Kapitel 4

Integraldarstellungen mit Hilfe des Laplaceoperators

Da der Laplaceoperator das Produkt der Diracoperatoren ∂ und $\overline{\partial}$ ist, kann man durch Iteration der beiden CAUCHY-POMPEIU Formeln (2.3) und (2.4) eine Integraldarstellung mit Hilfe des Laplaceoperators gewinnen.

4.1 Darstellungen 1. Ordnung

Es gelten die üblichen Voraussetzungen: G ist ein beschränktes Gebiet mit glattem Rand des $\mathbb{R}^{m+1}$. Weiterhin wird $m > 1$ vorausgesetzt; der Fall $m = 1$ ($\mathbb{C}$) ist bekannt, siehe z.B. [Beg97].

Satz 4.1.1 (einfache Integraldarstellung für Δ) *Ist $f \in \mathrm{C}^2(\overline{G},\mathcal{A})$, so gilt für $x \in G$:*

$$\begin{aligned} f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y [\partial f(y)] \\ &\quad + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} [\Delta f(y)] \mathrm{d}V_y. \end{aligned} \tag{4.1}$$

Beweis:

Wendet man (2.4) auf ∂f an, so erhält man

$$\partial f(x) = \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{y-x}{|y-x|^{m+1}} \mathrm{d}\overline{\vec{\sigma}}_y [\partial f(y)] - \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{y-x}{|y-x|^{m+1}} [\Delta f(y)] \mathrm{d}V_y.$$

Das setzt man wieder in (2.3) ein und benutzt den Satz von Fubini:

$$\begin{aligned}
f(x) &= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}}\int\limits_{G}\frac{\overline{y-x}}{|y-x|^{m+1}}[\partial f(y)]\mathrm{d}V_y \\
&= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y) \\
&\quad -\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\Bigg\{\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{z-y}{|z-y|^{m+1}}\mathrm{d}\overline{\vec{\sigma}}_z[\partial f(z)] \\
&\qquad\qquad -\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{z-y}{|z-y|^{m+1}}[\Delta f(z)]\mathrm{d}V_z\Bigg\}\mathrm{d}V_y \\
&= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y) \\
&\quad -\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{\overline{y-x}}{|y-x|^{m+1}}\frac{z-y}{|z-y|^{m+1}}\mathrm{d}V_y\mathrm{d}\overline{\vec{\sigma}}_z\,[\partial f(z)] \\
&\quad +\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{\overline{y-x}}{|y-x|^{m+1}}\frac{z-y}{|z-y|^{m+1}}\mathrm{d}V_y\,[\Delta f(z)]\mathrm{d}V_z \\
&= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\tilde{\Psi}_1(x,z)\mathrm{d}\overline{\vec{\sigma}}_z\,[\partial f(z)] \\
&\quad +\frac{1}{\omega_{m+1}}\int\limits_{G}\tilde{\Psi}_1(x,z)\,[\Delta f(z)]\,\mathrm{d}V_z
\end{aligned}$$

mit

$$\tilde{\Psi}_1(x,z) := \frac{1}{\omega_{m+1}}\int\limits_{G}\frac{\overline{y-x}}{|y-x|^{m+1}}\frac{z-y}{|z-y|^{m+1}}\mathrm{d}V_y.$$

Sei

$$\Psi_1(x,z) := \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{|y-z|^{1-m}}{1-m}.$$

Da

$$\left(\partial_y\frac{|y-z|^{1-m}}{1-m}\right) = \frac{1-m}{1-m}|y-z|^{-m-1}(y-z) = -\frac{z-y}{|z-y|^{m+1}}$$

gilt, folgt mit (2.3) für ε *hinreichend klein mit* $|x-z| > 2\varepsilon > 0$ *und*
$G_\varepsilon := G\setminus\{y\in G : |y-z|\le\varepsilon\}$

$$\begin{aligned}
\frac{|x-z|^{1-m}}{1-m} &= \frac{1}{\omega_{m+1}}\int\limits_{\partial G_\varepsilon}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{|y-z|^{1-m}}{1-m} + \frac{1}{\omega_{m+1}}\int\limits_{G_\varepsilon}\frac{\overline{y-x}}{|y-x|^{m+1}}\frac{z-y}{|z-y|^{m+1}}\mathrm{d}V_y \\
&= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{|y-z|^{1-m}}{1-m} - \frac{1}{\omega_{m+1}}\int\limits_{|y-z|=\varepsilon}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{|y-z|^{1-m}}{1-m} \\
&\quad +\frac{1}{\omega_{m+1}}\int\limits_{G_\varepsilon}\frac{\overline{y-x}}{|y-x|^{m+1}}\frac{z-y}{|z-y|^{m+1}}\mathrm{d}V_y.
\end{aligned}$$

Mit

$$\tilde{\Phi}_1(x,z) := \int\limits_{|y-z|=\varepsilon} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{|y-z|^{1-m}}{1-m} + \int\limits_{|y|=\varepsilon} \frac{\overline{y+z-x}}{|y+z-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{|y|^{1-m}}{1-m}$$

gilt

$$\begin{aligned} |\tilde{\Phi}_1(x,z)|_0 &\leq \frac{\mathrm{const}(x,z)}{\omega_{m+1}} \varepsilon^{1-m} \int\limits_{|y|=\varepsilon} \mathrm{d}\vec{\sigma}_y \\ &= \varepsilon \cdot \mathrm{const}(x,z) \\ &\overset{\varepsilon\to 0}{\to} 0. \end{aligned}$$

Da

$$\tilde{\Psi}_1(x,z) = \frac{1}{\omega_{m+1}} \int\limits_{G_\varepsilon} \frac{\overline{y-x}}{|y-x|^{m+1}} \frac{z-y}{|z-y|^{m+1}} \mathrm{d}V_y + \frac{1}{\omega_{m+1}} \int\limits_{|z-y|\leq\varepsilon} \frac{\overline{y-x}}{|y-x|^{m+1}} \frac{z-y}{|z-y|^{m+1}} \mathrm{d}V_y,$$

und

$$\int\limits_{|z-y|\leq\varepsilon} \left|\frac{\overline{y-x}}{|y-x|^{m+1}}\right|_0 \left|\frac{z-y}{|z-y|^{m+1}}\right|_0 \mathrm{d}V_y \leq \varepsilon \cdot \mathrm{const}(x,z) \overset{\varepsilon\to 0}{\to} 0,$$

gilt

$$\begin{aligned} \frac{|x-z|^{1-m}}{1-m} &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{|y-z|^{1-m}}{1-m} + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{\overline{y-x}}{|y-x|^{m+1}} \frac{z-y}{|z-y|^{m+1}} \mathrm{d}V_y \\ &= \Psi_1(x,z) + \tilde{\Psi}_1(x,z). \end{aligned}$$

Da

$$\begin{aligned} \Psi_1(x,z)\overline{\partial}_z &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \underbrace{\left(\frac{|y-z|^{1-m}}{1-m}\overline{\partial}_z\right)}_{-|y-z|^{-m-1}(\overline{y-z})} \\ &= -\frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{\overline{y-z}}{|y-z|^{m+1}} \\ &= \Phi_0(x,z) \\ &\overset{\text{Lemma 3.2.1}}{=} 0, \end{aligned}$$

$\Psi_1 \in \mathrm{C}^\infty$ *und* $f \in \mathrm{C}^2$ *gilt, folgt mit (2.2)*

$$\begin{aligned} \int\limits_{\partial G} \Psi_1(x,z)\mathrm{d}\overline{\vec{\sigma}}_z\,[\partial f(z)] &= \int\limits_{G} \Big\{[\Psi_1(x,z)\overline{\partial_z}][\partial f(z)] + \Psi_1(x,z)[\Delta f(z)]\Big\}\mathrm{d}V_z \\ &= \int\limits_{G} \Psi_1(x,z)[\Delta f(z)]\mathrm{d}V_z. \end{aligned}$$

Also

$$\begin{aligned} f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \left[\frac{|x-z|^{1-m}}{1-m} - \Psi_1(x,z) \right] \mathrm{d}\overline{\vec{\sigma}}_z [\partial f(z)] \\ &+ \frac{1}{\omega_{m+1}} \int\limits_{G} \left[\frac{|x-z|^{1-m}}{1-m} - \Psi_1(x,z) \right] [\Delta f(z)] \mathrm{d}V_z \\ &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|x-z|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_z [\partial f(z)] \\ &+ \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|x-z|^{1-m}}{1-m} [\Delta f(z)] \mathrm{d}V_z. \end{aligned}$$

Hiermit erhält man (4.1).

❏

4.2 Darstellungen höherer Ordnung

Lemma 4.2.1 *Für $k \in \mathbb{N}$, $k \geq 2$ und $x \neq 0$ gelten*

$$1.\ \Delta^{k-1}|x|^{2k-m-1} = 2^{k-1}\,(k-1)! \prod_{j=2}^{k} (2j-m-1)|x|^{1-m}$$

und

$$2.\ \Delta^{k}|x|^{2k-m-1} = 0.$$

Beweis (durch Induktion).

1. (a) *Induktionsanfang ($k = 2$):*

$$\begin{aligned} \Delta^{2-1}|x|^{2\cdot 2-m-1} &= \Delta|x|^{3-m} \\ &= \overline{\partial}\Big[(3-m)x|x|^{1-m}\Big] \\ &= \Big\{(3-m)\Big[(m+1)|x|^{1-m} + (1-m)x\overline{x}|x|^{-m-1}\Big]\Big\} \\ &= 2(3-m)|x|^{1-m} \\ &= 2^{2-1}\,(2-1)! \prod_{j=2}^{2} (2j-m-1)|x|^{1-m}. \end{aligned}$$

(b) Induktionsschritt:

$$\begin{aligned}
\Delta^{(k+1)-1}|x|^{2(k+1)-m-1} &= \Delta^k|x|^{2(k+1)-m-1} \\
&= \Delta^{k-1}\overline{\partial}\Big[\{2(k+1)-m-1\}x|x|^{2k-m-1}\Big] \\
&= \{2(k+1)-m-1\}\Delta^{k-1}\overline{\partial}\left(\frac{x}{|x|^{m+1}}|x|^{2k}\right) \\
&= \{2(k+1)-m-1\}\Delta^{k-1}\left(\frac{x}{|x|^{m+1}}2k\overline{x}|x|^{2k-2}\right) \\
&= \{2(k+1)-m-1\}2k\Delta^{k-1}|x|^{2k-m-1} \\
&\overset{\text{Ann.}}{=} \{2(k+1)-m-1\}2k\,2^{k-1}(k-1)!\prod_{j=2}^{k}(2j-m-1)|x|^{1-m} \\
&= 2^{(k+1)-1}\Big[(k+1)-1\Big]!\prod_{j=2}^{k+1}(2j-m-1)|x|^{1-m}.
\end{aligned}$$

2.

$$\begin{aligned}
\Delta^k|x|^{2k-m-1} &= 2^{k-1}\,(k-1)!\prod_{j=2}^{k}(2j-m-1)\Big[\Delta|x|^{1-m}\Big] \\
&= 2^{k-1}\,(k-1)!\prod_{j=2}^{k}(2j-m-1)\Big\{\overline{\partial}\Big[(1-m)x|x|^{-m-1}\Big]\Big\} \\
&= 2^{k-1}\,(k-1)!\prod_{j=1}^{k}(2j-m-1)\Big[(m+1)|x|^{-m-1}+(-m-1)x\overline{x}|x|^{-m-3}\Big] \\
&= 0.
\end{aligned}$$

❑

Satz 4.2.1 *Sei $G \subset \mathbb{R}^{m+1}$ ein beschränktes Gebiet mit glattem Rand, $f \in \mathrm{C}^{2k}(G;A) \cap \mathrm{C}^{2k-1}(\overline{G};A)$ und $1 \leq k$, dann gilt für $m > 1$ und gerade m oder ungerade m mit $0 \leq 2k < m+1$*

$$\begin{aligned}
f(x) = \sum_{\mu=1}^{k}\Bigg\{&\frac{1}{\omega_{m+1}}\int_{\partial G}\frac{(\overline{y-x})\,|y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod_{j=1}^{\mu-1}(2j-m-1)}\,\mathrm{d}\vec{\sigma}_y\Big[\Delta^{\mu-1}f(y)\Big] \\
&-\frac{1}{\omega_{m+1}}\int_{\partial G}\frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod_{j=1}^{\mu}(2j-m-1)}\,\mathrm{d}\overline{\vec{\sigma}}_y\left\{\partial\left[\Delta^{\mu-1}f(y)\right]\right\}\Bigg\} \qquad (4.2) \\
&+\frac{1}{\omega_{m+1}}\int_{G}\frac{|y-x|^{2k-m-1}}{2^{k-1}\,(k-1)!\,\prod_{j=1}^{k}(2j-m-1)}\left[\Delta^k f(y)\right]\mathrm{d}V_y.
\end{aligned}$$

Beweis:

Für $k = 1$ ist das die Gleichung (4.1):

$$\begin{aligned} f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y [\partial f(y)] \\ &\quad + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} [\Delta f(y)] \mathrm{d}V_y \end{aligned}$$

$$\begin{aligned} &= \sum_{\mu=1}^{1} \left\{ \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})\, |y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\, \prod_{j=1}^{\mu-1} (2j-m-1)} \,\mathrm{d}\vec{\sigma}_y \left[\Delta^{\mu-1} f(y)\right] \right. \\ &\qquad \left. - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\, \prod_{j=1}^{\mu} (2j-m-1)} \,\mathrm{d}\overline{\vec{\sigma}}_y \left\{ \partial \left[\Delta^{\mu-1} f(y) \right] \right\} \right\} \\ &\quad + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{2\cdot 1-m-1}}{2^{1-1}\,(1-1)!\, \prod_{j=1}^{1} (2j-m-1)} \left[\Delta^k f(y)\right] \mathrm{d}V_y. \end{aligned}$$

Der Beweis erfolgt durch Induktion über k, wobei aus Gründen der Übersichtlichkeit als Induktionsanfang $k = 2$ benutzt wird.

1. *Induktionsanfang ($k = 2$):*

   Benutzt man (4.1) für Δf, erhält man

$$\begin{aligned} \Delta f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \left[\Delta f(y)\right] - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \left\{ \partial \left[\Delta f(y) \right] \right\} \\ &\quad + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \left[\Delta^2 f(y)\right] \mathrm{d}V_y. \end{aligned}$$

   Das setzt man wieder in (4.1) ein:

$$\begin{aligned} f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \left[\partial f(y)\right] \\ &\quad + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \left\{ \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{z-y}}{|z-y|^{m+1}} \mathrm{d}\vec{\sigma}_z \left[\Delta f(z)\right] \right. \\ &\qquad - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|z-y|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_z \left\{ \partial \left[\Delta f(z) \right] \right\} \\ &\qquad \left. + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|z-y|^{1-m}}{1-m} \left[\Delta^2 f(z)\right] \mathrm{d}V_z \right\} \mathrm{d}V_y \end{aligned}$$

$$\begin{aligned}
= {} & \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \Big[\partial f(y)\Big] \\
& + \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \frac{\overline{z-y}}{|z-y|^{m+1}} \mathrm{d}V_y \mathrm{d}\vec{\sigma}_z \Big[\Delta f(z)\Big] \\
& - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \frac{|z-y|^{1-m}}{1-m} \mathrm{d}V_y \mathrm{d}\overline{\vec{\sigma}}_z \Big\{\partial \Big[\Delta f(z)\Big]\Big\} \\
& + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \frac{|z-y|^{1-m}}{1-m} \mathrm{d}V_y \Big[\Delta^2 f(z)\Big] \mathrm{d}V_z.
\end{aligned}$$

Mit

$$\varphi_0(x) := \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y),$$

$$\varphi_1(x) := -\frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \Big[\partial f(y)\Big],$$

$$\Phi_1(x,z) := \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \frac{\overline{z-y}}{|z-y|^{m+1}} \mathrm{d}V_y$$

und

$$\Psi_1(x,z) := \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \frac{|z-y|^{1-m}}{1-m} \mathrm{d}V_y$$

folgt

$$\begin{aligned}
f(x) = {} & \varphi_0(x) + \varphi_1(x) + \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \Phi_1(x,z) \mathrm{d}\vec{\sigma}_z \Big[\Delta f(z)\Big] \\
& - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \Psi_1(x,z) \mathrm{d}\overline{\vec{\sigma}}_z \Big\{\partial \Big[\Delta f(z)\Big]\Big\} + \frac{1}{\omega_{m+1}} \int\limits_{G} \Psi_1(x,z) \Big[\Delta^2 f(z)\Big] \mathrm{d}V_z.
\end{aligned}$$

Es gilt

$$\begin{aligned}
\overline{\partial_z} \Psi_1(x,z) &= \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \left[\overline{\partial_z} \frac{|z-y|^{1-m}}{1-m}\right] \mathrm{d}V_y \\
&= \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{1-m}}{1-m} \frac{\overline{z-y}}{|z-y|^{m+1}} \mathrm{d}V_y \\
&= \Phi_1(x,z) \\
&= \Psi_1(x,z) \overline{\partial_z}.
\end{aligned}$$

Seien

$$\tilde{\Phi}_1(x,z) := \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \frac{(y-z)|y-z|^{1-m}}{2(1-m)}$$

und

$$\tilde{\Psi}_1(x,z) := \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{|y-z|^{3-m}}{2(3-m)(1-m)}.$$

Dann kann man mit $x \neq z$ (4.1) benutzen:

$$\begin{aligned}\frac{|x-z|^{3-m}}{2(3-m)(1-m)} &= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{|y-z|^{3-m}}{2(3-m)(1-m)}\\ &\quad-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\frac{(y-z)|y-z|^{1-m}}{2(1-m)}\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}\,\frac{|y-z|^{1-m}}{1-m}\mathrm{d}V_y\\ &= \tilde{\Psi}_1(x,z)-\tilde{\Phi}_1(x,z)+\Psi_1(x,z).\end{aligned}\tag{4.3}$$

Von rechts nach $\overline{\partial_z}$ abgeleitet, ergibt

$$\frac{(\overline{z-x})|z-x|^{1-m}}{2(1-m)} = \left[\tilde{\Psi}_1(x,z)\overline{\partial_z}\right]-\left[\tilde{\Phi}_1(x,z)\overline{\partial_z}\right]+\Phi_1(x,z).$$

Also gilt

$$\Phi_1(x,y) = \frac{(\overline{y-x})|y-x|^{1-m}}{2(1-m)}+\left[(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]$$

und aus (4.3)

$$\Psi_1(x,y) = \frac{|x-y|^{3-m}}{2(3-m)(1-m)}+(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y).$$

Also gilt

$$\begin{aligned}f(x) &= \varphi_0(x)+\varphi_1(x)+\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{(\overline{y-x})|y-x|^{1-m}}{2(1-m)}\mathrm{d}\vec{\sigma}_y\left[\Delta f(y)\right]\\ &\quad-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|x-y|^{3-m}}{2(3-m)(1-m)}\mathrm{d}\overline{\vec{\sigma}}_y\left\{\partial\left[\Delta f(y)\right]\right\}\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|x-y|^{3-m}}{2(3-m)(1-m)}\left[\Delta^2 f(y)\right]\mathrm{d}V_y\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\left[(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\mathrm{d}\vec{\sigma}_y\left[\Delta f(y)\right]\\ &\quad-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\mathrm{d}\overline{\vec{\sigma}}_y\left\{\partial\left[\Delta f(y)\right]\right\}\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{G}(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\left[\Delta^2 f(y)\right]\mathrm{d}V_y.\end{aligned}$$

Die letzten drei Zeilen seien mit A bezeichnet. Sie sind zusammen null:
Mit (2.1) und (2.2) ergeben sich

$$\begin{aligned}&\int\limits_G\left\langle\left\{\left[(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\partial_y\right\}\left[\Delta f(y)\right]+\left[(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\left\{\partial\left[\Delta f(y)\right]\right\}\right\rangle\mathrm{d}V_y\\ =&\int\limits_{\partial G}\left[(\tilde{\Phi}_1-\tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\mathrm{d}\vec{\sigma}_y\left[\Delta f(y)\right],\end{aligned}$$

und

$$\int_G \left\{ \left[(\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\left\{\partial_y\left[\Delta f(y)\right]\right\} + (\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\left(\overline{\partial_y}\left\{\partial_y\left[\Delta f(y)\right]\right\}\right)\right\} \mathrm{d}V_y$$
$$= \int_{\partial G} (\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\mathrm{d}\overline{\vec{\sigma}}_y\left\{\partial_y\left[\Delta f(y)\right]\right\}.$$

Also gilt

$$\begin{aligned}
\omega_{m+1}A &= \int_G \left\langle \left\{\left[(\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\partial_y\right\}\left[\Delta f(y)\right] + \left[(\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\left\{\partial\left[\Delta f(y)\right]\right\}\right\rangle \mathrm{d}V_y \\
&\quad - \int_G \left\langle \left[(\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\overline{\partial_y}\right]\left\{\partial\left[\Delta f(y)\right]\right\} + (\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\left\langle\overline{\partial}\left\{\partial\left[\Delta f(y)\right]\right\}\right\rangle\right\rangle \mathrm{d}V_y \\
&\quad + \int_G (\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\left[\Delta^2 f(y)\right]\mathrm{d}V_y \\
&= \int_G \left[(\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,y)\Delta_y\right]\left[\Delta f(y)\right]\mathrm{d}V_y.
\end{aligned}$$

Mit $|y-z|^{3-m}\overline{\partial}_z = (3-m)(\overline{y-z})|y-z|^{1-m}$ *und*

$$\begin{aligned}
[(\overline{y-z})|y-z|^{1-m}]\partial_z &= -[(m+1)|z-y|^{1-m} + (\overline{z-y})(1-m)(z-y)|z-y|^{-m-1}] \\
&= -2|y-z|^{1-m}
\end{aligned}$$

gelten

$$\tilde{\Phi}_1(x,z)\overline{\partial}_z = -\frac{1}{\omega_{m+1}}\int_{\partial G} \frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\frac{|y-z|^{1-m}}{1-m},$$

also

$$\tilde{\Phi}_1(x,z)\Delta_z = \frac{1}{\omega_{m+1}}\int_{\partial G} \frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\frac{y-z}{|y-z|^{m+1}} = \tilde{\Phi}_0(x,z)$$

und

$$\tilde{\Psi}_1(x,z)\overline{\partial}_z = \frac{1}{\omega_{m+1}}\int_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{(\overline{z-y})|y-z|^{1-m}}{2(1-m)},$$

also

$$\tilde{\Psi}_1(x,z)\Delta_z = \frac{1}{\omega_{m+1}}\int_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\frac{|y-z|^{1-m}}{1-m} = \tilde{\Psi}_0(x,z).$$

Also gilt mit (2.1) und (2.2)

$$\begin{aligned}
(\tilde{\Phi}_1 - \tilde{\Psi}_1)(x,z)\Delta_z &= \frac{1}{\omega_{m+1}} \int\limits_G \Bigg[\left(\frac{|y-x|^{1-m}}{1-m} \overline{\partial}_y \right) \frac{y-z}{|y-z|^{m+1}} + \frac{|y-x|^{1-m}}{1-m} \underbrace{\left(\overline{\partial}_y \frac{y-z}{|y-z|^{m+1}} \right)}_{=0} \\
&\quad - \underbrace{\left(\frac{\overline{y-x}}{|y-x|^{m+1}} \partial_y \right)}_{=0} \frac{|y-z|^{1-m}}{1-m} - \frac{\overline{y-x}}{|y-x|^{m+1}} \left(\partial_y \frac{|y-z|^{1-m}}{1-m} \right) \Bigg] \mathrm{d}V_y \\
&= \frac{1}{\omega_{m+1}} \int\limits_G \left(\frac{\overline{y-x}}{|y-x|^{m+1}} \frac{y-z}{|y-z|^{m+1}} - \frac{\overline{y-x}}{|y-x|^{m+1}} \frac{y-z}{|y-z|^{m+1}} \right) \mathrm{d}V_y \\
&= 0.
\end{aligned}$$

Also ist $A = 0$.

Also gilt

$$\begin{aligned}
f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(1-1)-m-1}}{2^{1-1}\,(1-1)!\,\prod\limits_{j=1}^{0}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big[\Delta^{1-1} f(y)\Big] \\
&\quad - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\cdot 1-m-1}}{2^{1-1}\,(1-1)!\,\prod\limits_{j=1}^{1}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\{ \partial \Big[\Delta^{1-1} f(y)\Big] \Big\} \\
&\quad + \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(2-1)-m-1}}{2^{2-1}\,(2-1)!\,\prod\limits_{j=1}^{1}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big[\Delta^{2-1} f(y)\Big] \\
&\quad - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\cdot 2-m-1}}{2^{2-1}\,(2-1)!\,\prod\limits_{j=1}^{2}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\{ \partial \Big[\Delta^{2-1} f(y)\Big] \Big\} \\
&\quad + \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\cdot 2-m-1}}{2^{2-1}\,(2-1)!\,\prod\limits_{j=1}^{2}(2j-m-1)} \Big[\Delta^{2} f(y)\Big] \mathrm{d}V_y \\
&= \sum_{\mu=1}^{2} \Bigg\{ \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu-1}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big[\Delta^{\mu-1} f(y)\Big] \\
&\qquad - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\{ \partial \Big[\Delta^{\mu-1} f(y)\Big] \Big\} \Bigg\} \\
&\quad + \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\cdot 2-m-1}}{2^{2-1}\,(2-1)!\,\prod\limits_{j=1}^{2}(2j-m-1)} \Big[\Delta^{2} f(y)\Big] \mathrm{d}V_y.
\end{aligned}$$

2. Induktionsschritt:

Man nimmt an, (4.2) gilt für $k-1$, und wendet dies auf $\Delta f(x)$ an:

$$\begin{aligned}\Delta f(x) &= \sum_{\mu=1}^{k-1}\Bigg\{\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{(\overline{y-x})|y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu-1}(2j-m-1)}\mathrm{d}\vec{\sigma}_y\Big[\Delta^\mu f(y)\Big]\\ &\quad-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu}(2j-m-1)}\mathrm{d}\overline{\vec{\sigma}}_y\Big\{\partial\Big[\Delta^\mu f(y)\Big]\Big\}\Bigg\}\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{2(k-1)-m-1}}{2^{k-2}\,(k-2)!\,\prod\limits_{j=1}^{k-1}(2j-m-1)}\Big[\Delta^k f(y)\Big]\mathrm{d}V_y.\end{aligned}$$

Einsetzen in (4.1) ergibt

$$\begin{aligned}f(x) &= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y)-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\Big[\partial f(y)\Big]\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}\Bigg\langle\sum_{\mu=1}^{k-1}\Bigg\{\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{(\overline{z-y})|z-y|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu-1}(2j-m-1)}\mathrm{d}\vec{\sigma}_z\Big[\Delta^\mu f(z)\Big]\\ &\quad-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|z-y|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu}(2j-m-1)}\mathrm{d}\overline{\vec{\sigma}}_z\Big\{\partial\Big[\Delta^\mu f(z)\Big]\Big\}\Bigg\}\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|z-y|^{2(k-1)-m-1}}{2^{k-2}\,(k-2)!\,\prod\limits_{j=1}^{k-1}(2j-m-1)}\Big[\Delta^k f(z)\Big]\mathrm{d}V_z\Bigg\rangle\mathrm{d}V_y\\ &= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y)-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\Big[\partial f(y)\Big]\\ &\quad+\sum_{\mu=1}^{k-1}\Bigg\langle\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}\,\frac{(\overline{z-y})|z-y|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu-1}(2j-m-1)}\mathrm{d}V_y\mathrm{d}\vec{\sigma}_z\Big[\Delta^\mu f(z)\Big]\\ &\quad-\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}\,\frac{|z-y|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod\limits_{j=1}^{\mu}(2j-m-1)}\mathrm{d}V_y\mathrm{d}\overline{\vec{\sigma}}_z\Big\{\partial\Big[\Delta^\mu f(z)\Big]\Big\}\Bigg\rangle\\ &\quad+\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}\,\frac{|z-y|^{2(k-1)-m-1}}{2^{k-2}\,(k-2)!\,\prod\limits_{j=1}^{k-1}(2j-m-1)}\mathrm{d}V_y\Big[\Delta^k f(z)\Big]\mathrm{d}V_z.\end{aligned}$$

Nun setzt man

$$\Phi_{\mu+1}(x,z):=\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}\,\frac{(\overline{z-y})|z-y|^{2\mu-m-1}}{2^\mu\,\mu!\,\prod\limits_{j=1}^{\mu}(2j-m-1)}\mathrm{d}V_y$$

und

$$\Psi_\mu(x,z) := \frac{1}{\omega_{m+1}} \int_G \frac{|y-x|^{1-m}}{1-m} \frac{|z-y|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}V_y.$$

Dann gilt

$$\begin{aligned} f(x) &= \frac{1}{\omega_{m+1}} \int_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \Big[\partial f(y)\Big] \\ &\quad + \sum_{\mu=1}^{k-1} \left\langle \frac{1}{\omega_{m+1}} \int_{\partial G} \Phi_\mu(x,z) \mathrm{d}\vec{\sigma}_z \Big[\Delta^\mu f(z)\Big] - \frac{1}{\omega_{m+1}} \int_{\partial G} \Psi_\mu(x,z) \mathrm{d}\overline{\vec{\sigma}}_z \Big\{\partial \Big[\Delta^\mu f(z)\Big]\Big\} \right\rangle \\ &\quad + \frac{1}{\omega_{m+1}} \int_G \Psi_{k-1}(x,z) \Big[\Delta^k f(z)\Big] \mathrm{d}V_z. \end{aligned}$$

Es gilt

$$\begin{aligned} \Psi_\mu(x,z)\overline{\partial_z} &= \frac{1}{\omega_{m+1}} \int_G \frac{|y-x|^{1-m}}{1-m} \frac{(2\mu-m-1)(\overline{z-y})|z-y|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}V_y \\ &= \frac{1}{\omega_{m+1}} \int_G \frac{|y-x|^{1-m}}{1-m} \frac{(\overline{z-y})|z-y|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\,\prod_{j=1}^{\mu-1}(2j-m-1)} \mathrm{d}V_y \\ &= \Phi_\mu(x,z). \end{aligned}$$

Mit (4.1) und analog dem Beweis zu (4.1) (zuerst für $x \neq z$, ε klein genug mit $0 < 2\varepsilon < |x-z|$ und $G_\varepsilon = G \setminus \{y : |y-z| \leq \varepsilon\}$) ergibt sich

$$\begin{aligned} \frac{|z-x|^{2(\mu+1)-m-1}}{2^\mu\,\mu!\,\prod_{j=1}^{\mu+1}(2j-m-1)} &= \frac{1}{\omega_{m+1}} \int_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{|z-y|^{2(\mu+1)-m-1}}{2^\mu\,\mu!\,\prod_{j=1}^{\mu+1}(2j-m-1)} \\ &\quad - \frac{1}{\omega_{m+1}} \int_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \frac{(z-y)|z-y|^{2\mu-m-1}}{2^\mu\,\mu!\,\prod_{j=1}^{\mu}(2j-m-1)} \\ &\quad + \frac{1}{\omega_{m+1}} \int_G \frac{|y-x|^{1-m}}{1-m} \frac{\Big[\partial_z(\overline{z-y})|z-y|^{2\mu-m-1}\Big]}{2^\mu\,\mu!\,\prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}V_y \\ &= \frac{1}{\omega_{m+1}} \int_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \frac{|z-y|^{2(\mu+1)-m-1}}{2^\mu\,\mu!\,\prod_{j=1}^{\mu+1}(2j-m-1)} \\ &\quad - \frac{1}{\omega_{m+1}} \int_{\partial G} \frac{|y-x|^{1-m}}{1-m} \mathrm{d}\overline{\vec{\sigma}}_y \frac{(z-y)|z-y|^{2\mu-m-1}}{2^\mu\,\mu!\,\prod_{j=1}^{\mu}(2j-m-1)} \\ &\quad + \frac{1}{\omega_{m+1}} \int_G \frac{|y-x|^{1-m}}{1-m} \frac{2\mu|z-y|^{2\mu-m-1}}{2^\mu\,\mu!\,\prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}V_y. \end{aligned} \tag{4.4}$$

Mit

$$\tilde{\Psi}_\mu(x,z) := \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} d\vec{\sigma}_y \frac{|z-y|^{2(\mu+1)-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu+1}(2j-m-1)}$$

und

$$\tilde{\Phi}_\mu(x,z) := \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} d\overline{\vec{\sigma}}_y \frac{(z-y)|z-y|^{2\mu-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu}(2j-m-1)}$$

wird aus (4.4)

$$\frac{|z-x|^{2(\mu+1)-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu+1}(2j-m-1)} = \tilde{\Psi}_\mu(x,z) - \tilde{\Phi}_\mu(x,z) + \Psi_\mu(x,z) \qquad (\mu \in \mathbb{N}).$$

Differenzieren von rechts mit $\overline{\partial_z}$ liefert

$$\frac{(\overline{z-x})|z-x|^{2\mu-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu}(2j-m-1)} = \tilde{\Psi}_\mu(x,z)\overline{\partial_z} - \tilde{\Phi}_\mu(x,z)\overline{\partial_z} + \Phi_\mu(x,z),$$

also

$$\Phi_\mu(x,z) = \frac{(\overline{z-x})|z-x|^{2\mu-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu}(2j-m-1)} - \tilde{\Psi}_\mu(x,z)\overline{\partial_z} + \tilde{\Phi}_\mu(x,z)\overline{\partial_z}.$$

Also gilt

$$\begin{aligned}
f(x) \quad = \quad & \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{m+1}} d\vec{\sigma}_y f(y) - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{1-m}}{1-m} d\overline{\vec{\sigma}}_y \Big[\partial f(y)\Big] \\
& + \sum_{\mu=1}^{k-1} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2\mu-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu}(2j-m-1)} d\vec{\sigma}_y \Big[\Delta^\mu f(y)\Big] \\
& - \sum_{\mu=1}^{k-1} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2(\mu+1)-m-1}}{2^\mu\ \mu!\ \prod\limits_{j=1}^{\mu+1}(2j-m-1)} d\overline{\vec{\sigma}}_y \Big\{\partial\Big[\Delta^\mu f(y)\Big]\Big\} \\
& + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{2k-m-1}}{2^{k-1}\ (k-1)!\ \prod\limits_{j=1}^{k}(2j-m-1)} \Big[\Delta^k f(y)\Big] dV_y \\
& - \sum_{\mu=1}^{k-1} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} (-\tilde{\Psi}_\mu + \tilde{\Phi}_\mu)(x,z) d\overline{\vec{\sigma}}_z \Big\{\partial\Big[\Delta^\mu f(y)\Big]\Big\} \\
& + \sum_{\mu=1}^{k-1} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \Big\{\Big[-\tilde{\Psi}_\mu(x,z) + \tilde{\Phi}_\mu(x,z)\Big]\overline{\partial_z}\Big\} d\vec{\sigma}_z \Big[\Delta^\mu f(z)\Big] \\
& + \frac{1}{\omega_{m+1}} \int\limits_{G} (-\tilde{\Psi}_{k-1} + \tilde{\Phi}_{k-1})(x,z) \Big[\Delta^k f(z)\Big] dV_z.
\end{aligned}$$

Die letzten drei Zeilen seien mit A bezeichnet.
Mit

$$
\begin{aligned}
\Big[(\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\Big]\,\Delta_z &= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\left[\frac{(z-y)|z-y|^{2\mu-m-1}}{2^\mu\;\mu!\;\prod\limits_{j=1}^{\mu}(2j-m-1)}\overline{\partial_z}\partial_z\right] \\
&\quad -\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\left[\frac{|z-y|^{2(\mu+1)-m-1}}{2^\mu\;\mu!\;\prod\limits_{j=1}^{\mu+1}(2j-m-1)}\partial_z\overline{\partial_z}\right] \\
&= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\left[\frac{2\mu|z-y|^{2\mu-m-1}}{2^\mu\;\mu!\;\prod\limits_{j=1}^{\mu}(2j-m-1)}\partial_z\right] \\
&\quad -\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\left[\frac{\{2(\mu+1)-m-1\}(z-y)|z-y|^{2\mu-m-1}}{2^\mu\;\mu!\;\prod\limits_{j=1}^{\mu+1}(2j-m-1)}\overline{\partial_z}\right] \\
&= \frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{1-m}}{1-m}\mathrm{d}\overline{\vec{\sigma}}_y\left[\frac{(2\mu-m-1)(z-y)|z-y|^{2(\mu-1)-m-1}}{2^{\mu-1}\;(\mu-1)!\;\prod\limits_{j=1}^{\mu}(2j-m-1)}\right] \\
&\quad -\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{\overline{y-x}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y\left[\frac{2\mu|z-y|^{2\mu-m-1}}{2^\mu\;\mu!\;\prod\limits_{j=1}^{\mu}2j-m-1}\right] \\
&= \tilde{\Phi}_{\mu-1}(x,z) - \tilde{\Psi}_{\mu-1}(x,z),
\end{aligned}
$$

gilt

$$
\begin{aligned}
-\omega_{m+1}A &= \sum_{\mu=1}^{k-1}\left\langle\int\limits_{\partial G}(\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\mathrm{d}\overline{\vec{\sigma}}_z\Big\{\partial\Big[\Delta^\mu f(z)\Big]\Big\}\right. \\
&\quad \left. -\int\limits_{\partial G}\Big[(\tilde{\Phi}_\mu(x,z)-\tilde{\Psi}_\mu(x,z))\overline{\partial_z}\Big]\mathrm{d}\vec{\sigma}_z\Big[\Delta^\mu f(z)\Big]\right\rangle \\
&\quad -\int\limits_G(\tilde{\Phi}_{k-1}-\tilde{\Psi}_{k-1})(x,z)\Big[\Delta^k f(z)\Big]\mathrm{d}V_z
\end{aligned}
$$

$$\begin{aligned}
&\overset{(2.1),(2.2)}{=} \sum_{\mu=1}^{k-1} \int\limits_G \Big\langle \Big[(\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\overline{\partial_z}\Big]\Big\{\partial\Big[\Delta^\mu f(z)\Big]\Big\} + (\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\Big[\Delta^{\mu+1} f(z)\Big] \\
&\qquad - \Big[(\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\Delta_z\Big]\Big[\Delta^\mu f(z)\Big] - \Big[(\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\overline{\partial_z}\Big]\Big\{\partial\Big[\Delta^\mu f(z)\Big]\Big\}\Big\rangle \mathrm{d}V_z \\
&\qquad - \int\limits_G (\tilde{\Phi}_{k-1} - \tilde{\Psi}_{k-1})(x,z)\Big[\Delta^k f(z)\Big]\mathrm{d}V_z \\
&= \sum_{\mu=1}^{k-1} \Big\{ \int\limits_G (\tilde{\Phi}_\mu - \tilde{\Psi}_\mu)(x,z)\Big[\Delta^{\mu+1} f(z)\Big] - (\tilde{\Phi}_{\mu-1} - \tilde{\Psi}_{\mu-1})(x,z)\Big[\Delta^\mu f(z)\Big]\Big\}\mathrm{d}V_z \\
&\qquad - \int\limits_G (\tilde{\Phi}_{k-1} - \tilde{\Psi}_{k-1})(x,z)\Big[\Delta^k f(z)\Big]\mathrm{d}V_z \\
&= \int\limits_G \Big\{(\tilde{\Phi}_{k-1} - \tilde{\Psi}_{k-1})(x,z)\Big[\Delta^k f(z)\Big] - (\tilde{\Phi}_0 - \tilde{\Psi}_0)(x,z)\Big[\Delta f(z)\Big]\Big\}\mathrm{d}V_z \\
&\qquad - \int\limits_G (\tilde{\Phi}_{k-1} - \tilde{\Psi}_{k-1})(x,z)\Big[\Delta^k f(z)\Big]\mathrm{d}V_z \\
&= - \int\limits_G (\tilde{\Phi}_0 - \tilde{\Psi}_0)(x,z)\Big[\Delta f(z)\Big]\mathrm{d}V_z \\
&= 0.
\end{aligned}$$

Also ergibt sich

$$\begin{aligned}
f(x) &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(1-1)-m-1}}{2^{1-1}\,(1-1)\prod\limits_{j=1}^{1-1}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big[\Delta^{1-1} f(y)\Big] \\
&\quad + \sum_{\mu=2}^{k} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}\,(\mu-1)!\prod\limits_{j=1}^{\mu-1}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big[\Delta^{\mu-1} f(y)\Big] \\
&\quad - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\cdot 1-m-1}}{2^{1-1}\,(1-1)!\prod\limits_{j=1}^{1}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\{\partial\Big[\Delta^{1-1} f(y)\Big]\Big\} \\
&\quad - \sum_{\mu=2}^{k} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}\,(\mu-1)!\prod\limits_{j=1}^{\mu}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\{\partial\Big[\Delta^{\mu-1} f(y)\Big]\Big\} \\
&\quad + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{2k-m-1}}{2^{k-1}\,(k-1)!\prod\limits_{j=1}^{k}(2j-m-1)} \Big[\Delta^k f(y)\Big]\mathrm{d}V_y,
\end{aligned}$$

das ergibt die zu beweisende Gleichung (4.2).

❑

Wenn $m+1=2\alpha>2$ hat die Darstellung höherer Ordnung für den Laplaceoperator folgende Form.

Satz 4.2.2 (iterierte Integraldarstellung für Δ) *Für $0\le k$, $G\subset\mathbb{R}^{2\alpha}$ ein beschränktes Gebiet mit glattem Rand und*
$f\in C^{2(\alpha+k)}(G;A)\cap C^{2(\alpha+k)-1}(\overline{G};A)$ gilt

$$\begin{aligned}
f(x) &= \sum_{\mu=1}^{\alpha-1}\Bigg\{\frac{(-1)^{\mu-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{(\alpha-\mu)!\overline{(y-x)}|y-x|^{2(\mu-1-\alpha)}}{2^{2(\mu-1)}(\mu-1)!\,(\alpha-1)!}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\mu-1}f(y)\\
&\qquad+\frac{(-1)^{\mu-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{(\alpha-\mu-1)!|y-x|^{2(\mu-\alpha)}}{2^{2\mu-1}(\mu-1)!\,(\alpha-1)!}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\mu-1}f(y)\Bigg\}\\
&\quad+\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(y-x)}|y-x|^{-2}}{2^{2(\alpha-1)}(\alpha-1)!^2}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\alpha-1}f(y)\\
&\quad-\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\log|y-x|^2}{2^{2\alpha-1}(\alpha-1)!^2}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\alpha-1}f(y)\\
&\quad+\sum_{\mu=1}^{k}\Bigg\langle\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(y-x)}|y-x|^{2(\mu-1)}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{\mu-1}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{\mu-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+\mu-1)}(\mu-1)!\,(\alpha-1)!\,(\alpha+\mu-1)!}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\alpha+\mu-1}f(y)\\
&\qquad-\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|y-x|^{2\mu}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{\mu}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{\mu-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+\mu)-1}\mu!\,(\alpha-1)!\,(\alpha+\mu-1)!}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\alpha+\mu-1}f(y)\Bigg\rangle\\
&\quad+\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{G}\frac{|y-x|^{2k}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{k}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)-1}k!\,(\alpha-1)!\,(\alpha+k-1)!}\,\Delta^{\alpha+k}f(y)\,\mathrm{d}V_y.
\end{aligned}\tag{4.5}$$

Beweis:

Der Beweis erfolgt durch Induktion.
1. Induktionsanfang: Die Formel (4.2) lautet für $m+1=2\alpha$ und $k=\alpha-1$

$$\begin{aligned}
f(x) &= \sum_{\mu=1}^{\alpha-1}\Bigg\{\frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(y-x)}|y-x|^{2(\mu-1-\alpha)}}{2^{\mu-1}(\mu-1)!\prod\limits_{\nu=1}^{\mu-1}2(\nu-\alpha)}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\mu-1}f(y)\\
&\qquad-\frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|y-x|^{2(\mu-\alpha)}}{2^{\mu-1}(\mu-1)!\prod\limits_{\nu=1}^{\mu}2(\nu-\alpha)}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\mu-1}f(y)\Bigg\}\\
&\quad+\frac{1}{\omega_{2\alpha}}\int\limits_{G}\frac{|y-x|^{-2}}{2^{\alpha-2}(\alpha-2)!\prod\limits_{\nu=1}^{\alpha-1}2(\nu-\alpha)}\,\Delta^{\alpha-1}f(y)\,\mathrm{d}V_y.
\end{aligned}\tag{4.6}$$

Für $k = 1$ wird diese Gleichung zu

$$f(x) = \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\overline{y-x}}{|y-x|^{2\alpha}} \, d\vec{\sigma}_y \, f(y) - \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{|y-x|^{2(1-\alpha)}}{2(1-\alpha)} \, d\overline{\vec{\sigma}}_y \, \partial f(y) + \frac{1}{\omega_{2\alpha}} \int\limits_{G} \frac{|y-x|^{2(1-\alpha)}}{2(1-\alpha)} \, \Delta f(y) \, dV_y. \tag{4.7}$$

Gleichung (4.7) auf $\Delta^{\alpha-1} f(y)$ angewendet ergibt

$$\Delta^{\alpha-1} f(y) = \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\overline{z-y}}{|z-y|^{2\alpha}} \, d\vec{\sigma}_z \, \Delta^{\alpha-1} f(z) - \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)} \, d\overline{\vec{\sigma}}_z \, \partial \Delta^{\alpha-1} f(z) + \frac{1}{\omega_{2\alpha}} \int\limits_{G} \frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)} \, \Delta^{\alpha} f(z) \, dV_z. \tag{4.8}$$

Seien

$$\Phi_0(x,z) := \frac{1}{\omega_{2\alpha}} \int\limits_{G} \frac{|y-x|^{-2}}{2^{\alpha-2}(\alpha-2)! \prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)} \frac{\overline{(z-y)}}{|z-y|^{2\alpha}} \, dV_y$$

und

$$\Psi_0(x,z) := \frac{1}{\omega_{2\alpha}} \int\limits_{G} \frac{|y-x|^{-2}}{2^{\alpha-2}(\alpha-2)! \prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)} \frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)} \, dV_y.$$

Dann gilt $\overline{\partial}_z \Psi_0(x,z) = \Phi_0(x,z)$. Durch Einsetzen von (4.8) in (4.6) ergibt sich nach Vertauschung der Integrationsreihenfolgen

$$\begin{aligned} f(x) = \sum_{\mu=1}^{\alpha-1} \Bigg\{ & \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\overline{(y-x)} |y-x|^{2(\mu-1-\alpha)}}{2^{\mu-1}(\mu-1)! \prod\limits_{\nu=1}^{\mu-1} 2(\nu-\alpha)} \, d\vec{\sigma}_y \, \Delta^{\mu-1} f(y) \\ & - \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{|y-x|^{2(\mu-\alpha)}}{2^{\mu-1}(\mu-1)! \prod\limits_{\nu=1}^{\mu} 2(\nu-\alpha)} \, d\overline{\vec{\sigma}}_y \, \partial \Delta^{\mu-1} f(y) \Bigg\} \\ & + \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \Phi_0(x,z) \, d\vec{\sigma}_z \, \Delta^{\alpha-1} f(z) - \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \Psi_0(x,z) \, d\overline{\vec{\sigma}}_z \, \partial \Delta^{\alpha-1} f(z) \\ & + \frac{1}{\omega_{2\alpha}} \int\limits_{G} \Psi_0(x,z) \, \Delta^{\alpha} f(z) \, dV_z. \end{aligned} \tag{4.9}$$

Sei

$$I := \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \Phi_0(x,z) \, d\vec{\sigma}_z \, \Delta^{\alpha-1} f(z) - \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \Psi_0(x,z) \, d\overline{\vec{\sigma}}_z \, \partial \Delta^{\alpha-1} f(z) + \frac{1}{\omega_{2\alpha}} \int\limits_{G} \Psi_0(x,z) \, \Delta^{\alpha} f(z) \, dV_z.$$

Seien

$$\tilde{\Phi}_0(x,z) := \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\log|y-x|^2}{2^\alpha(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)}\, \mathrm{d}\vec{\sigma}_y\, \frac{\overline{(y-z)}}{|y-z|^{2\alpha}}$$

und

$$\tilde{\Psi}_0(x,z) := -\frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{(y-x)|y-x|^{-2}}{2^{\alpha-1}(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)}\, \mathrm{d}\overline{\vec{\sigma}}_y\, \frac{|y-z|^{2(1-\alpha)}}{2(1-\alpha)},$$

dann gelten mit (4.7)

$$\frac{\log|z-x|^2}{2^\alpha(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)} = \tilde{\Phi}_0(x,z) + \tilde{\Psi}_0(x,z) + \Psi_0(x,z)$$

und

$$\frac{\overline{(z-x)}|z-x|^{-2}}{2^{\alpha-1}(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)} = \tilde{\Phi}_0(x,z)\overline{\partial}_z + \tilde{\Psi}_0(x,z)\overline{\partial}_z + \Phi_0(x,z).$$

Also gilt mit

$$\begin{aligned} A_1 \;:=\; & -\frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \left\{ \tilde{\Phi}_0(x,z)\overline{\partial}_z + \tilde{\Psi}_0(x,z)\overline{\partial}_z \right\} \mathrm{d}\vec{\sigma}_z\, \Delta^{\alpha-1} f(z) \\ & +\frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \left\{ \tilde{\Phi}_0(x,z) + \tilde{\Psi}_0(x,z) \right\} \mathrm{d}\overline{\vec{\sigma}}_z\, \partial\Delta^{\alpha-1} f(z) \\ & -\frac{1}{\omega_{2\alpha}} \int\limits_{G} \left\{ \tilde{\Phi}_0(x,z) + \tilde{\Psi}_0(x,z) \right\} \Delta^\alpha f(z)\, \mathrm{d}V_z, \end{aligned}$$

$$\begin{aligned} I \;=\; & \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\overline{(z-x)}|z-x|^{-2}}{2^{\alpha-1}(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)}\, \mathrm{d}\vec{\sigma}_z\, \Delta^{\alpha-1} f(z) \\ & -\frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\log|z-x|^2}{2^\alpha(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)}\, \mathrm{d}\overline{\vec{\sigma}}_z\, \partial\Delta^{\alpha-1} f(z) \\ & +\frac{1}{\omega_{2\alpha}} \int\limits_{G} \frac{\log|z-x|^2}{2^\alpha(\alpha-1)!\prod\limits_{\nu=1}^{\alpha-1} 2(\nu-\alpha)}\, \Delta^\alpha f(z)\, \mathrm{d}V_z + A_1. \end{aligned}$$

Mit $\tilde{\Phi}_0(x,z)\partial_z = 0, \quad \tilde{\Psi}_0(x,z)\Delta_z = 0$ *und (2.3) folgt*

$$\begin{aligned}
A_1 &= -\frac{1}{\omega_{2\alpha}} \int\limits_G \Big\langle \underbrace{\left\{\tilde{\Phi}_0(x,z)\Delta_z\right\}}_{=\tilde{\Phi}_0(x,z)\partial_z\overline{\partial}_z=0} \left\{\Delta^{\alpha-1} f(z)\right\} + \left\{\tilde{\Phi}_0(x,z)\overline{\partial}_z\right\}\left\{\partial\Delta^{\alpha-1} f(z)\right\} \\
&\qquad + \underbrace{\left\{\tilde{\Psi}_0(x,z)\Delta_z\right\}}_{=0}\left\{\Delta^{\alpha-1} f(z)\right\} + \left\{\tilde{\Psi}_0(x,z)\overline{\partial}_z\right\}\left\{\partial\Delta^{\alpha-1} f(z)\right\} \\
&\qquad - \left\{\tilde{\Phi}_0(x,z)\overline{\partial}_z\right\}\left\{\partial\Delta^{\alpha-1} f(z)\right\} - \tilde{\Phi}_0(x,z)\left\{\Delta^{\alpha} f(z)\right\} \\
&\qquad - \left\{\tilde{\Psi}_0(x,z)\overline{\partial}_z\right\}\left\{\partial\Delta^{\alpha-1} f(z)\right\} - \tilde{\Psi}_0(x,z)\left\{\Delta^{\alpha} f(z)\right\} \Big\rangle \, \mathrm{d}V_z \\
&\quad -\frac{1}{\omega_{2\alpha}} \int\limits_G \left\{\tilde{\Phi}_0(x,z) + \tilde{\Psi}_0(x,z)\right\} \left\{\Delta^{\alpha} f(z)\right\} \, \mathrm{d}V_z \\
&= 0.
\end{aligned}$$

Also gilt

$$\begin{aligned}
f(x) &= \sum_{\mu=1}^{\alpha-1} \Bigg\{ \frac{(-1)^{\mu-1}}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{(\alpha-\mu)!\,\overline{(y-x)}|y-x|^{2(\mu-1-\alpha)}}{2^{2(\mu-1)}(\mu-1)!\,(\alpha-1)!} \, \mathrm{d}\vec{\sigma}_y \, \Delta^{\mu-1} f(y) \\
&\qquad + \frac{(-1)^{\mu-1}}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{(\alpha-\mu-1)!\,|y-x|^{2(\mu-\alpha)}}{2^{2\mu-1}(\mu-1)!\,(\alpha-1)!} \, \mathrm{d}\overline{\vec{\sigma}}_y \, \partial\Delta^{\mu-1} f(y) \Bigg\} \\
&\quad + \frac{(-1)^{\alpha-1}}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\overline{(y-x)}|y-x|^{-2}}{2^{2(\alpha-1)}(\alpha-1)!^{\,2}} \, \mathrm{d}\vec{\sigma}_y \, \Delta^{\alpha-1} f(y) \\
&\quad - \frac{(-1)^{\alpha-1}}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\log|y-x|^2}{2^{2\alpha-1}(\alpha-1)!^{\,2}} \, \mathrm{d}\overline{\vec{\sigma}}_y \, \partial\Delta^{\alpha-1} f(y) \\
&\quad + \frac{(-1)^{\alpha-1}}{\omega_{2\alpha}} \int\limits_{G} \frac{\log|y-x|^2}{2^{2\alpha-1}(\alpha-1)!^{\,2}} \, \Delta^{\alpha} f(y) \, \mathrm{d}V_y.
\end{aligned} \tag{4.10}$$

2. Induktionsannahme: Die Gleichung (4.5) gilt für $k \in \mathbb{N}$.

3. Induktionsschritt:

Gleichung (4.1) ist für $m+1 = 2\alpha$

$$\begin{aligned}
f(y) &= \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{\overline{z-y}}{|z-y|^{2\alpha}} \, \mathrm{d}\vec{\sigma}_z f(z) - \frac{1}{\omega_{2\alpha}} \int\limits_{\partial G} \frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)} \, \mathrm{d}\overline{\vec{\sigma}}_z \, \partial f(z) \\
&\quad + \frac{1}{\omega_{2\alpha}} \int\limits_{G} \frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)} \, \Delta f(z) \, \mathrm{d}V_z.
\end{aligned} \tag{4.11}$$

Also gilt

$$\begin{aligned}\Delta^{\alpha+k}f(y) &= \frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{z-y}}{|z-y|^{2\alpha}}\,\mathrm{d}\vec{\sigma}_z\,\Delta^{\alpha+k}f(z)-\frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)}\,\mathrm{d}\overline{\vec{\sigma}}_z\,\partial\Delta^{\alpha+k}f(z)\\ &\quad+\frac{1}{\omega_{2\alpha}}\int\limits_{G}\frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)}\,\Delta^{\alpha+k+1}f(z)\,\mathrm{d}V_z.\end{aligned}\tag{4.12}$$

Seien

$$\Psi(x,z):=\frac{1}{\omega_{2\alpha}}\int\limits_{G}\frac{|y-x|^{2k}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{k}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)-1}k!(\alpha-1)!(\alpha+k-1)!}\frac{\overline{z-y}}{|z-y|^{2\alpha}}\,\mathrm{d}V_y$$

und

$$\Phi(x,z):=\frac{1}{\omega_{2\alpha}}\int\limits_{G}\frac{|y-x|^{2k}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{k}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)-1}k!(\alpha-1)!(\alpha+k-1)!}\frac{|z-y|^{2(1-\alpha)}}{2(1-\alpha)}\,\mathrm{d}V_y$$

Setzt man Gleichung (4.12) in (4.5) ein, so wird das Gebietsintegral nach Vertauschung der Integrationsreihenfolge

$$\begin{aligned}I &= \frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\Psi(x,z)\,\mathrm{d}\vec{\sigma}_z\,\Delta^{\alpha+k}f(z)-\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\Phi(x,z)\,\mathrm{d}\overline{\vec{\sigma}}_z\,\partial\Delta^{\alpha+k}f(z)\\ &\quad+\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{G}\Phi(x,z)\,\Delta^{\alpha+k+1}f(z)\,\mathrm{d}V_z.\end{aligned}$$

Seien

$$\tilde{\Psi}(x,z):=\frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|y-x|^{2(k+1)}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{k+1}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)+1}(k+1)!(\alpha-1)!(\alpha+k)!}\,\mathrm{d}\vec{\sigma}_y\,\frac{\overline{(y-z)}}{|y-z|^{2\alpha}}$$

und

$$\tilde{\Phi}(x,z):=-\frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{(y-x)|y-x|^{2k}\Big\{\log|y-x|^2-\sum\limits_{\rho=1}^{k}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)}k!(\alpha-1)!(\alpha+k)!}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\frac{|y-z|^{2(1-\alpha)}}{2(1-\alpha)}.$$

Mit einer analogen (symmetrischen) Gleichung zu (4.11) folgt

$$\frac{|z-x|^{2(k+1)}\Big\{\log|z-x|^2-\sum\limits_{\rho=1}^{k+1}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)+1}(k+1)!(\alpha-1)!(\alpha+k)!}=\tilde{\Psi}(x,z)+\tilde{\Phi}(x,z)+\Phi(x,z).$$

Differenzieren von rechts nach ∂_z liefert

$$\frac{\overline{(z-x)}|z-x|^{2k}\Big\{\log|z-x|^2-\sum\limits_{\rho=1}^{k}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)}k!(\alpha-1)!(\alpha+k)!}=\tilde{\Psi}(x,z)\overline{\partial_z}+\tilde{\Phi}(x,z)\overline{\partial_z}+\Psi(x,z).$$

Mit

$$\begin{aligned}(-1)^{\alpha-1}B &:= \frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\left\{-\tilde{\Psi}(x,z)\overline{\partial_z}-\tilde{\Phi}(x,z)\overline{\partial_z}\right\}\mathrm{d}\vec{\sigma}_z\,\Delta^{\alpha+k}f(z)\\ &\quad-\frac{1}{\omega_{2\alpha}}\int\limits_{\partial G}\left\{-\tilde{\Psi}(x,z)-\tilde{\Phi}(x,z)\right\}\mathrm{d}\overline{\vec{\sigma}}_z\,\partial\Delta^{\alpha+k}f(z)\\ &\quad+\frac{1}{\omega_{2\alpha}}\int\limits_{G}\left\{-\tilde{\Psi}(x,z)-\tilde{\Phi}(x,z)\right\}\Delta^{\alpha+k+1}f(z)\,\mathrm{d}V_z\end{aligned}$$

folgt

$$\begin{aligned}I &= \frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(z-x)}|z-x|^{2k}\left\{\log|z-x|^2-\sum\limits_{\rho=1}^{k}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\right\}}{2^{2(\alpha+k)}k!(\alpha-1)!(\alpha+k)!}\mathrm{d}\vec{\sigma}_z\,\Delta^{\alpha+k}f(z)\\ &\quad-\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|z-x|^{2(k+1)}\left\{\log|z-x|^2-\sum\limits_{\rho=1}^{k+1}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\right\}}{2^{2(\alpha+k)+1}(k+1)!(\alpha-1)!(\alpha+k)!}\mathrm{d}\overline{\vec{\sigma}}_z\,\partial\Delta^{\alpha+k}f(z)\\ &\quad+\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{G}\frac{|z-x|^{2(k+1)}\left\{\log|z-x|^2-\sum\limits_{\rho=1}^{k+1}\frac{1}{\rho}-\sum\limits_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\right\}}{2^{2(\alpha+k)+1}(k+1)!(\alpha-1)!(\alpha+k)!}\Delta^{\alpha+k+1}f(z)\,\mathrm{d}V_z\\ &\quad+B.\end{aligned}$$

Nach Gleichung (2.1) gilt

$$\begin{aligned}(-1)^{\alpha-1}B &= \frac{1}{\omega_{2\alpha}}\int\limits_{G}\left\{-\tilde{\Psi}(x,z)\Delta_z-\tilde{\Phi}(x,z)\Delta_z\right\}\Delta^{\alpha+k}f(z)\mathrm{d}V_z\\ &\quad+\frac{1}{\omega_{2\alpha}}\int\limits_{G}\left\{-\tilde{\Psi}(x,z)\overline{\partial_z}-\tilde{\Phi}(x,z)\overline{\partial_z}\right\}\partial\Delta^{\alpha+k}f(z)\mathrm{d}V_z\\ &\quad-\frac{1}{\omega_{2\alpha}}\int\limits_{G}\left\{-\tilde{\Psi}(x,z)\overline{\partial_z}-\tilde{\Phi}(x,z)\overline{\partial_z}\right\}\partial\Delta^{\alpha+k}f(z)\mathrm{d}V_z\\ &\quad-\frac{1}{\omega_{2\alpha}}\int\limits_{G}\left\{-\tilde{\Psi}(x,z)-\tilde{\Phi}(x,z)\right\}\Delta^{\alpha+k+1}f(z)\mathrm{d}V_z\\ &\quad+\frac{1}{\omega_{2\alpha}}\int\limits_{G}\left\{-\tilde{\Psi}(x,z)-\tilde{\Phi}(x,z)\right\}\Delta^{\alpha+k+1}f(z)\mathrm{d}V_z.\end{aligned}$$

Mit

$$\tilde{\Psi}(x,z)\Delta_z=\tilde{\Psi}(x,z)\partial_z\overline{\partial_z}=0\overline{\partial_z}=0$$

und

$$\tilde{\Phi}(x,z)\Delta_z=\tilde{\Phi}(x,z)\overline{\partial_z}\partial_z=\tilde{\Psi}(x,z)\partial_z=0$$

folgt

$$B=0.$$

Also gilt

$$\begin{aligned}
f(x) \;=\; & \sum_{\mu=1}^{\alpha-1}\Bigg\{\frac{(-1)^{\mu-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{(\alpha-\mu)!\overline{(y-x)}|y-x|^{2(\mu-1-\alpha)}}{2^{2(\mu-1)}(\mu-1)!\,(\alpha-1)!}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\mu-1}f(y) \\
& \qquad +\frac{(-1)^{\mu-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{(\alpha-\mu-1)!|y-x|^{2(\mu-\alpha)}}{2^{2\mu-1}(\mu-1)!\,(\alpha-1)!}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\mu-1}f(y)\Bigg\} \\
& +\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(y-x)}|y-x|^{-2}}{2^{2(\alpha-1)}(\alpha-1)!^2}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\alpha-1}f(y) \\
& -\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\log|y-x|^2}{2^{2\alpha-1}(\alpha-1)!^2}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\alpha-1}f(y) \\
& +\sum_{\mu=1}^{k}\Bigg\langle\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(y-x)}|y-x|^{2(\mu-1)}\Big\{\log|y-x|^2-\sum_{\rho=1}^{\mu-1}\frac{1}{\rho}-\sum_{\sigma=0}^{\mu-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+\mu-1)}(\mu-1)!\,(\alpha-1)!\,(\alpha+\mu-1)!}\,\mathrm{d}\vec{\sigma}_y\,\Delta^{\alpha+\mu-1}f(y) \\
& \qquad -\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|y-x|^{2\mu}\Big\{\log|y-x|^2-\sum_{\rho=1}^{\mu}\frac{1}{\rho}-\sum_{\sigma=0}^{\mu-1}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+\mu)-1}\mu!\,(\alpha-1)!\,(\alpha+\mu-1)!}\,\mathrm{d}\overline{\vec{\sigma}}_y\,\partial\Delta^{\alpha+\mu-1}f(y)\Bigg\rangle \\
& +\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{\overline{(z-x)}|z-x|^{2k}\Big\{\log|z-x|^2-\sum_{\rho=1}^{k}\frac{1}{\rho}-\sum_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)}k!(\alpha-1)!(\alpha+k)!}\,\mathrm{d}\vec{\sigma}_z\,\Delta^{\alpha+k}f(z) \\
& -\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{\partial G}\frac{|z-x|^{2(k+1)}\Big\{\log|z-x|^2-\sum_{\rho=1}^{k+1}\frac{1}{\rho}-\sum_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)+1}(k+1)!(\alpha-1)!(\alpha+k)!}\,\mathrm{d}\overline{\vec{\sigma}}_z\,\partial\Delta^{\alpha+k}f(z) \\
& +\frac{(-1)^{\alpha-1}}{\omega_{2\alpha}}\int\limits_{G}\frac{|z-x|^{2(k+1)}\Big\{\log|z-x|^2-\sum_{\rho=1}^{k+1}\frac{1}{\rho}-\sum_{\sigma=0}^{k}\frac{1}{\alpha+\sigma}\Big\}}{2^{2(\alpha+k)+1}(k+1)!(\alpha-1)!(\alpha+k)!}\,\Delta^{\alpha+k+1}f(z)\,\mathrm{d}V_z.
\end{aligned}$$

Also gilt (4.5). ❏

Satz 4.2.3 *Für $1<\alpha$ sind die schwach-singulären Kernfunktionen*

$$K_{k,m}(x) \;=\; \begin{cases}
\dfrac{|x|^{2(k-\alpha)-1}}{\omega_{2\alpha-1}2^{k-1}(k-1)!\,\prod_{j=1}^{k}\{2(j-\alpha)+1\}}, & m=2\alpha,\ 1\le k, \\[2ex]
\dfrac{(-1)^k(\alpha-k-1)!\,|x|^{2(k-\alpha)}}{\omega_{2\alpha}2^{2k-1}(k-1)!\,(\alpha-1)!}, & m=2\alpha-1,\ 1\le k\le\alpha-1, \\[2ex]
\dfrac{(-1)^{\alpha-1}\log|x|^2}{\omega_{2\alpha}2^{2\alpha-1}(\alpha-1)!^{\,2}}, & m=2\alpha-1,\ \ k=\alpha, \\[2ex]
\dfrac{(-1)^{\alpha-1}|x|^{2(k-\alpha)}\Big\{\log|x|^2-\sum_{\rho=1}^{k-\alpha}\frac{1}{\rho}-\sum_{\sigma=0}^{k-\alpha-1}\frac{1}{\alpha+\sigma}\Big\}}{\omega_{2\alpha}2^{2k-1}(k-\alpha)!\,(\alpha-1)!\,(k-1)!}, & m=2\alpha-1,\ \alpha+1\le k,
\end{cases} \qquad (4.13)$$

Fundamentallösungen in $\mathcal{A}$ *für* Δ^k *in* $\mathbb{R}^{m+1}$.

Beweis:

Aus dem Beweis von Satz 4.2.2 weiß man, dass für $2 \leq k$

$$\Delta K_{k,m}(x) = K_{k-1,m}(x)$$

gilt.
Außerdem gilt

$$K_{1,m}(x) = \frac{|x|^{1-m}}{\omega_{m+1}(1-m)}.$$

❑

Satz 4.2.4 *Für* $f \in L_1(G;\mathcal{A})$ *ist*

$$T_{k,m}f(x) := \int\limits_G K_{k,m}(y-x)f(y)\,\mathrm{d}V_y \tag{4.14}$$

eine spezielle Lösung für $\Delta^k w = f$ *in* G *mit* $\Delta^k w = 0$ *in* $\mathbb{R}^{m+1} \setminus \overline{G}$.

Beweis:

Aus dem Beweis von Satz 4.2.3 folgt

$$\Delta^{k-1}T_{k,m}f(x) = \frac{1}{\omega_{m+1}} \int\limits_G \frac{|y-x|^{1-m}}{1-m}\, f(y)\,\mathrm{d}V_y,$$

also gilt

$$\overline{\partial}\Delta^{k-1}T_{k,m}f(x) = -\int\limits_G E(y-x)f(y)\,\mathrm{d}V_y.$$

Die Aussage folgt dann aus Satz 3.1.3.