Chapter 1

The Dirichlet problem for the inhomogeneous pluriholomorphic system

1.1 Preliminaries and Definition

The Dirichlet problem for the inhomogeneous pluriharmonic system in polydiscs was studied in many papers to various extend, see [7] and [10]. However only in [2] Chapter 5 and [4] the problem is solved in full scale: the solvability conditions and the unique solution are given explicitly . About the Dirichlet problem for the inhomogeneous pluriholomorphic system in polydiscs there is no such rich result, see again [7] and [10] .

Let $\mathbb{D}^n$ be the unit polydisc $\{z : z = (z_1, \cdots, z_n) \in \mathbb{C}^n \ , \ |z_k| < 1 \ , \ 1 \le k \le n\}$ and $\partial_0 \mathbb{D}^n$ be its essential boundary $\{z : z = (z_1, \cdots, z_n) \in \mathbb{C}^n \ , \ |z_k| = 1 \ , \ 1 \le k \le n\}$. Let f_{kl} , γ_0 be given functions with $f_{kl\overline{z}_j} \in L_1(\overline{\mathbb{D}^n}) \cap C^\alpha(\overline{\mathbb{D}^n}), \gamma_0 \in C^\alpha(\partial_0 \mathbb{D}^n)$, $\alpha \ge 1/2$. Consider the following inhomogeneous system of $n(n+1)/2$ independent equations

$$\frac{\partial^2 u}{\partial \overline{z}_k \partial \overline{z}_\ell} = f_{k\ell}(z), \quad 1 \le k, \ell \le n \tag{1.1}$$

with given right − hand sides, satisfying the compatibility conditions

$$f_{k\ell} = f_{\ell k} \ , \quad \frac{\partial f_{k\ell}}{\partial \overline{z}_j} - \frac{\partial f_{kj}}{\partial \overline{z}_\ell} = 0 \ , \ 1 \le k \ , \ \ell \ , \ j \le n \ . \tag{1.2}$$

Problem D . For $\gamma_0 \in C^\alpha(\partial_0 \mathbb{D}^n)$ find a $C^\alpha(\overline{\mathbb{D}^n})$ − solution of system (1.1), satisfying the Dirichlet condition

$$u(\zeta) = \gamma_0(\zeta), \quad \zeta \in \partial_0 \mathbb{D}^n \ . \tag{1.3}$$

It is known that any solution to (1.1) can be represented as, see [2],

$$u(z) = \varphi_0(z) + \sum_{k=1}^{n} \overline{z}_k \varphi_k(z) + u_0(z) \tag{1.4}$$

where $\varphi_k(z)(k=0,1,\cdots,n)$ are arbitrary analytic functions in $\mathbb{D}^n$, $u_0(z)$ is a special solution to (1.1) and has to be found. For this purpose we quote a theorem from [2].

Theorem 1 *Let* $D^n := \times_{k=1}^n D_k$, $\overline{D^n} := \times_{k=1}^n \overline{D_k}$ *where* D_k *is a smooth bounded plane domain in* $\mathbb{C}$, $1 \le k \le n$. *Let* w *have mixed derivatives with respect to each variable of first order in* $L_1(\overline{D^n})$. *Then* $w = \varphi + w_0$ *where* φ *is analytic in* D^n *and*

$$w_0 = \sum_{\nu=1}^n (-1)^{\nu+1} \sum_{1\le k_1<\cdots<k_\nu\le n} T_{k_\nu} T_{k_{\nu-1}} \cdots T_{k_1} w_{\overline{\zeta}_{k_1}\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}} .$$

Remark 1 T_k, $1 \le k \le n$ is the Pompeiu operator , given by

$$T_k f(z) = -\frac{1}{\pi}\int_{D_k} f(\zeta)\frac{d\xi_k d\eta_k}{\zeta_k - z_k}, \quad 1\le k\le n, \quad \zeta_k = \xi_k + i\eta_k, \quad \zeta = (\zeta_1,\cdots,\zeta_n) \in \partial_0 D^n$$

see [29] .

1.2 Special solution

Applying once this theorem to the system (1.1) one can easily obtain the special solution

$$u_{0\overline{z}_\ell} = \sum_{\nu=1}^n (-1)^{\nu+1} \sum_{1\le k_1<\cdots<k_\nu\le n} T_{k_\nu} T_{k_{\nu-1}} \cdots T_{k_1} f_{k_1\ell\,\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}} =: F_\ell, \quad 1\le \ell\le n. \tag{1.5}$$

For solvability of (1.5) we need the compatibility conditions

$$F_{\ell\overline{z}_k} = F_{k\overline{z}_\ell} \quad 1\le \ell,k\le n,$$

to be satisfied, which is actually equivalent to $f_{k\ell} = f_{\ell k}$. Repeating once more the above procedure we get a particular solution to (1.1)

$$u_0 = \sum_{\nu=1}^n (-1)^{\nu+1} \sum_{1\le \ell_1<\cdots<\ell_\nu\le n} T_{\ell_\nu} T_{\ell_{\nu-1}} \cdots T_{\ell_1} F_{\ell_1\overline{\zeta}_{\ell_2}\cdots\overline{\zeta}_{\ell_\nu}} . \tag{1.6}$$

Carefully combining (1.5) with (1.6), the final explicit form of u_0 can be found.

$$\begin{aligned} u_0 &= \sum_{\ell_1=1}^n T_{\ell_1} F_{\ell_1} + \sum_{\nu=2}^n (-1)^{\nu+1} \sum_{1\le \ell_1<\cdots<\ell_\nu\le n} T_{\ell_\nu} \cdots T_{\ell_1} F_{\ell_1\overline{\zeta}_{\ell_2}\cdots\overline{\zeta}_{\ell_\nu}} \\ &= \sum_{\ell_1=1}^n T_{\ell_1}\left[\sum_{\mu=1}^n (-1)^{\mu+1} \sum_{1\le k_1<\cdots<k_\mu\le n} T_{k_\mu}\cdots T_{k_1} f_{k_1\ell_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\mu}}\right] \\ &\quad + \sum_{\mu=2}^n (-1)^{\mu+1} \sum_{1\le k_1<\cdots<k_\mu\le n} T_{k_\mu}\cdots T_{k_1} f_{k_2k_1\overline{\zeta}_{k_3}\cdots\overline{\zeta}_{k_\mu}} \end{aligned}$$

$$= \sum_{k,\ \mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\in\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}}$$

$$+ \sum_{k,\ \mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}}$$

$$+\sum_{\nu=2}^{n}(-1)^{\nu+1} \sum_{1\le \ell_1<\cdots<\ell_\nu\le n} T_{\ell_\nu}\cdots T_{\ell_1} f_{\ell_2\ell_1\bar{\zeta}_{\ell_3}\cdots\bar{\zeta}_{\ell_\nu}}$$

Since

$$\{(k,\ell_1,\cdots,\ell_n) \quad | \quad 1\le \ell_1<\cdots<\ell_n\le n\ ; \quad k\ne \ell_\alpha \in I\!N\} = \phi$$

it follows that

$$\sum_{k,\ \mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}}$$

$$= \sum_{\mu=1}^{n-1}(-1)^{\mu+1}\sum_{k=1}^{n} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}} =: C$$

Let $\alpha\ne\beta$ and $\alpha,\ \beta\in\{1,\cdots,n\}$. Then

$$C = \sum_{\mu=1}^{n-1}(-1)^{\mu+1}\Big[\sum_{\substack{k=1 \\ k\notin\{\alpha,\ \beta\}}}^{n} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}}$$

$$+T_\alpha T_\beta \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ \alpha,\ \beta\ \notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1} f_{\ell_1\ \alpha\ \bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}\bar{\zeta}_\beta} + T_\beta T_\alpha \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ \alpha,\ \beta\ \notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1} f_{\ell_1\ \beta\ \bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}\bar{\zeta}_\alpha}\Big]$$

$$= \sum_{\mu=1}^{n-1}(-1)^{\mu+1}\Big[\sum_{\substack{k=1 \\ k\notin\{\alpha,\ \beta\}}}^{n} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar{\zeta}_{\ell_2}\cdots\bar{\zeta}_{\ell_\mu}}$$

$$+T_\alpha T_\beta \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ \alpha,\ \beta\ \notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1} f_{\beta\alpha\ \bar{\zeta}_{\ell_1}\cdots\bar{\zeta}_{\ell_\mu}} + T_\beta T_\alpha \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ \alpha,\ \beta\ \notin\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1} f_{\alpha\ \beta\ \bar{\zeta}_{\ell_1}\cdots\bar{\zeta}_{\ell_\mu}}\Big]$$

where the condition (1.2) is used. Paying attention to the fact that $\alpha,\ \beta\in\{1,\cdots,n\}$ and that for every $(\alpha,\ \beta)$- term there is one and only one $(\beta,\ \alpha)$- term and they are equal to each other by the condition (1.2) it is easy to see that

$$C = 2\sum_{\nu=2}^{n}(-1)^{\nu} \sum_{1\le \ell_1<\cdots<\ell_\nu\le n} T_{\ell_\nu}\cdots T_{\ell_1} f_{\ell_1\ell_2\bar{\zeta}_{\ell_3}\cdots\bar{\zeta}_{\ell_\nu}}.$$

Thus

$$u_0 = \sum_{k,\,\mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1\le \ell_1<\cdots<\ell_\mu\le n \\ k\in\{\ell_1,\cdots,\ell_\mu\}}} T_{\ell_\mu}\cdots T_{\ell_1}T_k f_{\ell_1 k\bar\zeta_{\ell_2}\cdots\bar\zeta_{\ell_\mu}}$$
$$+\sum_{\nu=2}^{n}(-1)^{\nu} \sum_{1\le\ell_1<\cdots<\ell_\nu\le n} T_{\ell_\nu}\cdots T_{\ell_1} f_{\ell_1\ell_2\bar\zeta_{\ell_3}\cdots\bar\zeta_{\ell_\nu}}.$$

Further by means of the compatibility condition (1.2) the special solution u_0 can be written as

$$u_0 = \sum_{\mu=1}^{n}(-1)^{\mu+1} \sum_{\substack{1\le\ell_1\le n \\ 1\le\ell_2<\cdots<\ell_\mu\le n}} T_{\ell_\mu}\cdots T_{\ell_2}T_{\ell_1}^2 f_{\ell_1\ell_1\bar\zeta_{\ell_2}\cdots\bar\zeta_{\ell_\mu}}$$
$$+\sum_{\nu=2}^{n}(-1)^{\nu} \sum_{1\le\ell_1<\cdots<\ell_\nu\le n} T_{\ell_\nu}\cdots T_{\ell_1} f_{\ell_1\ell_2\bar\zeta_{\ell_3}\cdots\bar\zeta_{\ell_\nu}} \ . \tag{1.7}$$

1.3 Boundary values of a holomorphic function in polydiscs - A necessary and sufficient condition

Now we specify the general solution (1.4) with the Dirichlet boundary condition (1.3):

$$\varphi_0(\zeta) + \sum_{k=1}^{n} \bar\zeta_k \varphi_k(\zeta) + u_0(\zeta) = \gamma_0(\zeta)\ , \qquad \zeta\in\partial_0\mathbb{D}^n \tag{1.8}$$

i.e. ,

$$\sum_{k}^{n} \bar\zeta_k\Big[\frac{1}{n}\zeta_k(\varphi_0(\zeta)+u_0(\zeta)-\gamma_0(\zeta))+\varphi_k(\zeta)\Big]=0\ ,\quad \zeta\in\partial_0\mathbb{D}^n\ . \tag{1.9}$$

The equality (1.9) holds if and only if

$$\zeta_k\varphi_0(\zeta)+n\varphi_k(\zeta)=\zeta_k\Big[\gamma_0(\zeta)-u_0(\zeta)\Big]\ ,\quad \zeta\in\partial_0\mathbb{D}^n\ , \tag{1.10}$$

holds for $1\le k\le n$. Since the left–hand side is the boundary value of a holomorphic function, the right–hand side is too. Thus from the Cauchy formula it follows that (1.10) holds if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_k} \eta_h\Big[\gamma_0(\eta)-u_0(\eta)\Big]\frac{d\eta_k}{\eta_k-\zeta_k} = \frac{1}{2}\eta_h\Big[\gamma_0(\eta)-u_0(\eta)\Big]\Big|_{\eta_k=\zeta_k}\ ,$$
$$\eta=(\eta_k,\eta')\in\partial_0\mathbb{D}^n\ ,\ |\zeta_k|=1,\ \ \eta'=(\eta_1,\cdots,\eta_{k-1},\eta_{k+1},\cdots,\eta_n)$$

for any $k,h\in\{1,\cdots,n\}$, i.e.,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_k} <\eta,z>\Big[\gamma_0(\eta)-u_0(\eta)\Big]\frac{d\eta_k}{\eta_k-\zeta_k} = \frac{1}{2}<\eta,z>\Big[\gamma_0(\eta)-u_0(\eta)\Big]\Big|_{\eta_k=\zeta_k}\ ,$$

$$\eta = (\eta_k, \eta') \in \partial_0 I\!D^{\,n}\ , \quad |\zeta_k| = 1 \tag{1.11}$$

Evidently, if the right–hand side of (1.10) is the boundary value of an analytic function in $I\!D^{\,n}$, then from (1.11) it follows that

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^{\,n}} < \eta, z > \Big[\gamma_0(\eta) - u_0(\eta)\Big]\frac{d\eta}{\eta - \zeta} = \frac{1}{2^n} < \zeta, z > \Big[\gamma_0(\zeta) - u_0(\zeta)\Big], \zeta \in \partial_0 I\!D^{\,n}\ . \tag{1.12}$$

This means (1.12) is a necessary solvability condition of the system (1.1) for the boundary condition (1.3). However, it is not sufficient for the above problem to be solvable. This can be shown by a simple example:

Let $n = 3$ and $f_{k\ell} = 0$, $\gamma_0(\zeta) = \zeta_1^{t_1}\zeta_2^{-t_2}\zeta_3^{-t_3}$, $t_i \in I\!N, i = 1, 2, 3$. Clearly the necessary condition (1.12) is satisfied for this example, but the condition (1.12) is not satisfied for ζ_2 and ζ_3. The reason is that $\gamma_0(\zeta) - u_0(\zeta)$ is not the boundary value of a function, which is holomorphic in $I\!D^{\,3}$.

Theorem 2 *Let $W(\partial_0 I\!D^{\,n})$ be the Wiener algebra on $\partial_0 I\!D^{\,n}$ and $\Gamma \in W(\partial_0 I\!D^{\,n})$. Then the condition*

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^{\,n}} \Gamma(\eta)\frac{d\eta}{\eta - \zeta} = \frac{1}{2^n}\Gamma(\zeta), \quad \zeta \in \partial_0 I\!D^{\,n} \tag{1.13}$$

is necessary and together with

$$\frac{1}{(2\pi i)^k}\int_{\partial_0 I\!D^{\,k}} \Gamma(\eta)\frac{d\eta^*}{\eta^* - \zeta^*} = \frac{1}{2^k}\Gamma(\zeta^{(k)}),$$

$$\eta = (\eta^*, \eta')\ ,\ \zeta^{(k)} = (\zeta^*, \eta') \in \partial_0 I\!D^{\,n}\ ;\ \zeta^*\ ,\ \eta^* \in \partial_0 I\!D^{\,k}\ ,\ k = 1, 2, \cdots, n-1, \tag{1.14}$$

is necessary and sufficient for Γ to be the boundary values of a holomorphic function in $I\!D^{\,n}$.

Proof By the definition of analytic functions we know that the function Γ is analytic in $I\!D^{\,n}$ if and only if

$$\frac{1}{2\pi i}\int_{\partial I\!D_{\,k}} \Gamma(\eta)\frac{d\eta_k}{\eta_k - \zeta_k} = \frac{1}{2}\Gamma(\eta)\Big|_{\eta_k = \zeta_k}\ , \quad \eta = (\eta_k, \eta') \in \partial_0 I\!D^{\,n}\ , \quad |\zeta_k| = 1. \tag{1.15}$$

Suppose condition (1.15) holds. That means

$$\frac{1}{2\pi i}\int_{\partial I\!D_{\,1}} \Gamma(\eta)\frac{d\eta_1}{\eta_1 - \zeta_1} = \frac{1}{2}\Gamma(\zeta^{(1)}),$$

$$\eta = (\eta_1, \eta')\ ,\ \zeta^{(1)} = (\zeta_1, \eta') \in \partial_0 I\!D^{\,n}\ ;\ \eta_1\ ,\ \zeta_1 \in \partial I\!D_{\,1}\ .$$

Thus the case $k = 1$ is proved.

For $\zeta^{(1)}$ from (1.15) it follows that

$$\frac{1}{2\pi i}\int_{\partial I\!D_{\,2}} \Gamma(\zeta^{(1)})\frac{d\eta_2}{\eta_2 - \zeta_2} = \frac{1}{2}\Gamma(\zeta^{(2)}),$$

$$\zeta^{(1)} = (\eta^*, \eta')\ ,\ \zeta^{(2)} = (\zeta^*, \eta') \in \partial_0 I\!D^{\,n}\ ;\ \eta^* = (\zeta_1, \eta_2)\ ,\ \zeta^* = (\zeta_1, \zeta_2)\ \in \partial_0 I\!D^{\,2}\ .$$

Further by the previous equality we have

$$\frac{1}{(2\pi i)^2}\int_{\partial_0 \mathbb{D}^2}\Gamma(\eta)\frac{d\eta^*}{\eta^*-\zeta^*}=\frac{1}{2^2}\Gamma(\zeta^{(2)}) ,$$

i.e., condition (1.14) is true for the case $k=2$.

We assume condition (1.14) is true for the case $k=n-1$, i.e.,

$$\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 \mathbb{D}^{n-1}}\Gamma(\zeta^{(n-2)})\frac{d\eta^*}{\eta^*-\zeta^*}=\frac{1}{2^{n-1}}\Gamma(\zeta^{(n-1)}),\quad \zeta^{(n-2)}=(\eta^*,\eta') ,$$

$$\zeta^{(n-1)}=(\zeta^*,\eta')\in\partial_0\mathbb{D}^n\ ;\eta^*=(\zeta_1,\cdots,\zeta_{n-2},\eta_{n-1})\ ,\ \zeta^*=(\zeta_1,\cdots,\zeta_{n-2},\zeta_{n-1})\ ,\ \in\partial_0\mathbb{D}^{n-1}\ .$$

Applying (1.15) for $\zeta^{(n-1)}$ we have

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}_n}\Gamma(\zeta^{(n-1)})\frac{d\eta_n}{\eta_n-\zeta_n}=\frac{1}{2}\Gamma(\zeta),\quad \zeta\in\partial_0\mathbb{D}^n\ .$$

Hence, from the assumption for $k=n-1$ it follows that

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Gamma(\eta)\frac{d\eta}{\eta-\zeta}=\frac{1}{2^n}\Gamma(\zeta)\ ,\zeta\in\partial_0\mathbb{D}^n\ .$$

So the case $k=n$ is proved.

Next from (1.13) and (1.14) we derive (1.15) , i.e., that the function $\Gamma(\zeta)$ is analytic for every $\zeta_k, 1\le k\le n\ ,\ |\zeta_k|\le 1$.

Assume condition (1.13) and (1.14) hold, i.e.,

$$\frac{1}{(2\pi i)^k}\int_{\partial_0 \mathbb{D}^k}\Gamma(\eta)\frac{d\eta^*}{\eta^*-\zeta^*}=\frac{1}{2^k}\Gamma(\zeta^{(k)}),$$

$$\eta=(\eta^*,\eta')\ ,\ \zeta^{(k)}=(\zeta^*,\eta')\in\partial_0\mathbb{D}^n\ ;\ \zeta^*\ ,\ \eta^*\in\partial_0\mathbb{D}^k\ ,\ k=1,2,\cdots,n-1,n.$$

Then

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Gamma(\eta)\frac{d\eta}{\eta-\zeta}=\frac{1}{2^n}\Gamma(\zeta),\zeta\in\partial_0\mathbb{D}^n\ ,$$

and

$$\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 \mathbb{D}^{n-1}}\Gamma(\eta)\frac{d\eta^*}{\eta^*-\zeta^*}=\frac{1}{2^{n-1}}\Gamma(\zeta^{(n-1)}),$$

$$\eta=(\eta^*,\eta')\ ,\ \zeta^{(n-1)}=(\zeta^*,\eta')\in\partial_0\mathbb{D}^n\ ;\ \zeta^*\ ,\ \eta^*\in\partial_0\mathbb{D}^{n-1}\ ,\ \eta'=\eta_n\in\partial\mathbb{D}_n.$$

But since

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n}\Gamma(\eta)\frac{d\eta}{\eta-\zeta}=\frac{1}{(2\pi i)^n}\int_{\partial \mathbb{D}_n}\int_{\partial_0 \mathbb{D}^{n-1}}\Gamma(\eta)\frac{d\eta^*}{\eta^*-\zeta^*}\frac{d\eta_n}{\eta_n-\zeta_n}$$

it follows that

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}_n}\Gamma(\zeta^{(n-1)})\frac{d\eta_n}{\eta_n-\zeta_n}=\frac{1}{2}\Gamma(\zeta)\ ,\ \zeta=(\zeta^*,\zeta_n)\ ,\ \zeta^{(n-1)}=(\zeta^*,\eta_n)\in\partial_0\mathbb{D}^n,$$

i.e., function $\Gamma(\eta)$ is analytic for $\eta_n\ ,\ |\eta_n|\le 1$. The rest can be proved in the same way.

This theorem can be proved also by applying the properties of the Wiener algebra and Fourier series method.

Remark 2 $C^\alpha(\partial_0\mathbb{D}^n)\subset W(\partial_0\mathbb{D}^n),\quad \alpha\ge 1/2$, see [16].

1.4 The classical problem

Lemma 1 *Equivalently to (1.12), condition (1.12) together with*

$$\frac{1}{(2\pi i)^k}\int_{\partial_0 \mathbb{D}^k} < \eta, z > \Big[\gamma_0(\eta) - u_0(\eta)\Big]\frac{d\eta^*}{\eta^* - \zeta^*} = \frac{1}{2^k} < \zeta^{(k)}, z > \Big[\gamma_0(\zeta^{(k)}) - u_0(\zeta^{(k)})\Big],$$

$$\eta = (\eta^*, \eta') \ , \ \zeta^{(k)} = (\zeta^*, \eta') \in \partial_0 \mathbb{D}^n \ ; \ \zeta^* \ , \ \eta^* \in \partial_0 \mathbb{D}^k \ , \ k = 1, 2, \cdots, n-1, \tag{1.16}$$

becomes necessary and sufficient for the problem D to be solvable.

Interestingly in the case $n = 1$ condition (1.16) vanishes automatically and therefore condition (1.12) alone is necessary and sufficient for the problem to be solvable.

Now by (1.10), if condition (1.12) holds, it is obvious that

$$n\varphi_k(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \zeta_k \Big[\gamma_0(\zeta) - u_0(\zeta)\Big]\frac{d\zeta}{\zeta - z} - z_k\varphi_0(z) \ .$$

Substituting it into (1.4) we have

$$u(z) = \Big[1 - \frac{< z, z >}{n}\Big]\varphi_0(z) + \frac{1}{(2\pi i)^n}\int_{\partial_0 \mathbb{D}^n} \frac{< \zeta, z >}{n}\Big[\gamma_0(\zeta) - u_0(\zeta)\Big]\frac{d\zeta}{\zeta - z} + u_0(z). \tag{1.17}$$

Theorem 3 *Let $f_{k\ell}(z) \in C^{1+\alpha}(\overline{\mathbb{D}^n}) \cap L_1(\overline{\mathbb{D}^n})$, $1/2 \leq \alpha$ and satisfy the compatibility conditions (1.2). A particular solution to the inhomogeneous system (1.1) is given by (1.7). If the problem D is solvable then the condition (1.12) must be satisfied. The problem D is solvable if and only if the condition (1.12) holds. The solution is given by (1.17). The corresponding homogeneous problem has an infinite number of nontrivial solutions. The problem is not well–posed.*

In order to get a unique solution we may introduce a proper boundary condition.

1.5 The modified problem

Problem M . Find a $C^{1+\alpha}(\overline{\mathbb{D}^n})$ – solution of system (1.1), which satisfies condition (1.3) and

$$Re < grad_{\overline{\zeta}} u, \zeta >= \gamma_1(\zeta) \ , \quad \zeta \in \partial_0 \mathbb{D}^n \ \ , \ Im \ u(0) = C_0 \ , \tag{1.18}$$

as well.

The representation (1.4) gives

$$u_{\overline{z}_k} = \varphi_k(z) + u_{0\overline{z}_k} \ , \quad z \in \mathbb{D}^n \ ,$$

or

$$< grad_{\overline{\zeta}} u \ , \ \zeta >= \sum_{k=1}^{n} \overline{\zeta}_k \Big(\varphi_k + u_{0\,\overline{\zeta}_k}\Big) \ , \quad \zeta \in \partial_0 \mathbb{D}^n \ .$$

So by the first part of the modified boundary condition (1.18) we have

$$Re \sum_{k=1}^{n} \overline{\zeta}_k \varphi_k(\zeta) = \gamma_1(\zeta) - Re \sum_{k=1}^{n} \overline{\zeta}_k u_{0\,\overline{\zeta}_k} \,, \quad \zeta \in \partial_0 \mathbb{D}^n \,. \tag{1.19}$$

Taking the real part of (1.8) and combining it with (1.19) leads to

$$Re\varphi_0(\zeta) = Re\Big[\gamma_0(\zeta) - u_0(\zeta) + < grad_{\overline{\zeta}} u_0, \zeta >\Big] - \gamma_1(\zeta) \,, \quad \zeta \in \partial_0 \mathbb{D}^n \,. \tag{1.20}$$

This is a simple Schwarz problem for analytic functions in $\mathbb{D}^n$ and this problem is solvable if and only if, see [2],

$$\sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big\{ Re\Big[\gamma_0(\zeta) - u_0(\zeta) + < grad_{\overline{\zeta}} u_0, \zeta >\Big] - \gamma_1(\zeta) \Big\}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0 \quad , \quad \zeta \in \partial_0 \mathbb{D}^n \,, \quad z \in \mathbb{D}^n \,, \tag{1.21}$$

is satisfied . For any real C then

$$\varphi_0(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big\{ Re\Big[\gamma_0(\zeta) - u_0(\zeta) + < grad_{\overline{\zeta}} u_0, \zeta >\Big] - \gamma_1(\zeta) \Big\} \Big[2\frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} + iC \tag{1.22}$$

is analytic in $\mathbb{D}^n$ and satisfies (1.20).

Having the second part of the modified boundary condition M in mind , by (1.17) and (1.22) we have got the unique solution of the system (1.1), i.e.

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{< \zeta, z >}{n} \Big[\gamma_0(\zeta) - u_0(\zeta)\Big] \frac{d\zeta}{\zeta - z} + u_0(z) + \Big[1 - \frac{< z, z >}{n}\Big]$$

$$\times \Big\{ \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big[Re\Big(\gamma_0(\zeta) - u_0(\zeta) + < grad_{\overline{\zeta}} u_0, \zeta >\Big) - \gamma_1(\zeta) \Big] \Big[2\frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} + iC_0 \Big\}. \tag{1.23}$$

Next we simplify the solvability condition (1.21) and the solution (1.23).
Let

$$I_{s1} := \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{< \zeta, z >}{n} u_0(\zeta) \frac{d\zeta}{\zeta - z}.$$

Since

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \overline{\zeta}^{\kappa} T f(\zeta) \frac{d\zeta}{\zeta - z} = 0, \quad f \in L_1(\overline{\mathbb{D}}), \quad 0 \le \kappa, \quad z \in \mathbb{D} \,, \tag{1.24}$$

see [2] Lemma 5.20 page 305, it is easy to see

$$I_{s1} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \sum_{k=1}^{n} \frac{\zeta_k \overline{z}_k}{n} T_k F_k \frac{d\zeta}{\zeta - z}$$

$$= \sum_{k=1}^{n} \frac{\overline{z}_k}{n} \frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 \mathbb{D}^{n-1}} \frac{1}{(2\pi i)} \int_{\partial \mathbb{D}_k} \zeta_k T_k^2 f_{kk}(\zeta) \frac{d\zeta}{\zeta - z} \,.$$

Substituting

$$T^2 f(\zeta) = \frac{1}{\pi}\int_{I\!D} f(\eta)\frac{\overline{\eta-\zeta}}{\eta-\zeta}d\eta_1 d\eta_2 \tag{1.25}$$

we have

$$I_{s1} = \sum_{k=1}^{n}\frac{\overline{z}_k}{n}\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 I\!D^{n-1}}\frac{1}{2\pi i}\int_{\partial I\!D_k}\zeta_k\Big[\frac{1}{\pi}\int_{I\!D_k} f_{kk}(\eta_k)\frac{\overline{\eta_k-\zeta_k}}{\eta_k-\zeta_k}d\eta_{1,k}d\eta_{2,k}\Big]\frac{d\zeta_k}{\zeta_k-z_k}\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{d\zeta_\tau}{\zeta_\tau-z_\tau}$$

$$= \sum_{k=1}^{n}\frac{\overline{z}_k}{n}\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 I\!D^{n-1}}\frac{1}{\pi}\int_{I\!D_k} f_{kk}(\eta_k)\Big[\frac{1}{2\pi i}\int_{\partial I\!D_k}\zeta_k\frac{\overline{\eta_k-\zeta_k}}{\eta_k-\zeta_k}\frac{d\zeta_k}{\zeta_k-z_k}\Big]d\eta_{1,k}d\eta_{2,k}\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{d\zeta_\tau}{\zeta_\tau-z_\tau}$$

$$= \sum_{k=1}^{n}\frac{\overline{z}_k}{n}\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 I\!D^{n-1}}\frac{1}{\pi}\int_{I\!D_k} f_{kk}(\eta_k)\Big[\frac{1}{2\pi i}\int_{\partial I\!D_k}\frac{1-\zeta_k\overline{\eta}_k}{\eta_k-z_k}\Big(\frac{1}{\zeta_k-\eta_k}-\frac{1}{\zeta_k-z_k}\Big)d\zeta_k\Big]d\eta_{1,k}d\eta_{2,k}$$

$$\times\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{d\zeta_\tau}{\zeta_\tau-z_\tau}$$

$$= \sum_{k=1}^{n}\frac{\overline{z}_k}{n}\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 I\!D^{n-1}}\frac{1}{\pi}\int_{I\!D_k} f_{kk}(\eta_k)\Big[\frac{1-\eta_k\overline{\eta}_k}{\eta_k-z_k}-\frac{1-z_k\overline{\eta}_k}{\eta_k-z_k}\Big]d\eta_{1,k}d\eta_{2,k}\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{d\zeta_\tau}{\zeta_\tau-z_\tau}$$

$$= \sum_{k=1}^{n}\frac{\overline{z}_k}{n}\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 I\!D^{n-1}}\frac{1}{\pi}\int_{I\!D_k} f_{kk}(\eta_k)(-\overline{\eta}_k)d\eta_{1,k}d\eta_{2,k}\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{d\zeta_\tau}{\zeta_\tau-z_\tau}$$

$$= -\sum_{k=1}^{n}\frac{\overline{z}_k}{n}\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0 I\!D^{n-1}}\frac{1}{\pi}\int_{I\!D_k}\overline{\zeta}_k f_{kk}(\zeta)d\xi_k d\eta_k\prod_{\substack{\tau=1\\ \tau\neq k}}^{n}\frac{d\zeta_\tau}{\zeta_\tau-z_\tau}\,.$$

By simple repetition of the Pompeiu formula for one variable it is easy to derive

$$\frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n} f(\zeta)\frac{d\zeta}{\zeta-z} = f(z)+\sum_{\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{1}{\pi^\nu}\int_{I\!D_{k_1}}\cdots\int_{I\!D_{k_\nu}} f_{\overline{\zeta}_{k_1}\cdots\overline{\zeta}_{k_\nu}}\prod_{\tau=1}^{\nu}\frac{d\xi_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\,, \tag{1.26}$$

where f is defined and properly differentiable in $I\!D^n$, continuous even on $\overline{I\!D^n}$. Direct application of the formula for $n-1$ variables shows that

$$I_{s1} = \sum_{\nu=1}^{n}\sum_{\substack{1\le k_1\le n\\ 1\le k_2<\cdots<k_\nu\le n}}\frac{\overline{z}_{k_1}}{n}\frac{1}{\pi^\nu}\int_{I\!D_{k_1}}\cdots\int_{I\!D_{k_\nu}}\overline{\zeta}_{k_1} f_{k_1 k_1\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}}d\xi_{k_1}d\eta_{k_1}\prod_{\tau=2}^{n}\frac{d\xi_{k_\tau}d\eta_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}.$$

The second term we have to simplify is

$$I_{s2} = \frac{1}{(2\pi i)^n}\int_{\partial_0 I\!D^n}\Big[Re u_0(\zeta)\Big]\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}.$$

Applying (1.21) and its special case ($z = 0$), we have

$$I_{s2} = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{u_0(\zeta)} \frac{d\zeta}{\zeta - z}$$

$$= \sum_{\nu=1}^{n} (-1)^{\nu+1} \sum_{1 \le k_1 < \cdots < k_\nu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{k_\nu} \cdots T_{k_1} F_{k_1 \bar{\zeta}_{k_2} \cdots \bar{\zeta}_{k_\nu}}} \frac{d\zeta}{\zeta - z}$$

$$= \sum_{1 \le k_1 \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{k_1} F_{k_1}} \frac{d\zeta}{\zeta - z}$$

$$+ \sum_{\nu=2}^{n} (-1)^{\nu+1} \sum_{1 \le k_1 < k_2 < \cdots < k_\nu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{k_\nu} \cdots T_{k_2} T_{k_1} f_{k_2 k_1 \bar{\zeta}_{k_3} \cdots \bar{\zeta}_{k_\nu}}} \frac{d\zeta}{\zeta - z}$$

$$= \sum_{k_1, \mu=1}^{n} (-1)^{\mu+1} \sum_{1 \le \ell_1 < \cdots < \ell_\mu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{\ell_\mu} \cdots T_{\ell_1} T_{k_1} f_{\ell_1 k_1 \bar{\zeta}_{\ell_2} \cdots \bar{\zeta}_{\ell_\mu}}} \frac{d\zeta}{\zeta - z}$$

$$+ \sum_{\nu=2}^{n} (-1)^{\nu+1} \sum_{1 \le k_1 < k_2 < \cdots < k_\nu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{k_\nu} \cdots T_{k_2} T_{k_1} f_{k_2 k_1 \bar{\zeta}_{k_3} \cdots \bar{\zeta}_{k_\nu}}} \frac{d\zeta}{\zeta - z}$$

$$= \sum_{k_1, \mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1 \le \ell_1 < \cdots < \ell_\mu \le n \\ k_1 \in \{\ell_1, \cdots, \ell_\mu\}}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{\ell_\mu} \cdots T_{\ell_1} T_{k_1} f_{\ell_1 k_1 \bar{\zeta}_{\ell_2} \cdots \bar{\zeta}_{\ell_\mu}}} \frac{d\zeta}{\zeta - z}$$

$$+ \sum_{k_1, \mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1 \le \ell_1 < \cdots < \ell_\mu \le n \\ k_1 \notin \{\ell_1, \cdots, \ell_\mu\}}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{\ell_\mu} \cdots T_{\ell_1} T_{k_1} f_{\ell_1 k_1 \bar{\zeta}_{\ell_2} \cdots \bar{\zeta}_{\ell_\mu}}} \frac{d\zeta}{\zeta - z}$$

$$+ \sum_{\nu=2}^{n} (-1)^{\nu+1} \sum_{1 \le k_1 < k_2 < \cdots < k_\nu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{k_\nu} \cdots T_{k_2} T_{k_1} f_{k_2 k_1 \bar{\zeta}_{k_3} \cdots \bar{\zeta}_{k_\nu}}} \frac{d\zeta}{\zeta - z}$$

$$= \sum_{\mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1 \le \ell_1 \le n \\ 1 \le \ell_2 < \cdots < \ell_\mu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{\ell_\mu} \cdots T_{\ell_2} S_{\ell_1} f_{\ell_1 \ell_1 \bar{\zeta}_{\ell_2} \cdots \bar{\zeta}_{\ell_\mu}}} \frac{d\zeta}{\zeta - z}$$

$$+ \sum_{\mu=1}^{n} (-1)^{\mu+1} \sum_{\substack{1 \le k_1 \le n \\ 1 \le \ell_1 < \cdots < \ell_\mu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{\ell_\mu} \cdots T_{\ell_1} T_{k_1} f_{\ell_1 k_1 \bar{\zeta}_{\ell_2} \cdots \bar{\zeta}_{\ell_\mu}}} \frac{d\zeta}{\zeta - z}$$

$$+ \sum_{\nu=2}^{n} (-1)^{\nu+1} \sum_{1 \le k_1 < k_2 < \cdots < k_\nu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{k_\nu} \cdots T_{k_2} T_{k_1} f_{k_2 k_1 \bar{\zeta}_{k_3} \cdots \bar{\zeta}_{k_\nu}}} \frac{d\zeta}{\zeta - z}$$

Since the second term turns out to be

$$\sum_{\mu=2}^{n} (-1)^{\mu} \sum_{1 \le \ell_1 < \ell_2 < \cdots < \ell_\mu \le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{T_{\ell_\mu} \cdots T_{\ell_1} f_{\ell_2 \ell_1 \bar{\zeta}_{\ell_3} \cdots \bar{\zeta}_{\ell_\mu}}} \frac{d\zeta}{\zeta - z}$$

it vanishes together with the third term.

Theorem 4 *Let $f \in L_1(\overline{\mathbb{D}})$. Then*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \overline{Sf(\zeta)}\frac{d\zeta}{\zeta - z} = \frac{z}{\pi}\int_{\mathbb{D}} \overline{f(\zeta)}\frac{z-\zeta}{1-z\overline{\zeta}}d\xi d\eta, \quad z \in \mathbb{D}\,, \tag{1.27}$$

where $S = T^2$.

Proof Using (1.25) we have

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \overline{Sf(\zeta)}\frac{d\zeta}{\zeta - z} = \frac{1}{\pi}\int_{\mathbb{D}} \overline{f(\eta)}\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \overline{\frac{\eta-\zeta}{\eta-\zeta}}\frac{d\zeta}{\zeta-z}\Big]d\eta_1 d\eta_2$$

$$= \frac{1}{\pi}\int_{\mathbb{D}} \overline{f(\eta)}\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\zeta(\zeta-\eta)}{1-\overline{\eta}\zeta}\frac{d\zeta}{\zeta-z}\Big]d\eta_1 d\eta_2 = \frac{1}{\pi}\int_{\mathbb{D}} \overline{f(\eta)}\Big[\frac{z(z-\eta)}{1-z\overline{\eta}}\Big]d\eta_1 d\eta_2\,.$$

So

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \overline{Sf(\zeta)}\frac{d\zeta}{\zeta - z} = \frac{z}{\pi}\int_{\mathbb{D}} \overline{f(\eta)}\frac{z-\zeta}{1-z\overline{\zeta}}d\xi d\eta\,.$$

Applying (1.27) and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \zeta^\kappa\overline{Tf(\zeta)}\frac{d\zeta}{\zeta - z} = \frac{z^\kappa}{\pi}\int_{\mathbb{D}} \overline{f(\eta)}\frac{z}{1-z\overline{\zeta}}d\xi d\eta, \quad 0 \le \kappa, \quad f \in L_1(\overline{\mathbb{D}}) \tag{1.28}$$

see [2] page 307, we obtain

$$I_{s2} = \sum_{\mu=1}^{n} \sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_1<\cdots<\ell_\mu\le n\\ 1\le\ell_{\mu+1}<\cdots<\ell_n\le n}} \frac{(-1)^{\mu+1}}{\pi^\mu(2\pi i)^{n-\mu}}\int_{\mathbb{D}_{\ell_1}}\cdots\int_{\mathbb{D}_{\ell_\mu}}\int_{\partial\mathbb{D}_{\ell_{\mu+1}}}\cdots\int_{\partial\mathbb{D}_{\ell_n}} \overline{f_{\ell_1\ell_1\overline{\zeta}_{\ell_2}\cdots\overline{\zeta}_{\ell_\mu}}}$$

$$\times\frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}}\prod_{\tau=2}^{\nu}\frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}}\prod_{\tau=1}^{\nu}d\xi_{\ell_\tau}d\eta_{\ell_\tau}\prod_{\tau=\mu+1}^{n}\frac{d\zeta_{\ell_\tau}}{\zeta_{\ell_\tau}-z_{\ell_\tau}}.$$

With (1.26) then

$$I_{s2} = \sum_{\mu=1}^{n} \sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_2<\cdots<\ell_\mu\le n}} \frac{(-1)^{\mu+1}}{\pi^\mu}\int_{\mathbb{D}_{\ell_1}}\cdots\int_{\mathbb{D}_{\ell_\mu}} \overline{f_{\ell_1\ell_1\overline{\zeta}_{\ell_2}\cdots\overline{\zeta}_{\ell_\mu}}}\frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}}$$

$$\times\prod_{\tau=2}^{\mu}\frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}}\prod_{\tau=1}^{\mu}d\xi_{\ell_\tau}d\eta_{\ell_\tau}$$

$$+\sum_{\mu=1}^{n}\sum_{\gamma=1}^{n-\mu}\sum_{1\le\rho_1<\cdots<\rho_\gamma\le n-\mu}\sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_2<\cdots<\ell_\mu\le n\\ 1\le\ell_{\mu+\rho_1}<\cdots<\ell_{\mu+\rho_\gamma}\le n}}\frac{(-1)^{\mu+1}}{\pi^{\mu+\gamma}}\int_{\mathbb{D}_{\ell_1}}\cdots\int_{\mathbb{D}_{\ell_\mu}}\int_{\mathbb{D}_{\ell_{\mu+\rho_1}}}\cdots\int_{\mathbb{D}_{\ell_{\mu+\rho_\gamma}}}$$

$$\times\overline{f_{\ell_1\ell_1\overline{\zeta}_{\ell_2}\cdots\overline{\zeta}_{\ell_\mu}\zeta_{\ell_{\mu+\rho_1}}\cdots\zeta_{\ell_{\mu+\rho_\gamma}}}}\frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}}\prod_{\tau=2}^{\mu}\frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}}\prod_{\tau=1}^{\gamma}\frac{1}{\zeta_{\ell_{\mu+\tau}}-z_{\ell_{\mu+\tau}}}\prod_{\tau=1}^{\mu+\gamma}d\xi_{\ell_\tau}d\eta_{\ell_\tau}\quad.$$

Switching the summation index from μ to $\lambda := \mu + \gamma$ the integral above can be written as

$$I_{s2} = \sum_{\nu=1}^{n} \sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_2<\cdots<\ell_\nu\le n}} \frac{(-1)^{\nu+1}}{\pi^\nu} \int_{\mathbb{D}_{\ell_1}} \cdots \int_{\mathbb{D}_{\ell_\nu}} \overline{f_{\ell_1\ell_1\bar\zeta_{\ell_2}\cdots\bar\zeta_{\ell_\nu}}} \frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}}$$

$$\times \prod_{\tau=2}^{\nu} \frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{\ell_\tau} d\eta_{\ell_\tau}$$

$$+\sum_{\nu=1}^{n-1}\sum_{\lambda=\nu+1}^{n} \sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_2<\cdots<\ell_\nu\le n\\ 1\le\ell_{\nu+1}<\cdots<\ell_\lambda\le n}} \frac{(-1)^{\nu+1}}{\pi^\lambda} \int_{\mathbb{D}_{\ell_1}} \cdots \int_{\mathbb{D}_{\ell_\lambda}} \overline{f_{\ell_1\ell_1\bar\zeta_{\ell_2}\cdots\bar\zeta_{\ell_\nu}\zeta_{\ell_{\nu+1}}\cdots\zeta_{\ell_\lambda}}} \frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}}$$

$$\times \prod_{\tau=2}^{\nu} \frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}} \prod_{\tau=\nu+1}^{\lambda} \frac{1}{\zeta_{\ell_\tau}-z_{\ell_\tau}} \prod_{\tau=1}^{\lambda} d\xi_{\ell_\tau} d\eta_{\ell_\tau}$$

$$= \sum_{\nu=1}^{n} \sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_2<\cdots<\ell_\nu\le n}} \frac{(-1)^{\nu+1}}{\pi^\nu} \int_{\mathbb{D}_{\ell_1}} \cdots \int_{\mathbb{D}_{\ell_\nu}} \overline{f_{\ell_1\ell_1\bar\zeta_{\ell_2}\cdots\bar\zeta_{\ell_\nu}}} \frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}} \prod_{\tau=2}^{\nu} \frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{\ell_\tau} d\eta_{\ell_\tau}$$

$$+\sum_{\lambda=2}^{n}\sum_{\nu=1}^{\lambda-1} \sum_{\substack{1\le\ell_1\le n\\ 1\le\ell_2<\cdots<\ell_\nu\le n\\ 1\le\ell_{\nu+1}<\cdots<\ell_\lambda\le n}} \frac{(-1)^{\nu+1}}{\pi^\lambda} \int_{\mathbb{D}_{\ell_1}} \cdots \int_{\mathbb{D}_{\ell_\lambda}} \overline{f_{\ell_1\ell_1\bar\zeta_{\ell_2}\cdots\bar\zeta_{\ell_\nu}\zeta_{\ell_{\nu+1}}\cdots\zeta_{\ell_\lambda}}} \frac{z_{\ell_1}(z_{\ell_1}-\zeta_{\ell_1})}{1-z_{\ell_1}\overline{\zeta_{\ell_1}}}$$

$$\times \prod_{\tau=2}^{\nu} \frac{z_{\ell_\tau}}{1-z_{\ell_\tau}\overline{\zeta_{\ell_\tau}}} \prod_{\tau=\nu+1}^{\lambda} \frac{1}{\zeta_{\ell_\tau}-z_{\ell_\tau}} \prod_{\tau=1}^{\lambda} d\xi_{\ell_\tau} d\eta_{\ell_\tau}.$$

For the second term with some $a_{\lambda\nu}$, we have

$$\sum_{\lambda=2}^{n}\sum_{\nu=1}^{\lambda-1} a_{\lambda\nu} = \sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1} a_{\nu\lambda}, \quad 2\le\lambda\le n, \quad 1\le\nu\le\lambda-1.$$

So adding the first term with the second one we get

$$I_{s2} = \sum_{\nu=1}^{n}\sum_{\lambda=1}^{\nu} \sum_{\substack{1\le k_1\le n\\ 1\le k_2<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \overline{f_{k_1k_1\bar\zeta_{k_2}\cdots\bar\zeta_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}} \frac{z_{k_1}(z_{k_1}-\zeta_{k_1})}{1-z_{k_1}\overline{\zeta_{k_1}}}$$

$$\times \prod_{\tau=2}^{\lambda} \frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau}-z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$= \sum_{k,\,\nu=1}^{n}\sum_{\lambda=1}^{\nu} \sum_{\substack{1\le k_1<\cdots<k_{\lambda-1}\le n\\ 1\le k_\lambda<\cdots<k_{\nu-1}\le n\\ k\notin\{k_1,\cdots,k_{\lambda-1}\}}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_k}\int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_{\nu-1}}} (z_k-\zeta_k)\overline{f_{kk\,\bar\zeta_{k_1}\cdots\bar\zeta_{k_{\lambda-1}}\zeta_{k_\lambda}\cdots\zeta_{k_{\nu-1}}}}$$

$$\times \frac{z_k}{1 - z_k \overline{\zeta}_k} \prod_{\substack{\tau=1 \\ k \neq k_\tau}}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}.$$

$$= \sum_{k,\,\nu=1}^{n} \sum_{\lambda=1}^{\nu} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n \\ k \in \{k_1, \cdots, k_\lambda\}}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} (z_k - \zeta_k) \overline{f_{k_1\, k\, \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta}_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}.$$

The third term to be calculated is

$$I_{s3} := \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} Re < grad_\zeta u_0(\zeta), \zeta > \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta}$$

$$= \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} Re \Big[\sum_{k=1}^{n} \overline{\zeta}_k u_{0 \overline{\zeta}_k} \Big] \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta}$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big[\overline{\zeta_k} F_k + \zeta_k \overline{F}_k \Big] \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} = I_{s3a} + I_{s3b}\ ,\ z \in \mathbb{D}^n\ .$$

From (1.24) it follows that

$$I_{s3a} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\zeta}_k F_k \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta}$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\zeta_k} T_{kk} f_{kk} \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} \qquad .$$

By direct application of (1.24), we have $I_{s3a} = 0$ and therefore $I_{s3} = I_{s3b}$. Hence from $I_{s3a}(z) = 0$, $z \in \mathbb{D}^n$, i.e. $I_{s3a}(0) = 0$ and

$$\sum_{k=1}^{n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \zeta_k \overline{F}_k \frac{d\zeta}{\zeta} = \overline{\sum_{k=1}^{n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \overline{\zeta}_k F_k \frac{d\zeta}{\zeta}} = \overline{I_{s3a}(0)}$$

it is obvious that

$$I_{s3} = \sum_{k=1}^{n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \zeta_k \overline{F}_k \frac{d\zeta}{\zeta - z}, \quad z \in \mathbb{D}^n.$$

Further (1.28) gives

$$I_{s3} = \sum_{k,\nu=1}^{n} \sum_{\substack{1 \le k_1 < \cdots < k_\nu \le n \\ 1 \le k_{\nu+1} < \cdots < k_n \le n \\ k \in \{k_1, \cdots, k_\nu\}}} \frac{(-1)^{\nu+1}}{\pi^\nu (2\pi i)^{n-\nu}} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \int_{\partial \mathbb{D}_{k_{\nu+1}}} \cdots \int_{\partial \mathbb{D}_{k_n}} z_k \overline{f_{k_1 k \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta_{k_\tau}}} d\xi_{k_\tau} d\eta_{k_\tau} \prod_{\tau=\nu+1}^{n} \frac{d\zeta_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}}$$

$$+\sum_{k,\nu=1}^{n}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ 1\le k_{\nu+1}<\cdots<k_n\le n\\ k\in\{k_{\nu+1},\cdots,k_n\}}}\frac{(-1)^{\nu+1}}{\pi^\nu(2\pi i)^{n-\nu}}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\int_{\partial\mathbb{D}_{k_{\nu+1}}}\cdots\int_{\partial\mathbb{D}_{k_n}}\zeta_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}}}$$
$$\times\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}d\xi_{k_\tau}d\eta_{k_\tau}\prod_{\tau=\nu+1}^{n}\frac{d\zeta_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\,.$$

Applying (1.26) once again leads to

$$I_{s3}=\sum_{k,\nu=1}^{n}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ k\in\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}z_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}}}\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}d\xi_{k_\tau}d\eta_{k_\tau}$$
$$+\sum_{k,\nu=1}^{n}\sum_{\mu=1}^{n-\nu}\sum_{1\le\sigma_1<\cdots<\sigma_\mu\le n-\nu}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ 1\le k_{\nu+\sigma_1}<\cdots<k_{\nu+\sigma_\mu}\le n\\ k\in\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^{\nu+\mu}}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\int_{\mathbb{D}_{k_{\nu+\sigma_1}}}\cdots\int_{\mathbb{D}_{k_{\nu+\sigma_\mu}}}$$
$$\times z_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}\zeta_{k_{\nu+\sigma_1}}\cdots\zeta_{k_{\nu+\sigma_\mu}}}}\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=1}^{\mu}\frac{1}{\zeta_{k_{\nu+\sigma_\tau}}-z_{k_{\nu+\sigma_\tau}}}\prod_{\tau=1}^{\nu+\mu}d\xi_{k_\tau}d\eta_{k_\tau}$$
$$+\sum_{k,\nu=1}^{n}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ k\notin\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\zeta_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}}}\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}d\xi_{k_\tau}d\eta_{k_\tau}$$
$$+\sum_{k,\nu=1}^{n}\sum_{\mu=1}^{n-\nu}\sum_{1\le\sigma_1<\cdots<\sigma_\mu\le n-\nu}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ 1\le k_{\nu+\sigma_1}<\cdots<k_{\nu+\sigma_\mu}\le n\\ k\notin\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^{\nu+\mu}}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\int_{\mathbb{D}_{k_{\nu+\sigma_1}}}\cdots\int_{\mathbb{D}_{k_{\nu+\sigma_\mu}}}$$
$$\times\zeta_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}\zeta_{k_{\nu+\sigma_1}}\cdots\zeta_{k_{\nu+\sigma_\mu}}}}\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=1}^{\mu}\frac{1}{\zeta_{k_{\nu+\sigma_\tau}}-z_{k_{\nu+\sigma_\tau}}}\prod_{\tau=1}^{\nu+\mu}d\xi_{k_\tau}d\eta_{k_\tau}\,.$$

By switching the summation index from μ to $\lambda:=\nu+\mu$ and taking $1\le\nu\le n$, $1\le\mu\le n-\nu$ into account it is easily seen that

$$I_{s3}=\sum_{k,\nu=1}^{n}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ k\in\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}z_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}}}\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}d\xi_{k_\tau}d\eta_{k_\tau}$$
$$+\sum_{k=1}^{n}\sum_{\nu=1}^{n-1}\sum_{\lambda=\nu+1}^{n}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ 1\le k_{\nu+1}<\cdots<k_\lambda\le n\\ k\in\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^\lambda}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\lambda}}z_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}\zeta_{k_{\nu+1}}\cdots\zeta_{k_\lambda}}}\prod_{\tau=1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}$$
$$\times\prod_{\tau=\nu+1}^{\lambda}\frac{1}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=1}^{\lambda}d\xi_{k_\tau}d\eta_{k_\tau}+\sum_{k,\nu=1}^{n}\sum_{\substack{1\le k_1<\cdots<k_\nu\le n\\ k\notin\{k_1,\cdots,k_\nu\}}}\frac{(-1)^{\nu+1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\zeta_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} d\xi_{k_\tau} d\eta_{k_\tau} + \sum_{k=1}^{n}\sum_{\nu=1}^{n-1}\sum_{\lambda=\nu+1}^{n} \sum_{\substack{1\le k_1<\cdots<k_\nu\le n \\ 1\le k_{\nu+1}<\cdots<k_\lambda\le n \\ k\notin\{k_1,\cdots,k_\nu\}}} \frac{(-1)^{\nu+1}}{\pi^\lambda} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\lambda}}$$

$$\times \zeta_k \overline{f_{k_1 k \overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\nu}\zeta_{k_{\nu+1}}\cdots\zeta_{k_\lambda}}} \prod_{\tau=1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=\nu+1}^{\lambda} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\lambda} d\xi_{k_\tau} d\eta_{k_\tau}.$$

Changing the summation order of the second and the fourth term for some $a_{\nu\lambda}$ gives

$$\sum_{\nu=1}^{n-1}\sum_{\lambda=\nu+1}^{n} a_{\nu\lambda} = \sum_{\lambda=2}^{n}\sum_{\nu=1}^{\lambda-1} a_{\nu\lambda} = \sum_{\lambda=1}^{n}\sum_{\nu=1}^{\lambda-1} a_{\nu\lambda} = \sum_{\nu=1}^{n}\sum_{\lambda=1}^{\nu-1} a_{\lambda\nu}, \quad 1 \le \nu \le n-1, \quad 1 \le \mu \le n-\nu.$$

Adding up the first term with the second and the third term with the fourth we reach

$$I_{s3} = \sum_{k,\nu=1}^{n}\sum_{\lambda=1}^{\nu} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n \\ k\in\{k_1,\cdots,k_\lambda\}}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} z_k \overline{f_{k_1 k \overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$+ \sum_{k,\nu=1}^{n}\sum_{\lambda=1}^{\nu} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n \\ k\notin\{k_1,\cdots,k_\lambda\}}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \zeta_k \overline{f_{k_1 k \overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}\ .$$

Thus

$$I_{s3} - I_{s2} = \sum_{k,\nu=1}^{n}\sum_{\lambda=1}^{\nu} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n \\ k\in\{k_1,\cdots,k_\lambda\}}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \zeta_k \overline{f_{k_1\, k\, \overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$+ \sum_{k,\nu=1}^{n}\sum_{\lambda=1}^{\nu} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n \\ k\notin\{k_1,\cdots,k_\lambda\}}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \zeta_k \overline{f_{k_1 k \overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}\ .$$

$$= \sum_{\nu=1}^{n} \sum_{\lambda=1}^{\nu} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{(-1)^{\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \sum_{k=1}^{n} \zeta_k \overline{f_{k_1 k \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{1 - z_{k_\tau} \overline{\zeta_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{1}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} \; .$$

Next we simplify the solvability conditions (1.21). Since the Cauchy kernel of (1.21) is real, the condition is actually equivalent to

$$Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big[\gamma_0(\zeta) - u_0(\zeta) + < grad_{\overline{\zeta}} u_0(\zeta), \zeta > - \gamma_1(\zeta) \Big]$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z_{k_\tau}}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0, \quad z \in \mathbb{D}^n, \quad \zeta \in \partial_0 \mathbb{D}^n. \qquad (1.29)$$

The remaining is to clarify the second and the third term. Applying (1.5) and (1.6) we find

$$I(z) = \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big[< grad_{\overline{\zeta}} u_0(\zeta), \zeta > - u_0(\zeta) \Big]$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z_{k_\tau}}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}^n, \quad \zeta \in \partial_0 \mathbb{D}^n$$

$$= \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Big[\sum_{\kappa=1}^{n} \overline{\zeta}_\kappa F_\kappa - u_0(\zeta) \Big] \prod_{\tau=1}^{\lambda} \frac{\overline{z_{k_\tau}}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta}$$

$$= \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n \\ 1 \le k_{\nu+1} < \cdots < k_n \le n}} \frac{1}{\pi (2\pi i)^{n-1}} \int_{\mathbb{D}_{k_1}} \int_{\partial \mathbb{D}_{k_2}} \cdots \int_{\partial \mathbb{D}_{k_n}} \Big[(\overline{\zeta}_{k_1} F_{k_1})_{\overline{\zeta}_{k_1}} + \sum_{\substack{\kappa=1 \\ \kappa \ne k_1}}^{n} \overline{\zeta}_\kappa F_{\kappa \overline{\zeta}_{k_1}} - u_{0 \overline{\zeta}_{k_1}} \Big]$$

$$\times \frac{\overline{z}_{k_1}}{1 - \overline{z}_{k_1} \zeta_{k_1}} d\xi_{k_1} d\eta_{k_1} \prod_{\tau=2}^{\lambda} \frac{\overline{z_{k_\tau}}}{\overline{\zeta_{k_\tau} - z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=2}^{n} \frac{d\zeta_{k_\tau}}{\zeta_{k_\tau}}$$

where

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{\overline{z}}{\overline{\zeta - z}} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{\overline{z}}{1 - \overline{z}\zeta} d\zeta$$

$$= \frac{1}{\pi} \int_{\mathbb{D}} f_{\overline{\zeta}}(\zeta) \frac{\overline{z}}{1 - \overline{z}\zeta} d\xi d\eta, \; f \in C^1(\overline{\mathbb{D}}), \; z \in \mathbb{D} \; , \qquad (1.30)$$

is used. Since

$$(\overline{\zeta}_{k_1} F_{k_1})_{\overline{\zeta}_{k_1}} + \sum_{\substack{\kappa=1 \\ \kappa \ne k_1}}^{n} \overline{\zeta}_\kappa F_{\kappa \overline{\zeta}_{k_1}} - u_{0 \overline{\zeta}_{k_1}} = F_{k_1} + \overline{\zeta}_{k_1} F_{k_1 \overline{\zeta}_{k_1}} + \sum_{\substack{\kappa=1 \\ \kappa \ne k_1}}^{n} \overline{\zeta}_\kappa F_{\kappa \overline{\zeta}_{k_1}} - F_{k_1} = \sum_{\kappa=1}^{n} \overline{\zeta}_\kappa F_{\kappa \overline{\zeta}_{k_1}}$$

and $F_{\kappa\overline{\zeta}_{k_1}} = f_{k_1\kappa}$, we have

$$I(z) = \sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\1\le k_{\lambda+1}<\cdots<k_\nu\le n\\1\le k_{\nu+1}<\cdots<k_n\le n}}\frac{1}{\pi(2\pi i)^{n-1}}\int_{\mathbb{D}_{k_1}}\int_{\partial\mathbb{D}_{k_2}}\cdots\int_{\partial\mathbb{D}_{k_n}}\overline{\zeta}_\kappa f_{k_1\kappa}$$
$$\times\frac{\overline{z}_{k_1}}{1-\overline{z}_{k_1}\zeta_{k_1}}d\xi_{k_1}d\eta_{k_1}\prod_{\tau=2}^{\lambda}\frac{\overline{z_{k_\tau}}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}}\prod_{\tau=\lambda+1}^{\nu}\frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=2}^{n}\frac{d\zeta_{k_\tau}}{\zeta_{k_\tau}}\,.$$

Using

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}f(\zeta)\frac{z}{\zeta-z}\frac{d\zeta}{\zeta}=\frac{-1}{2\pi i}\int_{\partial\mathbb{D}}f(\zeta)\frac{z}{1-z\overline{\zeta}}d\overline{\zeta}$$
$$=\frac{1}{\pi}\int_{\mathbb{D}}f_\zeta(\zeta)\frac{z}{1-z\overline{\zeta}}d\xi d\eta\,,\qquad z\in\mathbb{D}\quad,\quad f\in C^1(\overline{\mathbb{D}}) \tag{1.31}$$

and (1.30) again, we see

$$I(z) = \sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\1\le k_{\lambda+1}<\cdots<k_\nu\le n\\1\le k_{\nu+1}<\cdots<k_n\le n}}\frac{1}{\pi^\nu(2\pi i)^{n-\nu}}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\int_{\partial\mathbb{D}_{k_{\nu+1}}}\cdots\int_{\partial\mathbb{D}_{k_n}}$$
$$\times\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}\right)_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}}\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\lambda+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}\prod_{\tau=\nu+1}^{n}\frac{d\zeta_{k_\tau}}{\zeta_{k_\tau}}\,.$$

Applying the formula,

$$\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}f(\zeta)\frac{d\zeta}{\zeta}=f(z)+\sum_{k=1}^{n}\sum_{\mu=0}^{k}\sum_{\substack{1\le\nu_1<\cdots<\nu_\mu\le n\\1\le\nu_{\mu+1}<\cdots<\nu_k\le n}}\frac{(-1)^{k-\mu}}{\pi^k}\int_{\mathbb{D}_{\nu_1}}\cdots\int_{\mathbb{D}_{\nu_k}}$$
$$\times f_{\overline{\zeta}_{\nu_1}\cdots\overline{\zeta}_{\nu_\mu}\zeta_{\nu_{\mu+1}}\cdots\zeta_{\nu_k}}\prod_{\tau=1}^{\mu}\frac{1}{\zeta_{\nu_\tau}-z_{\nu_\tau}}\prod_{\tau=\mu+1}^{k}\frac{z_{\nu_\tau}}{1-z_{\nu_\tau}\overline{\zeta}_{\nu_\tau}}\prod_{\tau=1}^{k}d\xi_{k_\tau}d\eta_{k_\tau},\ f\in C^1(\overline{\mathbb{D}^n}) \tag{1.32}$$

see [2] page 262-263, we get

$$I(z) = \sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{1}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\Big[\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}\right)_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}}$$
$$+\sum_{\ell=1}^{n-\nu}\sum_{\mu=0}^{\ell}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\mu\le n-\nu\\1\le\sigma_{\mu+1}<\cdots<\sigma_\ell\le n-\nu\\\{\sigma_1,\cdots,\sigma_\ell\}=\{1,\cdots,\ell\}}}\sum_{\substack{1\le k_{\nu+\sigma_1}<\cdots<k_{\nu+\sigma_\mu}\le n\\1\le k_{\nu+\sigma_{\mu+1}}<\cdots<k_{\nu+\sigma_\ell}\le n}}\frac{(-1)^{\ell-\mu}}{\pi^\ell}\int_{\mathbb{D}_{k_{\nu+\sigma_1}}}\cdots\int_{\mathbb{D}_{k_{\nu+\sigma_\ell}}}$$
$$\times\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}\right)_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}\overline{\zeta}_{k_{\nu+\sigma_1}}\cdots\overline{\zeta}_{k_{\nu+\sigma_\mu}}\zeta_{k_{\nu+\sigma_{\mu+1}}}\cdots\zeta_{k_{\nu+\sigma_\ell}}}\prod_{\tau=1}^{\mu}\frac{1}{\zeta_{k_{\nu+\sigma_\tau}}-z_{k_{\nu+\sigma_\tau}}}$$

$$\times \prod_{\tau=\mu+1}^{\ell} \frac{z_{k_{\nu+\sigma_\tau}}}{1 - z_{k_{\nu+\sigma_\tau}}\overline{\zeta_{k_{\nu+\sigma_\tau}}}} \prod_{\tau=1}^{\ell} d\xi_{k_{\nu+\sigma_\tau}} d\eta_{k_{\nu+\sigma_\tau}} \Big] \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$= \sum_{\kappa=1}^{n} \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \Big[\Big(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \Big)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda}}$$

$$+ \sum_{\ell=1}^{n-\nu} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\ell\le n-\nu \\ \{\sigma_1,\cdots,\sigma_\ell\}=\{1,\cdots,\ell\}}} \sum_{1\le k_{\nu+\sigma_1}<\cdots<k_{\nu+\sigma_\ell}\le n} \frac{(-1)^\ell}{\pi^\ell} \int_{\mathbb{D}_{k_{\nu+\sigma_1}}} \cdots \int_{\mathbb{D}_{k_{\nu+\sigma_\ell}}}$$

$$\times \Big(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu} \zeta_{k_{\nu+\sigma_1}} \cdots \zeta_{k_{\nu+\sigma_\ell}}} \Big)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda}} \prod_{\tau=1}^{\ell} \frac{z_{k_{\nu+\sigma_\tau}}}{1 - z_{k_{\nu+\sigma_\tau}}\overline{\zeta_{k_{\nu+\sigma_\tau}}}} d\xi_{k_{\nu+\sigma_\tau}} d\eta_{k_{\nu+\sigma_\tau}}$$

$$+ \sum_{\ell=1}^{n-\nu} \sum_{\mu=1}^{\ell} \sum_{\substack{1\le\sigma_1<\cdots<\sigma_\mu\le n-\nu \\ 1\le\sigma_{\mu+1}<\cdots<\sigma_\ell\le n-\nu \\ \{\sigma_1,\cdots,\sigma_\ell\}=\{1,\cdots,\ell\}}} \sum_{\substack{1\le k_{\nu+\sigma_1}<\cdots<k_{\nu+\sigma_\mu}\le n \\ 1\le k_{\nu+\sigma_{\mu+1}}<\cdots<k_{\nu+\sigma_\ell}\le n}} \frac{(-1)^{\ell-\mu}}{\pi^\ell} \int_{\mathbb{D}_{k_{\nu+\sigma_1}}} \cdots \int_{\mathbb{D}_{k_{\nu+\sigma_\ell}}}$$

$$\times \Big(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu} \zeta_{k_{\nu+\sigma_1}} \cdots \zeta_{k_{\nu+\sigma_\ell}}} \Big)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda} \overline{\zeta}_{k_{\nu+\sigma_1}} \cdots \overline{\zeta}_{k_{\nu+\sigma_\mu}}} \prod_{\tau=1}^{\mu} \frac{1}{\zeta_{k_{\nu+\sigma_\tau}} - z_{k_{\nu+\sigma_\tau}}}$$

$$\times \prod_{\tau=\mu+1}^{\ell} \frac{z_{k_{\nu+\sigma_\tau}}}{1 - z_{k_{\nu+\sigma_\tau}}\overline{\zeta_{k_{\nu+\sigma_\tau}}}} \prod_{\tau=1}^{\ell} d\xi_{k_{\nu+\sigma_\tau}} d\eta_{k_{\nu+\sigma_\tau}} \Big] \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} . \tag{1.33}$$

Changing the summation from ℓ to $\alpha := \nu + \ell$ and changing the order of summation gives for the last two terms

$$\sum_{\kappa=1}^{n} \sum_{\alpha=3}^{n} \sum_{\nu=1}^{\alpha-1} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le h_1<\cdots<h_\lambda\le n\,,\ 1\le h_{\lambda+1}<\cdots<h_\alpha\le n\,,\{k_1,\cdots,k_\lambda\}=\{h_1,\cdots,h_\lambda\} \\ \{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_1},\cdots,k_{\nu+\sigma_{\alpha-\nu}}\}=\{h_{\lambda+1},\cdots,h_\alpha\}}} \frac{(-1)^{\alpha-\nu}}{\pi^\alpha} \int_{\mathbb{D}_{h_1}} \cdots \int_{\mathbb{D}_{h_\alpha}}$$

$$\times \Big(\overline{\zeta}_\kappa f_{h_1 \kappa \zeta_{h_{\lambda+1}} \cdots \zeta_{h_\alpha}} \Big)_{\overline{\zeta}_{h_2} \cdots \overline{\zeta}_{h_\lambda}} \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{h_\tau}}{1 - \overline{z}_{h_\tau}\zeta_{h_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{h_\tau}}{1 - z_{h_\tau}\overline{\zeta_{h_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{h_\tau} d\eta_{h_\tau}$$

$$+ \sum_{\kappa=1}^{n} \sum_{\alpha=3}^{n} \sum_{\nu=1}^{\alpha-1} \sum_{\lambda=1}^{\nu-1} \sum_{\mu=1}^{\alpha-\nu} \sum_{\substack{1\le h_1<\cdots<h_\lambda\le n\,,\ 1\le h_{\lambda+1}<\cdots<h_{\lambda+\mu}\le n\,,\ \{k_1,\cdots,k_\lambda,k_{\nu+\sigma_1}\cdots k_{\nu+\sigma_\mu}\}=\{h_1,\cdots,h_{\mu+\lambda}\} \\ 1\le h_{\lambda+\mu+1}<\cdots<h_\alpha\le n\,,\ \{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_{\mu+1}},\cdots,k_{\nu+\sigma_{\alpha-\nu}}\}=\{h_{\lambda+\mu+1},\cdots,h_\alpha\}}}$$

$$\times \frac{(-1)^{\alpha-\nu-\mu}}{\pi^\alpha} \int_{\mathbb{D}_{h_1}} \cdots \int_{\mathbb{D}_{h_\alpha}} \Big(\overline{\zeta}_\kappa f_{h_1 \kappa \zeta_{h_{\lambda+1}} \cdots \zeta_{h_\alpha}} \Big)_{\overline{\zeta}_{h_2} \cdots \overline{\zeta}_{h_\lambda}} \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\lambda+\mu} \frac{1}{\zeta_{h_\tau} - z_{h_\tau}}$$

$$\times \prod_{\tau=\lambda+\mu+1}^{\alpha} \frac{z_{h_\tau}}{1 - z_{h_\tau}\overline{\zeta_{h_\tau}}} \prod_{\tau=1}^{\alpha} d\xi_{h_\tau} d\eta_{h_\tau} .$$

Since

$$\sum_{\{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_{\mu+1}},\cdots,k_{\nu+\sigma_{\alpha-\nu}}\}=\{h_{\lambda+1},\cdots,h_\alpha\}} 1 = \binom{\alpha-\lambda}{\nu-\lambda}$$

for some a_λ with $1 \le \lambda \le \nu - 1$, we have

$$\sum_{\nu=1}^{\alpha-1}\sum_{\lambda=1}^{\nu-1}(-1)^{\alpha-\nu}\binom{\alpha-\lambda}{\nu-\lambda}a_\lambda = \sum_{\lambda=1}^{\alpha-2}\sum_{\nu=\lambda+1}^{\alpha-1}(-1)^{\alpha-\nu}\binom{\alpha-\lambda}{\nu-\lambda}a_\lambda = \sum_{\lambda=1}^{\alpha-2}\sum_{\nu=1}^{\alpha-\lambda-1}(-1)^{\alpha-\lambda-\nu}\binom{\alpha-\lambda}{\nu}a_\lambda$$

$$= \sum_{\lambda=1}^{\alpha-2}a_\lambda\Big[(1-1)^{\alpha-\lambda} - (-1)^{\alpha-\lambda}\binom{\alpha-\lambda}{0} - \binom{\alpha-\lambda}{\alpha-\lambda}\Big] = \sum_{\lambda=1}^{\alpha-2}\Big[(-1)^{\alpha-\lambda+1} - 1\Big]a_\lambda \; .$$

Similarly from

$$\sum_{\{k_1,\cdots,k_\lambda,k_{\nu+\sigma_1},\cdots,k_{\nu+\sigma_\mu}\}=\{h_1,\cdots,h_{\mu+\lambda}\}} 1 = \binom{\mu+\lambda}{\lambda} \, ,$$

$$\sum_{\{k_{\lambda+1},\cdots,k_\nu,k_{\nu+\sigma_{\mu+1}},\cdots,k_{\nu+\sigma_{\alpha-\nu}}\}=\{h_{\lambda+\mu+1},\cdots,h_\alpha\}} 1 = \binom{\alpha-\mu-\lambda}{\nu-\lambda} \, ,$$

for some $a_{\lambda\mu}$ with $1 \le \lambda \le \nu - 1$, $1 \le \mu \le \alpha - \nu$ it can be easily shown that

$$\sum_{\nu=1}^{\alpha-1}\sum_{\lambda=1}^{\nu-1}\sum_{\mu=1}^{\alpha-\nu}(-1)^{\alpha-\nu-\mu}\binom{\alpha-\mu-\lambda}{\nu-\lambda}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu} = \sum_{\lambda=1}^{\alpha-2}\sum_{\nu=\lambda+1}^{\alpha-1}\sum_{\mu=1}^{\alpha-\nu}(-1)^{\alpha-\nu-\mu}\binom{\alpha-\mu-\lambda}{\nu-\lambda}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$= \sum_{\lambda=1}^{\alpha-2}\sum_{\mu=1}^{\alpha-\lambda-1}\sum_{\nu=\lambda+1}^{\alpha-\mu}(-1)^{\alpha-\nu-\mu}\binom{\alpha-\mu-\lambda}{\nu-\lambda}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$= \sum_{\lambda=1}^{\alpha-2}\sum_{\mu=1}^{\alpha-\lambda-1}\sum_{\nu=1}^{\alpha-\mu-\lambda}(-1)^{\alpha-\nu-\lambda-\mu}\binom{\alpha-\mu-\lambda}{\nu}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$= \sum_{\lambda=1}^{\alpha-2}\sum_{\mu=1}^{\alpha-\lambda-1}\Big[(1-1)^{\alpha-\mu-\lambda} - (-1)^{\alpha-\mu-\lambda}\binom{\alpha-\mu-\lambda}{0}\Big]\binom{\mu+\lambda}{\lambda}a_{\lambda\mu}$$

$$= \sum_{\lambda=1}^{\alpha-2}\sum_{\mu=1}^{\alpha-\lambda-1}(-1)^{\alpha-\mu-\lambda+1}\binom{\mu+\lambda}{\lambda}a_{\lambda\mu} \, ,$$

where the rule of changing orders of summation is just the same as of multiple integrals. Thus we get the sum simplified to

$$\sum_{\kappa=1}^{n}\sum_{\alpha=3}^{n}\sum_{\lambda=1}^{\alpha-1}\Big[(-1)^{\alpha-\lambda+1} - 1\Big]\sum_{\substack{1\le h_1<\cdots<h_\lambda\le n\\ 1\le h_{\lambda+1}<\cdots<h_\alpha\le n}}\frac{1}{\pi^\alpha}\int_{I\!\!D_{h_1}}\cdots\int_{I\!\!D_{h_\alpha}}\Big(\overline{\zeta}_\kappa f_{h_1\kappa\zeta_{h_{\lambda+1}}\cdots\zeta_{h_\alpha}}\Big)_{\overline{\zeta}_{h_2}\cdots\overline{\zeta}_{h_\lambda}}$$

$$\times\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{h_\tau}}{1-\overline{z}_{h_\tau}\zeta_{h_\tau}}\prod_{\tau=\lambda+1}^{\alpha}\frac{z_{h_\tau}}{1-z_{h_\tau}\overline{\zeta_{h_\tau}}}\prod_{\tau=1}^{\alpha}d\xi_{h_\tau}d\eta_{h_\tau} + \sum_{\kappa=1}^{n}\sum_{\alpha=3}^{n}\sum_{\lambda=1}^{\alpha-2}\sum_{\mu=1}^{\alpha-\lambda-1}\binom{\mu+\lambda}{\lambda}\sum_{\substack{1\le h_1<\cdots<h_\lambda\le n\\ 1\le h_{\lambda+1}<\cdots<h_{\lambda+\mu}\le n\\ 1\le h_{\lambda+\mu+1}<\cdots<h_\alpha\le n}}$$

$$\times\frac{(-1)^{\alpha-\mu-\lambda+1}}{\pi^\alpha}\int_{I\!\!D_{h_1}}\cdots\int_{I\!\!D_{h_\alpha}}\Big(\overline{\zeta}_\kappa f_{h_1\kappa\zeta_{h_{\lambda+\mu+1}}\cdots\zeta_{h_\alpha}}\Big)_{\overline{\zeta}_{h_2}\cdots\overline{\zeta}_{h_{\lambda+\mu}}}\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{h_\tau}}{1-\overline{z}_{h_\tau}\zeta_{h_\tau}}$$

$$\times \prod_{\tau=\lambda+1}^{\lambda+\mu} \frac{1}{\zeta_{h_\tau} - z_{h_\tau}} \prod_{\tau=\lambda+\mu+1}^{\alpha} \frac{z_{h_\tau}}{1 - z_{h_\tau}\overline{\zeta_{h_\tau}}} \prod_{\tau=1}^{\alpha} d\xi_{h_\tau} d\eta_{h_\tau} \ , \ z \in \mathbb{D}^n \ , \tag{1.34}$$

namely $I(z)$ is simplified one step further,

$$I(z) = \sum_{\kappa=1}^{n} \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \left(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \right)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$+ \sum_{\kappa=1}^{n} \sum_{\nu=3}^{n} \sum_{\lambda=1}^{\nu-1} \left[(-1)^{\nu-\lambda+1} - 1 \right] \sum_{\substack{1 \le h_1 < \cdots < h_\lambda \le n \\ 1 \le h_{\lambda+1} < \cdots < h_\nu \le n}} \frac{1}{\pi^\nu} \int_{\mathbb{D}_{h_1}} \cdots \int_{\mathbb{D}_{h_\nu}} \left(\overline{\zeta}_\kappa f_{h_1 \kappa \zeta_{h_{\lambda+1}} \cdots \zeta_{h_\nu}} \right)_{\overline{\zeta}_{h_2} \cdots \overline{\zeta}_{h_\lambda}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{h_\tau}}{1 - \overline{z}_{h_\tau} \zeta_{h_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{h_\tau}}{1 - z_{h_\tau}\overline{\zeta_{h_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{h_\tau} d\eta_{h_\tau} + \sum_{\kappa=1}^{n} \sum_{\nu=3}^{n} \sum_{\lambda=1}^{\nu-2} \sum_{\mu=1}^{\nu-\lambda-1} \binom{\mu+\lambda}{\lambda} \sum_{\substack{1 \le h_1 < \cdots < h_\lambda \le n \\ 1 \le h_{\lambda+1} < \cdots < h_{\lambda+\mu} \le n \\ 1 \le h_{\lambda+\mu+1} < \cdots < h_\nu \le n}}$$

$$\times \frac{(-1)^{\nu-\mu-\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{h_1}} \cdots \int_{\mathbb{D}_{h_\nu}} \left(\overline{\zeta}_\kappa f_{h_1 \kappa \zeta_{h_{\lambda+\mu+1}} \cdots \zeta_{h_\nu}} \right)_{\overline{\zeta}_{h_2} \cdots \overline{\zeta}_{h_{\lambda+\mu}}} \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{h_\tau}}{1 - \overline{z}_{h_\tau} \zeta_{h_\tau}}$$

$$\times \prod_{\tau=\lambda+1}^{\lambda+\mu} \frac{1}{\zeta_{h_\tau} - z_{h_\tau}} \prod_{\tau=\lambda+\mu+1}^{\nu} \frac{z_{h_\tau}}{1 - z_{h_\tau}\overline{\zeta_{h_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{h_\tau} d\eta_{h_\tau} \ , \ z \in \mathbb{D}^n \ . \tag{1.35}$$

From the first and the second term it follows that

$$\sum_{\kappa=1}^{n} \sum_{\substack{1 \le k_1 \le n \\ 1 \le k_2 \le n}} \frac{1}{\pi^2} \int_{\mathbb{D}_{k_1}} \int_{\mathbb{D}_{k_2}} \left(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_2}} \right) \frac{\overline{z}_{k_1}}{1 - \overline{z}_{k_1} \zeta_{k_1}} \frac{z_{k_2}}{1 - z_{k_2}\overline{\zeta_{k_2}}} \prod_{\tau=1}^{2} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$+ \sum_{\kappa=1}^{n} \sum_{\nu=3}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{(-1)^{\nu-\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \left(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \right)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau}$$

$$= \sum_{\kappa=1}^{n} \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1 \le k_1 < \cdots < k_\lambda \le n \\ 1 \le k_{\lambda+1} < \cdots < k_\nu \le n}} \frac{(-1)^{\nu-\lambda+1}}{\pi^\nu} \int_{\mathbb{D}_{k_1}} \cdots \int_{\mathbb{D}_{k_\nu}} \left(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}} \right)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1 - \overline{z}_{k_\tau} \zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1 - z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} \ .$$

Adding this term to the last term of $I(z)$ in (1.35) leads to

$$I(z)=\sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\mu=0}^{\nu-\lambda-1}\binom{\mu+\lambda}{\lambda}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_{\lambda+\mu}\le n\\ 1\le k_{\lambda+\mu+1}<\cdots<k_\nu\le n}}\frac{(-1)^{\nu-\mu-\lambda+1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}$$
$$\times\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\lambda+\mu+1}}\cdots\zeta_{k_\nu}}\right)_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_{\lambda+\mu}}}\prod_{\tau=1}^{\lambda}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\lambda+1}^{\lambda+\mu}\frac{1}{\zeta_{k_\tau}-z_{k_\tau}}$$
$$\times\prod_{\tau=\lambda+\mu+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}\,,\qquad z\in\mathbb{D}^n\,. \tag{1.36}$$

As (1.29) is just needed on $\partial_0\mathbb{D}^n$, if we consider (1.36) for $z\in\partial_0\mathbb{D}^n$ instead of $z\in\mathbb{D}^n$ and take

$$(\zeta-z)^{-1}=-\overline{z}(1-\overline{z}\zeta)^{-1}\,,\qquad z\in\partial\mathbb{D}\,,$$

into account, then it can be written as

$$I(z)=\sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\mu=0}^{\nu-\lambda-1}\binom{\mu+\lambda}{\lambda}\sum_{\substack{1\le k_1<\cdots<k_{\lambda+\mu}\le n\\ 1\le k_{\lambda+\mu+1}<\cdots<k_\nu\le n}}\frac{(-1)^{\nu-\lambda-1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}$$
$$\times\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\lambda+\mu+1}}\cdots\zeta_{k_\nu}}\right)_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_{\lambda+\mu}}}\prod_{\tau=1}^{\lambda+\mu}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\lambda+\mu+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}\,,\ z\in\partial_0\mathbb{D}^n\,,$$

further via $\beta:=\lambda+\mu$

$$I(z)=\sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\beta=1}^{\nu-1}\sum_{\lambda=1}^{\beta}\binom{\beta}{\lambda}\sum_{\substack{1\le k_1<\cdots<k_\beta\le n\\ 1\le k_{\beta+1}<\cdots<k_\nu\le n}}\frac{(-1)^{\nu-\lambda-1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\beta+1}}\cdots\zeta_{k_\alpha}}\right)_{\overline{\zeta}_{h_2}\cdots\overline{\zeta}_{k_\beta}}$$
$$\times\prod_{\tau=1}^{\beta}\frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}}\prod_{\tau=\beta+1}^{\nu}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}\,,\ z\in\partial_0\mathbb{D}^n\,.$$

But from

$$\sum_{\beta=1}^{\nu-1}\sum_{\lambda=1}^{\beta}(-1)^{\nu-\lambda-1}\binom{\beta}{\lambda}a_\beta=\sum_{\beta=1}^{\nu-1}(-1)^{\nu-1}\sum_{\lambda=1}^{\beta}(-1)^{\lambda}\binom{\beta}{\lambda}a_\beta$$
$$=\sum_{\beta=1}^{\nu-1}(-1)^{\nu-1}\Big[(1-1)^\beta-(-1)^0\binom{\beta}{0}\Big]a_\beta=\sum_{\beta=1}^{\nu-1}(-1)^{\nu}a_\beta$$

where a_β are some definite terms with $1\le\beta\le\nu-1$, it follows that

$$I(z)=\sum_{\kappa=1}^{n}\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{(-1)^{\nu}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}\left(\overline{\zeta}_\kappa f_{k_1\kappa\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}\right)_{\overline{\zeta}_{k_2}\cdots\overline{\zeta}_{k_\lambda}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} \ , \ z \in \partial_0 I\!\!D^n \ . \tag{1.37}$$

Finally we have got explicitly the solvability conditions as well as the unique solution, i.e.,

$$Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 I\!\!D^n} \Big[\gamma_0(\zeta)-\gamma_1(\zeta)\Big] \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}} \frac{d\zeta}{\zeta}$$

$$= Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\mu=0}^{\nu-\lambda-1} \binom{\mu+\lambda}{\lambda} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_{\lambda+\mu}\le n \\ 1\le k_{\lambda+\mu+1}<\cdots<k_\nu\le n}} \frac{(-1)^{\nu-\mu-\lambda+1}}{\pi^\nu} \int_{I\!\!D_{k_1}} \cdots \int_{I\!\!D_{k_\nu}}$$

$$\times \sum_{\kappa=1}^{n} \Big(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+\mu+1}} \cdots \zeta_{k_\nu}}\Big)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_{\lambda+\mu}}} \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\lambda+\mu} \frac{1}{\zeta_{k_\tau}-z_{k_\tau}}$$

$$\times \prod_{\tau=\lambda+\mu+1}^{\nu} \frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} \ , \quad \zeta \in \partial_0 I\!\!D^n \ , \quad z \in I\!\!D^n \ , \tag{1.38}$$

or

$$Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{1}{(2\pi i)^n} \int_{\partial_0 I\!\!D^n} \Big[\gamma_0(\zeta)-\gamma_1(\zeta)\Big] \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{\overline{\zeta_{k_\tau}-z_{k_\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}} \frac{d\zeta}{\zeta}$$

$$= Re \sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{\substack{1\le k_1<\cdots<k_\lambda\le n \\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}} \frac{(-1)^\nu}{\pi^\nu} \int_{I\!\!D_{k_1}} \cdots \int_{I\!\!D_{k_\nu}} \sum_{\kappa=1}^{n} \Big(\overline{\zeta}_\kappa f_{k_1 \kappa \zeta_{k_{\lambda+1}} \cdots \zeta_{k_\nu}}\Big)_{\overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\lambda}}$$

$$\times \prod_{\tau=1}^{\lambda} \frac{\overline{z}_{k_\tau}}{1-\overline{z}_{k_\tau}\zeta_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}} \prod_{\tau=1}^{\nu} d\xi_{k_\tau} d\eta_{k_\tau} \ , \qquad z \in \partial_0 I\!\!D^n \ , \tag{1.39}$$

where the integral is understood as Cauchy principal value,

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 I\!\!D^n} \frac{<\zeta, z>}{n} \gamma_0(\zeta) \frac{d\zeta}{\zeta - z}$$

$$-\Big[1-\frac{<z,z>}{n}\Big] \frac{1}{(2\pi i)^n} \int_{\partial_0 I\!\!D^n} \Big[\gamma_1(\zeta)\Big(2\frac{\zeta}{\zeta-z}-1\Big)\frac{d\zeta}{\zeta} + iC_0\Big]$$

$$+\sum_{\nu=1}^{n} \sum_{\substack{1\le k_1\le n \\ 1\le k_2<\cdots<k_\nu\le n}} \frac{1}{\pi^\nu} \int_{I\!\!D_{k_1}} \cdots \int_{I\!\!D_{k_\nu}} f_{k_1 k_1 \overline{\zeta}_{k_2} \cdots \overline{\zeta}_{k_\nu}} \Big(\frac{\overline{z}_{k_1}\overline{\zeta}_{k_1}}{n} + \frac{\overline{\zeta}_{k_1}-\overline{z}_{k_1}}{\zeta_{k_1}-z_{k_1}}\Big) d\xi_{k_1} d\eta_{k_1} \prod_{\tau=2}^{\nu} \frac{d\xi_{k_\tau} d\eta_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}$$

$$+\sum_{\nu=2}^{n} \sum_{1\le k_1<\cdots<k_\nu\le n} \frac{1}{\pi^\nu} \int_{I\!\!D_{k_1}} \cdots \int_{I\!\!D_{k_\nu}} f_{k_1 k_2 \overline{\zeta}_{k_3} \cdots \overline{\zeta}_{k_\nu}} \prod_{\tau=1}^{\nu} \frac{d\xi_{k_\tau} d\eta_{k_\tau}}{\zeta_{k_\tau}-z_{k_\tau}}$$

$$+\Big[1-\frac{<z,z>}{n}\Big]\sum_{\nu=1}^{n}\sum_{\lambda=1}^{\nu}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{(-1)^{\lambda+1}}{\pi^\nu}\int_{\mathbb{D}_{k_1}}\cdots\int_{\mathbb{D}_{k_\nu}}$$

$$\times\sum_{k=1}^{n}\zeta_k\overline{f_{k_1k\bar\zeta_{k_2}\cdots\bar\zeta_{k_\lambda}\zeta_{k_{\lambda+1}}\cdots\zeta_{k_\nu}}}\prod_{\tau=1}^{\lambda}\frac{z_{k_\tau}}{1-z_{k_\tau}\overline{\zeta_{k_\tau}}}\prod_{\tau=\lambda+1}^{\nu}\frac{1}{\zeta_{k_\tau}-z_{k_\tau}}\prod_{\tau=1}^{\nu}d\xi_{k_\tau}d\eta_{k_\tau}\ ,\quad z\in\mathbb{D}^n. \tag{1.40}$$

Theorem 5 *Let $f_{k\ell}(z)\in C^{1+\alpha}(\overline{\mathbb{D}^n})\cap L_1(\overline{\mathbb{D}^n})$ and satisfy the compatibility conditions (1.2). A particular solution to the inhomogeneous system (1.1) is given by (1.7). If the problem M is solvable then condition (1.12) must be satisfied. The modified problem M is solvable if and only if condition (1.12) and (1.38) or (1.39) hold. Then the unique solution is given by (1.40). The corresponding homogeneous problem has no any nontrivial solutions. The modified problem M is well–posed.*