# Appendix D

# Some Explicit Calculations

In this appendix we explicitly work out some of the calculations appearing in Chapters 5 and 6.

## D.1 Correlation functions

In Chapters 5 and 6 we encounter the normal-normal correlation function

$$\langle \partial_i \mathbf{X}(\bar{\sigma}) \partial_j \mathbf{X}(\bar{\sigma}') \rangle = k_{\mathrm{B}} T \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{\delta_{ij} q^2 e^{-i\mathbf{q}\cdot(\bar{\sigma}-\bar{\sigma}')}}{\omega^2/\nu_0 + r_0 q^2 + q^4/\alpha_0}, \tag{D.1}$$

which, after performing the integration over the frequency $\omega$, reads

$$\begin{aligned} \langle \partial_i \mathbf{X}(\bar{\sigma}) \partial_j \mathbf{X}(\bar{\sigma}') \rangle &= \frac{\delta_{ij}}{2} k_{\mathrm{B}} T \alpha_0^{1/2} \nu_0^{1/2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{q e^{-i\mathbf{q}\cdot(\bar{\sigma}-\bar{\sigma}')}}{\sqrt{q^2 + r_0 \alpha_0}} \\ &= \frac{\delta_{ij}}{8\pi^2} k_{\mathrm{B}} T \alpha_0^{1/2} \nu_0^{1/2} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}q \frac{q^2 e^{-iq|\bar{\sigma}-\bar{\sigma}'|\cos\theta}}{\sqrt{q^2 + r_0 \alpha_0}}. \end{aligned} \tag{D.2}$$

The angular integral in (D.2) can be carried out with help of the identity [94]

$$\int_0^{2\pi} \mathrm{d}\theta \; e^{-iqR\cos\theta} = 2\pi J_0(qR), \tag{D.3}$$

where $J_0$ is the Bessel function and we arrive at

$$\begin{aligned} \langle \partial_i \mathbf{X}(\bar{\sigma}) \partial_j \mathbf{X}(\bar{\sigma}') \rangle &= \frac{\delta_{ij}}{4\pi} k_{\mathrm{B}} T \alpha_0^{1/2} \nu_0^{1/2} \int_0^{\infty} \mathrm{d}q \frac{q^2 J_0(q|\bar{\sigma}-\bar{\sigma}'|)}{\sqrt{q^2 + r_0 \alpha_0}} \\ &= \frac{\delta_{ij}}{4\pi} k_{\mathrm{B}} T \alpha_0^{1/2} \nu_0^{1/2} \mathcal{I}(|\bar{\sigma}-\bar{\sigma}'|, \sqrt{r_0 \alpha_0}), \end{aligned} \tag{D.4}$$

where we defined

$$\begin{aligned} \mathcal{I}(R,m) &= \int_0^\infty \mathrm{d}q \frac{q^2 J_0(q)}{\sqrt{q^2+t^2}} \\ &= \frac{1}{R^2}\int_0^\infty \mathrm{d}q \frac{q^2 J_0(q)}{\sqrt{q^2+m^2R^2}}. \end{aligned} \tag{D.5}$$

Consider the integral

$$\begin{aligned} \mathcal{F}(R,m) &= \int_0^\infty \mathrm{d}q \frac{J_0(qR)}{\sqrt{q^2+m^2}} \\ &= \int_0^\infty \mathrm{d}q \frac{J_0(q)}{\sqrt{q^2+m^2R^2}}. \end{aligned} \tag{D.6}$$

Using the formula [94]

$$\int_0^\infty \mathrm{d}q \frac{J_0(q)}{\sqrt{q^2+t^2}} = I_0(\tfrac{1}{2}t)K_0(\tfrac{1}{2}t), \tag{D.7}$$

where $I_0$ and $K_0$ are modified Bessel functions, one has

$$\mathcal{F}(R,m) = I_0(\tfrac{1}{2}mR)K_0(\tfrac{1}{2}mR). \tag{D.8}$$

Using the definition of the Bessel function $J_0$ [94]

$$z^2 J_0''(z) + zJ_0'(z) + z^2 J_0(z) = 0, \tag{D.9}$$

$\mathcal{I}(R,m)$ can be expressed as

$$\mathcal{I}(R,m) = -\frac{\mathrm{d}^2\mathcal{F}}{\mathrm{d}R^2} - \frac{1}{R}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}R}. \tag{D.10}$$

Using that, for large $R$ [94]

$$I_0(R) \sim \frac{e^R}{\sqrt{2\pi R}}\left(1+\frac{1}{8R}+\cdots\right);\ K_0(R) \sim \sqrt{\frac{\pi}{2R}}e^{-R}\left(1-\frac{1}{8R}+\cdots\right); \tag{D.11}$$

$$I_0'(R) \sim \frac{e^R}{\sqrt{2\pi R}}\left(1-\frac{3}{8R}+\cdots\right);\ K_0'(R) \sim -\sqrt{\frac{\pi}{2R}}e^{-R}\left(1+\frac{3}{8R}+\cdots\right); \tag{D.12}$$

$$I_0''(R) \sim \frac{e^R}{\sqrt{2\pi R}}\left(1-\frac{7}{8R}+\cdots\right);\ K_0''(R) \sim \sqrt{\frac{\pi}{2R}}e^{-R}\left(1+\frac{7}{8R}+\cdots\right), \tag{D.13}$$

we arrive at

$$\mathcal{I}(R,m) \sim \frac{1}{4mR^3}, \tag{D.14}$$

so that

$$\langle \partial_i \mathbf{X}(\bar{\sigma}) \partial_j \mathbf{X}(\bar{\sigma}') \rangle \sim \frac{k_{\mathrm{B}}T}{4\pi} \sqrt{\frac{\nu_0}{r_0}} \frac{\delta_{ij}}{|\bar{\sigma}-\bar{\sigma}'|^3}. \tag{D.15}$$

## D.2 Series expansion

Both in Chapter 5 and in Chapter 6 we calculate the one-loop contribution to the effective action, for a quantum membrane at finite temperature or for a finite stack of membranes, given by

$$S_1 = \frac{k_{\mathrm{B}}T}{2} d \int \mathrm{d}\tau \int \mathrm{d}^2\sigma \rho_0 f, \tag{D.16}$$

with

$$f = \frac{k_{\mathrm{B}}T}{\hbar} \sum_n \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[\frac{\omega_n^2}{\nu_0} + \frac{q^2}{\alpha_0}(\lambda_0 + q^2)\right], \tag{D.17}$$

and

$$\omega_n = 2\pi \frac{k_{\mathrm{B}}T}{\hbar} n, \; n = 0, \pm 1, \pm 2, \cdots \tag{D.18}$$

for the quantum membrane. To obtain the equivalent expression for the finite stack one must only replace $\frac{k_{\mathrm{B}}T}{\hbar} \to 1/L_{||}$ and $1/\nu_0 \to B_0, 1/\alpha_0 \to K_0$.

Let us separate the $n = 0$ term in the sum in Eq. (D.17), and consider

$$\bar{f} = 2\frac{k_{\mathrm{B}}T}{\hbar} \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[\frac{\omega_n^2}{\nu_0} + \frac{q^2}{\alpha_0}(\lambda_0 + q^2)\right], \tag{D.19}$$

the $n = 0$ term will be taken care of later. $\bar{f}$ can itself be split in a $\lambda_0$-independent part,

$$\bar{f}^{\lambda_0=0} = 2\frac{k_{\mathrm{B}}T}{\hbar} \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[\frac{\omega_n^2}{\nu_0} + \frac{q^4}{\alpha_0}\right], \tag{D.20}$$

so that

$$\bar{f} = \bar{f}^{\lambda_0=0} + \tilde{f}, \tag{D.21}$$

with

$$\tilde{f} = 2\frac{k_{\mathrm{B}}T}{\hbar} \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[1 + \frac{\lambda_0/\alpha_0 q^2}{\omega_n^2/\nu_0 + q^4/\alpha_0}\right]. \tag{D.22}$$

Using the formula [70]

$$\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2\pi^2}\right), \tag{D.23}$$

and ignoring powerlike divergences, one can readily write $\bar{f}^{\lambda_0=0}$ as

$$\begin{aligned} \bar{f}^{\lambda_0=0} &= 2\frac{k_{\rm B}T}{\hbar}\int \frac{{\rm d}^2q}{(2\pi)^2} \ln\left[1 - \exp\left(-\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)\right] \\ &= -\frac{\pi}{12}\left(\frac{k_{\rm B}T}{\hbar}\right)^2\sqrt{\frac{\alpha_0}{\nu_0}} \end{aligned} \tag{D.24}$$

To calculate $\tilde{f}$ we first expand the logarithm using the formula[70]

$$\ln(1+x) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m}, \tag{D.25}$$

arriving at

$$\tilde{f} = 2\frac{k_{\rm B}T}{\hbar}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\frac{\lambda_0}{\alpha_0}\right)^m \int\frac{{\rm d}^2q}{(2\pi)^2}\frac{q^{2m}}{(\omega_n^2/\nu_0 + q^4/\alpha_0)^m}. \tag{D.26}$$

The momentum integrals can now be calculated with help of the formula[94]

$$\int \frac{{\rm d}^2q}{(2\pi)^2}\frac{q^{2m}}{(aq^4+b^2)^m} = \frac{1}{4\sqrt{\pi}}\frac{\Gamma\left(\frac{m-1}{2}\right)}{2^m a^{\frac{m+1}{2}} b^{m-1}\Gamma\left(\frac{m}{2}\right)}, \tag{D.27}$$

yielding

$$\tilde{f} = \frac{1}{2\sqrt{\pi}}\frac{k_{\rm B}T}{\hbar}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2^m m}\lambda_0^m\left(\frac{\nu_0}{\alpha_0}\right)^{\frac{m-1}{2}}\frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\sum_{n=1}^{\infty}\frac{1}{\omega_n^{m-1}}, \tag{D.28}$$

where we have interchanged the order of the sums. The summation over the frequencies can be carried out by using the definition of the Riemann Zeta-function[94]

$$\zeta(m) \equiv \sum_{n=1}^{\infty}\frac{1}{n^m}, \tag{D.29}$$

finally leading to

$$\tilde{f} = \sqrt{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m 2^{2m} \pi^m} \lambda_0^m \left(\frac{\nu_0}{\alpha_0}\right)^{\frac{m-1}{2}} \left(\frac{\hbar}{k_{\mathrm{B}} T}\right)^{m-2} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \zeta(m-1). \tag{D.30}$$

The first two terms in the series (D.30) are singular, and need to be regularized. To do that, we derive the series expansion of $\tilde{f}$ in an alternative way, and compare the terms of the two expansions order by order. We shall now explicitly take the $n = 0$ term in (D.17) into account.

(D.17) can be rewritten as

$$f = \frac{k_{\mathrm{B}} T}{\hbar} \left\{ \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[\frac{q^2}{\alpha_0}(\lambda_0 + q^2)\right] + 2 \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[\frac{\omega_n^2}{\nu_0} + \frac{q^2}{\alpha_0}(\lambda_0 + q^2)\right]\right\}. \tag{D.31}$$

Using formula (D.23) again, the sum in the second term gives

$$\begin{aligned} 2\sum_{n=1}^{\infty} \ln\left[\frac{\omega_n^2}{\nu_0} + \frac{q^2}{\alpha_0}(\lambda_0 + q^2)\right] &= 2 \ln \sinh\left[\frac{\hbar}{k_{\mathrm{B}} T}\sqrt{\frac{\nu_0}{\alpha_0}} q (\lambda_0 + q^2)^{1/2}\right] \\ &- \ln\left[\frac{q^2}{2\alpha_0}(\lambda_0 + q^2)\right], \end{aligned} \tag{D.32}$$

and combining this result with the $n = 0$ term in (D.31) we obtain

$$\begin{aligned} f &= 2\frac{k_{\mathrm{B}} T}{\hbar} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[2 \sinh\left(\frac{\hbar}{k_{\mathrm{B}} T}\sqrt{\frac{\nu_0}{\alpha_0}} q (\lambda_0 + q^2)^{1/2}\right)\right] \\ &= \sqrt{\frac{\nu_0}{\alpha_0}} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q \sqrt{\lambda_0 + q^2} \\ &+ 2\frac{k_{\mathrm{B}} T}{\hbar} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \ln\left[1 - \exp\left(-\frac{\hbar}{k_{\mathrm{B}} T}\sqrt{\frac{\nu_0}{\alpha_0}} q \sqrt{\lambda_0 + q^2}\right)\right] \\ &\equiv f^{\infty} + \Delta f. \end{aligned} \tag{D.33}$$

The first term, $f^{\infty}$, corresponds to the $T = 0$ result for the quantum membrane, or to the infinite stack. It yields

$$f^{\infty} = \sqrt{\frac{\nu_0}{\alpha_0}} \left\{ \frac{\Lambda^4}{8\pi} + \frac{\lambda_0}{8\pi}\Lambda^2 + \frac{\lambda_0^2}{64\pi}\left[1 - 2\ln\left(\frac{4\Lambda^2}{\lambda_0}\right)\right]\right\}, \tag{D.34}$$

where $\Lambda$ is an ultraviolet cut-off. To calculate the finite temperature (or finite stack size) correction $\Delta f$, we expand the second term on the right hand side of (D.33) in a small-$\lambda_0$ series:

$$\begin{aligned}
\Delta f &= 2\frac{k_{\mathrm{B}}T}{\hbar}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\ln\left[1-\exp\left(-\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)\right] \\
&+ \lambda_0\sqrt{\frac{\nu_0}{\alpha_0}}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1} \\
&- \tfrac{1}{4}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\left(1+\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1}{q^2\left[\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1\right]^2}+\cdots
\end{aligned} \tag{D.35}$$

The first term on the RHS of (D.35) corresponds to $f^{\lambda_0=0}$ in our previous expansion (compare with Eq. (D.24)), and has already been calculated. The second integrals diverges in the infrared. To calculate it, we use dimensional regularization, and consider a membrane of $D=2+\epsilon$ dimensions. With help of the identity[94]

$$\int_0^\infty\frac{q^{1+\epsilon}}{e^{aq^2}-1}=\frac{1}{2a^{\epsilon+1}}\Gamma(1+\epsilon)\zeta(1+\epsilon)\overset{\epsilon\to 0}{\approx}\frac{1}{2a}\left(\frac{1}{\epsilon}-\ln a\right), \tag{D.36}$$

and identifying the pole $1/\epsilon$ with the logarithmic infrared divergence, $1/\epsilon = \ln L^{-2}$, where $L$ is the lateral size of the membrane, we arrive at

$$\lambda_0\sqrt{\frac{\nu_0}{\alpha_0}}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1}=\frac{\lambda_0}{4\pi}\frac{k_{\mathrm{B}}T}{\hbar}\ln\left(L^2\frac{k_{\mathrm{B}}T}{\hbar}\sqrt{\frac{\alpha_0}{\nu_0}}\right). \tag{D.37}$$

The third term in the expansion (D.35) can be rewritten as

$$\begin{aligned}
&-\tfrac{1}{4}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\left(1+\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1}{q^2\left[\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1\right]^2}= \\
&-\tfrac{1}{8\pi}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\left\{\int_0^\infty\frac{\mathrm{d}q}{q\left[\exp\left(\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1\right]}\right.
\end{aligned}$$

$$+\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}\int_0^\infty {\rm d}q\frac{q\exp\left(\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)}{\left[\exp\left(\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1\right]^2}\Bigg\}, \tag{D.38}$$

where the angle integration has already been carried out. The first term on the RHS of (D.38) is again IR divergent. Using dimensional regularization, as we did when calculating (D.37) and the formula[94]

$$\int_0^\infty \frac{q^{\epsilon-1}}{e^{aq^2}-1} = \frac{1}{2a^{\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\zeta\left(\frac{\epsilon}{2}\right) \overset{\epsilon\to 0}{\approx} -\frac{1}{2\epsilon}+\frac{1}{4}\left[\gamma+\ln\left(\frac{a}{2\pi}\right)\right], \tag{D.39}$$

it gives

$$-\tfrac{1}{8\pi}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\int_0^\infty \frac{1}{q\left[\exp\left(\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}q^2\right)-1\right]} = -\frac{1}{32\pi}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\left[\gamma+\ln\left(\frac{L^2}{2\pi}\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}\right)\right], \tag{D.40}$$

after identifying the pole $1/\epsilon$ with the logarithmic infrared divergence, $1/\epsilon = \ln L^{-1}$, where $L$ is the lateral size of the membrane.

The second term on the RHS of (D.38) can be derived from the first one by noticing that

$$\int_0^\infty \frac{aq^2e^{aq^2}}{q(e^{aq^2}-1)^2} = -a\frac{\rm d}{{\rm d}a}\int_0^\infty \frac{1}{q(e^{aq^2}-1)}, \tag{D.41}$$

and using formula (D.39).

The first three terms of the series expansion of $\Delta f$ thus yield

$$\begin{aligned}\Delta f &= -\frac{\pi}{12}\left(\frac{k_{\rm B}T}{\hbar}\right)^2\sqrt{\frac{\alpha_0}{\nu_0}}+\frac{\lambda_0}{4\pi}\frac{k_{\rm B}T}{\hbar}\ln\left(L^2\frac{k_{\rm B}T}{\hbar}\sqrt{\frac{\alpha_0}{\nu_0}}\right)\\ &- \frac{1}{32\pi}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\left[\gamma-1+\ln\left(\frac{L^2}{2\pi}\frac{\hbar}{k_{\rm B}T}\sqrt{\frac{\nu_0}{\alpha_0}}\right)\right]+\cdots\end{aligned} \tag{D.42}$$

The terms with powers of $\lambda_0$ greater than two can be extracted from the alternative series (D.30), since they are all finite and unique. The full one-loop contribution to effective action (D.17) is thus

$$f = \sqrt{\frac{\nu_0}{\alpha_0}}\left\{\frac{\Lambda^4}{8\pi}+\frac{\lambda_0}{8\pi}\Lambda^2+\frac{\lambda_0^2}{64\pi}\left[1-2\ln\left(\frac{4\Lambda^2}{\lambda_0}\right)\right]\right\}-\frac{\pi}{12}\left(\frac{k_{\rm B}T}{\hbar}\right)^2\sqrt{\frac{\alpha_0}{\nu_0}}$$

$$
\begin{aligned}
&+ \quad \frac{\lambda_0}{4\pi}\frac{k_{\mathrm{B}}T}{\hbar}\ln\left(L^2\frac{k_{\mathrm{B}}T}{\hbar}\sqrt{\frac{\alpha_0}{\nu_0}}\right) - \frac{1}{32\pi}\lambda_0^2\sqrt{\frac{\nu_0}{\alpha_0}}\left[\gamma - 1 + \ln\left(\frac{L^2}{2\pi}\frac{\hbar}{k_{\mathrm{B}}T}\sqrt{\frac{\nu_0}{\alpha_0}}\right)\right] \\
&+ \quad \sqrt{\pi}\sum_{m=3}^{\infty}\frac{(-1)^{m+1}}{m2^{2m}\pi^m}\lambda_0^m\left(\frac{\nu_0}{\alpha_0}\right)^{\frac{m-1}{2}}\left(\frac{\hbar}{k_{\mathrm{B}}T}\right)^{m-2}\frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\zeta(m-1). \qquad \text{(D.43)}
\end{aligned}
$$