Chapter 7

Fuzzy control

7.1 Introduction

The aim of this chapter is to define fuzzy control systems and to use this technique in the control of the robotic head for the object tracking. As important objective is also to carry out a comparison of the results obtained among this technique and the adaptive used in the previous chapter. The comparison not only seeks to show the technical results with behavior indexes but also to present comparisons in time of implementation and tunning.

Traditionally, an intelligent control system is defined as one in which classical control theory is combined with artificial intelligence (AI) and possibly OR (Operations Research). Stemming from this definition, two approaches to intelligent control have been in use. One approach combines expert systems in AI with differential equations to create the so called expert control, while the other integrates discrete event systems (Markov chains) and differential equations [85]. The first approach, although practically useful, is rather difficult to analyze because of the different natures of differential equations (based on mathematical relations) and AI expert systems (based on symbolic manipulations). The second approach, on the other hand, has well developed and solid theory, but is too complex for many practical applications. It is clear, therefore, that a new approach and a change of course are called for here. We begin with another definition of an intelligent control system. An intelligent control system is one in which a physical system or a mathematical model of it is being controlled by a combination of a knowledge-base, approximate (human-like) reasoning, and/or a learning process structured in a hierarchical fashion. Under this simple definition, any control system which involves fuzzy logic, neural networks, expert learning schemes, genetic algorithms, genetic programming or any combination of these would be designated as intelligent control. Among the many applications of fuzzy sets and fuzzy logic, fuzzy control is perhaps the most common. Most industrial fuzzy logic applications in Japan, the U.S., and Europe fall under fuzzy control. The reasons for the success of fuzzy control are both theoretical and practical [85]. From a theoretical point of view, a fuzzy logic rule-base, can be used to identify both a model, as a universal approximation, as well as a nonlinear controller. The most relevant information about any system comes in one of three waysa mathematical model, sensory input/output data, and human expert knowledge. The common factor in all these three sources is knowledge. For many years, classical control designers began their effort with a mathematical model and did not go any further in acquiring more knowledge about the system, i.e., designers put their entire trust in a mathematical model whose accuracy may sometimes be in question. Today, control engineers can use all of the above sources of information. Aside from a mathematical model whose utilization is clear, numerical (input/output) data can be used to develop an approximate model (input/output nonlinear mapping) as well as a controller, based on the acquired fuzzy IF-THEN rules. Some researchers and teachers of fuzzy control systems subscribe to the notion that fuzzy controls should always use a model free design approach and, hence, give the impression that a mathematical model is irrelevant. As indicated before, the authors, however, believe strongly that if a mathematical model does exist, it would be the first source of knowledge used in building the entire knowledge base. From a mathematical model, through simulation, for example, one can further build the knowledge base. Through utilization of the expert operators knowledge which comes in the form of a set of linguistic or semilinguistic IF-THEN rules, the fuzzy controller designer would get a big advantage in using every bit of information about the system during the design process. On the other hand, it is quite possible that a system, such as high dimensional large-scale systems, is so complex that a reliable mathematical tool either does not exist or is very costly to attain. This is where fuzzy control (or intelligent control) comes in. Fuzzy control approaches these problems through a set of local humanistic (expert-like) controllers governed by linguistic fuzzy IF-THEN rules. In short, fuzzy control falls into the category of intelligent controllers, which are not solely model-based, but also, knowledge-based. From a practical point of view, fuzzy controllers, which have appeared in industry and in manufactured consumer products, are easy to understand, simple to implement, and inexpensive to develop. Because fuzzy controllers emulate human control strategies, they are easily understood even by those who have no formal background in control. These controllers are also very simple to implement.

7.2 Basic definitions

A common definition of a fuzzy control system is that it is a system which emulates a human expert. In this situation, the knowledge of the human operator would be put in the form of a set of fuzzy linguistic rules. These rules would produce an approximate decision, just as a human would. Consider Figure 7.1, where a block diagram of this definition is shown. As shown, the human operator observes quantities by observing the inputs, i.e., reading a meter or

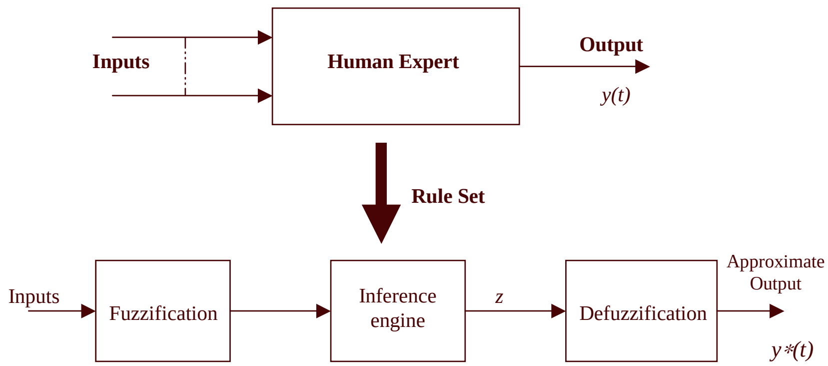


Figure 7.1: Conceptual Definition of a Fuzzy Control System.

measuring a chart, and performs a definite action (e.g., pushes a knob, turns on a switch, closes a gate, or replaces a fuse) thus leading to a crisp action, shown here by the output variable *y(t)*. The human operator can be replaced by a combination of a fuzzy rule-based system (FRBS) and a block called *defuzzifier*. The input sensory (crisp or numerical) data are fed into FRBS where physical quantities are represented or compressed into linguistic variables with appropriate membership functions. These linguistic variables are then used in the *antecedents* (IF-Part) of a set of fuzzy rules within an inference engine to result in a new set of fuzzy linguistic variables or *consequent* (THEN-Part). Variables are then denoted in this figure by *z*, and are combined and changed to a crisp (numerical) output *y*(t)* which represents an approximation to actual output y(t). It is therefore noted that a fuzzy controller consists of three operations: (1) fuzzification, (2) inference engine, and (3) defuzzification.

The fuzzification operation, or the *fuzzifier* unit, represents a mapping from a crisp point $x = (x_1 x_2 ... x_n)^T \in X$ into a fuzzy set $A \in X$, where X is the universe of discourse and T denotes vector or matrix transposition. There are normally two categories of fuzzifiers in use. The first is singleton and the second is nonsingleton. A singleton fuzzifier has one point (value) x_p as its fuzzy set support, i.e., the membership function is governed by the following relation:

$$\mu_A(x) = \begin{cases} 1, \ x = x_p \ \in \ X \\ 0, \ x \neq x_p \ \in \ X \end{cases} \tag{7.1}$$

The nonsingleton fuzzifiers are those in which the support is more than a point. Examples of these fuzzifiers are triangular, trapezoidal, Gaussian, etc. In these fuzzifiers, $\mu_A(1) = 1$ at $x = x_p$ where $x = x_p$ may be one or more than one point, and then $\mu_A(x)$ decreases from 1 as x moves away from $x = x_p$ or the core region to which $x = x_p$ belongs such that $\mu_A(x_p)$ remains 1 (see Section 1.5). For example, the following relation represents a Gaussian-type fuzzifier:

$$\mu_A(x) = \exp\left\{-\frac{(x-x_p)^T(x-x_p)}{\sigma^2}\right\} \tag{7.2}$$

where the variance, σ^2 , is a parameter characterizing the shape of $\mu_A(x)$.

7.2.1 Inference Engine

The cornerstone of any expert controller is its inference engine, which consists of a set of expert rules, which reflect the knowledge base and reasoning structure of the solution of any problem. A fuzzy (expert) control system is no exception and its rule base is the heart of the nonlinear fuzzy controller. A typical fuzzy rule can be composed as [87]

IF *A* is A_1 AND B is B_1 OR *C* is C_1 THEN *U* is U_1

(7.3)

where A, B, C and U are fuzzy variables, A_1, B_1,C_1 and $U_1$are fuzzy linquistic values (membership functions or fuzzy linguistic labels), AND, OR, and NOT are connectives of the rule. The rule in Equation 7.3 has three antecedents and one consequent. Typical fuzzy variables may in fact, represent physical or system quantities such as: temperature, position, output, elevation, etc. and typical fuzzy linguistic values (labels) may be hot, very close, low, etc. The portion very in a label very high is called a linquistic hedge. Other examples of a hedge are much, slightly, more, or less, etc. The above rule is known as Mamdani type rule. In Mamdani rules the antecedents and the consequent parts of the rule are expressed using linguistic labels. In general in fuzzy system theory, there are many forms and variations of fuzzy rules, some of which will be introduced here and throughout the chapter. Another form is Takagi-Sugeno rules in which the consequent part is expressed as an analytical expression or equation.

Two cases will be used here to illustrate the process of inferencing graphically. In the first case the inputs to the system are crisp values and we use max-min inference method. In the second case, the inputs to the system are also crisp, but we use the max-product inference method.

Consider the following rule whose consequent is not a fuzzy implication

IF x_1 is A_1^i AND x_2 is A_2^i THEN y^i is B^i, for *i*=1,2,*j*

(7.4)

Where A_1^i and A_2^i are the fuzzy sets representing the *i*th-antecedent pairs, and B^i are the fuzzy sets representing the *i*th-consequent, and *j* is the number of rules.

Case 1. Inputs x_1 and x_2 are crisp values, and max-min inference method is used. Based on the Mamdani implication method of inference, and for a set of disjunctive rules, i.e, rules connected by the OR connective, the aggregated output for the j rules presented in Equation 7.4 will be given by

$$\mu_{B^i}(y) = \max_i \left[\min\left[\mu_{A_1^i}(x_1), \mu_{A_2^i}(x_2)\right]\right], \text{for } i = 1, 2, ..., j \tag{7.5}$$

Figure 7.2 is a graphical illustration of Equation 7.5, for *j*=2, where A_1^1 and A_2^1 refer to the first and second fuzzy antecedents of the first rule, respectively, and B^1 refers to the fuzzy consequent of the first rule. Similarly, A_1^2 and A_2^2 refer to the first and second fuzzy antecedents of the second rule, respectively, and B^2refers to the fuzzy consequent of the second rule. Because the antecedent pairs used in general form presented in Equation 7.4 are connected by a logical *AND*, the minimum function is used. For each rule, minimum value of the antecedent propagates through and truncates the membership function for the consequent. This is done graphically for each rule. Assuming that the rules are disjunctive, the aggregation operation *max* results in an aggregated membership function comprised of the outer envelope of the individual truncated membership forms from each rule. To compute the final crisp value of the aggregated output, defuzzification is used, which will be explained in the next section.

Case 2: Inputs x_1 and x_2 are crisp values, and max-product inference method is used. Based on the Mamdani implication method of inference, and for a set of disjunctive rules, the aggregated output for the *l* rules presented in Equation 7.4 will be given by

$$\mu_{B^i}(y) = \max\left[\mu_{A_1^i}(x_1), \mu_{A_2^i}(x_2)\right], \text{for } i = 1, 2, ..., j \tag{7.6}$$

Figure 7.3 is a graphical illustration of Equation 7.6, for j=2, where A_1^1 and A_2^1 refer to the first and second fuzzy antecedents of the first rule, respectively, and B^1 refers to the fuzzy consequent of the first rule. Similarly, A_1^2 and A_2^2 refer to the first and second fuzzy antecedents of the second rule, respectively, and B^2 refers to the fuzzy consequent of the second rule. Since the antecedent pairs used in general form presented in Equation 7.4 are connected by a logical *AND*, the minimum function is used again. For each rule, minimum value of the antecedent propagates through and scales the membership function for the consequent. This is done graphically for each rule. Similar to the first case, the aggregation operation *max* results in an aggregated membership function comprised of the outer envelope of the individual truncated membership forms from each rule. To compute the final crisp value of the aggregated output, defuzzification is used.

7.2.2 Defuzzification

Defuzzification is the third important element of any fuzzy controller. In this section, only the *center of gravity defuzzifier*, which is the most common one, is

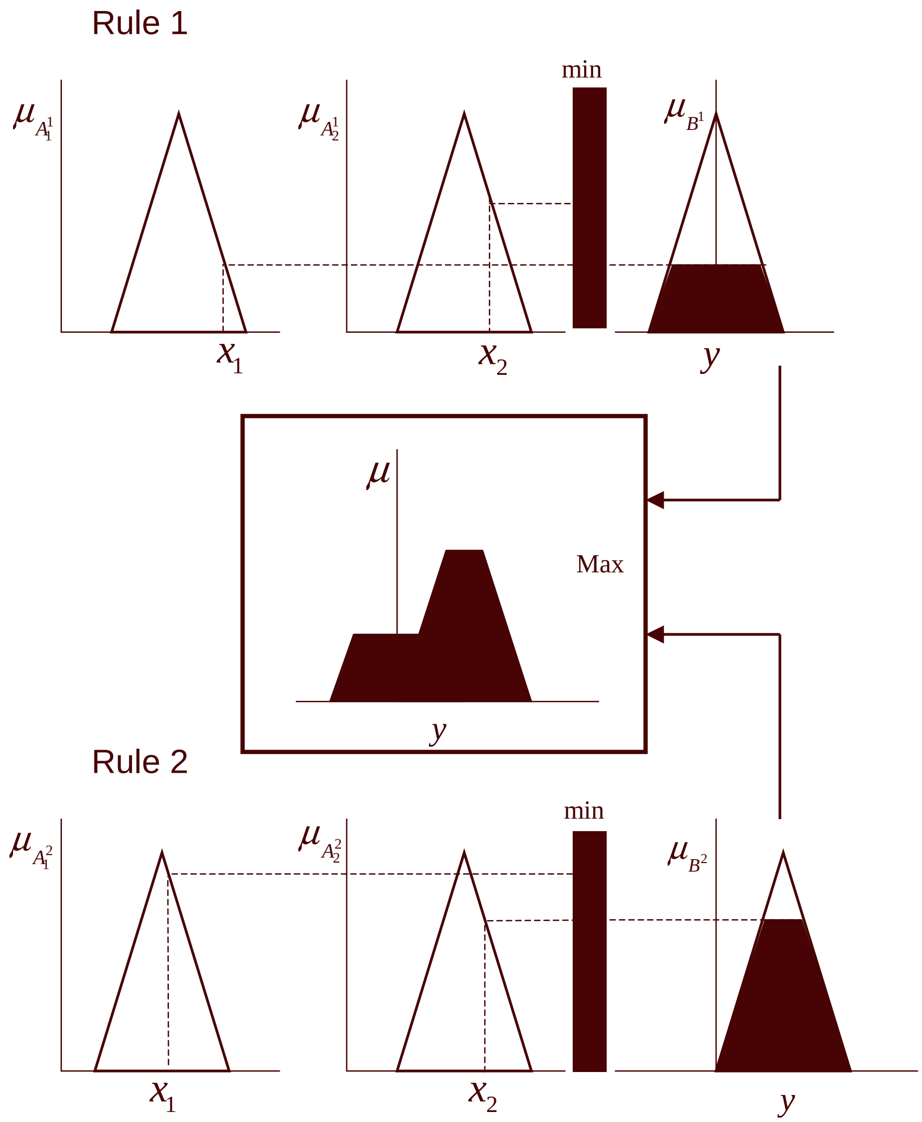


Figure 7.2: Max-min Inference method.

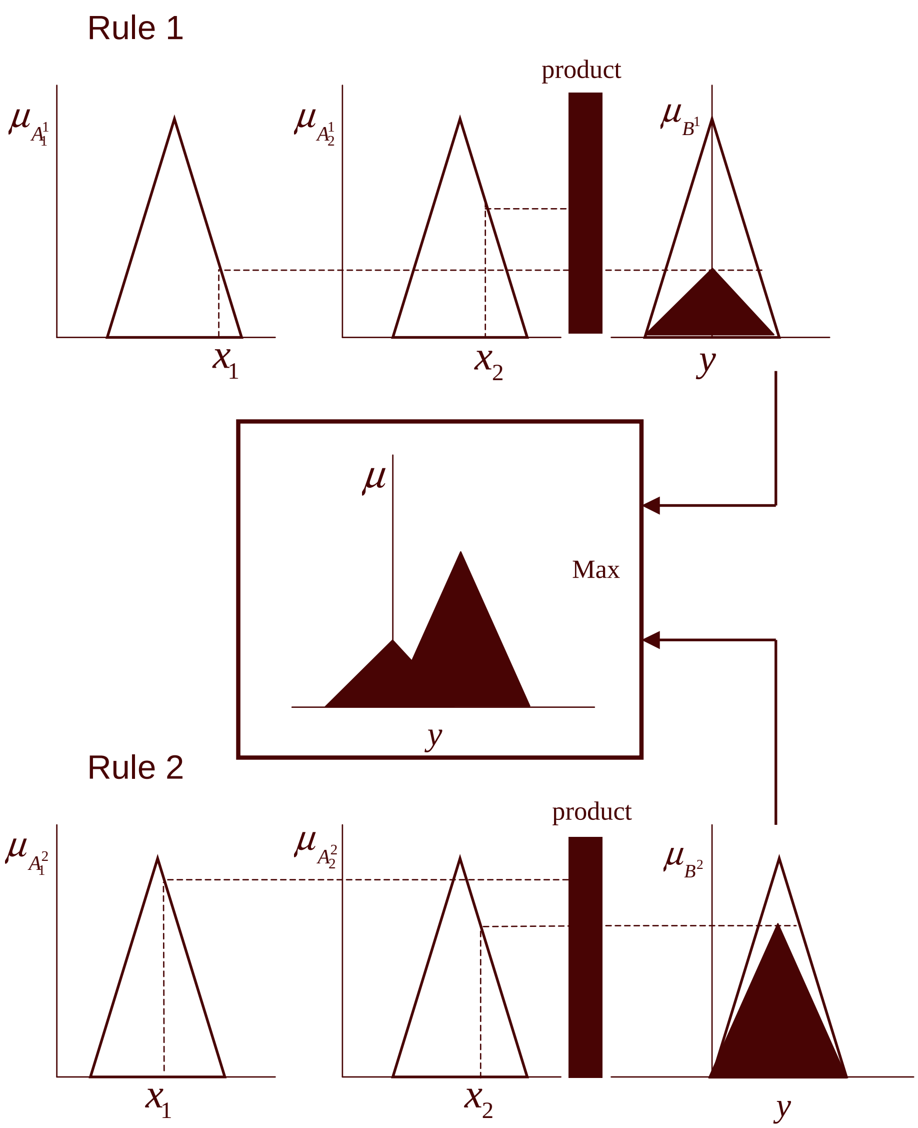


Figure 7.3: Max-product Inference method.

discussed. In this method the weighted average of the membership function or the center of gravity of the area bounded by the membership function curve is computed as the most typical crisp value of the union of all output fuzzy sets:

$$y_c = \frac{\int y\mu_A(y)dy}{\int \mu_A(y)dy} \tag{7.7}$$

7.3 Fuzzy control design

One of the first steps in the design of any fuzzy controller is to develop a knowledge base for the system to eventually lead to an initial set of rules. There are at least five different methods to generate a fuzzy rule base [88]:

1. Simulate the closed-loop system through its mathematical model,
2. Interview an operator who has had many years of experience controlling the system,
3. Generate rules through an algorithm using numerical input/output data of the system,
4. Use learning or optimization methods such as neural networks (NN) or genetic algorithms (GA) to create the rules, and,
5. In the absence of all of the above, if a system does exist, experiment with it in the laboratory or factory setting and gradually gain enough experience to create the initial set of rules.

7.3.1 Example 1

Consider the linearized model of the inverted pendulum Figure 7.4, described by the equation given below,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 15.79 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1.46 \end{bmatrix} u$$

with *l*=0.5m, *m*=100g, and initial conditions $x^T(0) = \begin{bmatrix} \theta(0) & \dot{\theta}(0) \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. It is desired to stabilize the system using fuzzy rules.

Clearly this system is unstable and a controller is needed to stabilize it. To generate the rules for this problem only common sense is needed, i.e., if the pole is falling in one direction then push the cart in the same direction to counter the movement of the pole. To put this into rules of the form Equation 7.4 we get the following:

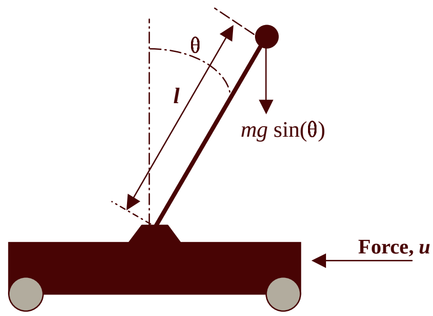


Figure 7.4: Inverted Pendulum.

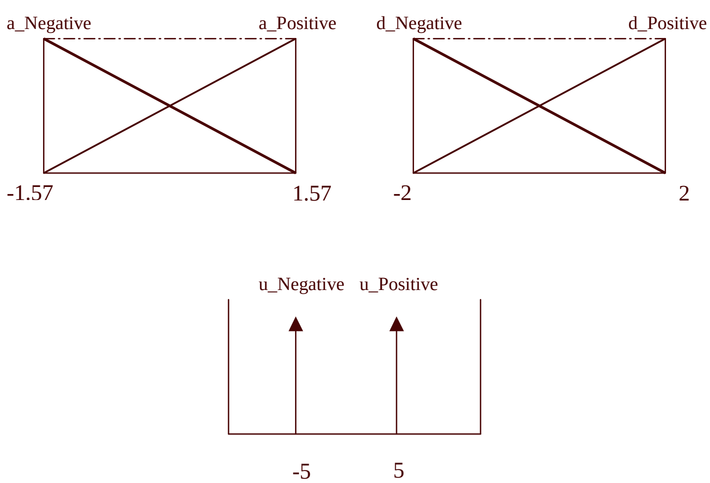


Figure 7.5: Membership Functions for the Inverted Pendulum Problem.

IF θ **a_Positive** and $\dot{\theta}$ is **d_Positive** THEN u is **u_Negative**

IF θ **a_Negative** and $\dot{\theta}$ is **d_Negative** THEN u is **u_Positive**

As shown in Figure 7.5, the membership functions for the inputs are half-triangular, while the membership function of the output is singleton. By simulating the system with fuzzy controller we get the response shown in Figure 7.6(b). It is clear that the system is stable. In this example only two rules were used, but more rules could be added in order to get a better response.

7.4 Analisis of fuzzy control systems

In this section, some results of Tanaka and Sugeno [89] with respect to analysis of feedback fuzzy control systems will be briefly discussed. This section would use Takagi-Sugeno models to develop fuzzy block diagrams and fuzzy closed-loop models. Consider a typical Takagi-Sugeno fuzzy plant model represented by implication P^iin Figure 7.7.

$$P^i : \text{IF } x(k) \text{ is } A_1^i \text{ AND...} x(k-n+1) \text{ is } A_n^i ...$$

$$\text{...AND } u(k) \text{ is } B_1^i \text{ AND...AND } u(k-m+1) \text{ is } B_m^i$$

$$\text{THEN } x^i(k+1) = a_0^i + a_1^i x(k) + ...$$

$$... + a_n^i x(k-n+1) + b_1^i u(k) + ... + b_n^i u(k-m+1) \tag{7.8}$$

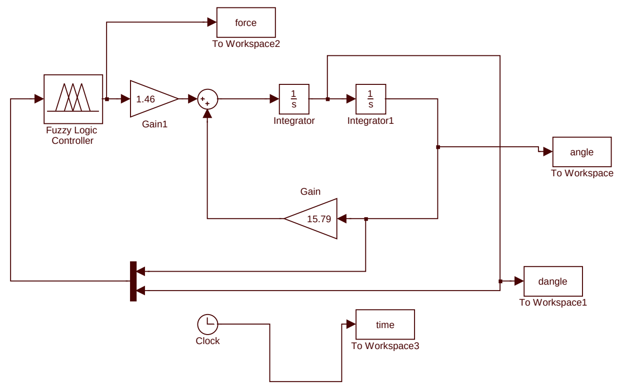
where P^i, (*i*=1,2,...,*j*) is the ith implication, *j*, is the total number of implications, a_p^i , (*p*=1,2,...,*n*) and b_q^i (*q*=1,2,...,*m*) are constant consequent parameters, k is time sample, *x(k)*,..., *x(k-n*+1*)* are input variables, *n* and m are the number of antecedents for states and inputs, respectively. The terms A_p^i and B_p^i are fuzzy sets with piecewise-continuous polynomial (PCP) membership functions. PCP is defined as follows.

Definition 1

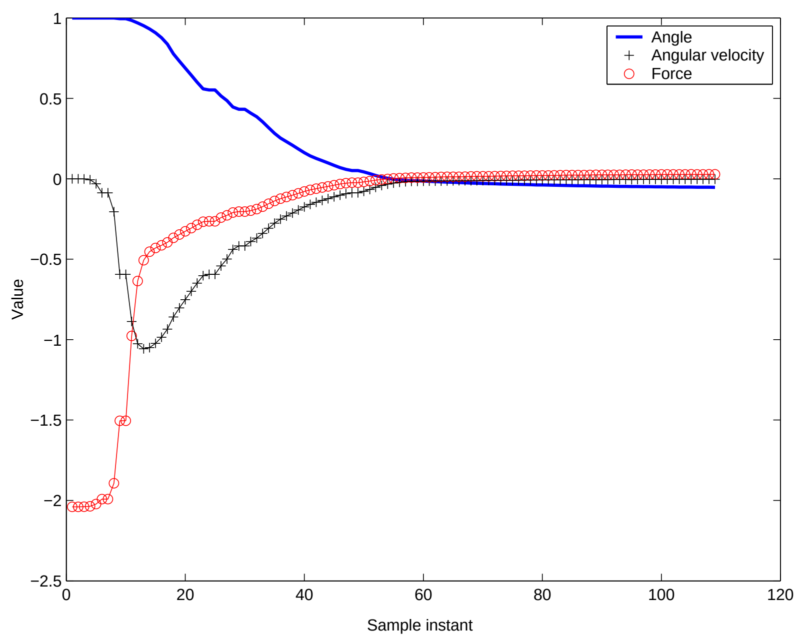
A fuzzy set A satisfying the following properties is said to be a piecewise continuous polynomial (PCP) membership function *A*(*x*) [4]:

$$A(x) = \begin{cases} \mu_1(x), x \in [p_0, p_1] \\ \vdots \\ \mu_s(x), x \in [p_{s-1}, p_s] \end{cases} \tag{7.9}$$

where $\mu_i(x) \in [0,1]$ for $x \in [p_{i-1,}p_i]$, *i*=1,2,...,*s* and $-\infty < p_0 < p_1 < ... < p_{s-1} < p_s < \infty$.



(a)



(b)

Figure 7.6: Simulation result for example 1, (a) Simulink diagram and (b) System response.

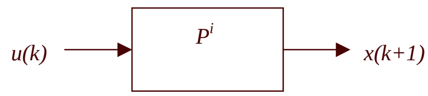


Figure 7.7: Single-Input, Single-Output Fuzzy Block Represented by ith Implication P^i.

$$\mu_i(x) = \sum_{j=0}^{n_i} c_j^i x^j \tag{7.10}$$

where c_j^i are known parameters of polynomials $\mu_i(x)$.
Given the inputs

$$\mathbf{x}(k) = \begin{bmatrix} x(k) & x(k-1) & \cdots & x(k-n+1) \end{bmatrix}^T$$

$$u(k) = \begin{bmatrix} u(k) & u(k-1) & \cdots & u(k-m+1) \end{bmatrix}^T \tag{7.11}$$

Using the above vector notation, Equation 7.11 can be represented in the following form,

$$P^i : \text{IF } \mathbf{x}(k) \text{ is } \mathbf{A}^i \text{ AND } \mathbf{u}(k) \text{ is } \mathbf{B}^i$$

$$\text{THEN } x^i(k+1) = a_0^i + \sum_{p=1}^{n} a_p^i x(k-p+1) + \sum_{q=1}^{m} b_q^i u(k-q+1) \tag{7.12}$$

where $\mathbf{A}^i \equiv \begin{bmatrix} A_1^i & A_2^i & \cdots & A_n^i \end{bmatrix}^T$, $\mathbf{B}^i \equiv \begin{bmatrix} B_1^i & B_2^i & \cdots & B_m^i \end{bmatrix}$, "*x*(*k*) is $\mathbf{A}^i$" are equivalent to antecedent "*x*(*k*) is A_1^i AND ... *x*(*k*-*n*+1) is A_n^i". The final defuzzified output of the inference is given by a weighted average of $x^i(k+1)$:

$$x(k+1) = \frac{\sum_{i=1}^{j} w^i x^i(k+1)}{\sum_{i=1}^{j} w^i} \tag{7.13}$$

where it is assumed that the denominator of Equation 7.13 is positive, and $x^i(k+1)$ is calculated from the ith implication, and the weight w^i wi refers to the overall truth value of the ith implication premise for the inputs in Equation 7.12.

Since the product of two PCP fuzzy sets can be considered as a series connection of two fuzzy blocks of the type in Figure 7.7, it is concluded that the convexity of fuzzy sets in succession is not preserved in general. Now let us consider a fuzzy control system whose plant model and controller are represented by fuzzy implications as depicted in Figure 7.8. In this figure, r(k) represents a reference input. The plant implication $P^i$is already defined by Equation 7.12, while the controllers dth implication is given by

$$C^d: \text{ IF } \mathbf{x}(k) \text{ is } \mathbf{D}^d \text{ AND } \mathbf{u}(k) \text{ is } \mathbf{F}^d$$

$$\text{THEN } f^d(k+1) = c_0^d + \sum_{p=1}^{n} c_p^d x(k-p+1) \tag{7.14}$$

where $\mathbf{D}^d \equiv \begin{bmatrix} D_1^d & D_2^d & \cdots & D_n^d \end{bmatrix}^T$, $\mathbf{F}^i \equiv \begin{bmatrix} F_1^i & F_2^i & \cdots & F_m^i \end{bmatrix}^T$, and of course u(k)=r(k)-f(k). The equivalent implication $S^{ij}$is given by

$$S^{id}: \text{IF } \mathbf{x}(k) \text{ is(} \mathbf{A}^i \text{ AND } \mathbf{D}^d \text{) AND } \mathbf{v}^*(k) \text{ is(} \mathbf{B}^i \text{ AND } \mathbf{F}^d \text{)}$$

$$\text{THEN } x^{id}(k+1) = a_0 - b^i c_0^d + b^i r(k) + \sum_{p=1}^{n} (a_p^d - b^i c_p^d) x(k-p+1) \tag{7.15}$$

where i=1,...,l_1, *d*=1,...,l_2, and l_1 and l_2 are the total number of implications for the plant and the controller, respectively. The term *v*(k)* is defined by

$$v^*(k) = [(k) - \eta(x(k)), r(k-1) - \eta(xr(k-1)), ...$$

$$..., r(k-m+1) - \eta(x(k-m+1))]^T \tag{7.16}$$

where $\eta(.)$ is the input-output mapping function of block C^d in Figure 7.8, i.e., $f(k) = \eta(x(k))$.

7.4.1 Example 2

Consider a fuzzy feedback control system of the type shown in Figure 7.8 with the following implications:

P^1: IF *x(k)* is A^1 THEN $x^1(k+1) = 1.85x(k) - 0.65x(k-1) + 0.35u(k)$
P^2: IF *x(k)* is A^2 THEN $x^2(k+1) = 2.56x(k) - 0.135x(k-1) + 2.22u(k)$
C^1: IF *x(k)* is D^2 THEN $f^1(k+1) = k_1^1 x(k) - k_2^1 x(k-1)$
C^2: IF *x(k)* is D^2 THEN $f^2(k+1) = k_1^2 x(k) - k_2^2 x(k-1)$

It is desired to find the closed-loop implications S^{id}, *i*=1,2, and *d*=1,2.

Noting that *u(k)=r(k)-f(k)* in Figure 7.8 and the implications in Equation 7.15, we have

S^{11}: IF *x*(k) is (A^1 AND D^1) THEN $x^{11}(k+1) = (1.85 - 0.35k_1^1)x(k) + ...$
$...(0.65 - 0.35k_2^1)x(k-1) + 0.35u(k)$
S^{12}: IF *x*(k) is (A^1 AND D^2) THEN $x^{12}(k+1) = (1.85 - 0.35k_1^2)x(k) + ...$
$...(0.65 - 0.35k_2^2)x(k-1) + 0.35u(k)$

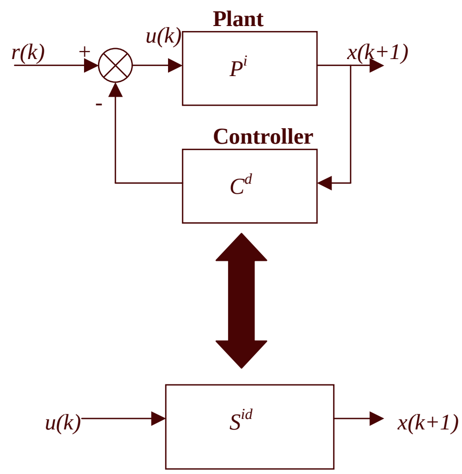


Figure 7.8: Fuzzy control system depicted by two implications and its equivalent implication.

S^{21}: IF x(k) is (A^2 AND D^1) THEN $x^{21}(k+1) = (2.56 - 2.22k_1^1)x(k) + ...$
...$(0.135 - 2.22k_2^1)x(k-1) + 2.22u(k)$
S^{22}: IF x(k) is (A^2 AND D^2) THEN $x^{22}(k+1) = (2.56 - 2.22k_1^2)x(k) + ...$
...$(0.135 - 2.22k_2^2)x(k-1) + 2.22u(k)$

7.5 Stability of fuzzy control systems

One of the most important issues in any control system fuzzy or otherwise is stability. Briefly, a system is said to be stable if it would come to its equilibrium state after any external input, initial conditions, and/or disturbances have impressed the system. The issue of stability is of even greater relevance when questions of safety, lives, and environment are at stake as in such systems as nuclear reactors, traffic systems, and airplane autopilots. The stability test for fuzzy control systems, or lack of it, has been a subject of criticism by many control engineers in some control engineering literature [90].

Almost any linear or nonlinear system under the influence of a closed-loop crisp controller has one type of stability test or as other. For example, the stability of a linear time-invariant system can be tested by a wide variety of methods such as Routh-Hurwitz, root locus, Bode plots, Nyquist criterion, and even through traditionally nonlinear systems methods of Lyapunov, Popov, and circle criterion. The common requirement in all these tests is the availability of a mathematical model, either in time or frequency domain. A reliable mathematical model for a very complex and large-scale system may, in practice, be unavailable or unfeasible. In such cases, a fuzzy controller may be designed based on expert knowledge or experimental practice. However, the issue of the

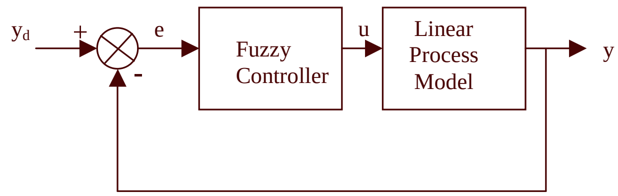


Figure 7.9: Class 1 of fuzzy control system stability problem.

stability of a fuzzy control system still remains and must be addressed. The aim of this section is to present an up-to-date survey of available techniques and tests for fuzzy control systems stability. From the viewpoint of stability a fuzzy controller can be either acting as a conventional (low-level) controller or as a supervisory (high-level) controller. Depending on the existence and nature of a systems mathematical model and the level in which fuzzy rules are being utilized for control and robustness, four classes of fuzzy control stability problems can be distinguished. These four classes are:

Class 1: Process model is crisp and linear and fuzzy controller is low level.

Class 2: Process model is crisp and nonlinear and the fuzzy controller is low level.

Class 3: Process model (linear or nonlinear) is crisp and a fuzzy tuner or an adaptive fuzzy controller is present at high level.

Class 4: Process model is fuzzy and fuzzy controller is low level.

Figures 7.9-7.12 show all four classes of fuzzy control systems whose stability is of concern. Here, we are concerned mainly with the first three classes. For the last class, traditional nonlinear control theory could fail and is beyond the scope of this section. It will be discussed very briefly. The techniques for testing the stability of the first two classes of systems (Figures 7.7 and 7.8) are divided into two main groupstime and frequency.

7.5.1 Time-Domain Methods

The state-space approach has been considered by many authors [91]-[99]. The basic approach here is to subdivide the state space into a finite number of cells based on the definitions of the membership functions. Now, if a separate rule is defined for every cell, a cell-to-cell trajectory can be constructed from the systems output induced by the new outputs of the fuzzy controller. If every cell of the modified state space is checked, one can identify all the equilibrium points, including the systems stable region. This method should be used with some care since the inaccuracies in the modified description could cause oscillatory phenomenon around the equilibrium points.

The second class of methods is based on Lyapunovs method. Several authors, [89], [94], [95], [97] [100]-[107], have used this theory to come up with criterion for stability of fuzzy control systems. The approach shows that the time

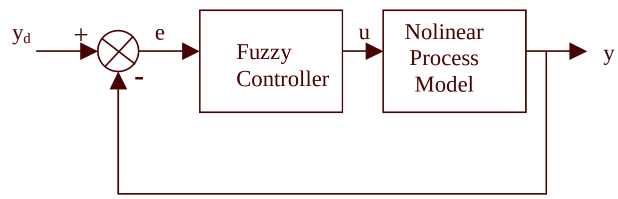


Figure 7.10: Class 2 of fuzzy control system stability problem.

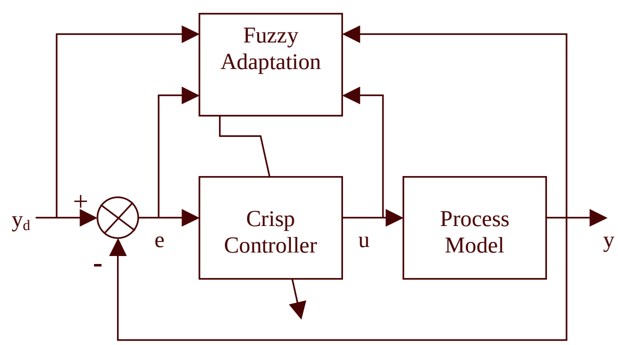


Figure 7.11: Class 3 of fuzzy control system stability problem.

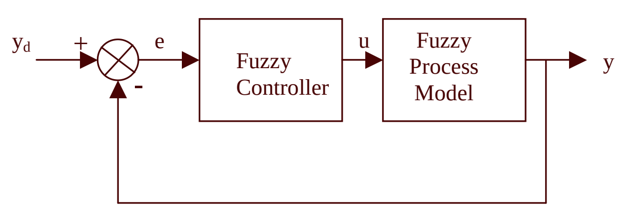


Figure 7.12: Class 4 of fuzzy control system stability problem.

derivative of the Lyapunov function at the equilibrium point is negative semi definite. Many approaches have been proposed. One approach is to define a Lyapunov function and then derive the fuzzy controllers architecture out of the stability conditions. Another approach uses Aisermans method [91] to find an adopted Lyapunov function, while representing the fuzzy controller by a nonlinear algebraic function *u=f(y)*, when y is the systems output. A third method calls for the use of so called facet functions, where the fuzzy controller is realized by boxwise multilinear *facet functions* with the system being described by a state space model. To test stability, a numerical parameter optimization scheme is needed.

The *hyperstability approach*, considered by other authors [108]-[110] has been used to check stability of systems depicted in Figure 7.9. The basic approach here is to restrict the input-output behavior of the nonlinear fuzzy controller by inequality and to derive conditions for the linear part of the closed-loop system to be satisfied for stability.

Bifurcation theory [97] can be used to check stability of fuzzy control systems of the class described in Figure 7.10. This approach represents a tool in deriving stability conditions and robustness indices for stability from small gain theory. The fuzzy controller, in this case, is described by a nonlinear vector function. The stability in this scheme could only be lost if one of the following conditions becomes true: (1) the origin becomes unstable if a pole crosses the imaginary axis into the right half-planestatic bifurcation, (2) the origin becomes unstable if a pair of poles would cross over the imaginary axis and assumes positive real parts Hopf bifurcation or (3) new additional equilibrium points are produced. The last time-domain method is the use of graph theory [13]. In this approach conditions for special nonlinearities are derived to test the BIBO stability.

7.5.2 Frequency-Domain Methods

There are three primary groups of methods which have been considered here. The harmonic balance approach, considered in references [111]-[113], among others, has been used to check the stability of the first two classes of fuzzy control systems (see Figures 7.9 and 7.10). The main idea is to check if permanent oscillations occur in the system and whether these oscillations with known amplitude or frequency are stable. The nonlinearity (fuzzy controller) is described by a complex-valued describing function and the condition of harmonic balance is tested. If this condition is satisfied, then a permanent oscillation exists. This approach is equally applicable to MIMO systems. The *circle criterion* [92],[110],[114],[115] and *Popov criterion* [116],[117] have been used to check stability of the first class of systems. In both criteria, certain conditions on the linear process model and static nonlinearity (controller) must be satisfied. It is assumed that the characteristic value of the nonlinearity remains within certain bounds, and the linear process model must be open-loop stable with proper transfer function. Both criteria can be graphically evaluated in simple manners. A summary of many stability approaches for fuzzy control systems has been presented in Jamshidi[88].

7.5.3 Lyapunov Stability

One of the most fundamental criteria of any control system is to ensure stability as part of the design process. In this section, some theoretical results on this important topic are detailed.

We begin with the ith Takagi-Sugeno implication of a fuzzy system:

$$P^i : \text{IF } x(k) \text{ is } A_1^i \text{ AND... } x(k-n+1) \text{ is } A_n^i$$

$$\text{THEN } x^i(k+1) = a_0^i + a_1^i x(k) + ... + a_n^i x(k-n+1) \tag{7.17}$$

with *i*=1,...,*j*. It is noted that this implication is similar to Equation 7.12 except since we are dealing with Lyapunov stability, the inputs *u(k)* are absent. The stability of a fuzzy control system with the presence of the inputs will be considered shortly. The consequent part of Equation 7.17 represents a set of linear subsystems and can be rewritten as [89]

$$\text{IF } x(k) \text{ is } A_1^i \text{ AND... } x(k-n+1) \text{ is } A_n^i \text{ THEN } \mathbf{x}(k+1) = \mathbf{A}_i \mathbf{x}(k) \tag{7.18}$$

where **x**(*k*) is defined by Equation 7.11 and the *n* x *n* matrix $\mathbf{A}_i$ is

$$\mathbf{A}_i = \begin{bmatrix} a_1^i & a_2^i & \cdots & a_{n-1}^i & a_n^i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{7.19}$$

The output of the fuzzy system described by Equations 7.17-7.19 is given by

$$\mathbf{x}(k+1) = \frac{\sum_{i=1}^{j} w^i \mathbf{A}_i \mathbf{x}(k)}{\sum_{i=1}^{j} w^i} \tag{7.20}$$

where w^i is the overall truth value of the ith implication and *j* is the total number of implications. Using this notation we then present the first stability result of fuzzy control systems [89].

Theorem 1

The equilibrium point of a fuzzy system Equation 7.20 is globally asymptotically stable if there exists a common positive definite matrix **P** for all subsystems such that

$$\mathbf{A}_i^T \mathbf{P} \mathbf{A}_i - \mathbf{P} < 0 \text{ for } i = 1, ..., j. \tag{7.21}$$

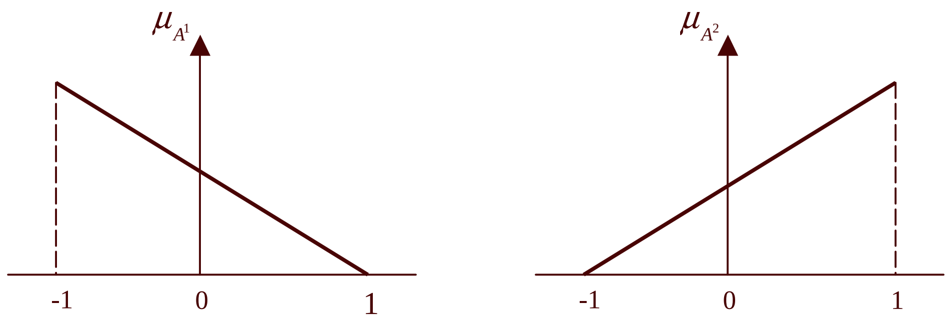


Figure 7.13: Fuzzy sets for example 3.

It is noted that the above theorem can be applied to any nonlinear system which can be approximated by a piecewise linear function if the stability condition (21) is satisfied. Moreover, if there exists a common positive definite matrix **P**, then all the $\mathbf{A}_i$matrices are stable. Since Theorem 1 is a sufficient condition for stability, it is possible not to find a **P** > 0 even if all the $\mathbf{A}_i$ matrices are stable. In other words, a fuzzy system may be globally asymptotically stable even if a **P** > 0 is not found. The fuzzy system is not always stable even if all the $\mathbf{A}_i$'s are stable.

Theorem 2

Let $\mathbf{A}_i$ be stable and nonsingular matrices for *i*=1,...,*j*. Then $\mathbf{A}_i\mathbf{A}_d$ are stable matrices for *i*,*d*=1,...,*j*, if there exists a common positive definite matrix **P** such that

$$\mathbf{A}_i^T\mathbf{P}\mathbf{A}_i - \mathbf{P} < 0 \text{ for } i = 1, ..., j. \tag{7.22}$$

7.5.4 Example 3

Consider the following fuzzy system:

P^1: IF x(k) is A^1 THEN $x^1(k+1) = 1.2x(k) - 0.6x(k-1)$

P^2: IF x(k) is A^2 THEN $x^2(k+1) = 1.2x(k) - 0.6x(k-1)$

Where $\mathbf{A}_i$are fuzzy sets shown in Figure 7.13. It is desired to check the stability of this system.

The two subsystems matrices are

$$\mathbf{A}_1 = \begin{bmatrix} 1.2 & -0.6 \\ 1 & 0 \end{bmatrix} \mathbf{A}_2 = \begin{bmatrix} 1 & -0.4 \\ 1 & 0 \end{bmatrix}$$

The product of matrix $\mathbf{A}_1\mathbf{A}_2$ is

$$\mathbf{A}_1\mathbf{A}_2 = \begin{bmatrix} 0.6 & -0.48 \\ 1 & -0.4 \end{bmatrix}$$

whose eigenvalues are $\lambda_{1,2} = 0.1 \pm j0.48$ which indicates that $\mathbf{A_1A_2}$ is a stable matrix. Thus, by Theorem 2 a common **P** exists, and if we use **P** with the following,

$$\mathbf{P} = \begin{bmatrix} 2 & -1.2 \\ -1.2 & 1 \end{bmatrix}$$

then both equations $\mathbf{A}_i^T\mathbf{P}\mathbf{A}_i - \mathbf{P} < 0$ for *i*=1,2 are simultaneously satisfied. This result was also verified using simulation. Figure 7.14 shows the surface response and the simulation result, which is clearly stable.

Thus far, the criteria which have been presented treat autonomous (either closed-loop or no input) systems. Consider the following non-autonomous fuzzy system:

$$P^i : \text{IF } x(k) \text{ is } A_1^i \text{AND....AND } x(k-n+1) \text{ is } A_n^i \text{ AND}$$

$$u(k) \text{ is } B_1^i \text{ AND...AND } u(k-m+1) \text{ is } B_m^i$$

$$\text{THEN } x^i(k+1) = a_0^i + a_1^i x(k) + ... + a_n^i x(k-n+1) + b_1^i u(k) + ... + b_m^i x(k-m+1) \quad (7.23)$$

Here, we use some results from Tahani and Sheikholeslam [107] to test the stability of the above system. We begin with a definition.

Definition 2

The nonlinear system

$$\mathbf{x}(k+1) = \mathbf{f}\left[\mathbf{x}(k), \mathbf{u}(k), k\right]$$

$$\mathbf{y} = \mathbf{g}\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \quad (7.24)$$

is totally stable if and only if for any bounded input **u**(*k*) and bounded initial state and bounded initial state $\mathbf{x}_0$, the state **x**(*k*) and the output **y**(*k*) of the system are bounded, i.e., we have

$$\text{for all } |\mathbf{x}_0| < \infty \text{ and for all } |\mathbf{u}(k)| < \infty \Rightarrow |\mathbf{x}(k)| < \infty \text{ and } |\mathbf{y}(k) < \infty| \quad (7.25)$$

Now we consider the following theorem:

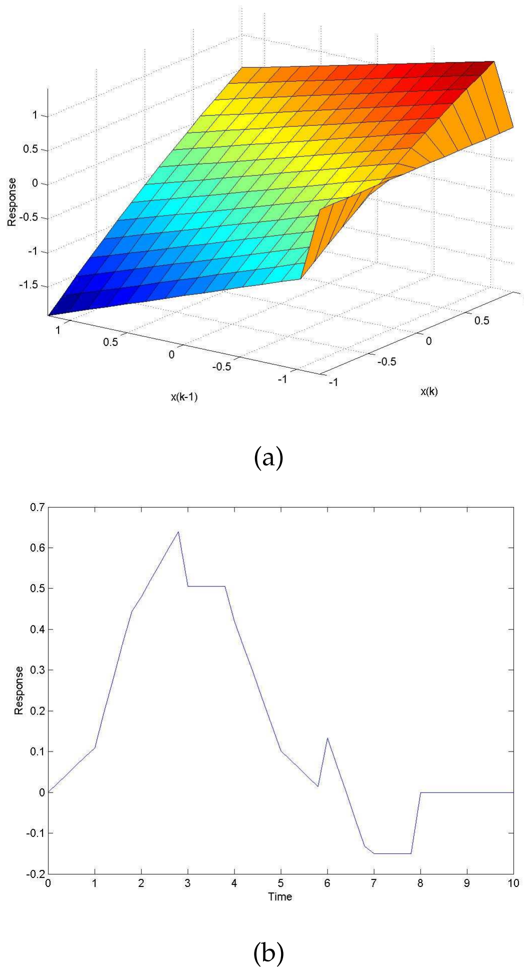


Figure 7.14: (a) Surface response and (b) simulation result for example 3.

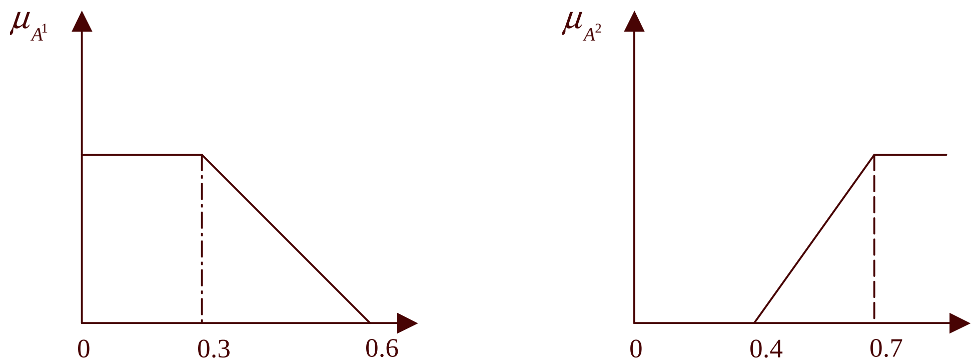


Figure 7.15: Fuzzy sets for example 3.

Theorem 3
The fuzzy system Equation 7.23 is totally stable if there exists a common positive definite matrix **P** such that the following inequalities hold

$$\mathbf{A}_i^T \mathbf{P} \mathbf{A}_i - \mathbf{P} < 0 \text{ for } i = 1, ..., j. \tag{7.26}$$

Where $\mathbf{A}_i$is defined by Equation 7.19. The proof of this theorem can be found in Sheikholeslam [118].

7.5.5 Example 4

P^1: IF x(k) is A^1 THEN $x^1(k+1) = 0.85x(k) - 0.25x(k-1) + 0.35u(k)$
P^2: IF x(k) is A^2 THEN $x^2(k+1) = 0.56x(k) - 0.25x(k-1) + 2.22u(k)$
where A^iAi are fuzzy sets shown in Figure 7.15. It is desired to check the stability of this system. Assume that the input $u(k)$ is bounded.

The two subsystems matrices are

$$\mathbf{A}_1 = \begin{bmatrix} 0.85 & -0.25 \\ 1 & 0 \end{bmatrix} \mathbf{A}_2 = \begin{bmatrix} 0.56 & -0.25 \\ 1 & 0 \end{bmatrix}$$

If we choose the positive definite matrix **P**

$$\mathbf{P} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

then it can be easily verified that the systems is totally stable The product of matrix $\mathbf{A}_1\mathbf{A}_2$is

$$\mathbf{A}_1\mathbf{A}_2 = \begin{bmatrix} 0.23 & -0.21 \\ 0.56 & -0.25 \end{bmatrix}$$

The eigenvalues of product of matrix $\mathbf{A}_1\mathbf{A}_2$ are $\lambda_{1,2} = 0.012 \pm j0.25$ which indicates that $\mathbf{A}_1\mathbf{A}_2$ is a stable matrix.

7.5.6 Stability via Interval Matrix Method

Some results on the stability of time varying discrete interval matrices by Han and Lee [119] can lead us to some more conservative, but computationally more convenient, stability criteria for fuzzy systems of the Takagi-Sugeno type shown by Equation 7.17. Before we can state these new criteria some preliminary discussion will be necessary.

Consider a linear discrete time system described by a difference equation in state form:

$$\mathbf{x}(k+1) = (\mathbf{A} + \mathbf{G}(k))\mathbf{x}(k), \mathbf{x}(0) = \mathbf{x}_0 \tag{7.27}$$

where **A** is an $n \times n$ constant asymptotically stable matrix, **x** is the $n \times 1$ state vector, and **G**(k) is an unknown n×n time varying matrix on the perturbation matrixs maximum modulus, i.e.,

$$|\mathbf{G}(k)| \leq \mathbf{G}_m \text{for all } k \tag{7.28}$$

where the $|\bullet|$represents the matrix with modulus elements and the inequality holds element-wise. Now, consider the following theorem.

Theorem 4

The time varying discrete time system Equation 7.27 is asymptotically stable if

$$\rho(|\mathbf{A}| + \mathbf{G}_m) < 1 \tag{7.29}$$

where $\rho(\bullet)$ stands for spectral radius of the matrix. The proof of this theorem is traightforward, based on the evaluation of the spectral norm $|x(k)|$ or $x(k)$ and showing that if condition Equation 7.29 holds, then $\lim_{k\to\infty} |x(k)| = 0$. The proof can be found in Han and Lee [119].

Definition 3

An interval matrix $\mathbf{AI}(k)$ is an $n \times n$ matrix whose elements consist of intervals [b_{ij},c_{ij}] for i,j=1,...,n, i.e.,

$$\mathbf{A}_1(k) = \begin{bmatrix} [b_{11}, c_{11}] & \cdots & [b_{1n}, c_{1n}] \\ \vdots & [b_{ij}, c_{ij}] & \vdots \\ [b_{n1}, c_{n1}] & \cdots & [b_{nn}, c_{nn}] \end{bmatrix} \tag{7.30}$$

Definition 4

The center matrix, $\mathbf{A}_c$ and the *maximum difference matrix*, $\mathbf{A}_m$ of $\mathbf{A}_I(k)$ in Equation 7.30 are defined by

$$\mathbf{A}_c = \frac{\mathbf{B} + \mathbf{C}}{2}$$

$$\mathbf{A}_m = \frac{\mathbf{C} - \mathbf{B}}{2} \tag{7.31}$$

where B=b_{ij} and C=c_{ij}. Thus, the interval matrix $\mathbf{A}_I(k)$ in 30 can also be rewritten as

$$\mathbf{A}_I(k) = [\mathbf{A}_c - \mathbf{A}_m, \mathbf{A}_c + \mathbf{A}_m] = \mathbf{A}_c + \Delta\mathbf{A}(k) \tag{7.32}$$

with $|\Delta\mathbf{A}(k) \leq \mathbf{A}_m|$.

Lemma 1

The interval matrix $\mathbf{A}_I(k)$ is asymptotically stable if matrix $\mathbf{A}_C$ is stable and

$$\rho(|\mathbf{A}_c| + \mathbf{A}_m) < 1 \tag{7.33}$$

The proof can be found in Han and Lee [119]. The above lemma can be used to check the sufficient condition for the stability of fuzzy systems of Takagi-Sugeno type given in Equation 7.18. Consider a set of m fuzzy rules like Equation 7.18,

$$\begin{gathered} \text{IF } x(k) \text{ is } A_1^1 \text{ AND... } x(k-n+1) \text{ is } A_n^1 \text{ THEN } \mathbf{x}(k+1) = \mathbf{A}_1\mathbf{x}(k) \\ \vdots \\ \text{IF} x(k) \text{is} A_1^m \text{AND...} x(k-n+1) \text{is} A_n^m \text{THEN} \mathbf{x}(k+1) = \mathbf{A}_m\mathbf{x}(k) \end{gathered} \tag{7.34}$$

where $\mathbf{A}_i$ matrices for i=1,...,m are defined by Equation 7.19. One can now formulate all the m matrices $\mathbf{A}_i$, i=1,...,m as an interval matrix of the form 30 by simply finding the minimum and the maximum of all elements at the top row of all the $\mathbf{A}_i$matrices. In other words, we have

$$\mathbf{A}_I(k) = \begin{bmatrix} [a_1^{mi}, a_1^{ma}] & [a_2^{mi}, a_2^{ma}] & \cdots & [a_{n-1}^{mi}, a_{n-1}^{ma}] & [a_n^{mi}, a_n^{ma}] \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{7.35}$$

where $a_i^{mi}$and a_i^{ma} for i=1,...,n are the minimum and maximum of the respective element of the first rows of $\mathbf{A}_i$ in Equation 7.19, taken element by element. Using the above definitions and observations, the fuzzy system Equation 7.34 can be rewritten by

$$\text{IF } x(k) \text{ is } A_1^i \text{ AND... } x(k-n+1) \text{ is } A_n^i \text{ THEN } \mathbf{x}(k+1) = \mathbf{A}_I^i\,\mathbf{x}(k) \tag{7.36}$$

where i=1,...,m and $\mathbf{A}_I^i$ is an interval matrix of form Equation 7.35 except that $a_i^{mi} = a_i^{ma} = a_i$. Now, finding the weighted average, one has

$$\mathbf{x}(k+1) = \frac{\sum_{i=1}^{j} w^i \mathbf{A}_I^i \mathbf{x}(k)}{\sum_{i=1}^{j} w^i} \tag{7.37}$$

Theorem 5

The fuzzy system Equation 7.37 is asymptotically stable if the interval matrix $\mathbf{A}_I(k)$ is asymptotically stable, i.e., the conditions in Lemma 1 are satisfied.

7.5.7 Example 5

Reconsider Example 10.3. It is desired to check its stability via the matrix interval approach

The systems two canonical matrices are written in the form of an interval matrix (30) as

$$\mathbf{A}_I(k) = \begin{bmatrix} [1, 1.2] & [-0.6, -0.4] \\ 1 & 0 \end{bmatrix}$$

The center and maximum difference matrices are

$$\mathbf{A}_c = \begin{bmatrix} 1.1 & 0.5 \\ 1 & 0 \end{bmatrix} \mathbf{A}_m = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0 \end{bmatrix}$$

Then, condition 33 would become,

$$\rho(|\mathbf{A}_c| + \mathbf{A}_m) = \rho \begin{bmatrix} 1.2 & 0.6 \\ 1 & 0 \end{bmatrix} = 1.58 > 1$$

Thus the stability of the fuzzy system under consideration is inconclusive. In fact, it was shown to be stable.

Consider the following fuzzy system:

P^1: IF x(k) is A^1 THEN $x^1(k+1) = 0.3x(k) + 0.5x(k-1)$

P^2: IF x(k) is A^2 THEN $x^2(k+1) = 0.2x(k) + 0.2x(k-1)$

where $A^i$are fuzzy sets shown in Figure 7.17. It is desired to check the stability of this system using matrix interval method.

The two subsystems matrices are

$$\mathbf{A}_1 = \begin{bmatrix} 0.3 & 0.5 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.2 & 0.2 \\ 1 & 0 \end{bmatrix}$$

The systems two canonical matrices are written in the form of an interval matrix 30 as

$$A_I(k) = \begin{bmatrix} [0.2, 0.3] & [0.2, 0.5] \\ 1 & 0 \end{bmatrix}$$

The center and maximum difference matrices are

$$\mathbf{A}_c = \begin{bmatrix} 0.25 & 0.35 \\ 1 & 0 \end{bmatrix} \mathbf{A}_m = \begin{bmatrix} 0.05 & 0.15 \\ 0 & 0 \end{bmatrix}$$

Then, condition 7.33 would become,

$$\rho(|\mathbf{A}_c| + \mathbf{A}_m) = \rho \begin{bmatrix} 0.3 & 0.5 \\ 1 & 0 \end{bmatrix} = 0.873 < 1$$