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Cascades of heteroclinic connections in hyperbolic balance laws

Julia Ehrt

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Abstract

The report investigates the relation between global attractors of hyperbolic balance laws and viscous balance laws on the circle. Hence it is thematically located at the crossroads of hyperbolic and parabolic partial differential equations with one-dimensional space variable and periodic boundary conditions. The two equations are given by:

$$u_t + f(u)_x = g(u). \tag{H}$$

and

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u) \tag{P}$$

where $x \in S^1$. The results of the work can be split into two areas: The description of the global attractor of equation (H) and the question regarding persistence of solutions on the global attractor of (P) when ε vanishes.

The key idea of the work is the introduction of finite dimensional sub-attractors. This tool allows to overcome several difficulties in the description of the global attractor of equation (H) and closes one of the last remaining gaps in its complete description: Theorem 2.6.1 yields a complete parameterization of all finite dimensional sub-attractors in the hyperbolic setting.

The second main result corrects a result on the persistence of heteroclinic connections by Fan and Hale [FH95] for the case $\varepsilon \to 0$ (*Connection Lemma* 3.2.8). The *Cascading Theorem* 3.2.9 then yields convergence of heteroclinic connections to a cascade of heteroclinics in case of non-persistence.

The report concludes with geometric investigations of the low dimensional sub-attractors.

"If the 20th century can be said to be the one in which the world was made safe for democracy, let it be said that the 21st century was the one in which the world was made safe for diversity."

Shashi Tharoor
UN Undersecretary-General

Meinen Eltern und allen Transmenschen auf dieser Welt gewidmet, die nicht das Glück hatten eine liebende Familie, gute Freund_innen und ein unterstützendes Umfeld zu haben.

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Chapter 1

Introduction

Parabolic differential equations with scalar spatial variable have been studied for a long time. In particular viscous balance laws can be described as exceptionally well understood: existence, uniqueness of solutions, long time behaviour, global attractors, heteroclinic orbits etc. have been analysed in detail for a range of boundary conditions.

The same is true for scalar hyperbolic partial differential equations. In particular for hyperbolic balance laws, where again questions of existence, uniqueness, the long time behaviour, global attractors and heteroclinics have been studied thoroughly.

However, when the two fields, viscous balance laws and hyperbolic balance laws come together many question marks appear.

This thesis is devoted to the study of solutions on the global attractors of viscous balance laws and their relation to solutions of hyperbolic balance laws when the viscosity is small or vanishes. Before going into further details we set the formal stage that clarifies the setting in which we will be working.

The hyperbolic balance law is given by

$$u_t(x,t) + [f(u(x,t))]_x = g(u(x,t)). \tag{H}$$

The viscous balance law is then given by

$$u_t(x,t) + [f(u(x,t))]_x = \varepsilon u_{xx}(x,t) + g(u(x,t)). \tag{P}$$

The subindex denotes the partial derivative with respect to the index. We solve for $x \in S^1$ with $S^1 := \mathbb{R}/(2\pi\mathbb{Z})$. This is equivalent to imposing periodic boundary conditions on a domain of length 2π. By an easy scaling argument all our results remain true for the situation of periodic boundary conditions in a domain of size L for any bounded and fixed $L \in \mathbb{R}$. u is a function mapping from $S^1 \times \mathbb{R} \to \mathbb{R}$.

The non-linearities f, g map from $\mathbb{R} \to \mathbb{R}$. Furthermore we make additional hypotheses that are assumed to hold throughout the whole work except if explicitly stated otherwise. We impose:

(H1) f is C^2 and strictly convex ($\exists \alpha \in \mathbb{R}$ $s.t.$ $f'' > \alpha > 0$) and $f'(0) = 0$.

(H2) g is C^1 and dissipative, i.e. there exists a constant $M > 0$ such that

$$ug(u) < M \tag{1.1}$$

for all $|u| > M$.

(H3) g has three simple zeros at $u_- < u_0 < u_+$, where we assume $u_0 = 0$.

A discussion of the assumptions will follow in the next chapter. They guarantee the existence and uniqueness of solutions and the existence of a global attractor in both equations. Roughly, (H1) is required in order to obtain unique admissible solutions for the hyperbolic equation, (H2) will guarantee the existence of global attractors for (H) and (P).

Viscous balance laws can be understood as a parabolic regularisation of hyperbolic balance laws. The latter are generalisations of conservation laws which do not possess a source term. The hyperbolic equation (H) is the limiting equation of the parabolic equation (P) when the viscosity vanishes.

Small or vanishing viscosity means that the viscosity parameter denoted by ε goes to zero. In terms of solutions there are two ways to look at this problem. From the perspective of the balance law, by asking what happens to solutions when viscosity is added. This is the transition form $\varepsilon = 0$ to $\varepsilon > 0$.

Or from the perspective of the viscous balance law, by asking what happens to viscous solutions when viscosity tends to zero, i.e. $\varepsilon \to 0$.

The answers to both questions are different in some cases but certainly there is a relation between these.

Both equations possess global attractors (see Chapter 2), denoted by $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$, which attract solutions in forward time. Thus the question about the relation between solutions can be understood as a question about the global attractors.

It is unknown whether

$$\lim_{\varepsilon \to 0} \mathcal{A}^\varepsilon = \mathcal{A}^0 \tag{1.2}$$

in the case of periodic boundary conditions. There are many ways how to understand equation (1.2):

- In the sense of sequences: all $u^0 \in \mathcal{A}^0$ are a limit of a sequence of $u^\varepsilon \in \mathcal{A}^\varepsilon$ and all converging sequences $u^\varepsilon \in \mathcal{A}^\varepsilon$ have a limit that is contained in $\mathcal{A}^0$.
- In the sense of sets: $\mathcal{A}^\varepsilon$ converges in the Hausdorff metric for sets in L^1, L^∞ or L^2 to $\mathcal{A}^0$.
- In the sense of solutions: all converging sequences of solutions $u^\varepsilon(\cdot, t) \in \mathcal{A}^\varepsilon$ converge to a solution $u^0 \in \mathcal{A}^0$, and all solutions $u^0 \in \mathcal{A}^0$ are a limit of a converging sequence of solutions $u^\varepsilon \in \mathcal{A}^\varepsilon$.
- In the sense of C^0-orbit equivalence: this would mean that the orbit structure on $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ is the same, hence there exists a C^0 bijective map mapping orbits of $\mathcal{A}^\varepsilon$ to orbits of $\mathcal{A}^0$.

For Neumann boundary conditions Härterich [Haer97] could prove a very interesting result. He proved under mild assumptions (i.e. f' does not vanish at zeros of g) that the dimension of the parabolic attractor $\mathcal{A}^\varepsilon$ stays finite even for $\varepsilon \to 0$ whereas the global attractor for $\varepsilon = 0$ is infinite dimensional. However, the problem here is that for Neumann boundary conditions the limiting equation is not well posed and the right hand side of equation (1.2) has no interpretation. In Section 2.2 we will see that the finite dimensionality of the limit does not hold for the S^1 case.

If we assume convergence of the limit in (1.2) in the sense of sub-sets of L^∞ or L^1 then it is a direct consequence of our Theorem 3.2.1 that

$$\lim_{\varepsilon\to 0} \mathcal{A}^\varepsilon \subset \mathcal{A}^0.$$

However this still does not answer the question about the relations of solutions. It is one of the main results of this work that heteroclinic solutions in $\mathcal{A}^\varepsilon$ do in general not persist for $\varepsilon \to 0$. This corrects an outstanding result of Fan and Hale [FH95] that states otherwise.

The *Connection Lemma* 3.2.8 states a purely algebraic necessary condition for the persistence of heteroclinics: if a heteroclinic connection between a source u_2 and a target u_1 persists, then the zero-number of the source is a multiple of the zero-number of the target. This excludes persistence for a lot of connections!

If a heteroclinic connection does not persist, the *Cascading Theorem* 3.2.9 yields convergence to a cascade of heteroclinics. This means the limit consists of heteroclinic connections of (H) separated by sections of equilibria. Because we have pointwise convergence of solutions, this implies that for small ε the heteroclinic carries a fast-slow dynamic structure.

This dynamical structure is the focus of Chapter 4 where we explore the geometry of the manifolds that form the global attractors. The tool of finite sub-attractors, introduced in Sections 2.4 and 2.5 proves extremely useful here, especially in combination with the main result of Chapter 2, Theorem 2.6.1, that provides an explicit parameterisation of all sub-attractors of equation (H).

This characterises the general ideas behind the main results of this work: the *Connection Lemma*, the *Cascading Theorem* and Theorem 2.6.1 on the sub-attractors of (H). The structure of the dissertation is as follows:

Chapter 2 will present a detailed review of what is known about the global attractors of equations (H) and (P), and will provide the necessary technical background. We begin with definitions of global attractors in Section 2.1 followed by three sections on the parabolic equation: after the existence of a global attractor is settled in Section 2.2, we apply the developed theory to our equation to classify all rotating waves of the parabolic equation in Section 2.3. This is possible by virtue of geometric singular perturbation theory, developed by Fenichel in the 70s [Fen79], in combination with rotated vector fields for ODEs.
Section 2.4 then solves the connection problem and allows us to fully classify the global attractor of the parabolic equation. This goes back to results of Fiedler, Rocha and Wolfrum [FRW04]. At the end of this section we introduce our new tool, the finite dimensional sub-attractors of order n in the parabolic setting.

Two sections on the hyperbolic equation then follow: Section 2.5 reviews questions on the existence and uniqueness of solutions, the existence of a global attractor and the connection problem. Many people have contributed to these results, the latest reference is [Haer99]. In the last part of Section 2.5 we introduce the sub-attractors of order n for the hyperbolic balance law.

Section 2.6 contains the main result of Chapter 2: Theorem 2.6.1. It yields an explicit parameterisation of and the flow on all sub-attractors of finite order and proves their finite dimensionality. It yields uniqueness of heteroclinic connections when the zero-numbers of source and target only differ by two. The zero-number limitation is unsatisfactory, because I believe it to be a purely technical constrain, however even this result will provide us with an important tool in the analysis of the geometric structure of heteroclinic connections of the parabolic equation.

Chapter 3 is devoted to the main theoretical results of this work on persitence of solutions: the *Cascading Theorem* and the *Connection Lemma* already described above.

Chapter 4 explores the implications of these theorems. The chapter proceeds by slowly increasing the dimension of the sub-attractors i.e. the number of zeros in the rotating waves that are involved. Section 4.1 deals with the parabolic sub-attractor of order 2: $\mathcal{A}_2^\varepsilon$. By virtue of the uniqueness for heteroclinics in the hyperbolic equation to homogeneous equilibria (Theorem 2.6.1 (e)), the result of this section yields convergence of sub-attractors for $\varepsilon \to 0$. Hence we can describe the solution manifolds of the parabolic equation on this subattractor and their geometry which has not been done rigorously before.

Section 4.2 investigates the relation of solutions between $\mathcal{A}_4^\varepsilon$ and $\mathcal{A}_4^0$. Here we use an additional assumption that the dimensionality of $\mathcal{A}^\varepsilon$ is preserved when performing the limit $\varepsilon \to 0$.

The section on geometry finishes with a proposition on how to construct the cascades of heteroclinics in Section 4.3. The suggested construction is a generalisation from the previous section's result, but it is not rigorous. It gives interesting insights on how the limits of heteroclinics might look.

We conclude in Chapter 5 with a discussion on the unanswered questions of this work and a discussion on the possibilities of finding answers to some of them.

Chapter 2

Global Attractors

The aim of this chapter is to introduce global attractors and to present an overview about what is known about attractors and their structure in the hyperbolic and parabolic cases. In the parabolic setting we will apply these known results and adapt them to our equation; in the hyperbolic setting we will push the limits a little further and obtain some new findings on the geometric representation of finite dimensional parts of the global attractor. We will combine the results of both equations to obtain our main result on the non-persistence of heteroclinic connections in Chapter 3.

This chapter is organised as follows: in the first section we give a definition of global attractors. The following two Sections 2.2 and 2.3 will present the general properties of global attractors for the parabolic equation and use them in the following Section 2.4 to solve the full connection problem. At the end of this section we introduce a new tool: sub-attractors of order n.

The fifth and sixth sections are devoted to the study of the attractor of the hyperbolic equation. In 2.5 we present the general properties of the global attractor and additionally define – in analogy to the parabolic setting – the sub-attractors for the hyperbolic equation.

Theorem 2.6.1 in Section 2.6 proves a complete explicit parameterisation of all finite dimensional sub-attractors and yields uniqueness of certain heteroclinic connections. This theorem thus closes one of the last remaining gaps of a full geometric description of the global attractor of equation (H) and is one of the main results of this work.
It will help us to better compare the attractors of the hyperbolic and parabolic setting for small ε and will bring us a step further towards understanding the question of whether the attractor of the parabolic equation converges to that of the hyperbolic equation for $\varepsilon \to 0$.

2.1 Preliminaries and Definitions

Although the attractors for the two equations show many similarities, we will present the results separately. The tools and methods involved in the two settings are quite different. Even the underlying spaces differ. We will see later that the parabolic equation “lives” in H^2, whereas the hyperbolic balance law “lives” in BV, thus the two equations have to be treated in different frameworks.

The functional analysis setup concerning existence, uniqueness, regularity etc. is standard material and has entered text books. I will not show proofs for most of the results, as they can be found in the works quoted. I do include these results for a better readability of this dissertation. Moreover the basic theory in each section will help us to understand from where the assumptions (H1)-(H3) we have made originally come.

Let us now address the definition of global attractors. In general there are several different ways to do this; some definitions are more suitable for the one or the other equation. The following definition, however, will serve us well as a starting point:

Definition 2.1.1 *Let*

$$u_t = \mathcal{F}(u, u_x, u_{xx}) \tag{2.1}$$

for $x \in S^1$ *define a semiflow denoted by* Φ *on a function space* X*. Then the global attractor of the above PDE* (2.1) *is defined - if it exists - as the subset* $\mathcal{A}$ *of the phase space* X *that consists of all global orbits of the equation.*

A global solution here is defined as a solution that exists for all times $t \in \mathbb{R}$ and stays bounded. It is far from obvious that such solutions exist, especially in backward time, because the PDE (2.1) only defines a semiflow. Hence it cannot be solved in backward time in general.

Thus we have to clarify what "exists for all times" means. We use the following (standard) definition:

Definition 2.1.2 *Let* $t \in \mathbb{R}^+$ *be a positive time and* $u_0(x)$ *be an initial condition. We say* $\Phi(u_0, -t)$ *exists if there is a* $\tilde{u} \in X$ *such that* $\Phi(\tilde{u}, t) = u_0$*. We call a solution* $u(x, t)$ *that exists for all* $t \in \mathbb{R}$ *a global solution.*

In other words, $\Phi(u_0, t)$ exists for negative times if u_0 lies on a forward orbit for some initial condition $\tilde{u}$. This does not imply that we solve the equation backwards because in general $\tilde{u}$ is not unique.

An alternative description of a global attractor yields the following definition:

Definition 2.1.3 *The global attractor of equation* (2.1) *is defined - if it exists - as the maximal compact invariant subset* $\mathcal{A}$ *of the phase space* X *of equation* (2.1)*, that attracts all bounded subsets* $\mathcal{B} \subset X$*.*

Both definitions are equivalent for the parabolic equation (P) and the hyperbolic equation (H) if we impose (H1)-(H3). However, this is far from obvious. The next sections will provide for references.

The difference between Definition 2.1.1 and Definition 2.1.3 clearly lies in the starting point of the definition. The first one uses global orbits that are collected to sets, the second focuses on attracting sets in phase space. The second makes it clear where the term "attractor" comes from.

2.2 The parabolic equation

In this section we will present general results on the solution theory of parabolic equations and the properties of the global attractor. The results are true for more general equations than equation (P).

We therefore introduce a more general form of a parabolic equation than equation (P), which we will use throughout this section:

$$u_t = \varepsilon u_{xx} + h(u, u_x) \tag{2.2}$$

where $h \in C^2$ and again $x \in S^1$. Obviously, if we set

$$h(u, u_x) = g(u) - f'(u)u_x$$

our equation (P) is of the above form.

For a more extensive overview than the one presented here of global attractors and patterns in general reaction diffusion equations, I refer to the article of Fiedler and Scheel [FS03] or the book of Chepyzhov and Vishik [CV02]. The latter even treats the non-autonomous case. The first half of the first article is exclusively devoted to one dimensional reaction diffusion equations under several boundary conditions including periodic boundary conditions.

It is known that the initial value problem (Cauchy problem) of PDE (2.2) together with Neumann, Dirichlet or periodic boundary conditions is well posed and has unique solutions for sufficiently regular initial conditions.

On the Sobolev space of twice weakly differentiable L^2-functions

$$X = W^{2,2}([0, 2\pi], \mathbb{R}) = H^2([0, 2\pi], \mathbb{R})$$

that satisfy the boundary conditions, the PDE generates a C^1 semiflow with the associated semigroup

$$\Phi^\varepsilon : X \times \mathbb{R}^+ \longrightarrow X$$

which assigns each pair $(u_0(x), t) \in X \times \mathbb{R}^+$ the solution $u(\cdot, t)$ at time t with initial condition u_0:

$$\Phi^\varepsilon(u^\varepsilon(\cdot, t_0), t) := u^\varepsilon(\cdot, t_0 + t).$$

The books of Henry [Hen81] or Pazy [Pazy83] which give a more detailed description are the standard references for the semigroup theory related to parabolic PDEs.

The existence and structure of global attractors for (2.2) were first described for separated boundary conditions such as Neumann or Dirichlet. In fact many publications focus up to this day on these two cases.

Dissipativity of the non-linearity is the key for the existence of global attractors. Dissipativity here is understood in the sense of Hale [Hale88] or Babin and Vishik [BV92]. A sufficient condition for dissipativity of h in the Neumann or Dirichlet case is:

$$uh(u, 0) < 0 \text{ for } |u| > M$$

for sufficiently large $M \in \mathbb{R}$.

In 1968 Zeleniak [Zel68] and later in 1978 Matano [Ma78] could achieve results not only regarding existence but also giving an efficient description of the attractor in the case of Neumann boundary conditions. They proved that any bounded solution tends to a single equilibrium for $t \to \infty$. This is due to the existence of a Lyapunov functional on the phase space X. In fact, this holds true in negative time direction as well, if the solution exists in negative time direction and stays bounded. This leads to the description of global attractors for the Neumann case as the set of equilibria and their connecting heteroclinic orbits (for a precise definition of heteroclinic orbit see equation (2.5)).

In the 90s Fiedler and Rocha proved in [FR96] that the connection problem can be solved exclusively with information about the stationary solutions of the PDE. In other words, once all equilibria are described, it is possible to decide which of the stationary solutions are connected. We do not go into further detail here for Neumann b.c. as we are only interested in the S^1 case.

In the S^1 case again dissipativity of h is sufficient for the existence of a global attractor on $X = H^2$. We quote the condition given by Matano and Nakamura in [MN97] that ensures existence:

(A) For each $K > 0$ there exists $C > 0$ such that $|h(p, q)| \leq C(1 + q^2)$ for $|p| \leq K$.

(B) There exists $M > 0$ such that $h(p, 0)p < 0$ for all $|p| > M$.

In other words, the non-linearity has to be positive for negative first argument and negative for positive first argument. In addition it has to grow sub-quadratically in the second variable.

It is easy to see that our PDE (P) is dissipative in this sense. The above condition (B) is the same as our condition (H2). Furthermore our non-linearity only grows linearly in the second variable u_x by definition. Hence we have existence of a global attractor.

In terms of the structure of the global attractor periodic boundary conditions are much more complicated to deal with than separated boundary conditions. This is due to the existence of rotating waves which cannot exist for separated boundary conditions.

If h depends in addition explicitly on x, the situation is even more complicated and few results are known. The problem is that the Morse-Smale property of the attractor is destroyed in this case. This is the main reason for not considering the x-dependent case.

For the homogenous case Angenent and Fielder [AF88] and Matano [Ma88] could show that, similar to the Neumann case, any solution of (2.2) tends to a set of functions $\Gamma(v) := \{v(\cdot + \theta) : \theta \in S^1\}$ for $t \to +\infty$. Here $v(x)$ is given by a solution of the ordinary differential equation

$$v_{xx} + cv_x + h(v, v_x) = 0 \tag{2.3}$$

for some value of $c \in \mathbb{R}$ and $x \in S^1$. This equation is usually called travelling or, in the S^1 case, rotating wave equation. Any non-homogenous solution v of (2.3) with non-zero c is a time periodic solution $u(x, t)$ of (2.2) if we define $u(x, t) := v(x - ct)$. This solution is called a rotating wave with wave-speed c. The orbit of this rotating wave is given by $\Gamma(v)$.

The above equation can be obtained by plugging a travelling wave ansatz $u(x, t) := v(x-ct)$ into the PDE (2.2) and then requiring the time-derivative to vanish. In fact, if $u(x, t)$ is a travelling wave, i.e. there is some $v(\cdot)$ and $c \neq 0$ such that

$$u(x, t) = v(x - ct), \tag{2.4}$$

then v solves the rotating wave equation (2.3) and vice versa. This is an "if and only if" relation. The above equation is commonly used to define the notion of rotating waves.

For $c = 0$ equation (2.3) turns into the stationary problem of (2.2). The non-homogenous equilibria then will be called frozen waves. For these $\Gamma(v)$ is an embedded circle of equilibria.

Finally the zeros u_i of $h(p, 0)$ solve equation (2.3) for $v(x) \equiv u_i$ and define the homogenous equilibria.

This leads to definition the following sets. Let

- $\mathcal{E}^\varepsilon$ denote the set of homogenous equilibria;
- $\mathcal{F}^\varepsilon$ denote the set of frozen waves;
- $\mathcal{R}^\varepsilon$ denote the set of rotating waves and
- $\mathcal{H}^\varepsilon$ denote the set of heteroclinic connections.

We define a heteroclinic connection as a solution $u(x, t)$ of (2.2) that has the property that

$$\begin{aligned} \lim_{t \to +\infty} \quad & u(x,t) \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon \\ \lim_{t \to -\infty} \quad & u(x,t) \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon. \end{aligned} \tag{2.5}$$

The result of Angenent and Fiedler or Matano quoted above means that any bounded solution of (2.2) converges towards either a rotating wave, a frozen wave or a homogenous equilibria in forward time direction. The same is true in backward time direction if the solution stays bounded. Thus, they have obtained the following theorem:

Theorem 2.2.1 *Let the non-linearity of equation* (2.2) *be dissipative and* C^2*. Then the global attractor* $\mathcal{A}^\varepsilon$ *of the PDE can be described as follows:*

$$\mathcal{A}^\varepsilon = \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon \cup \mathcal{H}^\varepsilon. \tag{2.6}$$

In particular, any time periodic orbit is a rotating wave and any orbit in $\mathcal{A}^\varepsilon \backslash (\mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon)$ *is a heteroclinic connection connecting* $u_1, u_2 \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ *with* $u_1 \neq u_2$*.*

In [FRW04] Fiedler, Rocha and Wolfrum were able to resolve the connection problem for the periodic case as well. Their idea was to use homotopies, such that every solution of the S^1 case solves a Neumann problem and vice versa. Then they could use their earlier results on the Neumann case and extend it to the periodic case.

The key ingredient is the concept of $k - (\mathcal{P})$-adjacency (see Definition 2.4.1 in Section 2.4), that was developed and used for the Neumann case in [FR96] and later in [Wol02a] and [Wol02b]. The whole approach relies heavily on nodal properties that have their origin in the fact that the linearisation of the PDE (2.2) is a Sturm-Liouville type problem. This goes back to Sturm [Stu1836]. A key observation is that the number of strict sign changes in a solution can only drop along trajectories, hence can be considered as a discrete Lyapunov function. This remains true for the difference of two solutions.

Information on the equilibria, the frozen and rotating waves is sufficient to determine which objects are connected to each other by heteroclinic orbits. The relation of the maxima of the rotating and frozen waves plays a key role in this analysis. Moreover, the direction of the connection is given by the Morse indices; the target always has smaller Morse index than the source. We will cover this in detail in Section 2.4 on the connection problem.

Let us conclude this section about the general properties of the global attractor by some additional remarks. We have seen that the attractor both for Neumann and periodic boundary conditions can be described in terms of stationary and periodic solutions and heteroclinic connections between these solutions. Moreover the existence of connections can in principle be computed if the travelling wave ODE (that turns into the stationary problem for $c = 0$) is well understood. However the problem still remains to describe the heteroclinic solutions in terms of their geometry in the phase space. So a proof about how solutions change in time within a heteroclinic connection is in general not known. In Chapter 4 we will prove some results in that direction for some low dimensional cases. At this point the works of Carr and Pego should be mentioned. In two long and very technical papers [CP89] and [CP90], using invariant manifold techniques, they proved that the dynamics on the heteroclinic connections in the simplest case ($f = 0$, g a cubic function) are exponentially slow for $\varepsilon \to 0$.

Their proof strongly relies on the the fact that the viscosity parameter ε is small and their approach is not suitable to describe the full heteroclinic connection via the manifold approach. The reason for this is that their description breaks down in neighbourhoods of points, where the connecting orbit (viewed as a manifold in the extended phase space) is not normally hyperbolic. In other words the linearisation in transverse direction cannot have eigenvalues with zero real part. But the Morse index (for a precise definition see 2.3.10) necessarily decreases along the heteroclinic connection (see Theorem C in [MN97]). At the point where it actually gets smaller at least one eigenvalue has to cross the imaginary line, hence this is a point on the heteroclinic connection where normal hyperbolicity breaks down.

Another important question relates to the dimension of the global attractor. A general result of Mallet-Paret [MP76] already shows that the Hausdorff dimension of the global attractor of (2.2) is finite if $\varepsilon > 0$.

However it might not stay finite for $\varepsilon \to 0$. Even in the simpler Neumann case there are examples where the dimension of the attractor approaches ∞ for $\varepsilon \to 0$. The most famous result is probably that of Chaffee and Infante [CI74], where already the number of isolated equilibria goes to infinity for $\varepsilon \to 0$, and so does the number of heteroclinic connections.

Härterich [Haer97] could prove under mild assumptions, that in the Neumann case the dimension of the attractor stays generically finite for viscous balance laws such as our equation (P). In fact if the zero of f' does not coincide with the middle zero of g in (P) then this is the case. However, the example stays artificial, because the limiting equation is not well posed for Neuman boundary conditions.

If the assumption of Härterich is violated and the middle zero of g and the zero of f' coincide then a Chaffee Infante type mechanism leads to a blow up of the dimension: more and more stationary solutions with increasing zero-number appear when ε approaches 0.

Any Neumann solution can be extended by an easy reflection to a periodic solution on the doubled domain. One might expect that the result of Härterich could be generalised to the periodic case, where there is a well posed limiting equation. However, this is not possible, because as we will see in Section 2.3 there is always a wave speed c, such that the zero of $(f' - c)$ and the middle zero of g coincide. This again leads for $\varepsilon \to 0$ to the generation of infinitely many rotating waves with that particular wave speed. The waves have increasing zero-numbers, similar to the Chaffee-Infante example. A consequence is a divergence of the attractor's dimension.

This is not as surprising as it might at first seem in this context. In the section on global attractors in the hyperbolic equation 2.5 we will see that for the hyperbolic equation where $\varepsilon = 0$, continua of linearly independent stationary solutions exist and the global attractor of the hyperbolic equation thus is infinite dimensional.

It has become clear that the rotating waves in the parabolic equation are important for the analysis of the attractor. The following section is devoted to the study and classification of these waves.

2.3 Rotating waves for the parabolic equation

In the description of the global attractor of the parabolic equation rotating waves play a key role.

We first state our version of the rotating wave equation. If we set

$$h(u, u_x) = g(u) - f'(u)u_x$$

in equation (2.3) then all rotating waves of the PDE (P) are solutions of the ODE

$$\varepsilon v_{xx} = (f(v) - c(\varepsilon)v)_x - g(v) \tag{2.7}$$

with boundary conditions

$$v(0) = v(2\pi) \tag{2.8}$$

$$v_x(0) = v_x(2\pi). \tag{2.9}$$

Hence they are periodic solutions of equation (2.7) with minimal x-period $\frac{2\pi}{n}$ for some $n \in \mathbb{N}$.

In the following we will describe all periodic solutions of (2.7), including those satisfying (2.8) and (2.9). This analysis makes use of three aspects of the equation. The first is its singular perturbed nature; the second is the fact, that (2.7) can be transformed to a planar rotated vector field and the third is the fact, that one of the equilibria of the ODE (2.7) undergoes a Hopf bifurcation for $c = 0$.

We begin by rewriting equation (2.7) as a first order system in Lienard coordinates. The equation then reads

$$\begin{aligned} \varepsilon v_x &= f(v) - c(\varepsilon)v + p \\ p_x &= -g(v). \end{aligned} \tag{2.10}$$

These coordinates are adapted to the geometry of the problem. However, sometimes it is more convenient to work with standard phase plane coordinates:

$$\begin{aligned} \varepsilon w_x &= q \\ q_x &= \frac{(f'(w) - c(\varepsilon))q}{\varepsilon} - g(w) \end{aligned} \tag{2.11}$$

We will use both sets of coordinates as each one has its own advantages. We will always use (v, p) when referring to the Lienard version and (w, q) when utilising phase plane coordinates.

The coordinates can be transformed into each other by the transformation:

$$w(v,p) = v \qquad v(w,q) = w \tag{2.12}$$

$$q(v,p) = f(v) - p \qquad p(w,q) = f(w) - q. \tag{2.13}$$

To not become confused by the rotating wave as a solution of the ODEs (2.10) or (2.11), and the (time) dependent rotating wave solution of the PDE (P) we will use the letters $v(x)$ or $w(x)$ when referring to the solution of the ODE, and we will use $u(x,t)$ when we refer to the solution of the PDE. Both solutions will be called "rotating wave". Sometimes we will drop the arguments for a better readability.

The ODEs are singularly perturbed in both coordinates. We use the theory developed by Fenichel in the '70s and '80s [Fen79] to analyse the properties of the two systems. Fenichel's idea was to split the dynamics into a slow part which is given by just putting $\varepsilon = 0$ and a fast part which is obtained by rescaling $\xi = \frac{x}{\varepsilon}$ and again putting $\varepsilon = 0$.

The slow dynamics then are confined to a manifold that consists of stationary solutions of the fast equation. Fenichel could prove that the manifold persists for $\varepsilon > 0$ if the manifold is normally hyperbolic i.e. the linearisation of the fast field on the slow manifold has no purely imaginary eigenvalues in the transverse direction.

In the Lienard case we obtain for the slow part after putting $\varepsilon = 0$,

$$\begin{aligned} 0 &= f(v) - c(0)v + p \\ p_x &= -g(v). \end{aligned}$$

Therefore the slow dynamic is confined in the manifold given by

$$\mathcal{F}_l := \{(v,p); p = cv - f(v), v \in \mathbb{R}\}$$

, which is just the graph of $cv - f$. The dynamics can be obtained by differentiation of the first equation $0 = (f'(v) - c)v_x + p_x$, which leads together with $p_x = -g(v)$ to:

$$v_x = \frac{g(v)}{f'(v) - c}. \tag{2.14}$$

This equation has in general one singularity depending on c, but for the appropriate choice of c this singularity can be removed.

For the fast dynamics we obtain upon rescaling and again putting $\varepsilon = 0$ the equations

$$\begin{aligned} v_\xi &= f(v) - c(0)v + p \\ p_\xi &= 0. \end{aligned}$$

This means that the fast vector field is given by horizontal lines and vanishes on the slow manifold. Every point on the slow manifold has at least one zero eigenvalue with an eigenvector that is tangential to the manifold.

An easy calculation yields that the second eigenvalue is non-zero except at a point $(v_0, cv_0 - f(v_0))$ depending on c, where $f'(v_0) - c = 0$. Due to the convexity of f the point v_0 is unique. At $(v_0, cv_0 - f(v_0))$ the fast vector field is tangential to the slow manifold.

This means $\mathcal{F}_l$ is normally hyperbolic except at the point $(v_0, cv_0 - f(v_0)$. The manifold persists outside a neighbourhood of this point. It is therefore not clear if the persisting unstable manifold $\mathcal{W}^u(u_-, cu_- - f(u_-))$ and the stable manifold $\mathcal{W}^s(u_+, cu_+ - f(u_+))$ coincide and form a heteroclinic connection for $\varepsilon > 0$. Later we will see that there is a unique wave-speed c such that they do in fact connect. Figure 2.1 shows a schematic plot of the vector field in Lienard coordinates for $c = 0$.

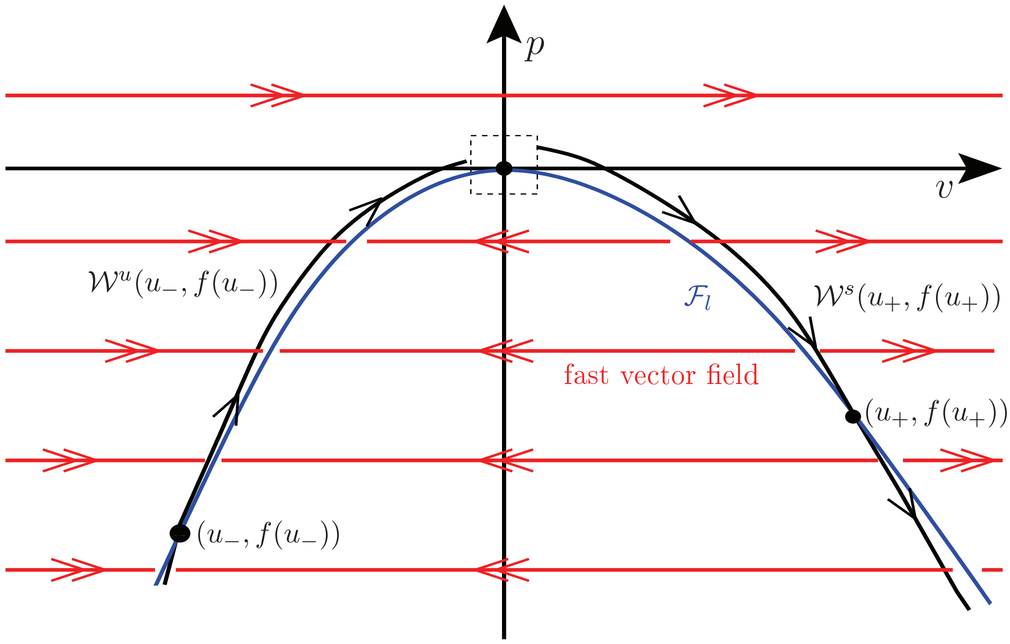

Figure 2.1: Phaseportrait of equation (2.10) in Lienard coordinates for $c = 0$. The dotted box is the area where the slow manifold does not necessarily persist . The unstable and stable manifolds of $(u_\pm, f(u_\pm))$ might not coincide.

For phase plane coordinates the situation is a little different. Here the slow manifold $\mathcal{F}_p$ is just given by the line with $q \equiv 0$. The fast vector field reads after rescaling

$$\begin{aligned} w_\xi &= q \\ q_\xi &= (f'(w) - c)q - \varepsilon g(w). \end{aligned} \tag{2.15}$$

Again the slow manifold is normally hyperbolic except in the unique point w_0 where $f'(w_0) - c = 0$. The phase portrait in phase plane coordinates is given in Figure 2.2.

Again the flow on the slow manifold is given by

$$w_x = \frac{g(w)}{f'(w) - c}. \tag{2.16}$$

To see this, we observe that $q_x = 0$ in the slow manifold. We use this in the second equation of (2.11) and plug in $\varepsilon w_x = q$ from the first equation to obtain the above expression.

The reason why we introduced phase plane coordinates at all, is that the system (2.11) is a **rotated vector field (mod $q = 0$)** with respect to the parameter c.

The notion of rotated vector fields was introduced by Duff [Duf53] and refined by Perko [Per75, Per93]. For exact definitions I refer to their papers or to Definition 4.1 in [Haer03].

The geometric interpretation of this is that the whole vector field rotates in the same direction when changing the parameter c except on the curve $q = 0$. A consequence of this is the following result:

Lemma 2.3.1 (Duff,Perko) *Consider a family of rotated vector fields. Suppose there is an equilibrium which for all values of c possesses a one-dimensional unstable manifold. Then*

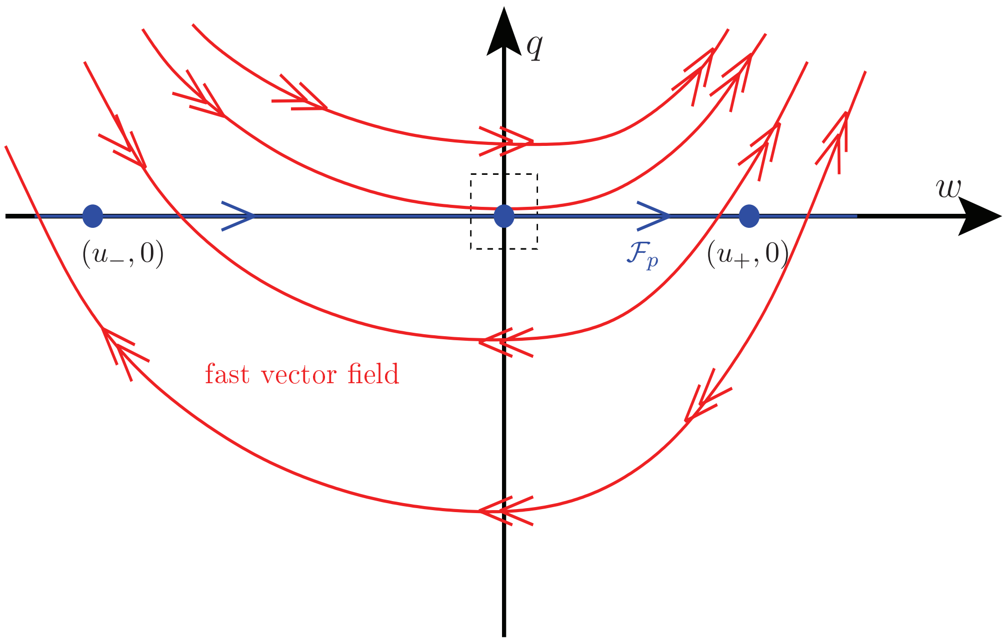

Figure 2.2: Phaseportrait of equation (2.11) when $c = 0$. The dotted box is the area where the slow manifold does not necessarily persist.

this manifold moves either clockwise or anti-clockwise as the parameter c is increased. The stable manifold moves in the same direction. Moreover, these directions are the same for all saddle equilibria of the system.

Before we state the main proposition of this section concerning the structure and existence of all periodic orbits of (2.7), we introduce the cyclicity set $\mathcal{C}$. This set was used already in [FRW04] in this form, but the idea was introduced earlier in similar problems, for example in [MN97].

Definition 2.3.2 *The cyclicity set $\mathcal{C}_p$ consists of all points $(w, q) \in \mathbb{R}^2$ that lie on a periodic orbit of equation* (2.11) *for some value of c or correspond to homogenous equilibria $(e, 0)$ of* (P) *that undergo a Hopf bifurcation for some value of c.*

We immediately observe that in our situation $\mathcal{C}_p$ is non-empty because the homogenous solution associated with the middle equilibrium $w \equiv 0$ undergoes a Hopf bifurcation at $c = 0$:
There are three homogenous equilibria of equation (P) that correspond to $(w_0, 0) = \{(u_-, 0), (0, 0), (u_+, 0)\}$. The characteristic polynomial of the linearisation of (2.11) in these equilibria is given by

$$\lambda_{1/2} = -\frac{f'(w_0) - c}{2\varepsilon} \pm \sqrt{\left(\frac{f'(w_0) - c}{2\varepsilon}\right)^2 - \frac{g'(w_0)}{\varepsilon}}. \tag{2.17}$$

For $w_0 = u_\pm$ both eigenvalues are real. For $w_0 = 0$ the eigenvalues are imaginary with the property that

$$sign(c) = sign(Re(\lambda))$$

and therefore undergo a Hopf bifurcation at $c = 0$.

According to Lemma 4.2 in [FRW04] the cyclicity set has in the case that it is not empty the following properties

Lemma 2.3.3 *The cyclicity set $\mathcal{C}_p$ is bounded and open. There exist C^2-functions*

$$\mathbf{c}, \mathcal{T} : \mathcal{C}_p \to \mathbb{R} \tag{2.18}$$

with the properties:

(i) For each non-stationary point $(w, q) \in \mathcal{C}_p$ the value $\mathbf{c}(w, q)$ defines the unique wave speed for which (w, q) lies on a periodic orbit of (2.11). Similarly, $\mathcal{T}(w, q)$ defines the minimal period of this orbit.

(ii) The wave speeds $\mathbf{c}$ are uniformally bounded.

(iii) The minimal periods $\mathcal{T}$ tend to infinity at the boundary $\partial\mathcal{C}_p$ of $\mathcal{C}_p$.

(iv) $\partial\mathcal{C}_p$ consists of saddles and of points which are homoclinic or heteroclinic to saddles for some parameter value of c.

We do not give a proof here but refer the reader to the paper quoted above. We now prove three Lemmata that will allow us to classify all periodic orbits of our system (2.11) and therefore all rotating waves.

Lemma 2.3.4 *Let $\varepsilon > 0$ be arbitrary. Then the following is true:*

a) The cyclicity set $\mathcal{C}_p$ is homeomorphic to a disc, i.e. it consists of one connected component and has no holes.

b) All periodic orbits $(w(x), q(x))$ have the property that $w(x) \neq 0$ except at exactly two points x_1, x_2 where $w(x_1) = w(x_2) = 0$.

c) All periodic orbits can be uniquely parameterised by their maxima $(\alpha, 0)$, with $\alpha > 0$.

Proof. We first prove c): we assume that $v_1 \neq v_2$ are two rotating waves with wave speeds c_1 and c_2 and identical maximum

$$\alpha = \max_{x \in S^1}\{v_1(x)\} = \max_{x \in S^1}\{v_2(x)\}. \tag{2.19}$$

We observe that the origin $(0, 0)$ has to lie in the interior of the area encircled by v_1 and v_2 respectively. This is a direct consequence of the Poincare-Bendixson Theorem for planar flows.
If the curves do not intersect or touch each other, then necessarily either

$$v_1(x) < \max_{x \in S^1}\{v_2(x)\}$$

or vice versa. This contradicts (2.19). See Panel c) in Figure 2.3 for illustration. The curves therefore have to touch or intersect.

We now distinguish two cases:

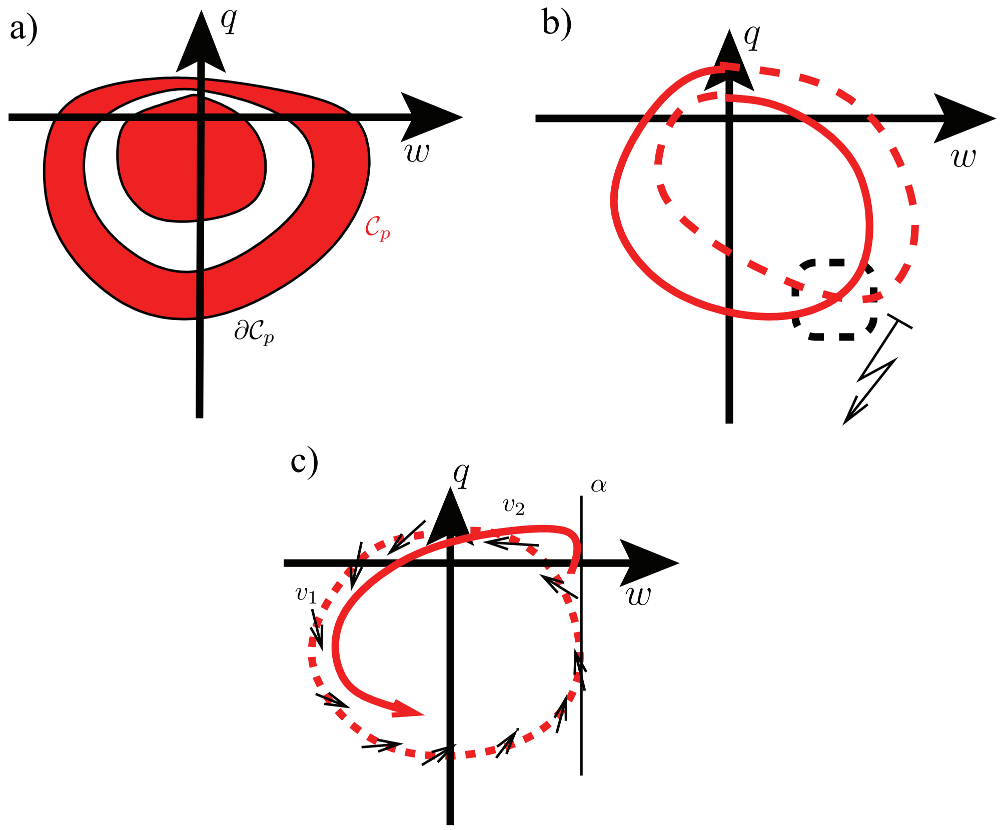

Figure 2.3: Illustration for the proof of Lemma 2.3.4.

(i) Assume $c_1 = c_2$. In this case the two curves have at least one point in common. Because trajectories of the same equation cannot intersect, we obtain $v_1 = v_2$.

(ii) Assume $c_1 \neq c_2$. We investigate the vector field of (2.11) for $c = c_2$ on the curve defined by v_1. Due to the fact that (2.11) is a rotated vector field with respect to c we obtain, that the vector field has to either point strictly to the outside or strictly to the inside of the area encircled by v_1. This excludes touching points. Assume the vector field points inwards, then the area encircled by v_1 is positive invariant. See again Panel c) for illustration.
Therefore v_1 enters at the intersection point but cannot intersect twice due to the positive invariance of the area encircled by v_1 – and thus cannot be closed. This contradicts that v_1 is a periodic orbit.

If the vector field points to the outside, the same argument holds (just reverse the "time direction" x).

This proves c).

For b) we observe that the number of zeros is necessarily even. The fact that $(0,0)$ lies in the area encircled by the periodic orbit excludes the no-zero case. The fact that the periodic orbit cannot intersect itself excludes the case of more than two zeros (see Panel b) in Figure 2.3). This proves b).

For a) we assume that $\mathcal{C}_p$ is not homeomorphic to a disc. The nesting property of the periodic orbits in c) excludes holes in $\mathcal{C}_p$. Hence $\partial_{\mathcal{C}_p}$ must consists of nested closed curves. Due to the boundedness of $\mathcal{C}_p$ and the fact that $(0,0) \in \mathcal{C}_p$, there must be a minimum of three curves. See Panel a) for such a situation.

According to 2.3.3 (iv) these curves must consist of saddles, homoclinic and heteroclinic connections. There are only three equilibria

$$(u_-, 0), (0, 0), (u_+, 0).$$

The second one $(0,0)$ is contained in the interior of the open set $\mathring{\mathcal{C}}_p$. Therefore it is enough to analyse homoclinic orbits of and heteroclinic orbits between $(u_-, 0), (u_+, 0)$.

Due to Lemma 2.3.1 there can at most be one wave-speed c_+ such that $(u_+, 0)$ has a homoclinic orbit. The same is true for at most one c_- and $(u_-, 0)$.

Moreover, Theorem 1.2 in [Haer03] states that there is a unique value

$$c^*(\varepsilon) = -\frac{1}{2}\frac{d}{dw}\left(\frac{g'(w)}{f''(w)}\right)\Big|_{w=0} \varepsilon + \mathcal{O}(\varepsilon^{3/2}) \tag{2.20}$$

for which there exists a heteroclinic connection that connects $(u_-, 0)$ with $(u_+, 0)$. Again the rotated vector field property is the key to the proof. Using the same argument there can be at most one value of c such that there is a heteroclinic connection from $(u_+, 0)$ to $(u_-, 0)$. (Note here that the fast orbits are given by curves defined through $q = f(w) - cw$).

From this we conclude that $\partial\mathcal{C}_p$ consists of maximal three curves, one given by the two heteroclinic connections, two by the homoclinic ones.

We now prove that the equilibrium $(0,0)$ is the only equilibrium inside each of the homoclinic connections, which completes the proof, because then, the two homoclinic curves cannot be nested.

However this is obvious, because the slow manifold given by $q = 0$ persists due to Fenichel for $w > u_+$ and $w < u_-$. This proves a).

□

The next Lemma gives a first-order description of all rotating waves. Here the singular perturbed nature of the problem yields the result.

Lemma 2.3.5 *Let $T > 0$ be given. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists a rotating wave w with minimal period T.*

Through a shift we can assume that $w(0) = 0$ and $w_x(0) > 0$, then $w(x)$ can be written in the following way:

$$w(x) = \phi(x) \qquad + o(\varepsilon) \quad \textit{for} \quad x \in [0, x_2 - \varepsilon\log\varepsilon] \cup [x_2 + \varepsilon\log\varepsilon, T] \tag{2.21}$$

$$w(x) = \psi(\frac{x}{\varepsilon}, x_2) + o(\varepsilon) \quad \textit{for} \quad x \in [x_2 - \varepsilon\log\varepsilon, x_2 + \varepsilon\log\varepsilon] \tag{2.22}$$

where x_2 is the second zero of w. $\phi(\cdot)$ is a solution of

$$\phi_x = \frac{g(\phi)}{f'(\phi)} \qquad \phi(0) = 0$$

and $\psi(\cdot, x_2)$ is a solution of

$$\psi_{\frac{x}{\varepsilon}} = f\left(\psi(\frac{x}{\varepsilon})\right) + \phi(x_2) \qquad \psi(0) = 0$$

Proof. For existence we observe that the centre in the origin $(0,0)$ undergoes a Hopf bifurcation. The two eigenvalues $\lambda_{1/2}$ were already given in equation (2.17). The imaginary part of $\lambda_{1/2}$ is given by

$$\nu := \mathrm{Im}(\lambda_{1/2}) = \frac{\sqrt{g'(0)}}{\sqrt{\varepsilon}}.$$

Thus, the limiting period at the Hopf bifurcation emerging limiting cycle is given by

$$T_{Hopf} = \frac{2\pi}{\nu} = \frac{2\pi\sqrt{\varepsilon}}{\sqrt{g(0)}}.$$

We already know that $\mathcal{T} \to \infty$ when $\partial\mathcal{C}$ is approached. As $\mathcal{T}$ is a C^2 function on $\mathcal{C}$ and in particular continuous we obtain existence if

$$\varepsilon < T^2\frac{g'(0)}{4\pi^2} := \varepsilon_0$$

by virtue of the intermediate value theorem.

It remains to prove equations (2.21),(2.22). For this we have to assume that

$$0 < \varepsilon_0 << T\frac{g'(0)}{4\pi}. \tag{2.23}$$

We need ε to be small to be able to apply Fenichel's his results. From Lemma 2.3.4 c) we know that the periodic orbits can be parameterised by their maxima.

Moreover, Lemma 2.3.4 b) proves that w_ε has exactly two zeros. Without loss of generality we shift the one with positive slope to $x_1 = 0$. We denote the other zero with x_2 and note that $w'(x)|_{x=x_2} < 0$ necessarily. We assume that the wave-speed $c(\varepsilon) = 0$ and prove equations (2.21,2.22). Then we will argue that the correct wave-speed is in fact small and hence does not destroy the approximation.

We start computing the trajectory of $(w(x), w_x(x))$ in $x = x_2$ and assume that

$$|w_x(x)|_{x=x_2}| >> \varepsilon_0. \tag{2.24}$$

This is always possible because we are free in the choice of ε_0. We use phase plane coordinates.

Due to equation (2.24) and (2.23) we can use the fast vector field to describe the solution up to the first order. In forward time direction the solution will converge exponentially to a ε-neighbourhood of the unstable manifold of $(u_-, 0)$. In backward time direction the solution will converge exponentially to a ε-neighbourhood of the stable manifold of $(u_+, 0)$. This part can be described due to Fenichel [Fen79] by the fast equations (2.15). This proves equation in (2.22).

The unstable manifold of $(u_-, 0)$ is transversally stable in forward time direction. So is the stable manifold of $(u_+, 0)$ in negative time direction. Thus in both cases the solution is given up to the first order by the slow equations (2.16) outside a neighbourhood of $(0,0)$ where the normal hyperbolicity of the slow manifold breaks down; but we already know that w is periodic, thus the two ends have to meet at $(0,0)$. This proves equation (2.21).

We now argue that this remains true for non-zero wave-speeds $c(\varepsilon)$. To do so we quote Lemma 4.3 in [Haer03]. The lemma states that the wave-speed c^* for which the heteroclinic connection between u_- and u_+ persists obeys

$$|c^*(\varepsilon)| < \sigma\varepsilon$$

for some $\sigma > 0$. The same equality holds for the wave-speed $c(\varepsilon)$ of the periodic orbit by virtue of the same argumentation as in [Haer03].

Härterich argues that $W^u(u_-)$ lies below the curve

$$\gamma(\phi) := -f(\phi) + \varepsilon \frac{g(\phi)}{f'(\phi) - c(\varepsilon)}$$

for $c(\varepsilon) < \sigma\varepsilon$ whereas $W^s(u_+)$ lies above γ. This order reverses for $c(\varepsilon) > \sigma\varepsilon$. Because our periodic solution $(w(x), w_x(x))$ converges exponentially to $W^u(u_-)$ and $W^s(u_+)$ as argued above a intermediate value argument yields the desired inequality. This implies that equations (2.21) and(2.22) hold as well for $c = c(\varepsilon)$.

Remark: I believe that in fact $c(\varepsilon)$ is given by equation (2.20). However in order to prove that one would have to go through the whole blow-up construction in Chapter 5 of [Haer03].

□

Remark 2.3.6 *A different description of the periodic orbit that is sometimes usefull is given by*

$$w(x) = \begin{cases} \phi(x - x_2) + \left[\psi\left(\frac{x_2}{\varepsilon}\right) - \phi(-x_2)\right] + o(\varepsilon) & \textit{for}\, x \in [0, x_2] \\ \phi(x - x_2) + \left[\psi\left(\frac{x_2 - 2\pi}{\varepsilon}\right) - \phi(2\pi - x_2)\right] + o(\varepsilon) & \textit{for}\, x \in [x_2, T] \end{cases} \tag{2.25}$$

Proof. A simple, straightforward calculation shows that this is true. The reason for this is the exponential convergence of ψ to the states $\phi(x_2)$ and $\phi(2\pi - x_2)$.

□

The next Lemma uses the above descriptions to prove hyperbolicity of all rotating waves in our equation which is a direct consequence of the monotonicity of $\mathcal{T}(w, q)$. This result forms the basis of a relation between the zeros of a solution and the number of its unstable eigenvalues.

Lemma 2.3.7 *Let T be arbitrary but fixed. Then there exists a $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the minimal period $\mathcal{T}(w, q)$ grows monotone with the maxima of the periodic orbits.*

Proof. We use the formula of the periodic orbit $w(x)$ obtained in equation (2.21). Let us assume we have two periodic orbits w_1 and w_2 with period T_1 and T_2 and the property that

$$\max_{x \in [0, T_1]} w_1 =: \alpha_1 < \alpha_1 + \delta_1 := \alpha_2 := \max_{x \in [0, T_2]} w_2$$

for some $\delta_1 > 0$. It is sufficient to prove $T_1 < T_2 - \delta_2$ for sufficiently small ε and some $\delta_2 > 0$.

Due to the fact that the periodic orbits are nested (Lemma 2.3.4 c)) $\alpha_1 < \alpha_2$ implies immediately

$$0 > \min_{x \in [0.T_1]} w_1 =: \beta_1 > \beta_2 := \min_{x \in [0, T_2]} w_2.$$

The solution $\phi(x)$ is strict monotonically growing because

$$\phi_x = \frac{g(\phi)}{f'(\phi)} > 0$$

due to the convexity of f and the fact that the zero of f' and the zero of g at $u = 0$ are simple. This implies invertibility of ϕ and monotonicity of ϕ^{-1}

We now have

$$T_1 = \phi^{-1}(\alpha_1) - \phi^{-1}(\beta_1) + o(\varepsilon \log \varepsilon) \tag{2.26}$$

$$T_2 = \phi^{-1}(\alpha_2) - \phi^{-1}(\beta_2) + o(\varepsilon \log \varepsilon) \tag{2.27}$$

The monotonicity of ϕ implies

$$\phi^{-1}(\alpha_2) = \phi^{-1}(\alpha_1 + \delta_1) = \phi^{-1}(\alpha_1) + (\phi^{-1})'(\alpha_1)\delta_1 + o(\delta_1) > \phi^{-1}(\alpha_1) + \delta_2$$

for some $\delta_2 > 0$ and

$$-\phi^{-1}(\beta_1) < -\phi^{-1}(\beta_2).$$

For sufficiently small $0 < \varepsilon$ we obtain the desired inequality for some δ_2 independent of ε.

□

Corollary 2.3.8 *Let $T \in \mathbb{R}^+$ be given. Then there is a unique periodic orbit with minimal period T, and it is hyperbolic as a rotating wave of* (P)*.*

Proof. The uniqueness is a direct consequence of the monotonicity of the $\mathcal{T}$-map. The hyperbolicity is also a direct consequence of the monotonicity of the $\mathcal{T}$-map. A periodic orbit is non-hyperbolic if, and only if, the time $\mathcal{T}$ map has a vanishing derivative. See for example Lemma 4.4 in [FRW04]. This would contradict monotonicity.

□

We are now set to construct rotating waves of the PDE (P) by using the periodic orbits constructed earlier in this section. We introduce the zero-number of a function $u : S^1 \to \mathbb{R}$.

Let therefore $u : S^1 \to \mathbb{R}$ then we define

$$z(u) := \sharp\{x \in S^1; u(x) = 0\}, \tag{2.28}$$

if the zero set of u is not countable we define $z(u) = \infty$.

Theorem 2.3.9 *Let $n \in 2\mathbb{N}$ be given, then there exists $0 < \varepsilon_n$ such that for all $0 < \varepsilon < \varepsilon_n$ there exists an up to shift unique rotating wave v_n^ε with the property*

$$z(v_n^\varepsilon) = n.$$

Proof. Every rotating wave with n zeros corresponds to a periodic solution of the rotating wave equation (2.7) with period $T_n = \frac{2\pi}{n}$.

Corollary 2.3.8 provides for the unique existence of a periodic orbit of the rotating wave equation with period

$$T_n = \frac{2\pi}{n}.$$

This proves the Theorem.

A qualitative bifurcation diagram of how periodic solutions are generated is given in Figure 2.4. The numbers at the branches indicate the numbers of zeros, the vertical axis shows the maximum of the rotating wave on the branch.

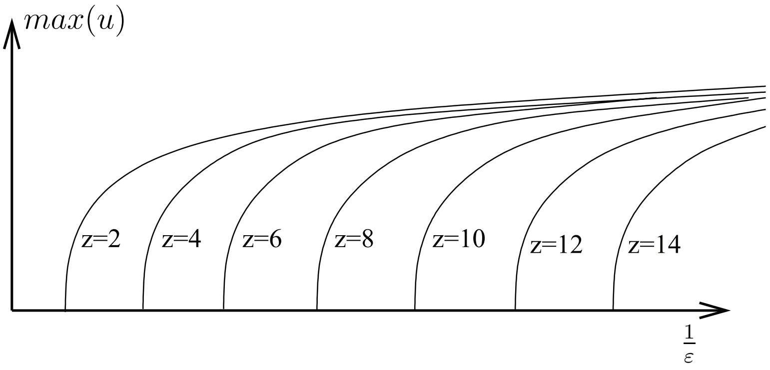

Figure 2.4: Schematic bifurcation diagram of the Hopf bifurcation generating branches of rotating waves (lines) with increasing zero-number.

□

The remainder of this section is devoted to the Morse index of solutions and the relation between the Morse index $i(u)$ and the zero-number $z(u)$ of a rotating or frozen wave u. The Morse index is the classical and generic tool to describe properties of solutions on the global attractor. However, in the hyperbolic setting it is rather uncommon to even introduce a Morse index. There the zero-number is more commonly used. This is the main reason why we have already introduced the zero-number here.

Let $L(u)$ define the linear operator obtained when the PDE (P) is linearised in the solution u, and let $\sigma(L(u))$ denote the spectrum of $L(u)$. We follow the definition given in [MN97] for the Morse index $i(u)$.

Definition 2.3.10 *For each $u \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ we define the Morse index $i(u)$ and the generalised Morse index $i_0(u)$ by*

$$i(u) := \sharp\{\lambda \in L(u); Re(\lambda) > 0\}$$

and

$$i_0(u) := \sharp\{\lambda \in L(u); Re(\lambda) \geq 0\}.$$

Here $\sharp$ counts eigenvalues repeatedly according to their multiplicity.

In terms of the Morse index we call a homogenous stationary solution u hyperbolic, if

$$i_0(u) = i(u).$$

We call a rotating wave u hyperbolic, if

$$i_0(u) = i(u) + 1.$$

Note that u_x is always an eigenfunction of $L(u)$ to $\lambda = 0$. The wave is called hyperbolic, if zero is a simple eigenvalue, hence u_x is the only eigenvector to $\lambda = 0$.

Remark 2.3.11 *The Morse index i corresponds the number of strong unstable eigendirections of the solution $u^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$, hence equals the dimension of the strong unstable manifold of u^ε in case of fixed points. For rotating waves u^ε the dimension of the strong unstable manifold is given by the Morse index $+1$.*

There is a one-to-one correspondence between the Morse index and the number of zeros in a solution.

Lemma 2.3.12 *Let* $u^\varepsilon \in \mathcal{R}^\varepsilon \cup \mathcal{F}^\varepsilon$ *then*

$$i(u^\varepsilon) = z(u^\varepsilon) - 1. \tag{2.29}$$

For $u^\varepsilon \equiv u_\pm$ *we have*

$$i(u^\varepsilon \equiv u_\pm) = 0. \tag{2.30}$$

To prove this Lemma we first quote a result that can be found for example in [FRW04] Lemma 5.3:

Lemma 2.3.13 *Let*

$$\dot{T} = \partial_\alpha T$$

be the derivative of the minimal period with respect to the maximum of the periodic orbits just as in Lemma 2.3.7. Then the Morse index of a rotating or frozen wave u is given by the following relations:

$$i(u) = z(u) - 1 \Longleftrightarrow \dot{T} > 0 \tag{2.31}$$

$$i(u) = z(u) \Longleftrightarrow \dot{T} < 0. \tag{2.32}$$

Proof of Lemma 2.3.12
We obtain from Lemma 2.3.7 $\dot{T} > 0$ which yields the result together with 2.3.13 for $u^\varepsilon \in \mathcal{R}^\varepsilon \cup \mathcal{F}^\varepsilon$.
For $u^\varepsilon \equiv u_\pm$ we use the fact that in Sturm-Liouville eigenvalue problems the eigenfunction to the leading eigenvalue λ_0 (eigenvalue with largest real part) has a sign, i.e. has no zeros. This can be found in [CL55] in Chapter 8, Theorem 3.1.
A small calculation shows that $\lambda_0 = g'(u_\pm) < 0$ with constant eigenfunction. Hence $i(u_\pm) = 0$.

□

The next section will apply the results on rotating waves to solve the connection problem on the attractor. The Morse index will play a key role in this.

2.4 The connection problem

With the results of the previous section we are now ready to solve the connection problem and to describe the structure of the global attractor.

The remaining question concerning the global attractor is which of the rotating waves are connected. Let therefore u^ε_a and u^ε_b be two rotating or frozen waves or homogenous equilibria of equation (P) with Morse indices

$$i(u^\varepsilon_a) = a - 1 \quad i(u^\varepsilon_b) = b - 1.$$

We want to know if there is a heteroclinic orbit with source u_a^ε and target u_b^ε, i.e. if there exists a solution $u^\varepsilon(x,t)$ with

$$\begin{aligned}\lim_{t\to-\infty} u^\varepsilon(\cdot,t) =& u_a^\varepsilon(\cdot,t)\\ \lim_{t\to\infty} u^\varepsilon(\cdot,t) =& u_b^\varepsilon(\cdot,t)\end{aligned}$$

where u_a^ε and u_b^ε are appropriately shifted. A key ingredient here is $k-(\mathcal{P})$-adjacency of rotating waves. The concept of k-adjacency was developed and used in [FR96] and later in [Wol02a] and [Wol02b] for the Neumann case. Fiedler, Rocha and Wolfrum presented in [FRW04] a version for the S^1 case which we will use:

Definition 2.4.1 ($k-(\mathcal{P})$-adjacency) *Let $u_a^\varepsilon, u_b^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$. Then u_a^ε and u_b^ε are called $k-(\mathcal{P})$-adjacent if the following holds:*

$$z(u_a^\varepsilon - u_b^\varepsilon) = k$$

for some $k \in \mathbb{N}$ and there does not exist a solution $a_c^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with the property

$$z(u_a^\varepsilon - u_c^\varepsilon) = z(u_b^\varepsilon - u_c^\varepsilon) = k \textit{ and} \tag{2.33}$$

$$\max_{x\in S^1} u_c^\varepsilon(x) \textit{ is strictly between } \max_{x\in S^1} u_a^\varepsilon(x) \textit{ and } \max_{x\in S^1} u_b^\varepsilon(x). \tag{2.34}$$

This notion of $k-(\mathcal{P})$-adjacency is the critical ingredient in Theorem 1.3 in [FRW04] answering the connection question. This theorem states that u_a^ε and u_b^ε are connected if, and only if, they are $k-(\mathcal{P})$-adjacent. The authors call a violation of $k-(\mathcal{P})$-adjacency the blocking principle because in this case there is another rotating wave u_c^ε that blocks the connection. If blocking does not occur, then the "principle of liberalism" states that the two solutions u_a^ε and u_b^ε are connected. We use these results to prove the following

Theorem 2.4.2 *Let $u_a^\varepsilon, u_b^\varepsilon \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with Morse indices $i(u_a^\varepsilon) = a-1$ and $i(u_b^\varepsilon) = b-1$. Then there exists a heteroclinic orbit connecting u_a^ε and u_b^ε, i.e. a heteroclinic orbit with source u_a^ε and target u_b^ε if, and only if, $a > b$.*

Proof. The "only if" has already been proven by Matano and Nakamura in [MN97]. The statement can be found in *Theorem C* on page 5. It is a direct consequence of the fact that due to the Sturm property of the problem the zero-number can only drop along trajectories and so does the Morse index.

For the "if" part we have to prove $k-(\mathcal{P})$-adjacency of u_a^ε and u_b^ε. The key observation lies in the fact that the number of zeros of the difference of two rotating waves is given by the minimum of the zero-numbers individually. In other words, we have for $\tilde{u}, \hat{u} \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ the following relation:

$$z(\tilde{u} - \hat{u}) = \min\{z(\tilde{u}), z(\hat{u})\}. \tag{2.35}$$

This is not true in general, but a direct consequence of the fact that in our situation all periodic orbits of the rotating wave equation are nested.

Now assume $a_c^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with the property

$$z(u_a^\varepsilon - u_c^\varepsilon) = z(u_b^\varepsilon - u_c^\varepsilon) = k \text{ and} \tag{2.36}$$

$$\max_{x\in S^1} u_c^\varepsilon(x) \text{ is strictly between } \max_{x\in S^1} u_a^\varepsilon(x) \text{ and } \max_{x\in S^1} u_b^\varepsilon(x) \tag{2.37}$$

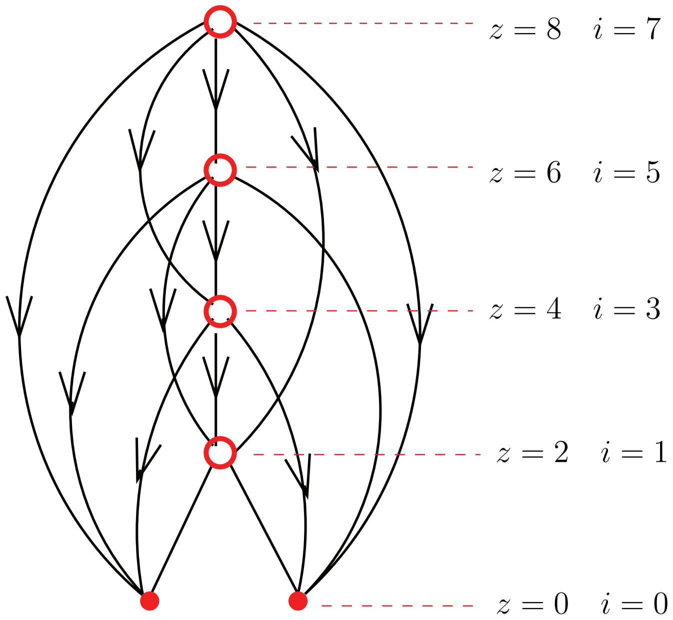

Figure 2.5: Structure of connections between rotating and frozen waves and homogenous equilibria of Morse index $i \leq 7$.

exists.

Due to Theorem 2.3.9 there is a unique rotating wave to each zero-number $k \in 2\mathbb{N}_0$ for fixed and sufficiently small $\varepsilon > 0$. From this we conclude that

$$a \neq k \neq b$$

otherwise $u_c^\varepsilon = u_a^\varepsilon$ or $u_c^\varepsilon = u_b^\varepsilon$.

In case $k > b$ equation (2.36) is violated. Hence, we have necessarily $k < b$. Due to the nested property of rotating waves this implies $\max u_c^\varepsilon > \max u_{a,b}^\varepsilon$, which violates (2.37). Thus u_a^ε and u_b^ε are k-($\mathcal{P}$)-adjacent and therefore connected.

In case $u_a^\varepsilon \in \mathcal{E}^\varepsilon$ or $u_b^\varepsilon \in \mathcal{E}^\varepsilon$ the same argument works, the zero properties are obvious because in this case u_a or u_b is constant.

□

This yields that all rotating and frozen waves are connected to rotating and frozen waves with lower Morse index and to $u \equiv u_\pm$. A representation of the connection structure of the global attractor for all rotating and frozen waves and homogeneous equilibria with Morse index $i \leq 7$ can be found in Figure 2.5. In the figure the arrows indicate the direction of the flow on the attractor.

What might be misleading in the figure is the fact that the connections between rotating or frozen waves look as if they were one-dimensional. This is not the case!

It is a classical result by Henry [Hen85] and Angenent [Ang86] for Neumann boundary conditions, that the unstable manifold $W^u(u_1)$ of an equilibrium u_1 and the stable manifold $W^s(u_2)$ of another equilibrium u_2 intersect transversally in our setting:

$$W^u(u_1) \pitchfork W^s(u_2). \tag{2.38}$$

Hence, the intersection is either empty or has maximal dimension.

Fielder, Rocha and Wolfrum were able to prove in [FRW04] that the same is true in the S^1 case, thus, the dimension of an intersection is given by the difference of the Morse indices of the source and the target. Note that in the S^1 case equation (2.38) has to be properly interpreted. To obtain the full two-dimensional connection manifold connecting u_1 with u_2 the target u_2 has to be properly shifted. We will discuss this in more detail in Section 4.1.

After we have solved the connection problem we introduce a new tool for our analysis: sub-attractors of order n.

We have already seen that for positive ε the attractor of the parabolic equation has finite dimension. However for small ε the dimension becomes very large. The idea of introducing sub-attractors is that we only want to consider a low dimensional part of the whole attractor when we investigate the limit $\varepsilon \to 0$. The clear advantage is that we do not have to deal with difficulties arising from the divergence of the global attractor's dimension in this limit.

For the parabolic attractor we define the sub-attractors of order n as the part of the whole attractor that consists of the two stable homogenous equilibria and the rotating waves with zero-number less or equal than n and all heteroclinic orbits between these objects. Note that in order to have existence of a rotating or frozen wave solution with zero-number n, ε has to be sufficiently small, according to Theorem 2.3.9.

Definition 2.4.3 *Let $n = 2\alpha$ for $\alpha \in \mathbb{N}$ and let ε_n be sufficiently small. Then we define for $0 < \varepsilon < \varepsilon_n$:*

- $\mathcal{E}_n^\varepsilon := \{u \in \mathcal{E}^\varepsilon; z(u) \leq n\} = \{u \equiv u_-, u \equiv u_+\}$;
- $\mathcal{F}_n^\varepsilon := \{u \in \mathcal{F}^\varepsilon; z(u) \leq n\}$;
- $\mathcal{R}_n^\varepsilon := \{u \in \mathcal{R}^\varepsilon; z(u) \leq n\}$;
- $\mathcal{H}_n^\varepsilon := \{u \in \mathcal{H}^\varepsilon; \lim_{t\to\pm\infty} u \in \mathcal{E}_n^\varepsilon \cup \mathcal{F}_n^\varepsilon \cup \mathcal{R}_n^\varepsilon\}$.

Then the finite dimensional subattractor of order n of the parabolic equation (P) *is given by*

$$\mathcal{A}_n^\varepsilon := \mathcal{E}_n^\varepsilon \cup \mathcal{F}_n^\varepsilon \cup \mathcal{R}_n^\varepsilon \cup \mathcal{H}_n^\varepsilon. \tag{2.39}$$

It is immediately clear that the subattractors are contained in each other for increasing n. In other words we have

$$\mathcal{A}_n^\varepsilon \subseteq \mathcal{A}_m^\varepsilon \quad \Leftrightarrow \quad n \leq m.$$

Figure 2.5 shows the sub-attractor of order $n = 8$: $\mathcal{A}_8^\varepsilon$.

From Definition 2.4.3 and Theorem 2.4.2 it is clear that $\mathcal{A}_n^\varepsilon$ contains all unstable manifolds $W^u(u^\varepsilon)$ of all waves u^ε with zero-number $z(u^\varepsilon) \leq n$. On the other hand if $u^\varepsilon \in \mathcal{A}_n^\varepsilon$ then u^ε must be contained in the closure of some unstable manifold $W^u(\tilde{u}^\varepsilon)$ of an element $\tilde{u}^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$. By construction $z(\tilde{u}^\varepsilon) \leq n$ must hold. This suggests another description of the sub-attractors $\mathcal{A}_n^\varepsilon$:

$$\begin{aligned}
\mathcal{A}_n^\varepsilon &= \bigcup_{m=1}^{n} \{W^u(u^\varepsilon); u^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon, z(u^\varepsilon) = m\} \cup \mathcal{E}_n^\varepsilon \qquad (2.40)\\
&= \bigcup_{m=1}^{n} \overline{\{W^u(u^\varepsilon); u^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon, z(u^\varepsilon) = m\}}\\
&= \overline{\{W^u(\mathcal{F}_n^\varepsilon \cup \mathcal{R}_n^\varepsilon)\}}\,.
\end{aligned}$$

This description clearly is very useful. Because $i(u^\varepsilon) = z(u^\varepsilon) - 1$ for every rotating or frozen wave we immediately conclude

$$\dim W^u(u^\varepsilon) = i(u^\varepsilon) = z(u^\varepsilon) - 1$$

and hence

$$\dim \mathcal{A}_n^\varepsilon = n$$

because all waves with given zero-number can be parameterised by S^1 (see Theorem 2.3.9). It is a theorem that in fact

$$\mathcal{A}_n^\varepsilon = \overline{\left\{W^u((\mathcal{F}_n^\varepsilon \cup \mathcal{R}_n^\varepsilon)\backslash(\mathcal{F}_{n-1}^\varepsilon \cup \mathcal{R}_{n-1}^\varepsilon))\right\}}.$$

We do not prove this here, but it is also a consequence of the Morse-Smale property of the attractor.

In the next section we will give an overview of the relevant results concerning the hyperbolic equation. The solution theory is somewhat more complicated, but the structure of the global attractor is very similar.

2.5 The hyperbolic equation

In the following we will provide the results concerning global attractors of scalar hyperbolic balance laws. As in the previous section on parabolic equations, some of the results presented hold for slightly more general equations. Nevertheless we do not introduce a more general equation such as $u_t = h(u, u_x)$ here because in contrast to the previous sections on the parabolic equation, the structure of the attractor and questions of the existence of unique solutions rely on the fact that we are investigating a balance law and not a completely general hyperbolic equation. Especially the convexity of f is a key feature. Without convexity none of the results presented holds true. Therefore we will state all results for equation (H) together with assumptions (H1),(H2) and (H3).

Some of the theorems quoted will be written for the case that g has n zeros located at $u_1, \ldots, u_n$. In this case we just set $n = 3$ and $u_1 = u_-$, $u_2 = 0$ and $u_3 = u_+$.

Before we investigate the question of global attractors we have to answer the question of existence and uniqueness of solutions. The initial value problem (Cauchy problem) of (H) can be solved by the method of characteristics. The classical solution $u(x, t)$ to a initial condition $u(x, 0) =: u_0(x)$ can be described in the following way:

$$u(\chi(t), t) := \underline{\mathrm{v}}(t)$$

where $\underline{\mathrm{v}}, \chi$ are curves that solve the following ODE:

$$\begin{aligned}\chi'(t) &= f'(\underline{\mathrm{v}})\\ \underline{\mathrm{v}}'(t) &= g(\underline{\mathrm{v}})\\ \chi(0) &= x_0\\ \underline{\mathrm{v}}(0) &= u_0(x_0)\end{aligned}$$

for all $x_0 \in S^1$. Unfortunately classical solutions in general only exist for finite time. This is even true for the simplest possible case where $g \equiv 0$, $f = \frac{1}{2}u^2$ and $u_0(x) \in C^\infty$. To see this one just has to choose an initial condition with sufficiently large negative slope somewhere. In fact, if the negative slope in the initial condition becomes large the time up to which a unique classical solution exists can become arbitrarily small.

Classical solvability breaks down due to the development of shocks. At the development point of a shock, characteristics meet each other in a finite angle. If both characteristics were to be continued they would intersect at that point transversally. Due to the convexity of f the values of $\underline{\mathrm{v}}$ on both characteristics differ from each other at the intersection point. Hence the classical solution develops a discontinuity at this point so that the solution is in particular not differentiable at this point as it would have to be in a classical solution.

However, there are weak solutions for times after classical solvability has reached its limit. To obtain weak solutions one has to multiply both sides of the differential equation (H) with C^1 test functions and integrate the equation over the whole domain. A weak solution then is defined as a solution that satisfies the resulting equation for all C^1 test functions, see equation (2.41).

Unfortunately the uniqueness of solutions is lost in this process. In general there are many weak solutions to the same initial condition. To overcome this obstacle a additional entropy condition can be imposed, that singles out a unique weak solution. This idea derives from the physical entropy in thermodynamics. Entropy conditions for hyperbolic balance laws in a weak framework were first considered by Volpert [Vol67] and Kruzhkov [Kr70].

We therefore define an entropy or admissible solution of the hyperbolic balance law (H) in the following way:

Definition 2.5.1 *We call $u \in BV([0,\infty) \times S^1, \mathbb{R})$ an entropy or admissible solution of equation* (H) *to the initial condition $u_0(x)$*

- *if $u(x,0) = u_0(x)$;*
- *if it solves equation* (H) *in the weak sense:*

$$\int_{S^1 \times \mathbb{R}^+} [u\varphi_t + f(u)\varphi_x - g(u)\varphi]\, dx\, dt = 0 \tag{2.41}$$

  for all $\varphi \in C_0^1(S^1 \times \mathbb{R}^+, \mathbb{R})$;
- *and if the entropy condition*

$$u(x+,t) \leq u(x-,t) \tag{2.42}$$

  holds for all $t > 0$.

Here $u(x+,t)$ defines the right hand, $u(x-,t)$ the left hand limit of u in x at time t and $BV([0,\infty) \times S^1, \mathbb{R})$ denotes the space of functions with bounded variation mapping from $[0,\infty) \times S^1$ to $\mathbb{R}$.

Let $\mathcal{P}$ be the set of all partitions $P = \{x_1, \cdots, x_{n_P}\}$. Then we define the space BV in the following way:

$$BV(S^1) := \left\{ u \in L^1(S^1) : \sup_{P \in \mathcal{P}} \sum_{i=1}^{n_P - 1} |u(x_i) - u(x_{i+1})| < \infty \right\}. \tag{2.43}$$

Volpert [Vol67] and later, and for more general initial conditions (L^∞), Kruzhkov [Kr70] were able to prove the following result on the existence of solutions:

Proposition 2.5.2 *If (H1) holds, then the Cauchy problem of equation* (H) *possesses a unique entropy solution u with the property $u : (0,\infty) \to L^1$ is continuous in time and $u(\cdot,t) \in BV(S^1)$ for every time $t > 0$.*

Therefore (H) together with (2.42) defines a semiflow on $BV(S^1)$. We denote that semiflow by

$$\begin{array}{rrcl} \Phi^h : & BV \times \mathbb{R}^+ & \to & BV \\ & u_0, t & \mapsto & \Phi^h(u_0,t) := u(\cdot,t) \end{array}$$

where u is the unique entropy solution to the initial condition u_0.

Note that Kruzhkov proved that the initial condition does not have to fulfil the entropy condition. Where the initial condition has up-jumps, i.e. $u(x+,t) > u(x-,t)$ for some x, these jumps are immediately smoothened by a rarefaction wave.

In order to compute weak solutions practically the notion of characteristics has to be generalised. Generalised characteristics were first introduced by Dafermos in [Daf77]:

Definition 2.5.3 *A Lipschitz curve $x = \chi(t)$, defined on the interval $[a,b] \subset \mathbb{R}$ is called a generalised characteristic associated with the solution u of* (H) *if it satisfies the inequality*

$$\dot{\chi} \in [f'(u(\chi+,t)), f'(u(\chi-,t))]$$

for almost all $t \in [a,b]$.

With this definition it is clear that generalised characteristics coincide with the classical characteristics $\chi(t)$ defined above, when the solution is differentiable. At points of non-differentiability of u i.e. at shocks, the generalised characteristic is only required to satisfy $\chi'(t) \in [f'(u-), f'(u+)]$ where $u-$ and $u+$ are the lower and upper states of the shock at $\chi(t)$. Filippov was able to show in [Fi88] that there is at least one forward and one backward characteristic through any point $(x,t) \in S^1 \times \mathbb{R}^+$.

It seems that there is a lot of freedom in computing forward characteristics. That this is in fact not the case is shown by a proposition to be found in [Fi88]:

Proposition 2.5.4 *Let $\chi : [a,b] \to \mathbb{R}$ be a generelized characteristic. Then the following holds for almost all $t \in [a,b]$:*

$$\dot{\chi}(t) = \begin{cases} f'(u(\chi(t)\pm,t)) & \text{if} \quad u(\chi(t)-,t) = u(\chi(t)+,t) \\ \frac{f(u(\chi(t)+,t)) - f(u(\chi(t)-,t))}{u(\chi(t)+,t) - u(\chi(t)-,t)} & \text{if} \quad u(\chi(t)-,t) > u(\chi(t)+,t) \end{cases}. \tag{2.44}$$

Hence, $\dot{\chi}(t)$ is uniquely defined even at the position of shocks. If the solution $u(x,t)$ possesses a shock at position x_0 then the shock speed is given by the Rankine-Hugoniot condition for shock speeds

$$c_{shock} = \frac{f(u(x_0+)) - f(u(x_0-))}{u(x_0+) - u(x_0-)}. \tag{2.45}$$

To distinguish between generalised characteristics and the characteristics of classical solutions the notion of genuine characteristics is important:

Definition 2.5.5 *A characteristic on the interval* $[a,b]$ *is called genuine, if*

$$u(\chi(t)-,t) = u(\chi(t)+,t) \textit{ for almost all } t \in [a,b].$$

The set of backward characteristics through a point $(\bar{x},\bar{t})$ spans a funnel between the

- *minimal backward characteristic* $\chi^-(t;\bar{x},\bar{t})$ and the
- *maximal backward characteristic* $\chi^+(t;\bar{x},\bar{t})$.

The properties of characteristics that are of importance for us are summarised in the next propositions. For proofs we refer to Dafermos' article [Daf77]. We will use these results in the following section.

Proposition 2.5.6 *Let* $(\bar{x},\bar{t}) \in S^1 \times \mathbb{R}$ *be arbitrary. Then the minimal backward characteristic* $\chi^-(t;\bar{x},\bar{t})$ *and the maximal backward characteristic* $\chi^+(t;\bar{x},\bar{t})$ *are genuine.*

Proposition 2.5.7 *Genuine characteristics intersect only at their end points; backward characteristics do not intersect in particular.*

We now direct our attention to the existence of global attractors for equation (H). Fan and Hale [FH95] were able to settle this question for the hyperbolic balance laws in 1995. As in the parabolic case, dissipativity of g is the key to the existence of a global attractor. It essentially guarantees that solutions stay bounded in forward time. (Note that convexity of f and the linear dependence on u_x already guarantee dissipativity of $f'(u)u_x$.)

Proposition 2.5.8 *(Fan and Hale) Assume (H1), (H2) and (H3) hold. Then*

$$\mathcal{A}^0 := \left\{u_0 \in BV(S^1) : \Phi^0(u_0,t) \textit{ exists for all } t \in \mathbb{R} \textit{ and is bounded}\right\} \tag{2.46}$$

is the global attractor of (H) *in* $L^p(S^1)$*, for any* $p \in [1,\infty]$*, i.e. it is invariant and attracts bounded sets in* $L^p(S^1)$*.*

This settles the existence of $\mathcal{A}^0$. We turn to the structure of the global attractor. Many people have worked on this and for a good overview over the latest results we refer to Härterich [Haer97].

Several authors proved Poincaré Bendixson type results for the scalar balance laws. See for example Fan and Hale [FH93], Sinestrari [Sin97] or Lyberopoulos [Lyb94]:

Proposition 2.5.9 *For $t \to \infty$ any solution of* (H) *tends either to a homogenous solution $u \equiv u_i$ for some $i \in \{1, ...n\}$ or it converges to a rotating wave solution*

$$u(x,t) = v(x - ct)$$

where the wave-speed c can only take the values $c = f'(u_{2i})$ for $i \in \{1, \cdots, \frac{n-1}{2}\}$.

In our case this implies

$$c = f'(u_2) = f'(0) = 0$$

which means that all waves are frozen waves. However the distinction is somewhat arbitrary, because a coordinate change $x \mapsto x - ct$ can freeze any wave, or make it rotate again. In this sense our assumptions $f'(0) = 0 = g'(0)$ fix a coordinate system in which all waves freeze.

For global solutions a theorem similar to 2.5.9 holds true in backward time. This leads to a description of the global attractor $\mathcal{A}^0$ as the unification of the homogenous steady states, the frozen waves and heteroclinic connections between all these objects similar to the parabolic case. Additionally the possible wave-speeds of all rotating waves are given a priori.

Following the definitions made in the parabolic section we define

- $\mathcal{E}^0$ to be the set of homogenous equilibria of (H);
- $\mathcal{F}^0$ to be the set of frozen waves of (H);
- $\mathcal{R}^0$ to be the set of rotating waves of (H);
- $\mathcal{H}^0$ to be the set of heteroclinic connections between objects in $\mathcal{E}^0$, $\mathcal{F}^0$ and $\mathcal{R}^0$ defined in the same way as in Section 2.2 equation (2.5).

Then the global attractor $\mathcal{A}^0$ of (H) can be described as

$$\mathcal{A}^0 = \mathcal{E}^0 \cup \mathcal{F}^0 \cup \mathcal{R}^0 \cup \mathcal{H}^0. \tag{2.47}$$

In our case we have $\mathcal{R}^0 = \emptyset$.

In [Sin95] Sinestrari was able to settle the description of all rotating or frozen waves. He proved that for any possible wave speed $c = f'(a_{2i})$ and for any closed set $Z \subset S^1$ there exists a unique rotating wave u_Z with the property

$$Z = \{y \in S^1 : u_Z(y) = u_{2i}\}.$$

The uniqueness automatically proves that these are all waves. Hence, only the connection-question remains.

For this it is convenient to introduce the map $\mathcal{Z}(\cdot)$ that assigns each function $u : S^1 \to \mathbb{R}$ its zero set:

$$\mathcal{Z}(u(\cdot,t)) := \{x \in S^1; u(x,t) = u_0 = 0\}. \tag{2.48}$$

This set plays a key role in understanding which rotating waves are connected to each other when they have the same wave-speed. Note that

$$z(u) = \sharp\mathcal{Z}(u).$$

If $\mathcal{Z}(u)$ is uncountable we define $z(u) = \infty$.

In addition we define

$$u_Z := u \in \mathcal{F}^0 \text{ such that } \mathcal{Z}(u_Z) = Z$$

to be the unique rotating wave with zero set Z.

Fan and Hale were able to show in Theorem 3.7 in [FH95] that if two rotating waves are connected by heteroclinic orbits, then the waves must have the same velocity. Moreover, if heteroclinic orbits connect a homogenous equilibrium $u \equiv u_j$ and a rotating wave with speed $f'(u_{2i})$, then $|j - 2i| = 1$.

On the attractor the zero-number z decays along trajectories, thus is a discrete Lyapunov function, just as in the parabolic setting.

In 1997 Sinestrari was able to prove that a necessary condition for a connection from the rotating wave u_{Z_1} to the wave u_{Z_2} was

$$Z_2 \subset Z_1. \tag{2.49}$$

Härterich was able to show that the above condition was not only necessary but also sufficient. This gives the following picture of the structure of the global attractor of equation (H), summarised in the three Theorems A, B and C in [Haer99]:

Theorem 2.5.10 (Theorem A) *For any rotating wave $u_{-\infty}$ there exist heteroclinic orbits which connect $u_{-\infty}$ to the homogenous states $u \equiv u_-$ and $u \equiv u_+$.*

Theorem 2.5.11 (Theorem B) *For any rotating wave $u_{+\infty}$ there exist (several) heteroclinic orbits that connect the spatially homogenous solution $u \equiv u_0 = 0$ to $u_{+\infty}$.*

Theorem 2.5.12 (Theorem C) *Suppose that for two rotating waves $u_{-\infty}$ and $u_{+\infty}$ the condition $\mathcal{Z}(u_\infty) \subset \mathcal{Z}(u_{-\infty})$ holds. Then there is a heteroclinic solution that approaches $u_{\pm\infty}$ as the time t tends to $\pm\infty$.*

These three Theorems in principle allow a full description of the connection problem on the global attractor. Härterich could even explicitly construct heteroclinic connections in the phase space, however up to now there has been no result on the uniqueness of these connections. The next section will provide a first result in that direction. Furthermore, we will present some examples of how to construct explicitly frozen waves and heteroclinic connections.

We conclude this section by defining the notion of sub-attractors in a similar way to the parabolic setting.

Definition 2.5.13 *Let $n = 2\alpha$ for $\alpha \in \mathbb{N}$. Then we define:*

- $\mathcal{E}_n^0 := \{u \equiv u_+, u \equiv u_-\}$;
- $\mathcal{F}_n^0 := \{u \in \mathcal{F}^0; z(u) \leq \alpha\}$;
- $\mathcal{H}_n^0 := \{u \in \mathcal{H}^0; \lim_{t\to\pm\infty} \in \mathcal{E}_n^0 \cup \mathcal{F}_n^0\}$.

Then we define the sub-attractor of order n of the hyperbolic balance law (H) *by*

$$\mathcal{A}_n^0 := \mathcal{E}_n^0 \cup \mathcal{F}_n^0 \cup \mathcal{H}_n^0. \tag{2.50}$$

Just as in the parabolic setting it is clear that the sub-attractors are contained in each other, hence we have

$$\mathcal{A}_n^0 \subset \mathcal{A}_m^0 \Leftrightarrow n < m.$$

At a first glance it seems strange to denote the hyperbolic sub-attractors by $\mathcal{A}_n^0$ and not $\mathcal{A}_\alpha^0$. However in the next section and in Chapter 3 we will see that this makes a lot of sense. Lemma 3.2.5 will yield that

$$\lim_{\varepsilon \to 0} (\mathcal{R}_n^\varepsilon \cup \mathcal{F}_n^\varepsilon) \subset \mathcal{F}_n^0$$

in the sense of solutions.

By analogy to the description of sub-attractors in the parabolic section we present an alternative representation of $\mathcal{A}_n^0$ in terms of unstable manifolds. With the same argumentation as for equation (2.40) we conclude that

$$\begin{aligned} \mathcal{A}_n^0 &= \bigcup_{\beta=1}^{\alpha} \{W^u(u^0); u^0 \in \mathcal{F}^0, z(u^0) = \beta\} \cup \mathcal{E}_n^0 \\ &= \bigcup_{\beta=1}^{\alpha} \overline{\{W^u(u^0); u^0 \in \mathcal{F}^0, z(u^0) = \beta\}} \\ &= \overline{\{W^u(\mathcal{F}_n^0)\}}. \end{aligned} \tag{2.51}$$

One of the results in the following section will be $\dim \mathcal{A}_n^0 = n$, which justifies the notation.

Definition 2.5.13 explicitly excludes frozen waves v where the zero set $\mathcal{Z}(v)$ includes whole intervals. These solutions do not have a counterpart in the parabolic equation. A rotating wave of the parabolic equation that vanishes on a whole interval has to be identically zero.

The last section of this chapter is devoted to the study of the sub-attractors $\mathcal{A}_n^0$. Theorem 2.6.1 yields results on the parameterisation and the dimension of the $\mathcal{A}_n^0$; moreover, it proves uniqueness of some heteroclinics in $\mathcal{A}_n^0$. Theorem 2.6.1 is the main result of the whole chapter and one of the main results of this dissertation.

2.6 Parameterisations for $\mathcal{A}_n^0$

Before we prove the main theorem of this section, Theorem 2.6.1 we will give two preparational examples: First we construct the unique frozen wave u_Z for a given $Z \subset S^1$, then we will give an explicit representation of a heteroclinic connection between two frozen waves v_1 and v_2.

We begin with the single frozen wave: In Sinestrari's result there are no restrictions regarding the zero set Z except closedness, so it could be finite, countable or uncountable. Even if whole intervals are contained it is still possible to define a rotating wave with this zero set.

As an example of a frozen wave we will construct a frozen wave solution for Burgers equation with a symmetric source term

$$f(u) := \frac{1}{2}u^2 \quad g(u) = u(1-u^2)$$

for the zero set $Z = [\frac{\pi}{2}, \pi] \cup \left\{\frac{3\pi}{2}\right\}$.

The rotating wave equation for $c = 0$ and the above f and g is given by

$$v_x = 1 - v^2$$

which has the fundamental solution $v(x) := \tanh(x - x_0)$.

To give a description of the travelling wave v_Z one just has to use appropriately shifted copies of $v(x)$ on $S^1 \setminus Z$ in a way such that the resulting shocks are stationary according to the Rankine-Hugoniot condition (2.45).

For the above given Z we define

$$v_Z(x) := \begin{cases} \tanh(x - \frac{\pi}{2}) & \text{for} \quad x \in [0, \frac{\pi}{2}] \\ 0 & \text{for} \quad x \in [\frac{\pi}{2}, \pi] \\ \tanh(x - \pi) & \text{for} \quad x \in [\pi, \frac{5}{4}\pi] \\ \tanh(x - \frac{3}{2}\pi) & \text{for} \quad x \in [\frac{5}{4}\pi, 2\pi]. \end{cases}$$

It is an easy exercise to show that this solution is in fact stationary. The uniqueness result guarantees that it is the only rotating wave with this zero set. Figure 2.6 shows a plot of v_Z. We remark that this frozen wave has infinitely many heteroclinic connections to other frozen waves. It has heteroclinic connections to every rotating wave

$$v_{\tilde{Z}}$$

with closed $\tilde{Z} \in \mathbb{P}(Z)$ where $\mathbb{P}(Z)$ is the powerset of Z. All directions are linearly independent and hence the unstable manifold of this wave is already infinite dimensional.

The second example concerns heteroclinic connections between two rotating waves. Again we use Burgers equation together with the source term $g = u(1-u^2)$. We construct the heteroclinic connection between the frozen waves v_{Z_1} and v_{Z_2} with $Z_1 = \{0, \pi\}$ and $Z_2 = \{0\}$.

Härterich could construct in [Haer99] a heteroclinic orbit by a sequence of solutions where the shocks travel along the profile defined by shifted copies of $v(x)$. The shocks travel towards each other such that they form a stationary shock when they collide. We set the collision time at $t = 0$ then the solution can be given in the following closed form:

$$v(x,t) = \begin{cases} \tanh(x - \frac{\pi}{8}) & \text{for} \quad x \in [0, \frac{3\pi}{8} + \Delta] \\ \tanh(x - \frac{5\pi}{8}) & \text{for} \quad x \in [\frac{3\pi}{8} + \Delta, \frac{7\pi}{8} - \Delta] \\ \tanh(x - \frac{7\pi}{8}) & \text{for} \quad x \in [\frac{7\pi}{8} - \Delta, 2\pi] \end{cases} \tag{2.52}$$

where Δ solves the following ODE in backward time

$$\dot{\Delta}(t) = \frac{\tanh(\pi/4 + \Delta) + \tanh(\pi/4 - \Delta)}{\tanh(\pi/4 + \Delta) - \tanh(\pi/4 - \Delta)}$$
$$\Delta(0) = \frac{\pi}{4}.$$

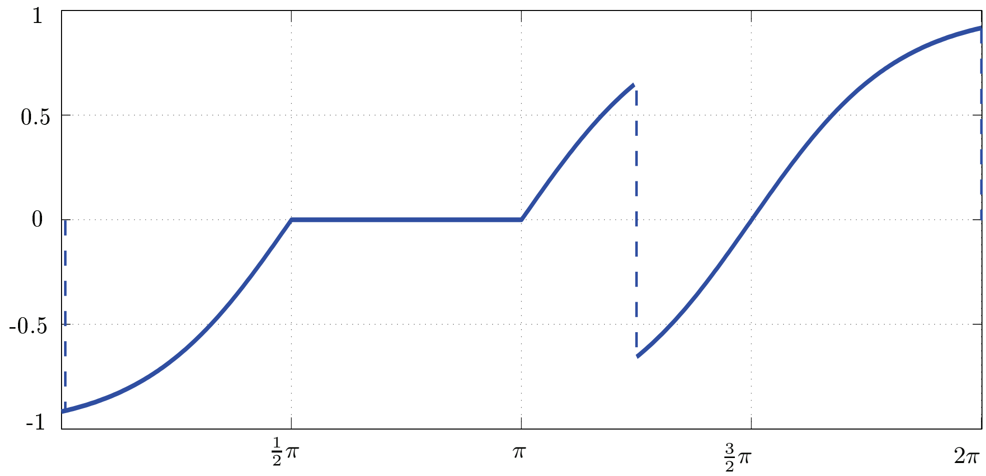

Figure 2.6: Stationary solution v_Z for $Z = [\frac{\pi}{2}, \pi] \cup \left\{\frac{3\pi}{2}\right\}$.

The solution is plotted in Figure 2.6.

We have now seen how heteroclinic connections can be constructed in principle for Burgers equation. A similar construction holds true for the general case.

The rest of this section is devoted to the preparation and proof of Theorem 2.6.1 on the explicit parameterisation of the sub-attractors $\mathcal{A}_n^0$. We first construct the manifold that will represent the local unstable manifold of a frozen wave.

Let $\phi(x)$ be the unique solution of the following equation:

$$\begin{aligned} v_x &= \frac{g(v)}{f'(v)} \\ v(0) &= 0. \end{aligned} \tag{2.53}$$

Furthermore $\phi(x)$ exists for all $x \in \mathbb{R}$ and

$$\lim_{x \to -\infty} \phi(x) = u_- \qquad \lim_{x \to \infty} \phi(x) = u_+.$$

Let $n = 2\alpha$ for some $\alpha \in \mathbb{N}$ be given. Then we choose a sequence of α zeros $0 < x_1 < x_2 < \cdots < x_\alpha < 2\pi$.

Due to Sinestrari there exists a unique frozen wave $v_\alpha^0(x)$ with

$$\mathcal{Z}(v_\alpha^0) = \{x_1, \cdots, x_\alpha\}.$$

Without loss of generality we assume $x_1 = 0$. All other cases can be generated by a shift along S^1.

Note that for every solution of equation (H) it is true that between two zeros there must be a shock and between two shocks with sign changing left- and right-hand states there must be a zero. This is even true in the case where f depends explicitly on x, see [Ehrt05]. It is in particular true for v_α^0. Hence there is a unique sequence of shocks $\hat{y}_1, \ldots, \hat{y}_\alpha$ with

$$x_1 < \hat{y}_1 < x_2 < \hat{y}_2 < \cdots < \hat{y}_{\alpha-1}, x_\alpha < \hat{y}_\alpha < x_1 + 2\pi$$

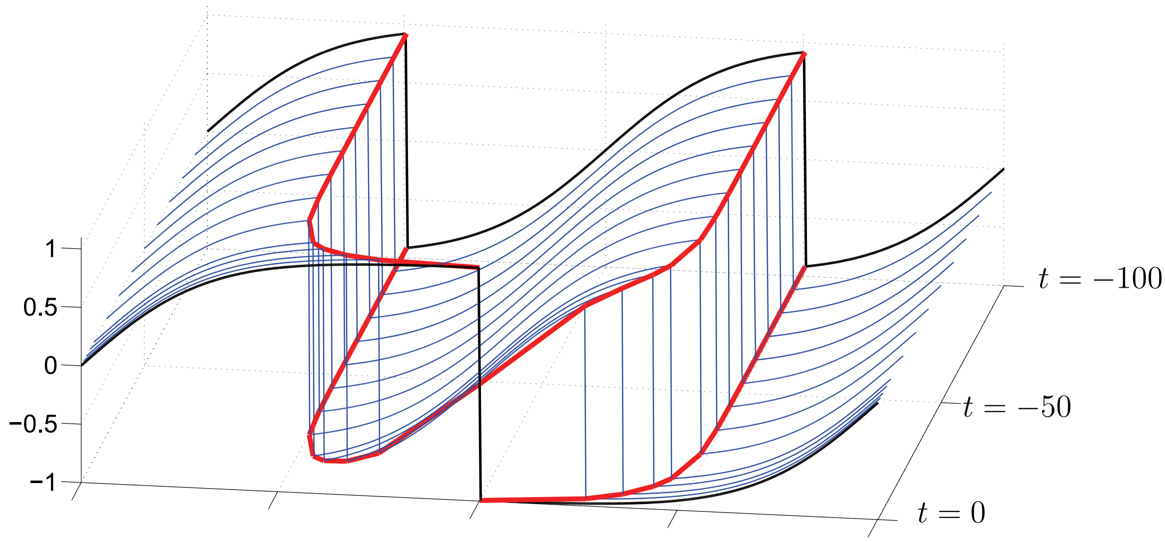

Figure 2.7: Heteroclinic orbit connecting u_{Z_1} and u_{Z_2}. The left and right states of the shocks are indicated in red.

such that v_α^0 is given by

$$v_\alpha^0 = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, \hat{y}_i] \\ \phi(x - x_{i+1}) & \text{for} \quad x \in [\hat{y}_i, x_{i+1}] \end{cases} \tag{2.54}$$

where we have set $x_{\alpha+1} = x_1 + 2\pi = 2\pi$. In case that $\hat{y}_\alpha \geq 2\pi$ we set $\hat{y}_0 := \hat{y}_\alpha(mod 2\pi)$. The sequence then reads $0 \leq \hat{y}_0 < x_1 \ldots$. In the following we will not always make this distinction but just identify $x + 2\pi$ with x without explicitly mentioning this. For convenience let us define the notation

$$\begin{aligned} \mathbf{x}_\alpha :=& \{x_1, \cdots, x_n\} \\ v^0_{\{\mathbf{x}_\alpha\}} :=& v^0_\alpha. \end{aligned}$$

We now define the solution $u_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}}$ with α shocks located between the zeros $\{x_1, \cdots, x_\alpha\}$ that consists piecewise of shifted copies of $\phi(x)$. In general $u_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}}$ is not stationary.
Let $0 \leq x_1 \leq y_1 < x_2 \leq \cdots < x_\alpha \leq y_\alpha < 2\pi$ then we define

$$u_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, y_i] \\ \phi(x - x_{i+1}) & \text{for} \quad x \in [y_i, x_{i+1}] \end{cases} \tag{2.55}$$

for $i = 1, \ldots, \alpha$ and again $x_{\alpha+1} = 2\pi$.

Finally let us define the general solution $\tilde{u}_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}}$ with α or less shocks that consists piecewise of shifted copies of $\phi(x)$ where all shocks are contained in $[0, 2\pi)$.
Let $0 \leq \tilde{y}_1 \leq \tilde{y}_2 \leq \cdots \leq \tilde{y}_\alpha < 2\pi$ then we define if $\tilde{y}_i < \tilde{y}_{i+1}$

$$\tilde{u}_{\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, \tilde{y}_i] \\ \phi(x - x_{i+1}) & \text{for} \quad x \in [\tilde{y}_i, x_{i+1}] \end{cases}, \tag{2.56}$$

and if $\tilde{y}_i = \tilde{y}_{i+1} = \cdots = \tilde{y}_{i+m}$

$$\tilde{u}_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for} \quad x \in [x_i, \tilde{y}_i] \\ \phi(x - x_{i+m+1}) & \text{for} \quad x \in [\tilde{y}_{i+m}, x_{i+m+1}] \end{cases} \tag{2.57}$$

Then the two sets of all these solutions with fixed $\{x_1, \cdots, x_\alpha\}$ are given by

$$A_{\{x_1,\ldots,x_\alpha\}} := \{u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}; 0 \le x_1 \le y_1 < x_2 \le \cdots < x_\alpha \le y_\alpha < 2\pi\} \tag{2.58}$$

and

$$\tilde{A}_{\{x_1,\ldots,x_\alpha\}} := \{\tilde{u}_{\{\mathbf{x}_\alpha,\tilde{\mathbf{y}}_\alpha\}}; 0 \le \tilde{y}_1 \le \cdots \le \tilde{y}_n < 2\pi\}. \tag{2.59}$$

Clearly we have

$$v^0_{\{\mathbf{x}_\alpha\}} \in A_{\{x_1,\ldots,x_\alpha\}} \subseteq \tilde{A}_{\{x_1,\ldots,x_\alpha\}}$$

and we have

$$\tilde{A}_{\{\tilde{x}_1,\ldots,\tilde{x}_\beta\}} \subseteq \tilde{A}_{\{x_1,\ldots,x_\alpha\}}$$

if $\tilde{x}_j \in \{x_1, \ldots, x_\alpha\}$ for all $1 \le j \le \beta$ and $\beta \le \alpha$. Note that there is no solution consisting piecewise of $\phi(x - x_j)$ for $x_j \in \{x_1, \ldots, x_\alpha\}$ that has more than α shocks. We argue as follows: We assume that the solution has a zero located at $x_1 = 0$ and another zero at x_2. Now we explicitly construct the set of all admissible solutions $u(x)$ that consist piecewise of shifted copies of $\phi(x - x_i - 2\pi k_j)$ for some $k_j \in \mathbb{Z}$ and $i \in \{1, 2\}$; with the additional property that $u(x_1 = 0) = 0$. Due to the monotonicity of ϕ it is clear that u possesses at least one shock.
Let us denote all shock positions by $0 < y_1 < \cdots < y_\beta \le 2\pi$. By construction we will see that $\beta \le 2$. Figure 2.6 illustrates the construction: all shocks are down shocks, therefore u must consist of sequences given by

$$...\phi(x + x_1), \phi(x), \phi(x - x_1), \phi(x - 2\pi), \phi(x - x_1 - 2\pi), \phi(x - 4\pi)... \tag{2.60}$$

Because $u(0) = 0$ we start at $x = 0$ with $u(x) = \phi(x)$ locally. At the first shock we can only jump to a solution $\phi(\cdot)$ that lies to the right of $\phi(\cdot)$ in the sequence in equation (2.60) without violating the entropy condition (2.42). This applies to all following shocks. In order to obtain a solution on S^1 we have to end at $x = 2\pi$ with $\phi(x - 2\pi)$. Hence we can jump twice at most. It is clear that the same argument works for arbitrary $\alpha \in \mathbb{N}$. The same applies to solutions that do not have a zero at all.

Let us now state the main theorem:

Theorem 2.6.1 *Let $n = 2\alpha$ and $\alpha \in \mathbb{N}$. Then the following is true:*

a) The local unstable manifold $W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}})$ of $v^0_{\{\mathbf{x}_\alpha\}}$ is given by $A_{\{x_1,\ldots,x_\alpha\}}$ defined in equation (2.58)*:*

$$W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}}) = A_{\{x_1,\ldots,x_\alpha\}} \tag{2.61}$$

where $v^0_{\{\mathbf{x}_\alpha\}}$ is the unique frozen wave of equation (H) *with zeros at $\{x_1, \ldots, x_\alpha\}$.*

b) The global unstable manifold $W^u(v^0_{\{\mathbf{x}_\alpha\}})$ of $v^0_{\{\mathbf{x}_\alpha\}}$ is then given by

$$W^u(v^0_{\{\mathbf{x}_\alpha\}}) = \left\{\Phi^0(u, t); u \in A_{\{x_1,\ldots,x_\alpha\}}, t \in \mathbb{R}^+\right\}. \tag{2.62}$$

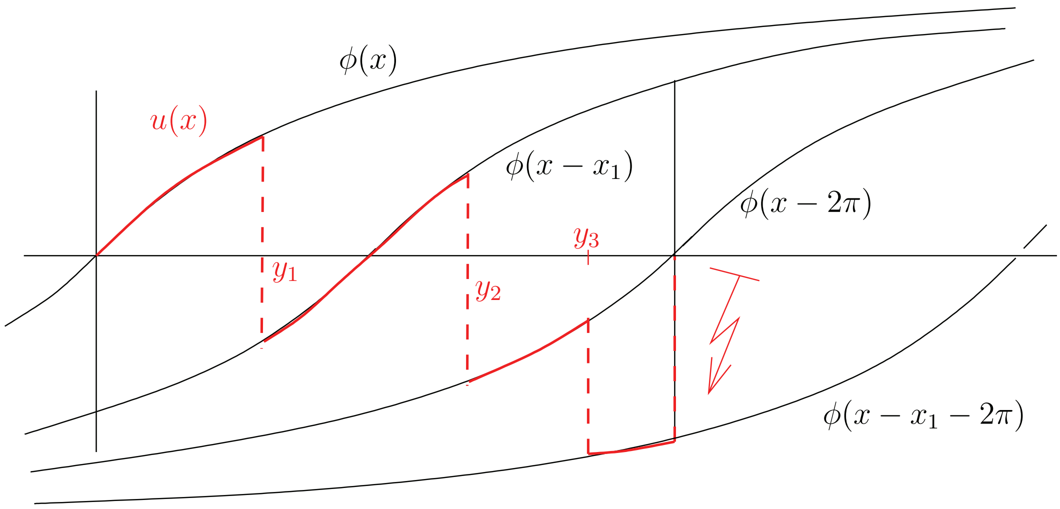

Figure 2.8: Schematic plot showing why a construction of $u_{\{\mathbf{x_2},\mathbf{y_3}\}}$ fails.

c) The semiflow on $\tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ defined in equation (2.59) can be described by the following equation for the shock parameters y_j:

$$\dot{y}_j(t) = \frac{f(\phi(y_j - x_j)) - f(\phi(y_j - x_{j+1})}{\phi(y_j - x_j) - \phi(y_j - x_{j+1})}. \tag{2.63}$$

d) The dimension of the sub-attractors $\mathcal{A}_n^0$ of order n is given by

$$\dim \mathcal{A}_n^0 = n.$$

e) Let v_1 be a frozen wave of equation (H) with

$$z(v_1) = 1.$$

Then there exist unique heteroclinic connections $\tilde{u}(x,t)$ and $\hat{u}(x,t)$ with

$$\lim_{t\to-\infty} \tilde{u}(\cdot,t) = \lim_{t\to-\infty} \hat{u}(\cdot,t) = v_1$$
$$\lim_{t\to\infty} \hat{u}(\cdot,t) = u_-$$
$$\lim_{t\to\infty} \tilde{u}(\cdot,t) = u_+.$$

f) Let $0 \le x_1 < x_2 < \cdots, x_\alpha < 2\pi$ and let v_1 and v_2 be frozen waves of equation (H) with the property

$$\mathcal{Z}(v_1) = \{x_1, \ldots, x_\alpha\}$$

and

$$\mathcal{Z}(v_2) = \{x_{k_1}, \ldots, x_{k_\beta}\}$$

where $k_{i+1} - k_i \in \{0,1\}$ for all $1 \le i \le \beta - 1$. Then there exists a unique heteroclinic connection $u(x,t)$ with the property

$$\lim_{t\to-\infty} u(\cdot,t) = v_1(\cdot)$$
$$\lim_{t\to\infty} u(\cdot,t) = v_2(\cdot).$$

Before we prove the theorem, we prove a important lemma:

Lemma 2.6.2 *Let $\{\mathbf{x}_\alpha\} := \{x_1, \ldots, x_\alpha\}$ with $0 \leq x_1 < \cdots < x_\alpha < 2\pi$ be given.*

(i) The set $\tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ is overflowing invariant under the semiflow of equation (H). *Overflowing means that if a solution $u_{\{\mathbf{x}_\alpha,\mathbf{y}\alpha\}} \in \tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ leaves $\tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ at time $\tilde{t} = 0$ then either $y_1 = x_1$ or $y_\alpha = x_1 + 2\pi$ in $u_{\{\mathbf{x}_\alpha,\mathbf{y}\alpha\}}$.*

(ii) The set $A_{\{x_1,\ldots,x_\alpha\}}$ is overflowing invariant under the semiflow of equation (H).

Proof. Let $u(x,0) \in \tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ such that $u(x,0) = u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}$ with $y_1 > 0$ and $y_\alpha < 2\pi$. Again we assume $x_1 = 0$.

Local forward invariance of $\tilde{A}_{\{x_1,\cdots,x_n\}}$ follows from the fact that the profiles ϕ that define $u_{\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}}$ are stationary. Hence u is stationary except near the points y_j, and so we only have to prove invariance locally at the shock points. Without loss of generality we only investigate the shock located at y_1.

Therefore let $u(x,0)$ be given by

$$u(x,0) = \begin{cases} \phi(x) & \text{for} \quad x < y_1 \\ \phi(x - x_2) & \text{for} \quad x > y_1 \end{cases} \tag{2.64}$$

At y_1 there is a unique forward characteristic $\chi(t)$ on which the shock evolves. It can be obtained by integrating the Rankine-Hugoniot condition 2.45. The other characteristics necessarily point towards $\chi(t)$ for $t > 0$. So to the left and right of χ the solution $u(x,t)$ must be stationary and is given by $\phi(x)$ for $x \leq \chi(t)$ and by $\phi(x - x_2)$ for $x > \chi(t)$. See the Figure 2.6 for illustration. $\chi(t)$ is uniquely determined by the differential equation:

$$\begin{aligned} \dot{\chi}(t) &= \frac{f(\phi(\chi(t)-)) - f(\phi((\chi(t) - x_2)+))}{\phi(\chi(t)-) - \phi((\chi(t) - x_2)+)} \\ \chi(0) &= y_0. \end{aligned} \tag{2.65}$$

The slope of $\chi(t)$ is bounded from above and hence, if t is sufficiently small we have obtained local forward invariance of the shock.

For the backward invariance we observe that a minimal characteristic $\chi^-(t)$ and a maximal backward characteristic $\chi^+(t)$ for $t < 0$ emanate from y_1. For the area between χ^- and χ^+ there are in principle many possibilities to define the solution such that we obtain $u(x,t)$ for $t \geq 0$. For backward invariance it is enough if we find one $u(x,t) \in \tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ for $t < 0$ with this property.

Let therefore $\chi(t)$ be the unique shock characteristic emanating from y_1 and hence

$$u(x,t) = \begin{cases} \phi(x) & \text{for} \quad x < y_1 \\ \phi(x - x_2) & \text{for} \quad x > y_1 \end{cases}$$

for $t \geq 0$.

Let now $t_0 < 0$ be given. Then we define

$$\tilde{u}(x,t_0) := \begin{cases} \phi(x) & \text{for} \quad x \in [\chi^-(t_0), \tilde{\chi}(t_0)) \\ \phi(x - x_2) & \text{for} \quad x \in (\tilde{\chi}(t_0), \chi^+(t_0)] \end{cases}$$

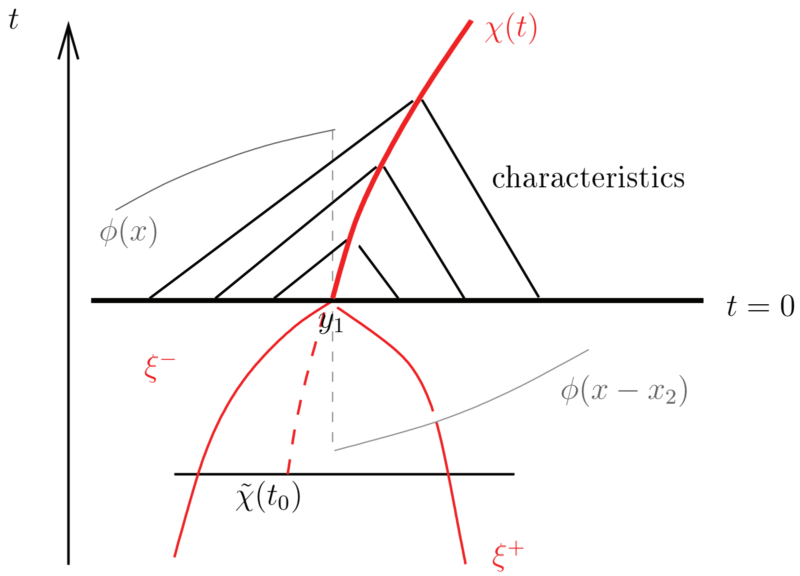

Figure 2.9: Illustration for the proof of Lemma 2.6.2.

for some $\tilde\chi(t_0) \in [\chi^-(t_0), \chi^+(t_0)]$. Local backward invariance follows if we can prove that there is exactly one $\tilde\chi(t_0)$ such that if we solve equation 2.65 with initial condition $\tilde\chi(t_0)$ we obtain

$$\tilde\chi(0) = \chi(0) = y_0.$$

Uniqueness of $\tilde\chi(t_0)$ is clear because the convexity of f implies together with the monotonicity of ϕ monotonicity of $\dot\chi$. Hence, this implies that $\tilde\chi(0)$ depends monotonically on $\tilde\chi(t_0)$, which implies uniqueness of $\tilde\chi(t_0)$ due to uniqueness of $\tilde\chi(0)$. Hence backward invariance follows.

As far as uniqueness is concerned, the backward solution is not unique in $\tilde A_{\{x_1,\ldots,x_\alpha\}}$ in general, due to the possibility of shock splittings in backward time direction. But it is clear that if we assume that this does not happen, we obtain uniqueness of the backward solution in $\tilde A_{\{x_1,\ldots,x_\alpha\}}$.

For the overflowing property we assume $u(x,0) \in \tilde A_{\{x_1,\ldots,x_\alpha\}}$ with $y_1 = 0$. Then the forward characteristic $\chi(t)$ in $x = y_1 = 0$ is given by the equation

$$\chi(t) = \frac{-f(\phi(y_\alpha - 2\pi)}{-\phi(y_\alpha - 2\pi)} < 0$$

for $t \in [0,\delta)$, δ positive and small and $\chi(0) = 2\pi$. Thus, after identification of 0 and 2π we obtain that the solution to be locally given by

$$\begin{array}{lll} \phi(x) & \text{for} & 0 < x < y_2 \\ \phi(x-2\pi) & \text{for} & \chi(t) < x < 2\pi \\ \phi(x-y_\alpha) & \text{for} & y_\alpha < x < \chi(t). \end{array}$$

In the case of $y_1 = y_2 = 0$ we have to replace y_2 by y_3 in the first line. If there is only one shock, we can drop the last line and replace y_2 with y_α in the first line.

This proves the overflowing property of $\tilde A_{\{x_1,\ldots,x_\alpha\}}$.

Due to the fact that $A_{\{x_1,\ldots,x_\alpha\}} \subset \tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ we conclude invariance of $A_{\{x_1,\ldots,x_\alpha\}}$ by virtue of the same construction immediately. The overflowing property works just as for $\tilde{A}_{\{x_1,\ldots,x_\alpha\}}$, here the boundary is given by the condition $y_j = x_j$ or $y_j = x_{j+1}$ for some $j \in \{1,\ldots,\alpha\}$.

□

Corollary 2.6.3 *For every $u(x,0) \in A_{\{x_1,\ldots,x_\alpha\}}$ there is a unique backward orbit in $A_{\{x_1,\ldots,x_\alpha\}}$.*

Proof. From the proof of the previous lemma we deduce that it is sufficient to show that shocks in u cannot split in backward time. This is clear by construction because any solution in $A_{\{x_1,\ldots,x_\alpha\}}$ has exactly α zeros and α shocks.

□

Proof of Theorem 2.6.1

In fact, we have already proven part **c)**. Equation (2.65) yields exactly equation (2.63) if we replace $\chi(t)\pm$ by the y_j. Hence we can integrate solutions along the (invariant) manifold $A_{\{x_1,\ldots,x_\alpha\}}$ by using equation (2.63) for every y_j $(1 \leq j \leq n)$. Note that y_j and y_{j+1} can meet. Thus y_j is only lipschitz not C^1.

For **a)** we prove that all solutions $u(\cdot,0) \in A_{\{x_1,\ldots,x_\alpha\}}$ converge in backward time to $v_{\{\mathbf{x}_\alpha\}}$, this shows

$$A_{\{x_1,\ldots,x_\alpha\}} \subset W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}}). \tag{2.66}$$

Then we show maximality of $A_{\{x_1,\ldots,x_\alpha\}}$ in the sense that all solutions $u(\cdot,t)$ converging to $v^0_{\{\mathbf{x}_\alpha\}}$ are contained in $A_{\{x_1,\ldots,x_\alpha\}}$ for sufficiently small $t < 0$ which proves

$$W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}}) \subset A_{\{x_1,\ldots,x_\alpha\}}. \tag{2.67}$$

The first part is a consequence of Lemma 2.6.2 and the convexity of f. Now we assume $u(\cdot,0) \in A_{\{x_1,\ldots,x_\alpha\}}$. Because of the overflowing invariance and backward uniqueness we conclude

$$u(\cdot,t) \in A_{\{x_1,\ldots,x_\alpha\}}$$

for all $t < 0$. In addition

$$\lim_{t\to-\infty} u(\cdot,t) \in \mathcal{F}^0 \cup \mathcal{E}^0$$

because this is true for all solutions that are globally bounded in backward time. Taking into account that $v^0_{\{\mathbf{x}_\alpha\}}$ is the only frozen wave in $A_{\{x_1,\ldots,x_\alpha\}}$ and hence $A_{\{x_1,\ldots,x_\alpha\}} \cap \mathcal{E}^0 \cup \mathcal{F}^0 = \{v^0_{\{\mathbf{x}_\alpha\}}\}$ we have obtained equation (2.66).

For the other direction we argue indirectly. Assume that there exists $u(x,t)$ with

$$u(x,t) \notin A_{\{x_1\ldots x_\alpha\}} \text{ for all } t<0 \text{ and } \lim_{t\to-\infty} u(x,t) = v^0_{\{\mathbf{x}_\alpha\}}.$$

Then for sufficiently small $\tilde{t} < 0$ there must be a $\tilde{x} \in S^1$ such that for all $1 \leq j \leq \alpha+1$

$$u(\tilde{x},\tilde{t}) \neq \phi(\tilde{x} - x_j). \tag{2.68}$$

Due to the fact that u connects to $v^0_{\{\mathbf{x}_\alpha\}}$ we can always choose $(\tilde{x},\tilde{t})$ such that $\tilde{u}(\tilde{x},\tilde{t})$ is smaller than the maximum and larger than the minimum of the stationary solution with one zero.

We now prove that $\tilde{u}$ cannot converge to $v^0_{\{\mathbf{x}_\alpha\}}$ in backward time which will yield the result: the idea is to use a stationary solution to calculate the backward characteristic of u that starts in $(\tilde{x}, \tilde{t})$ and thereby construct a contradiction.
Assuming equation (2.68) holds, then there is a stationary solution $u_s \in \mathcal{F}^0$ with the following properties:

$$
\begin{aligned}
u_s(\tilde{x}) &= u(\tilde{x}, \tilde{t}) \\
\mathcal{Z}(u_s) &= \{x_s\}
\end{aligned}
$$

where $x_s \notin \{x_1, \ldots, x_\alpha\}$.
We investigate the (genuine!) backward characteristic $(\chi(t), \underline{\mathrm{v}}(t))$ with

$$\chi(\tilde{t}) = \tilde{x} \text{ and } \underline{\mathrm{v}}(\tilde{t}) = u_s(\tilde{x}, \tilde{t}) = \tilde{u}(\tilde{x}, \tilde{t}).$$

Because u_s is stationary, the characteristic has the property that

$$\lim_{t \to -\infty} \chi(t) = x_s$$

and

$$\lim_{t \to -\infty} \underline{\mathrm{v}}(t) = 0.$$

Then

$$\lim_{t \to -\infty} u(x_s, t) = \lim_{t \to -\infty} u_s(x_s, t) = 0.$$

This contradicts $\lim_{t \to -\infty} u(\cdot, t) = v^0_{\{\mathbf{x}_\alpha\}}$ because $v^0_{\{\mathbf{x}_\alpha\}}(x_s) \neq 0$.
This yields the maximality of $A_{\{x_1, \ldots, x_\alpha\}}$ (equation (2.67)) and hence **a)** follows.

b) follows from the fact that due to unique forward solvability we obtain the global unstable manifold by using the semiflow to forward–solve the local unstable manifold. Dissipativity, or the fact that $A_{\{x_1, \ldots, x_\alpha\}} \subset \mathcal{A}^0$ ensures boundedness of the forward iteration, hence equation (2.62) follows.

For **d)** we use the fact that

$$\dim \left(W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}}) \right\} = \dim \left(W^u(v^0_{\{\mathbf{x}_\alpha\}}) \right\} \tag{2.69}$$

which is true due to forward uniqueness of solutions.

For $n = 2\alpha = 2$ the sub-attractor of order $n = 2$ consists of all frozen waves with one zero and heteroclinic connections to $u_\pm$. In other words

$$\mathcal{A}^0_2 = W^u_{loc}(\mathcal{F}^0_2) \cup \mathcal{E}^0_n$$

For fixed x_1 we have

$$\dim \left(W^u_{loc}(v^0_{\{x_1\}} \right) = 1.$$

From the uniqueness of frozen waves with given $x_1 \in S^1$ we deduce

$$\dim \mathcal{A}^0_2 = 2$$

For $n = 2\alpha > 2$ we use

$$\mathcal{A}^0_n = \left\{ W^u(u); u \in \mathcal{F}^0_n \right\} \cup \mathcal{E}^0_n. \tag{2.70}$$

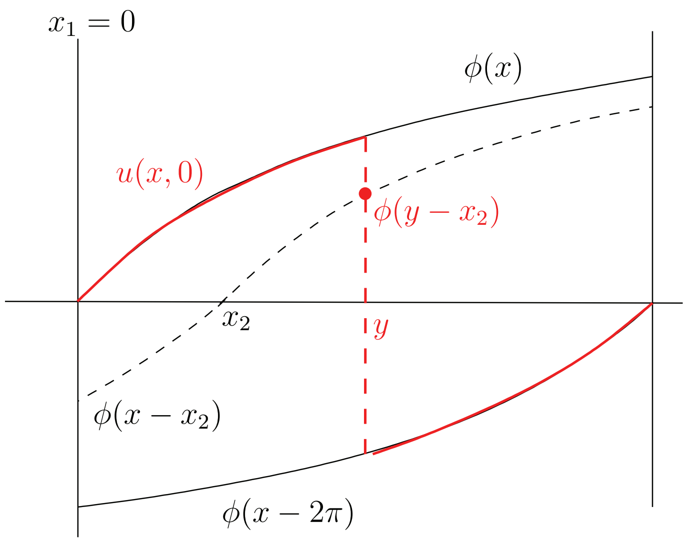

Figure 2.10: Unique shock-splitting of one shock in backward time in $A_{\{x_1,x_2\}}$.

First we prove

$$\dim\left\{W^u(u); u \in \mathcal{F}^0, z(u) = \alpha\right\} = 2\alpha = n.$$

For each fixed set of zeros $\{0 \leq x_1 < \cdots < x_\alpha < 2\pi\}$ we have by part a) of this theorem

$$\dim\left(W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}})\right) = \dim\left(A_{\{x_1,\ldots,x_\alpha\}}\right) = \alpha.$$

Moreover, all frozen waves v with zero-number $z(v) \leq \alpha$ can be parameterised by $(x_1, \ldots x_\alpha) \in \left(S^1\right)^\alpha = \mathbb{T}^\alpha$, hence

$$\dim \mathcal{F}^0_n = \dim \mathbb{T}^\alpha = \alpha.$$

Putting everything together we obtain by using equation (2.70)

$$\dim \mathcal{A}^0_n = \dim W^u_{loc}(\{\mathcal{F}^0_n\} = \dim W^u_{loc}(v^0_{\{\mathbf{x}_\alpha\}}) + \dim \mathbb{T}^\alpha = \alpha + \alpha = n$$

Hence **d)** is proven.

For **e)** we count dimensions to obtain uniqueness. For $\alpha = 1$ the unstable manifold of v_1 is one dimensional, thus the connection must be unique.

For **f)** we argue in the following way: the condition $k_{i+1} - k_i \in \{0, 1\}$ implies that at most every second zero can vanish, hence we can reduce the proof to the situation where

$$\mathcal{Z}(v_1) = \{0, x_2\}$$

and

$$\mathcal{Z}(v_2) = \{0\}.$$

Let us denote the unique shock position of v_2 by y and the two unique shock positions of v_1 by y_1, y_2.

It is a consequence of **c)** that in the class of solutions $A_{\{x_1,x_2\}}$ all stationary shocks are unstable. In order to obtain the solution v_2 with only one shock, the two shocks emanating form y_1 and y_2 consequently have to meet at the position y in such a way that the resulting shock is stationary.

We define $t = 0$ as the time at which the two shocks collide. So the question of uniqueness of heteroclinic connections reduces to the question of uniqueness of shock collisions in $A_{\{x_1,x_2\}}$, or in negative time direction the questions of uniqueness of the splitting of shocks at a given position; but this is clear.

Let $u(x,t)$ be the solution where two shocks meet at time $t = 0$ at position $x = y$ then the lower state of the left shock and the upper state of the right shock have to have the same value. By construction of $\tilde{A}_{\{x_1,x_2\}}$ it must be given by $\phi(y - x_2)$:

$$\lim_{x \searrow y} \lim_{t \nearrow 0} u(x,t) = \lim_{x \nearrow y} \lim_{t \nearrow 0} u(x,t) \stackrel{!}{=} \phi(y - x_2).$$

See Figure 2.10 for illustration. Note that the two limits in x and t are not interchangeable.

Hence uniqueness of the splitting follows by uniqueness of backward solutions in the case of $u \in A_{\{x_1,x_2\}}$ with two shocks. This proves **e)** and the Theorem is proven.

□

Note that for the situation of Theorem 2.6.1 e) we can explicitly parameterise the whole heteroclinic connection from v_1 to $u_\pm$. The stationary solution v_1 with $\mathcal{Z}(v_1) = \{x_1\}$ has one unique shock at position y_1. Then using Theorem 2.6.1 b) and c) we can parameterise the whole connection manifold $W^u(v_1)$ as follows: for any $k \in \mathbb{Z}$ and any $y_1 \in [2k\pi, 2(k+1)\pi)$ we define

$$u_{\{x_1,y_1\}}(x) := \begin{cases} \phi(x - x_1 + 2k\pi) & \text{for} \quad 0 \leq x \leq y_1 - 2k\pi \\ \phi(x - x_1 + 2(k-1)\pi) & \text{for} \quad 2\pi > x > y_1 - 2k\pi \end{cases}. \tag{2.71}$$

Then $W^u(v_1)$ is given by

$$W^u(v_1) := \{u_{\{x_1,y_1\}} \in BV; y_1 \in \mathbb{R}\}. \tag{2.72}$$

Corollary 2.6.4 *Again let $\alpha \in \mathbb{N}$ and $n = 2\alpha$. Then the set of heteroclinic connections between two frozen waves with zero-number $z \leq \alpha$ is completely contained in*

$$\tilde{\mathbf{A}}_{\mathbf{n}} := \left\{ \tilde{A}_{\{x_1,\ldots,x_\alpha\}}; 0 \leq x_1 < \cdots < x_\alpha < 2\pi \right\}. \tag{2.73}$$

Proof Let v_1, v_2 be two frozen waves with

$$\mathcal{Z}(v_1) = \{x_1, \ldots, x_\beta\}$$
$$\mathcal{Z}(v_2) \subset \mathcal{Z}(v_1)$$

for some given $0 \leq x_1 < \cdots, x_\beta < 2\pi$ and $\beta \leq \alpha$. Let $u(x,t)$ denote a heteroclinic connection between v_1 and v_2. Then

$$u(\cdot,t) \in A_{\{x_1,\ldots,x_\beta\}} \subset \tilde{A}_{\{x_1,\ldots,x_\beta\}} \subset \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

for some $x_{\beta+1}, \ldots, x_\alpha$ and t sufficiently small.

Now assume $u(\cdot,\tilde{t}) \notin \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$ for some $\tilde{t} \in \mathbb{R}$. Then we conclude

$$u(\cdot,t) \notin \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

for all $t > \tilde{t}$ due to the overflowing property of $\tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$.
This contradicts

$$\lim_{t\to\infty} u(\cdot, t) = v_2$$

because

$$v_2 \in \mathring{\tilde{A}}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

where $\mathring{\tilde{A}}$ denotes the interior of $\tilde{A}$ in the topology of the manifold $\tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$.

□

Theorem 2.6.1 and Corollary 2.6.4 are quite remarkable. They do not only provide a full parameterisation of the unstable manifolds of the $\mathcal{F}_n^0$, they also suggest that in analogy to the parabolic equation it is possible to define a Morse index i_h as the number of unstable eigendirections of a frozen wave.
In the hyperbolic setting, too there is the relation between zeros and the index but here it would be given by

$$i_h(u) = z(u). \tag{2.74}$$

This shows that for our purpose of comparing solutions for $\varepsilon > 0$ and $\varepsilon = 0$ sub-attractors $\mathcal{A}_n^\varepsilon$ and $\mathcal{A}_n^0$ are a good tools.

However, it is important to note that the connection properties of both sub-attractors are remarkably different. An example of how $\mathcal{A}_8^\varepsilon$ looks like has already been given in Figure 2.5. The situation for the $\mathcal{A}_n^0$ is far more complicated. Just from the definition of $\mathcal{A}_n^0$ and Theorem 2.6.1 we can conclude that $\mathcal{A}_n^0$ consists of a α-torus $\mathbb{T}^\alpha$ of frozen waves plus heteroclinics. Every point on this torus has heteroclinics to a sub-torus $\mathbb{T}^\beta$ and the number of connections in each point on $\mathbb{T}^\alpha$ is given by $\binom{\alpha}{\beta}$. In addition there are connections to $u_\pm$.

Surprisingly, Corollary 2.6.4 yields that all the heteroclinics that connect from the α-torus back to a sub-torus are contained in the set of all $\tilde{A}_{\{x_1,\ldots,x_\alpha\}}$ for given $\alpha \in \mathbb{N}$, which was denoted by $\tilde{\mathbf{A}}_\mathbf{n}$.

We will investigate this in Chapter 4 in greater detail for the low dimensional cases.

In the following Chapter 3 we will address the question whether the attractor of the parabolic equation converges to the attractor of the hyperbolic equation or, more precisely, whether every solution in $\mathcal{A}_n^\varepsilon$ has a counterpart in $\mathcal{A}_n^0$. The discussion above suggests that this is not the case, because although equation (2.74) shows similarities with equation (2.29) in the limiting process, every second zero in a rotating or frozen wave u^ε vanishes (see Remark 3.2.6). Hence for $\alpha \neq 1$ the number of unstable dimensions of the frozen and rotating waves in the parabolic and hyperbolic setting do not match.

Chapter 3

Persistence or non-persistence?

This chapter is devoted to the question of whether solutions on the global attractor of the parabolic equation (P) persist for $\varepsilon \to 0$ or not. In other words, the guiding question of this Chapter is whether

$$\lim_{\varepsilon \to 0} u^\varepsilon(\cdot, t) = u^0(\cdot, t)$$

for $u^\varepsilon \in \mathcal{A}^\varepsilon$ and some $u^0 \in \mathcal{A}^0$. We have already determined in the introduction that this is one of the aspects of the question whether$\mathcal{A}^\varepsilon$ converges to $\mathcal{A}^0$ for vanishing ε.

The main persistence results root in a result by Fan and Hale [FH95] on the persistence of heteroclinic orbits. We will present their theorem in the first section although it is in fact wrong. However, most of the proof is correct and delivers one of our key claims: pointwise convergence of solutions.
The corrected result will be presented in the beginning of Section 3.2. It is the starting point for the proof of the main results of this Chapter:

- The persistence result for rotating waves (Theorem 3.2.5).
- The *Connection Lemma* 3.2.8 that yields that whereas some heteroclinic connections persist, others do not.
- The *Cascading Theorem* 3.2.9, which states that in case a heteroclinic does not persist, it converges to a cascade of heteroclinic solutions and frozen waves.

Section 3.3 then addresses the question of which connections on the parabolic attractor do not persist. We prove the surprising result that persistence of heteroclinics and cascading appears for every choice of f and g as long as (H1)-(H3) are satisfied.

3.1 The result of Fan and Hale

The latest, most important and outstanding result on the question of persistence of heteroclinic orbits on the parabolic attractor so far is the result of Fan and Hale from 1995. In [FH95] Fan and Hale address the question of viscous regularisations of the hyperbolic equation.

In the first part of the paper they investigate the connection problem of the global attractor of the hyperbolic equation. These results were already presented in Section 2.5. In the second part of the publication Fan and Hale investigate the regularised equation, which is precisely our equation (P).

In Theorem 4.7 they state a persistence result for heteroclinic connections within this framework. Their theorem reads:

Theorem 3.1.1 *If $B = \{u^\varepsilon(x,t), 0 < \varepsilon \le \varepsilon_0\}$ is a set of connecting (heteroclinic) orbits of the parabolic equation* (P)*, then there is a sub-sequence $\{u^{\varepsilon_n}(x,t)\}$ of B converging to $u^0(x,t)$ as $\varepsilon \to 0$ a.e. in $S^1 \times \mathbb{R}$ where $u^0(x,t)$ is a connecting orbit of the hyperbolic equation* (H).

Unfortunately this theorem is wrong. The claim that convergence is a.e. on $S^1 \times \mathbb{R}$ is not true. As a result of this the limiting solution u_0 is not necessarily a heteroclinic connection.

Taking a closer look at the proof of their theorem one realises that it is almost completely correct. Only their conclusion using a diagonalising sequence argument at the very end of the proof is wrong. This argument does not work here. And it is not solely the argument that is wrong. We will see that in fact the claim is wrong as well!

We will see that the limit of a heteroclinic connection is a global solution, however, this does not imply that this global solution is a heteroclinic connection as one might expect. An additional complication is the fact that the limiting object in general depends on how the heteroclinic orbits $u^\varepsilon(\cdot,t)$ are parameterised in t and how sub-sequences are chosen. This means in general

$$\left|\Phi^0\left(\lim_{\varepsilon\to 0} u^\varepsilon(x,t), \tau\right) - u^\varepsilon(x,t+\tau)\right|$$

is not necessarily small for small $\varepsilon > 0$ (see page 59). The reason for this is that the limit is only pointwise on compact intervals $[-T,T]$, but not uniform.

In general the limit of a heteroclinic connection of the parabolic equation limits to a set of solutions of the hyperbolic equation. This set in fact is a subset of the global attractor of (H).

If we look at the dimensionality of the sub-attractors introduced in the last chapter, we see that $\dim \mathcal{A}_n^0 = \dim \mathcal{A}_n^\varepsilon$. However in $\mathcal{A}_n^0$ half of the dimensions consist of frozen waves whereas in $\mathcal{A}_n^\varepsilon$ the set of rotating waves is one dimensional. This already suggests that persistence of heteroclinics might fail just for dimensional reasons. We will see that the situation is even worse than that.

3.2 Cascade of heteroclinics

Let us state the corrected result of Fan and Hale first:

Theorem 3.2.1 (Global Solution) *Let $B := \{u^\varepsilon(x,t) \in \mathcal{H}^\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ for some $\varepsilon_0 <<$ 1. Then there exists a subset $\{u^{\varepsilon_n}(x,t)\}$ of B with the property that*

$$\lim_{n\to\infty} u^{\varepsilon_n}(x,t) = u^0(x,t)$$

a.e. on S^1 for all $t \in [-T,T]$. Moreover $u^0(x,t)$ is a global solution of equation (H).

For a better readability I include a full proof of the theorem. It closely follows the one in the paper of Fan and Hale [FH95], pages 1251-1253; but I have included some additional explanations and references. The theorem is proved by using the method of compensated compactness, which was developed in the 70s by Murat and Tartar see for example [Mu78] and [Tar79] and references therein. The theorems of Functional Analysis quoted in the proof can be found for example in the book of Werner [Wer].

The proof uses the **Div-Curl Lemma** by Murat [Mu78]:

Lemma 3.2.2 (Div-Curl-Lemma) *Assume that $\{v_k\}, \{w_k\}$ are two bounded sequences in $L^2(U, \mathbb{R}^n)$ where $U \subset \mathbb{R}^n$, such that*

(i) $\{div\ v_k\}$ is compact in $W^{-1,2}(U; \mathbb{R})$,

(ii) $\{curl\ w_k\}$ is compact in $W^{-1,2}(U; \mathbb{R}^{n\times n})$.

If $v_k \rightharpoonup v$ and $w_k \rightharpoonup w$ in $L^2(U, \mathbb{R}^n)$, then $v_k \cdot w_k \to v \cdot w$ in the sense of distributions.

Proof of Theorem 3.2.1 (Global Solution):
Due to the maximum principle and hypothesis (H3) all $u^\varepsilon \in B$ are globally bounded in L^∞. Hence there exists a sub-sequence in B denoted by $\{u^\varepsilon\}$ again and a function $u \in L^\infty(S^1 \times \mathbb{R}, \mathbb{R})$ such that

$$u^\varepsilon \overset{w*}{\rightharpoonup} u(x,t) \text{ in } L^\infty(S^1 \times \mathbb{R}, \mathbb{R}) \tag{3.1}$$

This is a direct consequence of the Theorem of Alaoglu-Banach in its sequential form, which states that the unit ball in the dual space of a vector space X is weak* sequentially compact if X is separable.

We use $X = L^1$ which is separable and thus $X' = L^\infty$ and obtain that there exists a sub-sequence $\{u^\varepsilon\}$ and a function $u(x,t)$ as claimed above such that

$$\int_{S^1\times\mathbb{R}} [u^\varepsilon(x,t) - u(x,t)]\, \varphi \mathrm{d}x\mathrm{d}t \longrightarrow 0 \quad \text{for } \varepsilon \to 0$$

for all $\varphi \in C^\infty(S^1 \times \mathbb{R}, \mathbb{R})$. Note that it is sufficient to test with smooth functions because C^∞ is dense in L^1.

Now let $g \in C(\mathbb{R})$ be arbitrary. Due to the global boundedness of the solutions in B we have by virtue of the same argument

$$g(u^\varepsilon(x,t)) \overset{w*}{\rightharpoonup} g(u(x,t)) := \bar{g}(x,t) \text{ in } L^\infty(S^1 \times \mathbb{R}, \mathbb{R}).$$

Then there is a family of Borel probability measures

$$\{\nu_{x,t} : (x,t) \in S^1 \times \mathbb{R}\},$$

such that we have the following representation:

$$g(u^\varepsilon(x,t)) \overset{w*}{\rightharpoonup} g(u(x,t)) := \bar{g}(x,t) \equiv \int_{\mathbb{R}} g(\lambda) \mathrm{d}\nu_{x,t}(\lambda) \tag{3.2}$$

in $L^\infty(S^1 \times \mathbb{R}, \mathbb{R})$. This is a consequence of the Theorem of Radon-Nikodym. For a detailed proof see Theorem 5 in [Tar79].
It is important to note that "probability measure" implies that the $\nu_{x,t}$ are signed.

In the following we will show that $\nu_{x,t}$ is in fact a point measure at (x,t) with weight $u(x,t)$. This will yield the pointwise convergence.
Let $\varphi \in C^2(\mathbb{R})$ be a convex function. Then we define

$$\psi(u) = \int^u \varphi'(s) f'(s) \mathrm{d}s. \tag{3.3}$$

Therefore we can write

$$\begin{aligned}
\varphi(u^\varepsilon(x,t)) &\overset{w*}{\rightharpoonup} = \bar{\varphi}(x,t) \equiv \int_{\mathbb{R}} \varphi(\lambda) \mathrm{d}\nu_{x,t}(\lambda) \\
\psi(u^\varepsilon(x,t)) &\overset{w*}{\rightharpoonup} = \bar{\psi}(x,t) \equiv \int_{\mathbb{R}} \psi(\lambda) \mathrm{d}\nu_{x,t}(\lambda)
\end{aligned}$$

in $L^\infty(S^1 \times \mathbb{R}, \mathbb{R})$.

Now we look at $\partial_t \varphi(u^\varepsilon) = \varphi'(u^\varepsilon)\partial_t u^\varepsilon$ and obtain by using (3.1), (3.3) and the PDE (P)

$$\varphi(u^\varepsilon)_t + \psi(u^\varepsilon)_x = \varepsilon \left(\phi(u^\varepsilon)_{xx} - \varphi''(u^\varepsilon)(u^\varepsilon_x)^2 \right) + \varphi'(u^\varepsilon) g(u^\varepsilon). \tag{3.4}$$

We claim that

$$\sup_{\varepsilon > 0} \int_0^T \int_{S^1} \varepsilon (u^\varepsilon_x)^2 \mathrm{d}x \mathrm{d}t < \infty \tag{3.5}$$

and therefore $\sqrt{\varepsilon}\partial_x \varphi(u^\varepsilon) \in L^2(S^1 \times [0,T])$.

To see (3.5) we use $\varphi(u) = u^2$ in (3.4) and integrate over $S^1 \times [0,T]$. We obtain

$$\begin{aligned}
\int_{S^1} (u^\varepsilon(x,T))^2 \mathrm{d}x = \int_{S^1} (u^\varepsilon(x,0))^2 \mathrm{d}x - 2 \int_0^T \int_{S^1} \varepsilon (u^\varepsilon_x)^2 \mathrm{d}x \mathrm{d}t \\
+ 2 \int_0^T \int_{S^1} u^\varepsilon g(u^\varepsilon) \mathrm{d}x \mathrm{d}t.
\end{aligned}$$

The left hand side of this equation and the first and last term of the right hand side are globally bounded for all $\varepsilon > 0$, hence we have obtained (3.5).

We conclude $\varepsilon \varphi_x(u^\varepsilon) \to 0$ in $L^2(S^1 \times [0,T])$ and thus

$$\varepsilon \varphi_{xx}(u^\varepsilon) \to 0 \text{ in } W^{-1,2}(S^1 \times [0,T]).$$

Furthermore $\varepsilon \varphi''(u^\varepsilon)(u^\varepsilon_x)^2$ and $u^\varepsilon \varphi(u^\varepsilon)$ are bounded in the space of signed Radon measures on $S^1 \times [0,T]$ with finite mass.

Now we can apply Corollary 1 of Chapter 1 of Evans [Ev90] which yields that the right hand side of (3.4) is compact in $W^{-1,2}(S^1 \times \mathbb{R}^+)$. Note that this remains true if φ is only piecewise C^2 and continuous. In this case we obtain a piecewise version of equation (3.4). The argumentation remains the same and we again obtain $\varepsilon \varphi_x(u^\varepsilon) \to 0$ in $L^2(S^1 \times [0,T])$.

We now want to apply the Div-Curl-Lemma 3.2.2. We define two sequences:

$$v_\varepsilon := (f(u^\varepsilon), u^\varepsilon) \tag{3.6}$$

$$w_\varepsilon := (\varphi(u^\varepsilon), -\psi(u^\varepsilon)). \tag{3.7}$$

Then by (3.2) we have

$$v_\varepsilon \cdot w_\varepsilon \xrightharpoonup{w*} \int_{\mathbb{R}} [f(\lambda)\varphi(\lambda) - \lambda\psi(\lambda)] \mathrm{d}\nu_{x,t}(\lambda).$$

The Div-Curl-Lemma provides

$$v_\varepsilon \cdot w_\varepsilon \xrightharpoonup{w*} \bar{v} \cdot \bar{w} = (\bar{f}, u) \cdot (\bar{\varphi}, -\psi).$$

Hence we obtain

$$\int_{\mathbb{R}} [f(\lambda)\varphi(\lambda) - \lambda\psi(\lambda)] \mathrm{d}\nu_{x,t}(\lambda) = \bar{f}(x,t) \int_{\mathbb{R}} \varphi(\lambda) \mathrm{d}\nu_{x,t}(\lambda) - u(x,t) \int_{\mathbb{R}} \psi(\lambda) \mathrm{d}\nu_{x,t}(\lambda)$$

which is equivalent to

$$\int_{\mathbb{R}} \left[(f(\lambda) - \bar{f}(x,t))\varphi(\lambda) + (u(x,t) - \lambda)\psi(\lambda) \right] \mathrm{d}\nu_{x,t}(\lambda) = 0. \tag{3.8}$$

We now choose $\varphi(\lambda) = |\lambda - u(x,t)|$ which is in fact only C^0 (but piecewise C^2), then

$$\psi(\lambda) = \int^{\lambda} f'(s)\varphi'(s) \mathrm{d}s = sign(\lambda - \bar{u}(x,t))(f(\lambda) - f(\bar{u}(x,t))).$$

With this equation (3.8) reduces to

$$\big(f(\bar{u}(x,t)) - \bar{f}(x,t) \big) \int_{\mathbb{R}} |\lambda - u(x,t)| \mathrm{d}\nu_{x,t}(\lambda) = 0.$$

Thus, one of the factors must be zero, this either leads to

$$f(\bar{u}(x,t)) - \bar{f}(x,t) = 0 \tag{3.9}$$

or to

$$supp\{\nu_{x,t}\} = \{u(x,t)\}.$$

Recalling the definition of $\bar{f}$ in equation (3.2) we observe that $supp\{\nu_{x,t}\} = \{u(x,t)\}$ again implies equation (3.9).

Now we choose $\varphi(\lambda) = f(\lambda) - f(u(x,t))$ and therefore $\psi(\lambda) = \int^{\lambda} (f'(s))^2 \mathrm{d}s$. In this case (3.8) takes the form

$$\int_{\mathbb{R}} \left[(f(\lambda) - f(u(x,t))^2 - (\lambda - \bar{u}(x,t)) \int_u^{\lambda} (f'(s))^2 \mathrm{d}s \right] \mathrm{d}\nu_{x,t}(\lambda) = 0. \tag{3.10}$$

We use Hölder's inequality for the first term of the integrant and obtain:

$$(f(\lambda) - f(u(x,t))^2 = \left(\int_u^{\lambda} f'(s) \cdot 1 \mathrm{d}s \right)^2 \leq (\lambda - u) \int_u^{\lambda} (f'(s))^2 \mathrm{d}s.$$

Hence the integrant of equation (3.10) is either zero or negative, therefore it must be zero. Here we have used the fact that $\nu_{x,t}$ is a probability measure. From the fact that the integrant of (3.10) is zero we conclude that either $f' \equiv const$ or $\lambda = u(x,t)$ and therefore

$$\lambda = u(x,t)$$

because $f'' > 0$ due to (H1).

We now have obtained that $\nu_{x,t}$ is a point measure at (x,t) with weight $u(x,t)$ but this implies that the convergence of u^ε to u is pointwise a.e. on S^1 for all times $t \in [0,T]$. The same argument works for negative times hence we have pointwise convergence a.e. on $S^1 \times [-T,T]$.

Because T was arbitrary, $u(x,t)$ is globally bounded for $t \in \mathbb{R}$. Otherwise there would exist a time t_0 and a point x_0, such that u would have to become large in a neighbourhood of (t_0, x_0). That is impossible because $|u^\varepsilon|$ is globally bounded by $\max\{|u_-|, |u_+|\}$. This yields that $u(x,t)$ is a global solution of the hyperbolic equation (H).

Due to the global boundedness of the u^ε it is clear, that u^0 is a solution of the hyperbolic equation. One just has to apply the Theorem of Dominated Convergence to the weak formulation of the parabolic equation and let $\varepsilon \to 0$.

□

Certainly the question arises why "global solution" does not imply "heteroclinic connection" in this case. The main obstacle for this is the occurrence of additional equilibria. Figure 3.1 illustrates why "global solution" does not necessarily imply heteroclinic connection when additional frozen waves occur in the limiting process. Both panels show the $(\varepsilon - t)$ plane, every point represents a solution profile $u(\cdot) \in L^\infty$. The t-variable is compactified. Both panels show the same set of heteroclinic orbits $u^\varepsilon(\cdot,\cdot)$ and its limit in ε. Panel a) depicts the convergence of a solution for $t \in [-T,T]$. Because $u^\varepsilon(\cdot, 0)$ converges to an equilibrium the length covered by $u^\varepsilon(\cdot, t)$, $t \in [-T,T]$ gets shorter with smaller ε and vanishes for $\varepsilon = 0$. This is true for all finite T.

In b) we have shifted $u^0(\cdot, 0)$ of Panel a) to the left to a solution $\hat{u}^0$ where we set $t = 0$. Again we investigate a sequence of $\hat{u}^\varepsilon(\cdot, 0)$ converging to the $\hat{u}^0(\cdot, t)$ now including time. What used to be a cone becomes trapezoidal. If we let t go to ∞ then the limiting functions $\hat{u}^0(\cdot, t)$ for $t \in [-T,T]$ will converge to the heteroclinic connection of the left and centre equilibrium. This idea will be used in Theorem 3.2.9. Note that the dashed curves in the figure are in general just curves and do not necessarily have to be straight lines as depicted. Note that in both cases we do not have persistence allthough in Panel b) we have convergence to a heteroclinic connection, but the heteroclinic orbit does not connect the limits of target and source of the parabolic connection.

Let us now state two corollaries to the Global Solution Theorem. The first one concerns rotating waves.

Corollary 3.2.3 *Let $B := \{u^\varepsilon(x,t) \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon; 0 < \varepsilon < \varepsilon_0\}$ for some $0 < \varepsilon_0 << 1$. Then there exists a subset $\{u^{\varepsilon_n}(x,t)\}$ of B with the property that*

$$\lim_{n\to\infty} u^{\varepsilon_n}(x,t) = u^0(x,t)$$

a.e. on $S^1 \times [-T,T]$. Moreover $u^0(x,t)$ is a global solution of (H).

Second, we make a statement on all possible limits of solutions in the set B:

Corollary 3.2.4 *Let B be defined as in Theorem 3.2.1 and let all parameterisations in t be fixed. Let $u \in BV(S^1 \times [-T,T], \mathbb{R})$ with*

$$u(x,t) := \lim_{n\to 0} u^{\varepsilon_n}(\cdot, \tau_n + t)$$

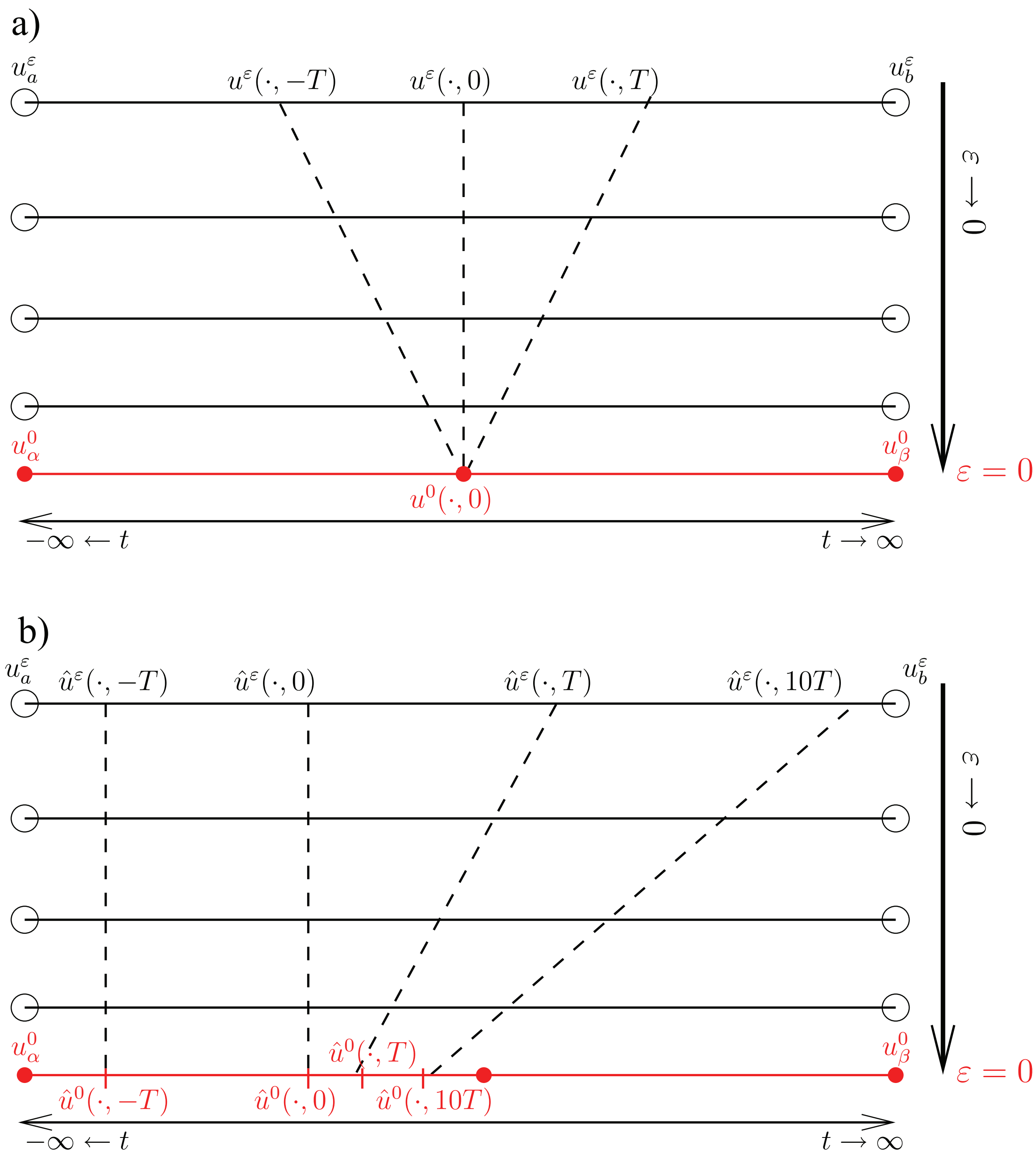

Figure 3.1: Convergence of a heteroclinic orbit connecting u_a^ε and u_b^ε for $\varepsilon \to 0$. The dots symbolise the frozen waves for $\varepsilon = 0$, the small circles rotating waves for $\varepsilon > 0$. For $\varepsilon = 0$ an additional frozen wave appears in the middle. In Panel a) convergence to the centre frozen wave is depicted. In Panel b) convergence for $\hat{u}^\varepsilon(\cdot,\cdot)$ is displayed . It is clear why there is no uniform convergence for all $t \in \mathbb{R}$ in this case.

a.e. on $S^1 \times [-T, T]$ *for sequences* $\{\varepsilon_n\} \to 0$ *and* $\{\tau_n\}$ *and all bounded* $T \in \mathbb{R}$*. Then*

$$u(\cdot, t) \in \mathcal{A}^0.$$

Proof. Certainly u must be globally bounded and due to the convergence of the limit be a solution of the hyperbolic equation. Therefore it must be a global solution and hence

$$u(\cdot, t) \in \mathcal{A}^0.$$

□

In the case of rotating waves Corollary 3.2.3 can be improved considerably. The limiting object is not only a global solution but again a frozen wave of the hyperbolic equation. This is the content of the following Theorem. Note that we use ODE theory here to obtain a much stronger result regarding convergence. We prove convergence for all ε.

Theorem 3.2.5 (Rotating Waves) *Let* $a = 2\alpha$ *for* $\alpha \in \mathbb{N}$ *and* u_a^ε *be the up to shift unique rotating or frozen wave of* (P) *with the property*

$$z(u_a^\varepsilon) = a$$
$$u_a^\varepsilon(0, 0) = 0$$

Then the limit

$$\lim_{\varepsilon \to 0} u_a^\varepsilon(x, t) = u_\alpha^0(x, t)$$

exists almost everywhere and $u_\alpha^0(x, t)$ *is a frozen wave of the hyperbolic equation* (H) *with*

$$z(u_\alpha^0) = \alpha. \tag{3.11}$$

Proof. We perform the proof in several steps:

(i) For the existence of the limit we assume $a = 2$, the other cases just work with the same argument.
We observe that, according to Lemma 2.3.5, the rotating wave v_a^ε associated to $u_a^\varepsilon(\cdot, t)$ and its derivative lie in a $o(\varepsilon)$ channel around $\phi(x)$ outside a $\varepsilon \log \varepsilon$-neighbourhood of some $x_2(\varepsilon)$.
Because $x_2(\varepsilon_1) - x_2(\varepsilon_2) < C|\varepsilon_1 - \varepsilon_2|$ for some constant C, the limit of u^ε for $\varepsilon \to 0$ exists outside any open neighbourhood of x_2 and is in fact uniform. This proves the existence.

(ii) It remains to prove that u_0 is a rotating wave. From Corollary 3.2.3 we obtain that $u_0(x, t)$ is a global solution and therefore lies on the attractor. Because it converges uniformly to ϕ outside a neighbourhood of x_2, the solution u_0 neither can be a homogenous solution, nor a heteroclinic connection. From equation 2.47 follows that it must be a rotating wave which is unique up to shifts. This proves the claim.

(iii) The relation between the zero-numbers of the parabolic wave and the hyperbolic one is obvious. All frozen waves for $\varepsilon = 0$ have positive derivative in all their zeros (see Section 2.6). We have seen that these persist. For $\varepsilon > 0$ all rotating wave profiles are continuous and thus have alternating signs in the derivative. Together with the already proved persistence this yields (3.11).

Remark 3.2.6 *The relation of the zero-number between solutions on the parabolic attractor $u^\varepsilon \in \mathcal{A}^\varepsilon$ and their limits is true for all elements $u \in \mathcal{A}^\varepsilon$. The zero-number drops by one half when taking the limit $\varepsilon \to 0$, counting multiplicity in the case of double zeros.*
This immediately implies

$$\lim_{\varepsilon \to 0} \mathcal{A}_n^\varepsilon \subset \mathcal{A}_n^0 \tag{3.12}$$

in the sense of solutions.

The zero-number property is true because all solutions $u \in \mathcal{A}_n^0$ have the property that the derivative in the zero is positive. Assume u has a zero at x_0 with negative slope, then using the backward characteristic emanating from x_0 we conclude that the limit in backward time $u_{-\infty}$ has also a zero at x_0. The sign of the derivative cannot change, hence it is negative. This contradicts the fact that $u_{-\infty}$ must be a frozen wave. Because the sign of the derivative in the zeros of all u^ε alternates, the zero-number drops by one half. Moreover it must be finite for $\varepsilon \neq 0$.

Coming back to rotating waves, we summarise that all rotating waves persist for $\varepsilon \to 0$. Moreover there is the relation between the zero-number of the rotating wave for $\varepsilon > 0$ and the number of zeros of the limiting frozen wave.

Definition 3.2.7 *Let $a := 2\alpha$ for some $\alpha \in \mathbb{N}$ be given and let ε_0 be sufficiently small. Then $u_a^\varepsilon(\cdot,\cdot)$ denotes the up to rotation unique rotating wave with zero-numbers $z = a$ for all $0 < \varepsilon < \varepsilon_0$.*
The set of rotating and frozen waves u_a^ε with a given zero-number $z(u_a^\varepsilon) = a$ shall be denoted by

$$B_a := \{u_a^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon : 0 < \varepsilon < \varepsilon_0\}.$$

Moreover we fix the notation of Theorem 3.2.5 by defining

$$u_\alpha^0(\cdot, t) := \lim_{\varepsilon \to 0} u_a^\varepsilon(\cdot, t).$$

As mentioned above, the persistence result that is valid for rotating waves, is not true for heteroclinic orbits although Theorem 3.2.1 yields convergence to a global solution. The next Lemma will provide a criterion when heteroclinic orbits cannot persist. In order to prove this criterion we define the set of heteroclinic orbits connecting two rotating waves u_a^ε and u_b^ε with zero-number a and b by

$$B_{ab} := \left\{ u^\varepsilon \in \mathcal{H}^\varepsilon : \lim_{t \to -\infty} u^\varepsilon(\cdot, t) = u_a^\varepsilon, \lim_{t \to \infty} u^\varepsilon(\cdot, t) = u_b^\varepsilon, \quad 0 < \varepsilon < \varepsilon_0 \right\}. \tag{3.13}$$

The rotating wave u_a^ε is called the source and u_b^ε the target.

Lemma 3.2.8 (Connection Lemma) *Let B_a, B_b and B_{ab} be defined as above with $a = 2\alpha$ and $b = 2\beta$ for $\alpha, \beta \in \mathbb{N}$ and let $u^\varepsilon \in B_{ab}$ with*

$$u^0(x, t) := \lim_{n \to \infty} u^{\varepsilon_n}(x, t) \tag{3.14}$$

a.e., where ε_n is a sequence for which the u^{ε_n} converge due to Theorem 3.2.1.
If

$$\lim_{t \to -\infty} u^0(\cdot, t) = u_\alpha^0(\cdot) \qquad \text{and} \qquad \lim_{t \to \infty} u^0(\cdot, t) = u_\beta^0(\cdot)$$

for all fixed time parameterisations of $u(\cdot, t)$. Then there exists a $k \in \mathbb{N}$ such that

$$a = kb. \tag{3.15}$$

In other words, if $a \neq kb$ for all $k \in \mathbb{N}$ then the limit of the heteroclinic orbits connecting the rotating waves u_a^ε and u_b^ε does not connect the limits of the rotating waves given by u_α^0 and u_β^0, thus the heteroclinic connection cannot persist.

Proof. For all $0 < \varepsilon < \varepsilon_0$ the rotating waves u_a^ε and u_b^ε are periodic solutions of the rotating wave equation. Their period is given by $T_a = \frac{2\pi}{\alpha}$ and $T_b = \frac{2\pi}{\beta}$.

If the heteroclinic connection $u^\varepsilon(\cdot, t)$ persits, u_α^0 and u_β^0 have to be connected by a heteroclinic orbit. According to Theorem 2.5.12 this implies

$$\mathcal{Z}(u_\beta^0) \subset \mathcal{Z}(u_\alpha^0). \tag{3.16}$$

Taking the limit $\varepsilon \to 0$ for the rotating waves, we obtain that the zeros of u_α^0 and u_β^0 must be periodic in x and the distance of neighbouring zeros is given by T_a and T_b respectively. Then equation 3.16 implies

$$T_b = kT_a$$

for some $k \in \mathbb{N}$.
Hence

$$\alpha = k\beta$$

which implies

$$a = kb$$

just as desired.

□

The fact that on the global attractor of the parabolic equation all rotating waves are connected to all waves with strictly lower Morse index, implies that the condition in the connection Lemma 3.2.8 is non-empty.

In case of non-persistence there is much more to say. In fact in the next theorem we will prove that the limit of the heteroclinic connections consists of a finite cascade of heteroclinic connections.

In order to prove this statement we have to circumvent the problem that the parameterisation in t gets "stuck" in an emerging stationary state. We have heuristically argued that this can happen (see Figure 3.1). We will make this argument rigorously here.

For this purpose we again use the set B_{ab} and assume that

$$\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = u^0(x, t) \text{ a.e. on } S^1 \times [-T, T]$$

but

$$\lim_{t \to \infty} u^0(\cdot, t) \neq u_\beta^0(\cdot).$$

Without loss of generality we can assume $u^0(\cdot, t) \in \mathcal{F}^0$ otherwise we could use $\lim_{t\to\infty} u^0(\cdot, t)$ for the argument. Additionally $u^0(x, t)$ shall not correspond to the target frozen wave u_β^0.

Then the following is true: there is a $\delta > 0$ such that for every sufficiently small $\varepsilon > 0$ there exists a $\tilde{T} > 0$ such that

$$||u^\varepsilon(\cdot, \tilde{T}) - u^0(\cdot, \tilde{T})|| > \delta. \tag{3.17}$$

It is obvious that on the one hand

$$u^0(\cdot, \tilde{T}) = u^0(\cdot, 0).$$

On the other

$$u^\varepsilon(\cdot, \tilde{T}) \to u^\varepsilon_\beta(\cdot)$$

for $\tilde{T} \to \infty$.

Hence

$$||u^\varepsilon(\cdot, \tilde{T}) - u^0(\cdot, \tilde{T})|| = ||u^\varepsilon(\cdot, \tilde{T}) - u^0(\cdot, 0)|| \to ||u^\varepsilon_b(\cdot) - u^0(\cdot)||$$

for $\tilde{T} \to \infty$.

But $||u^\varepsilon_b(\cdot) - u^0(\cdot)|| > \delta$. If this was false then $u^0(\cdot, T)$ would converge to the target equilibrium which was excluded. In fact we can choose any $0 < \delta < ||u^0_\beta(\cdot) - u^0(\cdot)||$.

We have not specified any norms here. This was not necessary as the argument holds for the L^1, the L^2 or the L^∞ norm.

This means nothing else than that, no matter how large T is chosen, for all $\varepsilon > 0$ there is always a part of the heteroclinic orbit $u(\cdot, t)$ for $t > T$ that lies outside the cone of convergence (see Figure 3.1). We therefore introduce a different parameterisation to circumvent this problem.

In order to do this we have to use the concept that all heteroclinic connections are embedded manifolds in the extended phase space. In other words, the graph of the map

$$\begin{aligned} u^\varepsilon : \mathbb{R} &\to L^2 \\ t &\mapsto u^\varepsilon(\cdot, t) \end{aligned}$$

given by $(u^\varepsilon(\cdot, t), t)$ defines an embedded manifold in $L^2 \times \mathbb{R}$. Due to the global boundedness of all u^ε , this graph is also a manifold in $L^\infty \times \mathbb{R}$.

We now introduce a different parameterisation by the transformation

$$\tau := \frac{t}{||u^\varepsilon_t(\cdot, t)||_{L^2}}$$

and define:

$$U^\varepsilon(\tau) := u^\varepsilon(\cdot, t).$$

In case there are $\tau_\pm$ with $U^\varepsilon(\tau_\pm) = u^\varepsilon_{a,b}$ then we define

$$\begin{aligned} U^\varepsilon(\tau) &:= U^\varepsilon(\tau_-) \text{ for } \tau < \tau_- \\ U^\varepsilon(\tau) &:= U^\varepsilon(\tau_+) \text{ for } \tau > \tau_+. \end{aligned}$$

This compensates for the fact that in the case u^ε_a and u^ε_b are frozen waves $U(\cdot)$ can have finite length.

We now have $U^\varepsilon : \mathbb{R} \to L^2$ and $graph(U^\varepsilon(\cdot)) = graph(u^\varepsilon(\cdot))$ but $graph(U(\cdot))$ is parameterised by arc length.

The map $U^\varepsilon(\cdot) : \mathbb{R} \to L^2$ is differentiable and we have

$$||\partial_\tau U(\tau)|| = \left|\left|\frac{u_t^\varepsilon}{||u_t^\varepsilon||}\right|\right| = 1,$$

hence the U^ε are a bounded sequence of equicontinuous functions and therefore have, after possibly taking a sub-sequence, a continuous limit U^0 for all $\tau \in [-\tau_-, \tau_+]$ for arbitrary but finite $\tau_\pm$. Hence $U^0(\cdot)$ is again a manifold that can be parameterised by τ and is locally connected.

The new parameterisation has the important property that it cannot get "stuck" as the parameterisation over the t could.

Before stating the Theorem, we add the following notion: we say that $U^{\tilde{\varepsilon}}(\tau_2)$ lies to the right of $U^{\tilde{\varepsilon}}(\tau_1)$ on U if and only if $\tau_2 > \tau_1$ for fixed $\tilde{\varepsilon}$.

Theorem 3.2.9 (Cascading) *Let again*

$$B_{ab} := \left\{ u^\varepsilon \in \mathcal{H}^\varepsilon : \lim_{t\to-\infty} u^\varepsilon = u_a^\varepsilon, \lim_{t\to\infty} u^\varepsilon = u_b^\varepsilon \quad 0 < \varepsilon < \varepsilon_0 \right\},$$

where $a = 2\alpha$ and $b = 2\beta$.

Then there exists a sub-sequence $\{\varepsilon_n\} \to 0$ of $\{\varepsilon\}$ such that the limit in n of the $u^{\varepsilon_n} \in B_{ab}$ consists entirely of frozen waves or of a cascade of heteroclinic connections interrupted by sections of frozen waves.

There are at most $\alpha - \beta$ different heteroclinic orbits and $\alpha - \beta + 1$ sections of frozen waves.

Proof. We begin the proof by defining the set of all possible limits of B_{ab} denoted by $\mathcal{U}_1$.

Let therefore Γ_1 be the set of all sub-sequences $\{\varepsilon_n\}$ and $\{\tau_n\}$ for which

$$\lim_{n\to\infty} U^{\varepsilon_n}(\tau_n + \tau)$$

converges to a BV-function $U^0_{\{\varepsilon_n\},\{\tau_n\}}(\tau)$ such that, if we define

$$u^{\varepsilon_n}(\cdot, 0) = U^{\varepsilon_n}(\tau_n)$$

then $u^{\varepsilon_n}(\cdot, t) \to u^0(\cdot, t)$ a.e. on $S^1 \times [-T, T]$ for all finite $T \in \mathbb{R}$. The set of all these functions shall be denoted by

$$\mathcal{U}_1 := \left\{ u_{\{\varepsilon_n\},\{\tau_n\}} \in BV; (\{\varepsilon_n\}, \{\tau_n\}) \in \Gamma_1 \right\} \tag{3.18}$$

then Corollary 3.2.4 yields $\mathcal{U}_1 \subset \mathcal{A}^0$.

The following proof is a finite induction with respect to the number of heteroclinic orbits in $\mathcal{U}_1$.

If $\mathcal{U}_1 \cap \mathcal{H}^0 = \emptyset$ then the limit of the heteroclinic connections in B_{ab} does not contain a heteroclinic connection of equation (H). Hence the Theorem is true.

Therefore assume $\exists(\{\varepsilon_n\}, \{\tau_n\}) \in \Gamma_1$ such that

$$u_1^0(x, 0) := \lim_{n\to\infty} U^{\varepsilon_n}(\tau_n)$$

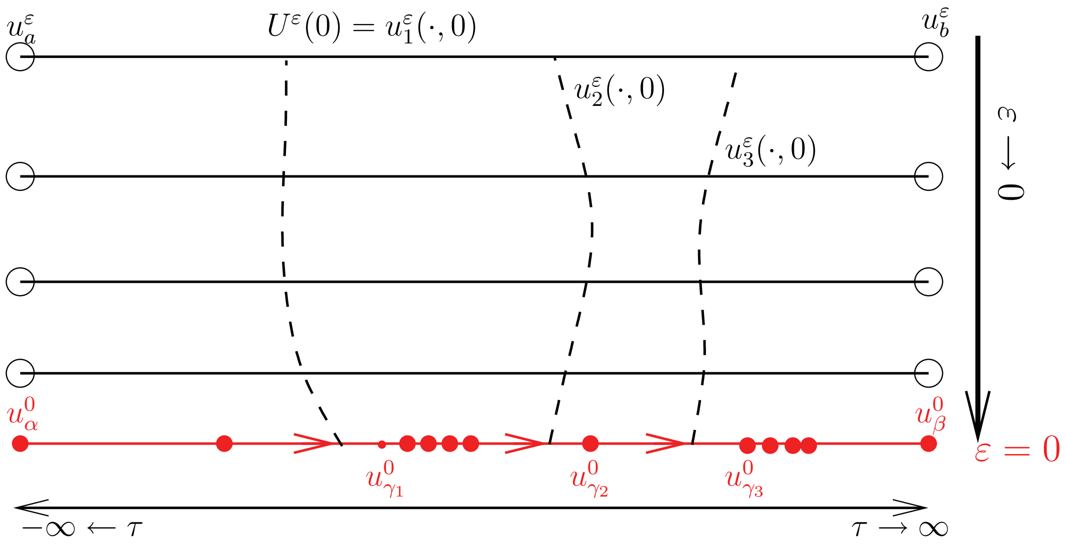

Figure 3.2: Schematic plot to illustrate the construction in the proof.

is not a frozen wave or equilibrium of equation (H) and thus lies on a heteroclinic connection. Without loss of generality we assume that

$$\lim_{t\to-\infty} u_1^0(\cdot,t) \neq u_\alpha^0(\cdot) \quad \text{or} \quad \lim_{t\to\infty} u_1^0(\cdot,t) \neq u_\beta^0(\cdot)$$

otherwise the heteroclinic orbit persists and we are finished.

Let us assume

$$\lim_{t\to\infty} u_1^0(\cdot,t) = u_{\gamma_1}^0(\cdot) \neq u_\beta^0$$

Due to the Sturm property the number of zeros can only drop along heteroclinics for $\varepsilon > 0$, the same is true for $\varepsilon = 0$ (see as well equation (2.49)). We conclude that

$$z(u_{\gamma_1}^0) \leq \alpha - 1$$

because the source of the heteroclinic can have at most α zeros. Here we have used the fact that the number of zeros in any $u^0(\cdot,t)$ has at most α many zeros, by Remark 3.2.6.

We go back to arc length parameterisation. Because $u_{\gamma_1}^0 \neq u_\beta^0$ there must be a sequence $\tilde{\tau}_n$ such that

$$\lim_{n\to\infty} U^{\varepsilon_n}(\tilde{\tau}_n) = u_{\gamma_1}^0.$$

We now define

$$\Gamma_2 \subset \Gamma_1$$

to be the subset of sub-sequences $(\{\varepsilon_m\},\{\tau_m\}) \in \Gamma_1$ with the property that they lie right of $\tilde{\tau}_n$:

$$(\{\varepsilon_m\},\{\tau_m\}) \in \Gamma_1 \Leftrightarrow \quad \begin{array}{l} \bullet\ (\{\varepsilon_m\},\{\tau_m\}) \in \Gamma_1 \\ \bullet\ \{\varepsilon_m\} \text{ sub-sequence of } \{\varepsilon_n\} \text{ and} \\ \bullet\ \tau_m > \tilde{\tau}_m \end{array}$$

In analogy to $\mathcal{U}_1$ we define

$$\mathcal{U}_2 := \left\{ u_{\{e_m\},\{\tau_m\}} \in BV; (\{e_m\},\{\tau_m\}) \in \Gamma_2 \right\}.$$

Due to the fact that $z(u^0_\gamma) \leq \alpha - 1$, we have the property that

$$z(u^0(\cdot, t)) \leq \alpha - 1$$

holds for all $u^0(\cdot, t) \in \mathcal{U}_2$. Again we use the fact that the zero-number decreases in both equations (P) and (H).

There are two cases:

- Either $\mathcal{U}_2 \cap \mathcal{H}^0 = \emptyset$. Then all parts of the heteroclinic connections that lie to the right of the $U^{\varepsilon_m}(0)$ converge to frozen waves.
- Or there are other heteroclinic connections in $\mathcal{U}_2$. Let $u^0_2(x, t) \in \mathcal{U}_2$ have the property that $z(\lim_{t\to-\infty} u^0_2(\cdot, t))$ is maximal among all heteroclinic connections. Then we have

  $$z(\lim_{t\to\infty} u^0_2(\cdot, t)) \leq \alpha - 2.$$

  We now repeat the above construction until the set $\mathcal{U}_k \cap \mathcal{H}^0 = \emptyset$. Because α is finite, k must be finite as well.

The same construction works in negative time direction with finitely many steps. Hence we have found a sub-sequence again denoted by $\{\varepsilon_n\}$ for which the set of heteroclinic connections $U^{\varepsilon_n}(\cdot)$ converges to a sequence of heteroclinic orbits intercepted by sections of frozen waves. There can be at most $\alpha - \beta$ heteroclinics, and consequently $\alpha - \beta + 1$ sections of stationary solutions in the limit, because the number of zeros has to drop at least by one in every heteroclinic connection.
Thus the theorem is proven.

$\square$

Remark 3.2.10 $\mathcal{F}^0_\alpha$ *can be parameterised completely by its zeros and therefore is a α dimensional torus $\mathbb{T}^\alpha$ embedded in $BV(S^1, \mathbb{R})$; see Section 2.6. All sections of frozen waves of a cascade of heteroclinics in the above Theorem 3.2.9 are contained in this manifold.*

As a Corollary to Theorem 3.2.9 we obtain two necessary conditions on the persistence of a heteroclinic orbits.

Corollary 3.2.11 (Persistence) *Let $u^\varepsilon(x, t)$ be a heteroclinic orbit connecting u^ε_a with u^ε_b. Then the following statements are true:*

(i) Let the set $\mathcal{U}_1$ defined in (3.18) *contain at least one solution $u^0(x, t)$ that is not stationary. If*

$$\lim_{t\to-\infty} u^0(\cdot, t) = u^0_\alpha(\cdot)$$

and

$$\lim_{t\to\infty} u^0(\cdot, t) = u^0_\beta(\cdot)$$

then the heteroclinic connection $u^\varepsilon(x, t)$ persists.

(ii) Let $\mathcal{U}_1 \cap \mathcal{F}^0 = \{u^0_\alpha, u^0_\beta\}$ then the heteroclinic orbit persists.

Now we have settled the question of persistence. The *Connection Lemma* 3.2.8 provides a necessary condition for the persistence of heteroclinic connections between given rotating or frozen waves. Corollary 3.2.11 yields two independent sufficient conditions for the persistence of a heteroclinic orbit.
In addition Theorem 3.2.9 gives a result on the structure of the limit of heteroclinic connections in case of non-persistence: a cascade of heteroclinic connections in $\mathcal{A}^0$.

In the following and last section of this chapter we will combine the results on persistence with the result from Section 2.4 on the connection problem of the global attractor of equation (P). A more detailed analysis of the geometrical properties of the global attractors will follow in Chapter 4.

3.3 Persistence and non-persistence!

In this section we will show that for all choices of f and g satisfying the assumptions (H1)-(H3) there exist heteroclinic connections in $\mathcal{H}^\varepsilon$ that do not persist for $\varepsilon \to 0$. The next chapter will yield that there are always connections that do persist, however these results of persistence are so far limited to low dimensional cases.

We have seen in Section 2.4 that on the global attractor of the parabolic equation a solution $u_1 \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ is connected to another solution $u_2 \in \mathcal{E}^\varepsilon \cup \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ if and only if

$$i(u_1) > i(u_2). \tag{3.19}$$

Figure 3.3 shows the sub-attractors $\mathcal{A}_4^\varepsilon$, $\mathcal{A}_6^\varepsilon$,and $\mathcal{A}_8^\varepsilon$ in the upper part and $\mathcal{A}_{10}^\varepsilon$ and $\mathcal{A}_{14}^\varepsilon$ in the lower part. In the first three illustrations the connections to the constant states $u \equiv u_\pm$ are also included, whereas we have omitted these connections in the two lower pictures. Equation (3.19) yields that the attractor possesses a gradient-like structure, hence the flow on all connections points downwards (see Figure 2.5).

Our main concern regards the question which of the connections do not persist. Lemma 3.2.8 yields a purely algebraic relation on the zeros to decide this. The only heteroclinic connections that possibly persist are the ones where the zero-number of the target wave is a natural fraction of the zero-number of the source. Hence the heteroclinics in the set B_{ab} defined in 3.13 possibly persist if there exists a $k \in \mathbb{N}$ such that

$$a = kb. \tag{3.20}$$

In Figure 3.3 the connections that satisfy equation (3.20) are drawn as solid lines, the connections that violate equation (3.20) are drawn with dashed lines. These connections are the ones where we know a priori that they do not persist. So all sub-attractors larger than $\mathcal{A}_4^\varepsilon$ contain connections that do not persist. All figures are independent of the choice of f and g.

The next Chapter will yield persistence of some connections, but only for limited low dimensional examples. The general question or whether connections that fulfil equation (3.20) persist or not, cannot be answered yet, but will be discussed in some detail in the Conclusion in Chapter 5.

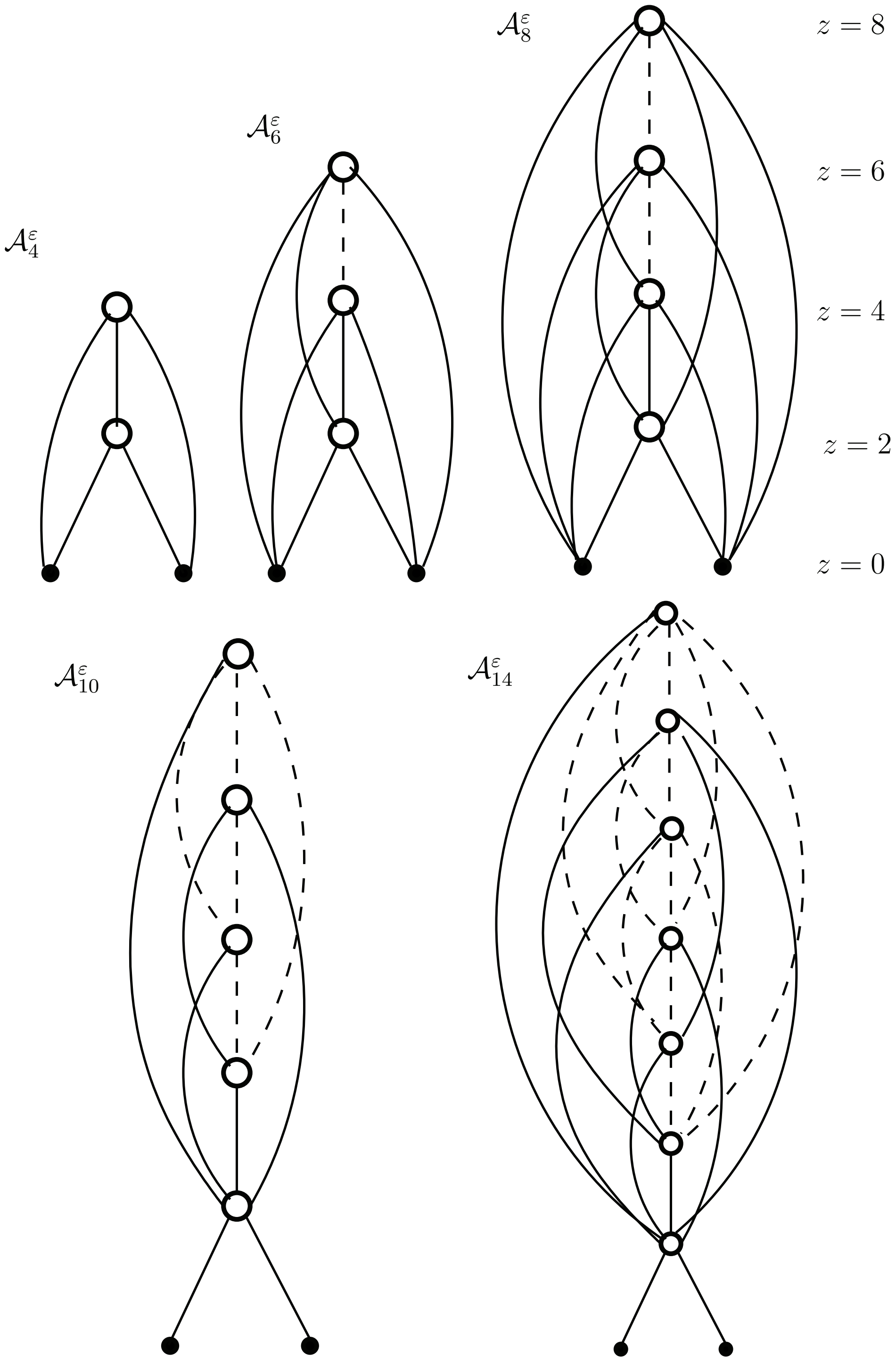

Figure 3.3: Depicted are the sub-attractors of order $n = 4, 6, 8, 10$ and 14. Heteroclinics that do not persist are drawn with dashed lines.

Chapter 4

The geometry of sub-attractors

In this chapter we will investigate the implications of our main results of the last chapters on the geometry of sub-attractors and the relation of solutions in $\mathcal{A}_n^0$ and $\mathcal{A}_n^\varepsilon$. We will investigate topological aspects of the manifold $\tilde{\mathbf{A}}_\mathbf{n}$ and use this knowledge to describe the geometry of the heteroclinic connections of the parabolic equation. Here, "geometric description" does not mean to draw further images on connection properties but to describe these connections as manifolds in $L^2 \times \mathbb{R}$ and, by doing so, shed some light on the topology of the $\mathcal{A}_n^\varepsilon$. Not all of the results presented are rigorous.

The main obstacle in making all results on the geometry and topology of the $\mathcal{A}_n^\varepsilon$ rigorous is that we have not addressed the spectral problem of the parabolic or hyperbolic equation. The problem is that the pointwise convergence of solutions (Theorem 3.2.1), the result on the dimensions of sub-attractors $\dim \mathcal{A}_n^0 = \dim \mathcal{A}_n^\varepsilon$ (Theorem 2.6.1) and the zero properties of solutions (Remark 3.2.6) only imply

$$\lim_{\varepsilon \to 0} \mathcal{A}_n^\varepsilon \subset \mathcal{A}_n^0$$

in the sense of sequences and solutions, but not

$$\lim_{\varepsilon \to 0} \mathcal{A}_n^\varepsilon = \mathcal{A}_n^0.$$

In other words we do not know whether the limiting procedure is surjective. In order to prove surjectivity we would need results on the convergence and persistence of the tangent vectors of the manifolds. If we restrict ourselves to neighbourhoods of the rotating and frozen waves (which would be sufficient in our case) we would need a result on the convergence of eigenvectors associated to the eigenvalue problem

$$\varepsilon \varphi_{xx} - f''(u^\varepsilon)(u_x^\varepsilon)^2 \varphi - f'(u^\varepsilon)\varphi_x + g'(u^\varepsilon)\varphi = \lambda \varphi \tag{4.1}$$

for waves $u^\varepsilon \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ to the eigenvectors of the hyperbolic problem and certain ε independent bounds on the associated spectral projections. The difficulties are manifold here:

- There is no generic way to explicitly compute the eigenvectors of (4.1)
- Equation (4.1) is only self-adjoint with respect to a scalar product that explicitly depends on ε, hence the spectral projections associated with these eigenvectors also depend explicitly on ε.

- The target manifold $W^u(u^0_\alpha)$ for $u^0_\alpha \in \mathcal{F}^0$ given by $A_{\{x_1,\ldots,x_\alpha\}}$ is not C^1 on $BV \times \mathbb{R}$ or $L^\infty \times \mathbb{R}$ but only Lipschitz.
- The dimensions of the unstable manifolds for $\varepsilon = 0$ and $\varepsilon > 0$ do not match:

  $$\lim_{\varepsilon\to 0}(\dim W^u(u^\varepsilon)) \neq \dim W^u(u^0)$$

  for $u^\varepsilon \to u^0$.
  We have seen already that in the case $z(u^\varepsilon) > 2$, u^ε a frozen wave

  $$\dim\left(W^u(u^\varepsilon)\right) = i(u^\varepsilon) = z(u^\varepsilon) - 1 \neq \dim W^u(u^0)) = z(u^\varepsilon)/2$$

We do not attempt to overcome all these difficulties here, but we will sometimes make the following assumption:

Assumption (D) *Let $u^\varepsilon_a \in \mathcal{F}^\varepsilon \cup \mathcal{R}^\varepsilon$ with $z(u^\varepsilon_a) = a$ and $\lim_{\varepsilon\to 0} u^\varepsilon_a = u^0_\alpha$. Then there exist for all $\varepsilon_0 > \varepsilon > 0$ neighbourhoods N^ε of u^ε_a in $\mathcal{A}^\varepsilon_n$ and a neighbourhood N^0 of u^0_α in $\mathcal{A}^0_n$ such that*

$$\lim_{\varepsilon\to 0} N^\varepsilon = N^0$$

i.e. for all $u^0 \in N^0$ there exists a sequence $u^\varepsilon \in N^\varepsilon$ such that $\lim_{\varepsilon\to 0} u^\varepsilon = u^0$ and all $u^0 \in N^0$ are limits of a sequence of $u^\varepsilon \in N^\varepsilon$.

Geometrically Assumption (D) states in particular, that the dimension of a neighbourhood N^ε of a rotating or frozen wave in $\mathcal{A}^\varepsilon_n$ does not change in the limiting process.

Now let us investigate the sub-attractors of the lower dimensions. We will do this for general f and g, but if explicit representations of solutions are plotted we use the special case where the source term g is odd and the transport term f is even and given by

$$f(u) := \frac{1}{2}u^2 \quad g(u) := u(1-u^2). \tag{4.2}$$

In principle explicit representations of solutions can be given for all f and g once the stationary problem of the hyperbolic equation given in equation (2.53) is solved.

Let me include a technical note: In the following we will compare the solutions of the hyperbolic and parabolic equations. Although the solution of the parabolic equation does not possess shocks in the sense of discontinuities, we will refer to the zeros that develop in the limit $\varepsilon \to 0$ discontinuities as well as shocks. In addition when we refer to a drift of zeros in the hyperbolic setting, we mean a drift with respect to the parameterisation on the respective manifold.

4.1 The sub-attractors $\mathcal{A}^0_2$ and $\mathcal{A}^\varepsilon_2$

According to the definition of $\mathcal{A}^0_2$ given in equation (2.50) in Chapter 2, the sub-attractor $\mathcal{A}^0_2$ consists of all frozen waves with zero-number $z = 1$, the two stable homogeneous equilibria $u \equiv u_\pm$ and all heteroclinic connections between these objects.
The frozen waves form a sub-manifold of $\mathcal{A}^0_2$ that can be represented as an S^1.

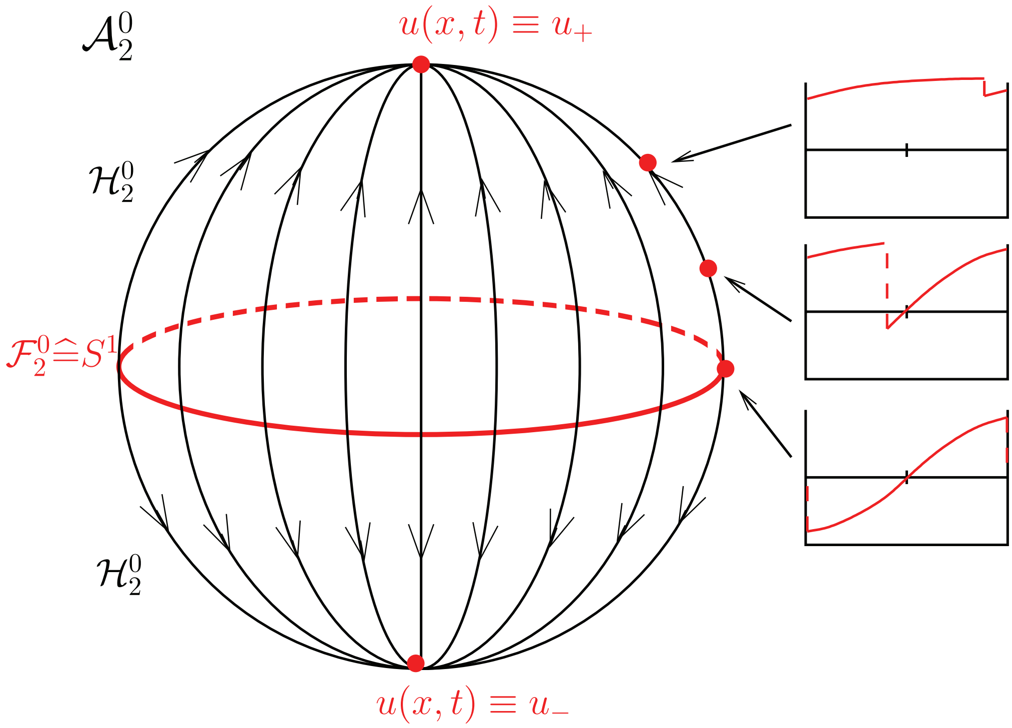

Figure 4.1: Geometric representation of the sub attractor $\mathcal{A}_2^0$.

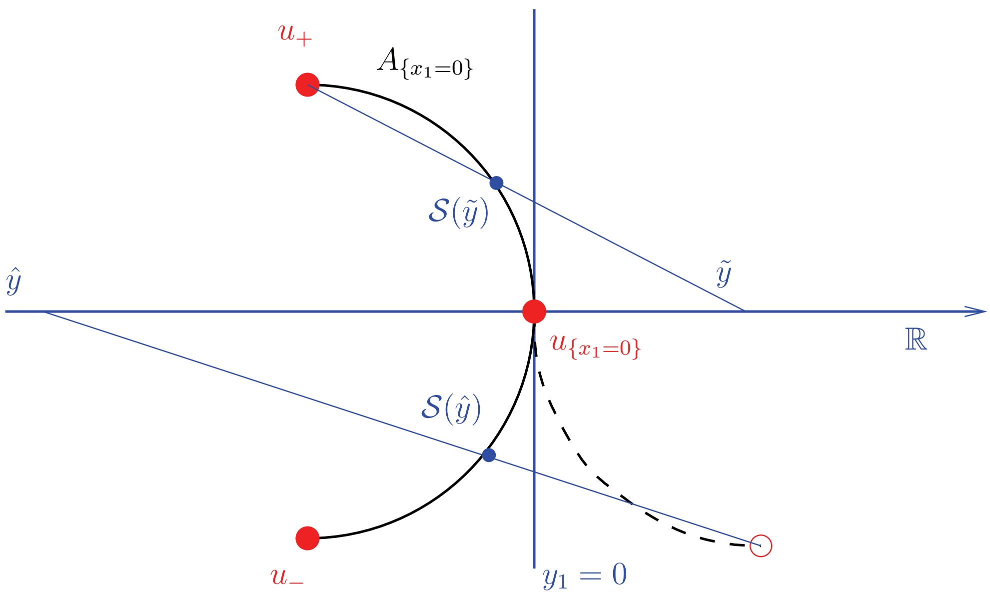

Figure 4.2: Stereographic projection for the case $x_1 = 0$.

Due to Theorem A (2.5.10) in Chapter 2 all frozen waves are connected to $u(x) \equiv u_\pm$. Theorem C (2.5.10) states that these are all heteroclinic connections in $\mathcal{A}_2^0$ and Theorem 2.6.1 e) yields uniqueness of these heteroclinics. Equation (2.71) provides together with equation (2.72) an explicit parameterisation of these connections $W^u(\mathcal{F}_2^0)$. Hence we can define an explicit embedding

$$\begin{aligned}\Sigma_2 : S^1 \times \mathbb{R} \to & BV(S^1, \mathbb{R}) \\ (x_1, y_1) \mapsto & \Sigma_2(x_1, y_1) := u_{\{x_1,y_1\}}\end{aligned}$$

where $u_{\{x_1,y_1\}}$ is defined in equation (2.71). The flow on $graph(\Sigma_2)$ can be computed explicitly and is given by equation (2.63) in Theorem 2.6.1 c).
By a stereographic projection $\mathcal{S}$ we can map the whole object onto the surface of a ball, thus obtaining a representation of $\mathcal{A}_2^0$ as an S^2, shown in Figure 4.1.

The stereographic projection $\mathcal{S}$ is outlined in Figure 4.2 where we have set $x_1 = 0$. In the figure the heteroclinic connection on the S^2 is depicted in black, the frozen wave is depicted in red. The explicit parameterisation of the heteroclinic by the shock position $y_1 \in \mathbb{R}$ is represented by the blue line. If we see Figure 4.2 as one slice of Figure 4.1 we can understand how solutions evolve along the heteroclinics on the S^2. The three diagrams on the right in Figure 4.1 show schematically how the shape of these solutions evolves.

Can we use this description to describe the parabolic sub-attractor $\mathcal{A}_2^\varepsilon$? There all rotating waves with Morse index $i = 1$ are given by

$$u^\varepsilon(x - c(\varepsilon)t) = u_1^\varepsilon(x + \theta)$$

with $\theta \in S^1$. In a co-moving coordinate system every rotating wave can be frozen. Every now frozen wave is connected by a heteroclinic orbit to u_+ and u_-. Due to Theorem 3.2.5 all rotating waves persist, hence we have converge to the red S^1 in Figure 4.1 for $\varepsilon \to 0$.

Due to Corollary 3.2.11 all heteroclinic connections persist as well. By uniqueness of the heteroclinic connections in $\mathcal{A}_2^0$ we obtain that

$$\lim_{\varepsilon \to 0} \mathcal{A}_2^\varepsilon = \mathcal{A}_2^0,$$

where this limit is understood as a limit of sequences and solutions. Furthermore there is a one-to-one correspondence between orbits on the sub-attractors, hence $\mathcal{A}_2^0$ and $\mathcal{A}_2^\varepsilon$ are C^0-orbit equivalent. Thus the above explicit description of $\mathcal{A}_2^0$ is a leading order description of $\mathcal{A}_2^\varepsilon$ in the appropriately co-rotating coordinate system.
This describes the geometry of these heteroclinic connections of the parabolic equation in first order in a completely rigorous way, because we have not used Assumption (D) here.

4.2 The sub-attractors $\mathcal{A}_4^0$ and $\mathcal{A}_4^\varepsilon$

We begin with the analysis of the sub-attractor of the hyperbolic equation. Theorem 2.6.1 yields $\dim \mathcal{A}_4^0 = 4$ and Corollary 2.6.4 states that all connections between rotating waves are contained in $\tilde{\mathbf{A}}_\mathbf{4}$ defined in equation (2.73). However, this does not yet explain the topology of the sub-attractor $\mathcal{A}_4^0$.

Following the definition of $\mathcal{A}_4^0 := \mathcal{E}_4^0 \cup \mathcal{F}_4^0 \cup \mathcal{H}_4^0$ we will first classify all homogeneous equilibria and frozen waves. It is clear that $\mathcal{E}_4^0 = \{u_-, u_+\}$. Due to Sinestrati the frozen waves can be uniquely parameterised by the position of their zeros x_1, x_2, so they form a two-torus:

$$\mathcal{F}_4^0 = \mathbb{T}^2 := S^1 \times S^1$$

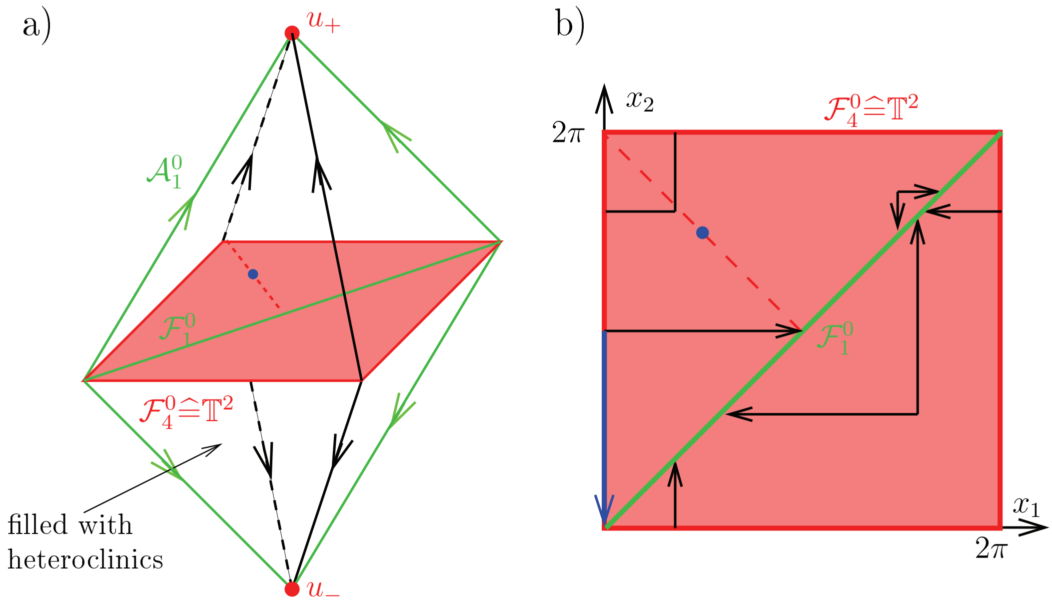

Figure 4.3: Heteroclinic connections in $\mathcal{A}_4^0$ with targets $u \equiv u_\pm$.

The torus also contains the frozen waves $\mathcal{F}_2^0 \widehat{=} S^1$ that possess only one zero.

Each element of this torus has a heteroclinic connection to the homogeneous equilibria $u \equiv u_\pm$. This can be depicted by a spindle with a quadratic horizontal section and $u_\pm$ located at the top and bottom. See Panel a) in Figure 4.3. The heteroclinic connections are plotted in black or green and the frozen waves in red. The edges of the red quadratic horizontal section have to be identified in order to obtain the torus. The sub-attractor $\mathcal{A}_2^0$ is contained in this picture as well and is depicted in green. Figure 4.1 is obtained after identification of the two corners involved that lie on the torus $\mathcal{F}_4^0$. Note that we have not plotted all heteroclinics in Figure 4.3. The complete spindle is filled with heteroclinics starting in $\mathcal{F}_4^0$ and ending at $u \equiv u_\pm$.

The more interesting part of $\mathcal{A}_4^0$ is the part of the attractor that consists of all frozen waves $\mathcal{F}_4^0$ and the heteroclinic connection between these waves. Theorem C (2.5.12) at the end of Section 2.5 yields that every frozen wave $\tilde{u}$ with zero-number $z(\tilde{u}) = 2$ is connected to two waves $\tilde{u}_a, \tilde{u}_b$ with zero-numbers $z(\tilde{u}_{a,b}) = 1$.

If we look at Panel a) in Figure 4.3, this means nothing else than that every point on the torus of frozen waves that is coloured in red has two heteroclinic connections to two points on the green curve on that torus. This is shown in Panel b) where we have parameterised the torus by the zeros (x_1, x_2) given as the horizontal and vertical axes. Some (but not all) heteroclinics are shown in black for illustration. The lines are vertical if the zero x_1 persists, horizontal if the zero x_2 persists. In principle there should be two heteroclinics emerging at every point. The one arrow coloured in blue represents the heteroclinic orbit shown in Figure 2.6 in Chapter 2 for Burgers equation.

The uniqueness result in Theorem 2.6.1 f) guarantees the uniqueness of these connections and equations (2.56) and (2.56) provide an explicit parameterisation of these connections.

To show the complete connection picture it is convenient to use another representation that divides out the S^1 symmetry. This representation is shown in Figure 4.4 and will prove useful for the comparison with the global attractor of the parabolic equation.

To understand the Figure it is best to start with the red vertical line. This line represents $\mathcal{F}_4^0/S^1$: the manifold that contains all frozen waves with zero-number $z = 2$ after having divided out the S^1 symmetry. The centre point (in blue) on this line is the π-periodic frozen wave with equidistant zeros. This is the only wave on the red line that is a limit of waves of the parabolic equation. No other waves on the red line can be obtained as limits of waves for $\varepsilon \to 0$. If they were, the rotating wave equation (2.7) would have to have self-intersecting solutions, which is impossible (compare with Figure 2.3 Panel b)).

The coordinates on the red manifold are given by the distance between the two zeros x_1 and x_2. On the bottom the distance is zero, in the middle at the blue dot it is π and then it goes to zero again towards the top. x_1 and x_2 change in such a way that the two shocks always remain in the same position (for Burgers equation (4.2) this means due to symmetries that $\frac{x_1+x_2}{2} = \pi$ along the red manifold). The three solution profiles drawn in red show how the solutions evolve along the manifold. The red manifold is also included in Panels a) and b) of Figure 4.3 as a red dashed line with a blue dot on the torus $\mathbb{T}^2$ in case of Burgers equation.

Each of the frozen waves has two connections to frozen waves with $z = 1$, one connection where the zero x_1 persists and one where x_2 persists. These are represented by the black arrows connecting to the green circle representing $\mathcal{F}_2^0$. To the left x_1 persists and to the right x_2 persist, this induces coordinates on the circle of frozen waves with zero-number $z = 1$. The green solution profiles in Figure 4.4 indicate how solutions evolve along the circle. A clockwise rotation along the S^1 in the figure corresponds to a shift of the solution to the right.

Now we are ready to include the S^1 symmetry in the figure that was divided out before. To do this we just have to rotate the whole figure along a circle in transverse direction attached to the blue dot representing the wave with two equidistant zeros. We obtain a filled torus where we have a figure similar to the one in Figure 4.4 in every slice.

Inside the torus the red line and the heteroclinic connections rotate once around the centre point with higher symmetry (blue point) and therefore form a spiral. Figure 4.5 shows a geometric representation of this. We have plotted half of the torus. The blue line corresponds to the frozen waves in $\mathcal{A}_4^0$ that are limits of waves of the parabolic equation with zero-number $z = 4$. The heteroclinics are shown only in the beginning and the end . They rotate with the red manifold and are always perpendicular to that manifold. There is a colour gradient included to illustrate the rotation of the heteroclinics. Note that the green S^1 does not rotate. Heteroclinics in the same colours correspond to each other. The green circle corresponds to the green circle in Figure 4.4. To obtain the full picture we have to identify all points on the surface of the torus with the green S^1, hence retract the torus surface to the S^1!

The result on uniqueness of the heteroclinic in Theorem 2.6.1 yields uniqueness of all heteroclinic connections described above. In particular all connections are one-dimensional. It follows that $\mathcal{A}_4^0 \backslash W^s(u_\pm)$ is in fact a three-dimensional manifold that can be represented as described.

Let us turn to the parabolic equation. We will focus on the part of rotating and frozen

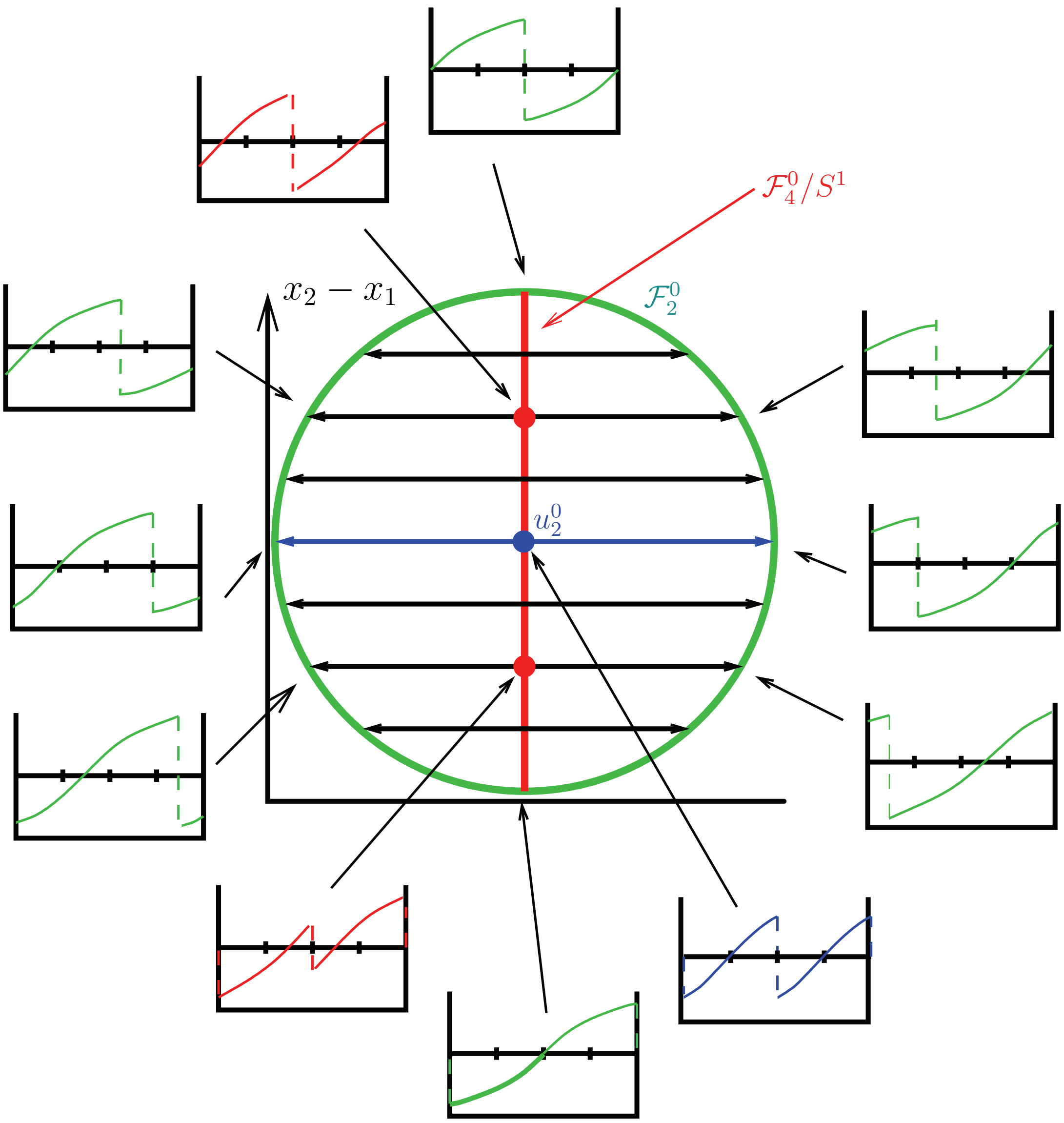

Figure 4.4: Heteroclinic connections in $\mathcal{A}_4^0$ from frozen waves with zero-number $z = 2$ to waves with zero-number $z = 1$. The S^1 symmetry is divided out.

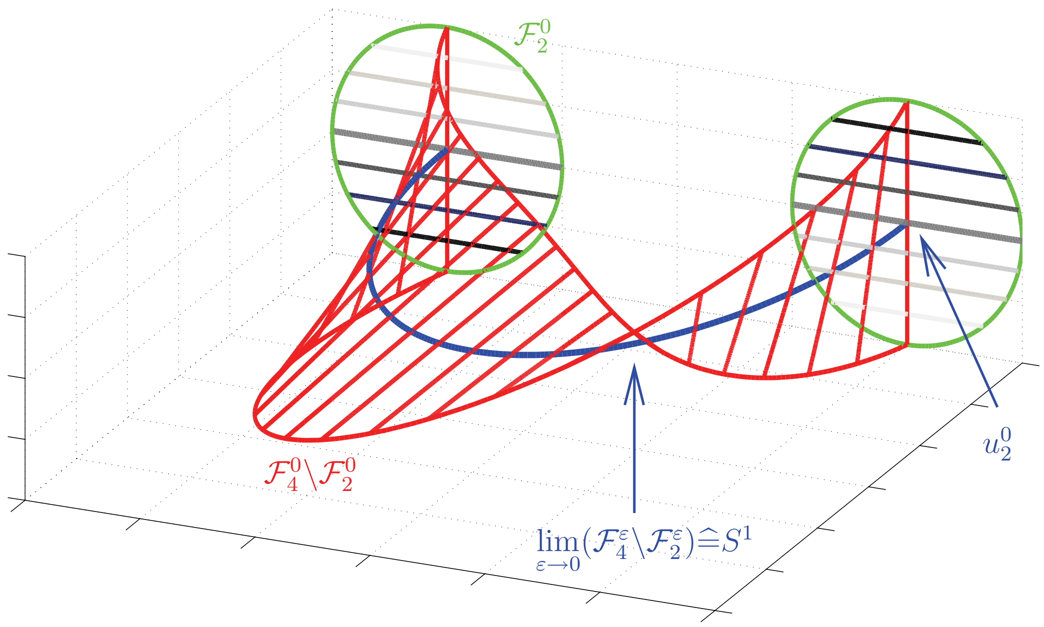

Figure 4.5: Torus representing $W^u(\mathcal{F}_4^0) \overline{\pitchfork} W^s(\mathcal{F}_2^0)$ after identification of the two ends of the cylinder and identification of the surface with the S^1 drawn in green.

waves with $z = 4$ to waves with $z = 2$. The connection between two individual waves u_4^ε with $z(u_4^\varepsilon) = 4$ and u_2^ε with $z(u_2^\varepsilon) = 2$ is due to the transversality result of stable and unstable manifolds in equation (2.38)

$$W^u(u_4^\varepsilon) \overline{\pitchfork} W^s(u_2^\varepsilon)$$

two-dimensional. This has to be properly interpreted. In the time dependent framework the above means that there exist two heteroclinic connections $\hat{u}$ and $\tilde{u}$ with

$$\lim_{t\to-\infty} \hat{u}^\varepsilon(x,t) = u_4^\varepsilon(x,t)$$
$$\lim_{t\to-\infty} \tilde{u}^\varepsilon(x,t) = u_4^\varepsilon(x,t)$$

that converge in forward time to appropriately shifted copies of $v_2^\varepsilon(\cdot)$ where we set $u_2^\varepsilon(0,0) = v_2^\varepsilon(0)$, i.e. there exist $\hat{\theta}, \tilde{\theta} \in S^1$ such that

$$\lim_{t\to\infty} \hat{u}^\varepsilon(x,t) = u_2^\varepsilon(x+\hat{\theta},t) \tag{4.3}$$
$$\lim_{t\to\infty} \tilde{u}^\varepsilon(x,t) = u_2^\varepsilon(x+\tilde{\theta},t). \tag{4.4}$$

$\hat{\theta}, \tilde{\theta} \in S^1$ are called the asymptotic phase. The transversality condition does not make any predictions on the phases, it only says that the connection is two-dimensional.

If we include the shift symmetry we obtain that

$$\dim\left(\mathcal{A}_4^\varepsilon\backslash W^s(u_\pm)\right) = \dim\left(W^u(\mathcal{R}_4^\varepsilon) \overline{\pitchfork} W^s(\mathcal{R}_2^\varepsilon)\right) = 3. \tag{4.5}$$

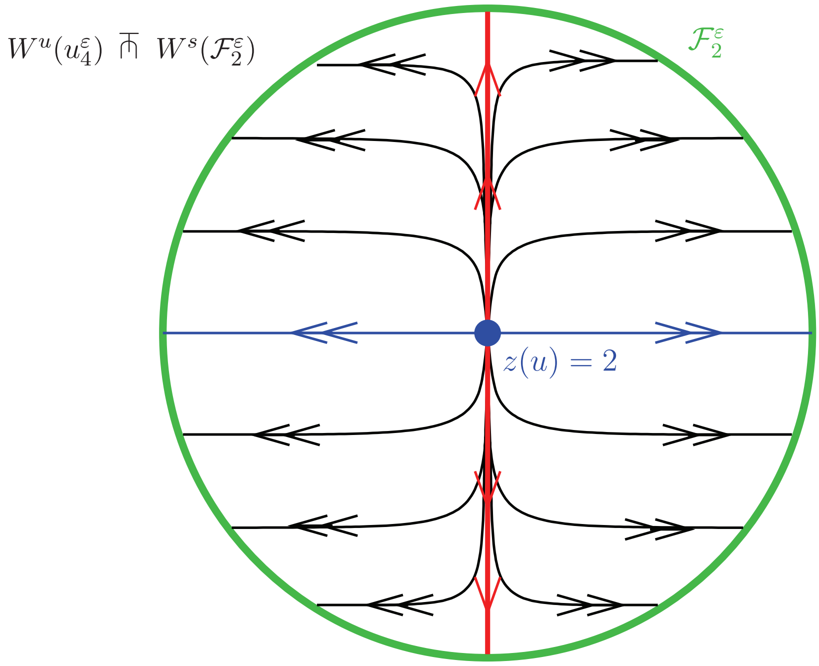

Figure 4.6: Heteroclinic connections in $\mathcal{A}_4^\varepsilon$ from frozen waves with zero-number $z = 2$ to waves with zero-number $z = 1$. The S^1 symmetry is divided out.

or the equivalent result in the case where the waves $u_{4,2}^\varepsilon$ are frozen.

Here we see already that the two-dimensional manifold $W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$ cannot persist completely, because

$$\dim\left(W^u(u_2^0)\right) \pitchfork W^s(u_1^0)) = 1$$

due to uniqueness!

From Remark 3.2.6 and Theorem 3.2.1 we obtain that

$$\lim_{\varepsilon \to 0} \mathcal{A}_4^\varepsilon \backslash W^s(u_\pm) \subset \mathcal{A}_4^0 \backslash W^s(u_\pm). \tag{4.6}$$

In addition we know that

$$\dim \mathcal{A}_4^\varepsilon \backslash W^s(u_\pm^\varepsilon) = \dim \mathcal{A}_4^0 \backslash W^s(u_\pm^0) = 3$$

due to equation (4.5) and Theorem 2.6.1, but this does not imply equality in equation (4.6). Here we use *Assumption (D)* in a neighbourhood $N^\varepsilon(u_4^\varepsilon)$ and $N^0(u_2^0)$. The local surjectivity of the limit in $N^0(u_2^0)$ translates to the existence of heteroclinics in $W^u(u_4^\varepsilon)$ that converge in a neighbourhood of u_4^ε.

Hence there is a heteroclinic connection in

$$W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$$

that locally persists to the connection drawn in blue in Figure 4.4. Corollary 3.2.11 then yields persistence of the full heteroclinic to the blue connection.

Because

$$\dim(W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)) = 2$$

but

$$\dim(W^u(u_2^0) \pitchfork W^s(u_1^0)) = 1$$

the heteroclinic orbit associated to the other linear independent direction $W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$ cannot persist.

This is remarkable because it shows that not only complete connection manifolds between rotating waves of the parabolic equation do not persist. Even within a connection manifold where target and source obey the connection condition (3.15) there are connections that do not persists. This is a result of our dimensional argument.

Can we deduce convergence of $W^u(u_2^\varepsilon) \pitchfork W^s(u_1^\varepsilon)$ to the manifold depicted in Figure 4.4?

Unfortunately the transversality condition of the stable and unstable manifolds in equation (2.38) does not necessarily imply that

$$\overline{W^u(u_4^\varepsilon)} \cap \mathcal{F}_2^\varepsilon = \mathcal{F}_2^\varepsilon \mathrel{\widehat{=}} S^1.$$

As far as I am aware there is no result on the asymptotic phase of the heteroclinic connections already mentioned in equations (4.3) and (4.4). However if we assume this to be true (which would be a consequence of *Assumption (D)*), then we could deduce that the heteroclinic orbit associated to the direction other than the persisting one would converge locally to the line of frozen waves with zero-number $z = 2$ depicted in red in Figure 4.4. In order to deduce global convergence to the red line we would have to choose the correct parameterisation of the red manifold. In other words we would have to choose the correct slice in the full three-dimensional manifold represented by the torus in Figure 4.5. Our parameterisation is such that the shocks have fixed positions on the whole (red) manifold in Figure 4.4. It thus represents the separatrix of the shock movement to the left and the right respectively.

Assuming that this is correctly chosen, then the unstable manifold in the case $\varepsilon > 0$ given by

$$W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$$

would converge pointwise to that depicted in Figure 4.4 and hence viscosity would induce a slow drift on the red manifold of waves with two zeros. This is shown in Figure 4.6. In this light it is plausible that our particular parameterisation of the frozen waves with two zeros is correct. The drift that is induced by the $\varepsilon > 0$ is such that the shocks remain in their position and remain stationary by construction. In all other parameterisations the shocks would have to adiabatically follow the drift of the zeros. There is no reason why this should be happening because the shocks are unstable in the hyperbolic framework and it is to be expected that they are unstable in the parabolic framework also. Note in addition that even if $W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$ would be represented by another parameterisation, hence we would have to choose another section of the torus in Figure 4.5 to obtain the correct illustration, qualitatively Figure 4.6 would remain the same.

If we summarise the results, we observe that the (two-dimensional) part of the unstable manifold of u_4^ε that connects to $\mathcal{F}_2^\varepsilon$ carries a dynamical slow-fast structure. This is a consequence of *Assumption (D)* together with the dimensional argument stating that not all heteroclinics in $W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$ can persist.

In addition we were able to argue that Figure 4.6 represents qualitatively $W^u(u_4^\varepsilon) \pitchfork W^s(u_2^\varepsilon)$.

At this moment, however, I would call the part on the parabolic setting a good educated guess or a conjecture that still needs a rigorous proof. Here I refer to discussion in the Conclusions in Chapter 5.

4.3 Heteroclinic Cascades

To get a parameterisation of a heteroclinic cascade we have to make *Assumption (D)*. However, I would like to mention that the existence of heteroclinic cascades is already a consequence of the *Connection Lemma*, the *Cascading Theorem* and the solved connection problem on the parabolic attractor.

To find a heteroclinic cascade one has to at least consider $\mathcal{A}_6^\varepsilon$ and $\mathcal{A}_6^0$ respectively. The set of frozen waves for $\mathcal{A}_6^0$ is then a three torus $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$. Even if we factor out the rotational symmetry and consider only connections between waves with $z = 3$ and $z = 2$ we have to consider a torus $\mathbb{T}^2$ where each element on the torus has three heteroclinic connections to the one-dimensional sub-torus given by a S^1. This object is four-dimensional.

We will therefore not try to iterate the procedure of the last two sections but only attempt to determine how solutions evolve along a specific connection. Here we only consider Burgers equation (4.2), but the same approach works for any equation.

Let us start this time with the parabolic equation and consider $u_6^\varepsilon \in \mathcal{F}_6^\varepsilon$ with $z(u_6^\varepsilon) = 6$ that connects to $u_4^\varepsilon \in \mathcal{F}_4^\varepsilon$ with $z(u_4^\varepsilon) = 4$.

The two waves converge for $\varepsilon \to 0$ to $u_3^0, u_2^0 \in \mathcal{F}^0$ with $z(u_3^0) = 3$ and $z(u_2^0) = 2$. Lemma 3.2.8 states that u_3^0 and u_2^0 are not connected. Hence, connections between u_6^ε and u_4^ε converge for $\varepsilon \to 0$ either to a line of equilibria or to a heteroclinic cascade. In the latter case following the result of the previous section it is clear that the connection itself then carries a slow-fast structure for $\varepsilon > 0$.

Panel a) in Figure 4.7 shows the possible targets of u_3^0 labeled u_{2a}^0, u_{2b}^0 and u_{2c}^0. None of the targets is a limit of a frozen wave of the parabolic equation.

We construct a connection between u_3^0 and u_2^0 consisting of heteroclinics and frozen waves in $\mathcal{A}_6^0$, based on the assumption that the development of solutions along the manifold of frozen waves is such that either shocks do not move or they move in the same way as the neighbouring zeros. This implies due to symmetry that neighbouring zeros drift at the same speed. In the case where the sign of their movements differs, the shock stays at its position, whereas in the other case the profile between the zeros stays unchanged.

Following the construction of the last section this assumption makes sense, however, I cannot prove that it must be like this.

Panel b) in Figure 4.7 shows possible ways how to construct connections between u_3^0 and u_2^0 based on the above assumptions. The smaller diagrams show the source and target of the desired connection and two intermediate steps $\tilde{u}, \hat{u}$ which are a target ($\hat{u}$) or a source ($\tilde{u}$) of a heteroclinic connection of the hyperbolic equation. Red arrows in the small diagrams correspond to the movement of zeros along the manifold, blue arrows indicate shock movements.

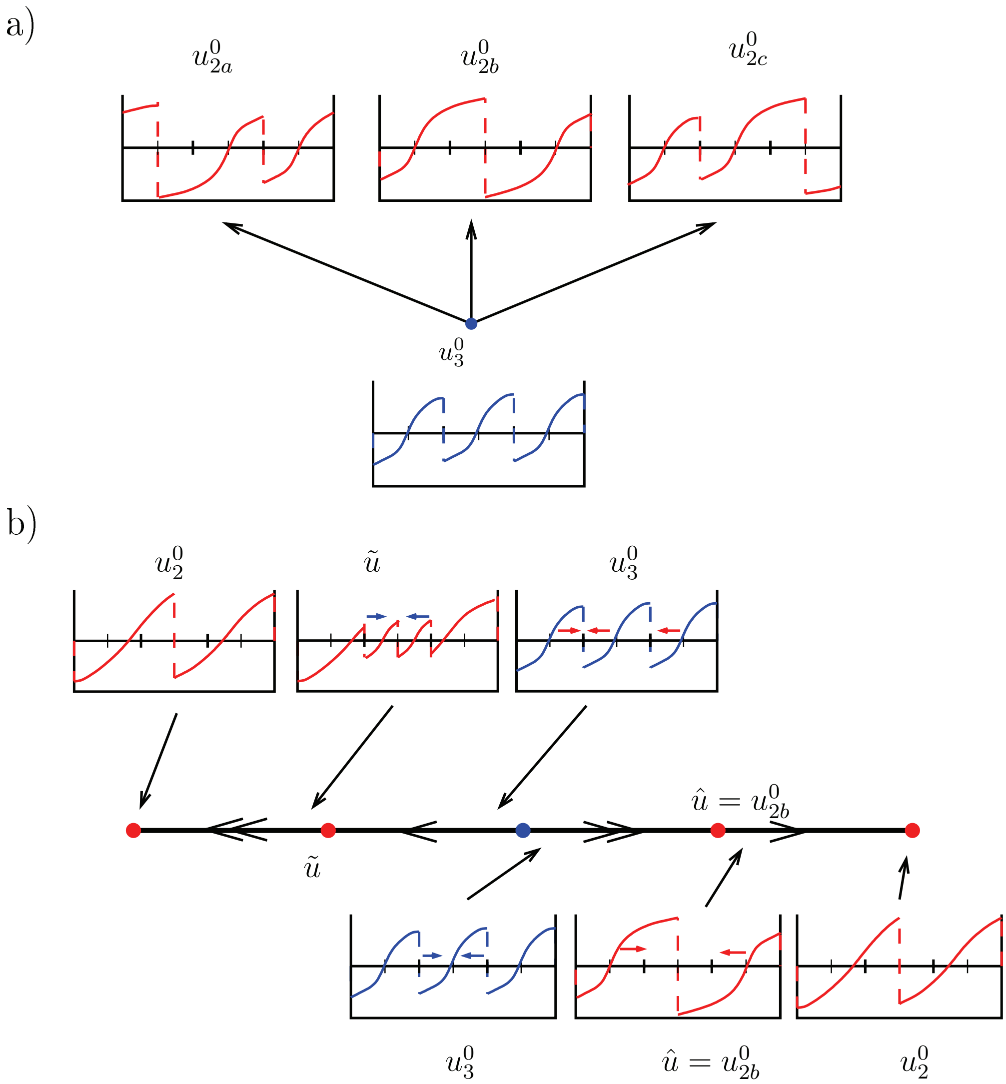

Figure 4.7: Heteroclinic connections in $\mathcal{A}_6^0$ from frozen waves with zero-number $z = 3$ to waves with zero-number $z = 2$.

We start the explanation of Panel b) with the connection to the right. We first use the heteroclinic connecting to u_{2b}^0. Then we let the zeros drift towards each other until they have reached the positions of the zeros in the target u_2^0.

The connection to the left starts with a line of frozen waves until two of the zeros are at the position of the target u_2^0. Both connections consist of a heteroclinic already described in Panel a) and a line of equilibria that is contained in $\mathcal{F}_6^0$.

I believe there exist heteroclinic connections from u_6^ε to u_4^ε that converge to the above constructed heteroclinics and lines of equilibria, but again a rigorous proof is lacking.

This approach can be adapted to waves with more and more zeros. Figure 4.8 shows the construction for a situation where the source u_{12}^ε has twelve zeros and the target u_2^ε has two. Hence there can be at most three heteroclinic connections in the cascade. The dynamical slow-fast structure of the heteroclinic is shown on the right. The respective limiting objects for $\varepsilon = 0$ are shown in the large square on the left. The blue crosses symbolise shocks, the

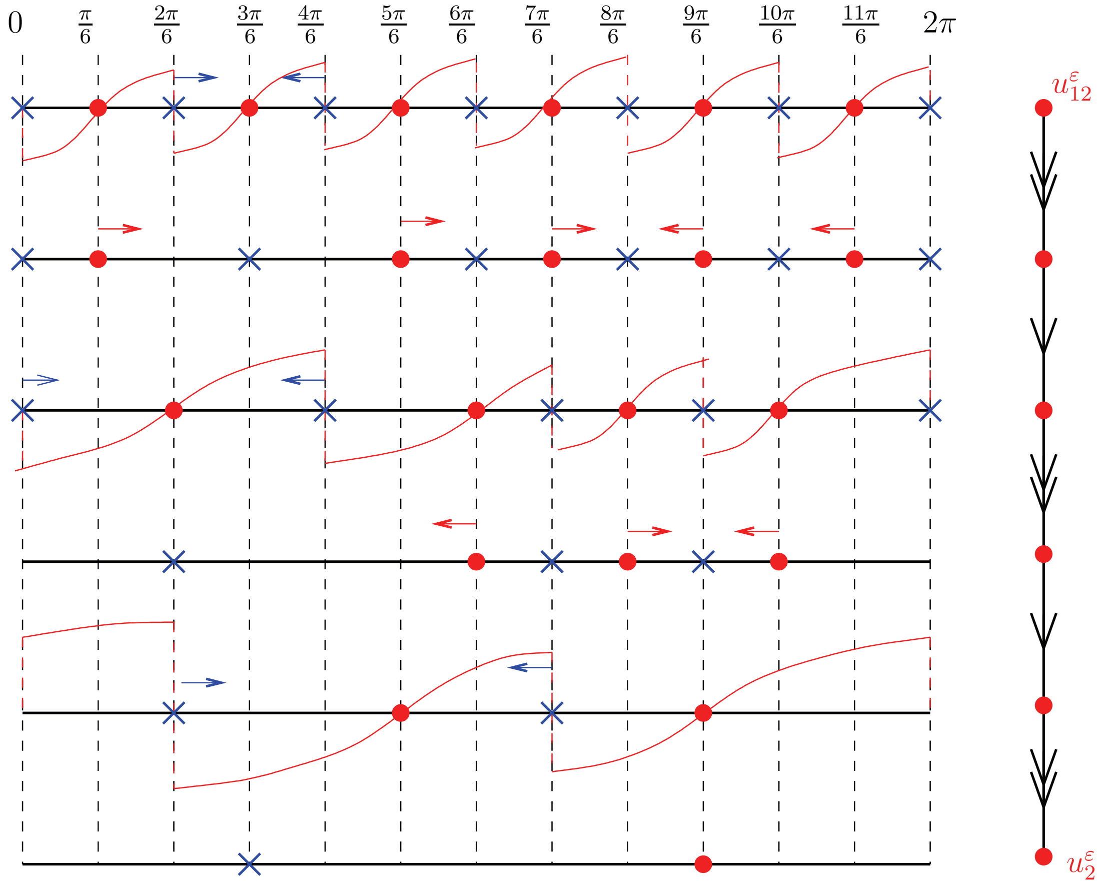

Figure 4.8: Heteroclinic connection between a wave with zero-number $z = 12$ and a wave with zero-number $z = 2$.

red dots zeros. As in Figure 4.7 the red arrows represent slow drifts of the zeros and blue arrows represent fast movements of shocks. Every second profile is plotted for illustration.

Chapter 5

Conclusions

The starting point of this dissertation was the question of the relation between solutions on the global attractor of the viscous balance law (P) and its hyperbolic limit (H). Both equations possess a global attractor that can be described by the set of equilibria, rotating waves and heteroclinic connections. Despite the fact that all equilibria and rotating or frozen waves of the parabolic equation persist to equilibria and frozen waves of the hyperbolic equation and the additional pointwise convergence of all solutions on the attractor, heteroclinic connections do in general not persist.

Even in the case of the finite dimensional sub-attractors this implies that the sub-attractors do not persist in the sense of solutions, i.e. $\mathcal{A}_n^\varepsilon$ is not C^0-orbit equivalent to $\mathcal{A}_n^0$. The only exception seems to be the sub-attractor of order two where we could prove rigorously

$$\lim_{\varepsilon \to 0} \mathcal{A}_2^\varepsilon = \mathcal{A}_2^0 \tag{5.1}$$

in the sense of solutions and sequences which implies C^0-orbit equivalence. For the higher dimensional cases the result on the dimensions of subattractors

$$\dim \mathcal{A}_n^0 = \dim \mathcal{A}_n^\varepsilon = n$$

and the consequence of the persistence theorem

$$\lim_{\varepsilon \to 0} \mathcal{A}_n^\varepsilon \subset \mathcal{A}_n^0$$

suggests that equality holds in the last equation in the sense of sequences. However we do not yet have a proof for this.

An important tool in the low dimensional case was the result of the explicit parameterisation of all sub-attractors $\mathcal{A}_n^0$ by $\mathbf{A_n}$ in the hyperbolic setting. This closes one of the last gaps in the full geometric description of the global attractor of equation (H). The missing link here lies in the geometric description of heteroclinics between frozen waves with uncountable zero set. However I believe our approach to be applicable in this case as well.

This would still not be sufficient to prove the convergence of the full parabolic attractor $\mathcal{A}^\varepsilon$ to the hyperbolic attractor $\mathcal{A}^0$ so this remains an open question.

Moreover it is unclear to me how we can prove the limiting cascade of heteroclinics for a connection between given target and source in the parabolic setting for large zero-numbers.

A rigorous proof of *Assumption (D)* would be a start in this direction. This would yield local persistence of manifolds and as a consequence prove global persistence of the fast connections on the parabolic attractor. It would also imply that condition (3.15) in the *Connection Lemma* was not only necessary but sufficient for the persistence of at least one heteroclinic connection between the respective target and source. For the slow parts converging to frozen waves the result would remain local.

In this sense the *Cascading Theorem* re-opens Pandora's box of possible limits of heteroclinics in equation (P), which Fan and Hale had seemingly closed in the mid '90s by their persistence result.

Although the slow manifolds on the parabolic attractor converge to frozen waves of the hyperbolic equation, these manifolds have to be considered as being far from equilibria. A local persistence result of stable or unstable manifolds of rotating or frozen waves would not be applicable. An approach that could yield a way out of this impasse towards the description of the slow parts of such cascades might be the description of heteroclinics by virtue of invariant manifold theory.

Carr and Pego already achieved this in a very explicit approach in the '90s (see [CP89],[CP90]) for the case of the dynamic Allan-Chan equation where

$$f(u) = 0$$

with Neumann boundary conditions. Their work has never been generalised to viscous balance laws. The transport term $f(u)$ here introduces several technical difficulties, some of which have been already mentioned in the beginning of Chapter 4. Especially the eigenvalue problem (4.1) becomes a lot more challenging. However, our results suggest strong similarities to the results of Carr and Pego.

Hence, there still remains much scope for exploration!

Chapter 6

Appendix: Notation

Here you will find a list of expressions and notation. Constants are only listed if they are of relevance throughout the document.

ε	viscosity parameter in the balance law
a,b	zero-number for rotating waves of the parabolic equation
α,β	zero-number for frozen waves of the hyperbolic equation
t	time variable
T	fixed time
x	spatial variable, large scale
$\mathbf{x}_\alpha$	set of zeros $\{x_1, \ldots, x_\alpha\}$
ξ	spatial variable on the small scale ($\xi = \frac{x}{\varepsilon}$)
$\partial_x, \partial_t, \partial_\xi$	partial derivative w. respect to x,t,ξ
∂_{xx}	second partial derivative with respect to x (t, ξ respectively)
$f(u)$	transport term
$g(u)$	source term
$u^\varepsilon(x,t)$	general notation of a solution of the PDE (P)
$u^0(x,t)$	general notation of a solution of the PDE (H)
c	wave speed
u_Z	rotating wave with zero set Z
$u^\varepsilon_a(\cdot,t)$	time-dependent rotating wave for $\varepsilon > 0$ with zero-number a, b
$u^0_{\alpha,\beta}(\cdot,t)$	rotating wave for $\varepsilon = 0$ with zero-number α,β
$v^\varepsilon(\cdot), v^0(\cdot)$	solutions of the rotating wave equation
v, p	rotating wave of (P) in Lienard coordinates
w, q	rotating wave of (P) in phase plane coordinates
ϕ	solution of the stationary problem of (H)

$\chi(t)$	characteristic
$\chi^{\pm}(t)$	maximal and minimal backward characteristic
$\underline{\mathrm{v}}$	value of a solution on a characteristic χ
$u(x\pm,\cdot)$	right and left hand limit of u in x
φ	test function
$\nu_{x,t}$	family of borel probability measures
X, X'	phase space and dual space
L^1, L^2, L^∞	space of integrable, squareintegrable and bounded functions
H^2	space of twice weakly differentiable L^2 functions
BV	space of functions with bounded variation
$L(u)$	linear operator representing the linearisation of (P) un u
$\sigma(L)$	spectrum of L
$A_{\{x_1,\ldots,x_\alpha\}}$	set of functions consisting piecewise of $\phi(x-x_j)$ and α shocks separated by the x_j
$\tilde{A}_{\{x_1,\ldots,x_\alpha\}}$	set of functions consisting piecewise of $\phi(x-x_j)$
$\mathbf{A}_\alpha$	set of all $A_{\{x_1,\ldots,x_\alpha\}}$ for fixed α
B_a	set of rotating or frozen waves for $0<\varepsilon<\varepsilon_0$ with $z=a$
B_{ab}	set of heteroclinics for $0<\varepsilon<\varepsilon_0$ between rotating or frozen waves with $z=a$ and $z=b$
$U^\varepsilon(\tau)$	parameterisation of heteroclinic orbit by arc length
$\mathcal{U}_1$,$\mathcal{U}_2$	set of limits of $U^{\varepsilon_n}(\tau_n)$
$W^u(v)$	unstable manifold of v
$W^s(v)$	stable manifold of v
$\mathcal{F}_l$	slow manifold in Lienard coordinates
$\mathcal{F}_p$	slow manifold in phase plane coordinates
$\mathcal{C}_l$	cyclicity set in Lienard coordinates
$\mathcal{C}_p$	cyclicity set in phase plane coordinates
$\mathbf{c}$	map assigning each periodic wave in the cyclicity set its wave speed
$\mathcal{T}$	map assigning each periodic wave in the cyclicity set its minimal period
S^n	n-sphere
$\mathbb{T}^n$	n-torus
Γ_1, Γ_2	sets of sequences $\{\varepsilon_n\}\{\tau_n\}$
$\mathcal{P}$	set of partitions $P=\{x_1,\ldots x_n\}$
$\mathbb{P}(\mathcal{Z})$	powerset of $\mathcal{Z}$
$i(u)$,$i_0(u)$	Morse index and generalised Morse index of u
$z(u)$	zero-number of u
$\mathcal{Z}(u)$	zeroset of u

$\mathcal{E}^\varepsilon$,$\mathcal{E}^0$	set of homogenous equilibria of (P) and (H)
$\mathcal{F}^\varepsilon$,$\mathcal{F}^0$	set of frozen waves of (P) and (H)
$\mathcal{R}^\varepsilon$,$\mathcal{R}^0$	set of rotating waves of (P) and (H)
$\mathcal{H}^\varepsilon$,$\mathcal{H}^0$	set of heteroclinic orbits of (P) and (H)
$\mathcal{A}^\varepsilon$,$\mathcal{A}^0$	global attractor of (P) and (H)
$\mathcal{E}_n^\varepsilon$,$\mathcal{E}_n^0$	subset of homogenous equilibria of order n (P) and (H)
$\mathcal{F}_n^\varepsilon$,$\mathcal{F}_n^0$	set of frozen waves of of order n (P) and (H)
$\mathcal{R}_n^\varepsilon$,$\mathcal{R}_n^0$	set of rotating waves of of order n (P) and (H)
$\mathcal{H}_n^\varepsilon$,$\mathcal{H}_n^0$	subset of heteroclinic orbits of order n (P) and (H)
$\mathcal{A}_n^\varepsilon$,$\mathcal{A}_n^0$	sub-attractor of order n for (P) and (H)

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