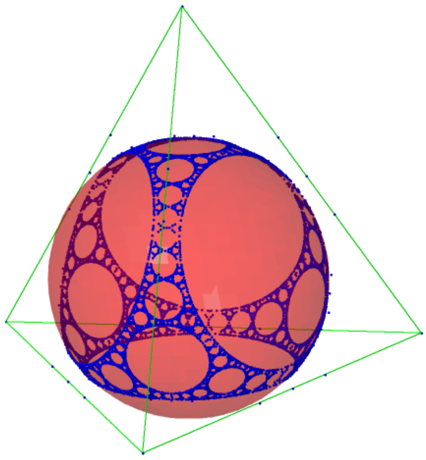
Freie Universität Berlin

Polyhedral Combinatorics of Coxeter Groups



Dissertation zur Erlangung des Grades
eines Doktors der Naturwissenschaften (Dr. rer. nat.)
am Fachbereich Mathematik und Informatik
der Freien Universität Berlin

von

Jean-Philippe Labbé

Berlin
Juli 2013

Supervisor and first reviewer:
Prof. Günter M. Ziegler

Second reviewer:
Prof. Volkmar Welker

Third reviewer:
Prof. Vic Reiner

Date of defense:
8 July 2013

Institut für Mathematik
Freie Universität Berlin

L'ÉLÈVE:
Les racines des mots sont-elles carrées?
LE PROFESSEUR:
Carrées ou cubiques. C'est selon.

La Leçon
Eugène Ionesco

Summary

This thesis presents recent developments concerning two open problems related to algebra and discrete geometry. In 1934, Harold S. M. Coxeter introduced Coxeter groups as abstractions of groups generated by reflections in a vector space [Cox34]. The present work lays out how a study of geometric and combinatorial properties of Coxeter groups contributed to the comprehension of the two open problems. Furthermore, this thesis presents some progress in proving the first problem, and an application of the approach introduced in this thesis for the second problem.

Open Problem (Lattice for infinite Coxeter groups, Dyer [Dye11])**.** *Is there, for each infinite Coxeter group, a complete ortholattice that contains the weak order?*

In Chapter 2, we study the asymptotical behaviour of roots of infinite Coxeter groups, which is part of a joint work with Christophe Hohlweg and Vivien Ripoll [HLR13]. In particular, we show that the directions of roots tend to the isotropic cone of the geometric representation of the root system. Moreover, using this framework, this thesis presents a proof that there is a complete ortholattice structure enclosing the weak order of infinite Coxeter groups of rank at most 3.

Open Problem (Existence of multiassociahedra, Jonsson [Jon05])**.** *Let k be an integer such that $k \geq 1$. Is there a polytope whose boundary complex corresponds to the simplicial complex of sets of diagonals of a convex polygon not containing $k+1$ mutually crossing diagonals?*

In Chapter 3 we introduce, for any finite Coxeter group and any nonnegative integer k, a spherical subword complex called *multi-cluster complex*. This family generalizes the concept of multitriangulations of type A and B to arbitrary finite Coxeter groups. For $k = 1$, this simplicial complex coincides with the finite cluster complex of the given type. We study combinatorial and geometric properties of multi-cluster complexes. In particular, we show that every spherical subword complex is the link of a face of a multi-cluster complex. This work was realized jointly with Cesar Ceballos and Christian Stump [CLS13]. Finally, this approach allows us to exhibit formulas counting the number of common vertices of permutahedra and generalized associahedra for arbitrary finite Coxeter groups and Coxeter elements.

Zusammenfassung

Diese Dissertation beschäftigt sich mit den neuesten Entwicklungen in zwei noch offenen Problemen der Algebra und diskreten Geometrie. 1934 führte Harold S. M. Coxeter die Coxeter-Gruppen als Abstraktion von Gruppen ein, die durch Spiegelungen in einen Vektorraum erzeugt werden. In dieser Dissertation verwenden wir geometrische und kombinatorische Eigenschaften der Coxeter-Gruppen um das Verständnis der beiden unten genannten Probleme zu verbessern. Genauer gesagt bietet diese Arbeit einen ersten Schritt zum Beweis des ersten Problems, und diskutiert einen möglichen Ansatz für die Lösung des zweiten Problems.

Offenes Problem (Verband Struktur für unendliche Coxeter-Gruppen, Dyer [Dye11])**.** *Gibt es einen vollständigen Orthoverband, der die schwache Ordnung der unendlichen Coxeter-Gruppen enthält?*

Im 2. Kapitel untersuchen wir die asymptotische Verhalten der Wurzeln von unendlichen Coxeter-Gruppen. Dies ist Teil einer gemeinsamen Arbeit mit Christophe Hohlweg und Vivien Ripoll [HLR13]. Insbesondere zeigen wir, dass die Richtungen der Wurzeln zu den isotropen Kegeln der geometrischen Darstellung des Wurzelsystems konvergieren. Darüber hinaus demonstrieren wir mit diesem Ansatz, dass ein vollständiger Orthoverband für die schwache Ordnung der unendlichen Coxeter-Gruppen von Rang höchstens 3 existiert.

Offenes Problem (Existenz der Multiassoziaeder, Jonsson [Jon05])**.** *Sei k eine ganze Zahl ≥ 1. Existiert ein Polytop, dessen Randkomplex dem Simplizialkomplex der Menge von Diagonalen eines konvexen Polygons entspricht, die keine $k+1$ sich paarweise schneidende Diagonalen enthalten?*

Im 3. Kapitel führen wir für jede Coxeter-Gruppe und jede nichtnegative ganze Zahl k einen sphärischen Teilwortkomplex ein, den sogenannten *Multi-Cluster Komplex.* Diese Familie verallgemeinert das Konzept von Multitriangulierungen der Typen A und B auf beliebige endliche Coxeter-Gruppen. Für $k = 1$ fällt dieser Simplizialkomplex mit dem endlichen Cluster Komplex des gegebenen Typs zusammen. Wir untersuchen kombinatorische und geometrische Eigenschaften von Multi-Cluster Komplexen. Insbesondere zeigen wir, dass jeder sphärische Teilwortkomplex der Link einer Seite in einem Multi-Cluster Komplex ist. Dieser Teil der Dissertation basiert auf einer gemeinsamen Arbeit mit Cesar Ceballos and Christian Stump [CLS13]. Abschließend ermöglicht es uns dieser Ansatz Formeln zu entwickeln, die die Anzahl der gemeinsamen Eckpunkte von Permutaedern und verallgemeinerten Assoziaedern für beliebige endliche Coxeter-Gruppen und Coxeter-Elemente berechnen.

Acknowledgements

I want to express my deep gratitude to my advisor Günter M. Ziegler, for his wise guidance, great availability and particularly for his trust. To my mentor, Tibor Szabó, thank you for your time and precious advices. To the reviewers of this thesis, Volkmar Welker and Vic Reiner, I am grateful for your interest in this work and your enthusiasm. I am thankful to my coauthors, Srečko Brlek, Michel Mendès France, Cesar Ceballos, Christian Stump, Christophe Hohlweg, Vivien Ripoll, Gilbert Labelle and Carsten Lange. It has been a great pleasure to work along with all of you during my doctoral studies. I am deeply thankful to my parents Madeleine and Jean for their unconditional support and for transmitting to me passion and curiosity. To my brother Sébastien, thank you for teaching me multiplication tables at six years old, for the innumerable memorable moments spent with you, and also with Renée and now Louis. To my travel partner Olaf, thank you for being there in good and bad moments. I am also thankful for all what your family did for me. I would like to thank the mathematicians Mario Doyon, Jean-Marie De Koninck, Jérémie Rostand, Claude Lévesque, Srečko Brlek, Christophe Hohlweg, François Bergeron, Christophe Reutenauer, Pierre Bouchard, Michel Mendès France, Vincent Pilaud, and Francisco Santos, who had a thriving influence through out my studies. I would like to thank all my collegues (past and present) from LaCIM, the Berlin Mathematical School, the research training group Methods for Discrete Structures, the workgroup Discrete Geometry with whom I spent a wonderful time, and especially to Giulia Battiston, Tommaso Benacchio, Giovanni De Gaetano, Stefan Keil, Kaie Kubjas, Barbara Jung, Emerson Leon, Julie Meißner, Jennifer Rasch, Annie Raymond, Christian Wald, Malik Younsi, Jérôme Fortier, Maxime Fortier Bourque, Quentin Rajon, Jérôme Tremblay, Franco Saliola, Marco Robado, Geoffrey Scott, and Salvatore Stella. I thank warmly my friends Christoph Eilers, Katharina Wermuth, Hanna Sartorius, Ellie Gregory, Fanny Berthier, Caroline De Freitas, Fannie Faivre, Fabien Vilain, and Jean-François Mosca. I would like to express my warmest thanks of all to Guillaume Lacombe, Vincent Ménard, Philippe Bolduc, Jean-Christophe Poulin, Lionel Campistron, Carolan Grégoire and Ana María Botero for filling my life with unforgettable moments spent with you. I would like to acknowledge the help of Elke Pose, Dorothea Kiefer, Nadja Wiesniewski, Tanja Fagel, Anja Bewersdorff, Dominique Schneider, Chris Seyffer and Mariusz Szmerlo, without whom my doctoral studies would not have gone so smoothly. This work would not have been possible without the financial support and the trust of the following organisations: Fonds Québécois de Recherche - Nature et Technologies, Berlin Mathematical School, Research Training Group - Methods for Discrete Structures.

Contents

Summary vii

Zusammenfassung ix

Acknowledgements xi

Notation xv

Introduction 1

1 Coxeter groups and discrete geometry 5
- 1.1 Basic notions on Coxeter groups . . . 5
- 1.2 Geometric representations of Coxeter groups . . . 7
- 1.3 Multi-triangulations . . . 12

2 A lattice for infinite Coxeter groups? 15
- 2.1 Geometries of infinite root systems . . . 16
- 2.2 Extended weak order of Coxeter groups . . . 18
- 2.3 Limit points of normalized roots and isotropic cone . . . 20
  - 2.3.1 Roots and normalized roots in ranks 2, 3, 4, and general setting . 21
  - 2.3.2 The limit points of normalized roots lie in the isotropic cone . . . 25
- 2.4 Complete ortholattice for rank ≤ 3 . . . 28
  - 2.4.1 The convex union is closed for rank ≤ 3 . . . 28
  - 2.4.2 The convex union is not closed for rank at least 4 . . . 34
- 2.5 Fractal description of the limit roots . . . 36

3 Subword complexes in discrete geometry 39
- 3.1 Subword complexes . . . 40
- 3.2 Cluster complexes . . . 41
- 3.3 Multi-cluster complexes . . . 42
- 3.4 General results on spherical subword complexes . . . 49
  - 3.4.1 Flips in spherical subword complexes . . . 49
  - 3.4.2 Isomorphic spherical subword complexes . . . 51
- 3.5 Proof of Theorem 3.8 . . . 52
- 3.6 Proof of Theorem 3.4 . . . 53
  - 3.6.1 Proof of condition (i) . . . 53
  - 3.6.2 Proof of condition (ii) . . . 56
- 3.7 Polytopality of spherical subword complexes . . . 56

3.7.1 Generalized associahedra . . . 57
3.7.2 Multi-associahedra of type A . . . 58
3.7.3 Multi-associahedra of type B . . . 59
3.7.4 Generalized multi-associahedra of rank 2 . . . 60
3.7.5 Generalized multi-associahedra . . . 60
3.8 Sorting words of the longest element and the SIN-property . . . 61
3.9 Common vertices of permutahedra and generalized associahedra . . . 64
3.9.1 Natural partial order and singletons . . . 64
3.9.2 Cylindric graphs of longest words and cuts . . . 66
3.9.3 Cylindric graphs of sorting words . . . 72
3.9.4 Formulas for the number of singletons . . . 74
3.9.5 Upper bounds . . . 76
3.9.6 Lower bounds . . . 82
3.9.7 Enumerative results . . . 85
3.10 Open problems . . . 85

A Some root systems of rank 3 & 4 **89**

B Subword complex vade-mecum **91**

Declaration of Authorship **93**

Index **95**

Bibliography **97**

Notation

$\mathbb{Z}, \mathbb{N}, \mathbb{R}$	the integers, the nonnegative integers, the real numbers
$\wp(X)$	the power set of X
(W, S)	a Coxeter system: a Coxeter group W with generators S
$W_{\langle s \rangle}$	the parabolic subgroup of W generated by $S \setminus \{s\}$
Γ	the Coxeter graph
V	a real vector space of dimension n
n	the dimension of V / cardinality of S
$B(\cdot, \cdot)$	the bilinear form of a geometric representation
$\mathbf{1}, e$	the identity transformation of V, the identity element of W
Δ	a simple system / basis of V
$s_\alpha(v)$	reflection of the vector $v \in V$ with respect to the vector α
Φ	the root system
$\widehat{\Phi}$	the normalized roots
$\Phi_{\langle s \rangle}$	the root system associated to $W_{\langle s \rangle}$
Φ^+	the positive roots
$\Phi_{\geq -1}$	the almost positive roots
(Φ, Δ)	a based root system
h	the Coxeter number of (finite irreducible) W
N	the cardinality of Φ^+ (when (W, S) is finite)
$\mathrm{inv}(w)$	the inversion set of $w \in W$
$\mathrm{dp}(\alpha)$	the depth of a root $\alpha \in \Phi^+$
c	a Coxeter element
$w_\circ$	the longest element of a finite Coxeter group W
ψ	the involution $\psi(s) = w_\circ^{-1} s w_\circ$
$\mathbf{w}$	a reduced expression for the element w
$\mathbf{w}(\mathbf{c})$	the c-sorting word for the element w
$\mathrm{rev}(\mathbf{w})$	the reverse of a word $\mathbf{w}$
$\mathrm{cone}(E)$	the polyhedral cone over the set of vectors $E \subset V$

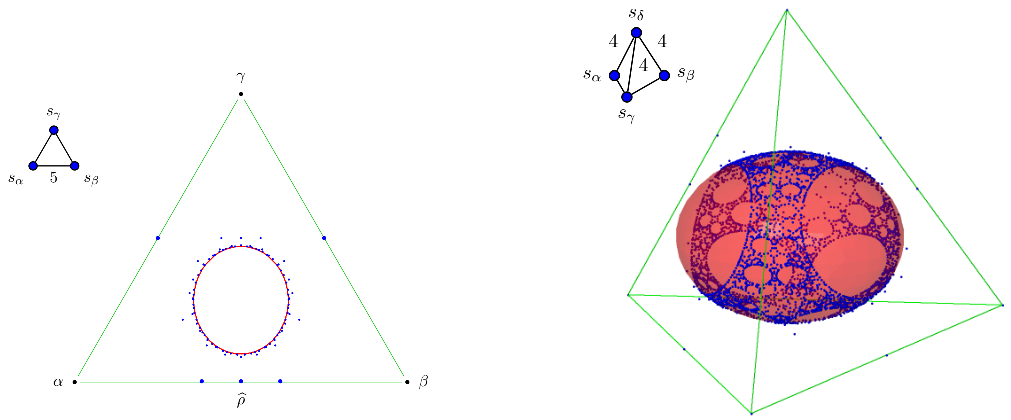
$\mathrm{conv}(E)$	the convex hull of the set of vectors $E \subset V$
$\mathrm{span}(E)$	the linear span over the set of vectors $E \subset V$
$\mathrm{int}(E)$	the interior of a set $E \subset V$
M	an alignment
$\overline{A}^{M}$	the closure of the set A with respect to the alignment M
A^c	the set $\Phi^+ \setminus A$
$\mathcal{B}(\Phi^+), \mathcal{C}(\Phi^+), \mathcal{S}(\Phi^+)$	biclosed sets, biconvex sets and separable sets of Φ^+
Δ_m	the boundary complex of the dual associahedron
$\Delta_{m,k}$	the simplicial complex of multitriangulations
$\Delta(Q, \pi)$	subword complex associated to the word Q and the element π
$\delta(Q)$	the Demazure product of Q
$Q_{\underset{s}{\circlearrowleft}}$	the rotation of the letter Q along the initial letter s
$\Vert_c$	the c-compatibility relation on almost positive roots
Lr_c	bijection between letters of $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ and almost positive roots
r_F	the root function associated to a facet F
$\Delta_c^k(W)$	the multi-cluster complex of type W
$\prec_{\mathbf{w}_\circ}$	the natural partial order on letters of $\mathbf{w}_\circ$
$\mathcal{Z}$	a bounded cylinder
$\overset{\circlearrowleft}{\mathbf{w}_\circ}$	a cyclic longest word
$\mathcal{Z}_{\mathbf{w}_\circ}$	the cylindric graph of $\mathbf{w}_\circ$
$\mathcal{L}_{\mathbf{w}_\circ}$	the loops of $\mathcal{Z}_{\mathbf{w}_\circ}$
T, $\mathcal{G}_T$	a tile of $\mathcal{Z}_{\mathbf{w}_\circ}$, the boundary graph of T
κ, κ^*	a cut of $\mathcal{Z}_{\mathbf{w}_\circ}$, its opposite cut
$\mathcal{T}_\kappa$	the support of a cut κ
$T^{\vee}_{\kappa,\kappa'}$	the split tile of two crossing cuts κ and κ'
$\mathcal{Z}_\diamond$	the cylindric graph of sorting words
$I_\kappa(T)$, $S_\kappa(T)$	inferior and superior poset of a tile T in the support of a cut κ

Conventions. For edges s—t labeled by ∞ in a Coxeter graph where the value $B(\alpha_s, \alpha_t)$ is strictly smaller than -1, we adopt Peter Abramenko & Kenneth S. Brown's notation [AB08, Section 10.3.3]. The corresponding edges are dashed and labeled with the value $B(\alpha_s, \alpha_t) < -1$, $\alpha_s, \alpha_t \in \Delta$. Words in the alphabet S are written as a sequence between brackets $(a_1, a_2, \ldots, a_r)$ and bold letters such as $\mathbf{w}$ denote them. Group elements are written as a concatenation of letters $a_1 a_2 \cdots a_r$ and normal script such as w denote them. Letters of words are considered with their embedding in the word: two letters representing the same generators are considered different, since they are at different positions.

Introduction

Coxeter groups are combinatorial abstractions of groups generated by reflections of a vector space. It is with no surprise that they arise in many fields of mathematics whenever symmetry is involved. For instance, in geometry, they are studied as the symmetry groups of regular polytopes; in algebra, they are studied in relation to root systems of semi-simple Lie algebras. Furthermore, by their exceptional definition, Coxeter groups possess a rich combinatorial structure. In this thesis, we study two seemingly unrelated topics—the weak order of infinite Coxeter groups and generalizations of triangulations of a polygon—where geometric and combinatorial properties of Coxeter groups play a prominent role. At first sight, the weak order of Coxeter groups—a combinatorial object defined from an abstract group—and triangulations—simplicial complexes on a plane—are not directly related. The former has become classical for algebraic combinatorialists for the study of Coxeter groups [BB05, Chapter 3], [Hum92, Section 5.9], whereas the latter are classical tools for discrete geometers and topologists; see the recent book [DLRS10]. The relation between the weak order and triangulations subtly lies in the notion of reflection order [Dye93]. On the one hand, reflection orders are important in the study of Bruhat order, Hecke algebras, and Kazhdan–Lusztig polynomials. In finite Coxeter groups, reflection orders correspond to maximal chains in the weak order, which correspond to reduced expressions of the longest element. However, in infinite Coxeter groups, there is no element of maximal length and maximal chains of the weak order are infinite. Therefore, reflection orders need a different description in the infinite case. On the other hand, reflection orders are related to sorting words of the longest element in finite Coxeter groups, which are related to finite cluster complexes in the theory of cluster algebras.

In Chapter 2, we study a conjecture of Matthew Dyer concerning an extension of the weak order of infinite Coxeter groups to a complete ortholattice, which would provide a geometric description for reflection orders. In studying this conjecture, a difficulty came up: We do not know much about the distribution of the roots of an infinite root system over the space. Taking up this challenge, the following pictures (Figures 1(a) and 1(b)) were obtained using the computer algebra system Sage [Sage]. They suggest that



(a) The first 100 normalized roots, around the isotropic cone Q, for the rank 3 Coxeter group with the depicted graph.

(b) The first 3890 normalized roots, around the isotropic cone Q, for the rank 4 Coxeter group with the depicted graph.

FIGURE 1: Root systems for two infinite Coxeter groups computed via the computer algebra system Sage

roots have a very interesting asymptotical behaviour. This motivated the study of the asymptotical behaviour of roots of infinite Coxeter groups, see [HLR13] and its sequel [DHR13]. Let us explain what we see in these pictures. First, we fix a geometric action of an infinite Coxeter group W on a finite dimensional real vector space V, which implies the data of a symmetric bilinear form B, and a simple system Δ, which is a basis for V. In order to visualize the roots, we normalize the positive cone by looking at the affine hyperplane V_1 spanned by the points corresponding to the simple roots: Figures 1(a) and (b) live in V_1 and the triangle resp. tetrahedron is the convex hull of the simple roots. The dots are the intersection of V_1 with the rays spanned by the roots. The closed curve resp. the closed surface depicts the isotropic cone $Q = \{v \in V : B(v,v) = 0\}$ of the quadratic form associated to B. We see on the pictures that the normalized roots tend to converge to points on Q, and that the set of limit points has an interesting structure: it seems to be equal to Q in Fig. 1(a), whereas in Fig. 1(b) it is similar to an Apollonian gasket. Using this framework, this thesis contains, in Section 2.4, a proof that the weak order of Coxeter groups of rank at most 3 can be extended to a complete ortholattice where the join and meet operations are defined geometrically. As we will see, in Chapter 2, geometric objects play an important role in understanding the combinatorial structure of Coxeter groups.

In Chapter 3, this thesis introduces a twofold generalization of the notion of triangulations of a convex polygon, the *multi-cluster complexes*, which subsumes naturally the following two generalizations. On the one hand, in cluster algebra theory, triangulations are generalized by *cluster complexes* of finite type. They were introduced by Sergey Fomin and Andrei Zelevinsky to encode exchange graphs of cluster algebras [FZ03].

Nathan Reading then showed that the definition of cluster complexes can be extended to all finite Coxeter groups [Rea07a, Rea07b]. The cluster complex of type A is isomorphic to the simplicial complex of sets of diagonals of a convex polygon which are mutually noncrossing (the boundary complex of the dual associahedron). On the other hand, in convex geometry, k-triangulations (or multitriangulations), with $k \geq 1$, are formed by maximal sets of diagonals of a convex polygon which do not contain $k+1$ diagonals that mutually cross. When $k = 1$, we get back the original notion of triangulations. In Section 3.9, we use this framework to derive formulas that enumerate the common vertices of W-permutahedra and c-generalized associahedra, see [HLT11], [MHPS12, Chapter 8]. To obtain this natural generalization, we present a new combinatorial description of cluster complexes using *subword complexes*. These were introduced by Allen Knutson and Ezra Miller, first in type A to study the combinatorics of determinantal ideals and Schubert polynomials [KM05], and then for all Coxeter groups in [KM04]. We provide, for any finite Coxeter group W and any Coxeter element $c \in W$, a subword complex which is isomorphic to the c-cluster complex of the corresponding type, and we thus obtain an explicit type-free characterization of c-clusters. The present approach allows us to define a new family of simplicial complexes by introducing an additional parameter k, such that one obtains c-cluster complexes for $k = 1$. In type A, this simplicial complex turns out to be isomorphic to the simplicial complex of multitriangulations of a convex polygon which was described by Christian Stump in [Stu11], and, in a similar manner, by Vincent Pilaud and Michel Pocchiola in the framework of sorting networks [PP12]. In type B, we obtain that this simplicial complex is isomorphic to the simplicial complex of centrally symmetric multitriangulations of a regular convex polygon. Multi-cluster complexes are different from *generalized cluster complexes* as defined by Sergey Fomin and Nathan Reading [FR05]. In the generalized cluster complex, the vertices are given by the simple negative roots together with several distinguished copies of the positive roots, while the vertices of the multi-cluster complex correspond to the positive roots together with several distinguished copies of the simple negative roots. Multi-cluster complexes turn out to be intimately related to Auslander–Reiten quivers and repetition quivers [GR97]. In particular, the Auslander–Reiten translate on facets of multi-cluster complexes in types A and B corresponds to cyclic rotation of (centrally symmetric) multitriangulations. Furthermore, multi-cluster complexes uniformize questions about multitriangulations, subword complexes, and cluster complexes. One important example concerns the open problem of realizing the simplicial complexes of (centrally symmetric) multitriangulations and spherical subword complexes as boundary complexes of convex polytopes. In Chapter 3, we will see that the combinatorial structure of Coxeter groups plays an unifying role in the study of these generalized triangulations.

Chapter 1

Coxeter groups and discrete geometry

This chapter introduces the concepts used in this thesis. Section 1.1 deals with classic notions of the theory of Coxeter groups. Section 1.2 describes geometric representations of Coxeter groups with examples. Finally, Section 1.3 recalls the notion of multitriangulations.

1.1 Basic notions on Coxeter groups

Throughout the text, (W,S) denotes a *Coxeter system*. The set $S \subseteq W$ is a set of generators for the group W. The generators in S are subject only to relations of the form $(st)^{m_{s,t}} = e$, where $m_{s,t} \in \{1,2,\ldots,\infty\}$ for each pair of generators $s,t \in S$, with $m_{s,s} = 1$ and $m_{s,t} \geq 2$ for $s \neq t$. We write $m_{s,t} = \infty$ if the product st has infinite order in W. The latter relations are called *braid relations* of order $m_{s,t}$. When $m_{s,t} = 2$, the generators s and t commute in W. The cardinality $|S|$ of S is called the *rank* of (W,S). In this thesis, we will always assume that Coxeter systems have finite rank n. Chapter 2 deals with Coxeter systems where the group W is infinite and Chapter 3 with finite ones. The *Coxeter graph* Γ of a Coxeter system (W,S) is the graph with vertices labeled by elements of S, edges between two vertices s and t whenever $m_{s,t} \geq 3$, and when $m_{s,t} > 3$, the edge between s and t comes with a label $m_{s,t}$. A Coxeter system is *irreducible* if the Coxeter graph Γ is connected. Denote by c a *Coxeter element*, i.e., the product of the generators in S in some order. For finite Coxeter systems, consider a bipartition of the set $S = S_- \sqcup S_+$ such that any two generators in S_ε commute (this is possible since the graph of a finite Coxeter group is a tree), the Coxeter element $c^* = c_-c_+$, where $c_\varepsilon = \prod_{s \in S_\varepsilon} s$, is called a *bipartite Coxeter element*. Coxeter elements of W are in bijection with acyclic orientations of the Coxeter graph Γ, see [Shi97]. A noncommuting pair $s, t \in S$ has the orientation $s \longrightarrow t$ if and only if *s comes before t in c*, i.e., s comes before t in any reduced expression for c. For finite irreducible Coxeter groups W, the smallest integer h for which $c^h = e \in W$ is called the *Coxeter number* of W. The Coxeter number does not depend on the choice of Coxeter element c, since Coxeter elements belong to the same conjugacy class. The *length function* on W is given by $\ell(w) = \min\{r : w = s_1 \cdots s_r,\ s_i \in S\}$. An expression for w of minimal length is called *reduced*. In finite Coxeter groups, the unique *longest element* is denoted by $w_\circ$, and its length is given by $\ell(w_\circ) = N := nh/2$. For reading convenience, a reduced expression $\mathbf{w}_\circ = (w_1, \cdots, w_N)$ of $w_\circ$ will be called a *longest word*. Let $\psi : S \to S$ be the involution given by $\psi(s) = w_\circ^{-1} s w_\circ$. Using the fact that $w_\circ$ is the longest element, ψ is verified to be an automorphism of the Coxeter graph. It is known that ψ is the identity on S if and only if $w_\circ = -\mathbf{1}$. For more details about the involution ψ, see [BB05, Exercise 10, Chapter 4]). Two reduced expressions (or words) coincide *up to commutations* if they can be obtained from each other by a sequence of braid relations of order 2. A generator $s \in S$ is called *initial* or *final* in a reduced expression $\mathbf{w}$ if $\ell(sw) < \ell(w)$ or $\ell(ws) < \ell(w)$, respectively. Let $T = \{wsw^{-1} : w \in W \text{ and } s \in S\}$ be the set of *reflexions* of W. The (left) *inversion set* $\text{inv}(w)$ of an element $w = w_1 w_2 \cdots w_r$, with $\ell(w) = r$ and $w_i \in S$, of W is

$$\{w_1, w_1 w_2 w_1^{-1}, \ldots, (w_1 \cdots w_{k-1}) w_k (w_1 \cdots w_{k-1})^{-1}\}.$$

The inversion set $\text{inv}(w)$ does not depend on the choice of reduced expression for w, see, for instance, [Hum92, Chapter 5.6, Exercise 1] for an equivalent formulation. Inversion sets have the following important property (see [BB05, Chapter 1]):

$$\ell(w) < \ell(ws) \Longleftrightarrow \text{inv}(w) \subset \text{inv}(ws).$$

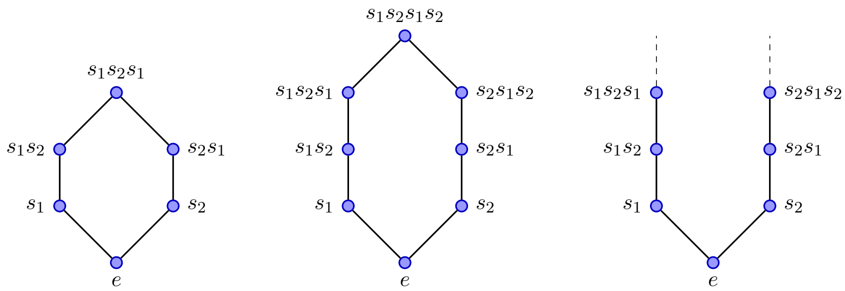
The *right weak order* $(W, \leq)$ is the poset (i.e. partially ordered set) whose cover relations are $w \leq ws$, where $\ell(w) < \ell(ws)$, $w \in W$, and $s \in S$. Alternatively, this poset is isomorphic to the (left) inversion sets ordered by inclusion where the length $\ell(w)$ of an element w is viewed as the cardinality of $\text{inv}(w)$, see [BB05, Proposition 3.1.3]. This poset is a complete meet-semilattice [BB05, Theorem 3.2.1] which is graded by the length function and with finitely many elements of fixed length.

Definition 1.1 ([BB05, Section 3.2])**.** A lattice L with bottom element $\hat{0}$ and top element $\hat{1}$ is called an *ortholattice* if there exists a map $x \mapsto x^\perp$ on L such that the following properties hold:

(i) $x \vee x^\perp = \hat{1}$, $x \wedge x^\perp = \hat{0}$, for all $x \in L$,
(ii) $x \leq y \Rightarrow x^\perp \geq y^\perp$, for all $x, y \in L$,
(iii) $x^{\perp\perp} = x$, for all $x \in L$.

For finite Coxeter groups, since there is a longest element, the join of any set of elements of W always exists. Therefore, when W is finite, the weak order forms a complete lattice and the translation $w \mapsto ww_\circ$ gives the structure of an ortholattice to the poset $(W, \leq)$.

Example 1.2. The Hasse diagram of the weak order of the Coxeter groups A_2, B_2 and $I_2(\infty)$ with generating set $S = \{s_1, s_2\}$ are depicted in Fig. 1.1. In type $I_2(\infty)$, the join of s_1 and s_2 does not exist.



(a) A_2 **(b)** B_2 **(c)** $I_2(\infty)$

FIGURE 1.1: Hasse diagrams of the Coxeter groups A_2, B_2 and $I_2(\infty)$.

In Chapter 2, we will see how the weak order poset can be defined on inversion sets geometrically, without the use of the weak order. Furthermore, in the case of infinite Coxeter groups, we investigate how the meet-semilattice could be extended to a complete ortholattice. We refer the reader to the books of Nicolas Bourbaki [Bou68], James E. Humphreys [Hum92], and Anders Björner & Francesco Brenti [BB05] for further definitions and for detailed introductions to Coxeter groups.

1.2 Geometric representations of Coxeter groups

Coxeter groups are modeled to be the abstract combinatorial counterparts of reflection groups, i.e., groups generated by reflections. Any finite Coxeter group can be represented geometrically as a finite reflection group. This property still holds for infinite Coxeter groups, for some adapted definition of reflection that we recall now. The *classical geometric representation* is defined as follows, see [Hum92, Section 5.3-5.4]. Consider a real vector space V of dimension n, with basis $\Delta = \{\alpha_s : s \in S\}$ and let B be the symmetric bilinear form defined by:

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{if } m_{s,t} < \infty, \\ -1 & \text{if } m_{s,t} = \infty. \end{cases}$$

For $\alpha \in V$ such that $B(\alpha, \alpha) \neq 0$, denote by s_α the map

$$s_\alpha(v) = v - 2\frac{B(\alpha, v)}{B(\alpha, \alpha)}\,\alpha, \quad \text{for any } v \in V. \tag{1.1}$$

Denote by $H_\alpha := \{v \in V : B(\alpha, v) = 0\}$ the orthogonal space of the line $\mathbb{R}\alpha$ with respect to the form B. Since $B(\alpha, \alpha) \neq 0$, we have $H_\alpha \oplus \mathbb{R}\alpha = V$. It is straightforward to check that s_α fixes H_α pointwise, that $s_\alpha(\alpha) = -\alpha$, and that s_α also preserves the form B, so it lies in the associated orthogonal group $\mathrm{O}_B(V)$. We call s_α the *B-reflection* associated to α (or simply reflection whenever B is clear). When B is a scalar product, this is of course the usual formula for a Euclidean reflection. Then any element s of S acts on V as the B-reflection associated to α_s (as defined in Equation (1.1)), i.e., $s(v) = v - 2B(\alpha_s, v)\ \alpha_s$ for $v \in V$. This action induces a faithful action of W on V, which preserves the form B; thus we denote by the same letter an element of W and its associated element of $\mathrm{O}_B(V)$.

The root system of W is a way to encode the reflections T of the Coxeter group, i.e., the conjugates of elements of S, which are called *simple reflections*. The elements of $\Delta = \{\alpha_s : s \in S\}$ are called *simple roots* of W, and the *root system* Φ of W is defined to be the orbit of Δ under the action of W. By construction, any root $\rho \in \Phi$ gives rise to the reflection s_ρ of W, which is conjugate to some $s_\alpha \in S$. Thus reflections in T correspond to pairs of opposed roots of Φ.

A *reflection subgroup* of W is a subgroup of W generated by reflections; so it can be built from a subset of Φ. It turns out that any such reflection subgroup is again a Coxeter group, with some canonical generators [Deo89],[Dye90]. So it is natural to desire to apply results valid for W to a reflection subgroup simply by restriction. A major drawback of the classical geometric representation we described above is that it is not "functorial" with respect to the reflection subgroups. It is possible that the representation of some reflection subgroups W' of W, induced (by restriction) by the geometric representation of W, is not the same as the geometric representation of W' as a Coxeter group; see Example 1.3 below.

Example 1.3 (Reflection subgroups of rank 2)**.** Let us consider the Coxeter group of rank 3 with $S = \{s_\alpha, s_\beta, s_\gamma\}$ and $m_{s_\alpha, s_\beta} = 5$, $m_{s_\beta, s_\gamma} = m_{s_\alpha, s_\gamma} = 3$ (whose Coxeter diagram is on Fig. 1(a) on page 2). Take the root $\rho = s_\alpha s_\beta(\alpha) = s_\beta s_\alpha(\beta)$, so that s_ρ corresponds to the longest element in the subgroup $\langle s_\alpha, s_\beta\rangle$: $s_\rho = s_\alpha s_\beta s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta s_\alpha s_\beta$. We compute $\rho = \frac{1+\sqrt{5}}{2}(\alpha + \beta)$. Consider the reflection subgroup W' generated by s_γ and s_ρ. The product $s_\gamma s_\rho$ has infinite order, so W' is an infinite dihedral group, with generators s_γ and s_ρ. But, if B denotes the bilinear form associated to the Coxeter group W, we get: $B(\gamma, \rho) = -\frac{1+\sqrt{5}}{2} \neq -1$. So, the restriction to W' of the geometric representation of W does not correspond to the classical geometric representation of W'

as an infinite dihedral group. In Example 2.21 we give a geometric interpretation of this fact, which is visible in Fig. 1(a).

To solve this problem, we relax the requirements on the bilinear form B used to represent the group W: we allow the values of some $B(\alpha, \beta)$ to be any real numbers less than or equal to -1 (when the associated product of reflections $s_\alpha s_\beta$ has infinite order). The notion of a based root system is better adapted here. It is used for instance in [How96], [Kra09], and [BD10].

Definition 1.4. Let V be a real vector space, equipped with a bilinear form B. Consider a finite subset Δ of V such that

(i) Δ is positively independent[1]: If $\sum_{\alpha\in\Delta} \lambda_\alpha \alpha = 0$ with all $\lambda_\alpha \geq 0$, then all $\lambda_\alpha = 0$,
(ii) for all $\alpha, \beta \in \Delta$, with $\alpha \neq \beta$, $B(\alpha, \beta) \in (-\infty, -1] \cup \left\{-\cos\left(\frac{\pi}{k}\right), k \in \mathbb{Z}_{\geq 2}\right\}$,
(iii) for all $\alpha \in \Delta$, $B(\alpha, \alpha) = 1$.

Such a set Δ is called a *simple system*. Denote by $S := \{s_\alpha : \alpha \in \Delta\}$ the set of B-reflections associated to elements in Δ (see Equation (1.1)). Let W be the subgroup of $\mathrm{O}_B(V)$ generated by S, and Φ be the orbit of Δ under the action of W. The pair (Φ, Δ) is a *based root system* in (V, B); its *rank* is the cardinality of Δ, i.e., the cardinality of S. We call the pair (V, B) a *geometric representation*[2] of W.

Remark 1.5.
- Condition (ii) is natural to ensure that subrepresentations are again geometric representation in the sense of this new definition. We saw in Example 1.3 that this does not work for the usual definition.
- In Condition (i), the relaxation is more subtle, but also necessary if we want a nice functorial behaviour on the subrepresentations. For instance, for some Coxeter group W there exists a reflection subgroup (as a Coxeter group) of rank strictly higher than that of W; see [HLR13, Example 5.1].
- Even if Δ is not anymore required to be a basis, the condition that it is positively independent is *necessary* to keep the usual properties of root systems, in particular the distinction between the set of *positive* roots and the set of *negative* roots.

This generalization of root system enjoys the following expected properties. See, for instance, [BD10, Kra09]).

- (W, S) is a Coxeter system, where the order of $s_\alpha s_\beta$ is k whenever $B(\alpha, \beta) = -\cos(\frac{\pi}{k})$, and the order of $s_\alpha s_\beta$ is ∞ if $B(\alpha, \beta) \leq -1$.

[1] Geometrically, this means we require that 0 does not lie in the affine hull of the points of Δ, i.e. the cone spanned by Δ is pointed.

[2] The triplet (V, Δ, B) is sometimes called a Coxeter datum in the literature, see, for instance, [Fu12b, Fu12a].

- The convex *cone* $\mathrm{cone}(\Delta)$ consisting of all positive linear combinations of elements of Δ allows to define the set of *positive roots* $\Phi^+ := \Phi \cap \mathrm{cone}(\Delta)$, and then $\Phi = \Phi^+ \sqcup (-\Phi^+)$ and $\mathbb{R}\rho \cap \Phi = \{\rho, -\rho\}$, for $\rho \in \Phi$.

If all $m_{s,t}$ are finite, then the only possible representation (supposing that Δ is a basis) is the classical one. In particular, when the form B is positive definite, then Φ is a finite root system and contains no more information than its associated finite Coxeter group.

Definition 1.6. A based root system (Φ, Δ) is an *affine based root system* when the form B is positive semidefinite, but not definite.

Traditionally, the Coxeter group itself is said to be affine if the root system of its classical geometric representation is affine.

Example 1.7 (Irreducible affine root systems)**.** The infinite dihedral group $I_2(\infty)$ has more than one geometric representation. If Φ is an infinite root system of rank 2, with simple roots α, β, then $B(\alpha, \beta) \leq -1$, and Φ is affine if and only if $B(\alpha, \beta) = -1$, i.e., when Φ corresponds to the classical geometric representation of W. We give a geometric description of these two cases in Fig. 2.3. However, if W is irreducible of rank ≥ 3, then Φ is affine if and only if W is affine, because there is no label ∞ in an irreducible affine Coxeter graph of rank ≥ 3.

All the desired properties of the root system and of positive and negative roots still hold for a based root system. In particular, the following statements are still valid in this new framework.

Proposition 1.8. *Let (Φ, Δ) be a based root system in (V, B), with associated Coxeter system (W, S).*

(i) *The set $\{B(\alpha, \rho) : \alpha \in \Delta,\ \rho \in \Phi^+ \text{ and } |B(\alpha, \rho)| < 1\}$ is finite.*
(ii) *Denote by Q the isotropic cone[3]:*

$$Q := \{v \in V : q(v) = 0\}, \text{ where } q(v) = B(v, v)$$

Let $\rho_1 \neq \rho_2$ be two roots in Φ^+. Denote by W' the dihedral reflection subgroup *of W generated by the two reflections s_{ρ_1} and s_{ρ_2}, and*

$$\Phi' := \{\rho \in \Phi : s_\rho \in W'\} .$$

Then there exists $\Delta' \subseteq \Phi^+ \cap \Phi'$ of cardinality 2 such that (Φ', Δ') is a based root system of rank 2, with associated Coxeter group W'. Moreover:

[3] In Section 1.3 and Chapter 3, the letter Q is used to denote a word in S. Since both notions will not be used simultaneously, no confusion should be possible.

(a) *Φ' is infinite if and only if the plane* $\mathrm{span}(\rho_1, \rho_2)$ *intersects* $Q \setminus \{0\}$*, if and only if* $|B(\rho_1, \rho_2)| \geq 1$*,*

(b) *Φ' is affine if and only if* $\mathrm{span}(\rho_1, \rho_2) \cap Q$ *is a line, if and only if* $B(\rho_1, \rho_2) = \pm 1$*,*

(c) *when Φ' is infinite,* $\Delta' = \{\rho_1, \rho_2\}$ *if and only if* $B(\rho_1, \rho_2) \leq -1$*.*

Proof. The nontrivial parts of the proofs of these statements in the context of a based root system are word for word the same as the proofs in the case of the root system of the classical geometric representation. The reader may find these for instance in [BB05, Section 4.5]. The last statement (ii)(c) is a consequence of [Dye90, Theorem 4.4]; see also [Fu12a, Theorem 1.8 (ii)]. □

Let (W, S) be a Coxeter group. Fix a matrix $A = (a_{s,t})_{s,t \in S}$ such that

$$\begin{cases} a_{s,t} = -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{if } m_{s,t} < \infty, \\ a_{s,t} \leq -1 & \text{if } m_{s,t} = \infty. \end{cases} \tag{1.2}$$

We associate to the matrix A a *canonical geometric representation* (V_A, B_A) of W as follows.

- V_A is a real vector space with basis $\Delta_A = \{\alpha_s : s \in S\}$ and B_A is the symmetric bilinear form defined by $B_A(\alpha_s, \alpha_t) = a_{s,t}$ for $s, t \in S$.
- Any element s of S acts on V as the B-reflection associated to α_s, i.e., $s(v) = v - 2B(\alpha_s, v)\, \alpha_s$ for $v \in V_A$.

Since Δ_A satisfies the requirement of Definition 1.4, W acts faithfully on V_A as the subgroup of $\mathrm{O}_{B_A}(V_A)$ spanned by the B-reflections associated to the α_s. Moreover, (Φ_A, Δ_A) is a based root system of (V_A, B_A), where Φ_A is the W-orbit of Δ_A. Giving a matrix A, as we did, is equivalent to fixing the values in Conditions (ii) and (iii) in Definition 1.4.

Example 1.9 (Continuation of Example 1.3)**.** In the case of Example 1.3, the restriction of the classical geometric representation of W to the reflection subgroup W' generated by s_γ and s_ρ gives the geometric representation that is associated to the canonical geometric representation of W' given by the matrix

$$A = \begin{pmatrix} 1 & -\frac{1+\sqrt{5}}{2} \\ -\frac{1+\sqrt{5}}{2} & 1 \end{pmatrix}.$$

Remark 1.10. By construction, the set Δ_A is a basis in the based root system (Φ_A, Δ_A) associated to the canonical geometric representation (V_A, B_A) defined above. This setting will actually be the one used throughout the text: We assume that the set of simple roots is a basis for the vector space. See [HLR13, Section 5] for a discussion of the case when Δ is not a basis.

1.3 Multi-triangulations

Definition 1.11 ([Tam51] [Sta63])**.** Let Δ_m be the simplicial complex whose vertices correspond to the diagonals of a convex m-gon and faces correspond to subsets of non-crossing diagonals. Its facets correspond to *triangulations* (i.e., maximal subsets of diagonals which are mutually noncrossing).

This simplicial complex is the boundary complex of the *dual associahedron* [Hai84, Lee89, Rea06, HL07]. We refer to the recent book [MHPS12] for a detailed treatment of the history of associahedra. The complex Δ_m can be generalized using a positive integer k with $2k+1 \leq m$: Define a *$(k+1)$-crossing* to be a set of $k+1$ diagonals which are pairwise crossing. A diagonal is called *k-relevant* if it is contained in some $(k+1)$-crossing, that is, if there are at least k vertices of the m-gon on each side of the diagonal.

Definition 1.12 (Jonsson [Jon05, Section 1])**.** The *simplicial complex of multitriangulations*, $\Delta_{m,k}$ is the simplicial complex of sets of k-relevant diagonals of a convex m-gon that do not contain a $(k+1)$-crossing. Its facets are given by *k-triangulations* of the convex m-gon (i.e., maximal subsets of diagonals which do not contain a $(k+1)$-crossing), without considering k-irrelevant diagonals.

The reason for restricting the set of diagonals is that including all non k-relevant diagonals would yield the join of $\Delta_{m,k}$ and an mk-simplex. Multitriangulations have been studied by several authors, see e.g. [CP92, Nak00, DKM02, Jon05, Kra06, JW07, Rub11, Stu11]. An interesting recent treatment of k-triangulations using complexes of star polygons can be found in [PS09]. In [Stu11], the following description of $\Delta_{m,k}$ is exhibited: Let $\mathcal{S}_{n+1}$ be the symmetric group generated by the n simple transpositions $s_i = (i\ i+1)$ for $1 \leq i \leq n$, where $n = m - 2k - 1$. The k-relevant diagonals of a convex m-gon are in bijection with (positions of) letters in the word

$$Q = (\underbrace{s_n, \ldots, s_1, \quad \cdots \quad s_n, \ldots, s_1,}_{k \text{ times } s_n,\ldots,s_1} \quad s_n, \ldots, s_1, \quad s_n, \ldots, s_2, \quad \cdots \quad s_n, s_{n-1}, \quad s_n)$$

of length $kn + \binom{n+1}{2} = \binom{m}{2} - mk$. If the vertices of the m-gon are cyclically labelled by the integers from 1 to m, the bijection sends the ith letter of Q to the ith k-relevant diagonal in lexicographic order. Under this bijection, a collection of diagonals forms a facet of $\Delta_{m,k}$ if and only if the complement of the corresponding subword in Q forms a reduced expression for the permutation $[n+1, \ldots, 2, 1] \in \mathcal{S}_{n+1}$. A similar approach which admits various possibilities for the word Q was described in [PP12] in the context of sorting networks.

Example 1.13. For $m = 5$ and $k = 1$, we get $Q = (q_1, q_2, q_3, q_4, q_5) = (s_2, s_1, s_2, s_1, s_2)$. By cyclically labeling the vertices of the pentagon with the integers $\{1, \ldots, 5\}$, the bijection sends the (position of the) letter q_i to the ith entry of the list of ordered diagonals $[1, 3], [1, 4], [2, 4], [2, 5], [3, 5]$. On one hand, two cyclically consecutive diagonals in the list form a triangulation of the pentagon. On the other hand, the complement of two cyclically consecutive letters of Q form a reduced expression for $[3, 2, 1] = s_1s_2s_1 = s_2s_1s_2 \in \mathcal{S}_3$; see Fig. 1.2.

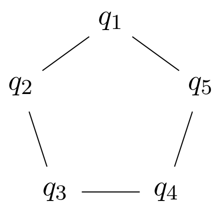


FIGURE 1.2: The simplicial complex Δ_5 of triangulations of the pentagon.

Example 1.14. Label the vertices of a convex 10-gon from 1 to 10 in clockwise direction. The set of relevant diagonals $\{[1, 8], [2, 5], [2, 8], [3, 6], [3, 7], [3, 8], [3, 10], [5, 8], [5, 10], [7, 10]\}$ forms a 2-triangulation of the 10-gon, see Fig. 1.3. This 2-triangulation is also centrally symmetric.

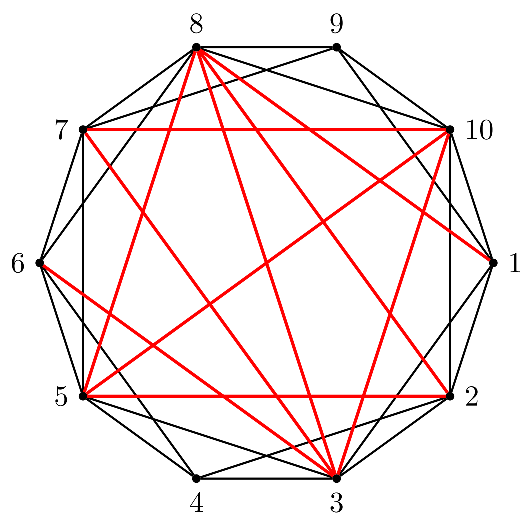


FIGURE 1.3: A 2-triangulation of the decagon. Relevant diagonals separate the remaining 8 vertices into two sets containing at least 2 vertices.

Chapter 2

A lattice for infinite Coxeter groups?

In this chapter, we investigate how the meet-semilattice structure of the weak order of *infinite* Coxeter groups could be extended to a complete ortholattice. This extension uses the notions of closure operators on biclosed, biconvex and separable sets. This is motivated by a conjecture of Matthew Dyer from [Dye11, Section 2], which states that biclosed sets ordered by inclusion form a complete ortholattice extending the weak order. This lattice would extend the complete lattice of the weak order in the finite case to the case of infinite Coxeter groups. If this conjecture is true, it would allow the use of new geometric computational methods in infinite Coxeter groups. This conjecture is motivated by the application of reflection orders to study questions related to Iwahori–Hecke algebras. For that, a substitute for the reduced expressions of the longest element in the infinite case is required. It would be possible to define this substitute using biclosed sets of an infinite Coxeter group. Also, in relation to cluster algebras and related structures, reflection orders in Coxeter groups recently received attention: This will be explored in more detail in the next chapter via the notion of *sorting words*.

This chapter is divided as follows. First, Section 2.1 presents different extensions of *convex geometries* to infinite root systems. Then, we discuss the extended weak order of Coxeter groups and relevant conjectures in Section 2.2. This motivated a study of the asymptotical behaviour of roots of infinite Coxeter groups (joint work with Christophe Hohlweg and Vivien Ripoll, see [HLR13]), from which we introduce the relevant results for our purpose in Section 2.3. Then, initial results concerning the conjectures and their limits in attempting to prove the conjectures in full generality are presented in Section 2.4. Finally, we conclude the chapter by a short discussion about the fractal behaviour of limit points of roots in Section 2.5.

2.1 Geometries of infinite root systems

In this section, we describe different generalizations of the concept of convex geometry.

Definition 2.1 (Edelman–Jamison [EJ85, Section 2])**.** Let X be a set and let M be a collection of subsets of X with the properties:

(i) $\varnothing \in M$ and $X \in M$,

(ii) $A_i \in M$ for $i \in I$ implies $\bigcap_{i\in I} A_i \in M$.

The collection M is called an *alignment* of X.

Definition 2.2 (Edelman–Jamison [EJ85, Section 2])**.** Let A be a subset of an alignment M. The *closure* $\overline{A}^M$ of A, with respect to the alignment M, is

$$\overline{A}^M := \bigcap_{\{B\in M \,:\, A\subseteq B\}} B.$$

Definition 2.3 (Edelman–Jamison [EJ85, Section 2])**.** An alignment M of a set X is a *convex geometry*, if given any $K \in M$ and two distinct element p and q in $X \setminus M$, then $q \in \overline{K \cup \{p\}}^M$ implies that $p \notin \overline{K \cup \{q\}}^M$.

Usually, it is assumed that X is finite. For more details about convex geometries where X is a finite root system, we refer to the article [Pil06]. We will be interested in the case when the set X is an infinite root system, in the sense of [Hum92, Chapter 5]. Thus, we will loosen the definition of convex geometry. Denote by A^c the set $\Phi^+ \setminus A$.

Definition 2.4 (Bourbaki [Bou68, Chap. 6, Sect. 7, Def. 4])**.** Let (Φ, Δ) be a based root system in (V, B) of a Coxeter group W. A subset A of Φ^+ is *closed* if given any $\alpha, \beta \in A$ and $\gamma \in \Phi^+$ with $\gamma = a\alpha + b\beta$, where $a, b \in \mathbb{R}^+$, we have $\gamma \in A$. A subset A of Φ^+ is *biclosed* if A and A^c are closed. Denote by $\mathcal{B}(\Phi^+)$ the collection of biclosed sets of Φ^+.

Remark 2.5. The original notion of closed sets dates back to Nicolas Bourbaki [Bou68, Chap. 6, Sect. 7, Def. 4], where it is assumed that that W is a finite Weyl group and $a, b \in \mathbb{Z}^+$. To work with arbitrary finite Coxeter groups, this condition needs to be changed to $\mathbb{R}^+$, for the results about biclosed sets to hold, see the discussion after Definition 4.1 in [BHS05]. Moreover, $\mathbb{Z}$-closedness and $\mathbb{R}$-closedness give rise to quite different convex geometries depending on the Coxeter group, see the theorem in [Pil06, Section 1]. The notion of closed set also appeared with different names such as completely closed, $\mathbb{R}$-closed, and 2-closed; see for instance [BHS05], [Pil06], and [Dye11].

The interest of closed sets lie in the following fundamental property. Recall that the inversion set $\mathrm{inv}(w)$ of w is the set $\Phi^+ \cap w^{-1}\Phi^-$, due to the bijection between positive roots and reflections. We will sometimes abuse the notation and consider the inversion sets as sets of roots or reflections, depending on the context.

Lemma 2.6 (Pilkington [Pil06, Proposition 1.2])**.** *Let Φ be a root system of a possibly infinite Coxeter group, and let $A \subseteq \Phi^+$. Then $A = \mathrm{inv}(w)$ for some $w \in W$ if and only if A is finite and biclosed.*

This shows that finite biclosed subsets of Φ^+ are in natural bijection with elements of W. Next, we give two other families of collections of Φ^+.

Definition 2.7 (Dyer [Dye11, Section 11.4])**.** A subset A of Φ^+ is *convex* if there exists a convex cone C of $\mathrm{cone}(\Phi^+)$ pointed at the origin such that $\Phi^+ \cap C = A$. A subset A of Φ^+ is *biconvex* if A and A^c are convex. Denote by $\mathcal{C}(\Phi^+)$ the collection of biconvex sets of Φ^+.

Definition 2.8. A subset A of Φ^+ is *separable* if there exists a hyperplane $H = \{v \in V : B(v, y) = 0\}$ for $y \in V$ such that

$$B(\alpha, y) > 0 \Longleftrightarrow \alpha \in A,$$
$$B(\alpha, y) < 0 \Longleftrightarrow \alpha \in A^c,$$

for all $\alpha \in \Phi^+$. Denote by $\mathcal{S}(\Phi^+)$ the collection of separable sets of Φ^+.

Remark 2.9. In general, the collections $\mathcal{B}(\Phi^+)$, $\mathcal{C}(\Phi^+)$ and $\mathcal{S}(\Phi^+)$ do not always form alignments in the sense of Definition 2.1. Indeed, already in the case of A_2, the sets $\{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2}\}$ and $\{\alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2}\}$ are separable, but their intersection $\{\alpha_{s_1} + \alpha_{s_2}\}$ is not separable; see Fig. 2.1 on page 18. Therefore, we will refer to the collections $\mathcal{B}(\Phi^+)$, $\mathcal{C}(\Phi^+)$ and $\mathcal{S}(\Phi^+)$ simply as *geometries* of Φ^+.

It is clear from their definitions that we have the sequence of inclusion $\mathcal{S}(\Phi^+) \subseteq \mathcal{C}(\Phi^+) \subseteq \mathcal{B}(\Phi^+)$. Restricting to finite sets, we get the reverse inclusion.

Lemma 2.10. *Let $\mathcal{S}_\circ(\Phi^+)$, $\mathcal{C}_\circ(\Phi^+)$ and $\mathcal{B}_\circ(\Phi^+)$ denote the collections of finite separable, finite biconvex and finite biclosed sets of a root system Φ, respectively. Then*

$$\mathcal{S}_\circ(\Phi^+) = \mathcal{C}_\circ(\Phi^+) = \mathcal{B}_\circ(\Phi^+) = \{\mathrm{inv}(w) : w \in W\}.$$

Proof. The last equality is a consequence of Lemma 2.6. Now take a finite biclosed set A. Again by Lemma 2.6, there exists an element $w \in W$ such that $\mathrm{inv}(w) = A$. The sets Φ^+ and and Φ^- can be strictly separated by a hyperplane H, so the hyperplane $w(H)$ will separate $\mathrm{inv}(w)$ and $\Phi^+ \setminus \mathrm{inv}(w)$ since $\mathrm{inv}(w) \subseteq w(\Phi^-)$ and $\Phi^+ \setminus \mathrm{inv}(w) \subseteq w(\Phi^+)$. See the beginning of the proof in [Dye11, Proposition 11.6] for a similar argument. □

Example 2.11. Let $(W, S) = (A_2, \{s_1, s_2\})$. Fig. 2.1 on page 18 shows the corresponding root system and the biclosed, biconvex and separable sets can be easily obtained.

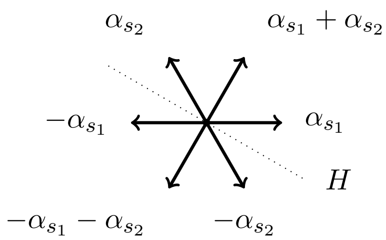


FIGURE 2.1: The root system of type A_2. The inversion sets $\varnothing$, $\{\alpha_{s_1}\}$, $\{\alpha_{s_2}\}$, $\{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2}\}$, $\{\alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2}\}$ and $\{\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2}\}$ are biclosed, biconvex and separable sets of Φ^+.

In Lemma 2.31 in Section 2.4, we give an example of root system of rank 3 with a biconvex set which is not separable. Moreover, some examples of root systems of Coxeter groups of rank 4 are known to have biclosed sets which are not biconvex [DH12]. Finally, we define an operation on sets of $\wp(\Phi^+)$.

Definition 2.12. Let M be a geometry of Φ^+. The M-*closure operator* sends a subset A of Φ^+ to the subset

$$\overline{A}^M := \bigcap_{\{B \in M \,:\, A \subseteq B\}} B.$$

of $\wp(\Phi^+)$. Note that $\overline{A}^M$ does not necessarily belong to M, since M is not a convex geometry.

We will study this closure operator more precisely in the next section. In particular, we will investigate how the union of two biconvex sets could be described geometrically.

2.2 Extended weak order of Coxeter groups

The weak order of Coxeter groups was extended by Matthew Dyer in [Dye11]. This extension uses the notion of geometries on root systems introduced in the previous section. In this extension, the geometry of the root system is used to describe the join (when it exists) and meet operations of the weak order and several of its properties. In the weak order, maximal chains going from the identity to any given element are in natural bijection with the reduced expressions of this element, which in turn correspond to admissible orders of the inversion set of the element; see [BB05, Proposition 3.1.2] and [Dye93, Proposition 2.13]. In the extended weak order, new elements are joined to the group W in the form of infinite sets of roots and their admissible orders are viewed as generalizations of reduced expressions. Moreover, the definition of the extended weak order allows one to determine the join of a set of elements (if it exists) in a geometric way. By Lemma 2.6, the elements of W are in bijection with finite biclosed sets $\mathcal{B}(\Phi^+)$ of Φ^+ via their inversion sets and that for $w \in W$ and $s \in S$,

$$\ell(w) < \ell(ws) \quad \Longleftrightarrow \quad \mathrm{inv}(w) \subset \mathrm{inv}(ws).$$

Therefore, biclosed sets inherit a complete meet-semilattice structure using the weak order. In attempting to describe the weak order in terms of biclosed sets, one could think of a *naive* set theoretical definition. But, in general, one can not use the intersection operation on inversion sets to obtain the meet of two elements in the poset.

Example 2.13. Let $(W,S) = (A_2, \{s_1, s_2\})$, $w = s_1s_2$ and $w' = s_2s_1$. The inversion set $\mathrm{inv}(w \wedge w')$ is not equal to $\mathrm{inv}(w) \cap \mathrm{inv}(w')$. Indeed the inversion sets of w, w' and $w \wedge w'$ are $\mathrm{inv}(w) = \{s_1, s_1s_2s_1\}$, $\mathrm{inv}(w') = \{s_2, s_1s_2s_1\}$, and $\varnothing$ respectively. But the intersection $\mathrm{inv}(w) \cap \mathrm{inv}(w') = \{s_1s_2s_1\}$ is not the empty set, and not even biclosed. Compare the root system on Fig. 2.1 and the weak order on Fig. 2.2.

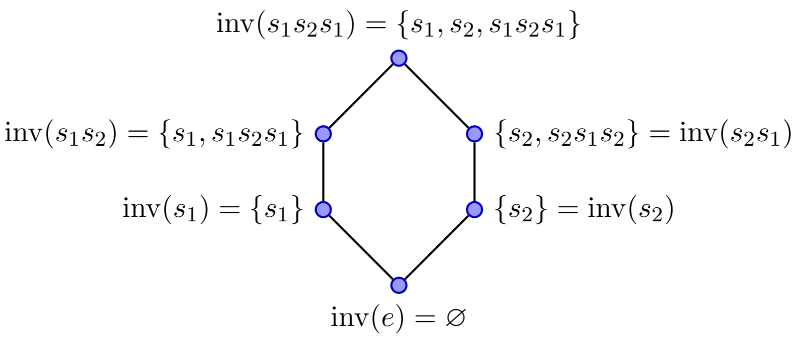


FIGURE 2.2: The weak order of the Coxeter group A_2.

The following result gives the “right” description of the weak order in terms of inversion sets and closure operations.

Theorem 2.14 (Dyer [Dye11, Theorem 1.5])**.** *Let W be a Coxeter group, M be either the geometry of biclosed sets $\mathcal{B}(\Phi^+)$ or of biconvex sets $\mathcal{C}(\Phi^+)$, and $\overline{\cdot}^M$ be the closure operator defined in Definition 2.12. The poset $(W, \leq)$ is a complete meet semilattice. The meet and the join (if it exists) of a non-empty subset X of W are given as follows:*

(i) *If $L := \bigwedge X$ then $\Phi^+ \setminus \mathrm{inv}(L) = \overline{\cup_{x\in X} \Phi^+ \setminus \mathrm{inv}(x)}^M$,*

(ii) *The join $J := \bigvee X$ exists in $(W, \leq)$ if and only if X has an upper bound in W, in which case $\mathrm{inv}(J) = \overline{\cup_{x\in X} \mathrm{inv}(x)}^M$.*

The top element $\hat{1}$ is the set Φ^+.

The preceding theorem was obtain by investigating generalizations of reduced expressions, reflection orders and admissible orders. Many properties about this generalization using the geometry of biclosed and biconvex sets are still open, see [Dye11, Section 2]. We present here the conjectures that motivated the work presented in this chapter.

Conjecture 2.15 (Dyer [Dye11, Conjecture 2.5])**.** *The poset $(\mathcal{B}(\Phi^+), \subseteq)$ of biclosed sets ordered by inclusion is a complete ortholattice. The join of a family X of biclosed subsets of Φ^+, is given by $\bigvee X = \overline{\cup_{\Gamma\in X}\Gamma}^{\mathcal{B}(\Phi^+)}$, and the ortholattice complement is the set complement in Φ^+.*

Remark 2.16. In [Dye11], it is stated that this conjecture is open for all infinite irreducible Coxeter groups except the infinite dihedral groups. Part of this conjecture is verified for particular cases of biclosed sets in the aforementioned article. This conjecture dates back to an observation of Matthew Dyer [Dye93, Remark 2.12].

As a first step to study the previous conjecture, Christophe Hohlweg stated the following conjecture.

Conjecture 2.17 (Hohlweg [Hoh10])**.** *Let $A, B \in \mathcal{C}(\Phi^+)$ be two biconvex sets of Φ^+. The $\mathcal{C}(\Phi^+)$-closure of the union of A and B is exactly $\Phi^+ \cap \mathrm{cone}(A \cup B)$. In other words,*

$$\overline{A \cup B}^{\mathcal{C}(\Phi^+)} := \bigcap_{\{C\in\mathcal{C}(\Phi^+) \mid (A\cup B)\subseteq C\}} C = \Phi^+ \cap \mathrm{cone}(A \cup B).$$

In Section 2.4, we prove Conjecture 2.17 for Coxeter groups of rank at most 3 and provide a counter example of rank 4.

2.3 Limit points of normalized roots and isotropic cone

In the literature, the term infinite root system seems to designate different objects, depending on whether associated to Lie algebras (see [Kac90, LN04]), Kac–Moody Lie algebras (see [MP89]) or Coxeter groups via their geometric representations (see [Hum92, Ch.5 & 6]). While all definitions of root systems are related to a given bilinear form, the bilinear forms considered in the case of Kac–Moody algebras or Lie algebras are different from the one in the classical definition of a root system for infinite Coxeter groups. In particular, a difference lies in the ability to change the value of the bilinear form on a pair of reflections whose product has infinite order. In this vein, more general geometric representations of a Coxeter group and of root systems (that we described in Section 1.2) have been introduced. These more general geometric representations were recently presented in [Kra09] and [BD10] (see also [How96]) but seem to go back to [Vin71], as stated in Daan Krammer's thesis. They have been the framework of several recent studies about infinite root systems of Coxeter groups (see for instance [BD10, Dye10, Dye11, Fu12a, Fu12b]).

In order to study the closure operation on biconvex sets of roots, we need a geometric description of the distribution of positive roots in cone(Φ^+). When W is finite, Φ is also finite and the distribution of the roots in the space V is well understood. However, when W is infinite, the root system is infinite and we have, as far as we know, very few tools to study the distribution of the roots over V. The asymptotical behaviour of roots is one of them. This section deals with a first step of this study: We show that the "lengths" of the roots tend to infinity, and that the limit points of the "directions" of the roots are included in the isotropic cone of the bilinear form associated to Φ.

2.3.1 Roots and normalized roots in ranks 2, 3, 4, and general setting

Let (Φ, Δ) be a based root system in (V, B), with associated Coxeter group W (as defined in Definition 1.4) and suppose that Δ is a basis for V. In order to get a first grip on what happens, we begin with some examples. Since $\Phi = \Phi^+ \sqcup (-\Phi^+)$, it is enough to look at the positive roots, which are inside the polyhedral cone cone(Δ).

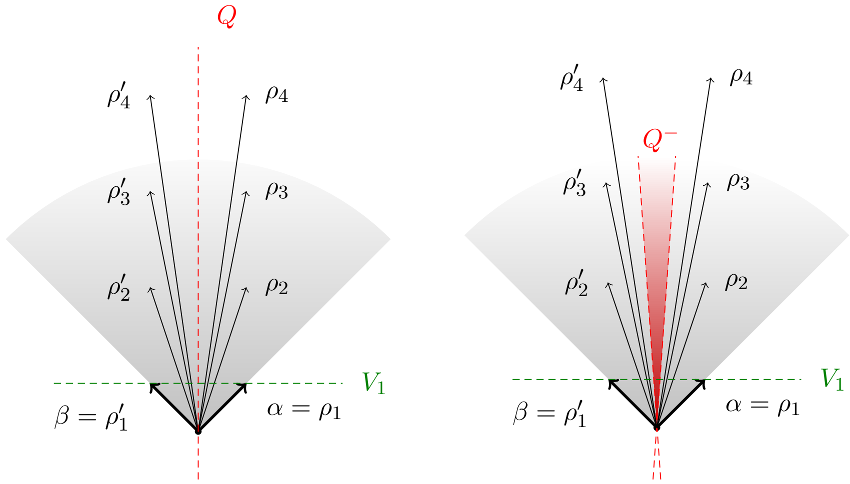
Example 2.18 (Rank 2: representations of infinite dihedral groups)**.** Let (Φ, Δ) be a based root system of rank 2. We get a Coxeter group W of rank 2, geometrically represented in a 2-dimensional vector space V (together with a bilinear form B), where V is generated by two simple roots α, β. Assume that W is an infinite dihedral group, so $B(\alpha, \beta) \leq -1$.

Suppose first that $B(\alpha, \beta) = -1$, i.e., that Φ is affine and with the classical geometric representation. Then the positive roots are $\rho_n = (n+1)\alpha + n\beta$, and $\rho'_n = n\alpha + (n+1)\beta$, for $n \in \mathbb{N}$.

If we fix a (Euclidean) norm on V (e.g., such that $\{\alpha, \beta\}$ is an orthonormal basis), then the norms of the roots tend to infinity, but their directions tend to the line generated by $\alpha + \beta$, as depicted in Fig. 2.3 (a). This line is precisely the isotropic cone of the bilinear form B, i.e., the set

$$Q := \{v \in V : q(v) = 0\} \text{ , where } q(v) = B(v, v).$$

In a general geometric representation of W, Φ can be non-affine, i.e., $B(\alpha, \beta) = x$ with $x < -1$ (see Definition 1.6). Then the isotropic cone Q consists of two lines (generated by $(-x \pm \sqrt{x^2 - 1})\alpha + \beta$). Fig. 2.3 (b) shows that, again, the norms of the roots diverge



(a) $B(\alpha, \beta) = -1$ (b) $B(\alpha, \beta) = -1.01 < -1$

FIGURE 2.3: The isotropic cone Q and the first positive roots of an infinite based root system of rank 2. (a): in the classical affine representation. (b): in a non-affine representation (the part Q^- denotes the set $\{v \in V : q(v) < 0\}$).

to infinity and their directions tend to the two directions of the lines of Q. See [How96, p.3] for a detailed computation.

Let us go back to the general case of an infinite based root system of rank n. In the example of dihedral groups, we saw that the roots themselves have no limit points; this phenomenon is actually general, so we are rather interested in the asymptotic behaviour of their directions. In order to talk properly about limits of directions, we normalize the roots. One simple way to do so is to intersect the line $\mathbb{R}\beta$ generated by a root β in Φ with the affine hyperplane V_1 spanned by the simple roots (seen as points), i.e., the affine hyperplane

$$V_1 := \{v \in V : \sum_{\alpha \in \Delta} v_\alpha = 1\} ,$$

where the v_α's are the coordinates of v in the basis Δ of simple roots. This yields the set of *normalized roots*, denoted by $\widehat{\Phi}$:

$$\widehat{\Phi} := \bigcup_{\beta \in \Phi} \mathbb{R}\beta \cap V_1.$$

Let us describe this set more precisely. Set

$$\begin{aligned} V_0 &:= \{v \in V : |v|_1 = 0\}, \quad \text{and} \\ V_0^+ &:= \{v \in V : |v|_1 > 0\}, \quad \text{where} \quad |v|_1 := \textstyle\sum_{\alpha \in \Delta} v_\alpha. \end{aligned}$$

Since all the positive roots are in the half-space V_0^+, the entire root system Φ is contained in $V \setminus V_0$. So the following normalization map can be applied to Φ:

$$\begin{array}{ccc} V \setminus V_0 & \to & V_1 \\ v & \mapsto & \widehat{v} := \frac{v}{|v|_1}. \end{array}$$

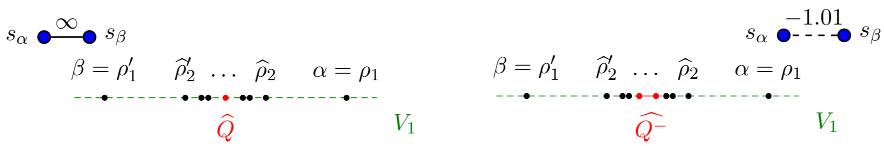
For any subset A of $V \setminus V_0$, write $\widehat{A}$ for the set $\{\widehat{a} : a \in A\}$. Because for $\rho \in \Phi$, $\mathbb{R}\rho \cap \Phi = \{\rho, -\rho\}$, it is then obvious that Φ^+ is in bijection with

$$\widehat{\Phi} = \widehat{\Phi^+} = \widehat{-\Phi^+} = \{\widehat{\rho} : \rho \in \Phi^+\}.$$

Remark 2.19. Obviously, we could also have considered other affine hyperplanes to "cut" the rays of Φ; it suffices that the chosen hyperplane be "transverse to Φ^+", and this is discussed in [HLR13, Section 5.2]. We could also have considered the roots abstractly, in the projective space $\mathbb{P}V$. Considering the intersection of cone(Φ^+) and an affine hyperplane, such as V_1, has the advantage of representing positive roots as points inside an n-simplex (here $n = \dim V$). This enables us to use convex geometry tools. As a convex polytope, conv(Δ) is compact, which is practical when studying sequences of roots. From now on, in examples, we only draw the normalized roots inside the n-simplex conv(Δ).

We now examine the relation between normalized roots and the isotropic cone Q.

Example 2.20 (Normalized roots in the dihedral case)**.** For the infinite dihedral case, the "normalized" version of Fig. 2.3 is Fig. 2.4. Here $\widehat{\Phi}$ is contained in the segment $[\alpha, \beta]$ and there are one or two limit points of normalized roots (depending on whether $B(\alpha, \beta) = -1$ or not), and the set of limit points is always equal to the intersection $Q \cap V_1 = \widehat{Q}$.



(a) $B(\alpha, \beta) = -1$ (b) $B(\alpha, \beta) = -1.01 < -1$

Figure 2.4: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots of an infinite based root system of rank 2. (a): in the (classical) affine representation. (b): in a non-affine representation. The Coxeter graphs follow Abramenko–Brown's notation [AB08, Section 10.3.3].

We give now some examples and pictures in rank 3 and 4.

Example 2.21 (Rank 3)**.** In Figures 1(a) (in the Introduction, on page 2) and 2.5 through 2.8, the normalized isotropic cone $\widehat{Q}$, the 3-simplex cone(Δ), and the first normalized roots are drawn. The normalized roots seem again to tend quickly towards

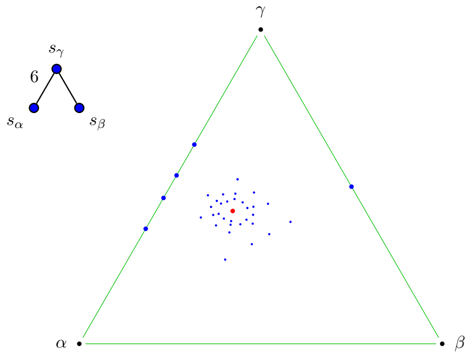


FIGURE 2.5: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots with depth ≤ 12 (see Definition 2.27) for the based root system of type $\widetilde{G}_2$ (affine).

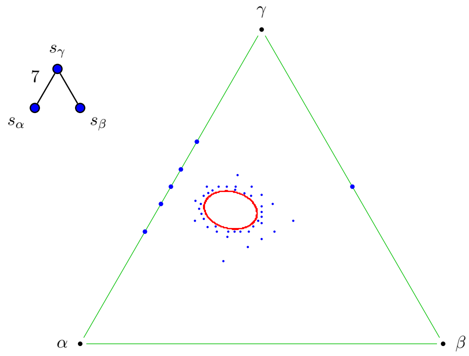


FIGURE 2.6: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots with depth ≤ 10 (see Definition 2.27) for the based root system with labels $2, 3, 7$.

$\widehat{Q}$. In the affine cases, $\widehat{Q}$ contains only one point, which is the intersection of the line $V^\perp$ (the radical of B) with V_1. In rank 3, there are three different types: $\widetilde{A}_2$, $\widetilde{B}_2$, and $\widetilde{G}_2$. The latter is drawn in Fig. 2.5. Otherwise, $\widehat{Q}$ is always a conic (because the signature of B is $(2,1)$), and moreover it is always an ellipse in the classical geometric representation (see [HLR13, Section 5.2] for more details).

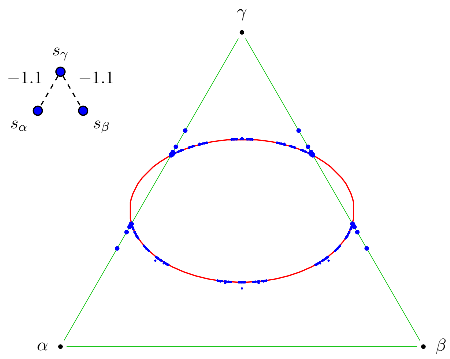


FIGURE 2.7: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots (with depth ≤ 10) for the based root system with labels $2, -1.1, -1.1$.

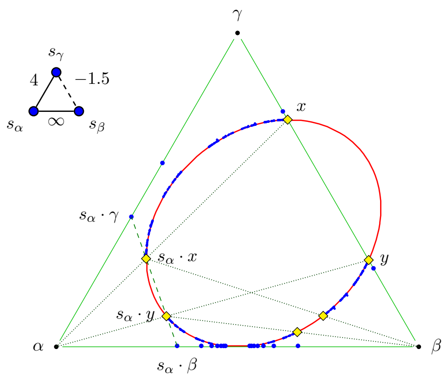


FIGURE 2.8: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots (with depth ≤ 8) for the based root system with labels $\infty, -1.5, 4$.

Some rank 2 root subsystems appear in Fig. 2.7 and 2.8; they correspond to dihedral reflection subgroups. The normalized roots corresponding to such a reflection subgroup, generated by two reflections s_{ρ_1} and s_{ρ_2}, lie in the line containing the normalized roots $\widehat{\rho_1}$ and $\widehat{\rho_2}$. Because of Proposition 1.8 (ii), the subgroup is infinite if and only if $\widehat{Q}$ intersects this line. In Fig. 1(a), for the group with labels $5, 3, 3$, the line joining γ and $\widehat{\rho} = \frac{\alpha+\beta}{2}$ intersects the ellipse in two points, as predicted by Example 1.3.

In general, the behaviour for standard parabolic dihedral subgroups is seen on the edges of the simplex, where three situations can occur. The ellipse $\widehat{Q}$ can either cut an edge

$[\alpha, \beta]$ in two points, or be tangent, or not intersect it, depending on whether $B(\alpha, \beta) < -1$, $= -1$, or > -1 respectively; see in particular Figures 2.7 and 2.8.

Remark 2.22. When $\widehat{Q}$ is included in the simplex, it seems that the limit points of normalized roots cover the whole ellipse, whereas in the other cases the behaviour is more complicated. We discuss this phenomenon in Section 2.5.

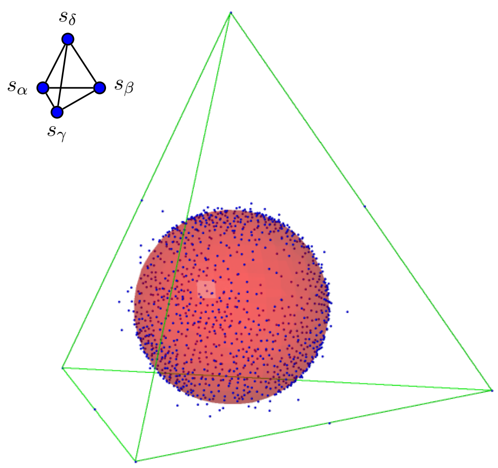


FIGURE 2.9: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots (with depth ≤ 8) for the based root system with diagram the complete graph with labels 3.

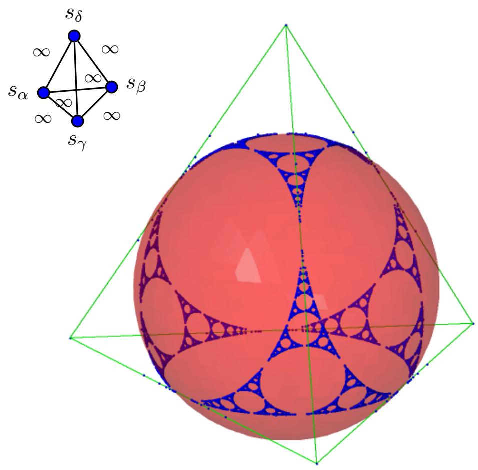


FIGURE 2.10: The normalized isotropic cone $\widehat{Q}$ and the first normalized roots (with depth ≤ 8) for the based root system with diagram the complete graph with labels ∞.

Example 2.23 (Rank 4)**.** Figures 1(b) (in Introduction), and 2.9-2.10 illustrate some based root systems of rank 4, together with the tetrahedron conv(Δ). Analogous properties seem to be true: The limit points are in $\widehat{Q}$, and the way in which $\widehat{Q}$ cuts a facet depends on whether the associated standard parabolic subgroup of rank 3 is infinite non affine, affine, or finite. Moreover, Remark 2.22 still holds: In Fig. 2.9 the limit points seem to cover the whole of $\widehat{Q}$, whereas in Figures 1(b) and 2.10, some Apollonian gasket shapes appear.

2.3.2 The limit points of normalized roots lie in the isotropic cone

Recall that we denote by q the quadratic form associated to B, and by Q the isotropic cone:

$$Q := \{v \in V : q(v) = 0\}, \text{ where } q(v) = B(v, v).$$

The following theorem, proved on page 27, formalizes our observations.

Theorem 2.24. *Consider an injective sequence of positive roots $(\rho_n)_{n\in\mathbb{N}}$, and suppose that $(\widehat{\rho_n})$ converges to a limit r. Then*

(i) *the norm $||\rho_n||$ tends to infinity (for any norm on V),*

(ii) *the limit r lies in $\widehat{Q} = Q \cap V_1$.*

In other words, the set $\widehat{Q}$ of accumulation points of normalized roots $\widehat{\Phi}$ is contained in the isotropic cone.

Remark 2.25. Matthew Dyer proved independently this property in the context of his work on imaginary cone [Dye13], extending a study of Victor Kac (in the framework of Weyl groups of Lie algebras), who states that the convex hull of the limit points correspond to the closure of the imaginary cone (see [Kac90], Lemma 5.8 and Exercise 5.12).

This theorem has the following consequence, which can of course be proved more directly using the fact that W is discrete in $\mathrm{GL}(V)$, see [Kra09] or [Hum92, Prop. 6.2].

Corollary 2.26. *The set of roots of a Coxeter group is discrete.*

Proof. Suppose ρ_n is an injective sequence converging to $\rho \in \Phi^+$. Then $\widehat{\rho_n}$ converges to $\widehat{\rho}$, so by Theorem 2.24, $\widehat{\rho} \in Q$. Therefore $q(\widehat{\rho}) = 0$ which gives a contradiction since $q(\rho) = 1$. □

The remainder of this subsection is devoted to the proof of Theorem 2.24. We first need to recall the notion of depth of a root.

Definition 2.27 ([BB05, Definition 4.6.1])**.** Let $\rho \in \Phi^+$. The *depth* $\mathrm{dp}(\rho)$ of ρ is the positive integer

$$1 + \min\{k : \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}),\ \text{for } \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}.$$

If $t \in T$ is the reflection corresponding to ρ, then $\ell(t) = 2\,\mathrm{dp}(\rho) - 1$.

The depth is a very useful tool that allows inductive proof in infinite root systems: if γ is a root of depth $d \geq 2$, then there is a root γ' of depth $d-1$ and a simple root $\alpha \in \Delta$ such that $\gamma = s_\alpha(\gamma')$, and moreover $B(\alpha, \gamma') < 0$, see [BB05, Lemma 4.6.2].

By construction, the number of positive roots of bounded depth is finite. Consider an injective sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive roots, as in Theorem 2.24. Then we obtain easily that $\mathrm{dp}(\rho_n)$ diverges to infinity as $n \to \infty$. So, to prove the first item of the theorem, it is sufficient to show that when the depth of a sequence of roots tends to infinity, so does the norm of the roots. This is done using the following lemma, which clarifies the relation between norm and depth.

Lemma 2.28. *Let (Φ, Δ) be a based root system. Let $||.||$ be the Euclidean norm for which Δ is an orthonormal basis for V. Then, we have the following properties.*

(i) *The set* $\{|B(\alpha,\rho)| : \alpha \in \Delta, \rho \in \Phi^+$ *and* $B(\alpha,\rho) \neq 0\}$ *is bounded away from 0,*

(ii) *Let* $\rho \in \Phi^+$. *The square of the norm* $||\rho||^2$ *of* $\rho \in \Phi$ *grows linearly with respect to its depth.*

Proof. The first point is a direct consequence of Proposition 1.8 (i). Let us now prove, by induction on $\mathrm{dp}(\rho)$, that

$$\forall \rho \in \Phi^+,\ ||\rho||^2 \geq \lambda(\mathrm{dp}(\rho) - 1) + 1,$$

where $\lambda = 4\kappa^2$ with $0 < \kappa \leq |B(\alpha,\rho)|$ for all $\alpha \in \Delta$ and $\rho \in \Phi^+$ is given by (i). If $\mathrm{dp}(\rho) = 1$, $\rho \in \Delta$ so $||\rho|| = 1 = \lambda(\mathrm{dp}(\rho) - 1) + 1$ by the choice of the norm $||\cdot||$. If $\mathrm{dp}(\rho) = r \geq 2$, then we can write $\rho = s_\alpha(\rho')$, with $\rho' \in \Phi^+$ and $\alpha \in \Delta$ such that $\mathrm{dp}(\rho') = r - 1$ and $B(\alpha,\rho') < 0$, by [BB05, Lemma 4.6.2]. We get

$$\begin{aligned} ||\rho||^2 &= ||\rho' - 2B(\alpha,\rho')\alpha||^2 \\ &= ||\rho'||^2 + 4B(\alpha,\rho')^2 - 4B(\alpha,\rho')\left\langle \alpha,\rho' \right\rangle, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean product of $||\cdot||$. But we know that $B(\alpha,\rho') < 0$, and $\langle \alpha,\rho'\rangle \geq 0$ since $\rho' \in \mathrm{cone}(\Delta)$ and Δ is an orthonormal basis for $||\cdot||$. So we obtain by induction hypothesis on ρ' and by (i):

$$||\rho||^2 \geq ||\rho'||^2 + 4B(\alpha,\rho')^2 \geq (r-2)\lambda + 4\kappa^2 + 1.$$

Since $\lambda = 4\kappa^2$, we have $||\rho||^2 \geq (r-2)\lambda + \lambda + 1 = (r-1)\lambda + 1$, which concludes the proof of (ii). □

We can now finish the proof of Theorem 2.24.

Proof of Theorem 2.24. As explained before Lemma 2.28, and by (ii) of this same lemma, the norm $||\rho_n||$ of any injective sequence $(\rho_n)_{n\in\mathbb{N}}$ in Φ^+ tends to infinity. Recall that for $v \in V_0^+$, $\widehat{v} = \frac{v}{|v|_1}$, where $|v|_1 = \sum_{\alpha\in\Delta} v_\alpha$. If ρ belongs to Φ^+, the coordinates ρ_α are nonnegative, so $|\rho|_1$ is the L_1-norm of ρ. In particular, by equivalence of the norms, $|\rho_n|_1$ tends to infinity as $||\rho_n||$ does. We get

$$q(\widehat{\rho_n}) = q\left(\frac{\rho_n}{|\rho_n|_1}\right) = \frac{q(\rho_n)}{(|\rho_n|_1)^2} = \frac{1}{(|\rho_n|_1)^2} \xrightarrow[n\to\infty]{} 0.$$

Supppose now that $\widehat{\rho_n}$ tends to a limit r. Then we obtain $q(r) = 0$, i.e., $r \in Q$, which concludes the proof of Theorem 2.24. □

From Theorem 2.24 and its proof, we also get these easy consequences.

Corollary 2.29. *Let* $\mathrm{dist}(\cdot,\cdot)$ *denote the Euclidean distance on* V_1. *The two following statements hold.*

(i) *for any* $x \geq 0$, *the set* $\{\rho \in \Phi : ||\rho|| \leq x\}$ *is finite,*
(ii) *for any* $\varepsilon > 0$, *the set* $\{\rho \in \Phi : \mathrm{dist}(\widehat{\rho}, \widehat{Q}) \geq \varepsilon\}$ *is finite.*

Definition 2.30. Let (Φ, Δ) be a based root system in (V, B). The set $E(\Phi)$ (or simply E when there is no possible confusion) consists of accumulation points (or limit points) of $\widehat{\Phi}$. We refer to the points of E as *limit roots* of the root system (Φ, Δ).

As $\widehat{\Phi}$ is included in the simplex $\mathrm{conv}(\Delta)$ (which is closed), Theorem 2.24 implies

$$E(\Phi) \subseteq Q \cap \mathrm{conv}(\Delta) = \widehat{Q} \cap \mathrm{cone}(\Delta).$$

The reverse inclusion is not always true: We saw some examples of this fact in Fig. 2.8 and 2.10. We address a more precise description of $E(\Phi)$ in Section 2.5.

In [HLR13], we discuss the case of dihedral, reducible and affine groups in more examples. We also describe an action of W on the set E which helps to understand the fractal behaviour appearing. Then, we exhibit a countable dense subset of E using dihedral reflection subgroups. Finally, we examine how this approach applies to parabolic subgroups, how to normalize using "transverse" hyperplanes and we examine the case where the based root system is not a basis.

2.4 Complete ortholattice for rank ≤ 3

In this section, we study Conjecture 2.17 with the intuition gained from the previous section. As we will see, the conjecture is false in general, which makes the study of Conjecture 2.15 less tractable in terms of biconvex sets.

2.4.1 The convex union is closed for rank ≤ 3

First, we establish that biconvex sets are not separable in general.

Lemma 2.31. *Let* $n \geq 3$. *There exists an infinite irreducible Coxeter group of rank* n *with based root system* (Φ, Δ) *for which the inclusion* $\mathcal{S}(\Phi^+) \subset \mathcal{C}(\Phi^+)$ *is strict.*

Proof. Assume $n = 3$ and consider the affine Coxeter group $\widetilde{A}_2$ whose normalized root system is illustrated in Fig. 2.11, where the simple system is $\Delta = \{\alpha, \beta, \gamma\}$. By a simple inspection, the roots contained in the convex cone $A = \mathrm{cone}\{\alpha, \alpha+\gamma, 2\alpha+2\beta+3\gamma, \alpha+$

$\beta\} \cap \Phi^+$ form a biconvex set. As the picture shows, the hyperplane H spanned by the roots γ and $\alpha+\beta$ separates the complement A^c and A, but it contains roots of both A^c and A. This shows that the biconvex set A is not separable. For $n > 3$, consider any irreducible Coxeter group whose Coxeter graph contains the Coxeter graph of $\widetilde{A}_2$. The set A defined above still forms a biconvex set which is not separable. □

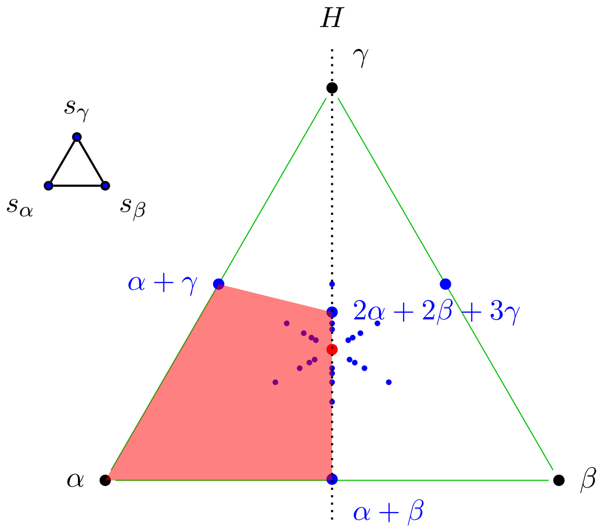


FIGURE 2.11: The root system of type $\widetilde{A}_2$, with roots of depth at most 10 the convex hull of $\{\alpha, \alpha+\gamma, 2\alpha+2\beta+3\gamma, \alpha+\beta\}$ is shadowed and the separating hyperplane H in shown by a dotted line.

Nevertheless, biconvex sets of rank smaller or equal to 3 still fulfill an important separation property. Before giving this property, we first give a useful lemma.

Lemma 2.32. *Let A be a non-empty biclosed set in $\mathcal{B}(\Phi^+)$. Then A contains at least one simple root, i.e. $A \cap \Delta \neq \varnothing$.*

Proof. Assume to the contrary, that A does not contain any simple root. Then its complement A^c contains them all. Since A is biclosed, the complement of A in Φ^+ is closed, and $\Phi^+ \subseteq A^c$. This implies that A is empty which contradicts the assumption. □

Lemma 2.33. *Let (W, S) be an Coxeter system of rank $n \leq 3$ and $A \in \mathcal{C}(\Phi^+)$ be a biconvex set. There exist an hyperplane $H = \{v \in V : B(v, y) = 0\}$ for some $y \in V$ such that*

$$B(\alpha, y) > 0 \Longrightarrow \alpha \in A,$$
$$B(\alpha, y) < 0 \Longrightarrow \alpha \in A^c,$$

for all $\alpha \in \Phi^+$. If $B(\alpha, y) = 0$, then α may belong to either A or A^c.

Proof. It is straightforward to check that it is true if $A = \varnothing$ or $A = \Phi^+$, which proves the rank 1 case. Now suppose that both A and A^c are nonempty. By Lemma 2.32, they contain at least one root. In the rank 2 case, nonempty biconvex sets consist of roots contained in a 2-dimensional cone with a simple root. By Lemma 2.10, if A or A^c is finite, A is separable and the desired hyperplane clearly exists. The remaining case is when A and A^c are both infinite: When A contains the roots in the cone $\text{cone}(\alpha, \alpha+\beta)$ located on one side of Q and A^c contains the ones inside the cone $\text{cone}(\beta, \alpha+\beta)$, see Fig. 2.3 and 2.4. In this case, the line $\mathbb{R}(\alpha+\beta)$ separates the two sets. Finally, assume that $n = 3$. Again by Lemma 2.32 and 2.10, it remains to check the case when A and A^c are infinite and such that $\Delta \cap A = \{a_1, a_2\}$ and $\Delta \cap (A^c) = \{a_c\}$, without loss of generality. Since A is biconvex, there exist cones C and D, such that $C \cap \Phi^+ = A$ and $D \cap \Phi^+ = A^c$. Therefore $C \cap D \cap \Phi^+ = \varnothing$.

If $\text{int}(C) \cap D = \varnothing$, we can look at the normalization of the cones on the plane V_1, to obtain 2-dimensional convex subsets of V_1. Since $\widehat{\text{int}(C)}$ and $\widehat{D}$ are convex, nonempty, disjoint subsets of V_1 and that $\widehat{\text{int}(C)}$ is open, there exists an hyperplane $H = \{v \in V : B(v, y) = 0\}$ for some $y \in V$ such that $\text{int}(C) \subset H^-$ and $D \subseteq H^+$ (by the Hahn–Banach separation theorem). Then, if a root is located strictly on one side of the hyperplane, it belongs either to A or A^c. For roots located on the hyperplane, it is possible that a root belongs to A and another to A^c.

If $\text{int}(C) \cap D \neq \varnothing$, we have to study more carefully what could happen. First, consider the closure of the normalized convex regions $\widehat{C}$ and $\widehat{D}$. Their intersection is also convex and homeomorphic to a 0, 1 or 2-dimensional ball (since $\widehat{C}$ and $\widehat{D}$ are homeomorphic to balls of dimension at most 2). If the intersection is 0 or 1-dimensional, we are in the previous case of $\text{int}(C) \cap D = \varnothing$.

Now assume that the intersection of $\widehat{C}$ and $\widehat{D}$ is 2-dimensional. The boundaries of the convex sets $\widehat{C}$ and $\widehat{D}$ in V_1 form two convex Jordan curves. We study how the Jordan curve divides V_1. Since $\widehat{C}$ and $\widehat{D}$ are convex, the intersection $\widehat{C} \cap \widehat{D}$ on V_1 is also convex, thus connected. Now we show that $\widehat{C} \setminus (\widehat{C} \cap \widehat{D})$ and $\widehat{D} \setminus (\widehat{C} \cap \widehat{D})$ are also connected and therefore the two Jordan curves form three bounded connected components on V_1: $\text{int}(\widehat{C})$, $\text{int}(\widehat{D})$ and $\text{int}(\widehat{C} \cap \widehat{D})$. Applying the rank 2 argument to standard parabolic subgroups, the root in $\text{cone}(a_c, a_1)$ and $\text{cone}(a_c, a_2)$ can be separated by 1-dimensional cones H_1 and H_2. The cone $\text{cone}(a_1, a_2)$ is completely included in C; see Fig. 2.12 for a representation of the normalized cone $\widehat{\Delta}$. Now take two points p, q in the same convex set but not in the intersection $\widehat{C} \cap \widehat{D}$. If p, q are in $\widehat{D}$, we can draw a straight line passing through p and a_c and another line passing through q and a_c. These two segments define a path from p to q completely in $\widehat{D} \setminus \widehat{C}$: Indeed, if there would be a point in $\widehat{C}$ it would force p or q to be in the intersection $\widehat{C} \cap \widehat{D}$. If p, q are in $\widehat{C}$, we can draw again straight lines passing through p and q from a_c, this time connecting them to the segment $[a_1, a_2]$. The same argument works to prove that these segments do not go through the intersection $\widehat{C} \setminus \widehat{D}$, since the segment $[a_1, a_2]$ is in $\widehat{C} \setminus \widehat{D}$. Therefore, there is a path connecting p and q is both cases.

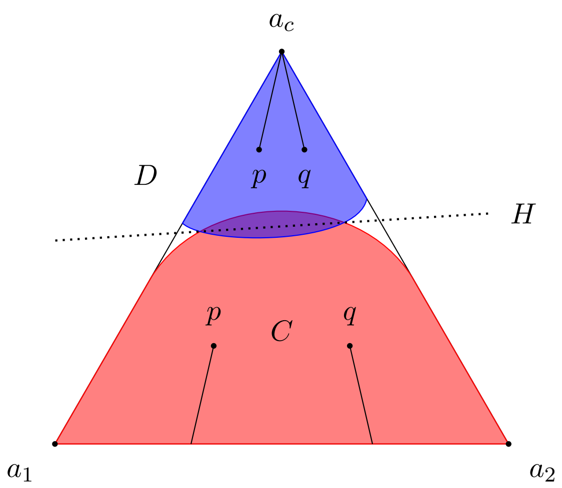


FIGURE 2.12: The set $A \in \mathcal{C}(\Phi^+)$ and its complement A^c. The line segments going from p, q to a_c or to the segment $[a_1, a_2]$ do not pass through the intersection $C \cap D$. The hyperplane H is obtained via the Hahn–Banach separation theorem.

Since there are three bounded connected regions determined by the boundaries of $\widehat{C}$ and $\widehat{D}$, their boundaries intersect on two closed connected regions $R_1, R_2 \subset V_1$. The boundary of the (closed) intersection $\widehat{C} \cap \widehat{D}$ is again a convex Jordan curve and removing R_1 and R_2 from it gives two disjoint closed connected curves: the boundary of $\widehat{C}$ included in D and the boundary of $\widehat{D}$ included in C. Then, taking away a segment $[r_1, r_2]$ (where $r_1 \in R_1$ and $r_2 \in R_2$) from the intersection clearly disconnects it and any connected path from the boundary of $\widehat{C}$ included in D to the boundary of $\widehat{D}$ included in C has to cross $[r_1, r_2]$ by the intermediate value theorem.

Now, consider H to be the hyperplane spanned by the vectors $\{r_1, r_2\}$, such that $r_1 \in R_1$ and $r_2 \in R_2$ and let $y \in V$ be such that $H = \{v \in V : B(v, y) = 0\}$. On V_1, the projection $\widehat{H}$ is a line passing through r_1 and r_2. The intersection $\widehat{H} \cap (\widehat{C} \cup \widehat{D})$ is contained in $\widehat{C} \cap \widehat{D}$ since the latter is convex and $\widehat{r}_1$ and $\widehat{r}_2$ belong to both the boundary of $\widehat{C}$ and $\widehat{D}$.

Since $\widehat{C} \setminus (\widehat{C} \cap \widehat{D})$ and $\widehat{D} \setminus (\widehat{C} \cap \widehat{D})$ are connected, and $\widehat{H} \cap (\widehat{C} \cup \widehat{D})$ is contained in $\widehat{C} \cap \widehat{D}$, it means that the interior of $\widehat{C} \setminus (\widehat{C} \cap \widehat{D})$ is located on one side of $\widehat{H}$. The same argument shows that $\widehat{C} \setminus (\widehat{C} \cap \widehat{D})$ is located on one side of $\widehat{H}$. It remains to show that $\widehat{H}$ separates them. Any path going from a point of $\widehat{C} \setminus (\widehat{C} \cap \widehat{D})$ to a point of $\widehat{D} \setminus (\widehat{C} \cap \widehat{D})$ has to go through their intersection from the boundary of $\widehat{C}$ included in D to the boundary of $\widehat{D}$ included in C. As we have seen, any such path cross the segment $[r_1, r_2]$. Finally, since

$C \cap D \cap \Phi^+ = \varnothing$, roots on boundaries of $\widehat{C}$ and $\widehat{D}$ are on the same side as the interior. Thus $\widehat{H}$ satisfies the desired conditions. □

Remark 2.34. For rank 1 and 2 the reverse of Lemma 2.33 is true, but not in rank 3; i.e. there could be $\alpha \in \Phi^+$ such that $B(\alpha, y) = 0$ and α can belong to A or A^c (see Lemma 2.31).

Theorem 2.35. *Let (W, S) be a Coxeter group of rank $n \leq 3$ and let $A, B \in \mathcal{C}(\Phi^+)$ be two biconvex sets. The closure of the union of A and B with respect to $\mathcal{C}(\Phi^+)$ is equal to the intersection of Φ^+ and $\operatorname{cone}(A \cup B)$. In other words,*

$$\overline{A \cup B}^{\mathcal{C}(\Phi^+)} := \bigcap_{\{C \in \mathcal{C}(\Phi^+)\ :\ (A \cup B) \subseteq C\}} C = \Phi^+ \cap \operatorname{cone}(A \cup B).$$

Proof. For $n = 1$ this is trivial. Next, we consider the case where W is of rank 2, i.e. $S = \{s_1, s_2\}$. Using the normalization in Fig. 2.4. The positive cone becomes a line segment $\widehat{\Phi^+}$, i.e. a 1-simplex. Looking at $\widehat{\Phi^+}$, roots are now points on the line segment. A set A of roots in $\widehat{\Phi^+}$ forms a convex set if there is a polytope (in dimension 1: a point or a line segment) that contains these roots and that any root in this polytope is again in A. Using Lemma 2.32, biconvex sets in $\widehat{\Phi^+}$ are either sets containing one of the point α_{s_1} or α_{s_2} or intervals that contains a least one of the roots α_{s_1} or α_{s_2}. Taking the convex hull of two such sets yields again a set of this form. By minimality of the convex hull the equality

$$\bigcap_{\{C \in \mathcal{C}(\Phi^+)\ :\ (A \cup B) \subseteq C\}} C = \Phi^+ \cap \operatorname{cone}(A \cup B)$$

follows. Finally, suppose W is of rank 3. Once again, the cone Φ^+ is normalized giving a a 2-simplex $\widehat{\Phi^+}$. Let A and B be two nonnested biconvex sets. Consider the set of roots $\Phi^+ \cap \operatorname{cone}(A \cup B)$. By definition it is convex, so it remains to show that its complement in Φ^+ is convex and by minimality of the convex hull the required equality will follow.

Now use Lemma 2.33 to separate the two biconvex sets A and B with hyperplanes H_A and H_B respectively; see Fig. 2.13. Suppose A and B are contained on the positive side of their hyperplanes. We can assume that H_A and H_B are not parallel, otherwise the sets A and B would be nested or the convex hull of the union of the two would be the complete cone and in both cases the result would follow. The 2-dimensional plane V_1 is then separated into four regions and two hyperplanes where roots are distributed. By definition, the closure of the convex set $\operatorname{conv}(\widehat{A} \cup \widehat{B})$ contains all roots in H_A^+ and H_B^+ and on the half-lines $H_A \cap H_B^+$ and $H_B \cap H_A^+$. Therefore, roots in the complement of $\Phi^+ \cap \operatorname{conv}(\widehat{A} \cup \widehat{B})$ are restricted to the closure of $H_A^- \cap H_B^-$, which is a convex set.

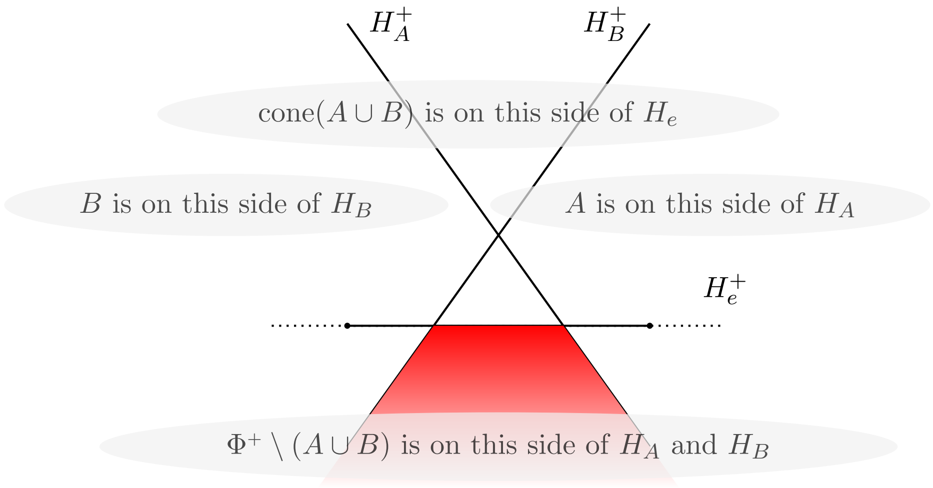


FIGURE 2.13: The hyperplanes H_A and H_B separate the sets A and B from their complements. The hyperplane H_e supports the convex union $\overline{A \cup B}$. Therefore, the roots in $\Phi^+ \setminus \overline{A \cup B}$ are restricted to a convex region (shadowed) which doesn't contain roots of $\overline{A \cup B}$.

Suppose that the convex set $\mathrm{conv}(\widehat{A} \cup \widehat{B})$ does not intersect $H_A^- \cap H_B^-$. Then form a cone C from the roots on H_A and H_B that are not in $\mathrm{conv}(\widehat{A} \cup \widehat{B})$ and the roots in $H_A^- \cap H_B^-$. This cone C is such that $\Phi^+ \cap C = \Phi^+ \setminus \mathrm{conv}(A \cup B)$. Therefore $\Phi^+ \cap \mathrm{conv}(\widehat{A} \cup \widehat{B})$ is biconvex.

If $\mathrm{conv}(\widehat{A} \cup \widehat{B})$ intersects $H_A^- \cap H_B^-$, then consider a supporting hyperplane H_e for an element e of the boundary of $\mathrm{conv}(\widehat{A} \cup \widehat{B})$ in $H_A^- \cap H_B^-$. Without loss of generality, all roots in $\Phi^+ \cap \mathrm{conv}(\widehat{A} \cup \widehat{B})$ are located (not necessarily strictly) in the halfplane H_e^+. Then form a cone C from the roots on H_A and H_B that are not in $\mathrm{conv}(\widehat{A} \cup \widehat{B})$ and the roots in $H_A^- \cap H_B^- \cap H_e^-$. This cone C is such that $\Phi^+ \cap C = \Phi^+ \setminus \mathrm{conv}(A \cup B)$. $\square$

Corollary 2.36. *Let W be a Coxeter group of rank $n \leq 3$ with based root system (Φ, Δ). The biconvex sets ordered by inclusion $(\mathcal{C}(\Phi^+), \subseteq)$ form a complete ortholattice, where the join of a collection of biconvex sets X is given by $\bigvee X = \Phi^+ \cap \mathrm{cone}(\bigcup_{\Gamma \in X} \Gamma)$. The ortholattice map is the set complementation in Φ^+.*

Proof. The closure operator $\overline{\cdot}^{\mathcal{C}(\Phi^+)}$ on the union of two biconvex sets is again a biconvex set. The closure operation thus corresponds to the join in the poset $(\mathcal{C}(\Phi^+), \subseteq)$, since the closure is inclusion minimal. The complementation in Φ^+ clearly satisfies the condition of Definition 1.1. $\square$

Example 2.37. Let $(W, S) = (I_2(\infty), \{s_1, s_2\})$ with corresponding based root system $(\Phi, \Delta = \{\alpha_{s_1}, \alpha_{s_2}\})$. The sets $\{\alpha_{s_1}\}$ and $\{\alpha_{s_2}\}$ are biconvex, see Fig. 2.3 and Fig. 2.4. Their join $\{\alpha_{s_1}\} \vee \{\alpha_{s_2}\}$ in $(\mathcal{C}(\Phi^+), \subseteq)$ is Φ^+. Therefore, the join of s_1 and s_2 does not exist in the weak order $(W, \leq)$ since Φ^+ is infinite, see Fig. 1.1 (c).

Example 2.38. Let $(W, S = \{s, t, u\})$ be the Coxeter group of rank 3 whose graph is complete with edges labeled by 5 with corresponding based root system $(\Phi, \Delta = \{\alpha_s, \alpha_t, \alpha_u\})$, see Fig. 2.14. Moreover, let $w_1 = uts$, $w_2 = utu$, $w_3 = sut$, and $w_4 = susu$. The join $w_1 \vee w_2$ exists in $(W, \leq)$. Indeed, $\text{cone}(\text{inv}(w_1) \cup \text{inv}(w_2))$ does not intersect the isotropic cone therefore by Corollary 2.29 it contains a finite number of roots and by Lemma 2.10 there exists an element $w \in W$ such that $\text{inv}(w) = \Phi^+ \cap \text{cone}(\text{inv}(w_1) \cup \text{inv}(w_2))$. This element w is $utsusus$. The join $w_3 \vee w_4$ does not exist in $(W, \leq)$ since $\text{cone}(\text{inv}(w_3) \cup \text{inv}(w_4))$ intersects the isotropic cone: evaluating the bilinear form $B(u(\alpha_s), su(\alpha_t)) \approx -2.11 < -1$ yields an infinite dihedral group.

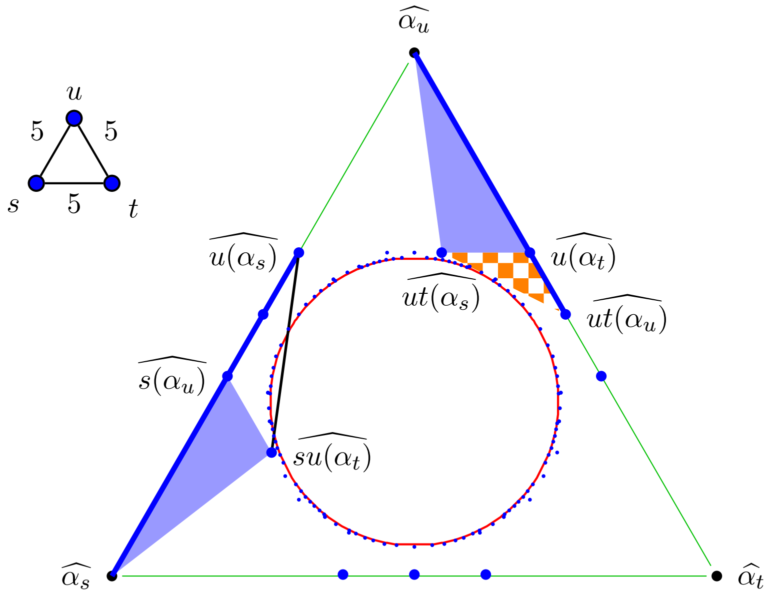


FIGURE 2.14: The inversion sets $\text{inv}(w_1)$ and $\text{inv}(w_3)$ are shown with shadowed triangles. The inversion sets $\text{inv}(w_2)$ and $\text{inv}(w_4)$ are shown using thicker lines on the boundary of the positive cone. The inversion set $\text{inv}(utsusus)$ is shown as the union of the triangle and the adjacent checkerboard triangle. The segment between $u(\alpha_s)$ and $su(\alpha_t)$ intersects the isotropic cone therefore the join $w_3 \vee w_4$ does not exist in $(W, \leq)$.

2.4.2 The convex union is not closed for rank at least 4

In this section, we present an example that disproves Conjecture 2.17 for rank ≥ 4. It was obtained looking at the geometric intuition behind separable sets of roots: Consider the two polytopes obtained by cutting a n-dimensional simplex with an hyperplane and looking at the two *pieces* left by the cut. Now, let P_1 and P_2 be two polytopes obtained in such a way from two different hyperplanes and consider the convex hull of their union. Can this new polytope be obtained by cutting the n-simplex by a unique hyperplane? The answer is no in general and the example below uses this fact to construct a polytope (see Fig. 2.15) included in the simplex which can not be obtained by cutting the simplex with a unique hyperplane.

Example 2.39. Consider the free Coxeter group on four generators, i.e. the Coxeter graph on four vertices with ∞ labels. See Fig. 2.10 for a normalized representation of the positive cone Φ^+. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the simple system. Now consider the

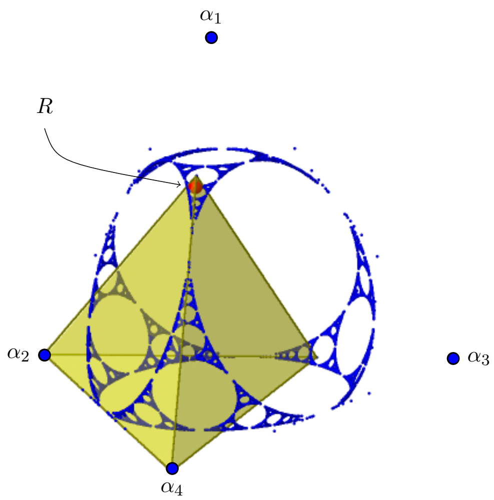


FIGURE 2.15: The convex hull cone$(A \cup B)$ of A and B in yellow. The root R is marked by a bigger sphere.

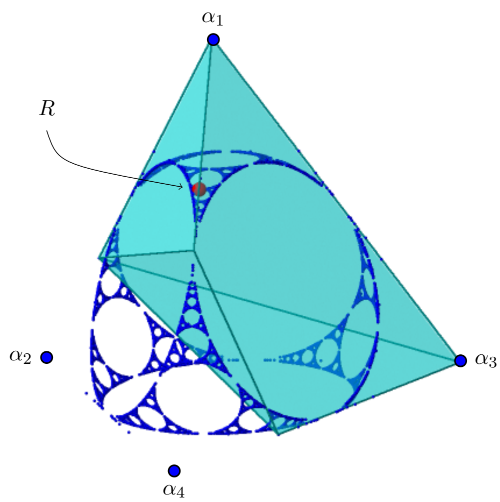


FIGURE 2.16: The complement $\Phi^+ \setminus \text{cone}(A \cup B)$ must at least contain the shown convex set. The root R is marked by a bigger sphere.

sets $A = \text{cone}\{\alpha_2, \alpha_2 + 2\alpha_3\} \cap \Phi^+$ and $B = \text{cone}\{\alpha_4, \alpha_4 + 2\alpha_1\} \cap \Phi^+$. It is easy to check that they are both biconvex sets. The cone cone$(A \cup B)$ is represented in Fig. 2.15. We now fix a specific root $R = \sigma_\delta(\sigma_{\alpha_4}(\delta)) = 1284\alpha_1 + 70\alpha_2 + 35\alpha_3 + 648\alpha_4$, where

$\delta = \sigma_{\alpha_2}(\alpha_3)$. On the one hand, an easy calculation yields

$$R = 105/2 \cdot (\alpha_2) + 35/2 \cdot (\alpha_2 + 2\alpha_3) + 6 \cdot (\alpha_4) + 642 \cdot (2\alpha_1 + \alpha_4) \in \mathrm{cone}(A \cup B).$$

Thus R is a positive combination of the rays of $\mathrm{cone}(A \cup B)$. On the other hand, the complement $\Phi^+ \setminus \mathrm{cone}(A \cup B)$ must at least contain the following set of roots

$$\{\alpha_1, \alpha_1 + 2\alpha_2, \alpha_3, \alpha_3 + 2\alpha_4, \sigma_\gamma(\sigma_{\alpha_3}(\gamma)) = 297\alpha_1 + 10\alpha_3 + 198\alpha_4\}, \qquad (\diamond)$$

where $\gamma = \sigma_{\alpha_1}(\sigma_{\alpha_4}(\alpha_1))$; see Fig. 2.16. Again an easy calculation yields

$$R = 277(\alpha_1) + 35(\alpha_1 + 2\alpha_2) + 25/11(\alpha_3) + 36/11(297\alpha_1 + 10\alpha_3 + 198\alpha_4) \in \Phi^+ \setminus \mathrm{cone}(A \cup B).$$

So R belongs to both $\mathrm{cone}(A \cup B)$ and $\Phi^+ \setminus \mathrm{cone}(A \cup B)$. Consider a biconvex set C that contains $A \cup B$. By minimality, C contains the root R. Since C is biconvex, $C \cap (C^c)$ is empty and R cannot be contained in the cone generated by the roots of Φ^+ not in C. Hence, C has to contain at least one of the five roots in $(\diamond)$. Thus the closure

$$\overline{A \cup B}^{\mathcal{C}(\Phi^+)} := \bigcap_{\{C \in \mathcal{C}(\Phi^+)\ :\ (A \cup B) \subseteq C\}} C$$

of $A \cup B$ is not equal to $\Phi^+ \cap \mathrm{cone}(A \cup B)$.

2.5 Fractal description of the limit roots

To end this chapter, we discuss the fractal behaviour of limit roots. As seen on the title page of this thesis, in Figures 1(b), 2.6, 2.7, 2.9 and 2.10. The roots seem to tend to $\widehat{Q}$ in two possible distinct ways: the limit roots E cover the whole isotropic cone or a subset of it, which appears to form a fractal shape.

For affine Coxeter groups, the isotropic cone $\widehat{Q}$ is 1-dimensional, i.e., a point in V_1. Using Theorem 2.24, the limit root is then unique and $\widehat{Q} = E$. Now, consider the Coxeter group represented in Fig. 2.8. The isotropic cone $\widehat{Q}$ goes out of the positive cone $\mathrm{cone}(\Phi^+)$. Acting by α on the isotropic cone $\widehat{Q}$, the portion of $\widehat{Q}$ outside $\widehat{\mathrm{cone}(\Phi^+)}$ is mapped inside $\widehat{\mathrm{cone}(\Phi^+)}$ and seems not to contain limit roots. This is indeed the case, otherwise it would be possible to send a root close to this portion of $\widehat{Q}$ outside $\widehat{\mathrm{cone}(\Phi^+)}$, which is not possible. This roughly explains why fractal shapes appear in higher dimension. See [HLR13, Section 3] for more details about the action of W on the set $\mathrm{conv}(\Delta) \cap \widehat{Q}$. In the sequel [DHR13, Section 4], Matthew Dyer, Christophe Hohlweg and Vivien Ripoll prove Conjecture 3.9 of [HLR13] explaining how the fractal phenomenon appears. It would be interesting to investigate in details the relations between this fractal behaviour and sphere packings produced by hyperbolic reflection groups; see, for instance, [Max82] and [DHR13, Section 7]. See the Appendix A, for more representations of normalized root systems of rank 3 and 4.

Chapter 3

Subword complexes in discrete geometry

In the preceding chapter, we looked at *infinite* Coxeter groups. The main objective of the present chapter is to introduce and study a natural generalization of multitriangulations to *finite* Coxeter groups: the *multi-cluster complex*. Also, we present an application to the enumeration of the common vertices of W-permutahedra and c-generalized associahedra [MHPS12, Problem 3.3, Chapter 8].

First, we review subword complexes of Coxeter groups in Section 3.1 and finite cluster complexes in Section 3.2. Section 3.3 presents the main results and defines the multi-cluster complex (Definition 3.7). Section 3.4 concerns flips on spherical subword complexes and exhibits two natural isomorphisms between subword complexes whose words differ by commutation or rotation of letters. In Section 3.5, we prove that the multi-cluster complex is independent of the choice of the Coxeter element (Theorem 3.8). Section 3.6 contains a proof that for $k = 1$ the multi-cluster complex is isomorphic to the cluster complex (Theorem 3.4). In Section 3.7, we discuss the generalizations of associahedra using subword complexes; we review known results about polytopal realizations, prove polytopality of multi-cluster cluster complexes of rank 2 (Theorem 3.42), and prove that the multi-cluster complex is universal in the sense that every spherical subword complex is the link of a face of a multi-cluster complex (Theorem 3.19). Section 3.8 introduces a combinatorial description of the sorting words of the longest element of finite Coxeter groups (Theorem 3.45) and an alternative definition of multi-cluster complexes in terms of the strong intervening neighbors property (Theorem 3.11). In Section 3.9, we derive formulas counting the common vertices of W-permutahedra and c-generalized associahedra using the approach presented in the previous sections.

Finally, in Section 3.10, we discuss open problems and questions arising in the context of multi-cluster complexes.

Sections 3.3 to 3.8 and Section 3.10 originate from joint work with Cesar Ceballos and Christian Stump contained in the article [CLS13], which also appeared in the thesis [Ceb12]. Moreover, in [CLS13, Section 8], we define a natural action on the vertices and facets of the multi-cluster complex and use this action to relate multi-cluster complexes to Auslander–Reiten and repetition quivers (not reproduced here).

3.1 Subword complexes

Subword complexes were introduced by Allen Knutson and Ezra Miller in order to study Gröbner geometry of Schubert varieties, see [KM05], and was further studied in [KM04].

Definition 3.1 (Knutson–Miller [KM05, Definition 1.8.1])**.** Let (W, S) be a finite Coxeter system and $Q = (q_1, \ldots, q_r)$ be a word in the generators S of W and let $\pi \in W$. The *subword complex* $\Delta(Q, \pi)$ is the simplicial complex whose faces are given by subwords P of Q for which the complement $Q \setminus P$ contains a reduced expression of π.

Here subwords come with their embedding into Q; two subwords P and P' representing the same word are considered to be different if they involve generators at different positions within Q. In Example 1.13 on page 13, we have seen an instance of a subword complex with $Q = (q_1, q_2, q_3, q_4, q_5) = (s_2, s_1, s_2, s_1, s_2)$ and $\pi = s_1s_2s_1 = s_2s_1s_2$. In this case, $\Delta(Q, \pi)$ has vertices $\{q_1, \ldots, q_5\}$ and facets

$$\{q_1, q_2\}, \{q_2, q_3\}, \{q_3, q_4\}, \{q_4, q_5\}, \{q_5, q_1\}.$$

Subword complexes are known to be vertex-decomposable and hence shellable [KM04, Theorem 2.5]. Moreover, they are topologically spheres or balls depending on the Demazure product of Q. Let Q' be the word obtained by adding $s \in S$ at the end of a word Q. The *Demazure product* $\delta(Q')$ is recursively defined by

$$\delta(Q') = \begin{cases} \delta(Q)s & \text{if } \ell(\delta(Q)s) > \ell(\delta(Q)), \\ \delta(Q) & \text{if } \ell(\delta(Q)s) < \ell(\delta(Q)), \end{cases}$$

where the Demazure product of the empty word is defined to be the identity element in W. A subword complex $\Delta(Q, \pi)$ is a sphere if and only if $\delta(Q) = \pi$, and a ball otherwise [KM04, Corollary 3.8].

3.2 Cluster complexes

In [FZ03], Sergey Fomin and Andrei Zelevinsky introduced the *cluster complex* associated to a finite crystallographic root system. This simplicial complex along with the *generalized associahedron* has become the object of intensive studies and generalizations in various contexts in mathematics, see, for instance, [CFZ02, MRZ03, Rea07a, HLT11]. Recall that the finite Coxeter group W acts naturally on the real vector space V with basis $\Delta = \{\alpha_s : s \in S\}$, whose elements are called *simple roots*. Let Φ denote a *root system* for W, and let $\Phi^+ \subseteq \Phi$ be the set of *positive roots* for the simple system Δ. Furthermore, let $\Phi_{\geq -1} = \Phi^+ \cup -\Delta$ be the set of *almost positive roots*. We denote by $W_{\langle s \rangle}$ the maximal standard parabolic subgroup generated by $S \setminus \{s\}$, and by $\Phi_{\langle s \rangle}$ the associated subroot system. For $s \in S$, the involution $\sigma_s : \Phi_{\geq -1} \longrightarrow \Phi_{\geq -1}$ is given by

$$\sigma_s(\beta) = \begin{cases} \beta & \text{if } -\beta \in \Delta \setminus \{\alpha_s\}, \\ s(\beta) & \text{otherwise.} \end{cases}$$

Nathan Reading showed that the definition of cluster complexes can be extended to all finite root systems and enriched with a parameter c being a Coxeter element [Rea07a]. These c-cluster complexes are defined using a family $\|_c$ of c-compatibility relations on $\Phi_{\geq -1}$, see [RS11, Section 5]. This family is characterized by the following two properties:

(i) for $s \in S$ and $\beta \in \Phi_{\geq -1}$,

$$-\alpha_s \|_c \beta \Longleftrightarrow \beta \in \left(\Phi_{\langle s \rangle}\right)_{\geq -1},$$

(ii) for $\beta_1, \beta_2 \in \Phi_{\geq -1}$ and s being initial in c,

$$\beta_1 \|_c \beta_2 \Longleftrightarrow \sigma_s(\beta_1) \|_{scs} \sigma_s(\beta_2).$$

A maximal subset of pairwise c-compatible almost positive roots is called *c-cluster*.

Definition 3.2 (Reading [Rea07a, Section 7])**.** The *c-cluster complex* $\Delta_c(W)$ is the simplicial complex whose vertices are the almost positive roots and whose facets are c-clusters.

It turns out that the c-cluster complexes associated to different Coxeter elements are isomorphic, see [MRZ03, Proposition 4.10] and [Rea07a, Proposition 7.2]. In crystallographic types, they are moreover isomorphic to the cluster complex as defined in [FZ03].

3.3 Multi-cluster complexes

In this section, we define the *multi-cluster complex* (see Definition 3.7), the central object which generalizes multitriangulations to all finite Coxeter groups.

Definition 3.3 (Reading [Rea07a, Section 2]). Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a reduced expression for a Coxeter element $c \in W$, and let $\mathbf{w} = (w_1, \ldots, w_{\ell(w)})$ be the lexicographically first subword of $\mathbf{c}^\infty$ that represents a reduced expression for the element $w \in W$. The word $\mathbf{w}(\mathbf{c})$ is called the *c-sorting word* for w.

The first theorem (proved in Section 3.6) gives a description of the cluster complex as a subword complex.

Theorem 3.4. *Let W be a finite Coxeter group, c a Coxeter element, and $\mathbf{w}_\circ(\mathbf{c})$ the c-sorting word of $w_\circ$. The subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ is isomorphic to the c-cluster complex $\Delta_c(W)$. The isomorphism is given by sending the letter c_i of $\mathbf{c}$ to the negative root $-\alpha_{c_i}$ and the letter w_i of $\mathbf{w}_\circ(\mathbf{c})$ to the positive root $w_1 \cdots w_{i-1}(\alpha_{w_i})$.*

As an equivalent statement, we obtain the following explicit description of the c-compatibility relation.

Corollary 3.5. *A subset C of $\Phi_{\geq -1}$ is a c-cluster if and only if the complement of the corresponding subword in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c}) = (c_1, \ldots, c_n, w_1, \ldots, w_N)$ represents a reduced expression for $w_\circ$.*

For finite crystallographic root systems, this description was obtained independently by Kiyoshi Igusa and Ralf Schiffler [IS10] in the context of cluster categories [IS10, Theorem 2.5]. They use results of William Crawley-Boevey and Claus M. Ringel saying that the braid group acts transitively on isomorphism classes of exceptional sequences of modules over a hereditary algebra, see [IS10, Section 2]. Kiyoshi Igusa and Ralf Schiffler then show combinatorially that the braid group acting on sequences of elements in any Coxeter group W of rank n also acts transitively on all sequences of n reflections whose product is a given Coxeter element [IS10, Theorem 1.4]. They then deduce Corollary 3.5 in crystallographic types from these two results, see [IS10, Theorem 2.5]. The approach presented in this thesis is valid uniformly for all finite Coxeter groups and is developed completely in terms of Coxeter group theory. Connections to the work of Kiyoshi Igusa and Ralf Schiffler are studied more closely in [CLS13, Section 8]. In the particular case of bipartite Coxeter elements, as defined in Section 1.1, a similar description as in Corollary 3.5 was as well obtained by Thomas Brady and Colum Watt in [BW08] in the context of the geometry of noncrossing partitions[1].

[1]We thank an anonymous referee of [CLS13] for pointing us to this result.

Example 3.6. Let W be the Coxeter group of type B_2 generated by $S = \{s_1, s_2\}$ and let $c = c_1c_2 = s_1s_2$. Then the word $\mathbf{cw}_\circ(\mathbf{c})$ is $(c_1, c_2, w_1, w_2, w_3, w_4) = (s_1, s_2, s_1, s_2, s_1, s_2)$. The corresponding list of almost positive roots is

$$[-\alpha_1,\ -\alpha_2,\ \alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + 2\alpha_2,\ \alpha_2].$$

The subword complex $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ is a hexagon whose facets are given by any two cyclically consecutive letters in $\mathbf{cw}_\circ(\mathbf{c})$. The corresponding c-clusters are

$$\{-\alpha_1, -\alpha_2\}, \{-\alpha_2, \alpha_1\}, \{\alpha_1, \alpha_1 + \alpha_2\}, \{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}, \{\alpha_1 + 2\alpha_2, \alpha_2\}, \{\alpha_2, -\alpha_1\}.$$

Inspired by results in [Stu11] and [PP12], we generalize the subword complex in Theorem 3.4 by considering the concatenation of k copies of the word $\mathbf{c}$. In type A, this generalization coincides with the description of the complex $\Delta_{m,k}$ given in a different language in [PP12].

Definition 3.7. Let W be a finite Coxeter group, c be a Coxeter element and $k \geq 1$. The *multi-cluster complex* $\Delta_c^k(W)$ is the spherical subword complex $\Delta(\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}), w_\circ)$.

Multi-cluster complexes offer a twofold generalization of the simplicial complex of non-crossing diagonals of a convex polygon that subsumes the generalization to multitriangulations of a convex polygon and to cluster complexes of finite Coxeter groups. The diagram depicted in Fig. 3.1 illustrates the different families and their relations. See Section 3.7 and Appendix B for more details about the different families and their properties.

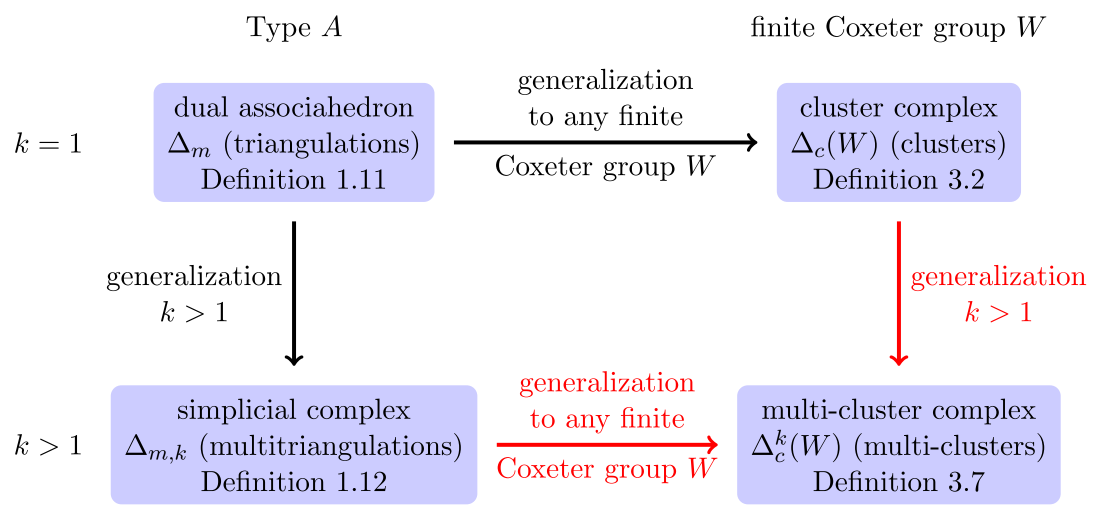


FIGURE 3.1: The twofold generalization of the multi-cluster complex. See Section 3.7 for references about the different objects.

Multi-cluster complexes are in fact independent of the Coxeter element c. In particular, we reobtain that all c-cluster complexes are isomorphic (see Section 3.5 for the proof).

Theorem 3.8. *Let W be a finite Coxeter group and c and c' be two distinct Coxeter elements of W. Then the multi-cluster complexes $\Delta_c^k(W)$ and $\Delta_{c'}^k(W)$ are isomorphic.*

Definition 3.9 (Eriksson–Eriksson [EE09, Section 3] and Speyer [Spe09, Proposition 2.1])**.** A word $Q = (q_1, \ldots, q_r)$ in S has the *intervening neighbors property*, if all non-commuting pairs $s, t \in S$ alternate within Q.

Recall that $\psi : S \to S$ is the involution given by $\psi(s) = w_\circ^{-1} s w_\circ$, it is extended to words as $\psi(Q) = (\psi(q_1), \ldots, \psi(q_r))$.

Definition 3.10. We say that Q has the *strong intervening neighbors property* (*SIN-property*) if $Q\psi(Q) = (q_1, \ldots, q_r, \psi(q_1), \ldots, \psi(q_r))$ has the intervening neighbors property and if in addition the Demazure product $\delta(Q)$ is $w_\circ$.

The following two results give alternative descriptions of multi-cluster complexes. The next theorem (proved in Section 3.8) characterizes all words that are equal to $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ up to commutations.

Theorem 3.11. *A word in S has the SIN-property if and only if it is equal to $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$, up to commutations, for some Coxeter element c and some nonnegative integer k.*

The following proposition gives a different description of the facets of the multi-cluster complex. It generalizes results in [BW08, Section 8] (see also [ABMW06, Section 2.6]) and in [IS10, Lemma 3.2]. In [BW08], the authors consider the case $k = 1$ with bipartite Coxeter elements. In [IS10], the authors consider the case $k = 1$ for crystallographic types with arbitrary Coxeter elements. Set $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = (q_1, q_2, \ldots, q_{kn+N})$. For an index $1 \leq i \leq kn + N$, set the reflection t_i to be $q_1 q_2 \ldots q_{i-1} q_i q_{i-1} \ldots q_2 q_1$. For example, in Example 3.6, we obtain the sequence

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (s_1, s_1s_2s_1, s_2s_1s_2, s_2, s_1, s_1s_2s_1).$$

Proposition 3.12. *A collection $\{q_{\ell_1}, \ldots, q_{\ell_{kn}}\}$ of letters in $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ forms a facet of $\Delta_c^k(W)$ if and only if*

$$t_{\ell_{kn}} \cdots t_{\ell_2} t_{\ell_1} = c^k.$$

The proof follows the lines of the proof of [IS10, Lemma 3.2]:

Proof. A direct calculation shows that $t_{\ell_1} \cdots t_{\ell_{kn}} q_1 q_2 \cdots q_{kn+N}$ equals the product of all letters in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ not in $\{q_{\ell_1}, \ldots, q_{\ell_{kn}}\}$. We get that $\{q_{\ell_1}, \ldots, q_{\ell_{kn}}\}$ is a facet of $\Delta_c^k(W)$

if and only if $t_{\ell_1} \cdots t_{\ell_{kn}} q_1 q_2 \cdots q_{kn+N} = w_\circ$. As $q_1 q_2 \cdots q_{kn+N} = c^k w_\circ$, the statement follows. □

We have seen in Section 1.3 that the multi-cluster complex of type A_{m-2k-1} is isomorphic to the simplicial complex whose facets correspond to k-triangulations of a convex m-gon,

$$\Delta_c^k(A_{m-2k-1}) \cong \Delta_{m,k}.$$

Thus, the multi-cluster complex extends the concept of multitriangulations to finite Coxeter groups and provides a unifying approach to multitriangulations and cluster complexes.

Remark 3.13. There is as well a "naive" way of extending the notion of cluster complexes. Consider the simplicial complex on the set of almost positive roots whose faces are given by the sets that do not contain any subset of $k+1$ pairwise incompatible roots. In type A, this complex gives rise to the simplicial complex of multitriangulations of a convex polygon as desired. However, this simplicial complex lacks basic properties of cluster complexes in general; in type B_3, it is not pure. In this case, the maximal faces have cardinality 6 or 7. A similar phenomenon was observed in [PP12, Remark 29], where the authors suggest that subword complexes of type A (viewed as pseudoline arrangements) are the right objects to define "multi-pseudotriangulations", and explain that the approach using pairwise crossings does not work.

The dictionary for type A is presented in Table 3.1. The general bijection between k-relevant diagonals of the m-gon and (positions of) letters of the word $Q = \mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ of type A_{m-2k-1} is given as follows. Label the vertices of the m-gon from 0 to $m-1$ in clockwise direction, and let $n = m-2k-1$ for simplicity. For $i \in \{1,2,\ldots,n\}$, denote by p_i the position of the generator s_i in $\mathbf{c}$, and let

$$\begin{aligned} a_i &= \left|\left\{j \in \{1,2,\ldots,n\} : j < i \text{ and } p_j < p_{j+1}\right\}\right| \bmod(m), \\ b_i &= -k-1-\left|\left\{j \in \{1,2,\ldots,n\} : j < i \text{ and } p_j > p_{j+1}\right\}\right| \bmod(m). \end{aligned}$$

The bijection sends the ℓth copy of a generator s_i in Q to the k-relevant diagonal obtained by rotating $\ell-1$ times the diagonal $[a_i, b_i]$ in clockwise direction. Under this bijection, a collection of k-relevant diagonals is a facet of $\Delta_{m,k}$ if and only if the corresponding subword in Q is a facet of $\Delta_c^k(A_{m-2k-1})$.

In type B we also obtain a previously known object, namely the simplicial complex $\Delta_{m,k}^{sym}$ of centrally symmetric k-triangulations of a regular convex $2m$-gon. The vertices of this complex are pairs of centrally symmetric k-relevant diagonals, and a collection of vertices

	$\Delta_{m,k}$	$\Delta_c^k(A_{m-2k-1})$
vertices:	k-relevant diagonals of a convex m-gon	letters of $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$
facets:	maximal sets of k-relevant diagonals without $(k+1)$-crossings	such that $Q \setminus P$ is a reduced expression for $w_\circ$
simplices:	sets of k-relevant diagonals without $(k+1)$-crossings	$P \subset Q$ such that $Q \setminus P$ contains a reduced expression for $w_\circ$
ridges:	flips between two k-triangulations	facet flips using Lemma 3.23

Table 3.1: The correspondence between the concepts of diagonals, multitriangulations and flips of multitriangulations in $\Delta_{m,k}$ and in the multi-cluster complex $\Delta_c^k(A_{m-2k-1})$.

form a face if and only if the corresponding diagonals do not contain a $(k+1)$-crossing. This simplicial complex was studied in algebraic and combinatorial contexts by [SW09], [RS10]. We refer to Section 3.7.3 for a proof of Theorem 3.14.

Theorem 3.14. *The multi-cluster complex $\Delta_c^k(B_{m-k})$ is isomorphic to the simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon.*

The description of the simplicial complex of centrally symmetric multitriangulations as a subword complex provides straightforward proofs of nontrivial results about centrally symmetric multitriangulations.

Corollary 3.15. *The following properties of centrally symmetric multitriangulations of a regular convex $2m$-gon hold.*

(i) *All centrally symmetric k-triangulations of a regular convex $2m$-gon contain exactly mk relevant (centrally) symmetric pairs of diagonals, of which k are diameters.*
(ii) *For any centrally symmetric k-triangulation T and any k-relevant symmetric pair of diagonals $d \in T$, there exists a unique k-relevant symmetric pair of diagonals d' not in T such that $T' = (T \backslash \{d\}) \cup \{d'\}$ is again a centrally symmetric k-triangulation. The operation of interchanging a symmetric pair of diagonals between T and T' is called* symmetric flip.
(iii) *All centrally symmetric k-triangulations of a $2m$-gon are connected by symmetric flips.*

The dictionary between the type B multi-cluster complex and the simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon is presented in Table 3.2.

The bijection between k-relevant symmetric pairs of diagonals of a regular convex $2m$-gon and (positions of) letters of the word $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^m$ of type B_n, where $n = m-k$

	$\Delta^{sym}_{m,k}$	$\Delta^k_c(B_{m-k})$
vertices:	k-relevant symmetric pairs of diagonals of a regular convex $2m$-gon	letters of $Q = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^m$
facets:	maximal sets of k-relevant centrally symmetric diagonals without $(k+1)$-crossings	$P \subset Q$ such that $Q \setminus P$ is a reduced expression for $w_\circ$
simplices:	sets of k-relevant symmetric pairs of diagonals without $(k+1)$-crossings	$P \subset Q$ such that $Q \setminus P$ contains a reduced expression for $w_\circ$
ridges:	symmetric flips between two centrally symmetric k-triangulations	facet flips using Lemma 3.23

TABLE 3.2: The generalization of the concept of diagonals, multitriangulations and flips of multitriangulations to the Coxeter group of type B_n.

and $(s_1s_2)^4 = (s_is_{i+1})^3 = \mathbf{1}$ for $1 < i < n$, is given as follows. Label the vertices of the $2m$-gon from 0 to $2m-1$ in clockwise direction. For $1 \leq i \leq n$, denote by p_i the position of the generator s_i in $\mathbf{c}$, and let

$$\begin{aligned} a_i &= \qquad |\{j : 1 \leq j < i \text{ and } p_j < p_{j+1}\}|, \\ b_i &= m - |\{j : 1 \leq j < i \text{ and } p_j > p_{j+1}\}|. \end{aligned}$$

The bijection sends the jth copy of a generator s_i in Q to the k-relevant symmetric pair of diagonals obtained by rotating $j-1$ times the symmetric pair $[a_i, b_i]_{\text{sym}} := \big\{[a_i, b_i], [a_i+m, b_i+m]\big\}$ in clockwise direction (observe that both diagonals coincide for $i = 1$). Under this bijection, a collection of k-relevant symmetric pairs of diagonals is a facet of $\Delta^{sym}_{m,k}$ if and only if the corresponding subword in Q is a facet of $\Delta^k_c(B_{m-k})$.

Example 3.16. Let $m = 5$ and $k = 2$, and let W be the Coxeter group of type B_3 generated by $S = \{s_1, s_2, s_3\}$ where $(s_1s_2)^4 = (s_2s_3)^3 = (s_1s_3)^2 = \mathbf{1}$. The multi-cluster complex $\Delta^2_c(B_3)$ is isomorphic to the simplicial complex of centrally symmetric 2-triangulations of a regular convex 10-gon. In the particular case where the Coxeter element $c = c_1c_2c_3 = s_1s_2s_3$, the bijection between 2-relevant symmetric pairs and the letters of the word $Q = \mathbf{c}^2\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_2, s_3)^5$ is given by

s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_3
$[6,1]$	$[6,2]$	$[6,3]$	$[7,2]$	$[7,3]$	$[7,4]$	$[8,3]$	$[8,4]$	$[8,5]$	$[9,4]$	$[9,5]$	$[9,6]$	$[10,5]$	$[10,6]$	$[10,7]$.
	$[1,7]$	$[1,8]$		$[2,8]$	$[2,9]$		$[3,9]$	$[3,10]$		$[4,10]$	$[4,1]$		$[5,1]$	$[5,2]$

For instance, the first appearance of the letter s_3 is mapped to the symmetric pair of diagonals $[6,3]_{\text{sym}} = \big\{[6,3],[1,8]\big\}$, while the third appearance of s_1 is mapped to the symmetric pair of diagonals $[8,3]_{\text{sym}} = \big\{[8,3]\big\}$. The centrally symmetric k-triangulations can be easily described using the subword complex approach. For example, the symmetric pairs of diagonals at positions $\{3, 5, 7, 9, 13, 15\}$ form a facet of $\Delta^{sym}_{m,k}$; the corresponding 2-triangulation is depicted in Fig. 1.3 on page 13. The symmetric flips are interpreted using Lemma 3.23.

Using algebraic techniques, Daniel Soll and Volkmar Welker proved that $\Delta^{sym}_{m,k}$ is a (mod 2)-homology-sphere [SW09, Theorem 10]. The subword complex description in Theorem 3.14 and the results by Allen Knutson and Ezra Miller [KM04, Theorem 2.5 and Corollary 3.8] imply the following stronger result.

Corollary 3.17. *The simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon is a vertex-decomposable simplicial sphere.*

This result together with the proof of [SW09, Conjecture 13] given in [RS09] implies the following conjecture by Daniel Soll and Volkmar Welker.

Corollary 3.18 ([SW09, Conjecture 17])**.** *For the term-order $\preceq$ defined in [SW09, Section 7], the initial ideal $\mathrm{in}_{\preceq}(I_{n,k})$ of the determinantal ideal $I_{n,k}$ defined in [SW09, Section 3] is spherical.*

We finish this section by describing all spherical subword complexes in terms of faces of multi-cluster complexes (see Section 3.7.5 for the proofs). This generalizes the universality of the multi-associahedron presented in [PS12a, Proposition 5.6] to finite Coxeter groups.

Theorem 3.19. *A simplicial sphere can be realized as a subword complex of a given finite type W if and only if it is the link of a face of a multi-cluster complex $\Delta^k_c(W)$.*

The previous theorem can be obtained for any family of subword complexes, for which arbitrary large powers of $\mathbf{c}$ appear as subwords. However, computations seem to indicate that the multi-cluster complex maximizes the number of facets among subword complexes $\Delta(Q, w_\circ)$ with word Q of the same size. We conjecture that this is true in general, see Conjecture 3.103. We also obtain the following corollary.

Corollary 3.20. *The following two statements are equivalent.*

(i) *Every spherical subword complex of type W is polytopal.*
(ii) *Every multi-cluster complex of type W is polytopal.*

3.4 General results on spherical subword complexes

Before proving the main results, we discuss several properties of spherical subword complexes in general which are not specific to multi-cluster complexes. Throughout this section, we let $Q = (q_1, \ldots, q_r)$ be a word in S and $\pi = \delta(Q)$ be its Demazure product.

3.4.1 Flips in spherical subword complexes

Lemma 3.21 (Knutson–Miller [KM04, Lemma 3.5]). *Let F be a facet of $\Delta(Q, \delta(Q))$. For any vertex $q \in F$, there exists a unique vertex $q' \in Q \setminus F$ such that $\big(F \setminus \{q\}\big) \cup \{q'\}$ is again a facet.*

Proof. This follows from the fact that $\Delta(Q, \delta(Q))$ is a simplicial sphere [KM04, Corollary 3.8]. □

Such a move between two adjacent facets is called *flip*. Next, we describe how to find the unique vertex $q' \notin F$ corresponding to $q \in F$. For this, we introduce the notion of root functions.

Definition 3.22. The *root function* $\mathsf{r}_F : Q \to \Phi$ associated to a facet F of $\Delta(Q, \pi)$ sends a letter $q \in Q$ to the root $\mathsf{r}_F(q) := w_q(\alpha_q) \in \Phi$, where $w_q \in W$ is given by the product of the letters in the prefix of $Q \setminus F = (q_{i_1}, \ldots, q_{i_\ell})$ that appears on the left of q in Q, and where α_q is the simple root associated to q.

Lemma 3.23. *Let F be a facet of $\Delta(Q, \delta(Q))$ and let $q \in F$ and $q' \in Q \setminus \{q\}$ be such that $\big(F \setminus \{q\}\big) \cup \{q'\}$ is again a facet. The vertex q' is the unique vertex not in F for which $\mathsf{r}_F(q') \in \{\pm \mathsf{r}_F(q)\}$.*

Proof. Since $q_{i_1} \ldots q_{i_\ell}$ is a reduced expression for $\pi = \delta(Q)$, the set $\{\mathsf{r}_F(q_{i_1}), \ldots, \mathsf{r}_F(q_{i_\ell})\}$ is equal to the inversion set $\mathrm{inv}(\pi) = \{\alpha_{i_1}, q_{i_1}(\alpha_{i_2}), \ldots, q_{i_1} \cdots q_{i_{\ell-1}}(\alpha_{i_\ell})\}$ of π, which only depends on π and not on the chosen reduced expression. In particular, any two elements in this set are distinct. Notice that the root $\mathsf{r}_F(q)$ for $q \in F$ is, up to sign, also contained in $\mathrm{inv}(\pi)$, otherwise it would contradict the fact that the Demazure product of Q is π. If we insert q into the reduced expression of π, we have to delete the unique letter q' that corresponds to the same root, with a positive sign if it appears on the right of q in Q, or with a negative sign otherwise. The resulting word is again a reduced expression for π. □

Remark 3.24. In the case of cluster complexes, this description can be found in [IS10, Lemma 2.7].

Example 3.25. As in Example 3.6, consider the Coxeter group of type B_2 generated by $S = \{s_1, s_2\}$ with $c = c_1c_2 = s_1s_2$ and $\mathbf{cw}_\circ(\mathbf{c}) = (c_1, c_2, w_1, w_2, w_3, w_4) = (s_1, s_2, s_1, s_2, s_1, s_2)$. Considering the facet $F = \{c_2, w_1\}$, we obtain

$$\begin{aligned} \mathsf{r}_F(c_1) &= \alpha_1, & \mathsf{r}_F(w_2) &= s_1(\alpha_2) = \alpha_1 + \alpha_2, \\ \mathsf{r}_F(c_2) &= s_1(\alpha_2) = \alpha_1 + \alpha_2, & \mathsf{r}_F(w_3) &= s_1s_2(\alpha_1) = \alpha_1 + 2\alpha_2, \\ \mathsf{r}_F(w_1) &= s_1(\alpha_1) = -\alpha_1, & \mathsf{r}_F(w_4) &= s_1s_2s_1(\alpha_2) = \alpha_2. \end{aligned}$$

Since $\mathsf{r}_F(c_2) = \mathsf{r}_F(w_2)$, the letter c_2 in F flips to w_2. As w_2 appears on the right of c_2, both roots have the same sign. Similarly, the letter w_1 flips to c_1, because $\mathsf{r}_F(c_1) = -\mathsf{r}_F(w_1)$. In this case, the roots have different signs because c_1 appear on the left of w_1.

The following lemma describes the relation between the root functions of two facets connected by a flip.

Lemma 3.26. *Let F and $F' = \big(F \setminus \{q\}\big) \cup \{q'\}$ be two adjacent facets of the subword complex $\Delta(Q, \delta(Q))$, and assume that q appears on the left of q' in Q. Then, for every letter $p \in Q$,*

$$\mathsf{r}_{F'}(p) = \begin{cases} t_q(\mathsf{r}_F(p)) & \text{if } p \text{ is between } q \text{ and } q', \text{ or } p = q', \\ \mathsf{r}_F(p) & \text{otherwise.} \end{cases}$$

Here, $t_q = w_q q w_q^{-1}$ where w_q is the product of the letters in the prefix of $Q\backslash F$ that appears on the left of q in Q. By construction, t_q is the reflection in W orthogonal to the root $\mathsf{r}_F(q) = w_q(\alpha_q)$.

Proof. Let p be a letter in Q, and w_p, w'_p be the products of the letters in the prefixes of $Q\backslash F$ and $Q\backslash F'$ that appear on the left of p. Then, by definition $\mathsf{r}_F(p) = w_p(\alpha_p)$ and $\mathsf{r}_{F'}(p) = w'_p(\alpha_p)$. We consider the following three cases:

- If p is on the left of q or $p = q$, then $w_p = w'_p$ and $\mathsf{r}_F(p) = \mathsf{r}_{F'}(p)$.
- If p is between q and q' or $p = q'$, then w'_p can be obtained from w_p by adding the letter q at its corresponding position. This addition is the result of multiplying w_p by $t_q = w_q q w_q^{-1}$ on the left, i.e. $w'_p = t_q w_p$. Therefore, $\mathsf{r}_F(p) = t_q(\mathsf{r}_{F'}(p))$.
- If p is on the right of q', consider the reflection $t_{q'} = w_{q'} q' w_{q'}^{-1}$ where $w_{q'}$ is the product of the letters in the prefix of $Q\backslash F$ that appears on the left of q'. By the same argument, one obtains that $w'_p = t_q t_{q'} w_p$. In addition, $t_q = t_{q'}$ because they correspond to the unique reflection orthogonal to the roots $\mathsf{r}_F(q)$ and $\mathsf{r}_F(q')$, which are up to sign equal by Lemma 3.23. Therefore, $w'_p = w_p$ and $\mathsf{r}_{F'}(p) = \mathsf{r}_F(p)$. □

3.4.2 Isomorphic spherical subword complexes

We now show how the study of general spherical subword complexes can be reduced to the study of spherical subword complexes satisfying the condition that $\delta(Q) = \pi = w_\circ$ and give two operations on the word Q giving isomorphic subword complexes.

Theorem 3.27. *Every spherical subword complex $\Delta(Q, \pi)$ is isomorphic to $\Delta(Q', w_\circ)$ for some word Q' such that $\delta(Q') = w_\circ$.*

Proof. Let $\mathbf{r}$ be a reduced word for $\pi^{-1}w_\circ = \delta(Q)^{-1}w_\circ \in W$. Moreover, define the word Q' as the concatenation of Q and $\mathbf{r}$. By construction, the Demazure product of Q' is $w_\circ$, and every reduced expression of $w_\circ$ in Q' must contain all the letters in $\mathbf{r}$. The reduced expressions of $w_\circ$ in Q' are given by reduced expressions of π in Q together with all the letters in $\mathbf{r}$. Therefore, the subword complexes $\Delta(Q, \pi)$ and $\Delta(Q', w_\circ)$ are isomorphic. □

Recall from Section 3.10, on page 44, the involution $\psi : S \to S$ given by $\psi(s) = w_\circ^{-1} s w_\circ$. This involution was used in [BHLT09] to characterize isometry classes of the c-generalized associahedra, and will also be in Section 3.9. Define the *rotated word* $Q_{\circlearrowleft_s}$ or the *rotation* of $Q = (s, q_2, \ldots, q_r)$ along the letter s as $(q_2, \ldots, q_r, \psi(s))$. The following two propositions are direct consequences of the definition of subword complexes.

Proposition 3.28. *If two words Q and Q' coincide up to commutations, then $\Delta(Q, \pi) \cong \Delta(Q', \pi)$.*

Proof. The isomorphism between $\Delta(Q, \pi)$ and $\Delta(Q', \pi)$ is induced by reordering the letters of Q to obtain Q'. □

Proposition 3.29. *Let $Q = (s, q_2, \ldots, q_r)$. Then $\Delta(Q, w_\circ) \cong \Delta(Q_{\circlearrowleft_s}, w_\circ)$.*

Proof. The isomorphism between $\Delta(Q, w_\circ)$ and $\Delta(Q_{\circlearrowleft_s}, w_\circ)$ is induced by sending q_i to q_i for $2 \leq i \leq r$ and the initial s to the final $\psi(s)$. The result follows from the fact that $s w_\circ = w_\circ \psi(s)$. □

Theorem 3.27 and Proposition 3.29 give an alternative viewpoint on spherical subword complexes. First, we can consider π to be the longest element $w_\circ \in W$. Second, $\Delta(Q, w_\circ)$ does not depend on the word Q but on the bi-infinite word

$$\begin{aligned} \widetilde{Q} &= \cdots \quad Q \qquad\quad \psi(Q) \qquad\quad Q \qquad \cdots \\ &= \ldots q_1, \ldots, q_r, \psi(q_1), \ldots, \psi(q_r), q_1, \ldots, q_r, \ldots. \end{aligned}$$

Taking any connected subword in $\widetilde{Q}$ of length r gives rise to an isomorphic spherical subword complex. This is a key observation that we will use in Section 3.9.

3.5 Proof of Theorem 3.8

In this section, we prove Theorem 3.8 which states that multi-cluster complexes given by distinct Coxeter elements are isomorphic. This result relies on the theory of sorting words (see Definition 3.3 on page 42), introduced by Nathan Reading in [Rea07a]. We use the following result.

Lemma 3.30 (Speyer [Spe09, Corollary 4.1])**.** *The longest element $w_\circ \in W$ can be expressed as a reduced prefix of $\mathbf{c}^\infty$ up to commutations.*

The next lemma unifies previously known results; the first statement is trivial, the second statement can be found in [Spe09, Section 4], and the third statement is equivalent to [HLT11, Lemma 1.6].

Lemma 3.31. *Let s be initial in c, and let $\mathbf{p} = (s, p_2, \dots, p_r)$ be a prefix of $\mathbf{c}^\infty$ up to commutations. Then,*

(i) *$(p_2, \dots, p_r)$ is a prefix of $(\mathbf{c}')^\infty$ up to commutations, where $\mathbf{c}'$ denotes a word for the Coxeter element $c' = scs$,*
(ii) *if $p = sp_2 \cdots p_r$ is reduced, then $\mathbf{p}$ is the c-sorting word for p up to commutations,*
(iii) *if $sp_2 \cdots p_r s'$ is reduced for some $s' \in S$, then $\mathbf{p}$ is a prefix of the c-sorting word for ps' up to commutations.*

Proposition 3.32. *Let s be initial in c, and let $\mathbf{w}_\circ(\mathbf{c}) = (s, w_2, \dots, w_N)$ be the c-sorting word of $w_\circ$ up to commutations. Then, $(w_2, \dots, w_N, \psi(s))$ is the scs-sorting word of $w_\circ$ up to commutations.*

Proof. By Lemma 3.30, the element $w_\circ$ can be written as a prefix of $\mathbf{c}^\infty$. By Lemma 3.31, this prefix is equal to the c-sorting of $w_\circ$, which we denote by $\mathbf{w}_\circ(\mathbf{c})$. Let $\mathbf{scs}$ denote the word for the Coxeter element scs. By Lemma 3.31 (i), the word $(w_2, \dots, w_N)$ is a prefix of $(\mathbf{scs})^\infty$, and by (ii) it is the scs-sorting word for $w_2 \cdots w_N$. By the definition of ψ, the word $(w_2, \dots, w_N, \psi(s))$ is a reduced expression for $w_\circ$. Lemma 3.31 (iii) with the word $(w_2, \dots, w_N)$ and $\psi(s)$ implies that $(w_2, \dots, w_N, \psi(s))$ is the scs-sorting word for $w_\circ$ up to commutations. □

Remark 3.33. In [RS11], Nathan Reading and David Speyer present a uniform approach to the theory of sorting words and sortable elements. This approach uses an anti-symmetric bilinear form, which is used to extend many results to infinite Coxeter groups. In particular, the previous proposition can be easily deduced from [RS11, Lemma 3.8].

We are now in the position to prove that all multi-cluster complexes for the various Coxeter elements are isomorphic.

Proof of Theorem 3.8. Let c and c' be two Coxeter elements such that $c' = scs$ for some initial letter s of c, and let $\mathbf{c}$ and $\mathbf{c}'$ denote reduced words for c and c', respectively. Moreover, let $Q_c = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$, and $Q_{c'} = (\mathbf{c}')^k\mathbf{w}_\circ(\mathbf{c}')$. By Proposition 3.28, we can assume that $Q_c = (s, c_2, \ldots, c_n)^k \cdot (s, w_2, \ldots, w_N)$, and by Proposition 3.32, we can also assume that $Q_{c'} = (c_2, \ldots, c_n, s)^k \cdot (w_2, \ldots, w_N, \psi(s))$. Therefore, $Q_{c'} = (Q_c)_{\circlearrowleft s}$, and Proposition 3.29 implies that the subword complexes $\Delta(Q_c, w_\circ)$ and $\Delta(Q_{c'}, w_\circ)$ are isomorphic. Since any two Coxeter elements can be obtained from each other by conjugation of initial letters (see [GP00, Theorem 3.1.4]), the result follows. □

3.6 Proof of Theorem 3.4

In this section, we prove that the subword complex $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ is isomorphic to the c-cluster complex. As in Theorem 3.4, we identify letters in $\mathbf{cw}_\circ(\mathbf{c}) = (c_1, \ldots, c_n, w_1, \ldots, w_N)$ with almost positive roots using the bijection $\mathsf{Lr}_c : \mathbf{cw}_\circ(\mathbf{c}) \xrightarrow{\sim} \Phi_{\geq -1}$ given by

$$\mathsf{Lr}_c(q) = \begin{cases} -\alpha_{c_i} & \text{if } q = c_i \text{ for some } 1 \leq i \leq n, \\ w_1 w_2 \cdots w_{i-1}(\alpha_{w_i}) & \text{if } q = w_i \text{ for some } 1 \leq i \leq N. \end{cases}$$

In [Rea07a], this map was used to establish a bijection between c-sortable elements and c-clusters. Note that under this bijection, letters of $\mathbf{cw}_\circ(\mathbf{c})$ correspond to almost positive roots and subwords of $\mathbf{cw}_\circ(\mathbf{c})$ correspond to subsets of almost positive roots. We use this identification to simplify several statements in this section. Observe, that for the particular facet F_0 of $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ corresponding to the prefix $\mathbf{c}$ of $\mathbf{cw}_\circ(\mathbf{c})$, we have that

$$\mathsf{Lr}_c(q) = \mathsf{r}_{F_0}(q) \text{ for every } q \in \mathbf{w}_\circ(\mathbf{c}) \subset \mathbf{cw}_\circ(\mathbf{c}),$$

where $\mathsf{r}_{F_0}(q)$ is the root function as defined in Definition 3.22. We interpret the two parts (i) and (ii) in the definition of c-compatibility (see Section 3.2), in Theorem 3.34 and Theorem 3.40. Proving these two conditions yields a proof of Theorem 3.4. The majority of this section is devoted to the proof of the initial condition. The proof of the recursive condition follows afterwards.

3.6.1 Proof of condition (i)

The following theorem implies that $\Delta(\mathbf{cw}_\circ(\mathbf{c}), w_\circ)$ satisfies the initial condition.

Theorem 3.34. *$\{-\alpha_s, \beta\}$ is a face of the subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ if and only if $\beta \in \big(\Phi_{\langle s\rangle}\big)_{\geq -1}$.*

We prove this theorem in several steps.

Lemma 3.35. *Let F be a facet of the subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ such that $c_i \in F$. Then*

(i) *for every $q \in F$ with $q \neq c_i$, $\mathsf{r}_F(q) \in \Phi_{\langle c_i\rangle}$,*
(ii) *for every $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, $\mathsf{r}_F(q) \in \Phi_{\langle c_i\rangle}$ if and only if $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i\rangle})_{\geq -1}$.*

Proof. For the proof of (i), notice that if $F = \mathbf{c}$, then the result is clear. Now suppose that the result is true for a given facet F with $c_i \in F$, and consider the facet $F' = \big(F \smallsetminus \{p\}\big) \cup \{p'\}$ obtained by flipping a letter $p \neq c_i$ in F. Since all the facets containing c_i are connected by flips which do not involve the letter c_i, it is enough to prove the result for the facet F'. By hypothesis, since $p \in F$ and $p \neq c_i$, we have $\mathsf{r}_F(p) \in \Phi_{\langle c_i\rangle}$. Then, the reflection t_p orthogonal to $\mathsf{r}_F(p)$ defined in Lemma 3.26 satisfies $t_p \in W_{\langle c_i\rangle}$. Using Lemma 3.26 we obtain that for every $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$,

$$\mathsf{r}_{F'}(q) \in \Phi_{\langle c_i\rangle} \Longleftrightarrow \mathsf{r}_F(q) \in \Phi_{\langle c_i\rangle}.$$

If $q \in F'$ and $q \neq c_i$, then ($q \in F$ and $q \neq c_i$) or $q = p'$. In the first case, $\mathsf{r}_F(q)$ is contained in $\Phi_{\langle c_i\rangle}$ by hypothesis, and consequently $\mathsf{r}_{F'}(q) \in \Phi_{\langle c_i\rangle}$. By Lemma 3.23, the second case $q = p'$ implies that $\mathsf{r}_F(q) = \pm\mathsf{r}_F(p)$. Again since $\mathsf{r}_F(p)$ belongs to $\Phi_{\langle c_i\rangle}$ by hypothesis, the root $\mathsf{r}_{F'}(q)$ belongs to $\Phi_{\langle c_i\rangle}$.

For the second part of the lemma, notice that the set $\{q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c}) : \mathsf{r}_F(q) \in \Phi_{\langle c_i\rangle}\}$ is invariant for every facet F containing c_i. In particular, if $F = \mathbf{c}$, this set is equal to $\{q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c}) : \mathsf{Lr}_c(q) \in (\Phi_{\langle c_i\rangle})_{\geq -1}\}$. Therefore, $\mathsf{r}_F(q) \in \Phi_{\langle c_i\rangle}$ if and only if $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i\rangle})_{\geq -1}$. ☐

Proposition 3.36. *If a facet F of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ contains c_i and $q \neq c_i$, then $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i\rangle})_{\geq -1}$.*

Proof. This proposition is a direct consequence of Lemma 3.35. ☐

Next, we consider the parabolic subgroup $W_{\langle c_i\rangle}$ obtained by removing the generator c_i from S.

Lemma 3.37. *Let c' be the Coxeter element of the parabolic subgroup $W_{\langle c_i\rangle}$ obtained from c by removing the generator c_i. Consider the word $\widehat{Q} = \mathbf{c}'\mathbf{w}_\circ(\mathbf{c})$ obtained by deleting* *the letter c_i from $Q = \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, and let $Q' = \mathbf{c}'\mathbf{w}_\circ(\mathbf{c}')$. Then, the subword complexes $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$ are isomorphic.*

Proof. Since every facet F of $\Delta(\widehat{Q}, w_\circ)$ can be seen as a facet $F \cup \{c_i\}$ of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ which contains c_i, for every $q \in F$, we have that $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$ by Proposition 3.36. This means that only the letters of $\widehat{Q}$ that correspond to roots in $(\Phi_{\langle c_i \rangle})_{\geq -1}$ appear in the subword complex $\Delta(\widehat{Q}, w_\circ)$. The letters in Q' are in bijection, under the map $\mathsf{Lr}_{c'}$, with the almost positive roots $(\Phi_{\langle c_i \rangle})_{\geq -1}$. Let φ be the map that sends a letter $q \in \widehat{Q}$ corresponding to a root in $(\Phi_{\langle c_i \rangle})_{\geq -1}$ to the letter in Q' corresponding to the same root. We will prove that φ induces an isomorphism between the subword complexes $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$. In other words, we show that F is a facet of $\Delta(\widehat{Q}, w_\circ)$ if and only if $\varphi(F)$ is a facet of $\Delta(Q', w'_\circ)$. Let $\widetilde{\mathsf{r}}_F$ and $\mathsf{r}'_{\varphi(F)}$ be the root functions associated to F and $\varphi(F)$ in $\widehat{Q}$ and Q', respectively. Then, for every $q \in \widehat{Q}$ such that $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$, we have

$$\widetilde{\mathsf{r}}_F(q) = \mathsf{r}'_{\varphi(F)}(\varphi(q)). \tag{$\star$}$$

If $F = \mathbf{c}'$, then $\varphi(F) = \mathbf{c}'$, and the equality ($\star$) holds by the definition of φ. Moreover, if ($\star$) holds for a facet F, then it is true for a facet F' obtained by flipping a letter in F. This follows by applying Lemma 3.26 and using the fact that the positive roots $(\Phi_{\langle c_i \rangle})_{\geq -1}$ in $\widehat{Q}$ and Q' appear in the same order, see [Rea07a, Prop. 3.2]. Finally, Lemma 3.23 and ($\star$) imply that the map φ sends flips to flips. Since $\mathbf{c}'$ and $\varphi(\mathbf{c}')$ are facets of $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$, respectively, and all facets are connected by flips, F is a facet of $\Delta(\widehat{Q}, w_\circ)$ if and only if $\varphi(F)$ is a facet of $\Delta(Q', w'_\circ)$. □

The next lemma states that every letter in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ is indeed a vertex of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$.

Lemma 3.38. *Every letter in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ is contained in some facet of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$.*

Proof. Write the word $Q = \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ as the concatenation of $\mathbf{c}$ and the c-factorization of $w_\circ$, i.e., $Q = \mathbf{c}\mathbf{c}_{K_1}\mathbf{c}_{K_2}\cdots\mathbf{c}_{K_r}$, where $K_i \subseteq S$ for $1 \leq i \leq r$ and c_I, with $I \subseteq S$, is the Coxeter element of W_I obtained from c by keeping only letters in I. Since $w_\circ$ is c-sortable, see [Rea07a, Corollary 4.4], the sets K_i form a decreasing chain of subsets of S, i.e., $K_r \subseteq K_{r-1} \subseteq \cdots \subseteq K_1 \subseteq S$. This implies that the word $\mathbf{c}\mathbf{c}_{K_1}\ldots\widehat{\mathbf{c}}_{K_i}\ldots\mathbf{c}_{K_r}$ contains a reduced expression for $w_\circ$ for any $1 \leq i \leq r$. Thus, all letters in $\mathbf{c}_{K_i}$ are indeed vertices. □

Proposition 3.39. *For every $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$ satisfying $\mathsf{Lr}_c(q) \in (\Phi_{\langle c_i \rangle})_{\geq -1}$, there exists a facet of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ that contains both c_i and q.*

Proof. Consider the parabolic subgroup $W_{\langle c_i \rangle}$ obtained by removing the letter c_i from S, and let $\widehat{Q}$ and Q' be the words as defined in Lemma 3.37. Since $\Delta(\widehat{Q}, w_\circ)$ and $\Delta(Q', w'_\circ)$ are isomorphic, applying Lemma 3.38 to $\Delta(Q', w'_\circ)$ completes the proof. □

Proof of Theorem 3.34. Taking $c_i = s$, $-\alpha_s = \mathsf{Lr}_c(c_i)$ and $\beta = \mathsf{Lr}_c(q)$, the two directions of the equivalence follow from Propositions 3.36 and 3.39. □

3.6.2 Proof of condition (ii)

The following theorem proves condition (ii).

Theorem 3.40. *Let $\beta_1, \beta_2 \in \Phi_{\geq -1}$, and let s be an initial letter of a Coxeter element c. Then, $\{\beta_1, \beta_2\}$ is a face of the subword complex $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ if and only if $\{\sigma_s(\beta_1), \sigma_s(\beta_2)\}$ is a face of the subword complex $\Delta(\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}'), w_\circ)$ with $c' = scs$.*

Proof. Let $Q = \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, s be initial in c, and $Q_{\circlearrowleft_s}$ be the rotated word of Q, as defined in Section 3.4.2. By Proposition 3.32 the word $Q_{\circlearrowleft_s}$ is equal to $\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}')$ up to commutations, and by Proposition 3.29 the subword complexes $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$ and $\Delta(\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}'), w_\circ)$ are isomorphic. For every letter $q \in \mathbf{c}\mathbf{w}_\circ(\mathbf{c})$, we denote by q' the corresponding letter in $\mathbf{c}'\mathbf{w}_\circ(\mathbf{c}')$ obtained from the previous isomorphism. We write $q_1 \sim_c q_2$ if and only if $\{q_1, q_2\}$ is a face of $\Delta(\mathbf{c}\mathbf{w}_\circ(\mathbf{c}), w_\circ)$. In terms of almost positive roots, this is written as

$$\mathsf{Lr}_c(q_1) \sim_c \mathsf{Lr}_c(q_2) \Longleftrightarrow \mathsf{Lr}_{scs}(q'_1) \sim_{scs} \mathsf{Lr}_{scs}\left(q'_2\right).$$

Note that the bijection Lr_{scs} can be described using Lr_c. Indeed, it is not hard to check that $\mathsf{Lr}_{scs}(q') = \sigma_s(\mathsf{Lr}_c(q))$ for all $q \in Q$. Therefore,

$$\mathsf{Lr}_c(q_1) \sim_c \mathsf{Lr}_c(q_2) \Longleftrightarrow \sigma_s(\mathsf{Lr}_c(q_1)) \sim_{scs} \sigma_s(\mathsf{Lr}_c(q_2)).$$

Taking $\beta_1 = \mathsf{Lr}_c(q_1)$ and $\beta_2 = \mathsf{Lr}_c(q_2)$, we get the desired result. □

3.7 Polytopality of spherical subword complexes

In this section, we discuss the polytopality of spherical subword complexes and present what is known in the particular cases of cluster complexes, simplicial complexes of multi-triangulations, and simplicial complexes of centrally symmetric multitriangulations. We then prove polytopality of multi-cluster complexes of rank 2. Finally, we show that every spherical subword complex is the link of a face of a multi-cluster complex, and consequently reduce the question of realizing spherical subword complexes to the question of realizing multi-cluster complexes.

Definition 3.41. A *generalized multi-associahedron* of type W is the dual of a polytopal realization of a multi-cluster complex of type W.

The existence of such realizations remains open in general; see Table 3.3. The subword complex approach provides new perspectives and methods for finding polytopal realizations. In a recent article, Christian Stump and Vincent Pilaud obtain a geometric construction of a class of subword complexes containing generalized associahedra purely in terms of subword complexes [PS12b].

simplicial complex	polytopal realization of the dual
of triangulations (classical)	associahedron [Hai84, Lee89, Lod04, Rea06] [HL07, GKZ08, CSZ11]
of multitriangulations [Jon05, Kra06, PS09, PP12, Stu11]	multi-associahedron (existence conjectured)
of centrally symmetric multitriangulations [SW09, RS10]	multi-associahedron of type B (existence conjectured)
cluster complex [FZ03, Rea06, Rea07a, Rea07b]	generalized associahedron [CFZ02, HL07, HLT11, Ste12, PS12b]
multi-cluster complex [CLS13]	generalized multi-associahedron (existence conjectured)

Table 3.3: Dictionary for generalized concepts of triangulations and associahedra.

3.7.1 Generalized associahedra

We have seen that for $k = 1$, the multi-cluster complex $\Delta^1_c(W)$ is isomorphic to the c-cluster complex. Sergey Fomin and Andrei Zelevinsky conjectured the existence of polytopal realizations of the cluster complex in [FZ03, Conjecture 1.12]. Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky then proved this conjecture by providing explicit inequalities for the defining hyperplanes of generalized associahedra [CFZ02]. Nathan Reading constructed c-Cambrian fans, which are complete simplicial fans coarsening the Coxeter fan, see [Rea06]. In [RS09], Nathan Reading and David Speyer prove that these fans are combinatorially isomorphic to the normal fan of the polytopal realization in [CFZ02]. Christophe Hohlweg, Carsten Lange and Hugh Thomas then provided a family of c-generalized associahedra having c-Cambrian fans as normal fans by removing certain hyperplanes from the permutahedron [HLT11]. Vincent Pilaud and Christian Stump recovered c-generalized associahedra by giving explicit vertex and hyperplane descriptions purely in terms of the subword complex approach introduced in the present paper [PS12b]. Moreover, Salvatore Stella completed the construction of Frédéric Chapoton, Sergey Fomin and Andrei Zelevinsky to all orientations of the Dynkin diagram and showed its relation to the one of Christophe Hohlweg, Carsten Lange and Hugh Thomas in the context of Cambrian fans, see [Ste12].

3.7.2 Multi-associahedra of type A

In type A_n for $n = m - 2k - 1$, the multi-cluster complex $\Delta_c^k(A_n)$ is isomorphic to the simplicial complex $\Delta_{m,k}$ of k-triangulations of a convex m-gon. This simplicial complex is conjectured to be realizable as the boundary complex of a polytope[2]. It was studied in many different contexts. See [PS09, Section 1] for a detailed description of previous work on multitriangulations. Apart from the most simple cases, very little is known about its polytopality. Nevertheless, this simplicial complex possesses very nice properties which makes this conjecture plausible. Indeed, the subword complex approach provides a simple description of the 1-skeleton of a possible multi-associahedron (see Lemma 3.23), and gives a new and very simple proof that it is a vertex-decomposable triangulated sphere [Stu11, Theorem 2.1]; see also [Jon03]. Below we survey the known polytopal realizations of $\Delta_{m,k}$ as boundary complexes of convex polytopes. The simplicial complex $\Delta_{m,k}$, or equivalently the multi-cluster complex $\Delta_c^k(A_n)$ for $n = m-2k-1$, is the boundary complex of

- a point if $k = 0$,
- an n-dimensional dual associahedron if $k = 1$,
- a k-dimensional simplex if $n = 1$,
- a $2k$-dimensional cyclic polytope on $2k + 3$ vertices if $n = 2$, see [PS09, Section 8],
- a 6-dimensional simplicial polytope if $n = 3$ and $k = 2$, see [BP09].

The case $n = 2$ is also a direct consequence of the rank 2 description in Section 3.7.4. Further unsuccessful attempts to realize $\Delta_{m,k}$ come from various directions in discrete geometry.

(a) A generalized construction of the polytope of pseudo-triangulations [RSS08] using rigidity of pseudo-triangulations [Pil10, Section 4.2 and Remark 4.82].

(b) A generalized construction of the secondary polytope. As presented in [GKZ08], the secondary polytope of a point configuration can be generalized using star polygons [Pil10, Section 4.3].

[2] As far as we know, the first reference to this conjecture appears in [Jon05, Section 1].

(c) The brick polytope of a sorting network [PS12a]. This new approach brought up a large family of spherical subword complexes that are realizable as the boundary of a polytope. In particular, it provides a new perspective on generalized associahedra [PS12b]. Unfortunately, this polytope fails to realize the multi-associahedron.

3.7.3 Multi-associahedra of type B

Theorem 3.14 stated that the multi-cluster complex $\Delta_c^k(B_{m-k})$ is isomorphic to the simplicial complex of centrally symmetric k-triangulations of a regular convex $2m$-gon. This simplicial complex was studied in [SW09, RS10]. We then present what is known about its polytopality. The new approach using subword complexes provides in particular very simple proofs of Corollaries 3.15, 3.17, and 3.18.

Proof of Theorem 3.14. Let $S = \{s_0, s_1, \ldots, s_{m-k-1}\}$ be the generators of B_{m-k}, where the generator s_0 is such that $(s_0 s_1)^4 = \mathbf{1} \in W$, and the other generators satisfy the same relations as in type A_{m-k-1}. Then, embed the group B_{m-k} in the group $A_{2(m-k)-1}$ by the standard folding technique: replace s_0 by s'_{m-k} and s_i by $s'_{m-k+i}s'_{m-k-i}$ for $1 \leq i \leq m-k-1$, where the set $S' = \{s'_1, \ldots, s'_{2(m-k)-1}\}$ generates the group $A_{2(m-k)-1}$. The multi-cluster complex $\Delta_c^k(B_{m-k})$ now has an embedding into the multi-cluster complex $\Delta_{c'}^k(A_{2(m-k)-1})$, where c' is the Coxeter element of type $A_{2(m-k)-1}$ corresponding to c in B_{m-k}; the corresponding subcomplex has the property that $2(m-k)$ generators (all of them except s'_{m-k}) always come in pairs. Using the correspondence between k-triangulations and the multi-cluster complex described in Section 1.3, the facets of $\Delta_c^k(B_{m-k})$ considered in $\Delta_{c'}^k(A_{2(m-k)-1})$ correspond to centrally symmetric multitriangulations. $\square$

Here, we present the few cases for which this simplicial complex is known to be polytopal. The multi-cluster complex $\Delta_c^k(B_{m-k})$ is the boundary complex of

- an $(m-1)$-dimensional dual cyclohedron (or type B associahedron) if $k=1$, see [Sim03, HL07],
- an $(m-1)$-dimensional simplex if $k = m-1$,
- a $(2m-4)$-dimensional cyclic polytope on $2m$ vertices if $k = m-2$, see [SW09].

The case $k = m-2$ also follows from the rank 2 description in Section 3.7.4.

3.7.4 Generalized multi-associahedra of rank 2

We now prove that multi-cluster complexes of rank 2 can be realized as boundary complexes of cyclic polytopes. In other words, we show the existence of rank 2 multi-associahedra. This case was known independently by Drew Armstrong [Arm11].

Theorem 3.42 (Type $I_2(m)$ multi-associahedra)**.** *The multi-cluster complex $\Delta_c^k(I_2(m))$ is isomorphic to the boundary complex of a $2k$-dimensional cyclic polytope on $2k+m$ vertices. The multi-associahedron of type $I_2(m)$ is the simple polytope given by the dual of a $2k$-dimensional cyclic polytope on $2k+m$ vertices.*

Proof. This is obtained by Gale's evenness criterion (see [Zie95, Section 0]) on the word $Q = (a, b, a, b, a, \dots)$ of length $2k+m$: Let F be a facet of $\Delta_c^k(I_2(m))$, and take two consecutive letters x and y in the complement of F. Since the complement of F is a reduced expression of $w_\circ$, x and y must represent different generators. Since the letters in Q are alternating, this implies that the number of letters between x and y is even. □

3.7.5 Generalized multi-associahedra

Recall from Section 3.1 that a subword complex $\Delta(Q, \pi)$ is homeomorphic to a sphere if and only if the Demazure product $\delta(Q)$ is π, and it is homeomorphic to a ball otherwise. This motivated the question whether spherical subword complexes can be realized as boundary complexes of polytopes [KM04, Question 6.4.]. We show that it is enough to consider multi-cluster complexes to prove polytopality for all spherical subword complexes, and we characterize simplicial spheres that can be realized as subword complexes in terms of faces of multi-cluster complexes.

Proposition 3.43. *Every spherical subword complex $\Delta(Q, w_\circ)$ is the link of a face of a multi-cluster complex $\Delta(\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}), w_\circ)$.*

Proof. Observe that any word Q in S can be embedded as a subword of $Q' = \mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ for k less than or equal to the size of Q, by assigning the ith letter of Q within the ith copy of $\mathbf{c}$. Since the Demazure product $\delta(Q)$ is equal to $w_\circ$, the word Q contains a reduced expression for $w_\circ$. In other words, the set $Q' \setminus Q$ is a face of $\Delta(Q', w_\circ)$. The link of this face in $\Delta(Q', w_\circ)$ consists of subwords of Q—viewed as a subword of Q'—whose complements contain a reduced expression of $w_\circ$. This corresponds exactly to the subword complex $\Delta(Q, w_\circ)$. □

We now prove that simplicial spheres realizable as subword complexes are links of faces of multi-cluster complexes.

Proof of Theorem 3.19. For any spherical subword complex $\Delta(Q, \pi)$, we have that the Demazure product $\delta(Q)$ equals π. By Theorem 3.27, $\Delta(Q, \pi)$ is isomorphic to a subword complex of the form $\Delta(Q', w_\circ)$. Using the previous lemma, we obtain that $\Delta(Q, \pi)$ is the link of a face of a multi-cluster complex. The other direction follows since the link of a subword (i.e., a face) of a multi-cluster complex is itself a subword complex, corresponding to the complement of this subword. □

Finally, we prove that the question of polytopality of spherical subword complexes is equivalent to the question of polytopality of multi-cluster complexes.

Proof of Corollary 3.20. On one hand, if every spherical subword complex is polytopal, then clearly every multi-cluster complex is polytopal. On the other hand, suppose that every spherical subword complex is polytopal. Every spherical subword complex is the link of a face of a multi-cluster complex. Since the link of a face of a polytope is also polytopal, Theorem 3.19 implies that every spherical subword complex is polytopal. □

3.8 Sorting words of the longest element and the SIN-property

In this section, we give a simple combinatorial description of the c-sorting words of $w_\circ$ and prove that a word Q coincides up to commutations with $\mathbf{c}^k \mathbf{w}_\circ(\mathbf{c})$ for some nonnegative integer k if and only if Q has the SIN-property as defined in Section 3.3. This gives us an alternative way of defining multi-cluster complexes in terms of words having the SIN-property. Recall the involution $\psi : S \to S$ from Section 3.5 defined by $\psi(s) = w_\circ^{-1} s w_\circ$. The sorting word of $w_\circ$ has the following important property.

Proposition 3.44. *The sorting word* $\mathbf{w}_\circ(\mathbf{c})$ *is, up to commutations, equal to a word with suffix* $(\psi(c_1), \ldots, \psi(c_n))$*, where* $c = c_1 \cdots c_n$*.*

Proof. As $w_\circ$ has a c-sorting word having $\mathbf{c} = (c_1, \ldots, c_n)$ as a prefix, the corollary is obtained by applying Proposition 3.32 n times. □

Given a word $\mathbf{w}$ in S, define the function $\phi_{\mathbf{w}} : S \to \mathbb{N}$ given by $\phi_{\mathbf{w}}(s)$ being the number of occurrences of the letter s in $\mathbf{w}$.

Theorem 3.45. *Let $\mathbf{w}_\circ(\mathbf{c})$ be the c-sorting word of $w_\circ$, and let s, t be neighbors in the Coxeter graph such that s comes before t in c. Then*

$$\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) = \begin{cases} \phi_{\mathbf{w}_\circ(\mathbf{c})}(t) & \text{if } \psi(s) \text{ comes before } \psi(t) \text{ in } c, \\ \phi_{\mathbf{w}_\circ(\mathbf{c})}(t) + 1 & \text{if } \psi(s) \text{ comes after } \psi(t) \text{ in } c. \end{cases}$$

Proof. Sorting words of $w_\circ$ have intervening neighbors; see [Spe09, Proposition 2.1] for an equivalent formulation. Therefore, s and t alternate in $\mathbf{w}_\circ(\mathbf{c})$, with s coming first. Thus, $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(t)$ if and only if the last t comes after the last s. Using Proposition 3.44, this means that s appears before t in $\psi(\mathbf{c})$ or equivalently $\psi(s)$ appear before $\psi(t)$ in c. Otherwise, the last s will appear after the last t. □

It is known that if ψ is the identity on S, or equivalently if $w_\circ = -\mathbf{1}$, then the c-sorting word of $w_\circ$ is given by $\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^{\frac{h}{2}}$, where h denotes the *Coxeter number* given by the order of any Coxeter element. In the case where ψ is not the identity on S (that is, when W is of types A_n ($n \geq 2$), D_n (n odd), E_6 and $I_2(m)$ (m odd), see [BB05, Exercise 10 of Chapter 4]), the previous theorem gives a simple way to obtain the sorting words of $w_\circ$.

Algorithm 3.46. *Let W be an irreducible finite Coxeter group, and let $c = c_1 c_2 \cdots c_n$ be a Coxeter element.*

(i) *Since the Coxeter diagram is connected, one can use Theorem 3.45 to compute $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s)$ for all s depending on $m := \phi_{\mathbf{w}_\circ(\mathbf{c})}(c_1)$,*

(ii) *using that the number of positive roots equals $nh/2$, one obtains m and thus all $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s)$ using*

$$2 \cdot \sum_{s \in S} \phi_{\mathbf{w}_\circ(\mathbf{c})}(s) = nh,$$

(iii) *using that $\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}_{K_1} \mathbf{c}_{K_2} \cdots \mathbf{c}_{K_r}$ where $K_i \subseteq S$ for $1 \leq i \leq r$ and c_I, with $I \subseteq S$, is the Coxeter element of W_I obtained from c by keeping only letters in I, we obtain that $\mathbf{c}_{K_i}$ is the product of all s for which $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) \geq i$.*

This algorithm provides an explicit description of the sorting words of the longest element $w_\circ$ of any finite Coxeter group using nothing other than Coxeter group theory. This answers a question raised in [HLT11, Remark 2.3] and simplifies a step in the construction of the c-generalized associahedron. In Section 3.9, we use this description to count the number of singletons. We now give two examples of how to use this algorithm.

Example 3.47. Let $W = A_4$ and $S = \{s_1, s_2, s_3, s_4\}$ with the relations $(s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_4)^3 = e$ and the other pairs of generators commute. Moreover, let $c = s_1 s_3 s_2 s_4$. Fix $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_1) = m$. Since s_1 comes before s_2 in c and that $\psi(s_1) = s_4$

comes after $\psi(s_2) = s_3$, the letter s_1 appears one more time than the letter s_2 in $\mathbf{w}_\circ(\mathbf{c})$, i.e., $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_2) = m-1$. Repeating the same argument gives $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_3) = m$ and $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_4) = m-1$. Summing up these values gives the equality $4m-2 = \frac{n\cdot h}{2} = \frac{4\cdot 5}{2} = 10$, and thus $m=3$. Finally, the c-sorting word is $\mathbf{w}_\circ(\mathbf{c}) = (s_1, s_3, s_2, s_4|s_1, s_3, s_2, s_4|s_1, s_3)$.

Example 3.48. Let $W = E_6$ and $S = \{s_1, s_2, \ldots, s_6\}$ with the relations $(s_1s_2)^3 = (s_2s_6)^3 = (s_4s_5)^3 = (s_5s_6)^3 = (s_3s_6)^3$ and the other pairs of generators commute. Moreover, let $c = s_3s_5s_4s_6s_2s_1$. Fix $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_6) = m$. Repeating the same procedure from the previous example and using that $\psi(s_6) = s_6$, $\psi(s_3) = s_3$, $\psi(s_2) = s_5$, $\psi(s_1) = s_4$, we get $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_1) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s_2) = m-1$, $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_3) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s_6) = m$, $\phi_{\mathbf{w}_\circ(\mathbf{c})}(s_4) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s_5) = m+1$. As the sum equals $\frac{nh}{2} = \frac{6\cdot 12}{2} = 36$, we obtain $m=6$. Finally, the c-sorting word is $(\mathbf{c}^5|s_3, s_5, s_4, s_6|s_5, s_4)$.

Remark 3.49. Propositions 3.32 and 3.44 have the following computational consequences. Denote by $\operatorname{rev}(\mathbf{w})$ the *reverse* of a word $\mathbf{w}$. First, up to commutations, we have

$$\mathbf{w}_\circ(\mathbf{c}) = \operatorname{rev}(\mathbf{w}_\circ(\psi(\operatorname{rev}(\mathbf{c})))).$$

Second, we also have, up to commutation,

$$\mathbf{c}^h = \mathbf{w}_\circ(\mathbf{c})\operatorname{rev}(\mathbf{w}_\circ(\operatorname{rev}(\mathbf{c}))).$$

Third, for all $s \in S$,

$$\phi_{\mathbf{w}_\circ(\mathbf{c})}(s) + \phi_{\mathbf{w}_\circ(\operatorname{rev}(\mathbf{c}))}(s) = \phi_{\mathbf{w}_\circ(\mathbf{c})}(s) + \phi_{\mathbf{w}_\circ(\mathbf{c})}(\psi(s)) = h.$$

We are now in the position to prove Theorem 3.11.

Proof of Theorem 3.11. Suppose that a word Q has the SIN-property; then it has complete support by definition, and it contains, up to commutations, some word $\mathbf{c} = (c_1, \ldots, c_n)$ for a Coxeter element c as a prefix. Moreover, the word $(\psi(c_1), \ldots, \psi(c_n))$ is a suffix of Q, up to commutations. Observe that a word has intervening neighbors if and only if it is a prefix of $\mathbf{c}^\infty$ up to commutations, see [EE09, Section 3]. In view of Lemma 3.30 and the equality $\delta(Q) = w_\circ$, the word Q has, up to commutations, $\mathbf{w}_\circ(\mathbf{c})$ as a prefix. If the length of Q equals $w_\circ$, the proof ends here with $k=0$. Otherwise, the analogous argument for $\operatorname{rev}(Q)$ gives that the word $\operatorname{rev}(Q)$ has, up to commutations, $\mathbf{w}_\circ(\psi(\operatorname{rev}(\mathbf{c})))$ as a prefix. By Remark 3.49, the word $\mathbf{w}_\circ(\psi(\operatorname{rev}(\mathbf{c})))$ is, up commutations, equal to the reverse of $\mathbf{w}_\circ(\mathbf{c})$. Therefore, Q has the word $\mathbf{w}_\circ(\mathbf{c})$ also as a suffix. Since $\mathbf{c} = (c_1, \ldots, c_n)$ is a prefix of Q and of $\mathbf{w}_\circ(\mathbf{c})$, and Q has intervening neighbors, Q coincides with $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ up to commutations. Moreover, if Q is equal to $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ up to commutations, it has intervening neighbors, and a suffix $(\psi(c_1), \dots, \psi(c_n))$, up to commutations, by Proposition 3.44. This implies that the word Q has the SIN-property. □

Remark 3.50. In light of Theorem 3.11 and Section 3.4.2, starting with a word Q having the SIN-property suffices to construct a multi-cluster complex, and choosing a particular connected subword in the bi-infinite word $\widetilde{Q}$, defined in Section 3.4.2, corresponds to choosing a particular Coxeter element.

3.9 Common vertices of permutahedra and generalized associahedra

The associahedron has been realized using many different constructions. We refer the reader to the recent book [MHPS12] for an extensive collection of them. In this section, we consider the construction of the c-generalized associahedron for arbitrary finite irreducible Coxeter groups presented in [HLT11] which is a generalization of a construction by Christophe Hohlweg and Carsten Lange for type A and B presented in [HL07]. This construction is also described in [MHPS12, Chapter 8]. In this construction, the c-generalized associahedron is obtained from the n-dimensional W-permutahedron by *taking away* certain facets. The n-dimensional W-permutahedron is a simple zonotope whose 1-skeleton corresponds to the Hasse diagram of the weak order of the Coxeter group W. In particular, the vertices of the W-permutahedron are labeled by elements of the group. The facets to take away are determined using the notions of sorting words and sortable elements, discussed in Section 3.5 and 3.8; they are facets that do not contain a vertex labeled by a c-*singleton* element. Consequently, c-singletons correspond to the vertices shared by the W-permutahedron and the resultant c-generalized associahedron, which is dual to the c-cluster complex of the corresponding type. In this section, we give explicit formulas counting singletons for all finite Coxeter groups and Coxeter elements. This answers a question raised in [HL07, Section 4.2] and [MHPS12, Problem 3.3, Chapter 8] about lower and upper bounds for common vertices.

3.9.1 Natural partial order and singletons

Let us recall the definition of sorting words; see Definition 3.3. Let $\mathbf{c} = (c_1, \dots, c_n)$ be a reduced expression for a Coxeter element $c \in W$, and let $\mathbf{w}(\mathbf{c}) = (w_1, \dots, w_N)$ be the lexicographically first subword of $\mathbf{c}^\infty$ that represents a reduced expression for the element $w \in W$. The word $\mathbf{w}_\circ(\mathbf{c})$ is called c-sorting word for $w_\circ$ (see Section 3.5 and 3.8 for more details about sorting words and their related structures).

Definition 3.51 (Hohlweg–Lange–Thomas [HLT11, Theorem 2.2])**.** Let W be a finite irreducible Coxeter group and c be a Coxeter element. A *c-singleton* is an element of W expressible by a prefix of the c-sorting word $\mathbf{w}_\circ(\mathbf{c})$ up to commutations.

Singletons admit different definitions. For example, they are sortable elements which are alone in their projection class $\pi_c^\downarrow$ (see [Rea07a]), therefore the name singleton. In Section 3.8, Theorem 3.45, Algorithm 3.46, and Remark 3.49 give a combinatorial description of the sorting word $\mathbf{w}_\circ(\mathbf{c})$.

Remark 3.52. In this section, we assume that Coxeter groups are finite and irreducible. The enumerative results of this section can easily be extended to reducible Coxeter groups by taking the product of the irreducible parabolic subgroups formulas.

Ádám Galambos and Vic Reiner gave a definition of *natural partial order* on crossings in a pseudoline arrangements, see [GR08, Definition 6]. This turns out to be an efficient way to encode prefixes of a longest word up to commutation as ideals of this natural partial order. To work with arbitrary finite irreducible Coxeter groups, we generalize the natural partial order.

Definition 3.53. Let W be an irreducible finite Coxeter group and $\mathbf{w}_\circ$ be a longest word. Define the *natural partial order* $\prec_{\mathbf{w}_\circ}$ on the letters of a longest word $\mathbf{w}_\circ$ as follows: $p \prec_{\mathbf{w}_\circ} q$ if and only if there exists a subword $u = (u_1, \dots, u_k)$, with $k \geq 1$, of $\mathbf{w}_\circ$ such that $u_1 = p$, $u_k = q$ and every pair u_i, u_{i+1} for $1 \leq i \leq k-1$ do not commute.

Lemma 3.54. *Let W be an irreducible finite Coxeter group, c a Coxeter element and $\mathbf{w}_\circ(\mathbf{c})$ the c-sorting word of $w_\circ$. Lower ideals of $\prec_{\mathbf{w}_\circ(\mathbf{c})}$ are in bijection with c-singletons.*

Proof. The set of c-singletons ordered with the (right) weak order is known to form a distributive lattice, see [HLT11, Proposition 2.5]. Given a finite distributive lattice L, there exists a unique (up to isomorphism) poset P for which the lattice of lower ideal $J(P)$ is isomorphic to L, see [Sta12, Theorem 3.4.1]. For a finite poset, anti-chains are in one-to-one correspondance with lower ideals, see [Sta12, Section 3.1]. We will prove that the lattice of lower ideals of $\prec_{\mathbf{w}_\circ(\mathbf{c})}$ is isomorphic to the lattice of c-singletons ordered by the weak order. Let w_c be a c-singleton, that is a prefix of $\mathbf{w}_\circ(\mathbf{c})$ up to commutation. This prefix is a subset of letters of $\mathbf{w}_\circ(\mathbf{c})$. If t is a letter in w_c and $s \prec_{\mathbf{w}_\circ(\mathbf{c})} t$, then there exists a subword $u = (u_1, \dots, u_k)$ of $\mathbf{w}_\circ(\mathbf{c})$, with $u_1 = s$ and $u_k = t$ such that every pair u_i, u_{i+1} for $1 \leq i \leq k-1$ do not commute. The subword w_c is a prefix of $\mathbf{w}_\circ(\mathbf{c})$ and therefore every letter in the subword u have to be in w_c. Indeed, otherwise, a certain u_i on the left of t in $\mathbf{w}_\circ(\mathbf{c})$ could not be moved to the right of t using only commutations, in order for t to be in a prefix w_c. In other words, letters of w_c form a lower ideal of $\prec_{\mathbf{w}_\circ(\mathbf{c})}$. The cover relations of the weak order on the c-singletons transfer to cover relations on lower ideals of $\prec_{\mathbf{w}_\circ(\mathbf{c})}$. The identity element correspond to the empty lower ideal and the c-singleton $\mathbf{w}_\circ(\mathbf{c})$ correspond to the complete poset $\prec_{\mathbf{w}_\circ(\mathbf{c})}$. Since both posets are finite, they are isomorphic. See [Sta12, Exercice 3.123] for a related problem. □

In this section, we enumerate ideals of the natural orders $\prec_{\mathbf{w}_\circ(\mathbf{c})}$, for all irreducible finite Coxeter groups and Coxeter elements.

3.9.2 Cylindric graphs of longest words and cuts

For all longest words $\mathbf{w}_\circ$ of a finite irreducible Coxeter group W, we introduce a graph called the *cylindric graph* that encodes the natural partial order $\prec_{\mathbf{w}_\circ}$ and *cuts* of a cylindric graph. Cuts encode conjugation by initial letters of $\mathbf{w}_\circ$ and lower ideals of the natural order. We start with a simple lemma.

Lemma 3.55. *Let $\mathbf{w}_\circ = (w_1, \cdots, w_N)$ be a longest word of W and define the* cyclic longest word

$$\overset{\circlearrowleft}{\mathbf{w}}_\circ := (w_1, \cdots, w_N, \psi(w_1), \cdots, \psi(w_N))\,.$$

Then, considering the word $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ cyclically, all consecutive subwords (up to commutations) of length N are reduced expressions of $w_\circ$.

Proof. By definition, there is a sequence of commutation and rotation of letters (see the definition before Proposition 3.28 on page 51) from any consecutive subword of $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ of length N, up to commutation, to $\mathbf{w}_\circ = (w_1, \cdots, w_N)$. These two actions leave the expression reduced. □

Definition 3.56. Let $\mathbf{w}_\circ = (w_1, \cdots, w_N)$ be a longest word of a finite irreducible Coxeter group W. The *cylindric graph* $\mathcal{Z}_{\mathbf{w}_\circ} = (V, E)$ of $\mathbf{w}_\circ$ is the oriented graph with vertices indexed by the $2N$ letters in $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ and edges given by $v_{w_i} \longrightarrow v_{w_j}$, for any noncommuting pair w_i, w_j such that w_i comes before w_j in a consecutive subword of $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ (considered cyclically) and no letter represented by w_i or w_j appear between them.

Let $\mathcal{Z}$ denote a bounded cylindric surface and refer to its boundaries as the bottom and top boundaries.

Example 3.57. Let $(W, S) = (A_4, \{s_1, s_2, s_3, s_4\})$. Consider the longest word $\mathbf{w}_\circ = (s_3, s_2, s_1, s_2, s_3, s_4, s_2, s_3, s_2, s_1)$. The cyclic word is then $\overset{\circlearrowleft}{\mathbf{w}}_\circ = (s_3, s_2, s_1, s_2, s_3, s_4, s_2, s_3, s_2, s_1 | s_2, s_3, s_4, s_3, s_2, s_1, s_3, s_2, s_3, s_4)$. The cylindric graph $\mathcal{Z}_{\mathbf{w}_\circ}$ is depicted in Fig. 3.2.

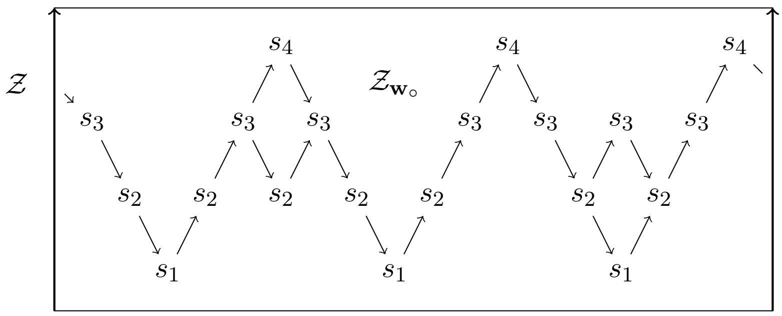


FIGURE 3.2: A cylindric graph $\mathcal{Z}_{\mathbf{w}_\circ}$ for the word $\mathbf{w}_\circ$ in the group A_4.

Definition 3.58. A *loop* L is a minimal length cyclic oriented closed chain in $\mathcal{Z}_{\mathbf{w}_\circ}$. Let $\mathcal{L}_{\mathbf{w}_\circ}$ be the set of loops of $\mathcal{Z}_{\mathbf{w}_\circ}$.

A priori, it is not clear if it is always possible to embed a cylindric graph on a cylinder as on Fig. 3.2. The following two lemmas give topological and combinatorial properties of cylindric graphs that guarantee the existence of natural embeddings on a cylinder $\mathcal{Z}$. The present approach relies on the existence of such natural embeddings.

Lemma 3.59. *Let W be a finite irreducible Coxeter group and $\mathbf{w}_\circ$ a longest word. There is a natural embedding of a cylindric graph $\mathcal{Z}_{\mathbf{w}_\circ}$ on a cylinder $\mathcal{Z}$, with the following properties:*

(i) *the projections of oriented edges of $\mathcal{Z}_{\mathbf{w}_\circ}$ on a boundary of $\mathcal{Z}$ all share the same orientation,*

(ii) *a loop of $\mathcal{L}_{\mathbf{w}_\circ}$ goes around $\mathcal{Z}$ exactly once.*

Proof. First we label the vertices of the Coxeter graph Γ. Select a longest chain τ in Γ and set k as the number of vertices in it. The integer k is equal to $n-1$ when Γ is of type D or E, it is equal to n otherwise. Label the vertices going through the chain τ using label $s_{1,1}$ to $s_{k,k}$. If $k = n-1$, label the remaining vertex with the special label $s_{n,j}$, where j is the integer used in the label of its neighbor. Let $\mathbf{c} = (s_{1,i_1}, s_{2,i_2}, \dots, s_{n,i_n})$, where i_j are given by the labeling, and let m be the smallest integer such that $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ is a subword of $\mathbf{c}^m$. Define the functions

$$\begin{aligned} \mathcal{E}^p_{\mathbf{c}} : \quad & S \to \mathbb{Z} \times \{1, \dots, n\} \\ & s_{i_1,i_2} \mapsto (x_{i_1} + 2p, i_2), \end{aligned}$$

where $x_{i_1} = x_{j_1} + 1$ whenever $s_{i_1,i_2} \longrightarrow s_{j_1,j_2}$ in the orientation given by $\mathbf{c}$, $x_1 = 1$ and $0 \le p \le m$. Looking at $E^0_{\mathbf{c}}(S)$, putting edges between noncommuting vertices and orienting them according to $\mathbf{c}$ gives an embedding of Γ on the plane such that projections of edges on the first coordinate all have the same orientation. The other functions give embeddings which are translations of $E^0_{\mathbf{c}}$. Form the graph $G_{\mathbf{c}}$ from the $m+1$ copies together with the oriented edges $(x_{i_1}+2r, i_2) \longrightarrow (x_{j_1}+2r+2, j_2)$, where $0 \leq r < m$, whenever s_{i_1,i_2} and s_{j_1,j_2} do not commute and $s_{j_1,j_2} \longrightarrow s_{i_1,i_2}$ in $\mathbf{c}$. This graph is embedded on the infinite strip $\mathbb{R} \times [0, n+1]$. Next, take the identification $0 = n+2m+1$ on the first coordinate of this strip, to obtain a cylinder $\mathcal{Z}$. Then, move each vertex in $E^0_{\mathbf{c}}(S)$ across the identification line to its corresponding vertex in $E^m_{\mathbf{c}}$ to obtain a graph $\mathcal{Z}_{\mathbf{c}}$ embedded on a cylinder $\mathcal{Z}$. This graph has property (i) by construction. Now, consider a minimal length oriented closed cycle C in $\mathcal{Z}_{\mathbf{c}}$ and its projection $P(C)$ on a boundary of $\mathcal{Z}$. Suppose it goes $k > 1$ times around the cylinder. Using basic algebraic topology arguments and property (i), one can show, by contradiction, that there exists a point p on the boundary whose fiber $P^{-1}(p)$ has fewer than k elements. This means that the oriented closed cycle is self-crossing, therefore one can remove a part of the oriented closed cycle and obtain a shorter closed cycle. This contradicts the minimality of the length of C and proves property (ii). The last step in the proof is to realize that $\mathcal{Z}_{\mathbf{w}_\circ}$ is a subgraph of $\mathcal{Z}_{\mathbf{c}}$. Using the embedding of $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ in $\mathbf{c}^m$, one gets the desired properties. □

Example 3.60 (Example 3.57 continued)**.** The cyclic longest word $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ is embedded in the word $\mathbf{c}^9 = (s_1, s_2, s_3, s_4)^9$. The natural embedding is depicted in Fig. 3.3.

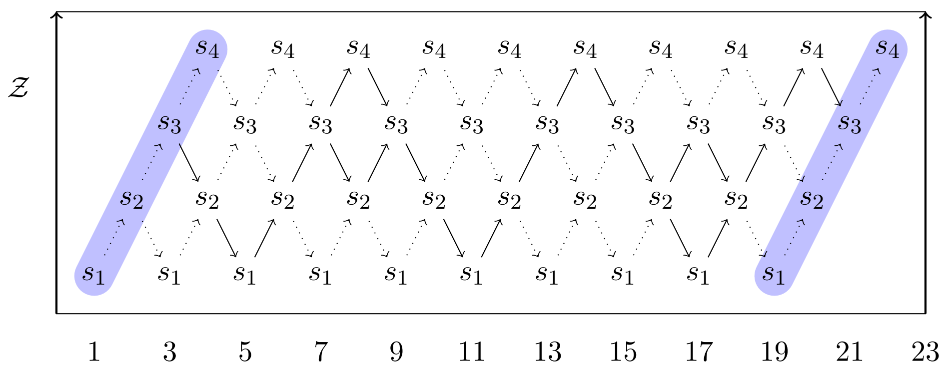


FIGURE 3.3: The natural embedding of $\mathcal{Z}_{\mathbf{w}_\circ}$ of Example 3.57 as a subgraph of $\mathcal{Z}_{\mathbf{c}}$, shown with dotted edges. The shadowed subgraph are identified to obtain the cylindric embedding.

Fix once and for all a natural embedding of $\mathcal{Z}_{\mathbf{w}_\circ}$ on $\mathcal{Z}$.

Definition 3.61. Let $\mathbf{w}_\circ$ be a longest word and $\mathcal{Z}_{\mathbf{w}_\circ}$ its cylindric graph. A *tile* T of $\mathcal{Z}_{\mathbf{w}_\circ}$ is a connected component of $\mathcal{Z} \setminus \mathcal{Z}_{\mathbf{w}_\circ}$ which do not contain a boundary of $\mathcal{Z}$. The *boundary graph* $\mathcal{G}_T$ of a tile T is the induced oriented subgraph of $\mathcal{Z}_{\mathbf{w}_\circ}$ whose vertices are on the boundary of T.

The next lemma allows to identify a tile of a cylindric graph with the unique source of its boundary.

Lemma 3.62. *Let W be a finite irreducible Coxeter group, and let $\mathbf{w}_\circ$ be a longest word. The cylindric graph $\mathcal{Z}_{\mathbf{w}_\circ}$ has the following properties;*

(i) *the graph $\mathcal{Z}_{\mathbf{w}_\circ}$ contains no sources nor sinks,*

(ii) *the boundary graph $\mathcal{G}_T$ of a tile T of $\mathcal{Z}_{\mathbf{w}_\circ}$ is an oriented cycle with exactly one source and one sink.*

Proof. (i) Let v be a vertex of $\mathcal{Z}_{\mathbf{w}_\circ}$ corresponding to a generator $s \in S$ and consider a consecutive longest word starting with the vertex v. Since all generators in S appear in any longest word, there exists a generator t not commuting with s that appears after it. Hence v is not a sink. The same argument for a longest word ending with the vertex v proves that v is not a source. (ii) As a consequence of (i), any oriented path in $\mathcal{Z}_{\mathbf{w}_\circ}$ can be extended by adding another oriented edge. Therefore, the boundary graph $\mathcal{G}_T$ of a tile T is necessarily a oriented cycle. Assume that the boundary graph $\mathcal{G}_T$ has two sources u_1 and u_2 that separates $\mathcal{G}_T$ into two component p_1 and p_2. Using Lemma 3.59, there has to be a chord going from p_1 to p_2 disconnecting the tile T, which is impossible. The proof for sinks works mutatis mutandis. □

Definition 3.63. Let $\mathbf{w}_\circ$ be a longest word and $\mathcal{Z}_{\mathbf{w}_\circ}$ its cylindric graph. A *cut* κ of $\mathcal{Z}_{\mathbf{w}_\circ}$ is a path joining the two boundaries of $\mathcal{Z}$ starting from the bottom which crosses $\mathcal{Z}_{\mathbf{w}_\circ}$ on edges and such that any loop of $\mathcal{Z}_{\mathbf{w}_\circ}$ is cut exactly once by κ.

A cut κ corresponds to a (possibly empty) sequence of conjugation by initial letters of $\mathbf{w}_\circ$ to obtain a word $\overset{\circlearrowleft}{\mathbf{w}}_{\circ,\kappa} = \mathbf{w}_{\circ,\kappa}\psi(\mathbf{w}_{\circ,\kappa})$. Moreover, this cut determines another cut κ^* which separates $\mathbf{w}_{\circ,\kappa}$ and $\psi(\mathbf{w}_{\circ,\kappa})$. We refer to this cut as the *opposite* of κ.

Example 3.64. Following Example 3.57, consider the cut κ depicted in Fig. 3.4. The cut κ corresponds to the word $\overset{\circlearrowleft}{\mathbf{w}}_{\circ,\kappa}$ obtained from $\overset{\circlearrowleft}{\mathbf{w}}_\circ$ by a sequence of conjugation by initial letters and the cut κ^* separates $\mathbf{w}_{\circ,\kappa}$ and $\psi(\mathbf{w}_{\circ,\kappa})$.

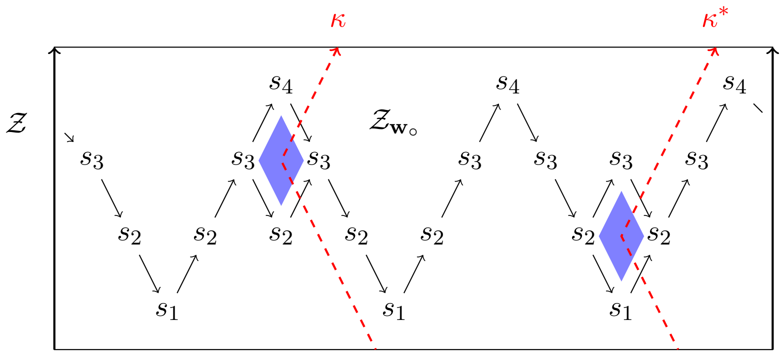


FIGURE 3.4: A cut κ and its opposite κ^*. The word $\mathbf{w}_{\circ,\kappa}$ is obtain from $\mathbf{w}_\circ$ conjugating by $s_2s_4s_3s_2s_1s_2s_3$. The supports $\mathcal{T}_\kappa$ and $\mathcal{T}_{\kappa^*}$ consist of only one shaded tile.

Definition 3.65. Denote by $P^{\kappa,+}_{\mathbf{w}_\circ}$ the induced subgraph of $\mathcal{Z}_{\mathbf{w}_\circ}$ whose vertices are letters of $\mathbf{w}_{\circ,\kappa}$. Similarly, denote by $P^{\kappa,-}_{\mathbf{w}_\circ}$ the induced subgraph of $\mathcal{Z}_{\mathbf{w}_\circ}$ whose vertices are letters of $\psi(\mathbf{w}_{\circ,\kappa})$.

Example 3.66. Following Example 3.57 and 3.64, the cut κ gives rise to the graphs $P^{\kappa,+}_{\mathbf{w}_\circ}$ and $P^{\kappa,-}_{\mathbf{w}_\circ}$, represented in Fig. 3.5.

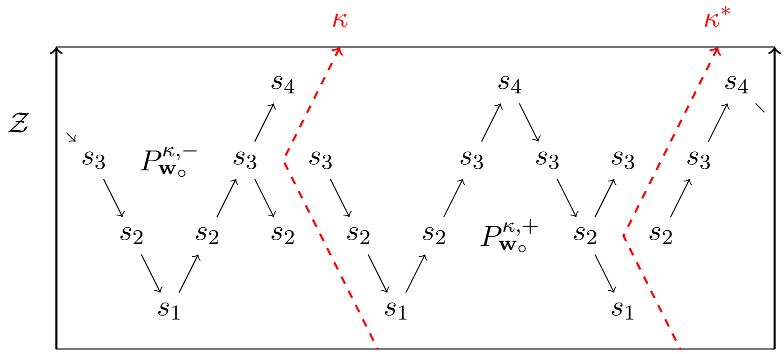


FIGURE 3.5: The Hasse diagrams of the partial order $\prec_{\mathbf{w}_{\circ,\kappa}}$ and $\prec_{\psi(\mathbf{w}_{\circ,\kappa})}$.

The next lemma is a consequence of the definition of $\mathcal{Z}_{\mathbf{w}_\circ}$.

Lemma 3.67. *The subgraphs $P^{\kappa,+}_{\mathbf{w}_\circ}$ and $P^{\kappa,-}_{\mathbf{w}_\circ}$ correspond to the Hasse diagrams of the natural partial order $\prec_{\mathbf{w}_{\circ,\kappa}}$ and $\prec_{\psi(\mathbf{w}_{\circ,\kappa})}$ respectively.*

By Lemma 3.54, counting c-singletons amounts to count lower ideals in $P^{\kappa,+}_{\mathbf{w}_\circ(\mathbf{c})}$ (for a well chosen κ). The main motivation of the present method lies in the following fact: lower ideals of $P^{\kappa,+}_{\mathbf{w}_\circ}$ are obtained by cuts of the cylinder which do not *cross* κ. Let us be more precise.

Definition 3.68. Given a cut κ of a cylindric graph $\mathcal{Z}_{\mathbf{w}_\circ}$, a cut κ' *crosses* κ if there exists a pair $(\ell, r) \in V^2$ of vertices of $\mathcal{Z}_{\mathbf{w}_\circ}$ satisfying the following conditions:

(i) ℓ belongs to $P^{\kappa,-\varepsilon}_{\mathbf{w}_\circ}$ and $P^{\kappa',\varepsilon}_{\mathbf{w}_\circ}$,

(ii) r belongs to $P^{\kappa,\varepsilon}_{\mathbf{w}_\circ}$ and $P^{\kappa',-\varepsilon}_{\mathbf{w}_\circ}$,

where $\varepsilon \in \{-,+\}$.

Example 3.69. Let $(W,S) = (A_5, \{s_1, s_2, s_3, s_4, s_5\})$ and $\mathcal{Z}_{\mathbf{w}_\circ(\mathbf{c})}$ be the cylindric graph where $\mathbf{c} = (s_2, s_4, s_1, s_3, s_5)$; see Fig. 3.6. The cut κ' crosses the cut κ: the circled relabeled vertices $\ell = s_2$ and $r = s_5$ satisfy the conditions.

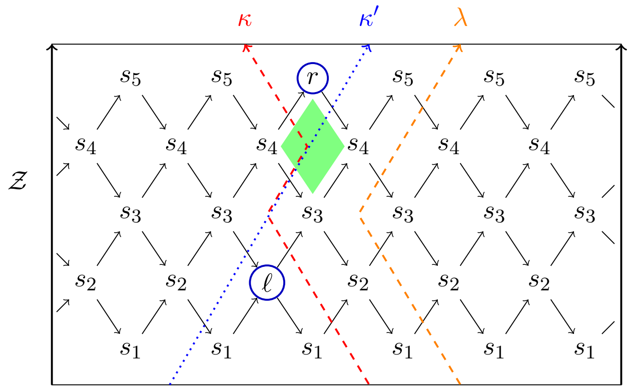


FIGURE 3.6: Two crossing cuts κ and κ' in type A_5. The split tile $T^{\vee}_{\kappa,\kappa'}$ is shaded. Since the cut λ do not cross κ, it corresponds to an ideal of $P^{\kappa,+}_{\mathbf{w}_\circ}$ represented by the prefix $(s_3, s_5, s_2, s_4, s_1, s_5)$ of the word $\mathbf{w}_{\circ,\kappa}$.

Finally, we define the remaining objects necessary to our approach.

Definition 3.70. The *support* $\mathcal{T}_\kappa$ of a cut κ is the sequence $(T_1, \ldots, T_k)$ of tiles of $\mathcal{Z}_{\mathbf{w}_\circ}$ which κ crosses in their relative interior. The support of a cut may be empty.

Definition 3.71. Let κ be a cut of $\mathcal{Z}_{\mathbf{w}_\circ}$, and T a tile in the support $\mathcal{T}_\kappa$. The oriented chain on the boundary graph $\mathcal{G}_T$ that κ crosses to enter T is called *inbound* boundary. Similarly, the oriented chain on the boundary graph $\mathcal{G}_T$ that κ crosses to leave T is called *outbound* boundary.

Definition 3.72. Let κ and κ' be two crossing cuts of a cylindric graph $\mathcal{Z}_{\mathbf{w}_\circ}$. The *split tile* $T^{\vee}_{\kappa,\kappa'}$ is the first tile in the support $\mathcal{T}_\kappa$ for which there exists a vertex v on its outbound boundary such that $v \in P^{\kappa,\varepsilon}_{\mathbf{w}_\circ} \cap P^{\kappa',-\varepsilon}_{\mathbf{w}_\circ}$.

Given two crossing cuts, the split tile exists by the definition of crossing cuts. Finally, we define the notion of commuting vertices before proceeding to cylindric graphs of sorting words.

Definition 3.73. Let v be a vertex of $\mathcal{Z}_{\mathbf{w}_\circ}$. A vertex u of $\mathcal{Z}_{\mathbf{w}_\circ}$ *commutes* with v if the following conditions are fulfilled:

(i) there exists a cut κ such that $u, v \in P^{\kappa,+}_{\mathbf{w}_\circ}$,

(ii) for all cuts κ such that $u, v \in P^{\kappa,+}_{\mathbf{w}_\circ}$, the set $\{u, v\}$ is an anti-chain of $P^{\kappa,+}_{\mathbf{w}_\circ}$.

Denote the set of commuting vertices of v by C_v. It is evident that anti-chains of $P^{\kappa,+}_{\mathbf{w}_\circ}$ correspond to cuts of $\mathcal{Z}$. Therefore, given two commuting letters u, v, there are cuts of $\mathcal{Z}$ which describe every anti-chain containing u and v. Using the support of κ, we say that the vertex u is *inferior* to v in a anti-chain when u appears in the boundary of a tile of $\mathcal{T}_\kappa$ before v. Similarly, v is *superior* to u in an anti-chain when v appears in the boundary of a tile of $\mathcal{T}_\kappa$ after u. By Lemma 3.62, two commuting vertices can not appear simultaneously on a outbound boundary of a tile of $\mathcal{T}_\kappa$. Thus, a set of commuting vertices C_v can be partitioned into the inferior commuting vertices C_v^I and the superior commuting vertices C_v^S.

Example 3.74. Following Example 3.69, the commuting vertices of vertices labeled by s_3 and s_5 are shown in Fig. 3.7 using shadows.

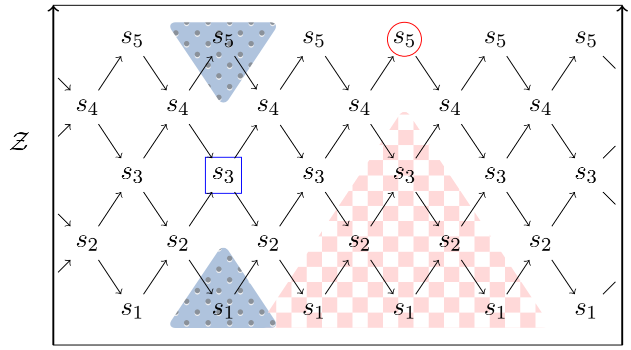


Figure 3.7: The commuting vertices for the circled vertex is checkerboard-shadowed. The commuting vertices of the boxed vertex is shaded with Polka dots.

3.9.3 Cylindric graphs of sorting words

Let us now restrict the study of cylindric graphs to the case of sorting words $\mathbf{w}_\circ(\mathbf{c})$. The next two lemmas form the base of our approach.

Lemma 3.75. *Let W be a finite irreducible Coxeter group and let c and c' be two distinct Coxeter elements. Then the cylindric graphs $\mathcal{Z}_{\mathbf{w}_\circ(\mathbf{c})}$ and $\mathcal{Z}_{\mathbf{w}_\circ(\mathbf{c'})}$ are isomorphic. Denote the cylindric graph of sorting words by $\mathcal{Z}_\diamond$. Different cuts of the cylindric graph $\mathcal{Z}_\diamond$ correspond to Coxeter elements.*

Proof. Using the two equalities of Remark 3.49, one gets that $\mathbf{c}^h = \mathbf{w}_\circ(\mathbf{c})\psi(\mathbf{w}_\circ(\mathbf{c}))$. Any two Coxeter elements can be obtained from each other by conjugation of initial letters (see [GP00, Theorem 3.1.4]), therefore cyclic longest words $\overset{\circlearrowleft}{\mathbf{w}}_\circ(\mathbf{c})$ are all equivalent by commutation of initial letters. Cuts of the cylindric graph $\mathcal{Z}_\diamond$ then correspond to a specific choice of Coxeter element c. In type A, this corresponds to a well-known fact about pseudoline arrangements on the Möbius strip [PP12, Section 2]. □

Example 3.76 (Example 3.69 continued)**.** The cut κ of $\mathcal{Z}_\diamond$ corresponds to the Coxeter element $c = s_3s_5s_2s_4s_1$. The cut κ' corresponds to the Coxeter element $c' = s_1s_2s_3s_4s_5$.

Lemma 3.77. *Let W be a finite irreducible Coxeter group and $\mathcal{Z}_\diamond$ its cylindric graph of sorting words. The boundary graphs $\mathcal{G}_T$ of tiles of $\mathcal{Z}_\diamond$ all have 4 vertices.*

Proof. Cylindric graphs are simple, therefore cycle graphs in $\mathcal{Z}_\diamond$ have at least 3 vertices. Since finite irreducible Coxeter graphs do not contain cycles and edges of $\mathcal{Z}_\diamond$ are between pairs of distinct noncommuting generators, cycle graphs in $\mathcal{Z}_\diamond$ contain at least 4 vertices. Since sorting words have the SIN-property, pairs of noncommuting generators alternate in $\overset{\circlearrowleft}{\mathbf{w}}_\circ(\mathbf{c})$. Thus all boundary graphs of tiles of $\mathcal{Z}_\diamond$ consist of 4 vertices. □

The previous lemma makes the following definitions possible.

Definition 3.78. Let T be a tile of the cylindric graph $\mathcal{Z}_\diamond$. The vertex on the outbound boundary of T which is between the source and the sink is called the *apex* of T.

Definition 3.79. Let κ be a cut of a cylindric graph $\mathcal{Z}_\diamond$, $\mathcal{T}_\kappa = (T_1, \ldots, T_k)$ its support, and a_i be the apex of the tile T_i and assume $a_i \in P^{\kappa,\varepsilon}_{\mathbf{w}_\circ}$. The *inferior poset* $I_\kappa(T_i)$ is the subposet of $P^{\kappa,-\varepsilon}_{\mathbf{w}_\circ}$ whose element are inferior commuting vertices of a_i. The *superior poset* $S_\kappa(T_i)$ is the poset induced by $\mathcal{Z}_\diamond$ whose elements are superior commuting vertices of v_i, where v_i is the unique neighbor of a_i on the boundary of T_i belonging to $P^{\kappa,\varepsilon}_{\mathbf{w}_\circ}$.

Theorem 3.80. *Let κ and κ' be two crossing cuts with split tile $T^\vee_{\kappa,\kappa'}$. The cut κ' correspond to a unique pair of ideals $(J,K) \in (I_\kappa(T^\vee_{\kappa,\kappa'}), S_\kappa(T^\vee_{\kappa,\kappa'}))$. Conversely, pairs of ideals $(J,K) \in (I_\kappa(T^\vee_{\kappa,\kappa'}), S_\kappa(T^\vee_{\kappa,\kappa'}))$ correspond to cuts κ' crossing κ, with the only restriction that J is nonempty (or not the whole $I_\kappa(T^\vee_{\kappa,\kappa'})$ depending on the cut κ).*

Proof. The cut κ' has to arrive on the the split tile $T^\vee_{\kappa,\kappa'}$ from the bottom. The corresponding paths correspond to ideals of the inferior poset $I_\kappa(T^\vee_{\kappa,\kappa'})$, with the exception that there should be an vertex between κ and κ'. By Lemma 3.77, the cut κ' has to go through the edge on the outbound boundary of $T^\vee_{\kappa,\kappa'}$ that κ does not cross. The existence of a vertex between κ and κ' above $T^\vee_{\kappa,\kappa'}$ is guaranteed from the definition of split tile. Then, κ' can go to the top boundary without further restrictions. Hence, after going through the split tile $T^\vee_{\kappa,\kappa'}$ the cut κ' corresponds to an ideal of $S_\kappa(T^\vee_{\kappa,\kappa'})$. □

Using the previous theorem, the set of cuts crossing κ can be expressed as a disjoint union of sets indexed by the split tiles.

Example 3.81. Suppose $(W,S) = (A_5, \{s_1, s_2, s_3, s_4, s_5\}$ and consider the cylindric graph shown on Fig. 3.8. The cut κ has 3 tiles in its support. The superior poset of the first tile and the inferior poset of the second tile are highlighted with different shadows.

Lemma 3.82. *Let W be a finite irreducible Coxeter group and $\mathcal{Z}_\diamond$ its cylindric graph of sorting words. The number of different cuts of $\mathcal{Z}_\diamond$ is $2^{n-1}h$, where h is the Coxeter number of W.*

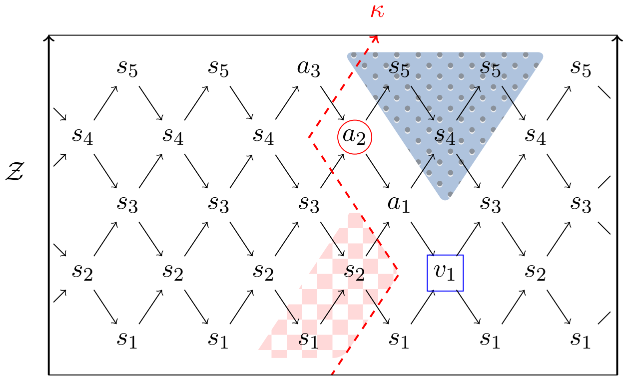


FIGURE 3.8: The inferior poset $I_\kappa(T_2)$ is shadowed in checkerboard. The superior poset $S_\kappa(T_1)$ is shadowed in with dots.

Proof. By Lemma 3.75, cuts of $\mathcal{Z}_\diamond$ correspond to the choice of a Coxeter element. There are 2^{n-1} such choices [Shi97]. The only remaining choice is to decide where the cuts begin: there are exactly h copies of the Coxeter graph in $\mathcal{Z}_\diamond$, hence the formula. □

Lemma 3.83. *Let W be a finite irreducible Coxeter group and $\mathcal{Z}_\diamond$ its cylindric graph of sorting words. In the cylindric graph $\mathcal{Z}_\diamond$, the number of lower ideals in $S_\kappa(T_i)$ is equal to 2^{n-2-i}.*

Proof. Notice that the definition of superior poset does not depend on the cut itself. Therefore, it suffices to look at the superior commuting vertices of the neighbors of apexes. Using the fact that lower ideals of $S_\kappa(T_i)$ correspond to paths going from T_i to the top boundary by crossing on the edge a_i—v_i, we get that the number of ideals is 2^{n-2-i}. Indeed by Lemma 3.77, at every tile that a path crosses after T_i it has 2 possibilities and there are $n-2-i$ remaining tiles to cross by Lemma 3.75. □

3.9.4 Formulas for the number of singletons

Theorem 3.84. *Let W be a finite irreducible Coxeter group of rank n and c be a Coxeter element. The number of c-singletons is*

$$2^{n-2}h - K_c + 1,$$

where h is the Coxeter number and K_c is the number of cuts crossing the corresponding cut κ_c of $\mathcal{Z}_\diamond$.

Proof. By Lemma 3.54 counting c-singletons correspond to counting ideals of the natural partial order $\prec_{\mathbf{w}_\circ(\mathbf{c})}$. Moreover, by Lemma 3.67 and Lemma 3.75 these ideals correspond to cuts κ' of $\mathcal{Z}_\diamond$ which do not cross the cut κ_c, corresponding to the Coxeter element c. Using Theorem 3.80, Lemma 3.82 and Lemma 3.83, we double count the cuts of $\mathcal{Z}_\diamond$. By Lemma 3.82, the number of different cuts of $\mathcal{Z}_\diamond$ is $2^{n-1}h$. Now let κ_c be a cut corresponding to a Coxeter element c. There are five distinct possibilities for a cut κ':

(i) it crosses κ_c,
(ii) it crosses the opposite cut κ_c^*,
(iii) it represents a c-singleton,
(iv) it represents a $\psi(c)$-singleton,
(v) it represents a c-singleton and a $\psi(c)$-singleton.

By [BHLT09, Theorem 2.3], the number of c-singletons and $\psi(c)$-singletons are equal, and thus cuts crossing κ_c and κ_c^* are also equal. There are exactly two cuts which represent both a c-singleton and a $\psi(c)$-singleton, namely κ and κ^*. Therefore, we have

$$2K_c + 2S_c = 2^{n-1}h + 2, \tag{$\star\star$}$$

where S_c is the number of c-singletons and K_c is the number of cuts crossing κ_c. □

Remark 3.85. Based on the duality of pseudoline arrangements, Carsten Lange obtained a formula for the number of c-singletons in type A and B which is closely related to the method presented here [Lan12].

Theorem 3.86. *Let W be a finite irreducible Coxeter group of rank n and c be a Coxeter element. The number of c-singletons is*

$$2^{n-2}(h+1) - \sum_{i=1}^{n-2} 2^{n-2-i}\iota(T_i),$$

where h is the Coxeter number, $\iota(T_i)$ is the number of ideals of the inferior poset $I_{\kappa_c}(T_i)$ and κ_c is a cut of $\mathcal{Z}_\diamond$ corresponding to c.

Proof. To obtain this formula, we develop the term K_c of the equality ($\star\star$) from the proof of Theorem 3.84. Let $\mathcal{T}_{\kappa_c} = (T_1, \dots, T_{n-2})$ be the support of κ_c. By Theorem 3.80, we obtain a formula for K_c when the split tile $T^\vee_{\kappa,\kappa'}$ is T_i, with $1 \leq i \leq n-2$. By Lemma 3.83, we obtain a formula for the number of ideals of $S_{\kappa_c}(T_i)$.

$$\begin{aligned} K_c &= \sum_{i=1}^{n-2} \left(\iota(I_{\kappa_c}(T_i)) - 1\right)\iota(S_{\kappa_c}(T_i)), \\ &= \sum_{i=1}^{n-2} \iota(I_{\kappa_c}(T_i))2^{n-2-i} - \sum_{i=1}^{n-2} 2^{n-2-i}, \\ &= \sum_{i=1}^{n-2} \iota(I_{\kappa_c}(T_i))2^{n-2-i} - 2^{n-2} + 1. \end{aligned}$$

Replacing this in $(\star\star)$, we get

$$\begin{aligned} 2\left(\sum_{i=1}^{n-2}\iota(I_{\kappa_c}(T_i))2^{n-2-i}-2^{n-2}+1\right)+2S_c &= 2^{n-1}h+2,\\ 2\sum_{i=1}^{n-2}\iota(I_{\kappa_c}(T_i))2^{n-2-i}-2^{n-1}+2+2S_c &= 2^{n-1}h+2,\\ 2\sum_{i=1}^{n-2}\iota(I_{\kappa_c}(T_i))2^{n-2-i}-2^{n-1}+2S_c &= 2^{n-1}h, \end{aligned}$$

and finally

$$S_c = 2^{n-2}(h+1)-\sum_{i=1}^{n-2}\iota(I_{\kappa_c}(T_i))2^{n-2-i}.$$

□

3.9.5 Upper bounds

To find the Coxeter elements that maximize the number of singletons, we introduce the *cut function*[3]

$$\mathcal{K}: S \to \mathbb{Z},$$

satifying $|\mathcal{K}(s)-\mathcal{K}(t)|=1$ for all noncommuting pairs $s,t\in S$.

Taking the values of $\mathcal{K}(S)$ modulo h yields a cut κ on the cylindric graph $\mathcal{Z}_\diamond$ corresponding to a certain Coxeter element. Indeed, given a cut function $\mathcal{K}$, label the vertices of degree 2 closer to the bottom boundary from 0 to $h-1$ (and assume they correspond to the generator s_1) in the direction of the orientation given by the embedding. The first edge that the corresponding cut κ will cross will be adjacent to the vertex j labeled by $j\equiv\mathcal{K}(s_1)$ modulo h. Then, κ will cross sequentially the edges whose sources have the highest value given by $\mathcal{K}$. The definition of crossing cuts can be reformulated in the case of a cylindric graph $\mathcal{Z}_\diamond$: Two cut functions $\mathcal{K}_1,\mathcal{K}_2$ are *crossing* if there exists s,t such that $\mathcal{K}_1(s)<\mathcal{K}_2(s)$ and $\mathcal{K}_1(t)>\mathcal{K}_2(t)$. The *width* $\mu_\mathcal{K}$ of $\mathcal{K}$ is $\max(\mathcal{K}(S))-\min(\mathcal{K}(S))$.

Example 3.87 (Example 3.69 and 3.76 continued)**.** The cut κ corresponding to the Coxeter element $c=s_3s_5s_2s_4s_1$ can be obtained from the cut function $\mathcal{K}(s_1,s_2,s_3,s_4,s_5)=(4,3,2,3,2)$. The cut κ' corresponding to the Coxeter element $c'=s_1s_2s_3s_4s_5$ can be obtained from the cut function $\mathcal{K}'(s_1,s_2,s_3,s_4,s_5)=(0,1,2,3,4)$. They are crossing since $\mathcal{K}(s_2)>\mathcal{K}'(s_2)$ and $\mathcal{K}(s_5)<\mathcal{K}'(s_5)$.

[3]The author is thankful to Cesar Ceballos for suggesting the cut function which simplified the approach.

Theorem 3.88. *Let W be a finite irreducible Coxeter group, c be a Coxeter element, and S_c denote the number of c-singletons. The following statements are equivalent.*

(i) *The Coxeter element c is bipartite.*
(ii) *The width $\mu_{\mathcal{K}_c}$ of the cut function $\mathcal{K}_c$ is 1.*
(iii) *For all Coxeter elements c' of W, $S_c \geq S_{c'}$.*

Proof. The equivalence of the first two statements is a consequence of the definition of cut function. We will show that cuts of width 1 are the only cuts that minimize the number of crossing cuts. Therefore they are the only cuts that maximize the number of singletons by the formula of Theorem 3.86. For this, we use the *m-reflection* defined as the linear function

$$\begin{aligned} \mathcal{M}_m : \quad & \mathbb{Z} \to \mathbb{Z} \\ & m+1 \mapsto m-1 \\ & m-1 \mapsto m+1. \end{aligned}$$

For a cut function $\mathcal{K}$ of width $\nu_{\mathcal{K}} > 1$ consider the cut function

$$\begin{aligned} \mathcal{K}' : S &\to \mathbb{Z} \\ s &\mapsto \begin{cases} \mathcal{K}(s) & \text{if } s < \max(\mathcal{K}(\Gamma)), \\ \mathcal{K}(s) - 2 & \text{if } s = \max(\mathcal{K}(\Gamma)). \end{cases} \end{aligned}$$

The cut function $\mathcal{K}'$ has width $\nu_{\mathcal{K}} - 1$; see Fig. 3.9 for an example. Denote by $\mu_{\mathcal{K}}$ the maximum $\max(\mathcal{K}(\Gamma))$. Denote $K_{\mathcal{K}\setminus\mathcal{K}'}$ the set of cuts crossing $\mathcal{K}$ but not $\mathcal{K}'$, and similarly, $K_{\mathcal{K}'\setminus\mathcal{K}}$ the set of cuts crossing $\mathcal{K}'$ but not $\mathcal{K}$. Now, we show that the set of cuts crossing $\mathcal{K}$ is bigger than the set of cuts crossing $\mathcal{K}'$. Therefore, the number of singletons corresponding to the cut $\mathcal{K}'$ will be greater than the singletons corresponding to the cut $\mathcal{K}$. To see this, consider the $(\mu - 1)$-reflections of the cuts in $K_{\mathcal{K}'\setminus\mathcal{K}}$. It sends cuts crossing $\mathcal{K}'$ but not $\mathcal{K}$ to cuts crossing $\mathcal{K}$ but not $\mathcal{K}'$. Indeed, if $\Lambda \in K_{\mathcal{K}'\setminus\mathcal{K}}$, the cut $\mathcal{M}_{\mu_{\mathcal{K}}-1}(\Lambda)$ belongs to $K_{\mathcal{K}\setminus\mathcal{K}'}$, and the injectivity of this map is clear. Now, consider the cut $\mathcal{M}_{\mu_{\mathcal{K}}-1}(\mathcal{K})$, this cut belongs to $K_{\mathcal{K}\setminus\mathcal{K}'}$ by construction and $\mathcal{K}$ does not cross $\mathcal{K}'$. Hence, the $(\mu_{\mathcal{K}} - 1)$-reflection map going from the set $K_{\mathcal{K}'\setminus\mathcal{K}}$ to the set $K_{\mathcal{K}\setminus\mathcal{K}'}$ is not surjective and thus there are more cuts crossing $\mathcal{K}$ than $\mathcal{K}'$. □

The next three theorems give formulas for the number of singletons for bipartite Coxeter elements. Before stating them, we give a useful combinatorial equality.

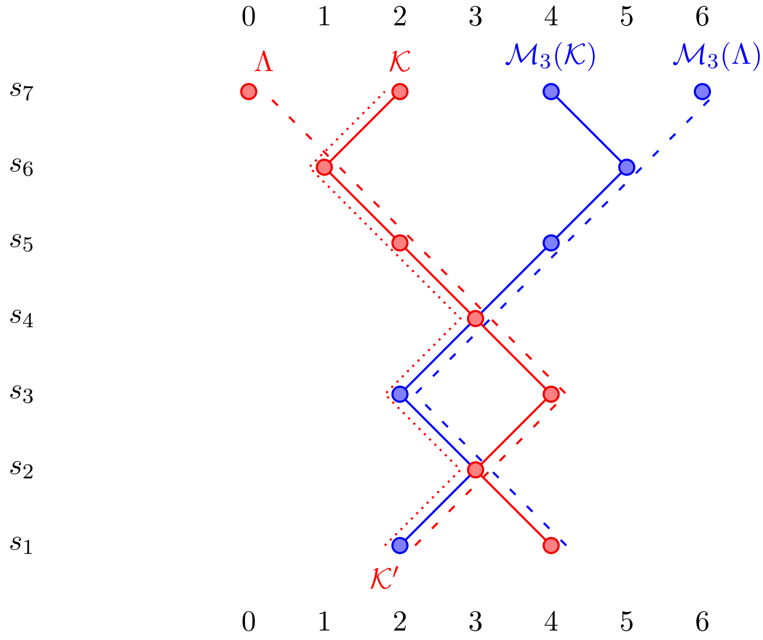


FIGURE 3.9: Examples of 3-reflections with the Coxeter system A_7. Let $\mathcal{K}$ be the cut function $\mathcal{K}(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (4, 3, 4, 3, 2, 1, 2)$. The cut function $\mathcal{K}$ is shown with a solid line, the cut $\mathcal{K}'(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (2, 3, 2, 3, 2, 1, 2)$ is shown with a dotted line, and a cut function $\Lambda(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (2, 3, 4, 3, 2, 1, 0)$ crossing $\mathcal{K}'$ but not $\mathcal{K}$ is shown with a dashed line. The 3-reflections of $\mathcal{K}$ and Λ are shown in solid and dashed lines respectively. The cut function $\mathcal{K}'$ has less crossings than the cut function $\mathcal{K}$. Some cuts are drawn with a slight deviation for visualization purposes.

Lemma 3.89. *Let $n \in \mathbb{N}$. Then,*

$$\sum_{i=0}^{n} \frac{1}{4^i}\binom{2i}{i} = \frac{2(n+1)}{4^{n+1}}\binom{2(n+1)}{n+1}.$$

Proof. This equality is easily obtained using Gosper's algorithm, for instance. See [PWZ96, Chapter 5] for more details about this method. □

Theorem 3.90. *Let W be a finite irreducible Coxeter group of rank $n > 1$, whose graph is a chain (Type A, B, F, H and $I_2(m)$). The number of singletons for a bipartite Coxeter element is*

$$2^{n-2}(h+3) - \begin{cases} \frac{n}{2}\binom{n}{n/2} & \text{for even } n, \\ (n-\frac{1}{2})\binom{n-1}{(n-1)/2} & \text{for odd } n. \end{cases}$$

Proof. Let W be a finite irreducible Coxeter group whose Coxeter graph is a chain and $n > 1$. The case $n = 1$ is trivial. Consider a cut κ of width 1 on $\mathcal{Z}_\diamond$, see Fig. 3.10 for an example.

From Theorem 3.86, it remains to find formulas for $\iota(T_i)$, for $1 \leq i \leq n-2$. From the shape of $\mathcal{Z}_\diamond$, we deduce that counting ideals of inferior posets of an apex of a tile T_{2j},

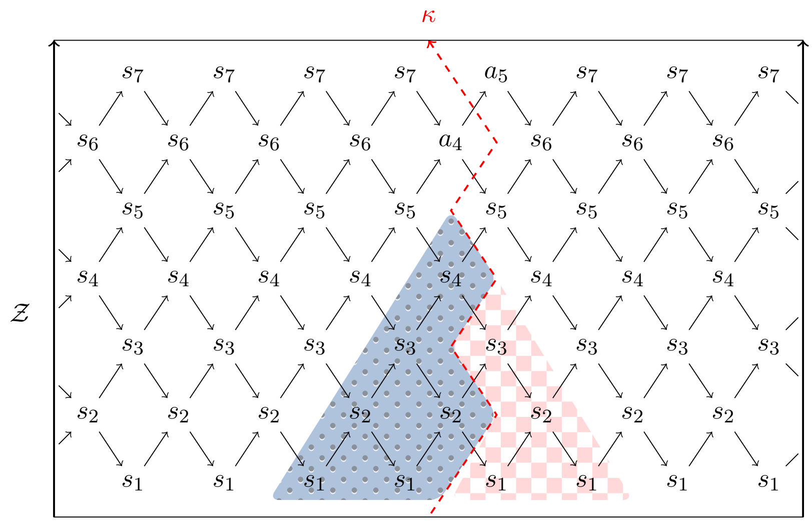


FIGURE 3.10: A cut of width 1 on the cylindric graph of sorting words $\mathcal{Z}_\diamond$ of type A_7. The inferior poset $I_\kappa(T_4)$ is shown over checkerboard shadow, it has $\binom{4}{2} = 6$ lower ideals. The inferior poset $I_\kappa(T_5)$ is shown over dotted shadow, it has $\binom{6}{3}/2 = 10$ lower ideals.

with $j \geq 1$, in the support of κ is equivalent to count sequences of letters $+$ and $-$ of length $2j$ where the number of instances of $-$ is always smaller or equal to the number of instances of $+$ in any initial segment of the sequences. The number of such sequences is $\binom{2j}{j}$, see, for instance, [Nil12, Corollary 6] for a detailed proof. By construction, the number of ideals of the inferior poset of an apex of a tile T_{2j-1}, with $j \geq 1$, in the support of κ is $\binom{2j}{j}/2$.

Assume n is even, the formula of Theorem 3.86 gives

$$\begin{aligned}
& 2^{n-2}(h+1) - \sum_{j=1}^{\frac{n-2}{2}} \frac{\binom{2j}{j} 2^{n-2-(2j-1)}}{2} - \sum_{j=1}^{\frac{n-2}{2}} \binom{2j}{j} \cdot 2^{n-2-2j}, \\
= \quad & 2^{n-2}(h+1) - 2 \cdot \sum_{j=1}^{\frac{n-2}{2}} \binom{2j}{j} \cdot 2^{n-2-2j}, \\
= \quad & 2^{n-2}(h+1) - 2^{n-1} \cdot \sum_{j=1}^{\frac{n-2}{2}} \frac{\binom{2j}{j}}{4^j}.
\end{aligned}$$

For odd n, a similar procedure gives

$$2^{n-2}(h+1) - 2^{n-1} \cdot \sum_{j=1}^{\frac{n-3}{2}} \frac{\binom{2j}{j}}{4^j} - \frac{1}{2} \cdot \binom{n-1}{\frac{n-1}{2}}.$$

Finally, applying Lemma 3.89 and simplifying, we get

$$2^{n-2}(h+3) - \begin{cases} \frac{n}{2}\binom{n}{n/2} & \text{for even } n, \\ (n-\frac{1}{2})\binom{n-1}{(n-1)/2} & \text{for odd } n. \end{cases}$$

□

Remark 3.91. In type A, this gives back the formula for the number of permutations satisfying the alternating scheme obtained in [GR08, Theorem 4]. Theorem 3.88 also provides the "right" condition for Conjecture 1 of Ádám Galambos and Vic Reiner to be true: the maximal reduced decompositions have to be restricted to sorting words. A general counter-example to Conjecture 1 is provided in type A_{41} in [DKK12, Section 6].

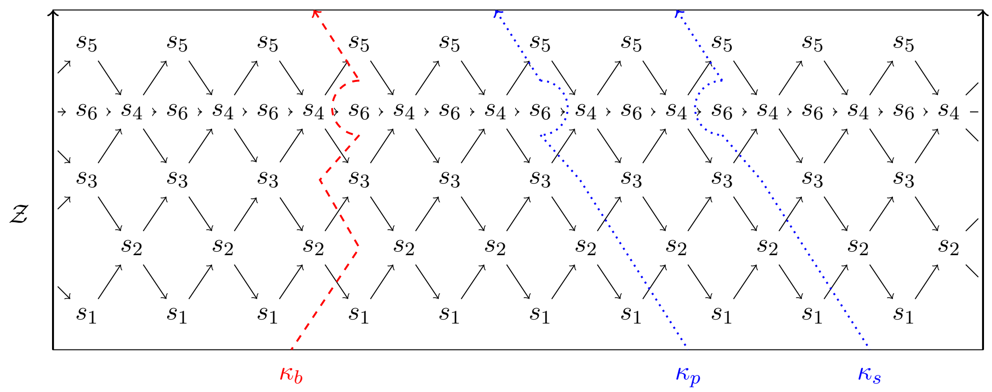


FIGURE 3.11: Cuts on the cylindric graph of sorting words $\mathcal{Z}_\diamond$ of type D_6: κ_b is cut of width 1, κ_p (pseudo-straight) and κ_s (straight) have width 4.

Theorem 3.92. *Let W be a Coxeter group of type D. The number of singletons for a bipartite Coxeter element is*

$$2^{n-2}(h+3) + \begin{cases} -\frac{n}{2}\binom{n}{n/2} + \frac{1}{2}\binom{n-2}{(n-2)/2} & \text{for even } n, \\ -(n-1)\binom{n-1}{(n-1)/2} - \binom{n-3}{(n-3)/2} & \text{for odd } n. \end{cases}$$

Proof. Using Lemma 3.59, select a natural embedding of the cylindric graph of the sorting word such that the vertex of degree 6 is closer to the top boundary, by which we mean that any cut going from bottom to top will cross less (not necessarily strictly) tiles after crossing an edge adjacent to a vertex of degree 6. This way, the cylindric graph $\mathcal{Z}_\diamond$ below the vertices of degree 6 is similar to that of type A, see Fig. 3.11. Consider a cut κ of width 1 on $\mathcal{Z}_\diamond$. From Theorem 3.86, it remains to find formulas for $\iota(T_i)$, for $1 \le i \le n-2$. Using the same arguments as in the proof of Theorem 3.90, we get that $\iota(T_{2j})$, with $1 \le j \le \lfloor (n-3)/2 \rfloor$, is equal to $\binom{2j}{j}$ and $\iota(T_{2j-1})$, with $1 \le j \le \lceil (n-3)/2 \rceil$, is equal to $\binom{2j}{j}/2$. Finally, $\iota(T_{n-2})$ is equal to $\iota(T_{n-3})$.

Assume n is even, the formula of Theorem 3.86 gives

$$\begin{aligned}
& 2^{n-2}(h+1) - \sum_{j=1}^{\frac{n-2}{2}} \frac{\binom{2j}{j}}{2} 2^{n-2-(2j-1)} - \sum_{j=1}^{\frac{n-4}{2}} \binom{2j}{j} \cdot 2^{n-2-2j} - \binom{n-2}{\frac{n-2}{2}} \frac{1}{2}, \\
= \quad & 2^{n-2}(h+1) - 2 \sum_{j=1}^{\frac{n-2}{2}} \binom{2j}{j} \cdot 2^{n-2-2j} + \binom{n-2}{\frac{n-2}{2}} \frac{1}{2}, \\
= \quad & 2^{n-2}(h+1) - 2^{n-1} \sum_{j=1}^{\frac{n-2}{2}} \frac{\binom{2j}{j}}{4^j} + \frac{1}{2} \binom{n-2}{\frac{n-2}{2}}.
\end{aligned}$$

For odd n, a similar procedure gives

$$2^{n-2}(h+1) - 2^{n-1} \cdot \sum_{j=1}^{\frac{n-3}{2}} \frac{\binom{2j}{j}}{4^j} - \binom{n-3}{\frac{n-3}{2}}.$$

Finally, applying Lemma 3.89 and simplifying, we get

$$2^{n-2}(h+3) + \begin{cases} -\frac{n}{2}\binom{n}{n/2} + \frac{1}{2}\binom{n-2}{(n-2)/2} & \text{for even } n, \\ -(n-1)\binom{n-1}{(n-1)/2} - \binom{n-3}{(n-3)/2} & \text{for odd } n. \end{cases}$$

□

Theorem 3.93. *The number of singletons for a bipartite Coxeter element is 182, 546 and 1840 for type* E_6*,* E_7*, and* E_8 *respectively.*

Proof. Similarly to the proof of Theorem 3.92, select a natural embedding of the cylindric graph of the sorting word such that the vertex of degree 6 is closer to the top boundary. Consider a cut κ of width 1 on $\mathcal{Z}_\diamond$. From Theorem 3.86, it remains to find formulas for $\iota(T_i)$, for $1 \le i \le n-2$. By inspection, one finds that $\iota(T_i) = i$ for $1 \le i \le n-5$, and $\iota(T_{n-4}) = \iota(T_{n-3})$. The latter number takes the values 2,3 and 6 for E_6 E_7 and E_8 respectively. Finally, $\iota(T_{n-2})$ is equal to 6, 12 and 20 for E_6 E_7 and E_8 respectively. For E_6, the formula of Theorem 3.86 gives

$$2^4(12+1) - 1 \cdot 2^{4-1} - 2 \cdot 2^{4-2} - 2 \cdot 2^{4-3} - 6 \cdot 2^{4-4} = 182.$$

For E_7, the formula of Theorem 3.86 gives

$$2^5(18+1) - 1 \cdot 2^{5-1} - 2 \cdot 2^{5-2} - 3 \cdot 2^{5-3} - 3 \cdot 2^{5-4} - 12 \cdot 2^{5-5} = 546.$$

For E_8, the formula of Theorem 3.86 gives

$$2^6(30+1) - 1 \cdot 2^{6-1} - 2 \cdot 2^{6-2} - 3 \cdot 2^{6-3} - 6 \cdot 2^{6-4} - 6 \cdot 2^{6-5} - 10 \cdot 2^{6-6} = 1840.$$

□

3.9.6 Lower bounds

The next proposition is a consequence of the proof of Theorem 3.88.

Proposition 3.94. *Let W be a finite irreducible Coxeter group, c be a Coxeter element, and S_c denote the number of c-singletons. If $S_c \leq S_{c'}$, for all Coxeter elements c' of W, then the width of $\mathcal{K}_c$ is maximal.*

The following theorem proves that the converse is true for Coxeter groups whose graphs are chains.

Theorem 3.95. *Let W be a finite Coxeter group of type A, B, F, H or $I_2(m)$, c be a Coxeter element, and S_c denote the number of c-singletons. The following statements are equivalent.*

- (i) *The Coxeter element c corresponds to an orientation of Γ with a unique sink and a unique source.*
- (ii) *For all Coxeter elements c' of W, $S_c \leq S_{c'}$.*

The minimal number of singletons is $2^{n-2}(h-n+3)$.

Proof. By Proposition 3.94, cuts giving rise to the minimal number of singleton have maximal width. Clearly, the corresponding orientations have a unique sink and a unique source. Consider such a cut, and refer to it as a *straight cut*. Using Theorem 3.86, it remains to compute the values of $\iota(T_i)$ for $1 \leq i \leq n-2$. But since apexes of tiles in the support of a straight cut all lie on one side of the cut, the inferior posets are very regular. Indeed, by inspection, it is easy to check that $\iota(T_i)$ is equal to 2^i for $1 \leq i \leq n-2$. Putting this in the formula of Theorem 3.86 gives

$$2^{n-2}(h+1) - \left(\sum_{i=1}^{n-2} 2^i \cdot 2^{n-2-i}\right) = 2^{n-2}(h+1) - 2^{n-2}(n-2).$$

Simplifying gives $2^{n-2}(h-n+3)$. Giving 2^n and $2^{n-2}(n+3)$ for type A_n and B_n respectively. Moreover, for type F_4, H_3, H_4, and $I_2(m)$, the formula gives 44, 20, 116 and $m+1$ respectively. □

In types D and E, a more precise description is needed.

Theorem 3.96. *Let W be a finite Coxeter group of type D or E. Orientations of the Coxeter graph Γ whose corresponding Coxeter elements minimize the number of singletons are shown in Fig. 3.12.*

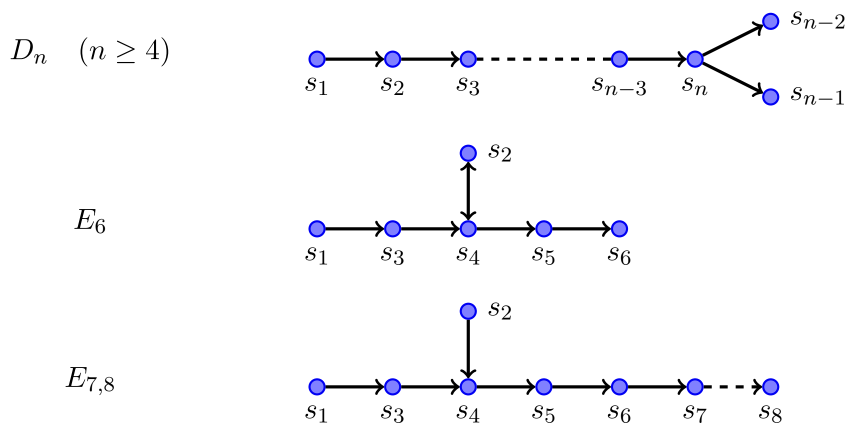


FIGURE 3.12: The orientations corresponding to the Coxeter elements minimizing the number of singletons in type D and E (the reverse orientations is also minimizing, see [BHLT09] for more details). In type E_6, the edge between s_2 and s_4 can be oriented at will.

Proof. By Proposition 3.94, the width of a minimizing cut should have maximal width. Therefore, consider a longest chain in Γ and orient its edges so that the chain as a unique sink and a unique source. Now, there is one edge between, say t_1 and t_2, left to orient. Before orienting it, we select a natural embedding of the cylindric graph of the sorting word that puts the edges between t_1 and t_2 closer to the top boundary, by which we mean that any cut going from bottom to top will cross less (not necessarily strictly) tiles after crossing an edge between t_1 and t_2.

Now, assume W is of type D and that edges between t_1 and t_2 in the cylindric graph are on the boundary of two tiles. This way, we make use of Theorem 3.86, to differentiate between the two possible orientations of the edge between t_1 and t_2. In Fig. 3.11, $t_1 = s_4$ and $t_2 = s_6$ and κ_p and κ_s are the two possible cuts of maximal width. The formula to count singletons remains exactly the same with the exception of the term with indices $n-3$ and $n-2$ in the sum. There is an orientation for which the apexes of the support are all located on one side of the cut, call this orientation *straight* (κ_s in Fig. 3.11). In the other one, the apex of the tile T_{n-3} is located on the other side of the others, call it *pseudo-straight* (κ_p in Fig. 3.11). Therefore, only $\iota(T_{n-3})$ and $\iota(T_{n-2})$ will change in the formula. For the straight orientation, by inspection $\iota(T_i)$ is equal to 2^i for $1 \leq i \leq n-3$ and $\iota(T_{n-2})$ is equal to 2^{n-3}. Putting this in the formula of Theorem 3.86 gives

$$2^{n-2}(h+1) - \left(\sum_{i=1}^{n-3} 2^i \cdot 2^{n-2-i}\right) - 2^{n-3} \cdot 2^0 = 2^{n-2}(h+1) - 2^{n-3}(2n-5).$$

Simplifying and replacing h by its value $2(n-1)$ gives $2^{n-2}n + 3 \cdot 2^{n-3}$. Now for the pseudo-straight orientation, by inspection $\iota(T_i)$ is equal to 2^i for $1 \leq i \leq n-4$. The number $\iota(T_{n-3})$ is equal to 1 since the inferior poset of the apex of T_{n-3} is empty. Finally, the number $\iota(T_{n-2})$ is 2^{n-2} because the apex of T_{n-3} is in the inferior poset of the apex of T_{n-2}. Putting this in the formula of Theorem 3.86 gives

$$2^{n-2}(h+1) - \left(\sum_{i=1}^{n-4} 2^i \cdot 2^{n-2-i}\right) - 1 \cdot 2^1 - 2^{n-2} \cdot 2^0 = 2^{n-2}(h+1) - 2(2^{n-3}(n-3)+1).$$

Simplifying and replacing h by its value $2(n-1)$ gives $2^{n-2}n + 2^{n-1} - 2$. As expected by [BHLT09, Example 2.5], the straight and pseudo-straight orientation give the same number for $n = 4$, and for $n > 4$ the pseudo-straight has more singletons. Finally, the straight orientation and its inverse (up to automorphism of the Coxeter graph) give rise to the minimal number of singletons.

Now, assume W is of type E. Again we use Theorem 3.86 to compare the straight and pseudo-straight orientations. For the straight orientation, by inspection $\iota(T_i)$ is equal to 2^i for $1 \leq i \leq n-4$, $\iota(T_{n-3})$ is equal to 2^{n-4}, and $\iota(T_{n-2})$ is equal to $3 \cdot 2^{n-4}$. Putting this in the formula of Theorem 3.86 gives

$$2^{n-2}(h+1) - \left(\sum_{i=1}^{n-4} 2^i \cdot 2^{n-2-i}\right) - 2^{n-3} - 3 \cdot 2^{n-4}.$$

Simplifying gives $2^{n-4}(4(h-n)+15)$. Giving 156, 472 and 1648 for type E_6, E_7 and E_8 respectively. Now for the pseudo-straight orientation, by inspection $\iota(T_i)$ is equal to 2^i for $1 \leq i \leq n-5$. The number $\iota(T_{n-4})$ is equal to 1 since the inferior poset of the apex of T_{n-4} is empty. The number $\iota(T_{n-3})$ is 2^{n-3} because the apex of T_{n-4} is in the inferior poset of the apex of T_{n-3}. Finally, the number $\iota(T_{n-2})$ is $4 \cdot 2^{n-4}$. Putting this in the formula of Theorem 3.86 gives

$$2^{n-2}(h+1) - \left(\sum_{i=1}^{n-5} 2^i \cdot 2^{n-2-i}\right) - 1 \cdot 2^2 - 2^{n-3} \cdot 2^1 - 4 \cdot 2^{n-4} \cdot 2^0 = 2^{n-2}(h+1) - 2^{n-2}(n-3) - 4.$$

Simplifying gives $2^{n-2}(h-n+4) - 4$. Giving 156, 476 and 1660 for type E_6, E_7 and E_8 respectively. The straight orientations in type D and E are represented in Fig. 3.12. In type E_6 the pseudo-straight and straight orientation both give rise to the minimal number of singletons. $\square$

3.9.7 Enumerative results

We summarize the results for the maximal and minimal values for the number of c-singletons for irreducible finite Coxeter groups in Table 3.4, along with the number of vertices of the c-generalized associahedron.

W	bipartite Coxeter elements	straight Coxeter elements	Coxeter–Catalan number
A_n	Theorem 3.90	2^n	$\frac{1}{n+1}\binom{2n}{n}$
B_n	Theorem 3.90	$2^{n-2}(n+3)$	$\binom{2n}{n}$
D_n	Theorem 3.92	$2^{n-2}n + 3 \cdot 2^{n-3}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$
E_6	182	156	833
E_7	546	472	4160
E_8	1840	1648	25080
F_4	48	44	105
H_3	21	20	32
H_4	120	116	280
$I_2(m)$	$m+1$	$m+1$	$m+2$

TABLE 3.4: Enumeration of c-singletons: maxima and minima for irreducible finite Coxeter groups.

3.10 Open problems

Finally, we discuss open problems and present several conjectures concerning subword complexes. We start with two open problems concerning counting formulas for multi-cluster complexes.

Open Problem 3.97. *Find* multi-Catalan numbers *counting the numbers of facets in the multi-cluster complexes.*

Although a formula in terms of invariants of the group for the number of facets of the generalized cluster complex defined by Sergey Fomin and Nathan Reading is known [FR05, Proposition 8.4], a general formula in terms of invariants of the group for the multi-cluster complex is yet to be found. An explicit formula for type A can be found in [Jon05, Corollary 17]. In type B, a formula was conjectured in [SW09, Conjecture 13] and proved in [RS09, Section 7]. In the dihedral type $I_2(m)$, the number of facets of the multi-cluster complex is equal to the number of facets of a $2k$-dimensional cyclic polytope on $2k+m$ vertices. These three formulas can be reformulated in terms of invariants of the Coxeter groups of type A, B and I_2 as follows:

$$\prod_{0\leq j<k}\prod_{1\leq i\leq n}\frac{d_i+h+2j}{d_i+2j},$$

where $d_1 \leq \cdots \leq d_n$ are the *degrees* of the corresponding group, and h is its Coxeter number. In general, this product is not an integer. The smallest example we are aware of is type D_6 with $k = 5$. Therefore, this product cannot count facets of the multi-cluster complex in general.

The next open problem raises the question of finding a family of simplicial complexes that includes the generalized cluster complexes of Sergey Fomin and Nathan Reading and the multi-cluster complexes.

Open Problem 3.98. *Construct a family of simplicial complexes which simultaneously contains generalized cluster complexes and multi-cluster complexes.*

The next open problem concerns a possible representation theoretic description of the multi-cluster complex in types ADE. For $k = 1$, in the cluster category, indecomposable objects V are indexed by almost positive roots and one can describe the compatibility between them by saying that $V \parallel_c V'$ if and only $\dim(\mathrm{Ext}^1(V, V')) = 0$, see [BMR+06].

Open Problem 3.99. *Describe the multi-cluster complex within the repetition quiver using similar methods.*

The following problem extends the diameter problem of the associahedron to the family of multi-cluster complexes, see [Pil10, Section 2.3.2] for further discussions in the case of multitriangulations.

Open Problem 3.100. *Find the diameter of the facet-adjacency graph of the multi-cluster complex* $\Delta_c^k(W)$.

Finally, we present several combinatorial conjectures on the multi-cluster complexes. We start with a conjecture concerning minimal nonfaces.

Conjecture 3.101. *All minimal nonfaces of the multi-cluster complex* $\Delta_c^k(W)$ *have cardinality* $k+1$.

Since $w_\circ$ is c-sortable, we have $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = \mathbf{c}^k\mathbf{c}_{K_1}\mathbf{c}_{K_2}\cdots\mathbf{c}_{K_r}$ with $K_r \subseteq \ldots \subseteq K_2 \subseteq K_1$. This implies that the complement of any k letters still contains a reduced expression for $w_\circ$. In other words, minimal nonfaces have at least cardinality $k+1$. Moreover, using the connection to multitriangulations and centrally symmetric triangulations, we see that the conjecture holds in types A and B. It also holds in the case of dihedral groups: it is not hard to see that the faces of the multi-cluster complex are given by subwords of $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c}) = (a, b, a, b, \ldots)$ that do not contain $k+1$ pairwise nonconsecutive letters (considered cyclically). The conjecture was moreover tested for all multi-cluster complexes of rank 3 and 4 with $k = 2$.

In types A and $I_2(m)$, there is a binary compatibility relation on the letters of $\mathbf{c}^k\mathbf{w}_\circ(\mathbf{c})$ such that the faces of the multi-cluster complex can be described as subsets avoiding $k+1$ pairwise incompatible elements. We remark that this is not possible in general: in type B_3 with $k=2$, as in Example 3.16, $\Delta_c^2(B_3)$ is isomorphic to the simplicial complex of centrally symmetric 2-triangulations of a regular convex 10-gon. Every pair of elements in the set $\mathcal{A} = \{[1,4]_{\text{sym}}, [4,7]_{\text{sym}}, [7,10]_{\text{sym}}\}$ is contained in a minimal nonface. But since $\mathcal{A}$ does not contain a 3-crossing, it forms a face of $\Delta_c^2(B_3)$.

Theorem 3.11 gives an alternative way of defining multi-cluster complexes as subword complexes $\Delta(Q,w_\circ)$, where the word Q has the SIN-property. It seems that this definition covers indeed all subword complexes isomorphic to multi-cluster complexes.

Conjecture 3.102. *Let Q be a word in S with complete support, and $\pi \in W$. The subword complex $\Delta(Q,\pi)$ is isomorphic to a multi-cluster complex if and only if Q has the SIN-property and $\pi = \delta(Q) = w_\circ$.*

The fact that $\pi = \delta(Q)$ is indeed necessary so that the subword complex is a sphere. It remains to show that $\pi = w_\circ$ and that Q has the SIN-property. One reason for this conjecture is that if Q does not have the SIN-property, then it seems that the subword complex $\Delta(Q,w_\circ)$ has fewer facets than required. Indeed, we conjecture that multi-cluster complexes maximize the number of facets among all subword complexes with a word Q of a given size.

Conjecture 3.103. *Let Q be any word in S with $kn+N$ letters (where N denotes the length of $w_\circ$) and $\Delta(Q,w_\circ)$ be the corresponding subword complex. The number of facets of $\Delta(Q,w_\circ)$ is less than or equal to the number of facets of the multi-cluster complex $\Delta_c^k(W)$. Moreover, if both numbers are equal, then the word Q has the SIN-property.*

The previous two conjectures hold for the dihedral types $I_2(m)$. In this case, the multi-cluster complex is isomorphic to the boundary complex of a cyclic polytope, which is a polytope that maximizes the number of facets among all polytopes in fixed dimension on a given number of vertices, see, e.g., [Zie95, Section 0]. Moreover, we present below a simple polytope theory argument in order to show that if a word does not satisfy the SIN-property, then the corresponding subword complex has strictly fewer facets than the multi-cluster complex. First note that Corollary 3.20 and Theorem 3.42 imply that all spherical subword complexes of type $I_2(m)$ are polytopal. By the upper bound theorem, a polytope has a many facets as a cyclic polytope if and only if it is neighborly, see, e.g., [Zie95, Section 8.4]. Therefore, it is enough to prove that if $Q = (q_1,\dots,q_r)$ with $r = 2k+m$ is a word in $S = \{a,b\}$ containing two consecutive letters that are equal, then the subword complex $\Delta(Q,w_\circ)$ is not neighborly. Since this is a $2k$-dimensional complex, this is equivalent to showing that there is a set of k letters of Q that do not form a face. By applying rotation of letters and Proposition 3.29, we can assume without loss of generality that the last two letters of Q are equal. Among the first $2k+1$ letters of Q, one of the generators a or b appears no more than k times. The set of these no more than k letters is not a face of the subword complex. The reason is that the reduced expressions in the complement of this set in Q have length at most $m-1$, which is one less than the length of $w_\circ$.

In view of Corollary 3.20, the following conjecture restricts the study of [KM04, Question 6.4].

Conjecture 3.104. *Let W be a finite Coxeter group. The multi-cluster complex of type W is the boundary complex of a simplicial polytope for all $k \geq 1$.*

In types A and B, this conjecture coincides with the conjecture on the existence of the corresponding multi-associahedra, see [Jon05, SW09], and Theorem 3.42 shows that this conjecture is true for dihedral groups.

Appendix A

Some root systems of rank 3 & 4

Here are some images of normalized root systems of rank 3 and 4 with small labels.

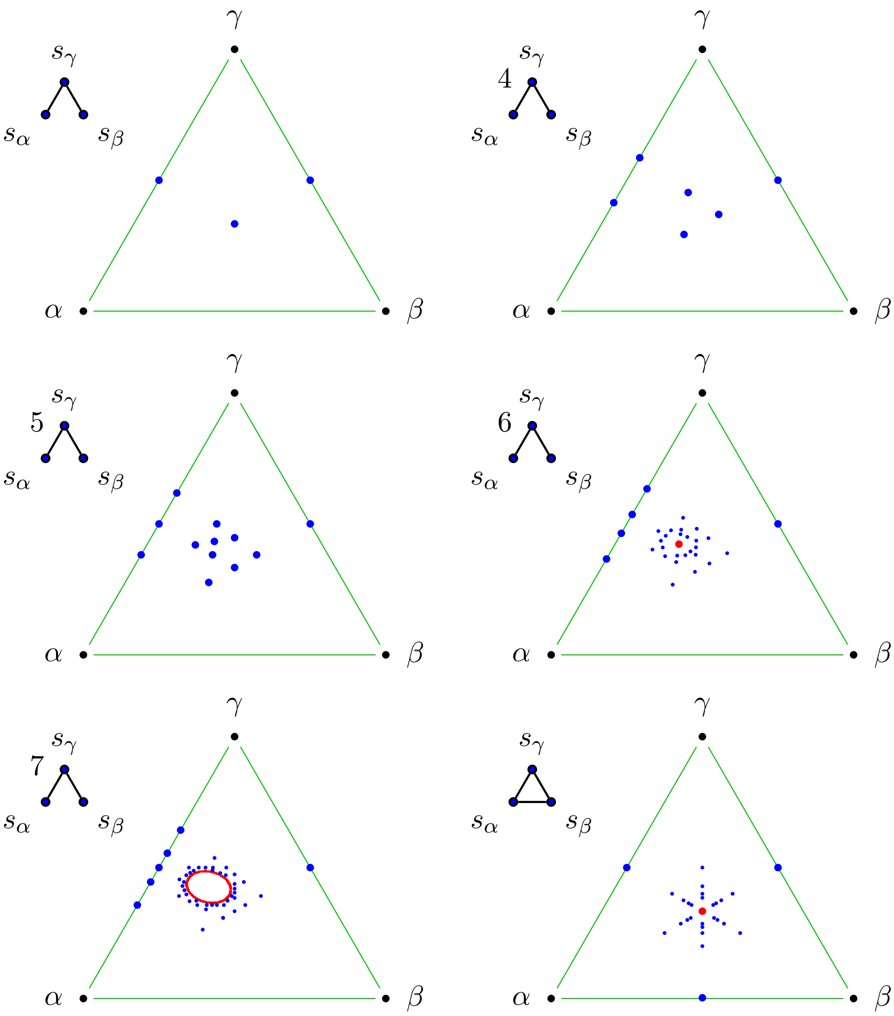


FIGURE A.1: In the first column: type A_3, H_3 and the triangle group $\{2, 3, 7\}$. In the second column: B_3, $\widetilde{I}_2(6)$ and type $\widetilde{A}_2$.

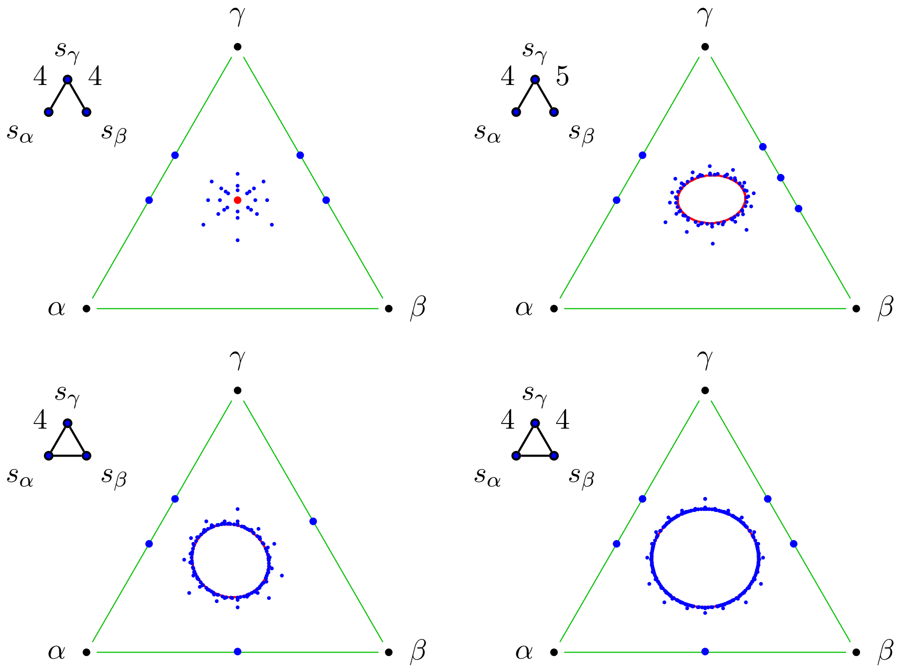


FIGURE A.2: In the first column: type $\widetilde{B}_2$ and the triangle group $\{3,3,4\}$. In the second column: the triangle group $\{2,4,5\}$ and the triangle group $\{3,4,4\}$.

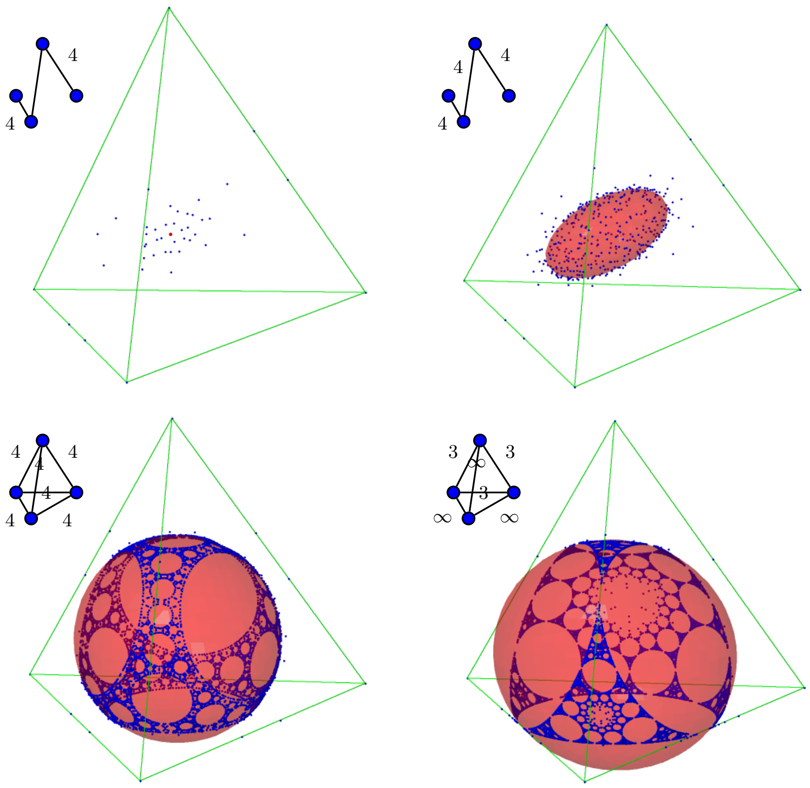


FIGURE A.3: In the top right image: type $\widetilde{C}_3$ and three different groups two of which give rise to fractal limits.

Appendix B

Subword complex vade-mecum

Spherical subword complex families

Spherical subword complexes $\Delta(Q, \delta(Q))$

If $\delta(Q) \neq w_\circ$, then $\Delta(Q, \delta(Q)) \cong \Delta(Q', w_\circ)$
for some Q' such that $\delta(Q') = w_\circ$ (Theorem 3.27)

If $\delta(Q) = w_\circ$ and Q does not have the SIN-property, then $\Delta(Q, \delta(Q))$ is the link of a face of $\Delta(Q', w_\circ)$ where Q' is a word with the SIN-property.
(Theorem 3.11 and Proposition 3.43)

Multi-cluster complexes $\Delta_c^k(W)$ (Q with SIN-property)

cluster complexes ($k = 1$): $\Delta_c(W)$	$k > 1$: $\Delta_c^k(W)$
dual associahedron Δ_m (triangulations)	Type A: $\Delta_c^k(A_{m-2k-1})$ simplicial complex $\Delta_{m,k}$ (multitriangulations)
dual cyclohedron Δ_m^{sym} (centrally sym. triangulations)	Type B: $\Delta_c^k(B_{m-k})$ simplicial complex $\Delta_{m,k}^{sym}$ (centr. sym. multitriang.)
$(m+2)$-gon	Type $I_2(m)$: $\Delta_c^k(I_2(m))$ $2k$-dim. cyclic polytope on $2k+m$ vertices

Bruhat order on subword complexes

The following result was obtained during a discussion with Cesar Ceballos. We denote by $\leq_B$ the Bruhat order of a Coxeter group and by $\partial\Delta(Q,\pi)$ the boundary of the subword complex $\Delta(Q,\pi)$.

Proposition B.1. *Let (W,S) be a Coxeter system, Q be a word in S, and $\pi,\tau \in W$ be such that the Demazure product $\delta(Q)$ strictly contains a reduced expression for π, i.e. $\pi <_B \delta(Q)$ and the element τ is a cover of π with respect to $\leq_B$. Then*

$$\Delta(Q,\tau) \subseteq \partial\Delta(Q,\pi),$$

with equality if $\tau = \delta(Q)$.

Proof. Let F be a facet of $\Delta(Q,\tau)$. Then $Q \setminus F$ is a reduced expression of τ. The word $Q \setminus F$ contains a subword which is a reduced expression for π, by the subword property [BB05, Theorem 2.2.2]. Moreover, this reduced subword is unique. Therefore F is contained in a unique facet G of $\Delta(Q,\pi)$. Thus, F is on the boundary of $\Delta(Q,\pi)$ since it is a vertex decomposable triangulated ball, see Section 3.1. □

Corollary B.2. *Given a word Q, the nonempty subword complexes*

$$\{\Delta(Q,\pi) : \pi \leq_B \delta(Q)\}$$

ordered by boundary-inclusion form a poset isomorphic to the interval $[\delta(Q), e]$ of the reverse Bruhat order $\geq_B$ on W.

Example B.3. Let $(W,S) = (A_2, \{s_1,s_2\})$ and $Q = (s_1,s_2,s_1,s_2,s_1)$. Fig. B.1 depicts the subword complexes $\{\Delta(Q,\pi) : \pi \in A_2\}$ ordered by boundary-inclusion.

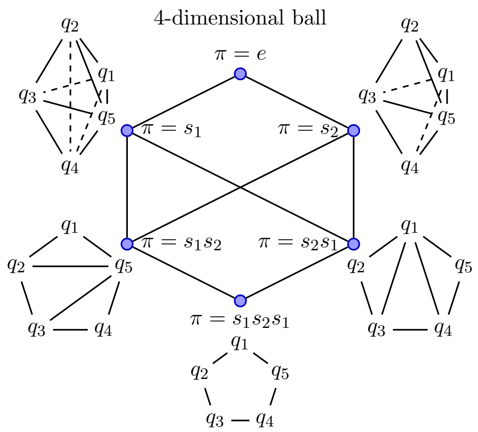


FIGURE B.1: Subword complexes ordered by boundary inclusions, see also Example 1.13.

Declaration of Authorship

I, Jean-Philippe Labbé, declare that this thesis titled, "Polyhedral Combinatorics of Coxeter groups" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ______________________________

Date: ______________________________

Index

A^c, 16

alignment, M, 16

biconvex sets, $\mathcal{C}(\Phi^+)$, 17

boundary

  inbound, 71

  outbound, 71

boundary graph, $\mathcal{G}_T$, 68

bounded cylinder, $\mathcal{Z}$, 66

braid relation, 5

c-cluster, 41

c-cluster complex, $\Delta_c(W)$, 41

c-singleton, 65

c-sorting word, $\mathbf{w}(\mathbf{c})$, 42, 64

commuting vertex, 71

  inferior, 71

  superior, 71

cone, cone(E), 10

convex geometry, 16

Coxeter element, c, 5

  bipartite, 5

Coxeter graph, Γ, 5

Coxeter number, h, 6

Coxeter system, (W, S), 5

  irreducible, 5

  rank, n, 5

cut, κ, 69

  crossing, 70

  function, $\mathcal{K}$, 76

  opposite, κ^* , 69

  straight, 82

  support, $\mathcal{T}_\kappa$, 71

  width, $\mu_\mathcal{K}$, 76

cyclic longest word, $\overset{\circlearrowleft}{\mathbf{w}}_\circ$, 66

cylindric graph, $\mathcal{Z}_{\mathbf{w}_\circ}$, 66

  of sorting words, $\mathcal{Z}_\diamond$, 72

Demazure product, δ, 40

dual associahedron, 12

flip, 49

generalized multi-associahedron, 57

generator

  final, 6

  initial, 6

geometric representation, 9

  canonial, (V_A, B_A), 11

  classical, 7

geometry, 17

inferior poset, $I_\kappa(T_i)$, 73

inversion set, inv(w), 6

k-crossing, 12

k-relevant diagonal, 12

k-triangulation, 12

length function, ℓ, 6
limit roots, $E(\Phi)$, 28
longest element, $w_\circ$, 6
longest word, $\mathbf{w}_\circ$, 6
loops, $\mathcal{L}_{\mathbf{w}_\circ}$, 67

M-closure operator, $\overline{\cdot}^M$, 18
multi-cluster complex, $\Delta_c^k(W)$, 43

natural partial order, $\prec_{\mathbf{w}_\circ}$, 65

ortholattice, 6

ψ, 6

reduced expression, 6
reflections, T, 6
  B-reflection, 8
  $s_\alpha(v)$, 8
  simple, 8
  subgroup, 8
reverse word, $\mathrm{rev}(\mathbf{w})$, 63
right weak order, $(W, \leq)$, 6
root function, r_F, 49
root system, Φ, 8, 41
  affine based, 10
  based, (Φ, Δ), 9
  rank, n, 9
roots
  almost positive, $\Phi_{\geq -1}$, 41
  biclosed sets, $\mathcal{B}(\Phi^+)$, 16
  depth, $\mathrm{dp}(\rho)$, 26
  normalized, $\widehat{\Phi}$, 22
  positive, Φ^+, 10, 41
  simple, Δ, 8, 41
rotated word, $Q_{\circlearrowleft s}$, 51

separable sets, $\mathcal{S}(\Phi^+)$, 17
simple system, Δ, 9
simplicial complex of multitriangulations, $\Delta_{m,k}$, 12
SIN-property, 44
subword complex, $\Delta(Q, \pi)$, 40
superior poset, $S_\kappa(T_i)$, 73

tile, T, 68
  apex, 73
  split tile, $T^\vee_{\kappa,\kappa'}$, 71
triangulation, 12

Bibliography

[AB08] Peter Abramenko and Kenneth S. Brown, *Buildings*, vol. 248, Springer, New York, 2008.

[Arm11] Drew Armstrong, *Personal communication*, 2011.

[ABMW06] Christos A. Athanasiadis, Thomas Brady, Jon McCammond, and Colum Watt, *h-vectors of generalized associahedra and noncrossing partitions*, Int. Math. Res. Not. **2006** (2006), 28 pp.

[BHLT09] Nantel Bergeron, Christophe Hohlweg, Carsten E.M.C. Lange, and Hugh Thomas, *Isometry classes of generalized associahedra*, Sém. Lothar. Combin. **B61Aa** (2009), 13 pp.

[BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, GTM, vol. 231, Springer, New York, 2005.

[BHS05] Dieter Blessenohl, Christophe Hohlweg, and Manfred Schocker, *A symmetry of the descent algebra of a finite Coxeter group*, Adv. Math. **193** (2005), no. 2, 416–437.

[BP09] Jürgen Bokowski and Vincent Pilaud, *On symmetric realizations of the simplicial complex of* 3*-crossing-free sets of diagonals of the octagon*, Proc. 21th Canadian Conference on Comput. Geom., 2009, pp. 41–44.

[BD10] Cédric Bonnafé and Matthew J. Dyer, *Semidirect product decomposition of Coxeter groups*, Comm. Algebra **38** (2010), no. 4, 1549–1574.

[Bou68] Nicolas Bourbaki, *Groupes et algèbres de Lie. Chapitre 4-6*, Paris: Hermann, 1968.

[BW08] Thomas Brady and Colum Watt, *Lattices in finite real reflection groups*, Trans. Amer. Math. Soc. **360** (2008), 1983–2005.

[BMR$^+$06] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

[CP92] Vasilis Capoyleas and János Pach, *A Turán-type theorem on chords of a convex polygon*, J. Combin. Theory Ser. B **56** (1992), no. 1, 9–15.

[Ceb12] Cesar Ceballos, *On associahedra and related topics*, Ph.D. thesis, Freie Universität Berlin, 2012, pp. xi+87.

[CLS13] Cesar Ceballos, Jean-Philippe Labbé, and Christian Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, to appear in *J. Algebraic Combin.* (2013), DOI: `10.1007/s10801--013--0437--x`, 35 pp.

[CSZ11] Cesar Ceballos, Francisco Santos, and Günter M. Ziegler, *Many non-equivalent realizations of the associahedron*, preprint, `arXiv:abs/1109.5544` (September 2011), 28 pp.

[CFZ02] Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2002), no. 4, 537–566.

[Cox34] Harold S. M. Coxeter, *Discrete groups generated by reflections*, Ann. Math. **35** (1934), no. 3, 588–621.

[DKK12] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb Koshevoy, *Condorcet domains of tiling type*, Discrete Appl. Math. **160** (2012), no. 7-8, 933–940.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos, *Triangulations*, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010.

[Deo89] Vinay V. Deodhar, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. **53** (1989), no. 6, 543–546.

[DKM02] Andreas Dress, Jack H. Koolen, and Vincent Moulton, *On line arrangements in the hyperbolic plane*, Eur. J. Combin. **23** (2002), no. 5, 549–557.

[Dye90] Matthew Dyer, *Reflection subgroups of Coxeter systems*, J. Algebra **135** (1990), no. 1, 57–73.

[Dye93] ———, *Hecke algebras and shellings of Bruhat intervals*, Compos. Math. **89** (1993), no. 1, 91–115.

[Dye13] ———, *Imaginary cone and reflection subgroups of Coxeter groups*, in preparation (2013).

[Dye11] ———, *On the weak order of Coxeter groups*, preprint, `arXiv:abs/1108.5557` (August 2011), 37 pp.

[Dye10] ———, *On rigidity of abstract root systems of Coxeter systems*, preprint, `arXiv:abs/1011.2270` (November 2010), 34 pp.

[DH12] Matthew Dyer and Christophe Hohlweg, *Personal communication*, 2012.

[DHR13] Matthew Dyer, Christophe Hohlweg, and Vivien Ripoll, *Asymptotical behaviour of roots of infinite Coxeter groups II*, preprint, `arXiv:abs/1303.6710` (March 2013), 63 pp.

[EJ85] Paul H. Edelman and Robert E. Jamison, *The theory of convex geometries*, Geom. Dedicata **19** (1985), no. 3, 247–270.

[EE09] Henrik Eriksson and Kimmo Eriksson, *Conjugacy of Coxeter elements*, Electron. J. Combin. **16** (2009), no. 2, R4 7pp.

[FR05] Sergey Fomin and Nathan Reading, *Generalized cluster complexes and Coxeter combinatorics*, Int. Math. Res. Notices **2005** (2005), no. 44, 2709–2757.

[FZ03] Sergey Fomin and Andrei Zelevinsky, *Y-Systems and generalized associahedra*, Ann. Math. **158** (2003), no. 3, 977–1018.

[Fu12a] Xiang Fu, *The dominance hierarchy in roots systems of Coxeter groups*, J. Algebra **366** (2012), no. 1, 187–204.

[Fu12b] ———, *Coxeter groups, imaginary cones and dominance*, preprint, `arXiv:abs/1108.5232` (October 2012), 25 pp.

[GR97] Peter Gabriel and Andrei V. Roiter, *Representations of finite-dimensional algebras*, Springer-Verlag, Berlin, 1997.

[GR08] Ádám Galambos and Victor Reiner, *Acyclic sets of linear orders via the Bruhat orders*, Soc. Choice Welfare **30** (2008), no. 2, 245–264.

[GP00] Meinolf Geck and Götz Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Oxford University Press, 2000.

[GKZ08] Israel M. Gelfand, Mikhael M. Kapranov, and Andrei Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., 2008.

[Hai84] Mark Haiman, *Constructing the associahedron*, unpublished manuscript, `http://math.berkeley.edu/~mhaiman/ ftp/assoc/manuscript.pdf` (1984), 12 pp.

[Hoh10] Christophe Hohlweg, *Personal communication*, Summer 2010.

[HLR13] Christophe Hohlweg, Jean-Philippe Labbé, and Vivien Ripoll, *Asymptotical behaviour of roots of infinite Coxeter groups*, Canad. J. Math. (2013), 29 pp.

[HL07] Christophe Hohlweg and Carsten E.M.C. Lange, *Realizations of the associahedron and cyclohedron*, Discrete Comput. Geom. **37** (2007), no. 4, 517–543.

[HLT11] Christophe Hohlweg, Carsten E.M.C. Lange, and Hugh Thomas, *Permutahedra and generalized associahedra*, Adv. Math. **226** (2011), no. 1, 608–640.

[How96] Robert B. Howlett, *Introduction to Coxeter groups*, Lectures at A.N.U., `http://www.maths.usyd.edu.au/res/Algebra/How/1997-6.html` (1996), 21 pp.

[Hum92] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics. 29 (Cambridge University Press), 1992.

[IS10] Kiyoshi Igusa and Ralf Schiffler, *Exceptional sequences and clusters*, J. Algebra **323** (2010), no. 8, 2183–2202.

[Jon03] Jakob Jonsson, *Generalized triangulations of the n-gon*, Unpublished manuscript, abstract in Mathematisches Forschungsinstitut Oberwolfach, Report No. 16/2003, 2003.

[Jon05] ———, *Generalized triangulations and diagonal-free subsets of stack polyominoes*, J. Comb. Theory, Ser. A **112** (2005), no. 1, 117–142.

[JW07] Jakob Jonsson and Volkmar Welker, *A spherical initial ideal for pfaffians*, Ill. J. Math. **51** (2007), no. 4, 1397–1407.

[Kac90] Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[KM04] Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, Adv. Math. **184** (2004), no. 1, 161–176.

[KM05] ———, *Gröbner geometry of Schubert polynomials*, Ann. Math. **161** (2005), no. 3, 1245–1318.

[Kra09] Daan Krammer, *The conjugacy problem for Coxeter groups*, Group. Geom. Dynam. **3** (2009), no. 1, 71–171.

[Kra06] Christian Krattenthaler, *Growth diagrams, and increasing and decreasing chains in fillings of ferrers shapes*, Adv. Appl. Math. **37** (2006), no. 3, 404–431.

[Lan12] Carsten Lange, *Counting singletons in type A*, Polyhedral Combinatorics Seminar (Freie Universität Berlin), April 30, 2012.

[Lee89] Carl W. Lee, *The associahedron and triangulations of the n-gon*, Eur. J. Combin. **10** (1989), no. 6, 551–560.

[Lod04] Jean-Louis Loday, *Realization of the Stasheff polytope*, Arch. Math. **83** (2004), no. 3, 267–278.

[LN04] Ottmar Loos and Erhard Neher, *Locally finite root systems*, Mem. Amer. Math. Soc. **171** (2004), no. 811, x+214.

[MRZ03] Robert Marsh, Markus Reineke, and Andrei Zelevinsky, *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4171–4186.

[Max82] George Maxwell, *Sphere packings and hyperbolic reflection groups*, J. Algebra **79** (1982), no. 1, 78–97.

[MP89] Robert V. Moody and Arturo Pianzola, *On infinite root systems*, Trans. Amer. Math. Soc. **315** (1989), no. 2, 661–696.

[MHPS12] Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff (eds.), *Associahedra, Tamari lattices and related structures*, Progress in Mathematics, vol. 299, Birkhäuser Boston Inc., Boston, MA, 2012.

[Nak00] Tomoki Nakamigawa, *A generalization of diagonal flips in a convex polygon*, Theor. Comput. Sci. **235** (2000), no. 2, 271–282.

[Nil12] Jonathan Nilsson, *Enumeration of basic ideals in type B Lie algebras*, J. Integer Seq. **15** (2012), no. 9, Art. 12.9.5, 30.

[PWZ96] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, $A = B$, A K Peters Ltd., Wellesley, MA, 1996.

[Pil10] Vincent Pilaud, *Multitriangulations, pseudotriangulations and some problems of realization of polytopes*, Ph.D. thesis, Université Paris 7 and Universidad de Cantabria, 2010, p. 312.

[PP12] Vincent Pilaud and Michel Pocchiola, *Multitriangulations, pseudotriangulations and primitive sorting networks*, Discrete Comput. Geom. **48** (2012), no. 1, 142–191.

[PS09] Vincent Pilaud and Francisco Santos, *Multitriangulations as complexes of star polygons*, Discrete Comput. Geom. **41** (2009), no. 2, 284–317.

[PS12a] ______, *The brick polytope of a sorting network*, Eur. J. Combin. **33** (2012), no. 4, 632–662.

[PS12b] Vincent Pilaud and Christian Stump, *Brick polytopes of spherical subword complexes: A new approach to generalized associahedra*, preprint, `arXiv:abs/1111.3349` (April 2012), 45 pp.

[Pil06] Annette Pilkington, *Convex geometries on root systems*, Comm. Algebra **34** (2006), no. 9, 3183–3202.

[Rea06] Nathan Reading, *Cambrian lattices*, Adv. Math. **205** (2006), no. 2, 313–353.

[Rea07a] ______, *Clusters, Coxeter-sortable elements and noncrossing partitions*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 5931–5958.

[Rea07b] ______, *Sortable elements and Cambrian lattices*, Algebra Univ. **56** (2007), no. 3–4, 411–437.

[RS09] Nathan Reading and David E. Speyer, *Cambrian fans*, J. Eur. Math. Soc. **11** (2009), no. 2, 407–447.

[RS11] ______, *Sortable elements in infinite Coxeter groups*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 699–761.

[RSS08] Günter Rote, Francisco Santos, and Ileana Streinu, *Pseudo-triangulations—a survey*, Surveys on discrete and computational geometry, Contemp. Math., vol. 453, Amer. Math. Soc., 2008, pp. 343–410.

[Rub11] Martin Rubey, *Increasing and decreasing sequences in fillings of moon polyominoes*, Adv. Appl. Math. **47** (2011), no. 1, 57–87.

[RS10] Martin Rubey and Christian Stump, *Crossings and nestings in set partitions of classical types*, Electron. J. Combin. **17** (2010), no. 1, R120.

[RS09] ______, *Crossings and nestings in set partitions of classical types*, preprint, `arXiv:abs/0904.1097` (April 2009), 22 pp.

[Shi97] Jian-Yi Shi, *The enumeration of Coxeter elements*, J. Algebraic Combin. **6** (1997), no. 2, 161–171.

[Sim03] Rodica Simion, *A type-B associahedron*, Adv. in Appl. Math. **30** (2003), no. 1–2, 2–25.

[SW09] Daniel Soll and Volkmar Welker, *Type-B generalized triangulations and determinantal ideals*, Discrete Math. **309** (2009), no. 9, 2782–2797.

[Spe09] David E. Speyer, *Powers of Coxeter elements in infinite groups are reduced*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1295–1302.

[Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

[Sta63] James D. Stasheff, *Homotopy associativity of H-Spaces*, Trans. Amer. Math. Soc. **108** (1963), no. 2, 275–292.

[Sage] William A. Stein et al., *Sage Mathematics Software (Version 5.6)*, The Sage Development Team, 2013, `http://www.sagemath.org`.

[Ste12] Salvatore Stella, *Polyhedral models for generalized associahedra via Coxeter elements*, J. Algebraic Combin. (2012), 1–38.

[Stu11] Christian Stump, *A new perspective on k-triangulations*, J. Comb. Theory, Ser. A **118** (2011), no. 6, 1794–1800.

[Tam51] Dov Tamari, *Monoides préordonnés et chaînes de malcev*, Ph.D. thesis, Paris, 1951, p. 81.

[Vin71] Ernest B. Vinberg, *Discrete linear groups that are generated by reflections*, Izv. Akad. Nauk SSSR. Ser. Mat. **35** (1971), 1072–1112.

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, GTM, vol. 152, Springer, New York, 1995.