CTRW, fractional diffusion and the effect of the central linear drift

4.1 Introduction

In a random walk we consider a walker (*particle*) starting at time $t = 0$ at a given point x in space and taking steps in a random direction at time instants t_n. The steps may be of fixed or of random length, depending on the considered model.

In Chapter 2, we have discussed the Ehrenfest model and represented it by a random walk with steps of length h to the left or to the right along the real axis. The steps happen at instants $t_n = n\tau$ and a linear drift acts toward the origin $x = 0$. We have shown that if $h \to 0$ and $\tau \to 0$, related to each other by proper scaling, the random walker typically exhibits an approximate kind of Brownian motion if there is no external force affecting the motion of the particle (i. e. the free motion).

We have studied also the generalized Ehrenfest model in which the walker (particle) as time proceeds from t_n to t_{n+1} can move one step to the left or one step to the right or stay in its position. We have shown in the simulation of this model that the random walker tends always to return to the origin because of the presence of the linear central force $-bx$, $b > 0$, which attracts the particle to the origin.

In Chapter 3, we have studied the approximate solution and the discrete simulation of the random walk of the time-fractional diffusion equation with central linear drift for which we sometimes use the abbreviation *FDECLD*. The time-FDECLD is a generalized form of the fractional *Fokker-Planck equation* in using the fractional derivative and a special form in using a central linear force. The simulation of the time-FDECLD showed the effect of the memory on the motion of the particle which always remembered its all previous

positions, including the initial position, and showed a tendency to return to them.

We devote this chapter to the discussion of the continuous time random walk (Montroll and Weiss 1965 [77]) for which the abbreviation *(CTRW)* is in common use. In CTRW a walker (particle) is staying fixed during a waiting time of random length and then makes an instantaneous space step that also may be of random length.

There are two different types of CTRW. The first type considers independent time and space steps. This means the time and space steps are identically independent distributed random variables and the waiting time and the space step are independent of each other. In what follows we use the abbreviation *iid* to refer to random variables which are identically independent distributed. Such random walk is also called decoupled or separable. In the long time and large distance limit the decoupled model of iid variables goes over into the fractional diffusion equation by a properly scaled passage to the limit of vanishing space and time steps. We leave aside the second type of interest, namely the coupled or non-separable case in which the steps in space and time are dependent on each other.

In this chapter we discuss in full details the decoupled CTRW and its relation to the fractional diffusion equation and henceforth to the fractional diffusion equation with central linear drift. We focus our interest on processes in which the probability distributions of the waiting times and jumps have fat tails characterized by power laws with exponent between 0 and 1 for the waiting times and between 0 and 2 for the jumps.

This chapter is organized as follows:

In Section 2, we give a quick review over the theory of CTRW and the renewal counting process. We also prove that the equation of motion of the CTRW or so called master equation is a straightforward consequence of the basic theory of the compound *renewal process*.

In Section 3, the time-fractional integral equation of CTRW is discussed with some examples of the memory function.

In Section 4, the relation between the fractional integral equation of the continuous time random walk (equation of motion of a free particle) and the space-time-FDE will be described. We consider the master equation in which the time derivative is replaced with one of fractional order.

In Section 5, we discuss the relation between the space-FDE as well as the space-FDECLD. We show that the fundamental solu-

tions of these equations belong to the class of α-stable probability densities functions. The transformation connecting the fractional diffusion equation and the fractional diffusion equation with central linear drift is also given and proved.

In Section 6, the simulation of the CTRW is interpreted for different fractional orders of the fractional diffusion equation and of the fractional diffusion equation with central linear drift.

In Section 7 the numerical results are displayed and interpreted.

4.2 The basic theory of CTRW and the compound renewal process

The theory of CTRW or compound (*cumulative*) renewal process was developed by Montroll and Weiss [77] in their study of statistical mechanics and by Cox [13], respectively. In what follows we first give a brief survey for the decoupled model. This model describes the motion of a particle starting at the origin ($x_0 = 0$), and waiting a period of time $T_k, k = 1, 2, \cdots$, at a particular location $x_{k-1}, k = 1, 2, \cdots$, before moving instantaneously to the next location with jump width X_k, $k = 1, 2, \cdots$. We call T_k, $k = 1, 2, \cdots$, the waiting times, where $T_k = t_k - t_{k-1}$, $\forall k$, , $t_k > t_{k-1}$. The waiting times T_k are iid and likewise the jumps X_k are iid. Furthermore the waiting times and the jumps are independent of each other. The new position at t_k is $x_k = x_{k-1} + X_k$ $\forall k = 1, 2, \cdots$, and the particle remains resting at $x = x_{k-1}$ in the time interval $t_k < t < t_{k+1}$.
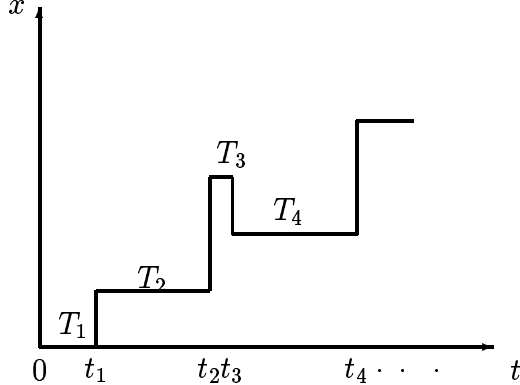
The sketch of the CTRW

As it is our intention to use the methods of the renewal theory, we consider the sequence $t_1, t_2, \cdots$ as renewal instants of a renewal counting process [13]. The renewal counting process $\{N(t), t \geq 0\}$

registers the successive occurrence of an event during the time interval $(0, t]$ where the time durations between consecutive events are positive iid random variables T_k, $k = 1, 2, \cdots$, such that T_i is the elapsed time between the $(i - 1)$st event and the ith event. The basic stipulations of the compound renewal process are:

(1) the process starts at $t = t_0 = 0$,
(2) $t_n = T_1 + T_2 + \cdots + T_n$, $n \geq 1$ refers to the time of the nth event.

The process is well known if we know the probability law for the waiting times $T_n = t_n - t_{n-1}$, $n \geq 1$, namely their probability density $\psi(t)$. So we introduce the cumulative distribution function $\Phi(t)$ defined as

$$\Phi(t) = Pr(T \leq t) = \int_0^t \psi(t')dt' \ , \ \psi(t) = \frac{d}{dt}\Phi(t) \ .$$

In renewal theory $\Phi(t)$ is often called the failure probability. We introduce also the function $\Psi(t)$ defined as

$$\Psi(t) = Pr(T > t) = \int_t^\infty \psi(t')dt' = 1 - \int_0^t \psi(t')dt' \ , \qquad (4.2.1)$$

which is called the survival probability and is equal to the probability of no events at or before the instant t. The waiting time density $\psi(t)$ is related to $\Psi(t)$ by

$$\psi(t) = -\frac{d\Psi(t)}{dt} \quad . \qquad (4.2.2)$$

Now we discuss shortly the function $m(t) = \langle N(t) \rangle = \sum_{k=1}^\infty Pr(t_k \leq t)$, which is called the first moment of the renewal process, in order to show the importance of the waiting density $\psi(t)$. The function $m(t)$ represents the average time of events in the interval $(0, t]$, then $m(t)$, $\Psi(t)$ and $\Phi(t)$ and their probability density functions satisfy the renewal equation (see [54] and [20])

$$m(t) = \Phi(t) + \int_0^t m(t-t')\psi(t')dt' = \int_0^t [1+m(t-t')]\psi(t')dt' \ , \ t \geq 0 \ .$$
$$(4.2.3)$$

To proceed further, we use the machinery of the Laplace transform defined for a (generalized) function $g(t)$ as

$$\widetilde{g}(s) = \int\limits_0^\infty e^{-st} g(t) dt \ . \tag{4.2.4}$$

Furthermore, we use the Laplace convolution of two (generalized) functions $g_1(t)$ and $g_2(t)$ which is defined as

$$(g_1 * g_2)(t) = \int\limits_0^\infty g_1(t') g_2(t - t') dt' \ , \ 0 \le t < \infty \ .$$

Therefore, by applying Laplace transform to equation (4.2.3) we get

$$\widetilde{m}(s) = \frac{\widetilde{\psi}(s)}{s(1 - \widetilde{\psi}(s))} \ , \tag{4.2.5}$$

where

$$\widetilde{\Phi}(s) = \frac{\widetilde{\psi}(s)}{s} \quad \text{and} \quad \widetilde{\Psi}(s) = \frac{1 - \widetilde{\psi}(s)}{s} \ .$$

For more information about the renewal process and its properties (see [85] and [13]).

Now we define the probability distribution function

$$F_k(t) = Pr(t_k = T_1 + T_2 + \cdots + T_k \le t) \ , k \ge 1 \ , \tag{4.2.6}$$

for the probability that the sum of the first k waiting times is less or equal to t. The corresponding probability density is $f_k = \frac{dF_k(t)}{dk}$. From the definition of the distribution function $F_k(t)$, we notice that it satisfies the normalization condition. This means that the probability density function for t_1 is $\psi(t) = f_1(t)$ and for t_2 is $(\psi * \psi)(t) = f_2(t)$ and consequently for t_k is $\psi^{*k}(t) = f_k(t)$. From the definition of $\Psi(t)$ in equation (4.2.1) as a residual probability function, we define the counting function $N(t)$ which equals the number of events in the interval $(0, t]$. The function $N(t)$ is a step function, see [13]. By using the definition of $f_k(t)$ we can write

$$p_k(t) = Pr(N(t) = k) = Pr(t_k \le t, t_{k+1} > t) = \int\limits_0^t f_k(t') \Psi(t - t') \, dt' = (f_k * \Psi)(t) \ .$$

$$\tag{4.2.7}$$

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Let us now introduce the concept of the compound renewal process. Such process is determined by the probability law of the waiting time T (i. e. $\psi(t)$) and also by the probability law of the jump X. Let us denote the jump density by $w(x)$ and introduce the distribution function of the jump defined as

$$W(x) = \int\limits_{-\infty}^{x} w(\xi)d\xi = Pr(X \leq x) ,$$

by restricting our definition to processes with one spatial dimension. Assume $x_k = X_1 + X_2 + \cdots + X_k$, $k \geq 1$, to be the sum of k independent random jumps in one dimension. Then the probability density of x_k is $w_k(x) = w^{*k}(x)$ $\forall k \geq 1$, where $w^{*0}(x) = \delta(x)$). To find the probability density at time t for the particle to be at the position x, we set

$$p(x, t) = \sum_{n=0}^{\infty} p_n(t) w_n(x) .$$

Using now equation (4.2.7), we get

$$p(x, t) = \sum_{n=0}^{\infty} \left((\psi^{*n} * \Psi)(t) \right) w^{*n}(x) . \tag{4.2.8}$$

This equation describes the compound renewal process as a subordination of a random walk to a renewal process (see [13] Chapter 8 formula (4)). In the Fourier-Laplace domain, it reads

$$\widehat{\widetilde{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s} \sum_{n=0}^{\infty} \widetilde{\psi}^n(s) \widehat{w}^n(\kappa) ,$$

where the Fourier transform of a (generalized) function $f(x)$, $x \in \mathbb{R}$, is defined as

$$\mathcal{F}\{f(x); \kappa\} = \widehat{f}(\kappa) = \int\limits_{-\infty}^{\infty} e^{i\kappa x} f(x) dx . \tag{4.2.9}$$

Since $|\widetilde{\psi}(s)\widehat{w}(\kappa)| < 1$ for all $\kappa \neq 0$ and $s \neq 0$, we have

$$\widehat{\widetilde{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s} \frac{1}{1 - \widetilde{\psi}(s)\,\widehat{w}(\kappa)} . \tag{4.2.10}$$

In what follows, we aim to prove that equation (4.2.10) coincides with the Fourier-Laplace transform of the integral equation of the

CTRW. To this end we rewrite equation (4.2.10) as

$$\widehat{\widetilde{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s} + \widetilde{\psi}(s)\, \widehat{w}(\kappa)\, \widehat{\widetilde{p}}(\kappa, s) \ .$$

Then first, by taking the inverse Fourier transform, we get

$$\widetilde{p}(x, s) = \delta(x)\, \widetilde{\Psi}(s) + \{ \int_0^x w(x - x')\widetilde{p}(x', s)\, dx' \}\widetilde{\psi}(s) \ ,$$

second, by taking the inverse Laplace transform, we get

$$p(x, t) = \delta(x)\Psi(t) + \int_0^t \{ \int_{-\infty}^{\infty} w(x - x')p(x', t')dx' \}\, \psi(t - t')dt' \ .$$

$$(4.2.11)$$

As we have seen, the inverse Fourier-Laplace transformation of equation (4.2.10) converts the probability density of the compound renewal process into the probability density of the decoupled model, equation (4.2.11). In this last equation $\psi(t)$, $w(x)$ and $p(x, t)$ are non-negative probability density functions. $p(x, t)$ represents the probability density of finding the walker at the position x at the instant t, starting with the initial condition $p(x, 0) = \delta(x)$, and $w(x)$ serves as the transition probability density from the point ξ to the point $\xi + x$. Finally, $\psi(t)$ represents the probability density of the waiting time between two successive jumps. Equation (4.2.11) is known by physicists as *the integral equation of the CTRW* (see for example: [77] [96], [46], [45] and [47]).

Many authors have treated this integral equation and its relation to the fractional diffusion equation in order to show its importance in the theory of anomalous diffusion (see for example: [1], [2], [57] and the references therein).

We are going now to give a simple example of a compound renewal process.

Example :
The **Poisson process** is characterized by an exponential waiting time with density

$$\psi(t) = \lambda e^{-\lambda t}\ , \lambda > 0\ , t > 0\ \ .$$

Its survival probability is

$$\Psi(t) = Pr(T > t) = e^{-\lambda t}\ , t \geq 0\ .$$

The exponential waiting time is an essential property of the memoryless renewal process (Markov Process). In the Laplace domain, we have

$$\widetilde{\psi}(s) = \frac{\lambda}{\lambda + s} \ \text{ and } \ \widetilde{\Psi}(s) = \frac{1}{\lambda + s} \ , \ s \geq 0 \ .$$

The moments of the waiting time are

$$\langle T \rangle = 1/\lambda \ , \langle T^2 \rangle = 1/\lambda^2, \cdots \langle T^n \rangle = 1/\lambda^n \ .$$

It is well known that the Poisson process counts the number of events occurring in an interval of length t and it is often defined by the equation

$$Pr(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \ , t \geq 0 \ . \tag{4.2.12}$$

The renewal function comes out as

$$\langle N(t) \rangle = m(t) = \lambda t \ , \ t \geq 0 \ .$$

In the Laplace domain it takes the form $\widetilde{m}(s) = \frac{\lambda}{s^2}$.

4.3 The time-fractional integral equation of the CTRW and the memory function

In this section we complete our survey of the integral equation of the CTRW by discussing the recent applications of the CTRW in economics and finance (see for examples [93], [67], [41] and [83]). Therefore, we consider again $p(x,t)$ as the probability density for finding the random walker (particle) at the position x at the instant t while being initially at $x = 0$ (i. e. $p(x,0) = \delta(x)$). We also keep the definition of the survival probability $\Psi(t)$ as in equation (4.2.1) and the definition of $w(x)$ as the jump density. The only change made here is that we write equation (4.2.10) representing the Fourier-Laplace transform of the integral equation of the CTRW in another form by introducing a new function $\nu(s)$ in the Laplace domain as follows (see [67]):
First, rewrite equation (4.2.10) as

$$s\widehat{\widetilde{p}}(\kappa, s) \left(1 - \widehat{w}(\kappa)\widetilde{\psi}(s) \right) = 1 - \widetilde{\psi}(s) \ .$$

By subtracting from each side $s\widehat{\widetilde{p}}(\kappa, s)\,\widetilde{\psi}(s)$ and then dividing by $s\widetilde{\psi}(s)$, one gets

$$\frac{s\left(1 - \widetilde{\psi}(s)\right)\widehat{\widetilde{p}}(\kappa, s)}{s\widetilde{\psi}(s)} - \frac{1}{s\widetilde{\psi}(s)} = \widehat{\widetilde{p}}(\kappa, s)(\widehat{w}(\kappa) - 1) - \frac{1}{s} \ .$$

Finally, by defining

$$\widetilde{\nu}(s) = \frac{1 - \widetilde{\psi}(s)}{s\,\widetilde{\psi}(s)} \ , \qquad (4.3.1)$$

and subtracting $\widetilde{\nu}(s)$ from both sides, we get the other version of equation (4.2.10)

$$\widetilde{\nu}(s)(s\,\widehat{\widetilde{p}}(\kappa, s) - 1) = (\widehat{w}(\kappa) - 1)\widehat{\widetilde{p}}(\kappa, s) \ , \qquad (4.3.2)$$

and then by taking the inverse Fourier-Laplace transform, we get another version of the integral equation of the CTRW (a master equation), namely

$$\int\limits_0^t \nu(t - t') \frac{\partial}{\partial t'} p(x, t') dt' = -p(x, t) + \int\limits_{-\infty}^{\infty} w(x - x') p(x', t) dx' \ , \quad (4.3.3)$$

where the auxiliary function $\nu(t)$ is connected to the survival probability $\Psi(t)$ by the convolution equation

$$\Psi(t) = \int\limits_0^t \nu(t - t')\psi(t') dt' \ .$$

The function $\nu(t)$ is called the memory function. By using it, the CTRW described by the integral equation (4.3.3) turns, in general, out to be a non-Markovian process. However, by special choices of $\nu(t)$, the process is a memoryless. Examples:

(a) if $\widetilde{\nu}(s) = 1$ (i.e. $\nu(t) = \delta(t)$), then by using equation (4.3.1) we get $\widetilde{\psi}(s) = \frac{1}{1+s}$, which leads by taking the inverse of Laplace-transform to $\psi(t) = e^{-t}$,

(b) if $\widetilde{\nu}(s) = \frac{1}{\lambda}$, $\lambda > 0$ (i.e. $\nu(t) = \delta(t)/\lambda$), then $\widetilde{\nu}(s) = \frac{\lambda}{\lambda+s}$ leading to $\psi(t) = \lambda e^{-\lambda t}$. This waiting time represents the general compound Poisson process. With this choice, equation (4.3.3) takes the form

$$\frac{\partial p(x, t)}{\partial t} = -\lambda\, p(x, t) + \lambda \int\limits_{-\infty}^{\infty} p(x, t) w(x - x')\, dx' \ , \qquad (4.3.4)$$

known as *Kolmogorov's forward equation* for the compound Poisson process. It is also called the *Kolmogorov-Feller* equation, and in the special case $\lambda = 1$, it gives the integral equation of example (a).

(c) If $\widetilde{\nu}(s) \neq const$, then we have a non-Markovian process (i.e. a process with memory). If we choose $\nu(t)$ to have a power-law time decay as

$$\nu(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} \ , \ 0 < \beta < 1 \ . \tag{4.3.5}$$

which in Laplace-domain reads $\widetilde{\nu}(s) = \frac{1}{s^{1-\beta}}$, then equation (4.3.3) turns out as

$$\frac{\partial^\beta p(x,t)}{\partial t^\beta} = -p(x,t) + \int\limits_{-\infty}^{\infty} p(x,t)w(x-x')\,dx' \ , \tag{4.3.6}$$

with the *Caputo fractional derivative* which is usually denoted by one of the following symbols.

$$\frac{\partial^\beta}{\partial t^\beta} = {}_t D^\beta_* \ , \tag{4.3.7}$$

(see Appendix A and [37]). The authors of [64] and [94] call equation (4.3.6) *the integral equation of the Mittag-Leffler process.* By applying the relation (4.3.1), the waiting time density of the memory process in Laplace-space now reads $\widetilde{\psi}(s) = \frac{1}{1+s^\beta}$, and by taking the inverse Laplace-transform we have

$$\psi(t) = -\frac{d}{dt}E_\beta(-t^\beta) \ . \tag{4.3.8}$$

The completely monotonicity of the Mittag-Leffler function $E_\beta(z)$, with $z = -x$ was first studied by Pollard [81]. The special case as $z = -t^\beta$ is studied in [37]. The results of [74] and [37]) are in agreement with [45]. For more information about the Mittag-Leffler function and its history, see ([17] and Appendix C). It is worth to say here that equation (4.3.6) is obtained by choosing at the beginning the survival probability $\Psi(t) = E_\beta(-t^\beta)$ [64].

4.4 The relation between the integral equation of the CTRW and the fractional diffusion equation

Henceforth, we shall give the relation between the integral equation of the CTRW in the Fourier-Laplace domain (4.2.10) and the fractional diffusion equation with proper use of the probability density of the jump. The space-time fractional diffusion equation is obtained from the standard diffusion equation by replacing the second order

space derivative with $\underset{x\ 0}{D}{}^{\alpha}$ (which is the symmetric fractional *Riesz* operator of order $\alpha \in (0,2]$, while the first order time derivative is replaced by the Caputo time derivative of order $\beta \in (0,1]$ (see equation (4.3.7) and Appendix A). We get

$$\underset{t\ *}{D}{}^{\beta} u(x,t) = \underset{x\ 0}{D}{}^{\alpha} u(x,t)\ ,\ 0 < \alpha \le 2,\ 0 < \beta \le 1\ , \qquad (4.4.1)$$

and as initial condition we take

$$u(x,0) = \delta(x)\ ,\ x \in \mathbb{R}\ ,\ t > 0\ .$$

The operator $\underset{x\ 0}{D}{}^{\alpha}$ is a linear pseudo differential operator with symbol $-|\kappa|^{\alpha}$ (see [25], [48]) and [49]. This means the Fourier representation of the symmetric fractional *Riesz* derivative operator $\underset{x\ 0}{D}{}^{\alpha}$ of order $\alpha \in (0,2]$ for a sufficiently smooth function $f(x)$, $x \in \mathbb{R}$, has the form

$$\mathcal{F}\{\underset{x\ 0}{D}{}^{\alpha} f(x); \kappa\} = -|\kappa|^{\alpha}\, \widehat{f}(\kappa)\ ,\ \kappa \in \mathbb{R}\ . \qquad (4.4.2)$$

The Caputo fractional derivative $\underset{t\ *}{D}{}^{\beta}$ and its relations with the Riemann-Liouville fractional derivative are explained in Appendix A. In the Laplace domain, it reads (see Gorenflo, Mainardi and et al [65] , [31], [37] and [66]).

$$\mathcal{L}\{\underset{t\ *}{D}{}^{\beta} f(t); s\} = s^{\beta}\, \widetilde{f}(s) - s^{\beta-1} f(0^{+})\ ,\ 0 < \beta \le 1,\ s > 0\ . \quad (4.4.3)$$

So far, in the Fourier-Laplace domain equation (4.4.1) takes the form (see [35])

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} + |\kappa|^{\alpha}}\ ,\ s > 0,\ \kappa \in \mathbb{R}\ , \qquad (4.4.4)$$

and the Laplace inversion leads to the characteristic function

$$\widehat{u}(\kappa, t) = E_{\beta}(-|\kappa|^{\alpha} t^{\beta})\ .$$

Our aim now is to search for probability densities $w(x)$ and $\psi(t)$ which by combination of Fourier and Laplace transforms approximately convert equation (4.2.10) to equation (4.4.4) by a proper limiting procedure using the asymptotic behaviors of $W(x)$ and $\Psi(t)$ near zero. So far, in what follows we shall state two Lemmata connecting the asymptotic behaviours of $w(x)$ and $\psi(t)$, near infinity to the asymptotic behaviour of their transforms $\widehat{w}(\kappa)$ and $\widetilde{\psi}(s)$, near zero.

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Lemma 1 : *Assume $w(x) \geq 0$, $w(x) = w(-x)$ for $x \in \mathbb{R}$,*
$\int_{-\infty}^{\infty} w(x)dx = 1$ *and either*

$$\sigma^2 := \int_{-\infty}^{\infty} x^2 w(x)dx < \infty \ , \qquad (4.4.5)$$

(relevant in the case $\alpha = 2$) or with $b > 0$ and some $\alpha \in (0, 2)$,

$$w(x) \sim b\,|x|^{-(\alpha+1)} \ for \ x \to \infty \ . \qquad (4.4.6)$$

The we have for $\kappa \in \mathbb{R}$ the asymptotic relation

$$1 - \widehat{w}(\kappa) \sim \mu|\kappa|^{\alpha} \ for \ \kappa \to 0 \ , \qquad (4.4.7)$$

with

$$\mu = \begin{cases} \frac{\sigma^2}{2} & if \ \alpha = 2 \ , \\ \frac{b\pi}{\Gamma(\alpha+1)sin(\alpha\pi/2)} & if \ 0 < \alpha < 2 \ , \end{cases} \qquad (4.4.8)$$

trivially $\widehat{w}(0) = 1$.

Proof : In the case $\alpha = 2$, the relation $\sigma^2 = -\frac{\partial^2 \widehat{w}(\kappa)}{\partial\kappa^2}\big|_{\kappa=0}$ immediately implies

$$1 - \widehat{w}(\kappa) \sim \mu\kappa^2 \ ,$$

hence equation (4.4.7) is satisfied for $\alpha = 2$ with $\mu = \frac{\sigma^2}{2}$. In the case $0 < \alpha < 2$ and $\gamma \neq 0$, we find by a simple calculation, using the symmetry of $w(x)$,

$$\widehat{w}(\kappa) - 1 = -4 \int_0^{\infty} sin^2\frac{\kappa x}{2}\,w(x)dx \ . \qquad (4.4.9)$$

Now with the formula (see [44])

$$\int_0^{\infty} \xi^{-(\alpha+1)} sin^2\xi \ d\xi = \frac{\Gamma(-\alpha)cos(\alpha\pi/2)}{2^{1-\alpha}} \ , \xi \in \mathbb{R} \ ,$$

with the identity

$$\Gamma(\xi)\Gamma(1-\xi) = \frac{\pi}{sin(\pi\xi)} \ , \ \xi \in \mathbb{R} \ ,$$

and finally with the property of $w(x)$ given in equation (4.4.6), we get

$$\widehat{w}(\kappa) \sim 1 - \frac{b\pi}{\Gamma(\alpha+1)sin(\alpha\pi/2)}\kappa^{\alpha} \ , \ \kappa \to 0 \ . \qquad (4.4.10)$$

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By using the definition of μ in equation (4.4.8) we get for $\kappa \in \mathbb{R}$ the asymptotic relation (4.4.7).

Lemma 2 : *Assume $\psi(t) \geq 0$ for $t > 0$, $\int_0^\infty \psi(t)dt = 1$, and either*

$$\rho := \int_0^\infty t\psi(t)dt < \infty , \tag{4.4.11}$$

(relevant in the case $\beta = 1$), or, with $c > 0$ and some $\beta \in (0,1)$,

$$\psi(t) \sim ct^{-(\beta+1)} \text{ for } t \to \infty . \tag{4.4.12}$$

Then we have the asymptotic relation

$$\widetilde{\psi}(s) = 1 - \lambda s^\beta + o(s^\beta) \text{ for } s \to 0 , \tag{4.4.13}$$

with

$$\lambda = \begin{cases} \rho , & \text{if } \beta = 1 , \\ \frac{c\Gamma(1-\beta)}{\beta} , & \text{if } 0 < \beta < 1 . \end{cases} \tag{4.4.14}$$

Proof : In the case $\beta = 1$ equation (4.4.13) is a consequence of the law of large numbers (see the book of Feller Vol. 2 chapter $XIII$ [20]) which states: If T_1, T_2, $\cdots$, T_n are independent random variables with a common Laplace transform ϕ and if $E(T_j) = \rho = \phi'(0)$, then the Laplace transform of the sum $T_1 + T_2 + \cdots + T_n$ is ϕ^n and near the origin $\phi(s) = 1 - \rho s + o(s)$ for a fixed $s > 0$.

For the case $0 < \beta < 1$, the proof is directly obtained by an application of a *Tauberian Theorem* which can also be found in the book of Widder [111] as Corollary (1 a). This corollary states: If $\alpha(t) \sim At^\nu / \Gamma(\nu + 1)$ as $t \to \infty$ for some $A \neq 0$ and some non negative ν, then

$$\int_0^\infty e^{-st}d\alpha(t) \sim \frac{A}{s^\nu} \text{ for } s \to 0^+ , \tag{4.4.15}$$

Substitute $\psi(t)$ from equation (4.4.12) into equation (4.2.2) and integrating the resulting equation we get $\Psi(t) = \frac{c}{\beta}t^{-\beta}$. Then as an application of the corollary of Widder, we set $\alpha'(t) = \Psi(t)$, and then integrate both sides, we get

$$\alpha(t) = \frac{A}{\Gamma(\nu + 1)}t^\nu = \frac{c}{\beta(1 - \beta)}t^{1-\beta} ,$$

which leads to $\nu = 1 - \beta$ and consequently $A = \frac{c\Gamma(1-\beta)}{\beta}$. Here A represents the constant of integration. Then we can write $\widetilde{\Psi}(s)$ as a direct application of equation (4.4.15), as

$$\widetilde{\Psi}(s) \sim A s^{\beta-1} \quad s \to 0^+ . \qquad (4.4.16)$$

Now to obtain $\widetilde{\psi}(s)$, we apply the Laplace transform to equation (4.2.2) and use the assumptions $\Psi(0) = 1$, $\Psi(\infty) = 0$. We get $\widetilde{\psi}(s) = 1 - s\widetilde{\Psi}(s)$. Finally, by replacing A by λ, defined in equation (4.4.14), and substituting back in the equation $\widetilde{\psi}(s) = 1 - s\widetilde{\Psi}(s)$, we get the required equation (4.4.13).

Now to prove that (4.4.7) and (4.4.13) lead to equation (4.2.10) (i.e. to the Fourier-Laplace transform of $p(x,t)$), we introduce two special scales for the waiting times and for the jumps. We define with a suitable positive scaling parameter τ, the new time instants as

$$t_n(\tau) = \tau T_1 + \tau T_2 + \cdots + \tau T_n \ \text{ for } n \in \mathbb{N} ,$$

and rewrite the asymptotic equation (4.4.13)

$$\widetilde{\psi}(s\tau) = 1 - \lambda(s\tau)^\beta + o(\tau^\beta) \ \text{ for } \tau \to 0 , \ s \ \text{ fixed } . \qquad (4.4.17)$$

Also we define the sum of jumps with a positive scaling parameter h as

$$x_n(h) = hX_1 + hX_2 + \cdots + hX_n , \ x_0(h) = 0 \text{ for } n \in \mathbb{N} ,$$

and rewrite the asymptotic equation (4.4.7) as

$$\widehat{w}(\kappa h) = 1 - \mu(|\kappa|h)^\alpha + o(h^\alpha) \ \text{ for } h \to 0 , \ \kappa \ \text{ fixed } . \qquad (4.4.18)$$

So the reduced waiting time density can be written as $\psi_\tau(t) = \frac{\psi(t/\tau)}{\tau}, t > 0$ and the reduced jump density as $w_h(x) = \frac{w(x/h)}{h}, x \in \mathbb{R}$. Then the Laplace transform of the reduced waiting density and the Fourier transform of the reduced jump density respectively are

$$\widetilde{\psi}_\tau(s) = \widetilde{\psi}(s\tau) , \ \widehat{w}_h(\kappa) = \widehat{w}(\kappa h) . \qquad (4.4.19)$$

Now by replacing $\psi(t)$ by $\psi_\tau(t)$ and $w(x)$ by $w_h(x)$ in equation (4.2.11) we get $p_{h,\tau}(x,t)$ which in Fourier-Laplace space reads

$$\widehat{\widetilde{p}}_{h,\tau}(\kappa, s) = \frac{1 - \widetilde{\psi}_\tau(s)}{s} + \widetilde{\psi}_\tau(s) \, \widehat{w}_h(\kappa) \widehat{\widetilde{p}}_{h,\tau}(\kappa, s) . \qquad (4.4.20)$$

Solving for $\widehat{\widetilde{p}}_{h,\tau}(\kappa, s)$ we get

$$\widehat{\widetilde{p}}_{h,\tau}(\kappa, s) = \frac{1 - \widetilde{\psi}_\tau(s)}{s} \frac{1}{1 - \widetilde{\psi}_\tau(s)\,\widehat{w_h}(\kappa)} \;. \tag{4.4.21}$$

Then by using equation (4.4.19), equation (4.4.17) and equation (4.4.18) in equation (4.4.21) we get

$$\widehat{\widetilde{p}}_{h,\tau}(\kappa, s) = \frac{\lambda \tau^\beta s^{\beta-1}}{\lambda(\tau s)^\beta + \mu(h\kappa)^\alpha + \lambda\mu(h\kappa)^\alpha(\tau s)^\beta + O(h^\alpha \tau^\beta)} \;. \tag{4.4.22}$$

Taking here the limit as $h \to 0$ and $\tau \to 0$ and introducing the scaling relation

$$\lambda \tau^\beta = \mu h^\alpha \;, \tag{4.4.23}$$

one obtains

$$\widehat{\widetilde{p}}(\kappa, s) \to \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha} = \widehat{\widetilde{u}}(\kappa, s) \;, \; s > 0 \;, \; \kappa \in \mathbb{R} \;. \tag{4.4.24}$$

In this kind of passing to the limit $\widehat{\widetilde{p}}(\kappa, s)$ and $\widehat{\widetilde{u}}(\kappa, s)$ are asymptotically equivalent in the Fourier-Laplace domain. Then the asymptotic equivalence in space-time domain between the integral equation of the CTRW (4.2.11) after rescaling the fractional diffusion equation (4.4.1) is provided by the continuity theorem for sequences of characteristic functions after having applied the analogous theorem for sequences of Laplace transforms (see [20]). Therefore we have *convergence in law* or *weak convergence* for the corresponding probability distributions (see e.g. [66]). Let us finally remark that another way of treating the connection between CTRW and the fractional diffusion is to exploit extensively the principle of subordination. Let us quote as a recent example [69]. In our work we have preferred the Fourier-Laplace method.

4.5 The space-fractional diffusion equation without and with central linear drift

Let us consider here the asymmetric Riesz pseudo-differential operator $\underset{x\ \theta}{D^\alpha}$ which has the symbol $-|\kappa|^\alpha i^{\theta sig(\kappa)}$ with a skewness θ such that $|\theta| \le min\{\alpha, (2 - \alpha)\}$. We have used the definition of the pseudo-differential operator A which acts with respect to the variable $x \in \mathbb{R}$ on a sufficiently well- behaved function $\phi(x)$. A is

defined through its Fourier representation, namely

$$\int\limits_{-\infty}^{\infty} e^{i\kappa x} A((\phi)(x)) dx = \widehat{A}(\kappa) \widehat{\phi}(\kappa) \ .$$

Here $\widehat{A}(\kappa)$ represents the symbol of A, given as $\widehat{A}(\kappa) = (A e^{-i\kappa x}) e^{+i\kappa x}$ (see [48] and [49]). We discuss now the space-FDE

$$\frac{\partial u(x,t)}{\partial t} = D^{\alpha}_{x\ \theta} u(x,t) \ , \ 0 < \alpha \leq 2 \ , \qquad (4.5.1)$$

with the initial condition $u(x,0) = f(x)$. Equation (4.5.1) is a generalization of equation (4.4.1) in the special case $\beta = 1$ by admitting the parameter θ of asymmetry (skewness) (see Feller [18] and Gorenflo and Mainardi [65], [35], [38] and [31]). We want here to prove that the function $\widehat{u}(\kappa, t)$ is the characteristic function of an α-stable probability density. To do so, we take the Fourier transform of equation (4.5.1) and solve the resulting ordinary differential equation. We get (with $sig(\kappa) = \kappa/|\kappa| = -1, 0,$ or 1, depends on $\kappa < 0, = 0,$ or > 0 respectively)

$$\widehat{g}_{\alpha}(\kappa, t; \theta) = exp[-t|\kappa|^{\alpha} e^{\frac{i\theta\pi}{2} sig(\kappa)}] \ , \ \widehat{u}(\kappa, 0) = \widehat{f}(\kappa) \ , \qquad (4.5.2)$$

where $\widehat{g}_{\alpha}(\kappa, t; \theta)$ is the Fourier transform of the Green function corresponding to the initial condition $g_{\alpha}(x, 0; \theta) = \delta(x)$ (see [38]). The solution of equation (4.5.1) can be written as

$$u(x,t) = \int\limits_{-\infty}^{\infty} g_{\alpha}(x - \zeta, t; \theta) f(\zeta) \, d\zeta \ , \ \forall t > 0 \ .$$

For the strictly stable distributions, we use the parameterization of Feller for the characteristic functions (see Appendix B)

$$\widehat{p}_{\alpha}(\kappa; \theta) = exp[-|\kappa|^{\alpha} e^{\frac{i\theta\pi}{2} sig(\kappa)}] \ , \qquad (4.5.3)$$

of the densities $p_{\alpha}(x; \theta)$ where the skewness parameter θ is restricted to the following region, depending on α, (see [18] and [101])

$$\begin{cases} |\theta| \leq \alpha \ , & \text{if } 0 < \alpha \leq 1 \ , \\ |\theta| \leq 2 - \alpha \ , & \text{if } 1 < \alpha \leq 2 \ . \end{cases} \qquad (4.5.4)$$

We recognize that $p_{\alpha}(x; \theta) = p_{\alpha}(-x; -\theta)$. If $\alpha = 2$, we have by (4.5.4), only with $\theta = 0$, the corresponding distribution is Gaussian. In the case $\alpha = 1$ with the special choice $\theta = 0$ gives the Cauchy

distribution. In general we also assume the restriction (4.5.4) to be satisfied.

If a linear force acts towards the origin and affects the motion of the free particle, then the equation of motion will be modelled by the diffusion equation with central linear drift (FPE). As we have mentioned before the space-FDECLD is a generalization to the FPE in using the fractional derivative and a special form of a central linear force. The Fokker-Plank equation is one of the most celebrated equations in physics because it is very useful for studying the dynamic behaviour of the solution of the stochastic differential equations driven by various kinds of noise, in our treatment we use the Lévy noise. The classical form of this equation is discussed and interpreted in Chapter 2. We discuss in this section the space-FDECLD which is obtained from the classical FPE by replacing the second order derivative with respect to ξ by $\underset{\xi}{D}\underset{\theta}{}^{\alpha}$. Therefore the space-FDECLD takes the form

$$\frac{\partial v(\xi, \tau)}{\partial \tau} = b\frac{\partial}{\partial \xi}(\xi v(\xi, \tau)) + a\underset{\xi}{D}\underset{\theta}{}^{\alpha} v(\xi, \tau) , \ 0 < \alpha \leq 2 , , \quad (4.5.5)$$

where $a > 0$ is the diffusion constant and $b > 0$ is the drift constant. The stochastic process modelled by the space-FDECLD is still a Markovian process. We aim now to prove that the characteristic function of the solution of the space-FDECLD also belongs to the class of α-stable probability densities and is related to the solution of equation (4.5.1). To this aim, we take the Fourier transform of both sides of equation (4.5.5), and get the ordinary differential equation

$$\frac{\partial \widehat{v}(\kappa, \tau)}{\partial \tau} = -b\kappa \frac{\partial \widehat{v}(\kappa, \tau)}{\partial \kappa} - a|\kappa|^{\alpha} e^{\frac{i\theta\pi}{2}sig(\kappa)}\widehat{v}(\kappa, \tau) . \quad (4.5.6)$$

To solve this equation, we use the method of characteristics which leads to the chain of equations

$$\frac{d\tau}{1} = \frac{d\kappa}{b\kappa} = \frac{d\widehat{v}(\kappa, \tau)}{\left(-a|\kappa|^{\alpha}\widehat{v}(\kappa, \tau)e^{i\theta\frac{\pi}{2}sig(\kappa)}\right)} . \quad (4.5.7)$$

Using the first equality sign of equation (4.5.7), and integrating both sides, we get

$$c_1 = \kappa e^{-b\tau} .$$

Using the second equality sign of equation (4.5.7), and integrating both sides, we get

$$c_2 = \widehat{v}(\kappa, \tau) \, exp \left[\frac{a}{b\alpha}|\kappa|^{\alpha}e^{i\theta\frac{\pi}{2}sig(\kappa)}\right] ,$$

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where c_1 and c_2 are the constants of integrations. Then solving these last two equations for c_1 and c_2 by using the initial condition

$$\widehat{v}(\kappa, 0) = 1 \, ,$$

we get the characteristic distribution of the solution of the space-FDECLD with the initial condition $v(\xi, 0) = \delta(\xi)$

$$\widehat{v}(\kappa, \tau) = \, exp\left[-|\kappa|^\alpha \frac{a}{b\alpha}(1 - e^{-b\alpha\tau})e^{i\theta\frac{\pi}{2}sig(\kappa)}\right] . \qquad (4.5.8)$$

Setting

$$\tau' = \frac{1}{b\alpha}(1 - e^{-b\alpha\tau}) = \tau - \frac{b\alpha}{2!}\tau^2 + \frac{(b\alpha)^2}{3!}\tau^3 \mp \cdots \, ,$$

we notice that $\tau' \to \tau$ as $b \to 0$ which is the same as in equation (4.5.2) (i. e. the fractional diffusion without drift). By this abbreviation we rewrite equation (4.5.8) to take the standard form

$$\widehat{v}(\kappa, \tau) = \, exp\left[-|\kappa|^\alpha \, a\tau' \, e^{i\theta\frac{\pi}{2}sig(\kappa)}\right] . \qquad (4.5.9)$$

For the strictly stable distributions of this equation, we use the notation (see Appendix B)

$$\widehat{h}_\alpha(\kappa, \theta) = \, exp[\, -|\kappa|^\alpha \, e^{\frac{i\theta\pi}{2}sig(\kappa)}] \, , a = 1 \, , \qquad (4.5.10)$$

clearly we notice that $\widehat{h}_\alpha(\kappa, \theta) = \widehat{p}_\alpha(\kappa, \theta)$ (see [109] and [104]).

Comparing the structures of equations (4.5.8) and (4.5.9) gives rise to the conjecture that the solution of the space-fractional diffusion equation and the solution of the space-fractional diffusion equation with central linear drift can be transformed to each other with some nonlinear rescaling for the space coordinate and for the time coordinate. Actually Biler et al [5] have given without proof the following *transformation theorem* after choosing, without loosing generality, the constant drift $b = 1$ and $\theta = 0$. We state here the transformations theory without this restriction for θ.

Theorem 1 *(on transformation):*
Consider the transformations (4.5.12) and (4.5.11) between the two pairs of independent variables (x, t) and (ξ, τ)

$$\xi = x(\alpha t + 1)^{-1/\alpha} \, , \quad \tau = \alpha^{-1}log(\alpha t + 1) \, , \qquad (4.5.11)$$

and

$$x = \xi e^\tau \, , \quad t = \frac{1}{\alpha}(e^{\alpha\tau} - 1) \, . \qquad (4.5.12)$$

By the transformation (4.5.12), a solution $u(x,t)$ of the space-FDE (4.5.1) goes over into a solution $v(\xi,\tau)$ of the space-FDECLD (4.5.5). By the transformation (4.5.11) the solution $v(\xi,\tau)$ of (4.5.5) goes over into the solution $u(x,t)$ of (4.5.1). These transformations are inverse to each other, and we have the relation

$$v(\xi,\tau) = (\alpha t + 1)^{1/\alpha} u(x,t) ,$$

with its inverse

$$u(x,t) = e^{-\tau} v(\xi,\tau) .$$

We notice here as $\tau = 0 \Longrightarrow \xi = x$, and so the solutions of the two considered equations with the same initial condition are equivalent.

Proof : The theorem is valid for $\alpha = 2$ (see [20]). For $0 < \alpha < 2$, we use the following lemma (see [72]) and [80]).

Lemma 3 : *If $x = a\xi$, $f(x) = g(\xi)$ and $a > 0$, then*

$$\underset{\xi\ \theta}{D}^\alpha g(\xi) = a^\alpha \underset{x\ \theta}{D}^\alpha f(x) .$$

Proof of Lemma 3: It is known that the operator $\underset{x\ \theta}{D}^\alpha$ has the symbol $\widehat{\underset{x\ \theta}{D}^\alpha} = -|\kappa|^\alpha i^{\theta sig(\kappa)}$ and in general:

$$\left(\widehat{\underset{x\ \theta}{D}^\alpha h}\right)(\kappa) = -|\kappa|^\alpha i^{\theta sig(\kappa)} \widehat{h}(\kappa) ,$$

and

$$\left(\underset{x\ \theta}{D}^\alpha h\right)(x) = \frac{1}{2\pi} \int e^{-i\kappa x} \left(\widehat{\underset{x\ \theta}{D}^\alpha h}\right)(\kappa) \, d\kappa .$$

In particular:

$$\left(\widehat{\underset{x\ \theta}{D}^\alpha g}\right)(\kappa) = -|\kappa|^\alpha i^{\theta sig(\kappa)} \widehat{g}(\kappa) = -|\kappa|^\alpha i^{\theta sig(\kappa)} \frac{1}{a} \widehat{f}(\frac{\kappa}{a}) ,$$

and

$$\left(\underset{x\ \theta}{D}^\alpha g\right)(x) = -\frac{1}{2\pi a} \int |\kappa|^\alpha i^{\theta sig(\kappa)} e^{-i\kappa x} \widehat{f}(\frac{\kappa}{a}) \, d\kappa .$$

By changing the variable $\kappa/a = \nu$ and substituting back in the last equation we get the statement of Lemma 3.

By applying this lemma we get

$$\underset{\xi\ \theta}{D}^\alpha v(\xi,\tau) = \underset{\xi\ \theta}{D}^\alpha \left(e^\tau v \left(x(\alpha t + 1)^{-1/\alpha}, \alpha^{-1} log(\alpha t + 1) \right) \right) = e^{(\alpha+1)\tau} \underset{x\ \theta}{D}^\alpha u(x,t) .$$

$$(4.5.13)$$

By applying the chain rule on the functions $u = u(x, t)$, $v = v(\xi, \tau)$, $x = x(\xi, \tau)$ and $t = t(\xi, \tau)$, we get

$$\frac{\partial v}{\partial \tau} = e^{\tau} u + e^{2\tau} \xi \frac{\partial u}{\partial x} + e^{(\alpha+1)\tau} \frac{\partial u}{\partial t} \ , \qquad (4.5.14)$$

and finally

$$\frac{\partial (\xi v)}{\partial \xi} = \xi e^{2\tau} \frac{\partial u}{\partial x} + v \ . \qquad (4.5.15)$$

We substitute now the results of equations (4.5.13- 4.5.15) in equation (4.5.5) and get equation (4.5.1). The other direction can be similarly proved.

After this discussion we note that for every solution of $u(x, t)$ there is a corresponding solution for v and vice versa. We note also that we have used for the space-FDE the initial condition $u(x, 0) = \delta(x)$ and also for the space-FDECLD $v(\xi, 0) = \delta(\xi)$, where δ is the Dirac function. Therefore, we can represent the fundamental solution of the fractional diffusion equation as the *Green function*(see[65])

$$u(x, t) = g_{\alpha}(x, t) \ ,$$

where $g_{\alpha}(x, t)$ is the Green function with Fourier transform (4.5.2). We refer here to the scaling property of the Green function defined as (see [65])

$$g_{\alpha}(ax, bt) = b^{-1/\alpha} g_{\alpha}\left(\frac{ax}{b^{1/\alpha}}, t\right) \ .$$

Consequently, introducing the similarity variable $x/t^{1/\alpha}$, we can write

$$g_{\alpha}(x, t) = t^{-1/\alpha} p_{\alpha}(xt^{-1/\alpha}) \ , \text{with } \widehat{p}(\kappa) = exp(-|\kappa|^{\alpha}) \ ,$$

where we have the following special cases

$$g_{2}(x, t; 0) = \frac{1}{2\sqrt{\pi}} t^{-1/2} exp[-\frac{x^2}{4t}] \ ,$$

$$g_{1}(x, t; 0) = \frac{1}{\pi} \frac{t}{x^2 + t^2} \ , \qquad (4.5.16)$$

$$g_{1/2}(x, t, -1) = \frac{1}{\sqrt{2\pi}} t^{-5} x^{-3/2} exp[\frac{-2t^2}{x}] \ .$$

So finally, the fundamental solution to the space-FDECLD (4.5.5) with the general initial condition $v(\xi, 0) = \delta(\xi - \xi_0)$ can be obtained

from the solution $u(x, t) = g_\alpha(x - x_0)$ of equation (4.5.1) with the initial condition $u(x, 0) = \delta(x - x_0)$, $x_0 = \xi_0$, as

$$v(\xi, \tau) = \frac{\alpha^{1/\alpha}}{(1 - e^{-\alpha\tau})^{1/\alpha}} p_\alpha \left(\frac{\alpha^{1/\alpha} (\xi - \xi_0 e^{-\tau})}{(1 - e^{-\alpha\tau})^{1/\alpha}} \right) . \qquad (4.5.17)$$

For $\tau \to \infty$ this solution becomes stationary

$$v(\xi, \tau) \to \alpha^{1/\alpha} p_\alpha(\alpha^{1/\alpha} \xi) \ ,$$

in contrast to the stationary solution $u(x, t)$ of equation (4.5.1) which leads to zero everywhere for $t \to \infty$. If we put $\alpha = 2$ and $\beta = 1$ in this formula, we get the well known solution of the classical diffusion with central linear drift (2.1.3).

The following figures show the analytical solution for different values of t for the standard diffusion equation (i.e. $\alpha = 2$) on the left and the analytical solution of the diffusion equation with central linear drift on the right. The figures show also that the coordinates representing the space and the time in the diffusion with drift are compressed and the width of the curve is also narrower than the corresponding one on the left.
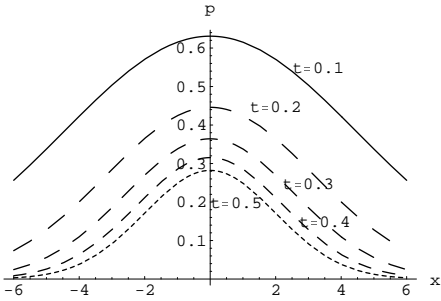
Figure 4.1: standard diffusion $g_2(x, t; 0)$
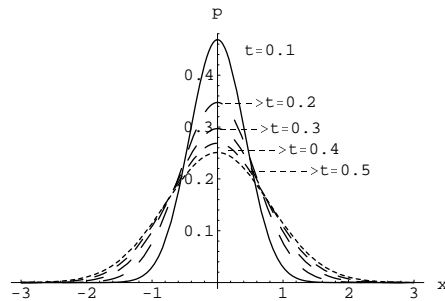
Figure 4.2: diffusion with central linear drift

4.6 The simulation of the space-fractional diffusion equation without and with central linear drift

For the purpose of simulations, we recall the notations of Section 2, to produce approximate particle paths for space-time FDE and henceforth to space-FDECLD. We simulate the both cases in the spatially symmetric case $\theta = 0$. We need for a given $0 < \alpha \leq 2$, a jump width satisfying Lemma (1), to create a sequence of iid random jumps $\{X_1, X_2, \cdots, X_n\}$. We need also for $0 < \beta \leq 1$, a waiting time density satisfying Lemma (2), to generate a sequence of iid random waiting times $\{T_1, T_2, \cdots, T_n\}$. The time instant t_n is defined as (see e. g. [21])

$$t_0 = 0 , \quad t_n = T_1 + T_2 + \cdots + T_n , T_n = t_n - t_{n-1} , \ n \geq 1 . \quad (4.6.1)$$

The position x_n of the walker at time t_n is defined as

$$x_n = x(t_n) = x(0) + X_1 + X_2 + \cdots + X_n , n \geq 1 . \qquad (4.6.2)$$

In equation (4.6.2) the walker starts at $x = x(0)$ at time $t = 0$. Consequently the random walker waits at a given location x_{n-1} for time T_n, $n \geq 1$, before taking a jump X_n which is independent on the waiting time T_n. Notice that we allow $x(0)$, in equation (4.6.2), to be different from zero. This will be interesting for the simulation of the space-FDECLD because we see how the particle is attracted to the origin.

Among the methods of random sequence generation with the given probability law, the method of inversion seems most simple and effective [55].

In the case $\beta = 1$, the waiting time distribution is exponential and the simulation of the waiting time T, in this case, is simply done by the equation

$$T = -log \ r \ , \ r \in [0, 1) , \qquad (4.6.3)$$

where r is a uniformly distributed random number.

In the case $0 < \beta < 1$, we use the results of Section 3. We look for the waiting time density $\psi(t)$ satisfying Lemma (2). This means it must have an asymptotic decay like $ct^{-(\beta+1)}$, $0 < \beta < 1$ and the survival probability $\Psi(t)$ should asymptotically behave like $\frac{c}{\beta}t^{-\beta}$, at infinity . The Mittag-Leffler function $E_\beta(-t^\beta)$ (see Appendix c) satisfies this asymptotic behaviour for the survival probability. Furthermore, it is a completely monotonic function and behaves

for $t \to 0$ like $(1 - \frac{t^\beta}{\Gamma(1+\beta)})$. It is shown in [45] and [67] that the Mittag-Leffler density is suitable for the description of the random walk. However it is difficult to invert the Mittag-Leffler function numerically. Therefore, we look for a simpler suitable function which has some of the properties of the Mittag-Leffler function. We chose (see [20] and [67])

$$\Psi_*(t) = \frac{1}{1 + \Gamma(1 - \beta)t^\beta} \ , \ t \geq 0 \ , \ 0 < \beta < 1 \ . \tag{4.6.4}$$

The function $\psi_*(t)$ behaves also asymptotically like the Mittag-Leffler function as $t \to \infty$ and shares with it the property of complete monotonicity in $t > 0$ (see [20] and [67]). For the simulation of the waiting time T, we generate a random number r, uniformly distributed in $[0, 1)$. Then with $r^* = (1 - r)$, we take

$$T = \left(\frac{1}{\Gamma(1 - \beta)} \left(\frac{1}{r^*} - 1 \right) \right)^{1/\beta} \ , \tag{4.6.5}$$

where $r^* = \Psi^*(T)$.

Before discussing the simulation of the CTRW of the space-FDE, with $0 < \alpha < 2$ and $\alpha \neq 1$, we consider the special cases $\alpha = 1$ and $\alpha = 2$ corresponding to the Cauchy distribution and the normal (Gaussian) distribution respectively. Their simulation is simply done by equating a uniformly random number $u \in [0, 1)$ by the cumulative density function

$$F(x) = \int_{-\infty}^{x} p(y)dy \ ,$$

with

$$p(x) = \begin{cases} \frac{1}{\pi(1+|x|^2)} & \text{if } \alpha = 1 \ , \\ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} & \text{if } \alpha = 2 \ . \end{cases} \tag{4.6.6}$$

Then the the random jump X is calculated as

$$X = \begin{cases} tan(\pi(u - \frac{1}{2})) & \text{if } \alpha = 1 \ , \\ \frac{1}{2}(1 + erf(u/2)) & \text{if } \alpha = 2 \ , \end{cases} \tag{4.6.7}$$

where $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ denotes the error function.

For simulating the random variable X of the remaining cases $0 < \alpha < 2$, $\alpha \neq 1$, we use the probability density function which has

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been successfully applied by Gorenflo & Mainardi [36] who modified a method of Chechkin & Gonchar [9]. This density is given by

$$w_*(x) = \frac{\alpha}{2} \frac{|x|^{\alpha-1}}{(1+|x|^\alpha)^2} \, ,$$

(4.6.8)

and the corresponding cumulative function is

$$W(x) = \int\limits_{-\infty}^{x} w_*(\xi)d\xi = \begin{cases} \frac{1}{2}\frac{1}{1+|x|^\alpha} & \text{if } x \le 0 \, , \\ 1 - \frac{1}{2}\frac{1}{1+|x|^\alpha} & \text{if } x > 0 \, . \end{cases}$$

(4.6.9)

This $W(x)$ satisfies the conditions of Lemma (1). Now, with a uniformly distributed random number $z = \in (0,1]$, we find by inverting $W(x)$, the jump

$$X = \begin{cases} \left(\frac{1}{2(1-z)} - 1\right)^{1/\alpha} & \text{if } z \ge 1/2 \, , \\ -\left(\frac{1}{2z} - 1\right)^{1/\alpha} & \text{if } z < 1/2 \, . \end{cases}$$

(4.6.10)

In the case of space-FDECLD, the situation is slightly different. Therefore, for the simulation of the random jump X, we use equation (4.6.10). For the waiting time T, we use equation (4.6.3). Then we calculate the position x_n, in equation (4.6.2), and the time instant t_n, in equation (4.6.1), of the space-FDE (4.5.1). Finally to simulate the path of the random walker of the space-FDECLD (4.5.5), we use theorem (1) to transfer the pair (x_n, t_n) to the pair (ξ_n, τ_n). These steps are done at every $n \ge 1$.

4.7 Numerical results

Figures [3-10] correspond to the simulation of the continuous random walk of a free diffusive particle corresponding to the space-time-FDE with different values of α and β.
The Cauchy random walk is simulated in fig [3] and the Brownian motion is simulated in in fig[5]. While in fig[4] and fig [6] we show the effect of the power law waiting time on the jumps of the Cauchy and the Gaussian distribution function of the jumps respectively.
Fig[7-10] represent the symmetric space-FDE approximated by using the density $w_*(x)$ for the jump. For the waiting time we use the exponential waiting time (i.e. as $\beta = 1$) in figures [7,8], and the power law waiting time $\psi_*(t)$ (i.e. as $0 < \beta < 1$) in figures [9,10]. In these Figures we have taken $x_0 = 0$ and the number of steps $n = 1000$.

Figures [11-20] display the simulation of the CTRW of the space-FDECLD with the initial condition $x_0 = \xi_0 = 0$, (in the first column) and the comparison between it and the CTRW of the space-FDECLD with the initial condition $x_0 = \xi_0 = 100$, (in the second column). The simulation shows that the space and the time coordinates of the space-FDECLD are compressed by using the transformation theorem (1). It also shows that in the long range the walker is attracted to the origin in both cases $x_0 = 0$ and $x_0 \neq 0$. We have taken the number of steps here to be 10000.
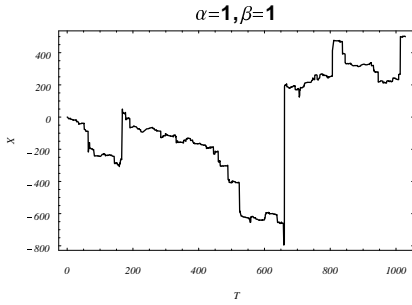
Figure 4.3: Cauchy
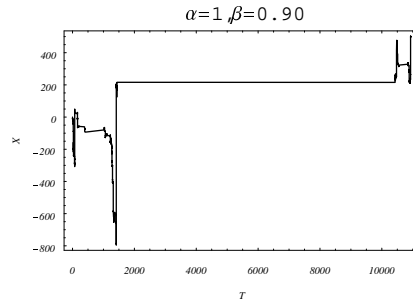
Figure 4.4: time-FDE
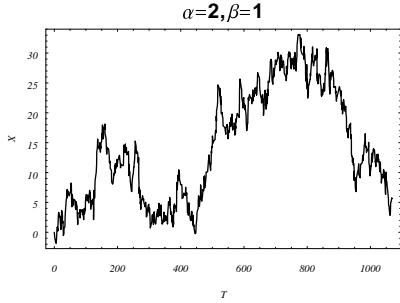
Figure 4.5: Gauss

Figure 4.6: time-FDE

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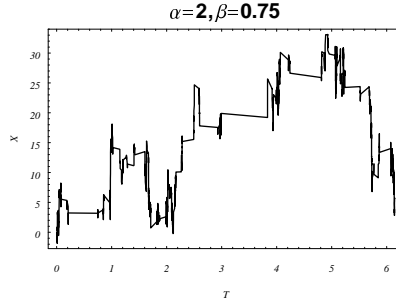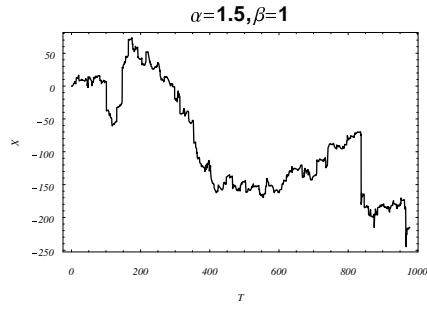
Figure 4.7: space-FDE
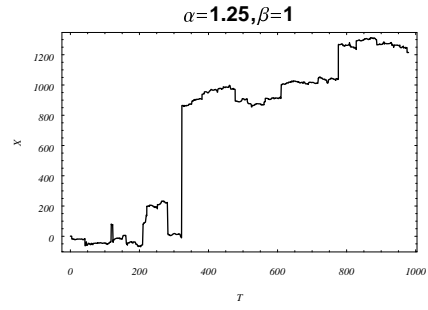
Figure 4.8: space-FDE
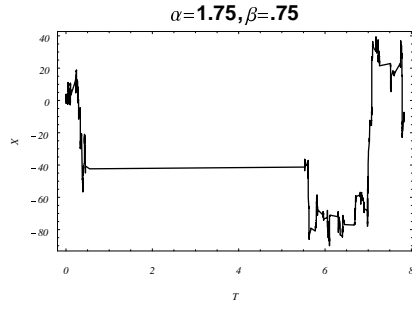
Figure 4.9: space-time-FDE
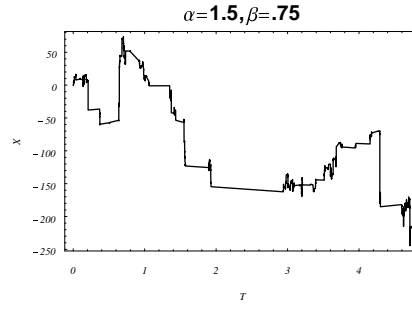
Figure 4.10: space-time-FDE
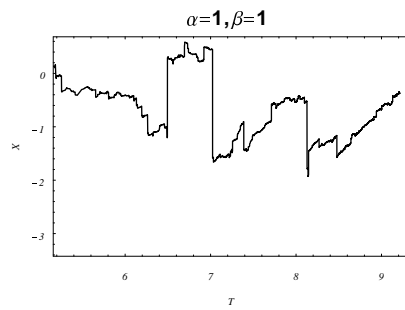
Figure 4.11: Cauchy, $\xi_0 = 0$

Figure 4.12: $\xi_0 = 0$, $\xi_0 = 100$

α=**2**,β=**1**
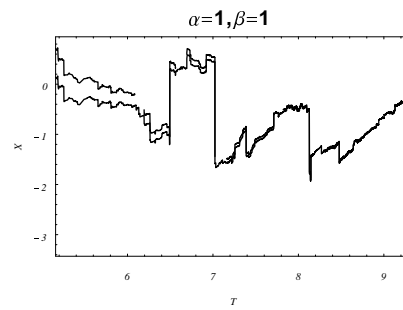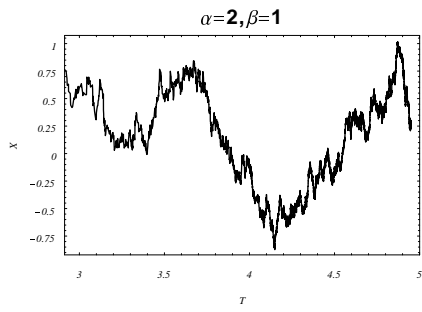
Figure 4.13: Gauss, $\xi_0 = 0$

α=**2**,β=**1**

Figure 4.14: $\xi_0 = 0$, $\xi_0 = 100$

α=**1.5**,β=**1**
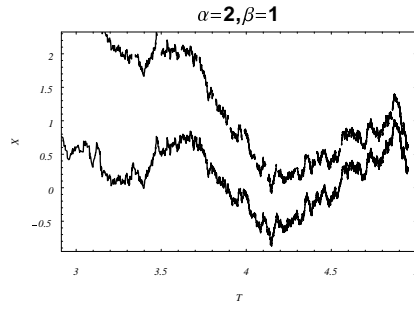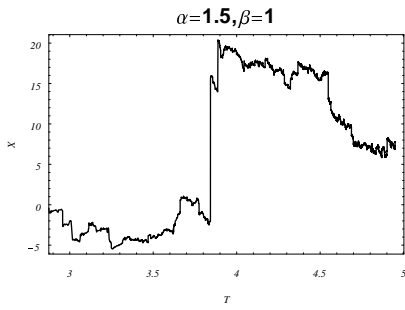
Figure 4.15: space-FDECLD

α=**1.5**,β=**1**

Figure 4.16: $\xi_0 = 0$,$\xi_0 = 100$

α=**1.75**,β=**1**
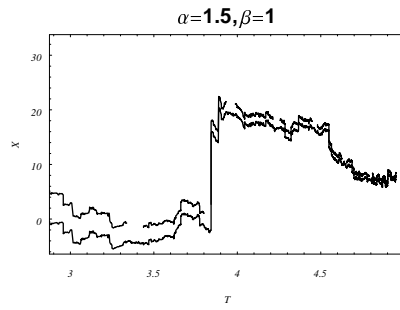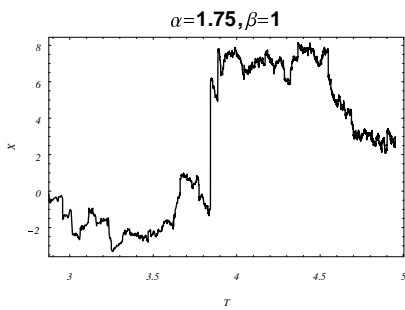
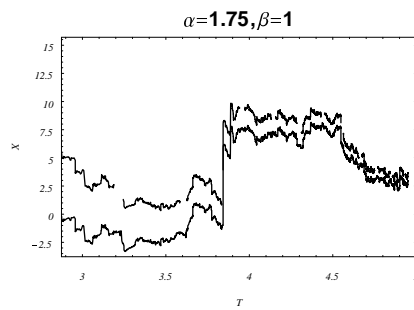Figure 4.17: space-FDECLD

α=**1.75**,β=**1**
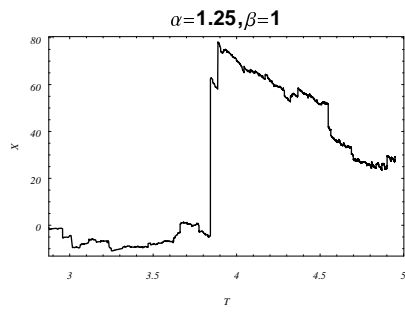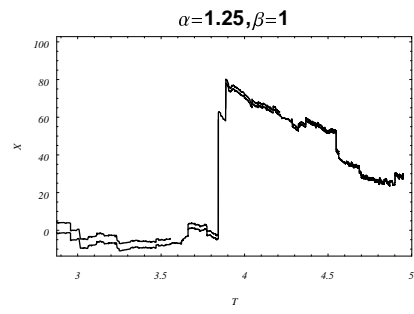
Figure 4.18: $\xi_0 = 0$,$\xi = 100$

Figure 4.19: space-FDECLD

Figure 4.20: $\xi_0 = 0, \xi = 100$