+1}{n+1} \\
&= \binom{l-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\,2(k+l+1).
\end{aligned}
$$

Damit hat $\Upsilon_{3,4}$ die gewünschte Form.

4. Nun wird Υ_4 betrachtet:

1. Fall: l ist gerade, d.h. es gilt $\left[\frac{l}{2}\right] = \left[\frac{1}{2}(l+1)\right] = \frac{l}{2}$.
In diesem Fall gilt

$$
\begin{aligned}
\Upsilon_4 &= \frac{|x|^{2(k+\left[\frac{l}{2}\right]+1)-m-1}}{\prod\limits_{j=\left[\frac{l}{2}\right]+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\left[\frac{l}{2}\right]+1}\{2(k+j)-m-1\}} \\
&\quad\times\left\{\overline{x}^{(l \bmod 2)-1}\left\langle 1\,\{m-1+2(l \bmod 2)\}\,\operatorname{sgn}(l \bmod 2) + 0\,\left\{2\left(k+\left[\frac{l}{2}\right]+1\right)-m-1\right\}\right\rangle\right\} \\
&= \frac{|x|^{2(k+\left[\frac{l}{2}\right]+1)-m-1}}{\prod\limits_{j=\left[\frac{l}{2}\right]+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\left[\frac{l}{2}\right]+1}\{2(k+j)-m-1\}}\,0.
\end{aligned}
$$

In diesem Fall hat Υ_4 also die gewünschte Form.
2. Fall: l ist ungerade, d.h. es gilt $\left[\frac{l}{2}\right] = \frac{l-1}{2}$ und $\left[\frac{1}{2}(l+1)\right] = \frac{l+1}{2}$.
Also gilt

$$
\begin{aligned}
\Upsilon_4 &= \frac{|x|^{2(k+\frac{l-1}{2}+1)-m-1}}{\prod\limits_{j=\frac{l-1}{2}+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\frac{l-1}{2}+1}\{2(k+j)-m-1\}} \\
&\quad\times\Bigg\langle\,\overline{x}^{(l \bmod 2)-1}\left\{\frac{l+3}{2}\{m-1+2(l \bmod 2)\}\,\operatorname{sgn}(l \bmod 2) + 1\,\left(2k+2\,\frac{l-1}{2}+2-m-1\right)\right\} \\
&\qquad + x^{3-(l \bmod 2)}|x|^{2\{(l \bmod 2)-2\}}\left\langle 0\,\left\{2\left(k+\left[\frac{l}{2}\right]+1\right)-m-1\right\}\right\rangle \\
&\qquad + \binom{\frac{l+1}{2}}{\frac{l-1}{2}}\,2\{(l \bmod 2)-1\}\,\operatorname{sgn}\{2(l \bmod 2)-2\} \\
&\qquad + \binom{\frac{l+1}{2}}{\frac{l-1}{2}}\,\operatorname{sgn}\{2-(l \bmod 2)\}\,(1-m)\,\overline{x}^{\,2-1-(l \bmod 2)}|x|^{2\{(l \bmod 2)-1\}} + 0\Bigg\rangle
\end{aligned}
$$

$$= \frac{|x|^{2(k+\frac{l-1}{2}+1)-m-1}}{\prod\limits_{j=\frac{l-1}{2}+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\frac{l-1}{2}+1}\{2(k+j)-m-1\}}\left\{\frac{l+3}{2}(m+1)+2k+l-m+\frac{l+1}{2}(1-m)\right\}$$

$$= \frac{|x|^{2(k+\frac{l-1}{2}+1)-m-1}}{\prod\limits_{j=\frac{l-1}{2}+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\frac{l-1}{2}+1}\{2(k+j)-m-1\}}$$

$$\times\left(\frac{1}{2}lm+\frac{1}{2}l+\frac{3}{2}m+\frac{3}{2}+2k+l-m+\frac{l-}{2}\frac{1}{2}lm+\frac{1}{2}-\frac{1}{2}m\right)$$

$$= \frac{|x|^{2(k+\frac{l-1}{2}+1)-m-1}}{\prod\limits_{j=\frac{l-1}{2}+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\frac{l-1}{2}+1}\{2(k+j)-m-1\}}\,2(k+l+1)$$

$$= \frac{|x|^{2(k+\frac{l-1}{2}+1)-m-1}}{\prod\limits_{j=\frac{l-1}{2}+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\frac{l-1}{2}+1}\{2(k+j)-m-1\}}\,2(k+l+1)\begin{pmatrix}\frac{l+1}{2}\\ \frac{l+1}{2}\end{pmatrix}\begin{pmatrix}\frac{l-1}{2}\\ \frac{l-1}{2}\end{pmatrix}$$

$$= \frac{|x|^{2(k+\frac{l-1}{2}+1)-m-1}}{\prod\limits_{j=\frac{l-1}{2}+2}^{k}(2j-m-1)\prod\limits_{j=1}^{\frac{l-1}{2}+1}\{2(k+j)-m-1\}}$$

$$\times 2(k+l+1)\begin{pmatrix} l-\frac{l-1}{2}-1+1\\ \frac{l-1}{2}+1\end{pmatrix}\begin{pmatrix}\frac{l-1}{2}+1-1\\ \frac{l-1}{2}\end{pmatrix}\overline{x}^{l-1-2\,\frac{l-1}{2}}x^{1-1}|x|^{2\left(k+\frac{l-1}{2}+1\right)-m-1}.$$

Auch in diesem Fall hat Υ_4 die gewünschte Form.

Daher ist F_{l+1} die Stammfunktion von F_l bzgl. ∂, d.h. der Satz 5.2.1 ist bewiesen. ❑