Exotic Components of the Toric Hilbert Scheme

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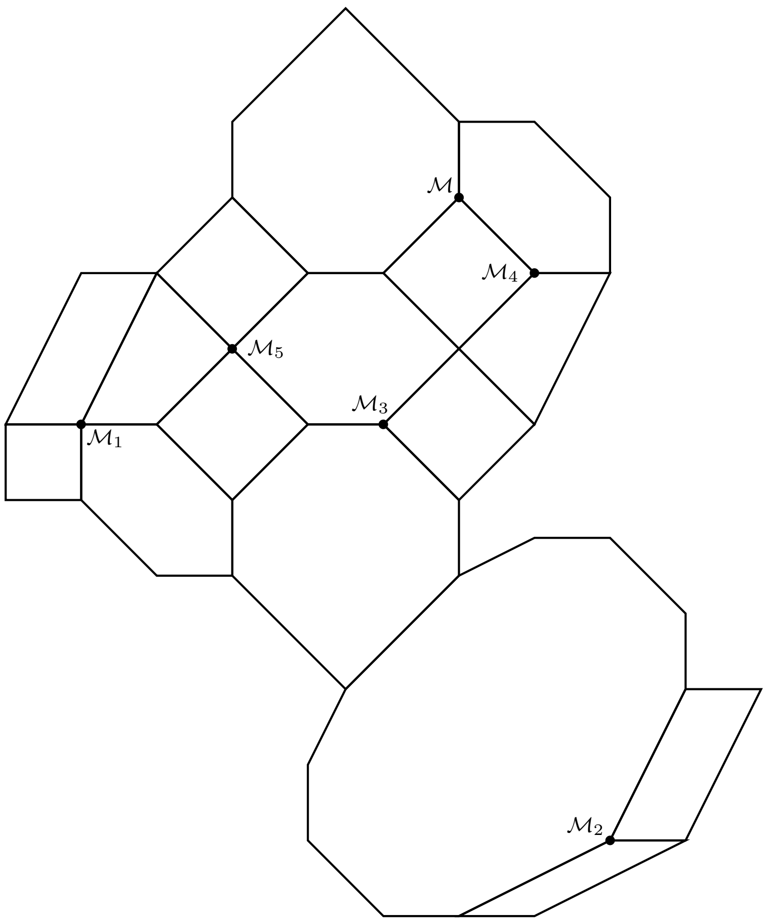

Dissertation, April 2010
Freie Universität Berlin
Fachbereich Mathematik und Informatik

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Eidesstattliche Erklärung

Gemäß §7 (4) der Promotionsordnung versichere ich hiermit, diese Arbeit selbstständig verfasst zu haben. Ich habe alle bei der Erstellung dieser Arbeit benutzten Hilfsmittel und Hilfen angegeben.

Berlin, den 13. April 2010

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Datum der Disputation: 14.07.2010

Abstract

We introduce an explicit construction of the non-coherent components of toric Hilbert schemes. In particular, we show that the polytope describing the normalisation of such a non-coherent component is in fact a state polytope of some homogeneous ideal. We give an explicit construction of this ideal, the so-called generalised universal family. We use this construction to compute examples with interesting characteristics, such as an embedded component in the coherent component. Furthermore, we construct a stratification of the toric Hilbert scheme by the maximal subtorus action that leaves the corresponding $\mathcal{A}$-graded ideals invariant. Finally, we present a correspondence between the toric Hilbert scheme and the moduli space of stable toric pairs constructed by Alexeev.

Contents

Chapter I

Introduction

In [Gro95] Grothendieck introduced the classical Hilbert scheme which parametrises all subschemes of $\mathbb{P}_{\Bbbk}^{n-1}$ with the same fixed Hilbert polynomial. Hereby, the total coordinate ring $S = \Bbbk[x_1, ..., x_n]$ is endowed with the classical $\mathbb{Z}$-grading. In toric geometry one works with multigraded rings, *i.e.* the degree of each x_i is an element in an abelian group A, for example $\mathbb{Z}^d$. In an analog of the classical case one can define the multigraded Hilbert function for some ideal I in the multigraded ring S, that counts the dimension over $\Bbbk$ of $(S/I)_a$ for each multidegree $a \in A$. Haiman and Sturmfels [HS04] showed that there is a multigraded Hilbert scheme, which parametrises all homogeneous ideals with a fixed multigraded Hilbert function in S.

Before that, a special case of multigraded Hilbert schemes, the toric Hilbert schemes, had already been studied by Arnol′d [Arn89], Korkina, Post and Roelofs [Kor92, KPR95], Sturmfels [Stu94], Peeva and Stillman [PS02, PS00], Maclagan and Thomas [MT03, MT02], and others. This toric Hilbert scheme is given by the multigraded Hilbert function of the toric ideal $I_{\mathcal{A}}$ in S. If the set of degrees is $\mathcal{A} = \{a_i := \deg(x_i)\} \subset \mathbb{Z}^d$ then the multigraded Hilbert function is 1 for a in the semigroup generated by $\mathcal{A}$ and 0 otherwise. At first all ideals with the same multigraded Hilbert function as the toric ideal $I_{\mathcal{A}}$ were called $\mathcal{A}$-graded ideals.

The toric Hilbert schemes are on the one hand more structured than the classical Hilbert schemes, but also more constricted than general multigraded Hilbert schemes, so that they can be studied more intensely. On the other hand, they still have many interesting characteristics one might not expect, for example the intersection behaviour of their components. Thus, they provide a very good class of examples to understand multigraded Hilbert schemes.

This thesis is structured into five chapters. We start in Chapters II and III by citing most of the facts about $\mathcal{A}$-graded ideals and the toric Hilbert scheme given mainly in [Stu94], [PS02, PS00] and [MT03, MT02]. By this we also have a consistent nomenclature to work with in the following chapters. Chapters IV and V present the new results of the current author. The last chapter depicts the moduli space of stable toric pairs by Alexeev and shows the correspondence to the toric Hilbert scheme established by the present author. Although this thesis is written to be self-explanatory and complete, we assume that the reader has basic knowledge of algebraic geometry and toric varieties as for example in the first chapters of [Har77] or [Ful93], respectively.

In Chapter II of this work we collect most of the facts known so far about $\mathcal{A}$-

graded ideals. In general, an initial ideal of some ideal I has the same multigraded Hilbert function as I (Lemma II.2.7), so that we get a set of $\mathcal{A}$-graded ideals by taking all initial ideals of $I_{\mathcal{A}}$ for all different weight vectors. An ideal isomorphic to such an initial ideal of the toric ideal is called *coherent*, where isomorphic means that the corresponding algebras are isomorphic as multigraded $\Bbbk$-algebras. Hence, the isomorphism classes of coherent $\mathcal{A}$-graded ideals are in bijection with the cones of the Gröbner fan of $I_{\mathcal{A}}$.

In Chapter III we give the two equivalent definitions of the toric Hilbert scheme by Maclagan and Thomas, and by Peeva and Stillman. They construct a projective scheme that parametrises all $\mathcal{A}$-graded ideals. The global equations of this scheme are in general very extensive, but Peeva and Stillman show that there is a cover of affine open charts around the monomial $\mathcal{A}$-graded ideals and that the calculations of the local equations for these affine charts are much more feasible (Section III.3). They show that the toric Hilbert scheme can have several irreducible components. However, there is a unique component containing the toric ideal (Theorem III.2.4), which in fact contains exactly all coherent $\mathcal{A}$-graded ideals. Therefore, this component is called the *coherent component* of the toric Hilbert scheme. Moreover, the normalisation of the coherent component is the toric variety associated to the Gröbner fan of the toric ideal (Theorem III.2.5). A third construction by Sturmfels [Stu94] shows that the underlying reduced structure of each irreducible component of the toric Hilbert scheme is given by binomial equations so that all these reduced components are projective toric varieties and therefore each of them contains a dense torus, the so-called *ambient torus* of that component. Thus, this implies the existence of a polytope P_V for each component V of the toric Hilbert scheme such that the normalisation of this component is the toric variety associated to the normal fan of P_V (Corollary III.2.3).

The main part of this work is Chapter IV where we give an explicit construction of the polytope P_V for an arbitrary non-coherent reduced component V of a toric Hilbert scheme. For this we use the local equations around a monomial $\mathcal{A}$-graded ideal $\mathcal{M}$ in new variables $\boldsymbol{y}$ to construct a *universal family* $J_{\mathcal{M}}(\mathfrak{p})$ for each non-coherent irreducible component containing $\mathcal{M}$ (Definition IV.1.21), where $\mathfrak{p}$ denotes an associated prime of the local equations defining the underlying reduced structure $V_{\mathfrak{p}}$ of such a non-coherent irreducible component.

Main Theorem 1 (Theorem IV.1.22)**.** *The universal family $J_{\mathcal{M}}(\mathfrak{p})$ parametrises the ambient torus of the reduced irreducible non-coherent component $V_{\mathfrak{p}}$ in the toric Hilbert scheme over the points in $(\Bbbk^*)^{\dim(V_{\mathfrak{p}})}$. To be precise, the closed points of this irreducible component of the toric Hilbert scheme intersected with its ambient torus are exactly those $\mathcal{A}$-graded ideals that are given by substituting a point $(\lambda_i)_{i=1,\dots,\dim(V_{\mathfrak{p}})} \in (\Bbbk^*)^{\dim(V_{\mathfrak{p}})}$ into $J_{\mathcal{M}}(\mathfrak{p})$.*

Thus, the ambient torus $(\Bbbk^*)^{\dim(V_{\mathfrak{p}})}$ of a non-coherent component is different from the torus $\mathrm{Spec}(\Bbbk[\boldsymbol{x}^{\pm 1}])$ of the coherent component. Then we construct a homogenised version $\widetilde{J_{\mathcal{M}}(\mathfrak{p})}$ of $J_{\mathcal{M}}(\mathfrak{p})$ by introducing a new set of variables $z_1, \dots, z_{\dim(V_{\mathfrak{p}})}$ with the same degrees as the $\boldsymbol{y}$ variables (Definition IV.3.1). This homogeneous family $\widetilde{J_{\mathcal{M}}(\mathfrak{p})}$ is called the *generalised universal family* and also gives the ambient torus of the non-coherent component. But more importantly, it gives the main result of that chapter:

Main Theorem 2 (Theorem IV.3.4)**.** *Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal and $\widetilde{J_{\mathcal{M}}(p)} \subseteq \Bbbk[\boldsymbol{x}, y_i, z_i \,|\, i = 1, ..., \dim(V_p)]$ a generalised universal family of a reduced component V_p containing $\mathcal{M}$. Then $\widetilde{J_{\mathcal{M}}(p)}$ is homogeneous with respect to a strictly positive grading and the normalisation of the component V_p is the toric variety defined by the normal fan of the state polytope* $\mathrm{state}(\widetilde{J_{\mathcal{M}}(p)})$*, i.e. the Gröbner fan of $\widetilde{J_{\mathcal{M}}(p)}$.*

Furthermore, for each $\mathcal{A}$-graded ideal I contained in some non-coherent component V_p the ambient torus orbit of I in this component corresponds to a face F_I of the state polytope of $\widetilde{J_{\mathcal{M}}(p)}$ giving V_p. In particular, vertices correspond to monomial $\mathcal{A}$-graded ideals and edges correspond to one-dimensional orbits of $\mathcal{A}$-graded ideals which are generated by one binomial $\boldsymbol{x}^{\boldsymbol{m}} - \lambda \boldsymbol{x}^{\boldsymbol{n}}$ for $\lambda \in \Bbbk^*$ and further monomials. The theorem allows one to construct all non-coherent components explicitly and study their properties.

Using our explicit construction we give various examples especially in Chapter V that demonstrate different very interesting intersection behaviours of the components of a toric Hilbert scheme which have not been found so far. Thus, we show that there can be a non-coherent component intersecting the coherent component, but the intersection is not a face of each of the corresponding polytopes (Example IV.1.4). On the other hand, there are also pairs of components whose intersection is given by a facet of each of the corresponding polytopes (Example V.2.1). Then again, embedded components do appear in toric Hilbert schemes (Example V.2.2). Moreover, there are even embedded components in the coherent component that can be given by facets of the state polytope of the toric ideal (Example V.2.3).

In toric geometry it is often of interest to form a quotient by a subtorus action on some variety. The torus $(\Bbbk^*)^n$ acts diagonally on $S = \Bbbk[x_1, ..., x_n]$ and if the variety is given by some ideal $I \subset S$ then a k-dimensional subtorus action on S/I is given by a k-dimensional grading D on S such that I is homogeneous with respect to D. This is equivalent to I being invariant under the subtorus action. Thus, in Section V.1 we study for each $\mathcal{A}$-graded ideal I the maximal rank of a subtorus of $(\Bbbk^*)^n$ under whose action I is invariant. It turns out that for every reduced irreducible component V_p containing I all $\mathcal{A}$-graded ideals corresponding to the $(\Bbbk^*)^{\dim(V_p)}$-orbit of I have the same space of possible subtorus actions. Moreover, we prove a correspondence between the restriction space V_I of the grading D inducing an invariant action, and the edges of the face F_I, where the restriction means that $\mathrm{Ker}(D)$ must contain V_I:

Main Theorem 3 (Theorem V.1.18)**.** *Let I be an $\mathcal{A}$-graded ideal, F_I the corresponding face of the polytope of some component containing I, and denote by $\mathcal{G}_{F_I}$ the set of binomials corresponding to the edges of F_I. Then the restriction space is given by*

$$V_I = \mathrm{span}\,\{\boldsymbol{m} - \boldsymbol{n} \,|\, \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}_{F_I}\}\,.$$

This means the maximal possible rank of a subtorus action induces a stratification on the toric Hilbert scheme. In this stratification each face of the polytopes P_V corresponds to a subset of a stratum and each stratum is then given by the collection of all faces with the same span of their edges (Remark V.1.20).

Finally, in the last chapter we establish a connection between the toric Hilbert scheme and the moduli space of stable toric pairs introduced by Alexeev in [Ale02]. For this, we first follow Alexeev's construction of stable toric pairs and their moduli space. A stable toric pair $(\mathbb{P}, \Theta)$ is a polarized stable toric variety $(\mathbb{P}, L)$ with a Cartier divisor Θ, that does not contain any torus orbits, and such that $L = \mathcal{O}_{\mathbb{P}}(\Theta)$. He shows that every polarized stable toric variety is isomorphic to a projective variety $\mathbb{P}[\Delta, t]$ over a complex of lattice polytopes Δ which is embedded on height 1 in a vector space of one dimension higher (Theorem VI.2.8). The variety $\mathbb{P}[\Delta, t]$ is obtained by gluing the irreducible projective varieties $\mathbb{P}(\delta)$ for all $\delta \in \Delta$ along the intersections of the polytopes, where $\mathbb{P}(\delta)$ is the projective spectrum of the semigroup algebra given by the lattice points of the cone over δ (Definitions VI.2.2 and VI.2.4).

If for one fiber $\mathbb{P}[\Delta, t]$ in a flat family of stable toric pairs the complex Δ is a cell decomposition of a polytope Q then for every fiber in that family the complex is some cell decomposition of Q (Lemma VI.3.6). Using this, Alexeev constructs a coarse moduli space $TP^{\mathrm{fr}}[Q]$ of stable toric pairs given by cell decompositions of the same polytope Q. Then he shows that there is a component in $TP^{\mathrm{fr}}[Q]$ that corresponds to all coherent subdivisions of Q. Note that the coherent subdivisions of a polytope Q are in bijection with the faces of the secondary polytope of Q (see [GKZ08]).

Assume that Q is a normal lattice polytope embedded at height 1, where normal means that the semigroup of lattice points of the cone over Q is generated by the lattice points of Q. We denote the lattice points of the embedded Q by $\mathcal{A} = \{a_1, ..., a_n\}$. Then we show that for each decomposition Δ of Q into normal polytopes we have $\mathbb{P}[\Delta, 1] = \Bbbk[x_1, ..., x_n]/I$ for a reduced $\mathcal{A}$-graded ideal I. Moreover, the main result of the last chapter establishes a correspondence between a subset of the moduli space of stable toric pairs $TP^{\mathrm{fr}}[Q]$ and a subset of the toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$.

Main Theorem 4 (Theorem VI.6.4). *There is a one-to-one correspondence between general points of orbits of reduced $\mathcal{A}$-graded ideals and unions of all strata of stable toric pairs with the same normal cell decomposition of Q.*

Furthermore, we show that both notions of coherence correspond to each other.

Main Theorem 5 (Theorem VI.6.6). *In the correspondence of Theorem VI.6.4 orbits of coherent $\mathcal{A}$-graded ideals correspond to subsets with coherent cell decompositions.*

This means that each face of the state polytope of the toric ideal of $\mathcal{A}$ giving a reduced initial ideal corresponds to a face of the secondary polytope of the convex hull of $\mathcal{A}$ giving a decomposition into normal polytopes.

Chapter II

$\mathcal{A}$-Graded Ideals

In this first chapter we mainly cite proofs as we are collecting already known facts about $\mathcal{A}$-graded ideals from [Stu94], [PS02], and [MT02]. We begin with the definition of $\mathcal{A}$-graded ideals and state some first properties.

II.1 Definitions and Properties

We will be using the lattice $M := \mathbb{Z}^d$ with its associated vector space over $\mathbb{Q}$ $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ and dual lattice $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^d$ with corresponding $\mathbb{Q}$-vector-space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. We will work over an algebraically closed field $\Bbbk$.

Let $\mathcal{A}$ be a collection of n vectors $a_1, ..., a_n$ in M such that $0 \in M$ is not contained in their positive hull. One can also define $\mathcal{A}$ as a linear map of lattices $\mathcal{A} : \mathbb{Z}^n \to M$ by $e_i \mapsto a_i$ such that

$$\mathrm{Ker}(\mathcal{A}) \cap \mathbb{N}^n = 0. \tag{II.1}$$

We will often make use of both notations. We denote by $\mathbb{N}\mathcal{A}$ the semigroup generated by $a_1, ..., a_n$ in M. This is the image of $\mathbb{N}^n$ under $\mathcal{A}$. Suppose that $\mathcal{A}$ has rank d. Otherwise we can restrict M to the sublattice $M \cap \mathcal{A}(\mathbb{Q}^n)$ in which $\mathcal{A}$ has full rank. Define a polynomial ring $S := \Bbbk[x_1, ..., x_n]$ over $\Bbbk$ with an M-grading given by $\mathcal{A}$. This just means that x_i has degree a_i. Furthermore, an element $r \in S$ is M-homogeneous if every term of r has the same degree in the M-grading induced by $\mathcal{A}$, and an ideal $I \subseteq S$ is M-homogeneous if for every $r \in I$ and its decomposition $r = \sum r_a$ into M-degree parts $r_a \in I$ holds for each a, where r_a is the sum over all terms in r with degree $a \in M$.

Definition II.1.1. An ideal $I \subseteq S$ is called *$\mathcal{A}$-graded* if it is M-homogeneous and

$$\dim_{\Bbbk} (S/I)_a = \begin{cases} 1 & \text{if } a \in \mathbb{N}\mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{II.2}$$

holds for its multigraded Hilbert function. It is called *weakly $\mathcal{A}$-graded* if the left-hand side is less than or equal to the right-hand side. We call a $\Bbbk$-algebra $\mathcal{A}$-graded if it is of the form S/I for some $\mathcal{A}$-graded ideal I.

A special $\mathcal{A}$-graded ideal is the *toric ideal* $I_{\mathcal{A}}$, which is the kernel of the homomorphism $S \to \Bbbk[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ that maps x_i to $\boldsymbol{t}^{a_i} = t_1^{a_i^1} \cdot \ldots \cdot t_d^{a_i^d}$ for $1 \le i \le n$. Therefore, $I_{\mathcal{A}} = \langle \boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} \mid \boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^n, \boldsymbol{a} - \boldsymbol{b} \in \mathrm{Ker}(\mathcal{A}) \rangle$ is generated by binomials and is by definition $\mathcal{A}$-graded, because it identifies all monomials of the same degree.

Remark II.1.2. Since $I_\mathcal{A}$ is $\mathcal{A}$-graded, we can also say that an M-homogeneous ideal $I \subset S$ is $\mathcal{A}$-graded if it has the same multigraded Hilbert function as $I_\mathcal{A}$.

Example II.1.3. Let $\mathcal{A} = \{\binom{1}{2}, \binom{1}{1}, \binom{2}{1}\} \subset \mathbb{Z}^2$ induce a multigrading on the ring $S = \Bbbk[x_1, x_2, x_3]$. Then the toric ideal for this $\mathcal{A}$ is $I_\mathcal{A} = \langle x_1x_3 - x_2^3 \rangle$. Another $\mathcal{A}$-graded ideal for this $\mathcal{A}$ is for example the monomial ideal $\langle x_1x_3 \rangle$.

$\Diamond$

Being a binomial ideal also holds for every other (weakly) $\mathcal{A}$-graded ideal. To see this, let $\boldsymbol{x^a}$ and $\boldsymbol{x^b}$ be of the same degree, say c. Since $\dim(S/I)_c \leq 1$ there exist $\alpha, \beta \in \Bbbk$ which are not both zero such that $\alpha \boldsymbol{x^a} - \beta \boldsymbol{x^b} \in I$. Hence every polynomial in I can be reduced to binomials. If one of them, for example $\boldsymbol{x^b}$, is not in I then $\alpha \neq 0$ so that we can set $\alpha = 1$ and get the following:

Lemma II.1.4. *Let I be an $\mathcal{A}$-graded ideal in S and $\boldsymbol{x^u}, \boldsymbol{x^v}$ two monomials of the same degree $a \in \mathbb{N}\mathcal{A}$ with $\boldsymbol{x^v} \notin I$. Then there is a unique $\alpha_{\boldsymbol{uv}} \in \Bbbk$ such that $\boldsymbol{x^u} - \alpha_{\boldsymbol{uv}} \cdot \boldsymbol{x^v} \in I$.* $\square$

Definition II.1.5. A binomial $\boldsymbol{x^u} - \boldsymbol{x^v} \in I_\mathcal{A}$ is called *primitive* (or *Graver*) if there are no proper monomial factors $\boldsymbol{x^{u'}}$ of $\boldsymbol{x^u}$ and $\boldsymbol{x^{v'}}$ of $\boldsymbol{x^v}$ with $\boldsymbol{x^{u'}} - \boldsymbol{x^{v'}} \in I_\mathcal{A}$. A degree $\boldsymbol{a} \in \mathbb{N}\mathcal{A}$ is called a *primitive* (or *Graver*) *degree* if there exists some primitive binomial $\boldsymbol{x^u} - \boldsymbol{x^v}$ with degree $\mathcal{A}\boldsymbol{u} = \mathcal{A}\boldsymbol{v} = \boldsymbol{a}$ and we denote the set of all primitive degrees by $\mathcal{P}d(\mathcal{A})$. The set of all primitive binomials is called the *Graver basis* and we denote it by $\mathcal{G}(\mathcal{A})$.

By [Stu96, Theorem 4.7] there are only finitely many primitive binomials and hence $\mathcal{P}d(\mathcal{A})$ and $\mathcal{G}(\mathcal{A})$ are finite.

Remark. From now on we will use alphabetical variables $a, b, c, ...$ instead of $x_1, x_2, x_3, ...$ in almost all examples to improve lucidity.

Example II.1.6. Let $\mathcal{A} = \{1, 2, 3\} \subset \mathbb{Z}^1$ define a grading on $S = \Bbbk[a, b, c]$. Then the toric ideal is

$$I_\mathcal{A} = \left\langle a^2 - b, ab - c \right\rangle.$$

The Graver basis in this case is

$$\mathcal{G}(\mathcal{A}) = \left\{a^2 - b, ab - c, a^3 - c, ac - b^2, b^3 - c^2\right\}$$

with primitive degrees $\mathcal{P}d(\mathcal{A}) = \{2, 3, 4, 6\}$ and the monomials of which the Graver basis elements consist are

$$\operatorname{mon}(\mathcal{G}(\mathcal{A})) = \left\{a^2, a^3, b, b^2, b^3, c, c^2, ab, ac\right\}.$$

$\Diamond$

These primitive binomials are useful to construct weakly $\mathcal{A}$-graded ideals.

Proposition II.1.7. *Let I be an ideal in S. Then I is weakly $\mathcal{A}$-graded if and only if for every primitive binomial $\boldsymbol{x^u} - \boldsymbol{x^v}$ there exists $(\alpha : \beta) \in \mathbb{P}^1$ such that $\alpha \boldsymbol{x^u} - \beta \boldsymbol{x^v} \in I$.*

Proof. See [PS02, Proposition 2.1]. $\square$

A refinement of this proposition gives a useful property of $\mathcal{A}$-graded ideals.

Lemma II.1.8. *Let I be an $\mathcal{A}$-graded ideal in S. Then there exist unique $\alpha_{\boldsymbol{uv}} \in \Bbbk$ such that*

$$I = \langle \boldsymbol{x^u} - \alpha_{\boldsymbol{uv}} \boldsymbol{x^v} \mid \boldsymbol{x^u} - \boldsymbol{x^v} \in \mathcal{G}(\mathcal{A}) \text{ and } \boldsymbol{x^v} \notin I \rangle .$$

In particular, there is a minimal set of generators of I consisting of primitive binomials with coefficients.

Proof. See [PS02, Corollary 2.3] or [Stu94, Lemma 2.2]. □

Proposition II.1.7 enables us to construct weakly $\mathcal{A}$-graded ideals: For every $\boldsymbol{x^u} - \boldsymbol{x^v} \in \mathcal{G}(\mathcal{A})$ pick some $(\alpha_{\boldsymbol{uv}} : \beta_{\boldsymbol{uv}}) \in \mathbb{P}^1$ and then set

$$I = \langle \alpha_{\boldsymbol{uv}} \boldsymbol{x^u} - \beta_{\boldsymbol{uv}} \boldsymbol{x^v} \mid \boldsymbol{x^u} - \boldsymbol{x^v} \in \mathcal{G}(\mathcal{A}) \rangle. \tag{II.3}$$

These ideals are all weakly $\mathcal{A}$-graded. However, since we are actually interested in $\mathcal{A}$-graded ideals, Lemma II.1.8 ensures that we also get all $\mathcal{A}$-graded ideals by this method. We only need a criterion to distinguish the $\mathcal{A}$-graded ones from the weakly $\mathcal{A}$-graded ones.

Obviously, we cannot check $\dim_{\Bbbk} (S/I)_a = 1$ for all $a \in \mathbb{N}\mathcal{A}$, so we are looking for smaller finite parts of $\mathbb{N}\mathcal{A}$ on which it is sufficient to check the condition. Let $\mathcal{R} \subset M$. We say that I is $\mathcal{A}$-graded on $\mathcal{R}$ if it has the same Hilbert function on $\mathcal{R}$ as the toric ideal, that is

$$\dim_{\Bbbk} (S/I)_a = \dim_{\Bbbk} (S/I_{\mathcal{A}})_a$$

for all $a \in \mathcal{R}$. Now we need to find some $\mathcal{R}$ such that being $\mathcal{A}$-graded on $\mathcal{R}$ implies being $\mathcal{A}$-graded. A first approach was done by Sturmfels in [Stu94, Section 5]. He defined the *zonotope*

$$Z_r(\mathcal{A}) := \left\{ \sum_{i=1}^{n} \lambda_i \cdot a_i \mid 0 \le \lambda_i \le r \text{ for } i = 1, \ldots, n \right\} \subset M_{\mathbb{Q}}$$

for $r > 0$. Then he showed the following:

Proposition II.1.9. *Let $a = \max\{||a_i||_2 \mid i = 1, \ldots, n\}$, $r = (n-d)^{2^n} \cdot a^{d2^n}$, and let I be generated as in (II.3). Then I is $\mathcal{A}$-graded if it is $\mathcal{A}$-graded on $Z_r(\mathcal{A}) \cap M$.*

Proof. See [Stu94, Proposition 5.1]. □

Since the r in Proposition II.1.9 grows double-exponentially it gives a rather high bound for the zonotope on which one has to check for $\mathcal{A}$-gradedness. Already in this paper Sturmfels mentioned that this bound seems too big and conjectured that $r = (n-d) \cdot a^d$ is large enough. This was then proven in [PS00]. Set

$$\begin{aligned}
\operatorname{mon}(\mathcal{G}(\mathcal{A})) &:= \{\boldsymbol{x^u} \mid \exists \boldsymbol{x^v} : \boldsymbol{x^u} - \boldsymbol{x^v} \in \mathcal{G}(\mathcal{A})\}, \\
T &:= \{\operatorname{lcm}(m_1, \ldots, m_n) \mid m_i \in \operatorname{mon}(\mathcal{G}(\mathcal{A})), 1 \le i \le n\}, \text{ and} \\
\mathcal{V} &:= \left\{ \deg(m) - \sum_{j \in J} a_j \mid m \in T, J \subseteq [n] \right\} \subset M.
\end{aligned}$$

Then $\mathcal{V}$ is a smaller bounding region as the next theorem shows.

Theorem II.1.10. *If a weakly $\mathcal{A}$-graded ideal I generated as in equation (II.3) is $\mathcal{A}$-graded on $\mathcal{V}$ then it is $\mathcal{A}$-graded.*

Proof. See [PS00, Theorem 5.2]. □

Hence, the zonotope $Z_r(\mathcal{A})$ with $r = (n-d)\cdot a^d$ suffices since it contains $\mathcal{V} \cap \mathbb{N}\mathcal{A}$.

Example II.1.11 (continuing **II.1.3**). For $\mathcal{A} = \{\binom{1}{2}, \binom{1}{1}, \binom{2}{1}\} \subset \mathbb{Z}^2$ we have $n = 3$, $d = 2$, and

$$a = \max\left\{\sqrt{5}, \sqrt{2}, \sqrt{5}\right\} = \sqrt{5}.$$

Thus, for the bound in Proposition II.1.9 we get

$$r = 1 \cdot 5^8 = 390{,}625$$

so that the zonotope $Z_r(\mathcal{A})$ has 762,940,625,001 lattice points. On the other hand, the bound resulting from Theorem II.1.10 amounts to

$$r = (3-2)\cdot \sqrt{5}^2 = 5$$

so that for this r the zonotope $Z_r(\mathcal{A})$ has only 141 lattice points. Finally, there is only one Graver degree which has just the two monomials b^3 and ac so that we get

$$T = \left\{b^3, ac, ab^3c\right\}.$$

Computing $\mathcal{V}$ results in the smallest region, *i.e.*

$$\mathcal{V} \cap \mathbb{N}\mathcal{A} = \{\binom{0}{0}, \binom{1}{1}, \binom{2}{1}, \binom{1}{2}, \binom{2}{2}, \binom{3}{3}, \binom{4}{3}, \binom{3}{4}, \binom{5}{4}, \binom{4}{5}, \binom{5}{5}, \binom{6}{6}\},$$

which consists of only 12 lattice points (See Figure II.1).

◇

II.2 Gröbner Degenerations and the State Polytope

Monomial $\mathcal{A}$-graded ideals play a central role in studying $\mathcal{A}$-graded ideals. Initial ideals with respect to a term order are monomial ideals. Therefore, we recall some basic facts about term orders and Gröbner bases. For more information on Gröbner bases we refer to [CLO07, Chapter 2] or [Stu96, Chapter I]. First of all we start with term orders.

A *term order* $\prec$ on a polynomial ring $S = \Bbbk[x_1, ..., x_n]$ is a total order on the monomials of S such that 1 is the unique minimal element and $m_1 \prec m_2$ implies $m_1 \cdot n \prec m_2 \cdot n$ for all monomials $m_1, m_2, n \in S$. For example, on $\Bbbk[x, y, z]$ we can define the so-called lexicographic term order $\prec_{\text{lex}}$ by $z \prec_{\text{lex}} y \prec_{\text{lex}} x$ which means a monomial is of higher order than another if the x exponent is greater. If they have the same x exponent, then the y exponents are compared and so on, *e.g.*

$$x^2y^4z^6 \prec_{\text{lex}} x^3yz \quad \text{and} \quad x^3y^2z^3 \prec_{\text{lex}} x^3y^2z^5.$$

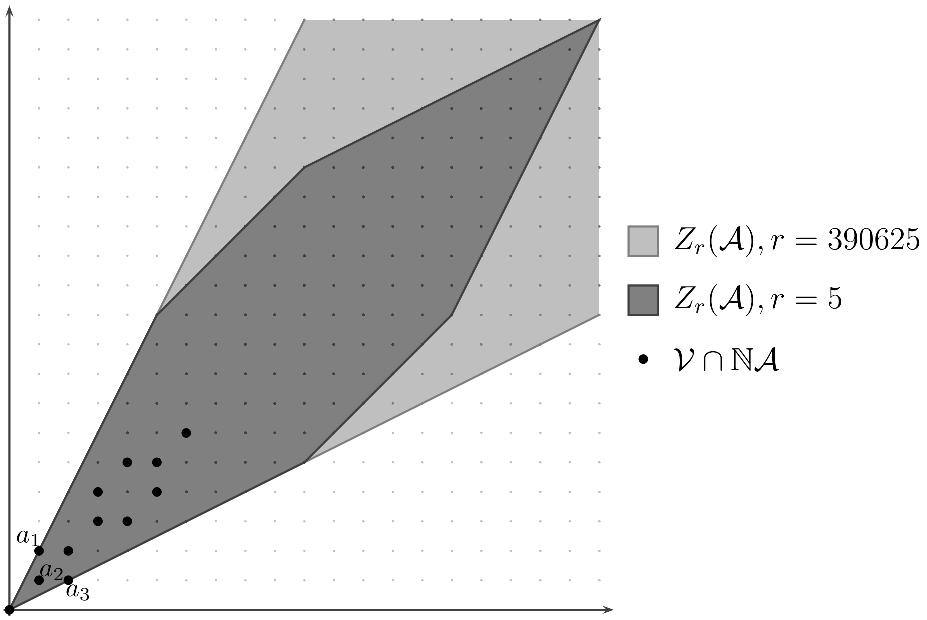

Figure II.1: The two zonotopes and $\mathcal{V} \cap \mathbb{N}\mathcal{A}$

Such a term order gives a unique *initial term* $\text{in}_\prec(f)$ for every $f \in S$, which is the monomial with the highest order in f with respect to $\prec$. Let $I \subset S$ be some ideal. Then the *initial ideal* of I is defined as $\text{in}_\prec(I) := \langle \text{in}_\prec(f) \,|\, f \in I \rangle$. Note that if $\{f_1, ..., f_r\}$ is a set of generators for I, the set $\{\text{in}_\prec(f_1), ..., \text{in}_\prec(f_r)\}$ need not be a generating set for $\text{in}_\prec(I)$. For example, if $I = \langle x - y, x - z \rangle \subset \Bbbk[x, y, z]$ with $z \prec y \prec x$, then $\text{in}_\prec(I) = \langle x, y \rangle$.

Definition II.2.1. A finite set $\{f_1, ..., f_r\} \subset I$ is a *Gröbner basis* of I with respect to $\prec$ if $\text{in}_\prec(I) = \langle \text{in}_\prec(f_1), ..., \text{in}_\prec(f_r) \rangle$. It is called *reduced*, if for every two distinct f_i, f_j no term of f_i is divisible by $\text{in}_\prec(f_j)$.

Note that every Gröbner basis is also a generating set of the ideal and the reduced Gröbner basis is unique for a given term order and an ideal. However, two term orders with different initial ideals might give the same reduced Gröbner basis but with different initial terms. Thus, if we also mark the initial terms in a reduced Gröbner basis we call it a *marked* reduced Gröbner basis. Then initial ideals of an ideal I are in one-to-one correspondence to marked reduced Gröbner bases of I. Gröbner bases can be computed with the *Buchberger Algorithm* (See [CLO07, Chapter 5]). In this computation new polynomials are generated, that might have initial terms not given so far, by using the following.

Definition II.2.2. Let $f, g \in S$ be two polynomials and $\prec$ a term order. Denote by h the least common multiple of the initial monomials of f and g, *i.e.* the initial terms without coefficients. Then the *S-polynomial of f and g* is defined as

$$S\,(f, g) := \frac{h}{\text{in}_\prec(f)} f - \frac{h}{\text{in}_\prec(g)} g.$$

Basically, the Buchberger algorithm takes a generating set of the ideal, computes the S-polynomial for a pair of polynomials in the generating set, and reduces this S-polynomial by the generating set. If the reduction is not zero it is added to the generating set. This is repeated until no new polynomials are added. Then this set will be reduced by itself, *i.e.* if any term is divisible be the initial term of another polynomial, then the appropriate multiple of that other polynomial is subtracted from the first. If our ideal is a binomial ideal and we start with a set of binomial generators then only binomials occur in the computation, so that we get:

Remark II.2.3. Every reduced Gröbner basis of a binomial ideal consists only of binomials.

Example II.2.4 (continuing **II.1.6**)**.** Let $S = \Bbbk[a,b,c]$ with $\mathcal{A} = \{1,2,3\}$ as before, and the toric ideal $I_{\mathcal{A}} = \langle a^2 - b, ab - c\rangle$. If we take the lexicographic term order $\prec_1$ on S with the order $c \prec_1 b \prec_1 a$ then

$$\mathrm{in}_{\prec_1}(I_{\mathcal{A}}) = \left\langle a^2, ab, ac, b^3 \right\rangle$$

is the initial ideal with respect to this term order and

$$\mathcal{G}_1 = \left\{a^2 - b, ab - c, ac - b^2, b^3 - c^2\right\}$$

is the reduced Gröbner basis to this term order. If we take on the other hand the lexicographic order $\prec_2$ with $a \prec_2 b \prec_2 c$ then

$$\mathrm{in}_{\prec_2}(I_{\mathcal{A}}) = \langle b, c\rangle$$

is the initial ideal with respect to this term order and

$$\mathcal{G}_2 = \left\{b - a^2, c - ab\right\}$$

is a Gröbner basis. If we replace $c - ab$ by $c - a^3$ it is the reduced Gröbner basis. ◊

A slightly different approach on constructing initial ideals is by weight vectors. Let $\omega \in \mathbb{Q}^n$ be a *weight vector*, that means a monomial $\boldsymbol{x}^{\boldsymbol{d}} = x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ has weight $\omega \cdot \boldsymbol{d} \in \mathbb{Q}$. Then for $f = \sum f_i \boldsymbol{x}^{\boldsymbol{d}_i}$ the initial term with respect to ω, $\mathrm{in}_\omega(f)$, is defined as the sum of all $f_i \boldsymbol{x}^{\boldsymbol{d}_i}$ where $\omega \cdot \boldsymbol{d}_i$ is maximal for all terms in f and for an ideal I the initial ideal with respect to ω is defined as $\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(f) \mid f \in I\rangle$. This need not be a monomial ideal, but for generic ω it is.

Proposition II.2.5. *For any ideal $I \subset S$ and any term order $\prec$ on S there exists a weight vector $\omega \in \mathbb{N}^n$ such that $\mathrm{in}_\omega(I) = \mathrm{in}_\prec(I)$.*

Proof. See [Stu96, Proposition 1.11]. □

We say that $\omega \in \mathbb{Q}^n$ is *generic for I* if $\mathrm{in}_\omega(I) = \mathrm{in}_\prec(I)$ for some term order $\prec$.

Definition II.2.6. For each positive weight vector $\omega \in \mathbb{Q}^n_{\geq 0}$ the ideal $\mathrm{in}_\omega(I)$ is called a *Gröbner degeneration* of I.

Note that for generic weight vectors the Gröbner degenerations are the initial ideals of I. They are useful, because passing from an ideal to one of its initial ideals preserves the Hilbert function.

Lemma II.2.7. *Let $I \subset S$ be a homogeneous ideal and $\prec$ a term order. Then I and $\mathrm{in}_\prec(I)$ have the same multigraded Hilbert function.*

Proof. See [CLO07, 9.§3 Proposition 9]. □

Thus, take a Gröbner degeneration $\mathrm{in}_\omega(I)$ and any term order $\prec$. Then $\mathrm{in}_\prec(\mathrm{in}_\omega(I))$ is an initial ideal of $\mathrm{in}_\omega(I)$ and I (see [Stu96, Proposition 1.8]), and hence every Gröbner degeneration of I has the same multigraded Hilbert function as I.

A finite set is called a *universal Gröbner basis* of I if it is a Gröbner basis for every term order $\prec$. For the existence of a universal Gröbner basis we consider the following theorem:

Theorem II.2.8. *Let $I \subset S$ be an ideal. There are only finitely many distinct marked reduced Gröbner bases of I or equivalently there are only finitely many distinct initial ideals of I.*

Proof. See [Stu96, Theorem 1.2]. □

Hence, the union of all reduced Gröbner bases of I is finite and thus a universal Gröbner basis for I. We will refer to the union of all reduced Gröbner basis of an ideal I as *the* universal Gröbner basis $\mathrm{UGB}(I)$.

Example II.2.9 (continuing **II.1.6**)**.** For $\mathcal{A} = \{1, 2, 3\}$ we already had two initial ideals of $I_\mathcal{A}$:

$$\begin{array}{lcl} \mathcal{M}_1 = \mathrm{in}_{\prec_1}(I_\mathcal{A}) = & \langle a^2, ab, ac, b^3\rangle & \text{for } c \prec_1 b \prec_1 a \\ \mathcal{M}_2 = \mathrm{in}_{\prec_2}(I_\mathcal{A}) = & \langle b, c\rangle & \text{for } a \prec_2 b \prec_2 c \end{array}$$

There are four more different initial monomial ideals:

$$\begin{array}{lcl} \mathcal{M}_3 = \mathrm{in}_{\prec_3}(I_\mathcal{A}) = & \langle a^2, c\rangle & \text{for } b \prec_3 a \prec_3 c \\ \mathcal{M}_4 = \mathrm{in}_{\prec_4}(I_\mathcal{A}) = & \langle a^2, ab, ac, c^2\rangle & \text{for } b \prec_4 c \prec_4 a \\ \mathcal{M}_5 = \mathrm{in}_{\prec_5}(I_\mathcal{A}) = & \langle a^2, ab, b^2\rangle & \\ \mathcal{M}_6 = \mathrm{in}_{\prec_6}(I_\mathcal{A}) = & \langle b, a^3\rangle & \text{for } c \prec_6 a \prec_6 b, \end{array}$$

where $\prec_5$ first compares the total degree and if they are equal uses $c \prec a \prec b$. The six marked reduced Gröbner bases are

$$\begin{array}{lcl} \mathcal{G}_1 & = & \left\{\underline{a^2} - b, \underline{ab} - c, \underline{ac} - b^2, \underline{b^3} - c^2\right\}, \\ \mathcal{G}_2 & = & \left\{\underline{b} - a^2, \underline{c} - a^3\right\}, \\ \mathcal{G}_3 & = & \left\{\underline{a^2} - b, \underline{c} - ab\right\}, \\ \mathcal{G}_4 & = & \left\{\underline{a^2} - b, \underline{ab} - c, \underline{ac} - b^2, \underline{c^2} - b^3\right\}, \\ \mathcal{G}_5 & = & \left\{\underline{a^2} - b, \underline{ab} - c, \underline{b^2} - ac\right\}, \text{ and} \\ \mathcal{G}_6 & = & \left\{\underline{b} - a^2, \underline{a^3} - c\right\}, \end{array}$$

respectively.

◊

For a toric ideal $I_{\mathcal{A}}$ the universal Gröbner basis $\mathrm{UGB}(I_{\mathcal{A}})$ is contained in the Graver basis $\mathcal{G}(\mathcal{A})$, since by [Stu96, Lemma 4.6] every element of a reduced Gröbner basis is primitive. If we consider two different weight vectors $\omega_1, \omega_2 \in \mathbb{Q}^n$, then it can happen that $\mathrm{in}_{\omega_1}(I) = \mathrm{in}_{\omega_2}(I)$. This thus defines an equivalence relation on the weight vectors. For a positive $\omega \in \mathbb{Q}^n_{\geq 0}$ the *Gröbner cone* $\mathcal{K}_\omega(I)$ is the set of all $\omega' \in \mathbb{Q}^n$ such that $\mathrm{in}_{\omega'}(I) = \mathrm{in}_\omega(I)$. This means each Gröbner cone is the set of all weight vectors that give the same Gröbner degeneration. Since every positive weight vector gives a Gröbner degeneration the set of all Gröbner cones covers the positive orthant in $\mathbb{Q}^n$.

Lemma II.2.10. *Let $I \subset S$ be an ideal and $\omega \in \mathbb{Q}^n_{\geq 0}$ be a positive weight vector. Then $\overline{\mathcal{K}_\omega(I)} \subseteq \mathbb{Q}^n$ is a convex polyhedral cone. In particular, if ω is generic for I with Gröbner basis $\mathcal{G}$ then we have*

$$\mathcal{K}_\omega(I) = \{\omega' \in \mathbb{Q}^n \mid \mathrm{in}_{\omega'}(g) = \mathrm{in}_\omega(g) \text{ for all } g \in \mathcal{G}\}\,.$$

Proof. See [Stu96, Proposition 2.3]. □

Note that these cones do not necessarily cover $\mathbb{Q}^n$ since for $\omega' \notin \mathbb{Q}^n_{\geq 0}$ arbitrary there need not exist an $\omega \in \mathbb{Q}^n_{\geq 0}$ such that $\mathrm{in}_{\omega'}(I) = \mathrm{in}_\omega(I)$.

In the toric case the lemma gives the following description: Let ω be generic for $I_{\mathcal{A}}$ and denote by $\mathcal{G} = \{g_i = \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{x}^{\boldsymbol{n}_i} \mid i = 1, ..., k\}$ the corresponding marked reduced Gröbner basis where $\mathrm{in}_\omega(g_i) = \boldsymbol{x}^{\boldsymbol{m}_i}$. Thus, for every $\omega' \in \mathbb{Q}^n$ we have $\mathrm{in}_{\omega'}(g_i) = \mathrm{in}_\omega(g_i)$ if and only if $\omega' \cdot \boldsymbol{m}_i > \omega' \cdot \boldsymbol{n}_i$. Hence, we get the following corollary:

Corollary II.2.11. *Let ω be generic for a toric ideal $I_{\mathcal{A}}$ with corresponding marked reduced Gröbner basis $\mathcal{G} = \{g_i = \underline{\boldsymbol{x}^{\boldsymbol{m}_i}} - \boldsymbol{x}^{\boldsymbol{n}_i} \mid i = 1, ..., k\}$. Then the Gröbner cone is given by*

$$\mathcal{K}_\omega(I_{\mathcal{A}}) = \{\omega' \in \mathbb{Q}^n \,|\, \omega' \cdot (\boldsymbol{m}_i - \boldsymbol{n}_i) > 0 \text{ for } i = 1, ..., k\}\,.$$ □

Hence, if we let ω vary over all weight vectors we get a collection of convex polyhedral cones.

Definition II.2.12. The collection $\mathrm{GF}(I)$ of the convex polyhedral cones

$$\left\{\overline{\mathcal{K}_\omega(I)} \,|\, \omega \in \mathbb{N}^n\right\}$$

and all their faces is called the *Gröbner fan* of I and its support the *Gröbner region* $\mathrm{GR}(I)$.

In the Gröbner fan the cones of maximal dimension are the cones $\overline{\mathcal{K}_\omega(I)}$ of weight vectors ω which are generic for I, because the restriction for $\mathrm{in}_\omega(I)$ to be a monomial ideal is that ω is greater on the leading term than on every other term for every element of the corresponding Gröbner basis of I. This means that $\mathcal{K}_\omega(I)$ is the intersection of open half-spaces in $\mathbb{Q}^n$ which is n-dimensional if it is non-empty. On the other hand, if ω is not generic then there is a $g \in \mathrm{UGB}(I)$ with two terms of the same order, so $\overline{\mathcal{K}_\omega(I)}$ is contained in some hyperplane and hence has dimension lower than n. By the next proposition it is a face of one of the n-dimensional cones.

Proposition II.2.13. *The Gröbner fan* $\mathrm{GF}(I)$ *is a fan. The* n*-dimensional cones are in one-to-one correspondence to the initial monomial ideals of* I*.*

Proof. See [Stu96, Proposition 2.4]. □

Example II.2.14 (continuing **II.1.6**)**.** The toric ideal $I_{\mathcal{A}} = \langle a^2 - b, ab - c\rangle$ has six initial monomial ideals and $n = 3$. Hence, the Gröbner fan has six three-dimensional cones which by Corollary II.2.11 are given by the closures of

$$\begin{aligned}
\mathcal{K}_1(I_{\mathcal{A}}) &= \left\{\omega \in \mathbb{Q}^3 \;\middle|\; \left(\begin{smallmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{smallmatrix}\right) \cdot \omega > 0\right\}, \\
\mathcal{K}_2(I_{\mathcal{A}}) &= \left\{\omega \in \mathbb{Q}^3 \;\middle|\; \left(\begin{smallmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{smallmatrix}\right) \cdot \omega > 0\right\}, \\
\mathcal{K}_3(I_{\mathcal{A}}) &= \left\{\omega \in \mathbb{Q}^3 \;\middle|\; \left(\begin{smallmatrix} 2 & -1 & 0 \\ -1 & -1 & 1 \end{smallmatrix}\right) \cdot \omega > 0\right\}, \\
\mathcal{K}_4(I_{\mathcal{A}}) &= \left\{\omega \in \mathbb{Q}^3 \;\middle|\; \left(\begin{smallmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & -3 & 2 \end{smallmatrix}\right) \cdot \omega > 0\right\}, \\
\mathcal{K}_5(I_{\mathcal{A}}) &= \left\{\omega \in \mathbb{Q}^3 \;\middle|\; \left(\begin{smallmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{smallmatrix}\right) \cdot \omega > 0\right\}, \text{ and} \\
\mathcal{K}_6(I_{\mathcal{A}}) &= \left\{\omega \in \mathbb{Q}^3 \;\middle|\; \left(\begin{smallmatrix} -2 & 1 & 0 \\ 3 & 0 & -1 \end{smallmatrix}\right) \cdot \omega > 0\right\},
\end{aligned}$$

where the condition > 0 for a matrix means that each entry is strictly positive.

◊

Note that for a toric ideal $I_{\mathcal{A}}$ the lineality space of every cone in $\mathrm{GF}(I_{\mathcal{A}})$ is given by the row space of $\mathcal{A}$. In fact, ω and $-\omega$ are in a Gröbner cone if and only if $\mathrm{in}_\omega(g) = g$ for all $g \in \mathcal{G}(\mathcal{A})$, because $I_{\mathcal{A}}$ does not contain any monomials, so that there is no $g \in \mathcal{G}(\mathcal{A})$ with $\mathrm{in}_\omega(g) \neq g$. The span of the $\boldsymbol{m} - \boldsymbol{n}$ for all $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}(\mathcal{A})$ is exactly $\mathrm{Ker}(\mathcal{A})$, so that this is equivalent to $\omega \in (\mathrm{Ker}(\mathcal{A}))^\perp$.

Consider two generic, *i.e.* full dimensional, cones $\overline{\mathcal{K}_{\omega_1}(I_{\mathcal{A}})}, \overline{\mathcal{K}_{\omega_2}(I_{\mathcal{A}})}$ in the Gröbner fan $\mathrm{GF}(I_{\mathcal{A}})$ of a toric ideal $I_{\mathcal{A}}$, which intersect in a common face F of codimension-one. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the corresponding marked reduced Gröbner bases of $I_{\mathcal{A}}$. Then $\omega \in \mathbb{Q}^n$ is in $\mathcal{K}_{\omega_1}(I_{\mathcal{A}})$ if the marked Gröbner basis with respect to the term order ω is $\mathcal{G}_1$. This is the case if and only if $\mathrm{in}_\omega(g) = \mathrm{in}_{\omega_1}(g)$ for all $g \in \mathcal{G}_1$. Since all $g \in \mathcal{G}_1$ are binomials, we have $g = \boldsymbol{x}^{\boldsymbol{m}_g} - \boldsymbol{x}^{\boldsymbol{n}_g}$ with leading term $\boldsymbol{x}^{\boldsymbol{m}_g}$ with respect to ω_1. Thus, $\mathrm{in}_\omega(g) = \mathrm{in}_{\omega_1}(g)$ is equivalent to $\omega_1 \cdot \boldsymbol{m}_g > \omega_1 \cdot \boldsymbol{n}_g$. Since $\mathcal{G}_1 \neq \mathcal{G}_2$, there exists some $g_0 = \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{n}_0} \in \mathcal{G}_1$ such that $\omega_1 \cdot \boldsymbol{m}_0 > \omega_1 \cdot \boldsymbol{n}_0$ and $\omega_2 \cdot \boldsymbol{m}_0 \leq \omega_2 \cdot \boldsymbol{n}_0$. But if we take a weight ω in the interior of F then $\omega \cdot \boldsymbol{m}_0 = \omega \cdot \boldsymbol{n}_0$, because ω lies in the closure of $\overline{\mathcal{K}_{\omega_1}(I_{\mathcal{A}})}$, and we therefore have $\omega \in (\boldsymbol{m}_0 - \boldsymbol{n}_0)^\perp$. This leads to:

Lemma II.2.15. *Let* $\overline{\mathcal{K}_{\omega_1}(I_{\mathcal{A}})}, \overline{\mathcal{K}_{\omega_2}(I_{\mathcal{A}})}$ *be two generic cones in* $\mathrm{GF}(I_{\mathcal{A}})$ *with common face* F *of codimension-one. Then there exists a unique* $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ *in* $\mathrm{UGB}(I_{\mathcal{A}})$ *such that*

$$\overline{\mathcal{K}_{\omega_1}(I_{\mathcal{A}})} \cap (\boldsymbol{m} - \boldsymbol{n})^\perp = F = \overline{\mathcal{K}_{\omega_2}(I_{\mathcal{A}})} \cap (\boldsymbol{m} - \boldsymbol{n})^\perp .$$

Proof. Let ω_0 be in the relative interior of F. By the above, there exists some $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathrm{UGB}(I_{\mathcal{A}})$ such that $\omega_0 \in \overline{\mathcal{K}_{\omega_i}(I_{\mathcal{A}})} \cap (\boldsymbol{m} - \boldsymbol{n})^\perp$ for $i = 1, 2$, and in addition we have $\omega \cdot (\boldsymbol{m} - \boldsymbol{n}) \geq 0$ for all $\omega \in \overline{\mathcal{K}_{\omega_1}(I_{\mathcal{A}})}$ and $\omega \cdot (\boldsymbol{n} - \boldsymbol{m}) \geq 0$ for all $\omega \in \overline{\mathcal{K}_{\omega_2}(I_{\mathcal{A}})}$. But this means $\overline{\mathcal{K}_{\omega_i}(I_{\mathcal{A}})} \cap (\boldsymbol{m} - \boldsymbol{n})^\perp$ is a face of $\overline{\mathcal{K}_{\omega_i}(I_{\mathcal{A}})}$ for $i = 1, 2$ and since F is a facet of both cones and a point in its relative interior is contained in $\overline{\mathcal{K}_{\omega_i}(I_\mathcal{A})} \cap (\boldsymbol{m}-\boldsymbol{n})^\perp$, the desired equality holds. Thus, it remains to prove the uniqueness. Let $g' = \boldsymbol{x}^{\boldsymbol{m}'} - \boldsymbol{x}^{\boldsymbol{n}'}$ be a different element of $\mathrm{UGB}(I_\mathcal{A})$ such that $F = (\boldsymbol{m}'-\boldsymbol{n}')^\perp \cap \overline{\mathcal{K}_{\omega_i}(I_\mathcal{A})}$ for $i = 1,2$. Then $(\boldsymbol{m}'-\boldsymbol{n}') = \lambda(\boldsymbol{m}-\boldsymbol{n})$ for some $\lambda \neq 0 \in \mathbb{Q}$. But g and g' are primitive and hence $\lambda = 1$. □

The Gröbner fan $\mathrm{GF}(I)$ need not be complete for arbitrary I, but there is a nice condition that ensures completeness. Moreover, it implies that $\mathrm{GF}(I)$ is the normal fan of some polytope, which is an even more special property.

Definition II.2.16. Let $I \subseteq S = \Bbbk[x_1, ..., x_n]$ be a homogeneous ideal with Gröbner fan $\mathrm{GF}(I)$ in $\mathbb{Q}^n$. A polytope in $\mathbb{Q}^n$, that has the Gröbner fan $\mathrm{GF}(I)$ as its normal fan, is called a *state polytope* for I.

In the toric case, because the lineality space of $\mathrm{GF}(I_\mathcal{A})$ is the row space of $\mathcal{A}$, a state polytope for $I_\mathcal{A}$ has to be $(n-d)$-dimensional and its affine hull has to be a translation of $\mathrm{Ker}(\mathcal{A})$.

Theorem II.2.17. *Let $I \subset S$ be a homogeneous ideal with respect to a strictly positive $\mathbb{Z}$-grading. Then there exists a state polytope* $\mathrm{state}(I)$*.*

Proof. See [Stu96, Theorem 2.5]. □

Sturmfels' proof is by construction of a state polytope $\mathrm{state}(I)$. We will call this polytope *the* state polytope. However, for a toric ideal $I_\mathcal{A}$ we can see the completeness directly. Since we assumed $\mathrm{Ker}(\mathcal{A}) \cap \mathbb{N}^n = 0$ (see (II.1)), there exists a strictly positive weight vector ω_0 in the row span of $\mathcal{A}$. This means that $\mathrm{in}_{\omega_0}(g) = g$ for all $g \in \mathcal{G}(\mathcal{A})$. Thus, $\mathrm{in}_{\lambda\omega_0+\omega}(I_\mathcal{A}) = \mathrm{in}_\omega(I_\mathcal{A})$ for all $\omega \in \mathbb{Q}^n$ and therefore every weight vector is equivalent to some strictly positive weight vector, which means that the Gröbner fan covers $\mathbb{Q}^n$.

Corollary II.2.18. *For every collection $\mathcal{A}$ of lattice points whose matrix satisfies* $\mathrm{Ker}\,(\mathcal{A}) \cap \mathbb{N}^n = 0$ *there exists a state polytope for the toric ideal $I_\mathcal{A}$.*

Proof. Just note that the strictly positive weight vector ω_0 in the row span of $\mathcal{A}$ induces a strictly positive $\mathbb{Z}$-grading on $I_\mathcal{A}$. □

Example II.2.19 (continuing **II.1.6**). A state polytope for $I_\mathcal{A}$ is a hexagon since it must be two-dimensional with six vertices. We use [Stu96, Theorem 2.5] to compute the state polytope. Then the vertices of $\mathrm{state}(I_\mathcal{A})$ correspond to the initial monomial ideals in the following way:

$$\mathcal{M}_1 \leftrightarrow \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \quad \mathcal{M}_2 \leftrightarrow \begin{pmatrix} 21 \\ 0 \\ 0 \end{pmatrix} \quad \mathcal{M}_3 \leftrightarrow \begin{pmatrix} 3 \\ 9 \\ 0 \end{pmatrix}$$

$$\mathcal{M}_4 \leftrightarrow \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \quad \mathcal{M}_5 \leftrightarrow \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} \quad \mathcal{M}_6 \leftrightarrow \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix}$$

See Figure II.2.

◊

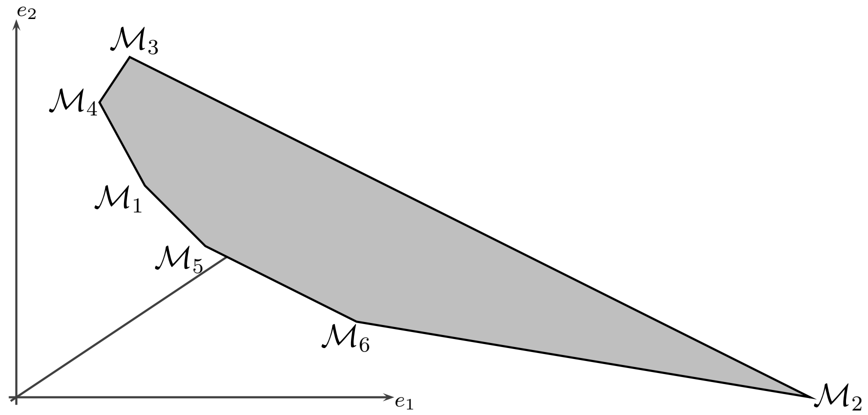

Figure II.2: A state polytope for $I_{\mathcal{A}}$

Remark. All computations with convex polyhedral objects have been done with *Polyhedra* [Bir09], a package for the computer algebra software *Macaulay2* [GS] written by Grayson and Stillman.

By definition, the faces of the state polytope are in one-to-one correspondence with the cones in the Gröbner fan. As the latter classify all Gröbner degenerations, we have the same correspondence between the Gröbner degenerations of I and the faces of state(I). Furthermore, the edges of the state polytope state$(I_{\mathcal{A}})$ of a toric ideal correspond to the codimension-one faces of the Gröbner cones of highest dimension. Thus, in combination with Lemma II.2.15 we get the following corollary.

Corollary II.2.20. *Every edge of the state polytope of a toric ideal $I_{\mathcal{A}}$ can be labeled by an element $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ of the universal Gröbner basis* UGB$(I_{\mathcal{A}})$ *such that the edge is parallel to $(\boldsymbol{m} - \boldsymbol{n})$.* □

Example II.2.21 (continuing **II.1.6**)**.** The edge directions in *state*$(I_{\mathcal{A}})$ are:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \leftrightarrow ac - b^2\ (\mathcal{M}_1 \text{ to } \mathcal{M}_5) \qquad \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \leftrightarrow a^2 - b\ (\mathcal{M}_5 \text{ to } \mathcal{M}_6)$$
$$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \leftrightarrow a^3 - c\ (\mathcal{M}_6 \text{ to } \mathcal{M}_2) \qquad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \leftrightarrow b - a^2\ (\mathcal{M}_2 \text{ to } \mathcal{M}_3)$$
$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \leftrightarrow c - ab\ (\mathcal{M}_3 \text{ to } \mathcal{M}_4) \qquad \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \leftrightarrow c^2 - b^3\ (\mathcal{M}_4 \text{ to } \mathcal{M}_1)$$

Hence, we can label the edges as in Figure II.3. ◊

For a toric ideal $I_{\mathcal{A}}$ there is a construction of a state polytope that simplifies the one in [Stu96, Theorem 2.5]. This construction uses certain degrees and monomials similar to those in Definition II.1.5, but this time for Gröbner bases instead of Graver bases.

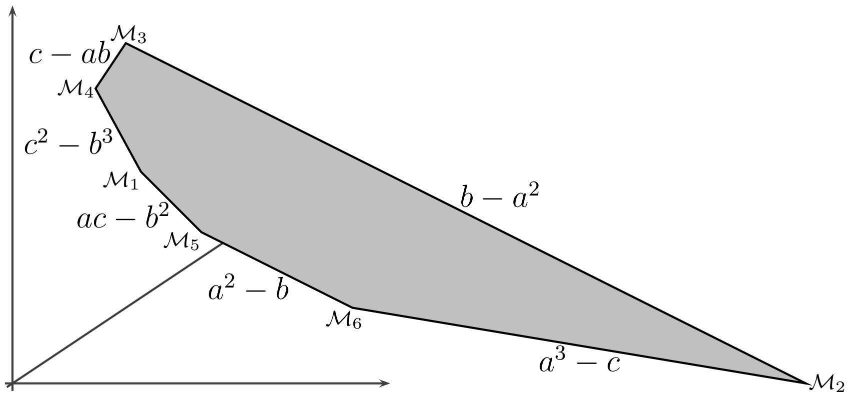

Figure II.3: The edges of $state(I_\mathcal{A})$ labeled by the elements of $\mathrm{UGB}(I_\mathcal{A})$

Definition II.2.22. We say an integral vector $\boldsymbol{a} \in \mathbb{N}\mathcal{A}$ is a *Gröbner degree* if there exists an element g of the universal Gröbner basis $\mathrm{UGB}(I_\mathcal{A})$ of degree $\boldsymbol{a}$. For such a Gröbner degree $\boldsymbol{a}$ the convex hull of $\mathcal{A}^{-1}(\boldsymbol{a}) \cap \mathbb{N}^n$ is called the *Gröbner fiber* of $\boldsymbol{a}$ and in general for an arbitrary degree $\boldsymbol{b} \in \mathbb{N}\mathcal{A}$ we call the convex hull of $\mathcal{A}^{-1}(\boldsymbol{b}) \cap \mathbb{N}^n$ the *fiber* of $\boldsymbol{b}$.

Note that equivalently the Gröbner fiber is the convex hull of the exponent vectors of all monomials in S of degree $\boldsymbol{a}$. These polytopes are used for the following construction.

Theorem II.2.23. *The Minkowski sum of all Gröbner fibers is a state polytope for $I_\mathcal{A}$.*

Proof. See [Stu96, Theorem 7.15]. □

Obviously, for $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathrm{UGB}(I_\mathcal{A})$ of Gröbner degree $\boldsymbol{a}$ the points $\boldsymbol{m}$ and $\boldsymbol{n}$ are lattice points in the same Gröbner fiber. On the other hand, if we take two arbitrary lattice points $\boldsymbol{m}$,$\boldsymbol{n}$ in a Gröbner fiber, we want to know when $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ is in the universal Gröbner basis. We have already seen that every element of a reduced Gröbner basis is primitive so that $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ has to be primitive. This means we have reduced to the case where $\boldsymbol{m}$ and $\boldsymbol{n}$ are lattice points in a fiber of a Graver degree and $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ is a Graver binomial.

Theorem II.2.24. *Let $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}(\mathcal{A})$ be a Graver binomial. Then g is in the universal Gröbner basis of $I_\mathcal{A}$ if and only if the line segment $[\boldsymbol{m}, \boldsymbol{n}]$ is an edge of the fiber of $\mathcal{A} \cdot \boldsymbol{m}$.*

Proof. This is basically [Stu96, Theorem 7.8]. We only have to note that the exponent vectors of the Graver binomials are the primitive vectorsl. □

As a consequence we get an improvement to Corollary II.2.20. The edge directions of a Minkowski sum of polytopes are the edge directions of the summands, hence the edge directions of the state polytope are all the edge directions of the Gröbner fibers (Theorem II.2.23). Thus, since for every element $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathrm{UGB}(I_\mathcal{A})$ the line segment $[\boldsymbol{m}, \boldsymbol{n}]$ is an edge of the corresponding Gröbner fiber (Theorem II.2.24), we have the following:

Corollary II.2.25. *The edges of the state polytope of $I_\mathcal{A}$ are each labeled by a unique element $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ of the universal Gröbner basis of $I_\mathcal{A}$ such that $(\boldsymbol{m} - \boldsymbol{n})$ is parallel to the edge and for every $g \in \mathrm{UGB}(I_\mathcal{A})$ there is at least one edge labeled by g.* □

Note that this is a slight refinement of Corollary 7.9 in [Stu96].

II.3 Varying $\mathcal{A}$-Graded Ideals

Now we have collected the tools to identify $\mathcal{A}$-graded ideals and recalled some facts about Gröbner degenerations, so we can focus on how to get new $\mathcal{A}$-graded ideals out of given ones.

If we fix some $\mathcal{A}$-graded ideal I and choose some weight vector $\omega \in \mathbb{N}^n$ we have the initial ideal $\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(f) \,|\, f \in I\rangle$. Then by Lemma II.2.7 $\mathrm{in}_\omega(I)$ has the same Hilbert function as I and is therefore also $\mathcal{A}$-graded.

On the other hand, we have assumed $\mathrm{Ker}(\mathcal{A}) \cap \mathbb{N}^n = 0$. This implies that there is a vector ω' in the row span of $\mathcal{A}$ which is strictly positive. Since $I_\mathcal{A}$ is homogeneous with respect to $\mathcal{A}$ it is homogeneous with respect to every vector in the row span of $\mathcal{A}$ so if we take the grading $\deg(x_i) = \omega'_i$, then $I_\mathcal{A}$ is homogeneous with respect to some strictly positive grading and by Theorem II.2.17 there is the state polytope $\mathrm{state}(I_\mathcal{A})$. Hence, the Gröbner fan of $I_\mathcal{A}$ covers all of $\mathbb{Q}^n$ and there is a strictly positive integer vector in the interior of each cone of the fan, in fact even in the common lineality space of all cones.

We denote the natural action of the torus $(\mathbb{k}^*)^n$ on S by $\lambda.x_i = \lambda_i \cdot x_i$ for $\lambda \in (\mathbb{k}^*)^n$. In other words the action corresponds to the grading given by the identity I_n on S. Then two $\mathcal{A}$-graded ideals I, I' are called *isomorphic as $\mathcal{A}$-graded ideals* if there exists a $\lambda \in (\mathbb{k}^*)^n$ such that $\lambda.I = I'$. This means the algebras S/I and S/I' are isomorphic as M-graded $\mathbb{k}$-algebras. Such isomorphisms lead to the notion of coherence:

Definition II.3.1. An $\mathcal{A}$-graded ideal is called *coherent* if it is isomorphic as an $\mathcal{A}$-graded ideal to some initial ideal $\mathrm{in}_\omega(I_\mathcal{A})$ of the toric ideal.

Lemma II.3.2. *Two different initial ideals of $I_\mathcal{A}$ are not isomorphic as $\mathcal{A}$-graded ideals.*

Proof. Denote the Graver basis by $\mathcal{G}(\mathcal{A}) = \{\boldsymbol{x}^{\boldsymbol{m}_1} - \boldsymbol{x}^{\boldsymbol{n}_1}, ..., \boldsymbol{x}^{\boldsymbol{m}_l} - \boldsymbol{x}^{\boldsymbol{n}_l}\}$ and the two different initial ideals of the toric ideal $I_\mathcal{A}$ by I_1 and I_2. Since I_1 is $\mathcal{A}$-graded it is generated in the Graver degrees so that without loss of generality we have

$$I_1 = \langle \boldsymbol{x}^{\boldsymbol{m}_1} - \boldsymbol{x}^{\boldsymbol{n}_1}, ..., \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j}, \boldsymbol{x}^{\boldsymbol{m}_{j+1}}, ..., \boldsymbol{x}^{\boldsymbol{m}_l}\rangle \,.$$

Assume that I_1 and I_2 are isomorphic. Then there exists some $\lambda \in (\mathbb{k}^*)^n$ such that $I_2 = \lambda.I_1$. But I_2 is also an initial ideal of the toric ideal so that it is generated by some of the $\boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{x}^{\boldsymbol{n}_i}$ and some of the monomials, too. Thus, I_2 contains only binomials with $+1$ and -1 as coefficients. Thus, we get

$$\lambda^{\boldsymbol{m}_i - \boldsymbol{n}_i} = 1 \quad \text{for } i = 1, ..., l$$

and hence that $I_2 = \lambda.I_1 = I_1$ which is a contradiction. □

If we pick a coherent $\mathcal{A}$-graded ideal I then there is some $\omega \in \mathbb{N}^n$ such that $I \cong \mathrm{in}_\omega(I_\mathcal{A})$. This weight vector ω is in the relative interior of a cone σ of the Gröbner fan and hence by definition $I \cong \mathrm{in}_{\omega'}(I_\mathcal{A})$ if and only if ω' is in the relative interior of σ. This means:

Lemma II.3.3. *The isomorphism classes as $\mathcal{A}$-graded ideals of coherent $\mathcal{A}$-graded ideals are in one-to-one correspondence with the cones of the Gröbner fan of $I_\mathcal{A}$.* □

Recall that if the weight vector ω is in the relative interior of an n-dimensional cone in the Gröbner fan, then $\mathrm{in}_\omega(I)$ is a monomial ideal. These n-dimensional cones correspond to the vertices of $\mathrm{state}(I)$ and thus we get the following equivalent correspondence:

Corollary II.3.4. *The isomorphism classes of coherent $\mathcal{A}$-graded ideals are in one-to-one correspondence with the faces of the state polytope of $I_\mathcal{A}$. In particular, monomial coherent $\mathcal{A}$-graded ideals correspond to vertices.* □

This leads to the notion of *adjacent* initial ideals of $I_\mathcal{A}$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two initial monomial ideals of $I_\mathcal{A}$ (or coherent monomial $\mathcal{A}$-graded ideals). Let v_1 and v_2 be the corresponding vertices of $\mathrm{state}(I_\mathcal{A})$. Assume that v_1 and v_2 are connected by an edge e of the state polytope. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be *adjacent* and there is some $\mathcal{A}$-graded ideal $I_{\mathcal{M}_1,\mathcal{M}_2}$ such that $\mathcal{M}_1$ and $\mathcal{M}_2$ are the only initial ideals of I. This ideal is $\mathrm{in}_\omega(I_\mathcal{A})$ for some ω in the interior of the cone in $\mathrm{GF}(I_\mathcal{A})$ that corresponds to the edge e. Let $g_0 = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ be the element of $\mathrm{UGB}(I_\mathcal{A})$ which labels e as in Corollary II.2.20 with $\boldsymbol{x}^{\boldsymbol{m}} \in \mathcal{M}_1, \boldsymbol{x}^{\boldsymbol{m}} \notin \mathcal{M}_2$ and $\boldsymbol{x}^{\boldsymbol{n}} \notin \mathcal{M}_1, \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{M}_2$.

Lemma II.3.5. *The ideal connecting $\mathcal{M}_1$ and $\mathcal{M}_2$ is given by*

$$\begin{aligned} I_{\mathcal{M}_1,\mathcal{M}_2} &= \mathrm{in}_\omega(I_\mathcal{A}) \\ &= \langle \boldsymbol{x}^{\boldsymbol{a}} \,|\, \boldsymbol{x}^{\boldsymbol{a}} \textit{ is a minimal generator of } \mathcal{M}, \boldsymbol{a} \neq \boldsymbol{m} \rangle + \langle \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \rangle \,. \end{aligned}$$

Proof. See [HT00, Theorem 3.6]. □

This means passing from one coherent monomial $\mathcal{A}$-graded ideal to another adjacent one involves "flipping" the term order on some $g_0 = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ in the corresponding two marked Gröbner bases of $I_\mathcal{A}$, whereas all other g in the two respective marked Gröbner bases keep their orientation. Note that the orientation of a $g \in \mathcal{G}(\mathcal{A})$ with both terms in the ideals might flip. The ideal $I = \mathrm{in}_\omega(I_\mathcal{A})$ is called the *wall ideal* for the flipping from $\mathcal{M}_1$ to $\mathcal{M}_2$ because it is the Gröbner degeneration of $I_\mathcal{A}$ by a weight vector ω in the interior of the common face of the cones in $\mathrm{GF}(I_\mathcal{A})$ corresponding to $\mathcal{M}_1$ and $\mathcal{M}_2$.

Example II.3.6 (continuing **II.1.6**). For $\mathcal{A} = \{1, 2, 3\}$ we had the two $\mathcal{A}$-graded ideals $\mathcal{M}_1 = \langle a^2, ab, ac, b^3\rangle$ and $\mathcal{M}_5 = \langle a^2, ab, b^2\rangle$. They are adjacent since they are connected by an edge of the state polytope (see Figure II.3). The ideal connecting $\mathcal{M}_1$ and $\mathcal{M}_5$ is

$$I_{\mathcal{M}_1,\mathcal{M}_5} = \left\langle ac - b^2, a^2, ab, b^3 \right\rangle .$$

◊

This "flipping" and adjacency have been extended to all monomial $\mathcal{A}$-graded ideals by Maclagan and Thomas ([MT02, Section 2]).

Definition II.3.7. Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal and $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ in $\mathcal{G}(\mathcal{A})$ with $\boldsymbol{x}^{\boldsymbol{m}}$ a minimal generator of $\mathcal{M}$ and $\boldsymbol{x}^{\boldsymbol{n}} \notin \mathcal{M}$. We define the *wall ideal* of g to be

$$\begin{aligned} W_{\boldsymbol{m}-\boldsymbol{n}} &:= \langle \boldsymbol{x}^{\boldsymbol{a}} \,|\, \boldsymbol{a} \neq \boldsymbol{m}, \boldsymbol{x}^{\boldsymbol{a}} \text{ is a minimal generator of } \mathcal{M} \rangle + \langle \boldsymbol{x}^{\boldsymbol{n}} - \boldsymbol{x}^{\boldsymbol{m}} \rangle \\ &= \left\langle \boldsymbol{x}^{\boldsymbol{a}} \,|\, \boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} \in \mathcal{G}(\mathcal{A}), \boldsymbol{x}^{\boldsymbol{a}} \in \mathcal{M}, \boldsymbol{x}^{\boldsymbol{b}} \notin \mathcal{M}, \boldsymbol{a} \neq \boldsymbol{m} \right\rangle + \langle \boldsymbol{x}^{\boldsymbol{n}} - \boldsymbol{x}^{\boldsymbol{m}} \rangle \end{aligned}$$

and the *flipping* of $\mathcal{M}$ along g to be

$$\mathcal{M}_{flip} := \left\langle \boldsymbol{x}^{\boldsymbol{a}} \,|\, \boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} \in \mathcal{G}(\mathcal{A}), \boldsymbol{x}^{\boldsymbol{a}} \in \mathcal{M}, \boldsymbol{x}^{\boldsymbol{b}} \notin \mathcal{M}, \boldsymbol{a} \neq \boldsymbol{m} \right\rangle + \langle \boldsymbol{x}^{\boldsymbol{n}} \rangle .$$

Note that both, the wall ideal and the flipping ideal, are M-homogeneous but not necessarily $\mathcal{A}$-graded.

One can construct the initial ideal of $W_{\boldsymbol{m}-\boldsymbol{n}}$ with respect to $\boldsymbol{x}^{\boldsymbol{m}} \prec \boldsymbol{x}^{\boldsymbol{n}}$, because only S-polynomials of $\boldsymbol{x}^{\boldsymbol{n}} - \boldsymbol{x}^{\boldsymbol{m}}$ with monomials have to be computed for the Gröbner basis. But these are multiples of $\boldsymbol{x}^{\boldsymbol{m}}$ and so the only binomial is $\boldsymbol{x}^{\boldsymbol{n}} - \boldsymbol{x}^{\boldsymbol{m}}$. Thus, for every element of $W_{\boldsymbol{m}-\boldsymbol{n}}$ there is a unique initial term and a Gröbner basis can be computed. Hence, the initial ideal of $W_{\boldsymbol{m}-\boldsymbol{n}}$ with respect to $\boldsymbol{x}^{\boldsymbol{m}} \prec \boldsymbol{x}^{\boldsymbol{n}}$ is well defined.

Lemma II.3.8. *The ideal $\mathcal{M}_{flip}$ is weakly $\mathcal{A}$-graded and is the initial ideal of $W_{\boldsymbol{m}-\boldsymbol{n}}$ with respect to $\boldsymbol{x}^{\boldsymbol{m}} \prec \boldsymbol{x}^{\boldsymbol{n}}$.*

Proof. See [MT02, Lemma 2.8 + 2.9]. □

If $\mathcal{M}$ is coherent and $g \in \mathrm{UGB}(I_{\mathcal{A}})$ corresponds to one of the codimension-one faces of the Gröbner cone given by $\mathcal{M}$ as in Lemma II.2.15, then $\mathcal{M}_{flip}$ and $W_{\boldsymbol{m}-\boldsymbol{n}}$ coincide with the ideals we had constructed before and are both again $\mathcal{A}$-graded. Since this is not the case for general monomial $\mathcal{A}$-graded ideals or for a g not corresponding to a codimension-one face, the notion of a flippable element of the Graver basis for a monomial ideal is defined due to [MT02] and we give their characterization of when this is the case:

Definition II.3.9. Let $\mathcal{M} \subset S$ be a monomial $\mathcal{A}$-graded ideal. A binomial $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}(\mathcal{A})$ is called *flippable* if $\boldsymbol{x}^{\boldsymbol{m}}$ is a minimal generator of $\mathcal{M}$, $\boldsymbol{x}^{\boldsymbol{n}} \notin \mathcal{M}$ and $\mathcal{M}_{flip}$ is again an $\mathcal{A}$-graded ideal.

Theorem II.3.10. *Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal and $g = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ be a Graver binomial. Then g is flippable for $\mathcal{M}$ if and only if $\mathcal{M}$ is the initial ideal of $W_{\boldsymbol{m}-\boldsymbol{n}}$ with respect to $\boldsymbol{x}^{\boldsymbol{n}} \prec \boldsymbol{x}^{\boldsymbol{m}}$.*

Proof. See [MT02, Theorem 2.11]. □

Example II.3.11 (continuing **II.1.6**)**.** For each of the six monomial $\mathcal{A}$-graded ideals there are precisely two elements in the Graver basis that are flippable. These two correspond to the binomials labeling the two edges emerging from the vertex of the state polytope corresponding to this monomial ideal (see again Figure II.3).

◊

Definition II.3.12. The *flip graph* of $\mathcal{A}$ has all monomial $\mathcal{A}$-graded ideals as vertices. There is an edge labeled by the Graver binomial g between two monomial $\mathcal{A}$-graded ideals $\mathcal{M}$ and $\mathcal{M}'$ if $\mathcal{M}'$ can be obtained from $\mathcal{M}$ by flipping along g. The set of all Graver binomials that appear as an edge in the flip graph is called $\mathrm{Flips}_{\mathcal{A}}$.

Because all coherent monomial $\mathcal{A}$-graded ideals are the vertices of the state polytope and we have seen that the edges of the state polytope correspond to flips along elements of $\mathrm{UGB}(I_{\mathcal{A}})$, the edge graph of the state polytope is a subgraph of the flip graph. Because of Corollary II.2.25 every element of the universal Gröbner basis of $I_{\mathcal{A}}$ appears at least on one edge of the state polytope and hence is flippable. Therefore, the universal Gröbner basis is contained in $\mathrm{Flips}_{\mathcal{A}}$.

Remark II.3.13. In general the universal Gröbner basis, the flips, and the Graver basis may be distinct sets, *i.e.* there exists an $\mathcal{A}$ such that

$$\mathrm{UGB}(I_{\mathcal{A}}) \subsetneq \mathrm{Flips}_{\mathcal{A}} \subsetneq \mathcal{G}(\mathcal{A}).$$

Maclagan and Thomas give various examples for combinations of proper subsets and equal sets in [MT02, Remark 2.14].

Example II.3.14 (continuing **II.1.6**)**.** For $\mathcal{A} = \{1, 2, 3\}$ the flip graph coincides with the edge graph of the state polytope of $I_{\mathcal{A}}$ and thus we have in fact

$$\mathrm{UGB}(I_{\mathcal{A}}) = \mathrm{Flips}_{\mathcal{A}} = \mathcal{G}(\mathcal{A}),$$

see Figure II.3. ♢

This allows one to investigate the combinatorics of $\mathcal{A}$-graded ideals by looking at a monomial $\mathcal{A}$-graded ideal and find the "neighbours" by determining the flips of that monomial ideal. The computer software TiGERS [HT00] written by Huber and Thomas enables one to start from a monomial $\mathcal{A}$-graded ideal and by computing all the flips to "walk" to all adjacent monomial $\mathcal{A}$-graded ideals. Hence, one can investigate the complete flip graph of the $\mathcal{A}$-graded ideals.

This walk along the flip graph enumerates all monomial $\mathcal{A}$-graded ideals if the flip graph is connected. Then one takes an initial monomial ideal $\mathcal{M}$ of $I_{\mathcal{A}}$ and can get from there to every monomial $\mathcal{A}$-graded ideal along $\mathrm{Flips}_{\mathcal{A}}$. But if the flip graph is not connected this is simply not possible. Thus, one needs to compute the set of all monomial $\mathcal{A}$-graded ideals first. This can for example be done with the following construction:

Construction II.3.15. Let $\mathcal{A}$ satisfy $\mathrm{Ker}(\mathcal{A}) \cap \mathbb{N}^n = 0$. We compute the toric ideal $I_{\mathcal{A}}$ and the Graver basis $\mathcal{G}(\mathcal{A})$. For each Graver degree $\boldsymbol{a}$ we select a standard monomial $s_{\boldsymbol{a}}$ and denote by $\mathcal{M}_{\boldsymbol{a}}$ the set of all other monomials of degree $\boldsymbol{a}$. Note that we do this degree by degree and thus do not choose an $s_{\boldsymbol{a}}$, that is already in the ideal generated by all the previous $\mathcal{M}_{\boldsymbol{a}'}$, because then the resulting ideal could not be $\mathcal{A}$-graded anyway. Then let $\mathcal{M}$ be the ideal generated by the $\mathcal{M}_{\boldsymbol{a}}$ for all Graver degrees. By Lemma II.1.8 with $\alpha_{uv} = 0$, we get all monomial $\mathcal{A}$-graded ideals and some monomial weakly $\mathcal{A}$-graded ideals as $\mathcal{M}$, if we do this for all combinations of standard monomials. Thus finally, we collect all $\mathcal{M}$'s, that have the same multigraded Hilbert function as $I_{\mathcal{A}}$. This can for example be checked with the computer algebra software Macaulay2 [GS].

Remark II.3.16. In [SST02, Section 1] the authors state the implementation of the above algorithm in Macaulay2 [GS]. This implementation is included in the package ToricHilbertSchemes [Bir10].

Furthermore, Santos has given an example in [San05] where the flip graph of $\mathcal{A}$ is disconnected. What this means for the $\mathcal{A}$-graded ideals will be shown in III.2.

Chapter III

Toric Hilbert Schemes

So far we have constructed or changed $\mathcal{A}$-graded ideals "by hand". But like the classical Hilbert scheme for the $\mathbb{Z}$-grading, which parametrises all subschemes of $\mathbb{P}^n$ with the same Hilbert polynomial, there is a scheme that parametrises all $\mathcal{A}$-graded ideals. This *toric Hilbert scheme* parametrises all subschemes of $\mathrm{Spec}(S)$ with the same multigraded Hilbert function as $S/I_{\mathcal{A}}$.

A parameter space for $\mathcal{A}$-graded ideals was first constructed by Sturmfels in [Stu94, Chapter 5]. This construction was then improved by Peeva and Stillman [PS02] who defined the toric Hilbert scheme. We will give the construction of the toric Hilbert scheme in a version of Maclagan and Thomas [MT02] and add some details of how this is the same as in [PS02].

III.1 Construction and Definition

Let $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A}) \subset \mathbb{N}\mathcal{A}$ be a primitive degree. We denote the number of elements in the fiber of $\boldsymbol{a}$ under the map $\mathcal{A}$ by $|\boldsymbol{a}|+1$ and the set of all such monomials by $\mathcal{G}m(\mathcal{A})_{\boldsymbol{a}}$, and assume that we have ordered them. Note that $\mathcal{G}m(\mathcal{A})_{\boldsymbol{a}}$ are exactly the monomials in the Graver fiber. The Graver basis of $\mathcal{A}$ is finite, so that we have $\mathcal{P}d(\mathcal{A}) = \{\boldsymbol{a}_1, ..., \boldsymbol{a}_l\}$. From this we define the following product of projective spaces over $\Bbbk$

$$\mathcal{P} := \mathbb{P}_{\Bbbk}^{|\boldsymbol{a}_1|} \times ... \times \mathbb{P}_{\Bbbk}^{|\boldsymbol{a}_l|}.$$

The toric Hilbert scheme will be a closed subscheme of $\mathcal{P}$ and the construction is motivated by Lemma II.1.8. For this we denote a homogeneous point in $\mathbb{P}_{\Bbbk}^{|\boldsymbol{a}_i|}$ by $\xi^{\boldsymbol{a}_i} = \left(\xi_0^{\boldsymbol{a}_i} : ... : \xi_{|\boldsymbol{a}_i|}^{\boldsymbol{a}_i}\right)$ where $\xi_j^{\boldsymbol{a}_i}$ corresponds to the j-th element $\boldsymbol{x}^{\boldsymbol{m}_j}$ in $\mathcal{G}m(\mathcal{A})_{\boldsymbol{a}}$. Equivalently, for some $\boldsymbol{m} \in \mathcal{G}m(\mathcal{A})_{\boldsymbol{a}}$ we also use the notation $\xi_{\boldsymbol{m}}^{\boldsymbol{a}}$ for the corresponding coordinate of $\xi^{\boldsymbol{a}}$. Hence, to every given point $\xi = (\xi^{\boldsymbol{a}_1}, ..., \xi^{\boldsymbol{a}_l}) \in \mathcal{P}$ we can associate an ideal $I_\xi \subset S$ by

$$I_\xi \;:=\; \sum_{i=1}^{l} \langle \xi_{\boldsymbol{n}}^{\boldsymbol{a}_i} \cdot \boldsymbol{x}^{\boldsymbol{m}} - \xi_{\boldsymbol{m}}^{\boldsymbol{a}_i} \cdot \boldsymbol{x}^{\boldsymbol{n}} \mid \boldsymbol{x}^{\boldsymbol{m}}, \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}m(\mathcal{A})_{\boldsymbol{a}_i} \rangle . \qquad \text{(III.1)}$$

Let $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ be a Graver binomial of degree $\boldsymbol{a}$. Then there exists some point $(\alpha : \beta) \in \mathbb{P}_{\Bbbk}^1$ such that $\alpha \boldsymbol{x}^{\boldsymbol{m}} - \beta \boldsymbol{x}^{\boldsymbol{n}} \in I_\xi$, except for the case when $\xi_{\boldsymbol{m}}^{\boldsymbol{a}} = \xi_{\boldsymbol{n}}^{\boldsymbol{a}} = 0$. But there has to be an $\boldsymbol{x}^{\boldsymbol{m}_0}$ in the Graver fiber of $\boldsymbol{a}$ such that $\xi_{\boldsymbol{m}_0}^{\boldsymbol{a}} \neq 0$, which implies in this case

$$\xi_{\boldsymbol{m}_0}^{\boldsymbol{a}} \cdot \boldsymbol{x}^{\boldsymbol{m}} = \xi_{\boldsymbol{m}_0}^{\boldsymbol{a}} \cdot \boldsymbol{x}^{\boldsymbol{m}} - \xi_{\boldsymbol{m}}^{\boldsymbol{a}} \cdot \boldsymbol{x}^{\boldsymbol{m}_0} \in I_\xi.$$

Thus, we have $1 \cdot \boldsymbol{x}^{\boldsymbol{m}} - 0 \cdot \boldsymbol{x}^{\boldsymbol{n}} \in I_\xi$ and because of Proposition II.1.7 I_ξ is weakly $\mathcal{A}$-graded. Note that $\boldsymbol{x}^{\boldsymbol{m}} \in I_\xi$ with $\deg(\boldsymbol{x}^{\boldsymbol{m}}) = \boldsymbol{a}$ if $\xi^{\boldsymbol{a}}_{\boldsymbol{m}} = 0$.

On the other hand, consider an $\mathcal{A}$-graded ideal I and fix a Graver degree $\boldsymbol{a}$. Since $\dim_{\Bbbk} (S/I)_{\boldsymbol{a}} = 1$, there is a monomial in $\mathcal{G}m(\mathcal{A})_{\boldsymbol{a}}$ that is not contained in I. Without loss of generality we may assume that it is $\boldsymbol{x}^{\boldsymbol{m}_0}$. It also follows from the one-dimensionality of every degree $\boldsymbol{a}$ part of S/I that for $i = 1, ..., |\boldsymbol{a}|$ there exist unique $\alpha_i \in \Bbbk$ such that $\alpha_i \cdot \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{m}_i} \in I$. Set

$$\xi^{\boldsymbol{a}} := \left(1 : \alpha_1 : ... : \alpha_{|\boldsymbol{a}|}\right).$$

If we do this for all $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A})$ we get a point $\xi(I) = (\xi^{\boldsymbol{a}_1}, ..., \xi^{\boldsymbol{a}_l}) \in \mathcal{P}$.

These two maps induce a bijection between the $\mathcal{A}$-graded ideals and a subset of $\mathcal{P}$. Indeed, let I_ξ be $\mathcal{A}$-graded. Then for every $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A})$ there is some i_0 with $\xi^{\boldsymbol{a}}_{i_0} \neq 0$ and we can write

$$\xi^{\boldsymbol{a}} = \left(\xi^{\boldsymbol{a}}_0/\xi^{\boldsymbol{a}}_{i_0} : ... : 1 : ... : \xi^{\boldsymbol{a}}_{|\boldsymbol{a}|}/\xi^{\boldsymbol{a}}_{i_0}\right).$$

This implies directly that each α_i in the construction above is exactly $\xi^{\boldsymbol{a}}_i/\xi^{\boldsymbol{a}}_{i_0}$ which means $\xi(I_\xi) = \xi$. On the other hand, take an $\mathcal{A}$-graded ideal I and fix some $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A})$. Then again we may assume that $\xi(I)^{\boldsymbol{a}} = \left(1 : \alpha_1 : ... : \alpha_{|\boldsymbol{a}|}\right)$. Therefore, we get $\xi^{\boldsymbol{a}}_0 \cdot \boldsymbol{x}^{\boldsymbol{m}_i} - \xi^{\boldsymbol{a}}_i \cdot \boldsymbol{x}^{\boldsymbol{m}_0} = -(\alpha_i \cdot \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{m}_i})$. Since

$$\begin{aligned} \xi^{\boldsymbol{a}}_j \cdot \boldsymbol{x}^{\boldsymbol{m}_i} - \xi^{\boldsymbol{a}}_i \cdot \boldsymbol{x}^{\boldsymbol{m}_j} &= \xi^{\boldsymbol{a}}_j(\xi^{\boldsymbol{a}}_i \cdot \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{m}_i}) - \xi^{\boldsymbol{a}}_i(\xi^{\boldsymbol{a}}_j \cdot \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{m}_j}) \\ &= \alpha_j(\alpha_i \cdot \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{m}_i}) - \alpha_i(\alpha_j \cdot \boldsymbol{x}^{\boldsymbol{m}_0} - \boldsymbol{x}^{\boldsymbol{m}_j}) \in I \end{aligned}$$

we have $I_{\xi(I)} \subset I$, but because $I_{\xi(I)}$ contains all generators of I we get $I_{\xi(I)} = I$ because of Lemma II.1.8. Note that the toric ideal $I_{\mathcal{A}}$ corresponds to the point in $\mathcal{P}$ with $\xi^{\boldsymbol{a}} = (1 : ... : 1)$ for all $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A})$.

To get the bijection between $\mathcal{A}$-graded ideals and a subset of $\mathcal{P}$ we have to give equations on $\mathcal{P}$ such that for exactly those points ξ fulfilling these equations I_ξ is an $\mathcal{A}$-graded ideal. The ideal I_ξ is weakly $\mathcal{A}$-graded so we fix a subset $\mathcal{R} \subset \mathbb{N}\mathcal{A}$ such that being $\mathcal{A}$-graded on $\mathcal{R}$ implies that the ideal is $\mathcal{A}$-graded. A good choice, for example, would be $\mathcal{V}$ from Theorem II.1.10. With this fixed $\mathcal{R}$ it follows that I_ξ is not $\mathcal{A}$-graded exactly if there exists some degree $\boldsymbol{a} \in \mathcal{R}$ such that I_ξ contains every monomial of degree $\boldsymbol{a}$. The homogeneous polynomials of degree $\boldsymbol{a}$ in I_ξ are linear combinations of binomials

$$\xi^{\boldsymbol{b}}_j \cdot \boldsymbol{x}^{\boldsymbol{m}_i} \cdot \boldsymbol{x}^{\gamma} - \xi^{\boldsymbol{b}}_i \cdot \boldsymbol{x}^{\boldsymbol{m}_j} \cdot \boldsymbol{x}^{\gamma}$$

where $\boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{x}^{\boldsymbol{m}_j}$ is a Graver binomial of degree $\boldsymbol{b}$ and the total degree of such a binomial is $\mathcal{A}(\boldsymbol{m}_i+\gamma) = \mathcal{A}(\boldsymbol{m}_j+\gamma) = \boldsymbol{a}$. This is equivalent to linear combinations of binomials

$$\xi^{\boldsymbol{b}}_j \cdot \boldsymbol{x}^{\boldsymbol{m}} - \xi^{\boldsymbol{b}}_i \cdot \boldsymbol{x}^{\boldsymbol{n}}$$

where $\boldsymbol{x}^{\boldsymbol{m}}$ and $\boldsymbol{x}^{\boldsymbol{n}}$ have degree $\boldsymbol{a}$ and $\boldsymbol{m} = \boldsymbol{n} - \boldsymbol{m}_j + \boldsymbol{m}_i$ for a Graver binomial $\boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{x}^{\boldsymbol{m}_j}$. Let $M_{\boldsymbol{a}}$ be a matrix whose $|\boldsymbol{a}|+1$ rows are labeled by the monomials of degree $\boldsymbol{a}$ and whose $n_{\boldsymbol{a}}$ columns are labeled by pairs of monomials $\boldsymbol{x}^{\boldsymbol{m}}, \boldsymbol{x}^{\boldsymbol{n}}$ of degree $\boldsymbol{a}$ such that there exists a Graver binomial $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{m}_i}$ of degree $\boldsymbol{b}$ with $\boldsymbol{m} = \boldsymbol{n} - \boldsymbol{m}_j + \boldsymbol{m}_i$, which is unique since it is primitive. The column of $M_{\boldsymbol{a}}$ corresponding to the pair $\boldsymbol{x}^{\boldsymbol{m}}, \boldsymbol{x}^{\boldsymbol{n}}$ consists of $\xi^{\boldsymbol{b}}_j$ in the $\boldsymbol{x}^{\boldsymbol{m}}$ row, $-\xi^{\boldsymbol{b}}_i$ in the $\boldsymbol{x}^{\boldsymbol{n}}$ row, and zero elsewhere.

Since homogeneous polynomials of degree $\boldsymbol{a}$ in S are in one-to-one correspondence with vectors in $\Bbbk^{|\boldsymbol{a}|+1}$, the homogeneous polynomials of degree $\boldsymbol{a}$ in I_ξ correspond to the image of the map $\sigma_{\boldsymbol{a}} : \Bbbk^{n_{\boldsymbol{a}}} \to \Bbbk^{|\boldsymbol{a}|+1}, x \mapsto M_{\boldsymbol{a}}x$. Thus, I_ξ is not $\mathcal{A}$-graded exactly if there is an $\boldsymbol{a} \in \mathcal{R}$ such that $\sigma_{\boldsymbol{a}}$ is surjective, which means there is a maximal minor of $M_{\boldsymbol{a}}$ that is not vanishing. Therefore, we get the following description due to Maclagan and Thomas [MT02].

Definition III.1.1. Choose an $\mathcal{R}$ as before and let $\mathcal{P}$ and $M_{\boldsymbol{a}}$, for all $\boldsymbol{a} \in \mathcal{R}$, be as before. Then we define the *toric Hilbert scheme* $\mathcal{H}_\mathcal{A} \subset \mathcal{P}$ to be the scheme given by the ideal generated by the maximal minors of $M_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in \mathcal{R}$. By using (III.1) the closed points in $\mathcal{H}_\mathcal{A}$ parametrise all $\mathcal{A}$-graded ideals.

Example III.1.2. Let again $\mathcal{A} = \{1, 2, 3\}$. Then we had the Graver basis

$$\mathcal{G}(\mathcal{A}) = \left\{a^2 - b, ab - c, a^3 - c, ac - b^2, b^3 - c^2\right\}$$

and the primitive degrees $\mathcal{P}d(\mathcal{A}) = \{2, 3, 4, 6\}$. Thus, we can divide the Graver monomials into

$$\mathcal{G}m(\mathcal{A})_2 = \left\{a^2, b\right\} \qquad \mathcal{G}m(\mathcal{A})_3 = \left\{a^3, ab, c\right\}$$
$$\mathcal{G}m(\mathcal{A})_4 = \left\{a^4, a^2b, ac, b^2\right\} \qquad \mathcal{G}m(\mathcal{A})_6 = \left\{a^6, a^4b, a^3c, a^2b^2, abc, b^3, c^2\right\}$$

Hence, in this example we get

$$\mathcal{P} := \mathbb{P}^1_\Bbbk \times \mathbb{P}^2_\Bbbk \times \mathbb{P}^3_\Bbbk \times \mathbb{P}^6_\Bbbk.$$

Exemplarily, we will compute the matrices $M_{\boldsymbol{a}}$ for the degrees 2 and 3, and give their maximal minors. So let the degree be 2. There are 2 monomials of degree 2 so that M_2 has two rows, the first labelled by a^2 and the second by b. Then there are two (ordered) pairs of monomials (a^2, b) and (b, a^2) such that there exists a Graver binomial which in this case are $a^2 - b$ and $b - a^2$ respectively satisfying the exponent condition because for example for the first pair $\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}2\\0\\0\end{pmatrix}$. Thus, in the first column we get ξ_b^2 in the a^2 row and $-\xi_{a^2}^2$ in the b row. The second column equals the first with permuted signs so that we get

$$M_2 = \begin{array}{c} (a^2,b)\ (b,a^2) \\ \begin{pmatrix} \xi_b^2 & -\xi_b^2 \\ -\xi_{a^2}^2 & \xi_{a^2}^2 \end{pmatrix} \end{array} \begin{array}{c} a^2 \\ b \end{array} ,$$

where we have noted down the labels of the rows and columns.

For degree 3 we have the three monomials a^3, ab, and c labelling the rows and there are six pairs of monomials $(a^3, c), (c, a^3), (ab, c), (c, ab), (a^3, ab), (ab, a^3)$ with the same exponent difference as the Graver basis element $a^3 - c$ for the first two, $ab - c$ for the second two, and $a^2 - b$ for the last two. Therefore, the matrix for degree 3 is

$$M_3 = \begin{array}{c} \scriptstyle (a^3,c)\quad (c,a^3)\quad (ab,c)\quad (c,ab)\quad (a^3,ab)\quad (ab,a^3) \\ \begin{pmatrix} \xi_c^3 & -\xi_c^3 & 0 & 0 & \xi_b^2 & -\xi_b^2 \\ 0 & 0 & \xi_c^3 & -\xi_c^3 & -\xi_{a^2}^2 & \xi_{a^2}^2 \\ -\xi_{a^3}^3 & \xi_{a^3}^3 & -\xi_{ab}^3 & \xi_{ab}^3 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} a^3 \\ ab \\ c \end{array} ,$$

where we again labelled the rows and columns. Then there is just one maximal minor of M_2 which is 0 and the ideal generated by the maximal minors of M_3 is

$$\left\langle \xi_b^2 \xi_{a^3}^3 \xi_c^3 - \xi_{a^2}^2 \xi_{ab}^2 \xi_c^3 \right\rangle . \qquad \Diamond$$

Remark. From now on we will use ξ and I_ξ or I and $I(\xi)$ interchangeably for $\xi \in \mathcal{H}_\mathcal{A}$ and $\mathcal{A}$-graded ideals I. So for example we write $I_\mathcal{A}$ for the toric ideal as well as for the point in $\mathcal{H}_\mathcal{A}$ that corresponds to $I_\mathcal{A}$.

This construction was evolved from the definition of Peeva and Stillman [PS02] in which they use the Fitting ideals (see [Eis95, Section 20.2]). Let $\mathcal{P}$ be as above and consider the subset $\mathcal{Y} \subset \mathcal{P} \times \mathbb{A}^n_\Bbbk$ given by

$$I(\mathcal{Y}) = \langle \xi^{\boldsymbol{a}}_{\boldsymbol{n}} \cdot \boldsymbol{x}^{\boldsymbol{m}} - \xi^{\boldsymbol{a}}_{\boldsymbol{m}} \cdot \boldsymbol{x}^{\boldsymbol{n}} \mid \forall \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}) \text{ and } \boldsymbol{x}^{\boldsymbol{m}}, \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}m(\mathcal{A})_{\boldsymbol{a}} \rangle$$

with projection $\phi : \mathcal{Y} \to \mathcal{P}$ and the grading on $\mathcal{O}_\mathcal{Y}$ induced by the grading of $\mathcal{A}$ on S. Hence, $\phi^\#$ is M-homogeneous and therefore we can write

$$\mathcal{Y} = \operatorname{Spec}_\mathcal{P} \left(\bigoplus_{\boldsymbol{a} \in \mathbb{N}\mathcal{A}} L_{\boldsymbol{a}} \right)$$

where $L_{\boldsymbol{a}}$ are coherent $\mathcal{O}_\mathcal{P}$-modules and $L_0 = \mathcal{O}_\mathcal{P}$ (see [PS02, Definition 3.1]). We define an ideal of $\mathcal{O}_\mathcal{P}$

$$\operatorname{dets}(\phi) = \sum_{\boldsymbol{a} \in \mathbb{N}\mathcal{A}} \operatorname{Fitt}_0(L_{\boldsymbol{a}}),$$

where $\operatorname{Fitt}_0(L_{\boldsymbol{a}})$ is the 0-th Fitting ideal of $L_{\boldsymbol{a}}$.

Before we explain what the 0-th Fitting ideal is in this case, we state the definition of the toric Hilbert scheme by Peeva and Stillman:

Definition III.1.3. The *toric Hilbert scheme* is defined as

$$\mathcal{H}_\mathcal{A} = V\left(\operatorname{dets}(\phi)\right) \subset \mathcal{P}.$$

For further details see [PS02, Section 3+4]. To see that both definitions are the same we have to examine the Fitting ideal for a fixed degree $\boldsymbol{a}$. For this we have to construct a free resolution of $\mathcal{O}_\mathcal{P}$-modules

$$F \overset{f_{\boldsymbol{a}}}{\to} G \to L_{\boldsymbol{a}} \to 0$$

and choose bases for F and G. Then $\operatorname{Fitt}_0(L_{\boldsymbol{a}})$ is the ideal generated by the maximal minors of the matrix representation of $f_{\boldsymbol{a}}$ (see [Eis95, Corollary-Definition 20.4]). This can be done by taking $G := \Bbbk^{|\boldsymbol{a}|+1}$ with $e_i \mapsto \boldsymbol{x}^{\boldsymbol{m}_i}$, the i-th element of the fiber of $\boldsymbol{a}$. Then the kernel of this map is given by the generators of $I(\mathcal{Y})$ and thus $M_{\boldsymbol{a}}$ is a matrix representation of $f_{\boldsymbol{a}}$. It is left to the reader to verify by set theoretic arguments that it suffices to take $\mathcal{R}$ instead of $\mathbb{N}\mathcal{A}$ as the index set for $\operatorname{dets}(\phi)$, which implies that the definitions are the same.

In Sturmfels' construction [Stu94, Chapter 5] the parameter space of $\mathcal{A}$-graded ideals is not given by determinantal equations. He considers the product space $\mathcal{P}' := \prod \mathbb{P}^{|\boldsymbol{a}|}$ for all $\boldsymbol{a} \in Z_r(\mathcal{A}) \cap \mathbb{N}\mathcal{A}$ where $Z_r(\mathcal{A})$ is the zonotope with edge length $r = (n-d)^{2^n} \cdot a^{d2^n}$. With the same notation for $\xi \in \mathcal{P}'$ as before he defined the closed subscheme $\mathcal{P}_\mathcal{A} \subset \mathcal{P}'$ by the equations

$$\xi^{\boldsymbol{a}}_{\boldsymbol{m}_1} \cdot \xi^{\boldsymbol{a}+\boldsymbol{b}}_{\boldsymbol{m}_2+\boldsymbol{n}} = \xi^{\boldsymbol{a}}_{\boldsymbol{m}_2} \cdot \xi^{\boldsymbol{a}+\boldsymbol{b}}_{\boldsymbol{m}_1+\boldsymbol{n}},$$

whenever $\deg(\boldsymbol{m}_1) = \deg(\boldsymbol{m}_2) = \boldsymbol{a}$ and $\deg(\boldsymbol{n}) = \boldsymbol{b}$. This is also a description of all $\mathcal{A}$-graded ideals.

Theorem III.1.4. *There exists a natural bijection between the set of $\mathcal{A}$-graded ideals in S and the set of closed points of $\mathcal{P}_\mathcal{A}$.*

Proof. See [Stu94, Theorem 5.3]. □

III.2 Components

In [HS04] Haiman and Sturmfels give general constructions of different multigraded Hilbert schemes. In particular, their work shows that the toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$ by Peeva and Stillman and the parameter space $\mathcal{P}_{\mathcal{A}}$ by Sturmfels are in fact the same, *i.e.* $\mathcal{H}_{\mathcal{A}} \cong \mathcal{P}_{\mathcal{A}}$. For this use [HS04, Propositions 5.2 + 5.3] in combination with [HS04, Theorem 3.16]. Thus, we get the following:

Lemma III.2.1. *The toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$ is given by binomial equations.* □

If we use the primary decomposition theorem from the work of Eisenbud and Sturmfels on binomial ideals [ES96, Theorem 7.1] it follows that every irreducible component of $\mathcal{H}_{\mathcal{A}}$ is generated by binomial ideals. Since the radical of a binomial ideal is again a binomial ideal (see [ES96, Theorem 3.1]) the reduced structure of each irreducible component, *i.e.* the variety given by the radicals of a covering of local rings, is given by binomial equations. This argument proves the following lemma:

Lemma III.2.2. *The underlying reduced structure of each component of the toric Hilbert scheme is a (not necessarily normal) projective toric variety.* □

Corollary III.2.3. *For each irreducible component V of the toric Hilbert scheme there is a polytope P_V such that the projective variety of the normal fan of P_V is the normalisation of V.* □

Note that we do not mean a distinguished polytope as we are not interested in the polarisation, just in the scheme structure.

We have seen in Chapter II that the faces of the state polytope are in bijection with the isomorphism classes of coherent $\mathcal{A}$-graded ideals. On the other hand, the closure of the orbit of $I_{\mathcal{A}}$ under the action of the n-torus $T = (\Bbbk^*)^n$ in $\mathcal{H}_{\mathcal{A}}$ is the set of coherent $\mathcal{A}$-graded ideals.

Theorem III.2.4. *There exists exactly one irreducible component containing $I_{\mathcal{A}}$. If* $\mathrm{char}(\Bbbk) = 0$*, then this component is reduced and the point $I_{\mathcal{A}}$ on $\mathcal{H}_{\mathcal{A}}$ is smooth.*

Proof. See [PS02, Theorem 5.3]. □

This means that all coherent $\mathcal{A}$-graded ideals are contained in this component. Therefore, we call the irreducible component containing $I_{\mathcal{A}}$ the *coherent component*. There is a refinement of the above and the theorem, which takes the correspondence between coherent $\mathcal{A}$-graded ideals and the Gröbner fan into account.

Theorem III.2.5. *The toric ideal $I_{\mathcal{A}}$ lies on a unique irreducible component of the toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$, the* coherent component*. The normalisation of the coherent component is the projective toric variety defined by the Gröbner fan of $I_{\mathcal{A}}$.*

Proof. See [SST02, Theorem 4.1]. □

Thus, the coherent component of the toric Hilbert scheme is already well described.

Example III.2.6. We continue Example II.1.6 from Chapter II, so that we have $\mathcal{A} = \{1, 2, 3\} \subset \mathbb{Z}^1$. We already computed that $\text{state}(I_\mathcal{A})$ is a hexagon (see Figure II.2). Thus, the normalisation of the coherent component of the toric Hilbert scheme $\mathcal{H}_\mathcal{A}$ is the toric variety associated to the normal fan of a hexagon. $\diamondsuit$

Lemma III.2.7. *Let $I \subset S$ be an $\mathcal{A}$-graded ideal which contains no monomials. Then I is torus isomorphic to the toric ideal $I_\mathcal{A}$.*

Proof. See [Stu94, Lemma 4.3]. □

Hence, for an $\mathcal{A}$-graded ideal which does not lie on the coherent component it follows that it contains monomials.

Now we will extend characteristics of the coherent component and the state polytope to non-coherent components and their polytopes. For this, let $(\mathcal{H}_\mathcal{A})_{\text{red}}$ be the underlying reduced scheme of $\mathcal{H}_\mathcal{A}$. By Lemma III.2.2 and Corollary III.2.3 each component of $(\mathcal{H}_\mathcal{A})_{\text{red}}$ is a projective toric variety and there is a polytope P_V for every component V of $(\mathcal{H}_\mathcal{A})_{\text{red}}$ such that the toric variety of P_V is the normalisation of V. Note that every component of $(\mathcal{H}_\mathcal{A})_{\text{red}}$ is the underlying reduced scheme of an irreducible component of $\mathcal{H}_\mathcal{A}$. The polytope for the coherent component is the state polytope of $I_\mathcal{A}$.

Definition III.2.8. We call the dense torus of an irreducible component V of $(\mathcal{H}_\mathcal{A})_{\text{red}}$ the *ambient torus* of V.

Note that the ambient torus of a non-coherent component is in general different from the n-torus $T = \text{Spec}(\Bbbk[\boldsymbol{x}^{\pm 1}])$ which is the ambient torus of the coherent component.

Now fix some component V of $(\mathcal{H}_\mathcal{A})_{\text{red}}$ with corresponding polytope P_V. Then there are similar results as for the state polytope of $I_\mathcal{A}$ in Chapter II.

Lemma III.2.9. *Vertices of P_V correspond exactly to the monomial $\mathcal{A}$-graded ideals in V.*

Proof. See [MT02, Lemma 3.4]. □

The edges of the state polytope correspond to flips of coherent monomial $\mathcal{A}$-graded ideals along elements of the universal Gröbner basis of $\mathcal{A}$. Almost the same holds also for non-coherent components.

Theorem III.2.10. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be monomial $\mathcal{A}$-graded ideals corresponding to vertices p_1 and p_2 of P_V. $\mathcal{M}_1$ and $\mathcal{M}_2$ are connected by a single flip if and only if there is an edge e of P_V connecting p_1 and p_2.*

Proof. See [MT02, Theorem 3.6]. □

We have seen that flips are done along elements of the Graver basis of $I_\mathcal{A}$, so we can label each edge of P_V by the Graver basis element of the corresponding wall ideal. The only difference now in the general case is that the edges of P_V are labeled by elements of $\mathcal{G}(\mathcal{A})$ but not necessarily of $\text{UGB}(\mathcal{A})$. This description of the toric Hilbert scheme in terms of polytopes and flips leads to the following:

Theorem III.2.11. *The toric Hilbert scheme $\mathcal{H}_\mathcal{A}$ is connected if and only if the flip graph of $\mathcal{A}$ is connected.*

Proof. See [MT02, Theorem 3.1]. ☐

Lemma III.2.12. *Any torus acting on the underlying reduced structure of an irreducible component of a Toric Hilbert Scheme $\mathcal{H}_{\mathcal{A}}$ acts diagonally by scaling each coordinate.*

Proof. The global equations of $\mathcal{H}_{\mathcal{A}}$ are a binomial ideal by Lemma III.2.1. From [ES96, Theorem 7.1] it follows that the associated primary ideals are also binomial ideals. Thus, the irreducile components are given by binomial equations. Using [ES96, Theorem 3.1] we get that also their radicals are binomials ideals. Hence, the underlying reduced irreducible structure of an irreducible component is given by binomial equations. Therefore, any torus must act diagonally. ☐

III.3 Local Equations

There are explicit global equations given in the construction of the toric Hilbert scheme by Peeva and Stillman. Unfortunately, $\mathcal{R} \subseteq \mathbb{N}\mathcal{A}$ may have to be chosen very large and the matrices $M_{\boldsymbol{a}}$ may get even larger, so we get a huge amount of maximal minors. Thus, these global equations are rather hard to compute. But Peeva and Stillman showed that the local equations of the toric Hilbert scheme around a monomial $\mathcal{A}$-graded ideal can be calculated efficiently.

Fix some monomial $\mathcal{A}$-graded ideal $\mathcal{M}$. Then for every degree $\boldsymbol{a} \in \mathbb{N}\mathcal{A}$ there is a unique monomial $s_{\boldsymbol{a}}$ which is not in $\mathcal{M}$. This is called the $\mathcal{M}$-*standard monomial* of degree $\boldsymbol{a}$.

Definition III.3.1. Let $\mathcal{U}_{\mathcal{M}} \subset \mathcal{H}_{\mathcal{A}}$ be the affine open subscheme

$$\mathcal{U}_{\mathcal{M}} := \mathcal{H}_{\mathcal{A}} \cap \left\{ \xi^{\boldsymbol{a}}_{s_{\boldsymbol{a}}} \neq 0 \,\middle|\, \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}) \right\}.$$

This means we have chosen an affine chart for every $\mathbb{P}^{|\boldsymbol{a}|}$ in $\mathcal{P}$ and intersected these with the toric Hilbert scheme. Since we have seen in the construction that $\boldsymbol{x}^{\boldsymbol{m}} \in \mathcal{M}$ if $\xi^{\boldsymbol{a}}_{\boldsymbol{m}} = 0$, it follows that $\mathcal{M}$ is contained in $\mathcal{U}_{\mathcal{M}}$.

Remark III.3.2. The affine open subscheme $\mathcal{U}_{\mathcal{M}}$ corresponds exactly to those $\mathcal{A}$-graded ideals with the same standard monomials as $S/\mathcal{M}$.

Now consider an arbitrary $\mathcal{A}$-graded ideal I. Take any initial monomial ideal $\mathcal{M}$ of I. Then the $\mathcal{M}$-standard monomial $s_{\boldsymbol{a}}$ of degree $\boldsymbol{a}$ of $\mathcal{M}$ forms a vector space basis of $(S/I)_{\boldsymbol{a}}$ for every $\boldsymbol{a} \in \mathbb{N}\mathcal{A}$. Since therefore $s_{\boldsymbol{a}}$ is not contained in I, which means $\xi^{\boldsymbol{a}}_{s_{\boldsymbol{a}}} \neq 0$ for all $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A})$, we have $I \in \mathcal{U}_{\mathcal{M}}$. Thus, we get the following:

Lemma III.3.3. *The set $\{\mathcal{U}_{\mathcal{M}}\}$ is an affine open cover for $\mathcal{H}_{\mathcal{A}}$, where $\mathcal{M}$ runs over all monomial $\mathcal{A}$-graded ideals.* ☐

For a fixed $\mathcal{M}$ with $\mathcal{M}$-standard monomials $s_{\boldsymbol{a}}$ we set

$$Z := \{ \xi^{\boldsymbol{a}}_{\boldsymbol{m}} \mid \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}), \deg(\boldsymbol{x}^{\boldsymbol{m}}) = \boldsymbol{a}, \boldsymbol{x}^{\boldsymbol{m}} \neq s_{\boldsymbol{a}} \}.$$

Now, in the original coordinates of $\mathcal{P}$, the affine chart $\mathcal{U}_{\mathcal{M}}$ is obtained by setting $\xi^{\boldsymbol{a}}_{s_{\boldsymbol{a}}} = 1$, since they are all not equal to 0. This means $\mathcal{U}_{\mathcal{M}}$ is given by an ideal in $\Bbbk[Z]$. This ideal is the restriction of $\operatorname{dets}(\phi)$ to $\Bbbk[Z]$. Alternatively, this ideal can be obtained by describing the points (*i.e.* $\mathcal{A}$-graded ideals) in $\mathcal{U}_{\mathcal{M}}$ in the following way: Because of Lemma II.1.4 and Lemma II.1.8 these are all generated by

$$\langle \boldsymbol{x}^{\boldsymbol{m}} - \alpha^{\boldsymbol{a}}_{\boldsymbol{m}} \cdot s_{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}), \deg(\boldsymbol{x}^{\boldsymbol{m}}) = \boldsymbol{a}, \boldsymbol{x}^{\boldsymbol{m}} \neq s_{\boldsymbol{a}} \rangle$$

for some $\alpha^{\boldsymbol{a}}_{\boldsymbol{m}} \in \Bbbk$. Thus, as in the construction of $\mathcal{H}_{\mathcal{A}}$ we get the following:

Lemma III.3.4. *Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal. For the affine open chart $\mathcal{U}_{\mathcal{M}} = \mathcal{H}_{\mathcal{A}} \cap \left\{ \xi^{\boldsymbol{a}}_{s_{\boldsymbol{a}}} \neq 0 \,\middle|\, \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}) \right\}$ and the corresponding set of variables $Z = \{ \xi^{\boldsymbol{a}}_{\boldsymbol{m}} \mid \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}), \deg(\boldsymbol{x}^{\boldsymbol{m}}) = \boldsymbol{a}, \boldsymbol{x}^{\boldsymbol{m}} \neq s_{\boldsymbol{a}} \}$ set*

$$\begin{aligned} G &:= \langle \boldsymbol{x}^{\boldsymbol{m}} - \xi^{\boldsymbol{a}}_{\boldsymbol{m}} \cdot s_{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}), \deg(\boldsymbol{x}^{\boldsymbol{m}}) = \boldsymbol{a}, \boldsymbol{x}^{\boldsymbol{m}} \neq s_{\boldsymbol{a}} \rangle \subseteq \Bbbk[Z] \otimes_{\Bbbk} S \quad \textit{and} \\ F &:= \textstyle\sum_{\boldsymbol{a} \in \mathbb{N}\mathcal{A}} \operatorname{Fitt}_0 \left((\Bbbk[Z][x_1, ..., x_n]/G)_{\boldsymbol{a}} \right) \subseteq \Bbbk[Z]. \end{aligned}$$

Then

$$\mathcal{U}_{\mathcal{M}} = \operatorname{Spec}\left(\Bbbk[Z]/F\right).$$

In particular, the ideal F is generated by the maximal minors of matrices of the form

$$\begin{pmatrix} 1 & & & 0 & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & 0 & & & & & \\ & & & 1 & & & \\ r^{\boldsymbol{a}}_0 & r^{\boldsymbol{a}}_1 & \cdots & r^{\boldsymbol{a}}_{|\boldsymbol{a}|-1} & r^{\boldsymbol{a}}_{|\boldsymbol{a}|} & \cdots & r^{\boldsymbol{a}}_t \end{pmatrix},$$

(where we assume that $s_{\boldsymbol{a}} = \boldsymbol{x}^{\boldsymbol{m}_{|\boldsymbol{a}|}}$). Therefore, we have

$$F = \sum_{\boldsymbol{a} \in \mathbb{N}\mathcal{A}} (G : s_{\boldsymbol{a}})$$

and thus F is a binomial ideal.

Proof. See [PS02, Corollary 4.5]. □

Note that the matrix in Lemma III.3.4 comes from the construction of Maclagan and Thomas. In this case there exist only pairs $(1, -\xi^{\boldsymbol{a}}_{\boldsymbol{m}})$ with $\boldsymbol{x}^{\boldsymbol{m}} \neq s_{\boldsymbol{a}}$ which have to be written into $M_{\boldsymbol{a}}$. If we convert this matrix into its reduced column echelon form we get on the one hand the desired form above and on the other hand that $r^{\boldsymbol{a}}_{|\boldsymbol{a}|}, ..., r^{\boldsymbol{a}}_t$ are the generators of $(G : s_{\boldsymbol{a}})$.

Lemma III.3.5. *Let $V \subseteq \mathcal{H}_{\mathcal{A}}$ be an irreducible component containing $\mathcal{M}$. Then $(\mathcal{U}_{\mathcal{M}} \cap V)_{red}$ contains the ambient torus of V_{red}.*

Proof. We may assume that V is already reduced. Let ξ be a point on the ambient torus of V. Then the corresponding $\mathcal{A}$-graded ideal is

$$I_\xi = \langle \xi^{\boldsymbol{a}}_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{m}} - \xi^{\boldsymbol{a}}_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{n}} \mid \boldsymbol{x}^{\boldsymbol{m}}, \boldsymbol{x}^{\boldsymbol{n}} \in \mathcal{G}m(\mathcal{A})_{\boldsymbol{a}}, \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}) \rangle.$$

By Lemma III.2.12 the ambient torus acts diagonally on the coefficients $\xi^{\boldsymbol{a}}_{\boldsymbol{n}}, \xi^{\boldsymbol{a}}_{\boldsymbol{m}}$ by the construction of the toric Hilbert scheme. Hence, if $\boldsymbol{x}^{\boldsymbol{m}_0} \in I_\xi$ then $\boldsymbol{x}^{\boldsymbol{m}_0} \in I$ holds for every I in the ambient torus orbit of I_ξ. Moreover, $\boldsymbol{x}^{\boldsymbol{m}_0} \in I_\xi$ implies $\boldsymbol{x}^{\boldsymbol{m}_0} \in I'$ for every I' in the closure of the ambient torus orbit.

On the other hand, since $\mathcal{M}$ lies on V, *i.e.* the closure of the ambient torus, $\boldsymbol{x}^{\boldsymbol{m}_0} \notin \mathcal{M}$ implies $\boldsymbol{x}^{\boldsymbol{m}_0} \notin I$ for every I in the ambient torus. Thus, the monomials of $S/\mathcal{M}$ are standard monomials for all ideals in the ambient torus of V. Hence by Remark III.3.2 the ambient torus lies in $\mathcal{U}_{\mathcal{M}}$. □

Now suppose that $\mathcal{M}$ is a coherent monomial $\mathcal{A}$-graded ideal. We will give an efficient description of $\Bbbk[Z]/F$ constructed by Peeva and Stillman.

Construction III.3.6 (Local coherent equations). The ideal $\mathcal{M}$ has a unique minimal set of monomial generators. We call this set $G_{\mathcal{M}} = \{f_i \,|\, 1 \leq i \leq p_{\mathcal{M}}\}$ where $p_{\mathcal{M}}$ is the number of generators of $\mathcal{M}$. Then for every f_i there is an $\mathcal{M}$-standard monomial of degree $\deg(f_i)$ which we call s_i. Note that $f_i - s_i$ is primitive because if it were not, f_i would not be a minimal generator of $\mathcal{M}$. Consider the ring $\Bbbk[x_1, ..., x_n, y_1, ..., y_{p_{\mathcal{M}}}]$ and the ideal $J_{\mathcal{M}}$ generated by the set

$$\overline{G_{\mathcal{M}}} = \{\, f_i - y_i \cdot s_i \mid 1 \leq i \leq p_{\mathcal{M}}\}\,.$$

We fix a term order $\prec_{\boldsymbol{x}}$ on S such that $\mathcal{M} = \mathrm{in}_{\prec_{\boldsymbol{x}}}(I_{\mathcal{A}})$ and an arbitrary term order $\prec_{\boldsymbol{y}}$ on $\Bbbk[y_1, ..., y_{p_{\mathcal{M}}}]$. Denote by $\prec$ the product term order of $\prec_{\boldsymbol{x}}$ and of $\prec_{\boldsymbol{y}}$ on $\Bbbk[x_1, ..., x_n, y_1, ..., y_{p_{\mathcal{M}}}]$, which means

$$\boldsymbol{x}^{\boldsymbol{a}} \cdot \boldsymbol{y}^{\boldsymbol{b}} \succ \boldsymbol{x}^{\boldsymbol{a}'} \cdot \boldsymbol{y}^{\boldsymbol{b}'} \Leftrightarrow \boldsymbol{x}^{\boldsymbol{a}} \succ_{\boldsymbol{x}} \boldsymbol{x}^{\boldsymbol{a}'} \text{ or } \boldsymbol{x}^{\boldsymbol{a}} = \boldsymbol{x}^{\boldsymbol{a}'}, \boldsymbol{y}^{\boldsymbol{b}} \succ_{\boldsymbol{y}} \boldsymbol{y}^{\boldsymbol{b}'}.$$

For each pair of binomials u and v in $\overline{G_{\mathcal{M}}}$ we form their S-polynomial $s(u,v)$ which is a homogeneous binomial with respect to the M-grading induced by $\mathcal{A}$ on $\Bbbk[x_1, ..., x_n]$. Then we choose a reduction of $s(u,v)$ by $\overline{G_{\mathcal{M}}}$ to $(e(\boldsymbol{y}) - h(\boldsymbol{y})) \cdot s_{u,v}$, where e and h are monomials in $\Bbbk[y_1, ..., y_{p_{\mathcal{M}}}]$. Since $\prec_{\boldsymbol{x}}$ is a term order with $\mathcal{M} = \mathrm{in}_{\prec_{\boldsymbol{x}}}(I_{\mathcal{A}})$, $s_{u,v}$ is the $\mathcal{M}$-standard monomial in the degree of $s(u,v)$. Set $r(u,v) := e(\boldsymbol{y}) - h(\boldsymbol{y}) \in \Bbbk[y_1, ..., y_{p_{\mathcal{M}}}]$ and define

$$I_{\mathcal{M}} := \left\langle r(u,v) \;\middle|\; u, v \in \overline{G_{\mathcal{M}}} \right\rangle \subset \Bbbk[y_1, ..., y_{p_{\mathcal{M}}}].$$

Note that the construction of $I_{\mathcal{M}}$ does not use the Graver basis, but needs the term order $\prec_{\boldsymbol{x}}$ for the reduction.

Theorem III.3.7. *Let $\mathcal{M}$ be a coherent monomial $\mathcal{A}$-graded ideal. Then*

$$\mathcal{U}_{\mathcal{M}} = \mathrm{Spec}\,(\Bbbk[Z]/F) \cong \mathrm{Spec}\,(\Bbbk[y_1, ..., y_{p_{\mathcal{M}}}]/I_{\mathcal{M}})\,.$$

Proof. See [PS00, Theorem 3.2]. □

Remark III.3.8. From now on most of the computations in the examples have been carried out by using MACAULAY2, a computer algebra software by Grayson and Stillman [GS], and the package TORICHILBERTSCHEMES [Bir10] containing algorithms from [SST02] and self-written code.

Example III.3.9. Let $\mathcal{A} = \{1, 3, 4, 7\} \subset \mathbb{Z}^1$ and $S = \Bbbk[a, b, c, d]$. Then the toric ideal is $I_{\mathcal{A}} = \langle a^3 - b, ab - c, bc - d\rangle$ and the Graver basis has 27 elements. There are 53 monomial $\mathcal{A}$-graded ideals of which 2 are non-coherent. We pick one of the coherent ones and apply Construction III.3.6 to it:

$$\mathcal{M} = \left\langle a^3, ab, b^2, bc, ad, a^2c^2, bd^2, ac^5, d^4 \right\rangle$$

This is the initial monomial ideal of the toric ideal $I_{\mathcal{A}}$ with respect to the weight vector $\omega = (265, 342, 1, 40)$. The standard monomials in the degrees of the generators are

$$\{s_1, ..., s_9\} = \left\{b, c, a^2c, d, c^2, bd, ac^4, d^3, c^7\right\}$$

respectively. Hence, the ideal $J_{\mathcal{M}}$ is generated by

$$\begin{aligned}\overline{G_{\mathcal{M}}} &= \{a^3 - y_1 b, ab - y_2 c, b^2 - y_3 a^2 c, bc - y_4 d, ad - y_5 c^2,\\ &\quad a^2c^2 - y_6 bd, bd^2 - y_7 ac^4, ac^5 - y_8 d^3, d^4 - y_9 c^7\}.\end{aligned}$$

Our term order $\prec$ is given by $\prec_\omega$ on $\mathbb{k}[a,b,c,d]$ and arbitrary on $\mathbb{k}[y_1,...,y_9]$. If we form S-polynomials and reduce them using this term order we get

$$s(a^3 - y_1 b, ab - y_2 c) = y_2 a^2 c - y_1 b^2.$$

Here $y_1 b^2$ is the leading term and thus $b^2 - y_3 a^2 c$ reduces the S-polynomial to $y_2 a^2 c - y_1 y_3 a^2 c = (y_2 - y_1 y_3) a^2 c$, so that we get

$$r(a^3 - y_1 b, ab - y_2 c) = (y_2 - y_1 y_3).$$

We continue with the remaining pairs but omit pairs where the leading terms are coprime since they give no equations. The leading terms are underlined:

$$\begin{aligned}
s(a^3 - y_1 b, ad - y_5 c^2) &= \underline{y_5 a^2 c^2} - y_1 bd \rightsquigarrow y_5 y_6 bd - y_1 bd = (y_5 y_6 - y_1) bd,\\
s(a^3 - y_1 b, ac^5 - y_8 d^3) &= \underline{y_8 a^2 d^3} - y_1 bc^5 \rightsquigarrow \underline{y_5 y_8 ac^2 d^2} - y_1 bc^5\\
&\rightsquigarrow y_5^2 y_8 c^4 d - \underline{y_1 bc^5}\\
&\rightsquigarrow y_5^2 y_8 c^4 d - y_1 y_4 c^4 d = (y_5^2 y_8 - y_1 y_4) c^4 d,\\
s(ab - y_2 c, bc - y_4 d) &= \underline{y_4 ad} - y_2 c^2 \rightsquigarrow y_4 y_5 c^2 - y_2 c^2 = (y_4 y_5 - y_2) c^2\\
&\vdots \qquad .
\end{aligned}$$

Continuing with the remaining pairs in the same way gives the local equations of the toric Hilbert scheme around $\mathcal{M}$ as

$$\begin{aligned}I_{\mathcal{M}} &= \langle y_2 - y_1 y_3, y_1 - y_5 y_6, y_1 y_4 - y_5^2 y_8, y_2 - y_4 y_5, y_5 - y_8 y_9, y_4 - y_7 y_8,\\ &\quad y_5 - y_6 y_7, y_4 y_9 - y_5 y_7, y_4 y_6 - y_5 y_8, y_3 y_6 - y_4, y_3 y_5^2 - y_2 y_7\}.\end{aligned}$$ ◊

We state one of the lemmata used by Peeva and Stillman for the proof since it shows how this construction and that of Lemma III.3.4 are connected.

Lemma III.3.10. *In* $\mathbb{k}[y_1,...,y_{p_{\mathcal{M}}}]$ *we have*

$$\begin{aligned}I_{\mathcal{M}} &= \sum_{\boldsymbol{a}\in\mathbb{N}\mathcal{A}} ((J_{\mathcal{M}})_{\boldsymbol{a}} : s_{\boldsymbol{a}})\\ &= \sum_{\boldsymbol{a}\in\mathbb{N}\mathcal{A}} \mathrm{Fitt}_0\left((\mathbb{k}[y_1,...,y_{p_{\mathcal{M}}}][x_1,...,x_n]/J_{\mathcal{M}})_{\boldsymbol{a}}\right).\end{aligned}$$

Proof. See [PS00, Lemma 3.5]. □

The binomials $f_i - y_i \cdot s_i$ in $\overline{G_{\mathcal{M}}}$ can also be written as $\boldsymbol{x}^{\boldsymbol{m}} - \xi^{\boldsymbol{a}}_{\boldsymbol{m}} \cdot s_{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \mathcal{P}d(\mathcal{A})$ and some monomial $\boldsymbol{x}^{\boldsymbol{m}}$ of degree $\boldsymbol{a}$. Thus, the variables $y_1,...,y_{p_{\mathcal{M}}}$ correspond to certain variables $\xi^{\boldsymbol{a}}_j \in Z$. We denote this set of variables by $Z_{small} \subset Z$. Hence, we can interpret $I_{\mathcal{M}}$ also as an ideal in $\mathbb{k}[Z_{small}]$.

In the non-coherent case the description of $\mathcal{U}_{\mathcal{M}}$ is not so easy because we do not have a term order for which $\mathcal{M}$ is the initial ideal of $I_{\mathcal{A}}$. We can still compute $\mathbb{k}[Z]/F$ using Lemma III.3.4, but Construction III.3.6, which would be much more efficient cannot be used, since the proof uses the term order $\prec_{\boldsymbol{x}}$ and the reduction with respect to it. Note that, although we do not have a term order $\prec_{\boldsymbol{x}}$, we can still construct $I_{\mathcal{M}}$ as defined in Lemma III.3.10. However, Theorem III.3.7 does not hold in the non-coherent case. But Peeva and Stillman enhanced this construction to the non-coherent case. They calculated the local ring of $\mathcal{H}_{\mathcal{A}}$ in the point $\mathcal{M}$.

Theorem III.3.11. *The local ring of $\mathcal{H}_{\mathcal{A}}$ at $\mathcal{M}$ is*

$$\mathcal{O}_{\mathcal{H}_{\mathcal{A}},[\mathcal{M}]} \cong \Bbbk[Z]_{\langle Z\rangle}/F \cong \Bbbk[Z_{small}]_{\langle Z_{small}\rangle}/I_{\mathcal{M}} = \Bbbk[y_1, ..., y_{p_{\mathcal{M}}}]_{\langle y_1,...,y_{p_{\mathcal{M}}}\rangle}/I_{\mathcal{M}}.$$

Proof. See [PS00, Theorem 4.4]. □

There is even a similar construction to Construction III.3.6 that uses F. Mora's tangent cone algorithm (see [Mor82]) in a simplified way instead of the Gröbner reduction by $\overline{G_{\mathcal{M}}}$. We end this chapter by giving the construction of $I_{\mathcal{M}}$ for a non-coherent monomial $\mathcal{A}$-graded ideal $\mathcal{M}$.

Construction III.3.12 (Local non-coherent equations). The first steps are exactly as in Construction III.3.6. There is again a minimal set of monomials $G_{\mathcal{M}} := \{f_i \mid 1 \le i \le p_{\mathcal{M}}\}$ generating $\mathcal{M}$, where $p_{\mathcal{M}}$ is the number of generators of $\mathcal{M}$. Let s_i be the $\mathcal{M}$-standard monomial of degree $\deg(f_i)$. Still $f_i - s_i$ is primitive by the same argument as before. Consider the ring $\Bbbk[x_1, ..., x_n, y_1, ..., y_{p_{\mathcal{M}}}]$ and the ideal $J_{\mathcal{M}}$ generated by the set

$$\overline{G_{\mathcal{M}}} = \{\, f_i - y_i \cdot s_i \mid 1 \le i \le p_{\mathcal{M}}\}\,.$$

Now we can not fix a term order as before. We have to use the second reduction process from Peeva and Stillman in [PS00].

Fix an order $\prec$ on the monomials of $\Bbbk[x_1, ..., x_n, y_1, ..., y_{p_{\mathcal{M}}}]$ with $y_i \prec 1 \prec x_j$ for all i, j. Note that this is not a term order since 1 is not the minimal element. Then f_i is the initial term of each element in $\overline{G_{\mathcal{M}}}$. Let m be a monomial in $\Bbbk[x_1, ..., x_n, y_i, ..., y_{p_{\mathcal{M}}}]$. Then the *remainder* $R(m, \overline{G_{\mathcal{M}}})$ is constructed as follows. If m is not divisible by any of the monomials f_i then $R(m, \overline{G_{\mathcal{M}}}) = m$. Otherwise, $m = f_i \cdot u$ for some i and monomial u. Then we reduce m to $m_1 := u \cdot y_i \cdot s_i$. We repeat this reduction until either we get at some point an m_p that is not further reducible by that method, in which case we set $R(m, \overline{G_{\mathcal{M}}}) = m_p$, or we obtain a loop

$$m \to m_1 \to m_2 \to ... \to m_i \to ... \to m_j \to ...$$

where m_i divides m_j. Then we set $R(m, \overline{G_{\mathcal{M}}}) = 0$. This reduction is extended to polynomials by linearity. Note that the remainder of any monomial is either 0 or $\boldsymbol{y}^{\boldsymbol{e}} \cdot s_a$ for some standard monomial s_a and $\boldsymbol{e} \in \mathbb{N}^{p_{\mathcal{M}}}$.

For each pair of binomials u and v in $\overline{G_{\mathcal{M}}}$ form their S-polynomial $s(u, v)$ and set

$$r(u, v) := R(s(u, v), \overline{G_{\mathcal{M}}})/s_{u,v},$$

where $s_{u,v}$ is the standard monomial in the degree of $s(u, v)$. Then

$$I_{\mathcal{M}} := \left\langle r(u, v) \;\middle|\; u, v \in \overline{G_{\mathcal{M}}} \right\rangle \subset \Bbbk[y_1, ..., y_{p_{\mathcal{M}}}].$$

Example III.3.13 (continuing **III.3.9**). Now we consider a non-coherent monomial $\mathcal{A}$-graded ideal for $\mathcal{A} = \{1, 3, 4, 7\}$:

$$\mathcal{M} = \langle a^3, ab, b^2, bc, ad, a^2c^2, ac^4, bd^3, d^4 \rangle$$

The standard monomials in the degrees of the generators are

$$\{s_1, ..., s_9\} = \{b, c, a^2c, d, c^2, bd, bd^2, c^6, c^7\},$$

respectively. Hence, the ideal $J_{\mathcal{M}}$ is generated by

$$\begin{aligned} \overline{G_{\mathcal{M}}} &= \{a^3 - y_1 b, ab - y_2 c, b^2 - y_3 a^2 c, bc - y_4 d, ad - y_5 c^2, \\ &\quad a^2c^2 - y_6 bd, ac^4 - y_7 bd^2, bd^3 - y_8 c^6, d^4 - y_9 c^7\}. \end{aligned}$$

For $ab - y_2c$ and $b^2 - y_3a^2c$ the S-polynomial is

$$s(ab - y_2c, b^2 - y_3a^2c) = y_2bc - y_3a^3c.$$

Then we get

$$R(y_2bc, \overline{G_{\mathcal{M}}}) = y_2y_4d$$

by using $bc - y_4d$, and

$$R(y_3a^3c, \overline{G_{\mathcal{M}}}) = y_1y_3y_4d$$

by using $a^3 - y_1b$ and then $bc - y_4d$. Thus, we get the remainder

$$R(s(ab - y_2c, b^2 - y_3a^2c), \overline{G_{\mathcal{M}}}) = y_2y_4d - y_1y_3y_4d$$

and hence

$$r(ab - y_2c, b^2 - y_3a^2c) = y_2y_4 - y_1y_3y_4.$$

The next pairs result in:

$$\begin{aligned} s(bc - y_4d, a^2c^2 - y_6bd) &= y_4a^2cd - y_6b^2d \\ &\rightsquigarrow y_4y_5ac^3 - y_3y_5y_6ac^3 = (y_4y_5 - y_3y_5y_6)ac^3 \\ s(ad - y_5c^2, bd^3 - y_8c^6) &= y_8ac^6 - y_5bc^2d^2 \\ &\rightsquigarrow y_4y_7y_8cd^3 - y_4y_5cd^3 = (y_4y_7y_8 - y_4y_5)cd^3 \\ &\vdots \end{aligned}$$

If we continue this for the remaining pairs we get the equations of the local ring in $\mathcal{M}$ of the toric Hilbert scheme as

$$\begin{aligned} I_{\mathcal{M}} &= \langle y_2y_4 - y_1y_3y_4, y_4y_5 - y_3y_5y_6, y_4y_9 - y_8, y_5 - y_7y_8, y_5y_7 - y_6, \\ &\quad y_1 - y_5y_6, y_4y_6 - y_2y_7, y_3y_6 - y_4, y_2 - y_4y_5, y_1y_4 - y_2y_6, \\ &\quad y_1y_3 - y_2, y_3y_5^2 - y_4y_8\rangle. \end{aligned}$$

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Chapter IV

A Polytope of a Non-Coherent Component

We want to understand the geometry of all components. Until now we have seen that the coherent component is (up to normalisation) the toric variety associated to the state polytope of the toric ideal $I_{\mathcal{A}}$. For the non-coherent components it was only known until now that there exists such a polytope describing the normalisation. Using the local equations and various facts given so far, we will derive an explicit construction of the polytope corresponding to the normalisation of the underlying reduced structure of a given non-coherent component of the toric Hilbert scheme.

IV.1 Universal Families

As before, we have our set of points $\mathcal{A} \in \mathbb{Z}^d$ with toric ideal $I_{\mathcal{A}}$. Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal. At first, assume that $\mathcal{M}$ is coherent. Then by Theorem III.3.7 we can compute the local equations $I_{\mathcal{M}} \subseteq \Bbbk[y_1, ..., y_l]$ of $\mathcal{H}_{\mathcal{A}}$ around $\mathcal{M}$ where l is the number of generators of $\mathcal{M} = \langle \boldsymbol{x}^{\boldsymbol{m}_1}, ..., \boldsymbol{x}^{\boldsymbol{m}_l} \rangle$.

Definition IV.1.1. We call the ideal

$$J_{\mathcal{M}} = \langle \boldsymbol{x}^{\boldsymbol{m}_1} - y_1 \cdot s_1, ..., \boldsymbol{x}^{\boldsymbol{m}_l} - y_l \cdot s_l \rangle$$

from Construction III.3.6 the *universal family* of $\mathcal{U}_{\mathcal{M}}$ with *defining ideal* $I_{\mathcal{M}}$.

Note that by Lemma III.3.4 the $\mathcal{A}$-graded ideals that correspond to the points in $\mathcal{U}_{\mathcal{M}} \subseteq \mathcal{H}_{\mathcal{A}}$ are precisely given by $J_{\mathcal{M}}$ for all $(y_1, ..., y_l)$ in the variety of $I_{\mathcal{M}}$.

We now give a construction of a new universal family that describes the ambient torus of the underlying reduced structure of an non-coherent component containing $\mathcal{M}$. This is done in several steps. Firstly, we remove redundant variables from $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ (Construction IV.1.2). Then we construct the primary decomposition of the resulting defining ideal $I'_{\mathcal{M}}$ to get the primary ideals q defining the irreducible components containing $\mathcal{M}$ (Propositions IV.1.6 and IV.1.11). Because the underlying reduced structure of each component is a projective toric variety we take the radical $p = \sqrt{q}$ for each of these primary ideals (Definition IV.1.13). Then we set the variables that are generators of p to zero in the universal family $J'_{\mathcal{M}}$ (Construction IV.1.15). Now the prime ideal p has become a pure binomial ideal, *i.e.* containing no monomials, and we perform a change of the $\boldsymbol{y}$ coordinates in $J'_{\mathcal{M}}$ so that p becomes trivial and we get a universal family that describes the ambient torus of the reduced structure of that non-coherent component without any defining ideal (Construction IV.1.17). We will now go through this construction step by step.

As seen in Construction III.3.6, the ideal $I_{\mathcal{M}}$ is a binomial ideal, which also follows from Lemma III.2.1. Hence, the generators of $I_{\mathcal{M}}$ may contain binomials of the form

$$y_i - \prod_{j \neq i} y_j^{b_j}$$

for some exponents b_j. Then we call the single variable y_i a *redundant variable*, since we can remove y_i from $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ by substituting y_i by the product $\prod_{j \neq i} y_j^{b_j}$.

Construction IV.1.2 (Removing redundant variables). Let $J_{\mathcal{M}}$ be the universal family of a neighbourhood $\mathcal{U}_{\mathcal{M}}$ with defining ideal $I_{\mathcal{M}}$. Let y_i be a redundant variable given by

$$y_i - \prod_{j \neq i} y_j^{b_j} \in I_{\mathcal{M}}.$$

Then we *remove the redundant variable* from $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ with the maps

$$\Phi_i : \mathbb{k}[\boldsymbol{y}] \to \mathbb{k}[y_j \,|\, j \neq i],\ y_j \mapsto \begin{cases} y_j & \text{if } j \neq i \\ \prod_{j \neq i} y_j^{b_j} & \text{if } j = i \end{cases}$$

and $\Psi_i = \mathrm{Id}_{\mathbb{k}[\boldsymbol{x}]} \otimes \Phi_i : \mathbb{k}[\boldsymbol{x}, \boldsymbol{y}] \to \mathbb{k}[\boldsymbol{x}, y_j \,|\, j \neq i]$, respectively.

We repeat this until there are no more redundant variables. Then we denote by $\mathfrak{r} \subseteq \{1, ..., l\}$ the indices of the *remaining variables* in $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ and write $I'_{\mathcal{M}}$ and $J'_{\mathcal{M}}$ for the ideals obtained by removing the redundant variables in $I_{\mathcal{M}}$ and $J_{\mathcal{M}}$ respectively. This means we have

$$\begin{aligned} I'_{\mathcal{M}} &\subseteq \mathbb{k}[y_i \,|\, i \in \mathfrak{r}] \quad \text{and} \\ J'_{\mathcal{M}} &= \langle \boldsymbol{x}^{\boldsymbol{m}_j} - p_j(\boldsymbol{y}) \cdot s_j \mid j = 1, ..., l \rangle, \end{aligned}$$

where $p_j(\boldsymbol{y})$ is the monomial into which y_j has been converted by removing all redundant variables.

Remark IV.1.3. The points in $\mathcal{U}_{\mathcal{M}}$ are still completely described by substituting a solution of $I'_{\mathcal{M}}$ into $J'_{\mathcal{M}}$.

Example IV.1.4. This continues Example III.3.9 from Chapter III. Let the grading be $\mathcal{A} = \{1, 3, 4, 7\} \subset \mathbb{Z}^1$ and consider the monomial $\mathcal{A}$-graded ideal

$$\mathcal{M} = \left\langle a^3, ab, b^2, bc, ad, a^2c^2, bd^2, ac^5, d^4 \right\rangle.$$

This ideal is coherent with weight vector $\omega = (265, 342, 1, 40)$ and by using Construction III.3.6 we get the universal family

$$\begin{aligned} J_{\mathcal{M}} = \langle & a^3 - y_1 b, ab - y_2 c, b^2 - y_3 a^2 c, bc - y_4 d, ad - y_5 c^2, a^2c^2 - y_6 bd, \\ & bd^2 - y_7 ac^4, ac^5 - y_8 d^3, d^4 - y_9 c^7 \rangle \end{aligned}$$

with the defining ideal

$$\begin{aligned} I_{\mathcal{M}} &= \langle y_5 - y_8y_9, y_4 - y_7y_8, y_5 - y_6y_7, y_4y_9 - y_5y_7, y_1 - y_5y_6, y_4y_6 - y_5y_8, \\ &\quad y_3y_6 - y_4, y_2 - y_4y_5, y_1y_3 - y_2, y_2y_7 - y_3y_5^2\rangle . \end{aligned}$$

There are four redundant variables, y_1, y_2, y_4, and y_5, given by

$$y_4 - y_7y_8,\ y_5 - y_8y_9,\ y_1 - y_5y_6,\ y_2 - y_4y_5.$$

If we remove the redundant variables from $I_{\mathcal{M}}$ we get

$$\begin{aligned} I'_{\mathcal{M}} &= \langle 0, 0, y_8y_9 - y_6y_7, y_7y_8y_9 - y_7y_8y_9, y_6y_8y_9 - y_6y_8y_9, y_6y_7y_8 - y_8^2y_9, \\ &\quad y_3y_6 - y_7y_8, 0, y_3y_8^2y_9^2 - y_7^2y_8^2y_9\rangle \\ &= \langle y_6y_7 - y_8y_9, y_3y_6 - y_7y_8\rangle , \end{aligned}$$

and the universal family becomes

$$\begin{aligned} J'_{\mathcal{M}} = \langle a^3 - y_6y_8y_9b, ab - y_7y_8^2y_9c, b^2 - y_3a^2c, bc - y_7y_8d, ad - y_8y_9c^2, \\ a^2c^2 - y_6bd, bd^2 - y_7ac^4, ac^5 - y_8d^3, d^4 - y_9c^7\rangle . \end{aligned}$$

◇

Still $I'_{\mathcal{M}}$ gives only a parametrisation of the neighbourhood $\mathcal{U}_{\mathcal{M}}$ of $\mathcal{M}$ and not of the different components that contain $\mathcal{M}$. However, if we decompose $I'_{\mathcal{M}}$ into its primary ideals, then each of these primary ideals determines exactly one of the possibly embedded components of $\mathcal{H}_{\mathcal{A}}$ intersecting $\mathcal{U}_{\mathcal{M}}$.

Lemma IV.1.5. *Let $\mathcal{M}$ be coherent and $I'_{\mathcal{M}} = \bigcap \mathfrak{q}_i$ be a minimal primary decomposition, then every component $V(\mathfrak{q}_i) \subset \mathcal{U}_{\mathcal{M}}$ contains $\mathcal{M}$.*

Proof. Fix a primary ideal $\mathfrak{q}_i$ and take a point $\mu \in V(\mathfrak{q}_i)$. This gives an $\mathcal{A}$-graded ideal

$$I = (J_{\mathcal{M}})_{(\boldsymbol{y})=\mu} = \langle \boldsymbol{x}^{\boldsymbol{m}_j} - p_j(\mu)s_j \mid j = 1, ..., l\rangle$$

on the component V given by $\mathfrak{q}_i$. Recall the action of the n-torus $T = (\Bbbk^*)^n$ on $S = \Bbbk[x_1, ..., x_n]$ by

$$\lambda.x_i = \lambda_i x_i$$

for $\lambda \in T$ which maps $\mathcal{A}$-graded ideals to $\mathcal{A}$-graded ideals. Hence, T acts on $\mathcal{H}_{\mathcal{A}}$ and the orbit of a point under the T-action lies in the same irreducible component as the point. Thus, the T-orbit of I lies in V. Furthermore, $\mathcal{M}$ was coherent so that there exists some $\omega \in \mathbb{N}^n$ such that $\mathcal{M} = \mathrm{in}_\omega(I_{\mathcal{A}})$. Finally, because $\{\boldsymbol{x}^{\boldsymbol{m}_j} - s_j \mid j = 1, ..., l\}$ is the reduced Gröbner basis with respect to ω we get that

$$\mathcal{M} = \langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j = 1, ..., l\rangle \subseteq \mathrm{in}_\omega(I)$$

which is in fact an equality because both ideals are $\mathcal{A}$-graded. This implies that $\mathcal{M}$ lies in the closure of the T-orbit of I by [Eis95, Theorem 15.17] and thus it lies on V. □

So let the primary decomposition be

$$I'_{\mathcal{M}} = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_k.$$

Then $V(\mathfrak{q}_i)$ is isomorphic to an irreducible, affine subset of $\mathcal{H}_{\mathcal{A}}$, in fact of $\mathcal{U}_{\mathcal{M}}$, containing $\mathcal{M}$. In particular, the closed points of $V(\mathfrak{q}_i)$ substituted into $J'_{\mathcal{M}}$ give exactly all $\mathcal{A}$-graded ideals in that component intersected with $\mathcal{U}_{\mathcal{M}}$ which are the closed points of that component. This gives us the following proposition:

Proposition IV.1.6. *Let $\mathcal{M}$ be a coherent monomial $\mathcal{A}$-graded ideal with local equations $I'_{\mathcal{M}}$ and universal family $J'_{\mathcal{M}}$, both after removing redundant variables. Let $I'_{\mathcal{M}} = q_1 \cap ... \cap q_k$ be a primary decomposition. Then $\overline{V(q_i)} \subseteq \mathcal{H}_{\mathcal{A}}$ is an irreducible component containing $\mathcal{M}$ for $i = 1, ..., k$ and one of them is the coherent component. Furthermore, $\overline{V(q_i)}$ is the coherent component if and only if q_i contains no monomials.*

Proof. The first statement follows from Remark IV.1.3 and that $V(q_i)$ is an affine open subset of an irreducible component containing $\mathcal{M}$, which by Lemma III.3.5 is dense, so that $\overline{V(q_i)}$ is in fact the component. Furthermore, $\mathcal{M}$ is coherent so one component must be the coherent one. For the last part, since q_i is generated by binomial differences, it contains no monomials exactly if $(1, ..., 1)$ is in $V(q_i)$. But this point corresponds to the $\mathcal{A}$-graded ideal

$$\langle \boldsymbol{x}^{\boldsymbol{m}_1} - s_1, ..., \boldsymbol{x}^{\boldsymbol{m}_l} - s_l \rangle ,$$

which is the toric ideal $I_{\mathcal{A}}$, because $\{\boldsymbol{x}^{\boldsymbol{m}_1} - s_1, ..., \boldsymbol{x}^{\boldsymbol{m}_l} - s_l\}$ is a Gröbner basis of $I_{\mathcal{A}}$ with respect to a term order giving $\mathcal{M}$ as initial ideal. Thus $V(q_i)$ contains the orbit of $I_{\mathcal{A}}$ under the action of the torus $T = (\mathbb{k}^*)^n$ which is the torus of the coherent component. Since $I_{\mathcal{A}}$ only lies on the coherent component, the closure of $V(q_i)$ closure is the coherent component. □

Since the coherent component is already completely described by the state polytope of the toric ideal, we can ignore the primary ideal that corresponds to the coherent component and just consider the remaining q_j's containing at least one monomial generator.

Example IV.1.7 (continuing **IV.1.4**)**.** A primary decomposition of the defining ideal after removing the redundant variables is

$$I'_{\mathcal{M}} = \langle y_6, y_8 \rangle \cap \left\langle y_7^2 - y_3 y_9, y_6 y_7 - y_8 y_9, y_3 y_6 - y_7 y_8 \right\rangle . \tag{IV.1}$$

Therefore, $\mathcal{M}$ lies on two components, which are both reduced. Since $\mathcal{M}$ is coherent one of the components is the coherent component which must be given by the latter primary ideal, because it contains no monomials. ◊

Now assume that $\mathcal{M}$ is a non-coherent $\mathcal{A}$-graded ideal and we have computed $J_{\mathcal{M}}$ and the local equations $I_{\mathcal{M}}$ using Construction III.3.12.

Definition IV.1.8. Let $\mathcal{M}$ be a non-coherent monomial $\mathcal{A}$-graded ideal. Then we call the ideal

$$J_{\mathcal{M}} = \langle \boldsymbol{x}^{\boldsymbol{m}_1} - y_1 \cdot s_1, ..., \boldsymbol{x}^{\boldsymbol{m}_l} - y_l \cdot s_l \rangle$$

from Construction III.3.12 the *universal family* of $\mathcal{U}_{\mathcal{M}}$ with *defining ideal* $I_{\mathcal{M}}$.

Remember that this time $I_{\mathcal{M}}$ only describes the local ring of the toric Hilbert scheme at $\mathcal{M}$, *i.e.*

$$\mathcal{O}_{\mathcal{H}_{\mathcal{A}},[\mathcal{M}]} \cong \mathbb{k}[Z]_{\langle Z \rangle}/F \cong \mathbb{k}[y_1, ..., y_{p_{\mathcal{M}}}]_{\langle y_1, ..., y_{p_{\mathcal{M}}} \rangle}/I_{\mathcal{M}}.$$

Now we are interested in the components of $\mathcal{U}_{\mathcal{M}}$ that contain $\mathcal{M}$. To get a description of these components we will use two facts from commutative algebra. The first one describes localisation of primary ideals.

Lemma IV.1.9. *Let $\mathcal{S} \subset R = \Bbbk[\boldsymbol{y}]$ be multiplicatively closed and $q \subset R$ be p-primary. Then*

- $\mathcal{S} \cap p \neq \emptyset \Rightarrow \mathcal{S}^{-1}q = \mathcal{S}^{-1}R$
- $\mathcal{S} \cap p = \emptyset \Rightarrow \mathcal{S}^{-1}q$ *is* $\mathcal{S}^{-1}p$ *primary and* $\left(\mathcal{S}^{-1}q\right) \cap R = q$

Proof. See [AM69, Proposition 4.8]. □

The second one is about primary decomposition in localisations.

Lemma IV.1.10. *Let $I = \bigcap_{i=1}^{n} q_i \subset R$ be a minimal primary decomposition, $\mathcal{S} \subset R$ multiplicatively closed, and $p_i = \sqrt{q_i}$, and $p_{m+1}, ..., p_n$ the associated primes with $p_j \cap \mathcal{S} \neq \emptyset$. Then*

$$\mathcal{S}^{-1}I = \bigcap_{i=1}^{m} \mathcal{S}^{-1}q_i \quad \text{and} \quad \left(\mathcal{S}^{-1}I\right) \cap R = \bigcup_{s \in \mathcal{S}} (I : s) = \bigcap_{i=1}^{m} q_i$$

are minimal primary decompositions.

Proof. See [AM69, Proposition 4.9]. □

So in our case we have $\mathcal{U}_{\mathcal{M}} = \operatorname{Spec}(\Bbbk[Z]/F)$, but we do not know F. Instead we have $\Bbbk[Z]_{\langle Z\rangle}/F \cong \Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}$. Recall

$$\begin{aligned} G :=& \ \langle \boldsymbol{x}^{\boldsymbol{m}} - \xi_{\boldsymbol{m}}^{\boldsymbol{a}} \cdot s_{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathcal{P}d(\mathcal{A}), \deg(\boldsymbol{x}^{\boldsymbol{m}}) = \boldsymbol{a}, \boldsymbol{x}^{\boldsymbol{m}} \neq s_{\boldsymbol{a}} \rangle \subseteq \Bbbk[Z] \otimes_{\Bbbk} S \quad \text{and} \\ F :=& \ \textstyle\sum_{\boldsymbol{a} \in \mathbb{N}\mathcal{A}} \operatorname{Fitt}_0\left((\Bbbk[Z][x_1, ..., x_n]/G)_{\boldsymbol{a}}\right) \subseteq \Bbbk[Z]. \end{aligned}$$

from Lemma III.3.4, and that by Z_{small} we denoted those variables of Z that correspond to $y_1, ..., y_{p_{\mathcal{M}}}$ as in Theorem III.3.11. If we take some $\boldsymbol{x}^{\boldsymbol{m}} - \xi_{\boldsymbol{m}}^{\boldsymbol{a}} \cdot s_{\boldsymbol{a}} \in G$ with $\xi_{\boldsymbol{m}}^{\boldsymbol{a}} \notin Z_{small}$, then there is a reduction of $\boldsymbol{x}^{\boldsymbol{m}} - \xi_{\boldsymbol{m}}^{\boldsymbol{a}} \cdot s_{\boldsymbol{a}}$ by $\overline{G_{\mathcal{M}}}$ to

$$\left(R\left(\boldsymbol{x}^{\boldsymbol{m}}, \overline{G_{\mathcal{M}}}\right) - \xi_{\boldsymbol{m}}^{\boldsymbol{a}}\right) \cdot s_{\boldsymbol{a}}$$

as in Construction III.3.12, where $R\left(\boldsymbol{x}^{\boldsymbol{m}}, \overline{G_{\mathcal{M}}}\right)$ is a monomial in Z_{small} which might be zero. Set

$$Z_{red} := \left\{ \xi_{\boldsymbol{m}}^{\boldsymbol{a}} - R\left(\boldsymbol{x}^{\boldsymbol{m}}, \overline{G_{\mathcal{M}}}\right) \middle| \, \xi_{\boldsymbol{m}}^{\boldsymbol{a}} \notin Z_{small} \right\}.$$

Note that $Z_{red} \subseteq F$ and $\Bbbk[Z]/Z_{red} \cong \Bbbk[Z_{small}] \cong \Bbbk[y_1, ..., y_{p_{\mathcal{M}}}]$. Hence, there is an ideal $F' \subseteq \Bbbk[\boldsymbol{y}]$ such that

$$\Bbbk[Z]/F \cong \Bbbk[\boldsymbol{y}]/F'.$$

But this means we get on the one hand

$$\mathcal{U}_{\mathcal{M}} = \operatorname{Spec}(\Bbbk[\boldsymbol{y}]/F')$$

and on the other hand

$$\begin{aligned} \Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}/F' &\cong \Bbbk[Z]_{\langle Z\rangle}/F \\ &\cong \Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}/I_{\mathcal{M}}, \end{aligned}$$

where the isomorphism between $\Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}/F'$ and $\Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}/I_{\mathcal{M}}$ is the identity. However, recall that

$$I_{\mathcal{M}} = \left\langle r(u,v) \;\middle|\; u,v \in \overline{G_{\mathcal{M}}} \right\rangle \subseteq \Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}$$

and that in fact $r(u,v) \in \Bbbk[\boldsymbol{y}]$ by construction. Thus, set the ideal

$$\widetilde{I_{\mathcal{M}}} := \left\langle r(u,v) \;\middle|\; u,v \in \overline{G_{\mathcal{M}}} \right\rangle \subseteq \Bbbk[\boldsymbol{y}]$$

and the multiplicatively closed set $\mathcal{S} = \Bbbk[\boldsymbol{y}] \setminus \langle \boldsymbol{y}\rangle$. Then we have

$$\mathcal{S}^{-1}\widetilde{I_{\mathcal{M}}} = I_{\mathcal{M}} = \mathcal{S}^{-1}F'.$$

Now let

$$\widetilde{I_{\mathcal{M}}} = \bigcap_{i=1}^{k} \mathfrak{q}_i$$

be a minimal primary decomposition in $\Bbbk[\boldsymbol{y}]$ with prime ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Note that, although $\widetilde{I_{\mathcal{M}}}$ is similar to $I_{\mathcal{M}}$ in the coherent case before, the primary decomposition of it does not only give the components containing $\mathcal{M}$. Therefore, we have to distinguish them further.

Assume that $\mathfrak{q}_i \cap \mathcal{S} = \emptyset$ for $i = 1, ..., m$ and $\mathfrak{q}_i \cap \mathcal{S} \neq \emptyset$ for $i = m+1, ..., k$ for some m. Then by using Lemmas IV.1.9 and IV.1.10 we have that

$$I_{\mathcal{M}} = \bigcap_{i=1}^{m} \mathcal{S}^{-1}\mathfrak{q}_i \quad \text{and} \quad I_{\mathcal{M}} \cap \Bbbk[\boldsymbol{y}] = \bigcap_{i=1}^{m} \mathfrak{q}_i$$

are minimal primary decompositions. On the other hand,

$$I_{\mathcal{M}} \cap \Bbbk[\boldsymbol{y}] = \bigcup_{s\in\mathcal{S}} \left(\widetilde{I_{\mathcal{M}}} : s\right) = \bigcup_{s\in\mathcal{S}} (F' : s)$$

is the saturation of $I_{\mathcal{M}}$ considered as an ideal in $\Bbbk[\boldsymbol{y}]$ with respect to $\Bbbk[\boldsymbol{y}]\setminus\langle \boldsymbol{y}\rangle$ and moreover the saturation of F'. But the latter are the functions that do not vanish on the point $(\mathbf{0})$, the point that corresponds to $\mathcal{M}$. Hence, $V\left(I_{\mathcal{M}} \cap \Bbbk[\boldsymbol{y}]\right) \subseteq \mathcal{U}_{\mathcal{M}}$ is the intersection of $\mathcal{U}_{\mathcal{M}}$ with all components of $\mathcal{H}_{\mathcal{A}}$, that contain M. Thus, $V\left(\mathfrak{q}_i \cap \Bbbk[\boldsymbol{y}]\right) \subseteq \mathcal{U}_{\mathcal{M}}$ is isomorphic to an irreducible subset of $\mathcal{U}_{\mathcal{M}}$ containing $\mathcal{M}$ and all $\mathfrak{q}_i \cap \Bbbk[\boldsymbol{y}]$ give exactly the reduced components of $\mathcal{H}_{\mathcal{A}}$, that contain $\mathcal{M}$, intersected with $\mathcal{U}_{\mathcal{M}}$. Note that again there might be embedded components.

These primary ideals give the following description of $\mathcal{U}_{\mathcal{M}}$.

Proposition IV.1.11. *Let $\mathcal{M}$ be a non-coherent monomial $\mathcal{A}$-graded ideal with universal family $J'_{\mathcal{M}}$ and defining ideal $I'_{\mathcal{M}}$, both after removing redundant variables. Let $I'_{\mathcal{M}} = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_k$ be the primary decomposition in $\Bbbk[y_i \,|\, i \in \mathfrak{r}]$. Then $\overline{V(\mathfrak{q}_j)} \subseteq \mathcal{H}_{\mathcal{A}}$ is an irreducible component containing $\mathcal{M}$ if and only if none of the generators of $\mathfrak{q}_j$ is a unit in $\Bbbk[y_i \,|\, i \in \mathfrak{r}]_{\langle y_i | i\in\mathfrak{r}\rangle}$.*

Proof. First of all note that removing redundant variables still maps $I_{\mathcal{M}}$ to an isomorphic description in the local ring since

$$y_i - \prod_{j\neq i} y_j^{b_j}$$

is not a unit in $\Bbbk[\boldsymbol{y}]_{\langle \boldsymbol{y}\rangle}$. Let q_i be one of the primary ideals in a minimal primary decomposition of $I_{\mathcal{M}}$ in $\Bbbk[\boldsymbol{y}]$ with prime ideal $p_j = \sqrt{q_j}$ and denote the multiplicatively closed set $\mathcal{S} := \Bbbk[y_i \,|\, i \in \mathfrak{r}] \setminus \langle y_i | i \in \mathfrak{r}\rangle$. Then $V(q_j)$ is the intersection of a reduced irreducible component containing $\mathcal{M}$ with $\mathcal{U}_{\mathcal{M}}$ if and only if p_j is an associated prime of $\bigcup_{s\in\mathcal{S}} (I_{\mathcal{M}} : s)$. But by using the above, p_j is an associated prime of $\bigcup_{s\in\mathcal{S}} (I_{\mathcal{M}} : s)$ exactly if none of the generators of q_j is a unit in $\mathcal{S}$. Again, by Lemma III.3.5 the closure $\overline{V(q_j)}$ is an irreducible component of $\mathcal{H}_{\mathcal{A}}$. □

Remark. Note that all components of $\mathcal{U}_{\mathcal{M}}$ containing $\mathcal{M}$ are in fact non-coherent and therefore all primary ideals contain a monomial generator.

Example IV.1.12 (continuing **IV.1.4**)**.** Now we consider the non-coherent $\mathcal{A}$-graded ideal

$$\mathcal{M}_0 = \left\langle a^3, ab, b^2, bc, ad, a^2c^2, ac^4, bd^3, d^4 \right\rangle .$$

Using Construction III.3.12 we get the universal family

$$J_{\mathcal{M}_0} = \big\langle a^3 - y_1 b, ab - y_2 c, b^2 - y_3 a^2 c, bc - y_4 d, ad - y_5 c^2,$$

$$a^2c^2 - y_6 bd, ac^4 - y_7 bd^2, bd^3 - y_8 c^6, d^4 - y_9 c^7 \big\rangle$$

with the defining ideal

$$\begin{aligned} I_{\mathcal{M}_0} \;=\; & \langle y_4y_9 - y_8, y_5 - y_7y_8, y_5y_7 - y_6, y_5y_6 - y_1, y_4y_6 - y_2y_7, \\ & y_3y_6 - y_4, y_2 - y_4y_5, y_1y_4 - y_2y_6, y_1y_3 - y_2, y_3y_5^2 - y_4y_8\rangle . \end{aligned}$$

There are five redundant variables, y_1, y_2, y_5, y_6, and y_8, given by

$$y_8 - y_4y_9,\; y_5 - y_7y_8,\; y_6 - y_5y_7,\; y_1 - y_5y_6,\; y_2 - y_4y_5.$$

If we remove the redundant variables from $I_{\mathcal{M}_0}$ we get

$$\begin{aligned} I'_{\mathcal{M}_0} \;=\; & \big\langle 0, 0, 0, 0, y_4^2y_7^2y_9 - y_4^2y_7^2y_9, y_3y_4y_7^2y_9 - y_4, 0, \\ & y_4^3y_7^3y_9^2 - y_4^3y_7^3y_9^2, y_3y_4^2y_7^3y_9^2 - y_4^2y_7y_9, y_3y_4^2y_7^2y_9^2 - y_4^2y_9 \big\rangle \\ =\; & \big\langle y_3y_4y_7^2y_9 - y_4 \big\rangle , \end{aligned}$$

and the universal family becomes

$$\begin{aligned} J'_{\mathcal{M}_0} = \; & \langle a^3 - y_4^2y_7^3y_9^2 b, ab - y_4^2y_7y_9 c, b^2 - y_3a^2c, bc - y_4 d, ad - y_4y_7y_9c^2, \\ & a^2c^2 - y_4y_7^2y_9 bd, ac^4 - y_7bd^2, bd^3 - y_4y_9c^6, d^4 - y_9c^7\rangle . \end{aligned}$$

For the defining ideal

$$I'_{\mathcal{M}_0} = \big\langle y_3y_7^2y_9 - 1\big\rangle \cap \langle y_4\rangle$$

is a minimal primary decomposition. The first primary ideal contains an element of $\Bbbk[y_3, y_4, y_7, y_9] \setminus \langle y_3, y_4, y_7, y_9\rangle$, so only $\langle y_4\rangle$ determines an irreducible component containing $\mathcal{M}_0$.

◊

From now on the construction is the same for coherent and non-coherent monomial ideals, since the primary ideals in Propositions IV.1.6 and IV.1.11 giving non-coherent components have exactly the same properties. We want to construct the polytope that defines the reduced underlying structure of a non-coherent component, so we fix one of these primary ideals q_i and take the radical $p_i := \sqrt{q_i}$. Then $V(p_i)$ is an affine open chart of the underlying reduced scheme of the non-coherent component given by q_i. To be more precise, it is isomorphic to an affine open chart, where the isomorphism is given by the universal family $J'_{\mathcal{M}}$. In other words, if the $\boldsymbol{y}$'s in $J'_{\mathcal{M}}$ are considered as coefficients, then those coefficients satisfying p_i give exactly the $\mathcal{A}$-graded ideals that correspond to the points of that component.

Definition IV.1.13. Let p be an associated prime of $I'_{\mathcal{M}}$ with corresponding primary ideal q and irreducible component $\overline{V(q)} \subseteq \mathcal{H}_{\mathcal{A}}$ as in Proposition IV.1.6 or IV.1.11. Then we denote the reduced scheme of the corresponding component by

$$V_p := \left(\overline{V(q)}\right)_{\text{red}} \subset (\mathcal{H}_{\mathcal{A}})_{\text{red}} .$$

Remark IV.1.14. Since $p = \sqrt{q}$ we have

$$V_p = \overline{V(p)} \subset (\mathcal{H}_{\mathcal{A}})_{\text{red}} .$$

We now give a construction via torus invariant isomorphisms to get a universal family $J_{\mathcal{M}}(p)$ that gives an open affine chart of the component for all values of the remaining $\boldsymbol{y}$-variables. This means we will perform a change of coordinates on the $\boldsymbol{y}$'s in $J'_{\mathcal{M}}$ and p, that makes p trivial. Since the solutions of p give all $\mathcal{A}$-graded ideals in the torus of that component we will get a new universal family where every set of values for $\boldsymbol{y}$ gives a point on that component.

Since p is a binomial prime ideal, a minimal generating set is of the form

$$p = \left\langle y_i, \boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-} \right\rangle$$

for some $i \in \mathfrak{r}$ and some $\boldsymbol{b}^+, \boldsymbol{b}^- \in \mathbb{N}^{\mathfrak{r}}$. Then the first step is to remove the y_i's in p and $J'_{\mathcal{M}}$ by just setting them to zero.

Construction IV.1.15 (Removing single variables). Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal and p an associated prime of $I'_{\mathcal{M}}$. Denote by $\mathfrak{r}'$ the indices of the variables, that are not contained in p, and by J_p the set of all $j \in \{1, ..., p_{\mathcal{M}}\}$ such that $y_i \nmid p_j(\boldsymbol{y})$ for all $y_i \in p$, *i.e.* all indices where $p_j(\boldsymbol{y})$ remains unchanged and is not set to zero. Then we remove the single variables by applying the map

$$\Psi : \mathbb{k}[y_i \,|\, i \in \mathfrak{r}] \to \mathbb{k}[y_i \,|\, i \in \mathfrak{r}'],\; y_j \mapsto \begin{cases} y_j & \text{if } j \in \mathfrak{r}' \\ 0 & \text{if } j \notin \mathfrak{r}' \end{cases}$$

to p, where we get

$$p' := \Psi(p) = \left\langle \boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-} \right\rangle ,$$

and by applying $\mathrm{Id}_{\mathbb{k}[\boldsymbol{x}]} \otimes \Psi$ to $J'_{\mathcal{M}}$, where we get

$$J''_{\mathcal{M}} := \mathrm{Id}_{\mathbb{k}[\boldsymbol{x}]} \otimes \Psi(J'_{\mathcal{M}}) = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot s_j \,\middle|\, j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \,\middle|\, j \in J_p \right\rangle$$

in $\mathbb{k}[\boldsymbol{x}, y_i \,|\, i \in \mathfrak{r}']$, for some $\boldsymbol{b}_j \in \mathbb{N}^{\mathfrak{r}'}, j \notin J_p$.

Remark IV.1.16. For the affine chart of V_p containing $\mathcal{M}$ we get the isomorphism $V_p \cap (\mathcal{U}_\mathcal{M})_{\mathrm{red}} \cong \operatorname{Spec}\left(\Bbbk[y_i \mid i \in \mathfrak{r}']/p'\right)$, where $(\mathcal{U}_\mathcal{M})_{\mathrm{red}}$ is the underlying reduced scheme of $\mathcal{U}_\mathcal{M}$.

Because p' is prime, if we take a generator $\boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-}$, the difference of the exponent vectors $\boldsymbol{b} := \boldsymbol{b}^+ - \boldsymbol{b}^-$ is coprime. Hence, there is an isomorphism $A \in \mathrm{GL}(\mathfrak{r}', \mathbb{Z})$ such that $A \cdot \boldsymbol{b} = e_1$, the first vector of the canonical basis. This is equivalent to a torus invariant isomorphism

$$\Phi_A : \Bbbk\left[y_i^{\pm 1} \mid i \in \mathfrak{r}'\right] \to \Bbbk\left[y_i'^{\pm 1} \mid i \in \mathfrak{r}'\right], \quad y_i \mapsto \boldsymbol{y}'^{A_i},$$

where A_i denotes the i-th column of A. This means that on the spectrum of these rings Φ_A gives an isomorphism on their tori. Using Φ_A, we can map p' to some prime ideal $\Phi_A(p')$ in $\Bbbk[y_i' \mid i \in \mathfrak{r}']$ by sending the binomial $\boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-}$ with $\boldsymbol{b} = \boldsymbol{b}^+ - \boldsymbol{b}^-$ to the binomial

$$\boldsymbol{y}'^{A(\boldsymbol{b})^+} - \boldsymbol{y}'^{A(\boldsymbol{b})^-},$$

which differs only by a unit from

$$\boldsymbol{y}'^{(A\boldsymbol{b})^+} - \boldsymbol{y}'^{(A\boldsymbol{b})^-},$$

where $A\boldsymbol{b} = (A\boldsymbol{b})^+ - (A\boldsymbol{b})^-$ is the unique decomposition into two positive vectors. If we extend Φ_A by the identity on the x_i, we can apply it to the universal family $J''_\mathcal{M} = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot s_j \mid j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle$ to get

$$\Phi_A(J''_\mathcal{M}) := \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{A\boldsymbol{b}_j} \cdot s_j \;\middle|\; j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle$$

in $\Bbbk[\boldsymbol{x}, \boldsymbol{y}'^{\pm 1}]$. Here the $\boldsymbol{y}'$ terms have become Laurent monomials, as they might have negative exponents.

Construction IV.1.17 (Change of coordinates). Let $\boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-} \in p'$ be an element of a minimal generating set. Fix a matrix $A \in \mathrm{GL}(\mathfrak{r}', \mathbb{Z})$ with torus invariant morphism

$$\Phi_A : \Bbbk\left[y_i^{\pm 1} \mid i \in \mathfrak{r}'\right] \to \Bbbk\left[y_i'^{\pm 1} \mid i \in \mathfrak{r}'\right], \quad y_i \mapsto \boldsymbol{y}'^{A_i},$$

such that $\Phi_A(\boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-}) = y_1' - 1$. Then compute the new universal family

$$\Phi_A(J''_\mathcal{M}) := \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{A\boldsymbol{b}_j} \cdot s_j \;\middle|\; j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle$$

where we set y_1' to be 1 and the new prime ideal

$$\Phi_A(p') = \left\langle \boldsymbol{y}'^{(A\boldsymbol{b})^+} - \boldsymbol{y}'^{(A\boldsymbol{b})^-} \;\middle|\; \boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-} \in p' \right\rangle$$

from which we also remove $y_1' - 1$ since it has become zero.

Lemma IV.1.18. *Let $J_\mathcal{M} = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot s_j \mid j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle$ be a universal family of a local chart of $\mathcal{H}_\mathcal{A}$ and $p \subseteq \Bbbk[y_i \mid i \in \mathfrak{r}]$ be a binomial prime ideal with no monomial generators that gives a reduced irreducible component on this chart. For a generator $\boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-}$ of p choose an isomorphism Φ_A as above and set the universal family $\Phi_A(J_\mathcal{M}) \subseteq \Bbbk[\boldsymbol{x}, \boldsymbol{y}'^{\pm 1}]$ and the prime ideal $\Phi_A(p) \in \Bbbk[y_i' \mid i \in \mathfrak{r}]$ as before. Then the prime ideal $\Phi_A(p)$ gives the intersection of the same irreducible component with its ambient torus.*

Proof. Consider Φ_A on the Laurent polynomials of both rings:

$$\Phi_A : \Bbbk\left[y_i^{\pm 1} \,\middle|\, i \in \mathfrak{r}\right] \to \Bbbk\left[y_i'^{\pm 1} \,\middle|\, i \in \mathfrak{r}\right], \quad y_i \mapsto \boldsymbol{y}'^{A_i}$$

This induces the second isomorphism of

$$V(p) \cong \Bbbk\left[y_i^{\pm 1} \,\middle|\, i \in \mathfrak{r}\right]/p \cong \Bbbk\left[y_i'^{\pm 1} \,\middle|\, i \in \mathfrak{r}\right]/\Phi_A(p).$$

This means that the points $(\lambda_i)_{i\in\mathfrak{r}}$ in $V\left(\Phi_A(p)\right)$ substituted into $\Phi_A\left(J_{\mathcal{M}}\right)$ parametrise $V(p) \cap T_p$. Hence, also $V\left(\Phi_A(p)\right)$ gives the ambient torus of V_p. □

The advantage of Lemma IV.1.18 is that the binomial $\boldsymbol{y}^{\boldsymbol{b}^+} - \boldsymbol{y}^{\boldsymbol{b}^-}$, that we have used to get A, is sent to $y_1' - 1$ under Φ_A. Hence, we can substitute 1 for y_1' in $\Phi_A(p)$ and $\Phi_A(J_{\mathcal{M}})$ and by this remove one more variable and one generator of $\Phi_A(p)$. The resulting prime ideal and universal family again satisfy the conditions for Lemma IV.1.18 and thus we can repeat this reduction until p has become the zero ideal. Thus, we can remove p with the following construction.

Construction IV.1.19 (Computing the universal family). Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal. Compute the universal family $J_{\mathcal{M}}$ and the defining ideal $I_{\mathcal{M}}$ as in Proposition III.3.6 or III.3.12, if $\mathcal{M}$ is coherent or non-coherent, respectively. Then reduce the redundant variables in $J_{\mathcal{M}}$ and $I_{\mathcal{M}}$ according to Construction IV.1.2 to $J'_{\mathcal{M}}$ and $I'_{\mathcal{M}}$. If $\mathcal{M}$ is coherent use Proposition IV.1.6 and if $\mathcal{M}$ is non-coherent use Proposition IV.1.11 to determine the primary ideals $q_1, \ldots, q_m$ that determine the non-coherent components containing $\mathcal{M}$. Let $p = \sqrt{q_i}$ be one of the associated primes. Then use Construction IV.1.15 to remove the single variables in p from $J'_{\mathcal{M}}$ and p to get $J''_{\mathcal{M}}$ and p', respectively. Pick a minimal generator $\boldsymbol{y}^{\boldsymbol{b}_1^+} - \boldsymbol{y}^{\boldsymbol{b}_1^-}$ of p' and use the corresponding isomorphism Φ_{A_1} as in Construction IV.1.17 to get $\Phi_{A_1}(p')$ and $\Phi_{A_1}(J''_{\mathcal{M}})$. Repeat this until the image of the prime ideal under the repeated isomorphisms is $\Phi_{A_k}(\ldots(\Phi_{A_1}(p'))) = (0)$. Denote the ideal resulting from applying $\Phi_{A_k} \circ \ldots \circ \Phi_{A_1}$ to $J''_{\mathcal{M}}$ by $J_{\mathcal{M}}(p)$.

Remark IV.1.20. The k steps in Construction IV.1.19 can also be done in one step. For this one uses one isomorphism over $\mathbb{Z}$ that maps to the torus. For lucidity we have shown the removal of p step by step. Although, when implementing this construction one should use the single isomorphism over $\mathbb{Z}$ to the torus. Furthermore, one can combine algorithmically Construction IV.1.15 and the repeated steps of Construction IV.1.17 into one single morphism to the torus over $\mathbb{Z}$.

Definition IV.1.21. Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal, $J_{\mathcal{M}}$ the universal family of $\mathcal{U}_{\mathcal{M}}$ with defining ideal $I_{\mathcal{M}}$, p a prime ideal as in Proposition IV.1.6 or IV.1.11 defining the underlying reduced scheme V_p of a non-coherent irreducible component containing $\mathcal{M}$. Then we call the ideal resulting from removing the single variables in p as in Construction IV.1.15 and the reduction of $J_{\mathcal{M}}$ by every generator of p as in Construction IV.1.19

$$J_{\mathcal{M}}(p) = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot s_j \,\middle|\, j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \,\middle|\, j \notin J_p \right\rangle$$

in $\Bbbk\left[\boldsymbol{x}, y_i^{\pm 1} \,\middle|\, i \in \mathfrak{r}(p)\right]$ *the universal family of the component* V_p, where $\mathfrak{r}(p)$ denotes the remaining variables and $\boldsymbol{b}_j \in \mathbb{Z}^{\#\mathfrak{r}(p)}$ are the resulting exponents.

Theorem IV.1.22. *Let $J_{\mathcal{M}}$ be a universal family with a prime ideal $\mathfrak{p}$ which together give an affine chart of a reduced irreducible component $V_{\mathfrak{p}}$ of the toric Hilbert scheme. Let $J_{\mathcal{M}}(\mathfrak{p})$ be the universal family of this component. Then $(\Bbbk^*)^{\#\mathfrak{r}(\mathfrak{p})} = \operatorname{Spec}\left(\Bbbk\left[y_i^{\pm 1} \mid i \in \mathfrak{r}(\mathfrak{p})\right]\right)$ is isomorphic to the reduced irreducible component $V_{\mathfrak{p}}$ intersected with its ambient torus by substituting these points into $J_{\mathcal{M}}(\mathfrak{p})$. To be precise, the closed points of this irreducible component of the toric Hilbert scheme intersected with its ambient torus are exactly those $\mathcal{A}$-graded ideals that are given by substituting a point $(\lambda_i)_{i\in\mathfrak{r}(\mathfrak{p})} \in (\Bbbk^*)^{\#\mathfrak{r}(\mathfrak{p})}$ into $J_{\mathcal{M}}(\mathfrak{p})$.*

Proof. The theorem follows directly from Lemma IV.1.18. Denote by A_i the matrix of the i-th reduction and let $V_{\mathfrak{p}}^T$ be the intersection of the reduced irreducible component, given by $\mathfrak{p}$, with its torus. Now we use the lemma at every step of the reduction to get the isomorphism between $V_{\mathfrak{p}}^T$ and $V(\Phi_{A_k}(...(\Phi_{A_1}(\mathfrak{p})))) \cap T$ via $\Phi_{A_k}(...(\Phi_{A_1}(J_{\mathcal{M}})))$. A minimal generating set of $\mathfrak{p}$ is finite, so after h reduction steps we get $\Phi_{A_h}(...(\Phi_{A_1}(\mathfrak{p}))) = (0)$ because we have removed all generators, thus $(\Bbbk^*)^{\#\mathfrak{r}(\mathfrak{p})} = V(0)$ is isomorphic to $V_{\mathfrak{p}}^T$ via $J_{\mathcal{M}}(\mathfrak{p}) = \Phi_{A_h}(...(\Phi_{A_1}(J_{\mathcal{M}})))$. □

Corollary IV.1.23. *The ambient torus of a non-coherent irreducible component of $\mathcal{H}_{\mathcal{A}}$ is given by one universal family $J_{\mathcal{M}}(\mathfrak{p})$ in $\Bbbk\left[\boldsymbol{x}, y_i^{\pm} \mid i \in \mathfrak{r}(\mathfrak{p})\right]$. Hence, the dimension of this component is $\#\mathfrak{r}(\mathfrak{p})$. This means that the dimension of every non-coherent component is bounded by the number of elements in the Graver basis, since $\#\mathfrak{r}(\mathfrak{p})$ is bounded by the number of generators of $J_{\mathcal{M}}$ and all $\boldsymbol{x}^{\boldsymbol{m}_j} - s_j$ are Graver.*

Furthermore, since $J_{\mathcal{M}}(\mathfrak{p})$ gives the ambient torus of the irreducible component, the closure of the torus is the whole component. □

Remark IV.1.24. The ambient torus of a reduced non-coherent component $V_{\mathfrak{p}}$ is given by $T_{\mathfrak{p}} := \operatorname{Spec}\left(\Bbbk\left[y_i^{\pm 1} \mid i \in \mathfrak{r}(\mathfrak{p})\right]\right)$, *i.e.* the points of $T_{\mathfrak{p}}$ correspond to the points in the ambient torus of $V_{\mathfrak{p}}$ via $J_{\mathcal{M}}(\mathfrak{p})$. We refer to $T_{\mathfrak{p}}$ as *the ambient torus of the non-coherent component $V_{\mathfrak{p}}$*. Note that the ambient torus of the coherent component is the n-torus $T = \operatorname{Spec}(\Bbbk[\boldsymbol{x}^{\pm 1}])$ to which $T_{\mathfrak{p}}$ is the analog for a non-coherent component.

Example IV.1.25 (continuing **IV.1.4**)**.** For the coherent ideal

$$\mathcal{M} = \left\langle a^3, ab, b^2, bc, ad, a^2c^2, bd^2, ac^5, d^4 \right\rangle$$

the primary ideal in the primary decomposition of $I'_{\mathcal{M}}$, that does not give the coherent component, is

$$\mathfrak{q} = \langle y_6, y_8 \rangle .$$

This is already a prime ideal, so $\mathfrak{p} = \mathfrak{q}$ and thus if we apply Construction IV.1.15 to remove the single variables we get $\mathfrak{p}' = (0)$ and

$$J_{\mathcal{M}}(\mathfrak{p}) = J''_{\mathcal{M}} = \left\langle b^2 - y_3a^2c, bd^2 - y_7ac^4, d^4 - y_9c^7, a^3, ab, bc, ad, a^2c^2, ac^5 \right\rangle ,$$

since there are no binomials in $\mathfrak{p}$. For the non-coherent ideal

$$\mathcal{M}_0 = \left\langle a^3, ab, b^2, bc, ad, a^2c^2, ac^4, bd^3, d^4 \right\rangle$$

the primary ideal of $I'_{\mathcal{M}_0}$ that determines a non-coherent component is

$$\mathfrak{q}_0 = \langle y_4 \rangle .$$

This is also already a prime ideal which contains no binomials, so $p_0 = q_0$ and if we apply Construction IV.1.15 to remove the single variables we get $p_0' = (0)$ and

$$J_{\mathcal{M}_0}(p_0) = J''_{\mathcal{M}_0} = \left\langle b^2 - y_3 a^2 c, ac^4 - y_7 bd^2, d^4 - y_9 c^7, a^3, ab, bc, ad, a^2c^2, bd^3 \right\rangle . \quad \diamondsuit$$

We now give a slight variation of Theorem IV.1.22 that avoids Laurent monomials.

Corollary IV.1.26. *Let* $J_{\mathcal{M}}(p) = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot s_j \,\middle|\, j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle$ *in* $\Bbbk\left[\boldsymbol{x}, y_i^{\pm 1} \,\middle|\, i \in \mathfrak{r}(p)\right]$ *be the universal family of* V_p*. Then*

$$J_{\mathcal{M}}(p)' = \left\langle \boldsymbol{y}^{\boldsymbol{b}_j^-} \cdot \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j^+} \cdot s_j \,\middle|\, j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle$$

in $\Bbbk\left[\boldsymbol{x}, y_i^{\pm} \,\middle|\, i \in \mathfrak{r}(p)\right]$ *is also a universal family for* V_p *giving an isomorphism between* $(\Bbbk^*)^{\#\mathfrak{r}(p)}$ *and* V_p *intersected with its ambient torus, where* $\boldsymbol{b}_j = \boldsymbol{b}_j^+ - \boldsymbol{b}_j^-$ *is the unique decomposition into two positive vectors.*

Proof. Just note that it is an isomorphism of tori. Thus we do not change the isomorphism by multiplying the j-th generator of $J_{\mathcal{M}}(p)$ with $\boldsymbol{y}^{\boldsymbol{b}_j^-}$, because this is just multiplication with a unit. □

The universal family $J_{\mathcal{M}}(p)$ has the advantage that it defines the ambient torus of the non-coherent irreducible component on its own, which means we do not need any equations on the coefficients in $\boldsymbol{y}$ anymore.

We now apply Construction IV.1.19 to a monomial $\mathcal{A}$-graded ideal where all steps of the construction of $J_{\mathcal{M}}(p)$ have to be done.

Example IV.1.27. Let $\mathcal{A} = \{\binom{0}{6}, \binom{2}{4}, \binom{3}{0}, \binom{3}{7}, \binom{4}{2}, \binom{6}{1}\} \subset \mathbb{Z}^2$. Then the toric ideal is

$$\begin{aligned} I_{\mathcal{A}} &= \langle bc^2 - e^2, ac^2 - be, b^2 - ae, cd - af, c^8e^3 - f^6, c^3e^6 - df^5, be^7 - d^2f^4, \\ &\quad a^2ce^6 - d^3f^3, a^3e^6 - d^4f^2, a^4bce^4 - d^5f, d^6 - a^5be^4 \rangle . \end{aligned}$$

The Graver basis has 381 elements and there are 9588 monomial $\mathcal{A}$-graded ideals, which were found by using Construction II.3.15. We choose the non-coherent monomial $\mathcal{A}$-graded ideal

$$\begin{aligned} \mathcal{M} &= \langle bc^2, ae, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, ab^3f^2, c^8e^3, \\ &\quad b^5f^2, df^5, bf^6, d^2f^4, af^6, b^4ce^4, d^4f^2, b^9c, ad^3f^3, ab^9, d^6e \rangle \end{aligned}$$

which has 22 generators. Thus, the universal family $J_{\mathcal{M}}$ of this ideal is in $\Bbbk[a, b, c, d, e, f, y_1, ..., y_{22}]$, and the defining ideal $I_{\mathcal{M}}$ lies in $\Bbbk[y_1, ..., y_{22}]$ and has 40 generators. The equations give 14 redundant variables (all variables except $y_4, y_{11}, y_{12}, y_{14}, y_{17}, y_{20}, y_{21}$, and y_{22}), so if we remove them we get

$$\begin{aligned} J'_{\mathcal{M}} &= \langle bc^2 - y_{11}y_{14}e^2, ae - y_{21}y_{22}b^2, ac^2 - y_4^2y_{12}y_{21}^3y_{22}^3be, cd - y_4af, \\ &\quad abcf - y_4y_{12}y_{21}^2y_{22}^2de^2, a^2cf - y_4y_{12}y_{21}^3y_{22}^3bde, b^3cf - y_4y_{12}y_{21}y_{22}de^3, \\ &\quad a^2bf^2 - y_{12}y_{21}^2y_{22}^2d^2e^2, a^3f^2 - y_{12}y_{21}^3y_{22}^3bd^2e, ab^3f^2 - y_{12}y_{21}y_{22}d^2e^3, \\ &\quad c^8e^3 - y_{11}f^6, b^5f^2 - y_{12}d^2e^4, df^5 - y_4^3y_{12}y_{14}y_{21}^3y_{22}^3c^3e^6, bf^6 - y_{14}c^6e^5, \\ &\quad d^2f^4 - y_{12}^2y_{22}be^7, af^6 - y_4^2y_{12}y_{14}y_{21}^3y_{22}^3c^4e^6, b^4ce^4 - y_{17}d^3f^3, \\ &\quad d^4f^2 - y_{12}y_{22}b^6e^3, b^9c - y_4y_{21}d^5f, ad^3f^3 - y_{20}b^6ce^3, \\ &\quad ab^9 - y_{21}d^6, d^6e - y_{22}b^{11} \rangle \end{aligned}$$

and the defining ideal

$$\begin{aligned} I'_{\mathcal{M}} &= \big\langle y_{17}y_{20} - y_{21}y_{22}, y_4y_{20} - y_{12}y_{22}, y_{12}y_{17} - y_4y_{21}, y_4^2y_{12}y_{21}^2y_{22}^2 - y_{11}y_{14}, \\ &\quad y_4y_{11}^2y_{14}^3y_{17}y_{21}y_{22} - y_{12}, y_{11}^3y_{14}^4y_{21}y_{22} - y_{12}^3y_{21}y_{12}^2\big\rangle \end{aligned}$$

for which we can construct the primary decomposition

$$\begin{aligned} I'_{\mathcal{M}} &= \big\langle y_{17}y_{20} - y_{21}y_{22}, y_4y_{20} - y_{12}y_{22}, y_{12}y_{17} - y_4y_{21}, y_4^2y_{12}y_{21}^2y_{22}^2 - y_{11}y_{14}, \\ &\quad y_{11}^2y_{14}^3y_{17}^2y_{22} - 1, y_{11}^3y_{14}^4y_{21}y_{22} - y_{12}^3y_{21}y_{22}^2\big\rangle \\ &\cap \langle y_4, y_{11}, y_{12}, y_{17}y_{20} - y_{21}y_{22}\rangle \\ &\cap \langle y_{11}, y_{12}, y_{20}, y_{21}\rangle \\ &\cap \langle y_4, y_{12}, y_{14}, y_{17}y_{20} - y_{21}y_{22}\rangle\,. \end{aligned}$$

The first primary ideal contains an element of $\Bbbk[\boldsymbol{y}]\backslash\langle\boldsymbol{y}\rangle$, so it does not define a non-coherent component containing $\mathcal{M}$. Hence, $\mathcal{M}$ lies on 3 non-coherent components and all three of them are reduced. Therefore, $\mathfrak{p} = \langle y_4, y_{11}, y_{12}, y_{17}y_{20} - y_{21}y_{22}\rangle$ defines an affine chart of a reduced irreducible component containing $\mathcal{M}$. Now we apply Construction IV.1.15 to remove the single variables in $\mathfrak{p}$ which gives $\mathfrak{p}' = \langle y_{17}y_{20} - y_{21}y_{22}\rangle$ and the new universal family

$$\begin{aligned} J''_{\mathcal{M}} &= \big\langle ae - y_{21}y_{22}b^2, bf^6 - y_{14}c^6e^5, b^4ce^4 - y_{17}d^3f^3, ad^3f^3 - y_{20}b^6ce^3, \\ &\quad ab^9 - y_{21}d^6, d^6e - y_{22}b^{11}\big\rangle + \big\langle bc^2, ac^2, cd, abcf, a^2cf, b^3cf, \\ &\quad a^2bf^2, a^3f^2, ab^3f^2, c^8e^3, b^5f^2, df^5, d^2f^4, af^6, d^4f^2, b^9c\big\rangle\,. \end{aligned}$$

There is one more binomial generator $y_{17}y_{20} - y_{21}y_{22}$ in $\mathfrak{p}'$ left, which has the exponent vector $(0, 1, 1, -1, -1)^t$ in the remaining variables $y_{14}, y_{17}, y_{20}, y_{21}, y_{22}$. Hence, we have to apply one isomorphism from Construction IV.1.17 to remove that binomial. Our choice for this is

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(5, \mathbb{Z}).$$

This means we get the isomorphism

$$\Phi_A : \Bbbk[y_{14}^{\pm 1}, y_{17}^{\pm 1}, y_{20}^{\pm 1}, y_{21}^{\pm 1}, y_{22}^{\pm 1}] \to \Bbbk[y_0^{\pm 1}, ..., y_4^{\pm 1}],$$

that maps

$$y_{14} \mapsto y_1, \quad y_{17} \mapsto \frac{y_0y_3y_4}{y_2}, \quad y_{20} \mapsto y_2, \quad y_{21} \mapsto y_3, \quad \text{and } y_{22} \mapsto y_4.$$

Hence, $\Phi_A(\mathfrak{p}') = y_0 - 1$ and the universal family is mapped to

$$\begin{aligned} \Phi_A(J''_{\mathcal{M}}) &= \Big\langle ae - y_3y_4b^2, bf^6 - y_1c^6e^5, b^4ce^4 - \frac{y_0y_3y_4}{y_2}d^3f^3, \\ &\quad ad^3f^3 - y_2b^6ce^3, ab^9 - y_3d^6, d^6e - y_4b^{11}\big\rangle + \\ &\quad \big\langle bc^2, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, \\ &\quad ab^3f^2, c^8e^3, b^5f^2, df^5, d^2f^4, af^6, d^4f^2, b^9c\big\rangle\,. \end{aligned}$$

But now we have to set y_0 to 1 so we get that the universal family of the component V_p is

$$\begin{aligned} J_{\mathcal{M}}(p) &= \Big\langle ae - y_3y_4b^2, bf^6 - y_1c^6e^5, b^4ce^4 - \frac{y_3y_4}{y_2}d^3f^3, \\ &\quad ad^3f^3 - y_2b^6ce^3, ab^9 - y_3d^6, d^6e - y_4b^{11}\Big\rangle + \\ &\quad \langle bc^2, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, \\ &\quad ab^3f^2, c^8e^3, b^5f^2, df^5, d^2f^4, af^6, d^4f^2, b^9c\rangle \end{aligned}$$

in $\Bbbk[a, b, c, d, e, f, y_1, y_2, y_3, y_4]$.

$\Diamond$

IV.2 Isomorphic Universal Families

So far, we have constructed universal families $J_{\mathcal{M}}(p)$ for every monomial $\mathcal{A}$-graded ideal $\mathcal{M}$ which gives the ambient torus of some reduced non-coherent component V_p containing $\mathcal{M}$. Thus, if we want to describe all non-coherent components, we have to compute all universal families for each monomial $\mathcal{A}$-graded ideal. But this means that we would construct for one non-coherent irreducible component different universal families, one for each monomial $\mathcal{A}$-graded ideal in that component. Hence, we have to find a method to check for two universal families if they define the same non-coherent component.

Consider two monomial $\mathcal{A}$-graded ideals $\mathcal{M}_1, \mathcal{M}_2$ with two prime ideals p_1, p_2 giving reduced irreducible components V_{p_1}, V_{p_2} of $\mathcal{H}_{\mathcal{A}}$ that contain $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then we have the two universal families

$$\begin{aligned} J_1 := \; J_{\mathcal{M}_1}(p_1) &= \; \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \mid j \in J_{p_1} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{p_1} \right\rangle \text{ and} \\ J_2 := \; J_{\mathcal{M}_2}(p_2) &= \; \left\langle \boldsymbol{x}^{\boldsymbol{u}_j} - \boldsymbol{y}'^{\boldsymbol{c}_j} \cdot \boldsymbol{x}^{\boldsymbol{v}_j} \mid j \in J_{p_2} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{u}_j} \mid j \notin J_{p_2} \right\rangle, \end{aligned} \tag{IV.2}$$

where again $\mathcal{M}_1 = \langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j = 1, ..., p_{\mathcal{M}_1}\rangle$ and $\mathcal{M}_2 = \langle \boldsymbol{x}^{\boldsymbol{u}_j} \mid j = 1, ..., p_{\mathcal{M}_2}\rangle$. Recall that J_{p_i} are all indices in $\{1, ..., p_{\mathcal{M}_i}\}$ whose variables were not set to zero by the removal of single variables (Construction IV.1.15), and the $\boldsymbol{b}_j, \boldsymbol{c}_j$ are the exponents of $\boldsymbol{y}$ and $\boldsymbol{y}'$, respectively, of the remaining binomials after Construction IV.1.19. Note that we have already renumbered the coefficients, such that the new variables are $\boldsymbol{y} = (y_1, ..., y_r)$ and $\boldsymbol{y}' = (y'_1, ..., y'_s)$ with $r = \mathfrak{r}(p_1)$ and $s = \mathfrak{r}(p_2)$.

Lemma IV.2.1. *Let J_1 and J_2 be two universal families as in (IV.2) which parametrise the ambient torus of the same reduced non-coherent irreducible component of $\mathcal{H}_{\mathcal{A}}$, then $J_{1\,(\boldsymbol{y}=(\boldsymbol{1}))} = J_{2\,(\boldsymbol{y}'=(\boldsymbol{1}))}$. This means the $\mathcal{A}$-graded ideals given by the identity in J_1 and J_2 are the same.*

Proof. If J_1 and J_2 parametrise the same $\mathcal{A}$-graded ideals then there is some $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_s)$ such that $J_{1\,(\boldsymbol{y}=(\boldsymbol{1}))} = J_{2\,(\boldsymbol{y}'=(\boldsymbol{\lambda}))}$. Now let $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ be a Graver binomial in $J_{1\,(\boldsymbol{y}=(\boldsymbol{1}))}$. Because they are equal, $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in J_{2\,(\boldsymbol{y}'=(\boldsymbol{\lambda}))}$ holds as well. But we have $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{\lambda}^{\boldsymbol{w}} \cdot \boldsymbol{x}^{\boldsymbol{n}} \in J_{2\,(\boldsymbol{y}'=(\boldsymbol{\lambda}))}$ for some $\boldsymbol{w} \in \mathbb{Z}^s$, so we must have $\boldsymbol{\lambda}^{\boldsymbol{w}} = 1$ which is satisfied for $\boldsymbol{\lambda} = (\boldsymbol{1})$. Because this holds for every Graver binomial, we get $J_{1\,(\boldsymbol{y}=(\boldsymbol{1}))} = J_{2\,(\boldsymbol{y}'=(\boldsymbol{\lambda}))} = J_{2\,(\boldsymbol{y}'=(\boldsymbol{1}))}$, since by Lemma II.1.8 every $\mathcal{A}$-graded ideal is determined by its coefficients of the Graver binomials. $\square$

Remark IV.2.2. Note, that in fact $\boldsymbol{\lambda} = (\mathbf{1})$ since the parametrisation by J_2 is an isomorphism. Furthermore, if the two families J_1 and J_2 are isomorphic then $r = s$ since the dimension of the component is the number of remaining variables in the universal family.

On the other hand, consider two universal families J_1 and J_2 with equal ideals for the identity $J_{1\,(\boldsymbol{y}=(\mathbf{1}))} = J_{2\,(\boldsymbol{y}'=(\mathbf{1}))}$ and let $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j}$ be in $J_{1\,(\boldsymbol{y}=(\mathbf{1}))}$ with $\boldsymbol{x}^{\boldsymbol{m}_j}, \boldsymbol{x}^{\boldsymbol{n}_j} \notin J_i$. Since the fibers of the universal families in the point $(\mathbf{1})$ are the same there is a unique $\boldsymbol{b}'_j \in \mathbb{Z}^s$ such that $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{\boldsymbol{b}'_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \in J_2$. In fact, this existence is not so trivial so that we will construct $\boldsymbol{b}'_j$ explicitly. For this we denote by

$$\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j} = \sum_{i \in J_{p2}} p_i(\boldsymbol{x}) \cdot (\boldsymbol{x}^{\boldsymbol{u}_i} - \boldsymbol{x}^{\boldsymbol{v}_i})$$

a decomposition into the generators given by $J_{2\,(\boldsymbol{y}'=(\mathbf{1}))}$ where the $p_i(\boldsymbol{x})$ are polynomials in $\boldsymbol{x}$. Then we split the p_i into monomials and rearrange the summands to get the telescoping series

$$\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j} = \sum_{i_k} m_{i_k}(\boldsymbol{x}) \cdot (\boldsymbol{x}^{\boldsymbol{u}_{i_k}} - \boldsymbol{x}^{\boldsymbol{v}_{i_k}}) \,,$$

i.e. $m_{i_k}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{u}_{i_k}} - m_{i_{k-1}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{v}_{i_{k-1}}} = 0$. We may assume that all m_{i_k} are positive, because otherwise we interchange $\boldsymbol{x}^{\boldsymbol{u}_{i_k}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{i_k}}$. If we insert the appropriate terms from $\Bbbk[\boldsymbol{y}'^{\pm 1}]$ on the right hand side we get a telescoping series in $\Bbbk[\boldsymbol{x}, \boldsymbol{y}'^{\pm 1}]$

$$\sum_{i_k} \left(\left(\prod_{\nu < k} \boldsymbol{y}'^{\boldsymbol{c}_{i_\nu}} \right) m_{i_k}(\boldsymbol{x}) \cdot (\boldsymbol{x}^{\boldsymbol{u}_{i_k}} - \boldsymbol{y}'^{\boldsymbol{c}_{i_k}} \cdot \boldsymbol{x}^{\boldsymbol{v}_{i_k}}) \right), \tag{IV.3}$$

where we take the negative $\boldsymbol{y}'$ exponent if we interchanged the $\boldsymbol{x}$ terms. But then this telescoping sum equals $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{\boldsymbol{b}'_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j}$, where

$$\boldsymbol{b}'_j = \sum_{\nu} \boldsymbol{c}_{i_\nu}.$$

Hence, the exponent $\boldsymbol{b}'_j$ is a linear combination of the $\boldsymbol{c}_i$. Because $\boldsymbol{x}^{\boldsymbol{m}_j}, \boldsymbol{x}^{\boldsymbol{n}_j} \notin J_i$ we have that $\boldsymbol{b}'_j$ is unique. This means when rearranging the generators of J_2 to get the same $\boldsymbol{x}$ binomials as in J_1 we get the new $\boldsymbol{y}'$ exponents $\boldsymbol{b}'_j$ as linear combinations of the $\boldsymbol{c}_i$. Thus, we get the following proposition:

Proposition IV.2.3. *Let J_1 and J_2 be two universal families given by equation (IV.2) with $J_{1\,(\boldsymbol{y}=(\mathbf{1}))} = J_{2\,(\boldsymbol{y}'=(\mathbf{1}))}$. Set $n_1 := \#(J_{p1})$ and $n_2 := \#(J_{p2})$. Then there is an $n_1 \times n_2$ matrix $B_{1,2}$ and an $n_2 \times n_1$ matrix $B_{2,1}$ such that for the binomials $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{\boldsymbol{b}'_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \in J_2$ and $\boldsymbol{x}^{\boldsymbol{u}_j} - \boldsymbol{y}^{\boldsymbol{c}'_j} \cdot \boldsymbol{x}^{\boldsymbol{v}_j} \in J_1$ we have*

$$\begin{aligned} \left(\boldsymbol{b}'_j\right)_{j \in J_{p1}} &= \left(\boldsymbol{c}_i\right)_{i \in J_{p2}} \cdot B_{2,1} \\ \left(\boldsymbol{c}'_j\right)_{j \in J_{p2}} &= \left(\boldsymbol{b}_i\right)_{i \in J_{p1}} \cdot B_{1,2}. \end{aligned}$$

□

Remark IV.2.4. If, on the other hand, we start with the $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{\boldsymbol{b}'_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \in J_2$, then by the same argument as before we can reconstruct the $\boldsymbol{x}^{\boldsymbol{u}_j} - \boldsymbol{y}'^{\boldsymbol{c}_j} \cdot \boldsymbol{x}^{\boldsymbol{v}_j}$, so that in fact

$$J_2 = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{\boldsymbol{b}'_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \,\middle|\, j \in J_{p1} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{p1} \right\rangle.$$

Using the above we can give a complete description of when two universal families give the same non-coherent component of the toric Hilbert scheme.

Theorem IV.2.5. *Two universal families J_1 and J_2 parametrise the ambient torus of the same non-coherent component of the toric Hilbert scheme if and only if $J_{1\,(\boldsymbol{y}=(\mathbf{1}))} = J_{2\,(\boldsymbol{y}'=(\mathbf{1}))}$, $r = s$ for $\boldsymbol{y} = (y_1, ..., y_r)$ and $\boldsymbol{y}' = (y'_1, ..., y'_s)$, and there exists an isomorphism $\Phi \in \mathrm{GL}(r, \mathbb{Z})$ such that $\Phi(\boldsymbol{b}_j) = \boldsymbol{b}'_j$ for $j \in J_{p_1}$ in the notation of Proposition IV.2.3.*

Proof. If J_1 and J_2 parametrise the ambient torus of the same non-coherent component, then by Corollary IV.1.23 their dimensions must be the same which are r and s, so we have $r = s$. We also get $J_{1\,(\boldsymbol{y}=(\mathbf{1}))} = J_{2\,(\boldsymbol{y}'=(\mathbf{1}))}$ by Lemma IV.2.1 so that we can write

$$\begin{aligned} J_1 &= \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \mid j \in J_{p_1} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{p_1} \right\rangle \text{ and} \\ J_2 &= \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}'^{\boldsymbol{b}'_j} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \,\middle|\, j \in J_{p_1} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{p_1} \right\rangle. \end{aligned}$$

Since these are all but one condition on the right hand side, it remains to show the equivalence to the $\mathbb{Z}$-isomorphism Φ. But J_1 and J_2 parametrise the ambient torus of the same non-coherent component exactly if there is an isomorphism of their parametrising tori $\mathrm{Spec}(\Bbbk[\boldsymbol{y}^{\pm 1}])$ and $\mathrm{Spec}(\Bbbk[\boldsymbol{y}'^{\pm 1}])$ which gives exactly the same points on $\mathcal{H}_{\mathcal{A}}$ by the two universal families J_1 and J_2. This means, if and only if there exists some $\Phi' : \Bbbk[\boldsymbol{y}^{\pm 1}] \to \Bbbk[\boldsymbol{y}'^{\pm 1}]$ such that

$$\Phi'(J_1) = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \Phi'\left(\boldsymbol{y}^{\boldsymbol{b}_j}\right) \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \mid j \in J_{p_1} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{p_1} \right\rangle = J_2.$$

This is equivalent to an isomorphism $\Phi \in \mathrm{GL}(r, \mathbb{Z})$ such that $\Phi(\boldsymbol{b}_j) = \boldsymbol{b}'_j$. □

Remark IV.2.6. Construction IV.1.19 and Theorem IV.2.5 allows us to compute all non-coherent components of a toric Hilbert Scheme for a given $\mathcal{A}$. To do this, one has to compute for each monomial $\mathcal{A}$-graded ideal $\mathcal{M}$ the universal families of all non-coherent components containing $\mathcal{M}$. Then one collects all isomorphic universal families into one component. By doing this, one has for each component already the list of all monomial $\mathcal{A}$-graded ideals contained in this component.

Example IV.2.7 (continuing **IV.1.4**)**.** We have computed the universal family

$$J_1 := J_{\mathcal{M}}(p) = \left\langle b^2 - y_3 a^2 c, bd^2 - y_7 ac^4, d^4 - y_9 c^7, a^3, ab, bc, ad, a^2c^2, ac^5 \right\rangle$$

for the coherent $\mathcal{M} = \langle a^3, ab, b^2, bc, ad, a^2c^2, bd^2, ac^5, d^4 \rangle$ and the universal family

$$J_2 := J_{\mathcal{M}_0}(p_0) = \left\langle b^2 - y_3 a^2 c, ac^4 - y_7 bd^2, d^4 - y_9 c^7, a^3, ab, bc, ad, a^2c^2, bd^3 \right\rangle$$

for the non-coherent $\mathcal{M}_0 = \langle a^3, ab, b^2, bc, ad, a^2c^2, ac^4, bd^3, d^4 \rangle$. When substituting $(\mathbf{1})$ we get

$$J_{1\,(\boldsymbol{y}=(\mathbf{1}))} = \left\langle b^2 - a^2c, bd^2 - ac^4, d^4 - c^7, a^3, ab, bc, ad, a^2c^2, ac^5 \right\rangle$$

and

$$J_{2\,(\boldsymbol{y}=(\mathbf{1}))} = \left\langle b^2 - a^2c, ac^4 - bd^2, d^4 - c^7, a^3, ab, bc, ad, a^2c^2, bd^3 \right\rangle.$$

Furthermore, since we have both, $ac^5 = c(ac^4 - bd^2) + d \cdot bc \in J_{1\,(\boldsymbol{y}=(\mathbf{1}))}$ as well as $bd^3 = d(bd^2 - ac^4) + c^4 \cdot ad \in J_{2\,(\boldsymbol{y}=(\mathbf{1}))}$, we get $J_{1\,(\boldsymbol{y}=(\mathbf{1}))} = J_{2\,(\boldsymbol{y}=(\mathbf{1}))}$.

The number of remaining variables in J_1 and in J_2 are 3, hence $r = s$. Finally, we have

$$\boldsymbol{b}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{b}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$
$$\boldsymbol{b}'_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{b}'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{b}'_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

so that

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism of the two universal families. Hence, J_1 and J_2 describe the same non-coherent component and both $\mathcal{M}$ and $\mathcal{M}_0$ lie on that component. When applying Construction IV.1.19 to the remaining 51 $\mathcal{A}$-graded monomial ideals we get that only six of them are also contained in a non-coherent component, which is in fact the same component:

$$\begin{aligned}
\mathcal{M}_1 &= \left\langle a^3, ab, b^2, bc, ad, a^2c^2, ac^4, bd^3, c^7 \right\rangle, \\
\mathcal{M}_2 &= \left\langle a^3, ab, a^2c, bc, ad, b^3, b^2d, bd^2, ac^5, d^4 \right\rangle, \\
\mathcal{M}_3 &= \left\langle a^3, ab, b^2, bc, ad, a^2c^2, bd^2, ac^5, c^7 \right\rangle, \\
\mathcal{M}_4 &= \left\langle a^3, ab, a^2c, bc, ad, b^3, b^2d, ac^4, bd^3, d^4 \right\rangle, \\
\mathcal{M}_5 &= \left\langle a^3, ab, a^2c, bc, ad, b^3, b^2d, ac^4, bd^3, c^7 \right\rangle, \text{ and} \\
\mathcal{M}_6 &= \left\langle a^3, ab, a^2c, bc, ad, b^3, b^2d, bd^2, ac^5, c^7 \right\rangle,
\end{aligned}$$

where all of them but $\mathcal{M}_6$ are coherent. Thus, the component V_p given by $J_{\mathcal{M}}(p)$ contains 8 monomial $\mathcal{A}$-graded ideals.

$\diamondsuit$

In this example it is quite clear that the two universal families define the same non-coherent component already from the equality of the two ideals when (**1**) had been substituted. One could assume now that having the same $\mathcal{A}$-graded ideal given by the identity may suffice for two universal families to define the same component. This would mean that Lemma IV.2.1 would in fact be an if and only if. The following is a counterexample to this.

Example IV.2.8 (continuing **IV.1.27**)**.** Consider the $\mathcal{A}$-graded monomial ideal

$$\begin{aligned}
\mathcal{M} = \big\langle & e^2, be, ae, cd, a^2c^2, b^3d, ac^{12}, b^3c^9, b^2c^{12}, ab^3c^8, d^3f^3, ab^2c^{11}, d^2ef^4, d^4f^2, \\
& b^6c^6, d^5f, d^6, a^3d^2f^4, b^9c^5, b^{12}c^3, a^8df^5, b^{15}c^2, b^{18} \big\rangle
\end{aligned}$$

with 23 generators. The defining ideal of $\mathcal{U}_{\mathcal{M}}$ after removing redundant variables is

$$\begin{aligned}
I'_{\mathcal{M}} = \langle & y_{15}y_{18} - y_{20}y_{21}, y_{12}y_{13}y_{15} - y_2y_{12}, y_{11}y_{13}y_{15} - y_2y_{11}, \\
& y_{20}y_{21}^2y_{23} - y_{11}y_{15}, y_{18}y_{20}^2y_{21} - y_{11}y_{15}, \\
& y_{13}y_{15}y_{20}y_{21} - y_2y_{20}y_{21}, y_{11}y_{15}y_{20}y_{21} - y_{12}y_{13} \rangle
\end{aligned}$$

in $\mathbb{k}[y_1, ..., y_{23}]$ which has a primary decomposition into 12 primary ideals. Two of them are

$$\begin{aligned}
\mathfrak{q}_1 &= \left\langle y_{11}, y_{12}, y_{18}^2, y_{18}y_{21}, y_{21}^2, y_{15}y_{18} - y_{20}y_{21}, y_{13}y_{15} - y_2 \right\rangle \text{ and} \\
\mathfrak{q}_2 &= \left\langle y_{11}, y_{12}, y_{18}, y_{21} \right\rangle,
\end{aligned}$$

of which the former is not reduced. The radical of the primary ideal $\mathfrak{q}_1$ is then $\mathfrak{p}_1 = \langle y_{11}, y_{12}, y_{18}, y_{21}, y_2 - y_{13}y_{15}\rangle$. The two reduced components corresponding to these prime ideals are given by the universal families

$$\begin{aligned} J_1 &= \big\langle be - y_1y_3ac^2, d^2ef^4 - y_1b^5c^8, b^6c^6 - y_3ad^2f^4, b^{12}c^3 - y_4a^6df^5, \\ &\quad b^{18} - y_5a^{11}f^6\big\rangle + \big\langle e^2, ae, cd, a^2c^2, b^3d, ac^{12}, b^3c^9, b^2c^{12}, ab^3c^8, \\ &\quad d^3f^3, ab^2c^{11}, d^4f^2, d^5f, d^6, a^3d^2f^4, b^9c^5, a^8df^5, b^{15}c^2\big\rangle \end{aligned}$$

and

$$\begin{aligned} J_2 &= \big\langle be - y_2ac^2, d^2ef^4 - y_1b^5c^8, b^6c^6 - y_3ad^2f^4, b^{12}c^3 - y_4a^6df^5, \\ &\quad b^{18} - y_5a^{11}f^6\big\rangle + \big\langle e^2, ae, cd, a^2c^2, b^3d, ac^{12}, b^3c^9, b^2c^{12}, ab^3c^8, \\ &\quad d^3f^3, ab^2c^{11}, d^4f^2, d^5f, d^6, a^3d^2f^4, b^9c^5, a^8df^5, b^{15}c^2\big\rangle, \end{aligned}$$

where we have mapped $y_{13}, y_{15}, y_{20}, y_{23}$ to y_1, y_3, y_4, y_5 and in J_1 have replaced y_2 by y_1y_3. Not only are these two not isomorphic by construction, also the first one is four-dimensional and the second five-dimensional. But substituting $\boldsymbol{y} = (\mathbf{1})$ in J_1 and J_2 gives the same ideal

$$\begin{aligned} &\langle be - ac^2, d^2ef^4 - b^5c^8, b^6c^6 - ad^2f^4, b^{12}c^3 - a^6df^5, b^{18} - a^{11}f^6\rangle + \\ &\langle e^2, ae, cd, a^2c^2, b^3d, ac^{12}, b^3c^9, b^2c^{12}, ab^3c^8, d^3f^3, \\ &ab^2c^{11}, d^4f^2, d^5f, d^6, a^3d^2f^4, b^9c^5, a^8df^5, b^{15}c^2\rangle. \end{aligned}$$

Furthermore, as can be seen from $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the reduced component $V_{\mathfrak{p}_1}$ is an embedded component in $V_{\mathfrak{p}_2}$.

◊

IV.3 The Polytope

This section is about the construction of the polytope of a non-coherent component. Recall, that the coherent component is given by the state polytope of the toric ideal $I_{\mathcal{A}}$. For a non-coherent component we will show that the polytope is again a state polytope. Unfortunately, $J_{\mathcal{M}}(\mathfrak{p})$ and $J_{\mathcal{M}}(\mathfrak{p})'$ are not necessarily homogeneous with respect to a strictly positive grading. But the non-coherent component given by the universal family is the projective closure, so that we want to homogenise the universal family. Thus, we define a last little modification we will be using to construct the polytope whose normal fan is the normalisation of the reduced non-coherent component $V_{\mathfrak{p}}$.

Definition IV.3.1. Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal, $\mathfrak{p}$ a prime ideal as in Proposition IV.1.6 or IV.1.11, and

$$J_{\mathcal{M}}(\mathfrak{p}) = \big\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{y}^{\boldsymbol{b}_j} \cdot s_j \,\big|\, j \in J_{\mathfrak{p}}\big\rangle + \big\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{\mathfrak{p}}\big\rangle$$

in $\Bbbk\left[\boldsymbol{x}, y_i^{\pm 1} \,\middle|\, i \in \mathfrak{r}(\mathfrak{p})\right]$ the universal family of the component $V_{\mathfrak{p}}$. Then we consider an additional set of variables $\{z_i \,|\, i \in \mathfrak{r}(\mathfrak{p})\}$ and define the *generalised universal family of the component* $V_{\mathfrak{p}}$ as

$$\widetilde{J_{\mathcal{M}}(\mathfrak{p})} = \left\langle \boldsymbol{z}^{\boldsymbol{b}_j^+}\boldsymbol{y}^{\boldsymbol{b}_j^-} \cdot \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{z}^{\boldsymbol{b}_j^-}\boldsymbol{y}^{\boldsymbol{b}_j^+} \cdot s_j \,\middle|\, j \in J_{\mathfrak{p}}\right\rangle + \big\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_{\mathfrak{p}}\big\rangle$$

in $\Bbbk[\boldsymbol{x}, y_i, z_i \,|\, i \in \mathfrak{r}(\mathfrak{p})]$.

Remark. Note that $\widetilde{J_{\mathcal{M}}(p)}$ is just a homogenisation of $J_{\mathcal{M}}(p)'$.

Example IV.3.2 (continuing IV.1.4)**.** The generalised universal family for $\mathcal{M}$ and $p = \langle y_6, y_8 \rangle$ is

$$\widetilde{J_{\mathcal{M}}(p)} = \left\langle z_1 b^2 - y_1 a^2 c, z_2 bd^2 - y_2 ac^4, z_3 d^4 - y_3 c^7, a^3, ab, bc, ad, a^2c^2, ac^5 \right\rangle$$

after replacing y_3, y_7, y_9 by y_1, y_2, y_3, respectively.

The generalised universal family for $\mathcal{M}_0$ and $p_0 = \langle y_4 \rangle$ is

$$\widetilde{J_{\mathcal{M}_0}(p_0)} = \left\langle z_1 b^2 - y_1 a^2 c, z_2 ac^4 - y_2 bd^2, z_3 d^4 - y_3 c^7, a^3, ab, bc, ad, a^2c^2, bd^3 \right\rangle$$

after replacing y_3, y_7, y_9 by y_1, y_2, y_3, respectively.

$\diamondsuit$

Example IV.3.3 (continuing IV.1.27)**.** The universal family was

$$\begin{aligned} J_{\mathcal{M}}(p) \quad = \quad & \Big\langle ae - y_3y_4b^2, bf^6 - y_1c^6e^5, b^4ce^4 - \frac{y_3y_4}{y_2}d^3f^3, \\ & ad^3f^3 - y_2b^6ce^3, ab^9 - y_3d^6, d^6e - y_4b^{11} \Big\rangle + \\ & \langle bc^2, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, \\ & ab^3f^2, c^8e^3, b^5f^2, df^5, d^2f^4, af^6, d^4f^2, b^9c \rangle \end{aligned}$$

in $\Bbbk[a, b, c, d, e, f, y_1, y_2, y_3, y_4]$ so that the generalised universal family is

$$\begin{aligned} \widetilde{J_{\mathcal{M}}(p)} \quad = \quad & \langle z_3z_4ae - y_3y_4b^2, z_1bf^6 - y_1c^6e^5, y_2z_3z_4b^4ce^4 - y_3y_4z_2d^3f^3, \\ & z_2ad^3f^3 - y_2b^6ce^3, z_3ab^9 - y_3d^6, z_4d^6e - y_4b^{11} \rangle + \\ & \langle bc^2, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, \\ & ab^3f^2, c^8e^3, b^5f^2, df^5, d^2f^4, af^6, d^4f^2, b^9c \rangle \end{aligned}$$

in $\Bbbk[a, ..., f, y_1, ..., y_4, z_1, ..., z_4]$.

We have done all the preliminary work now, so that we can start directly with the theorem and the rest of the section will construct the proof of it in four steps.

Theorem IV.3.4. *Let $\mathcal{M}$ be a monomial $\mathcal{A}$-graded ideal and consider a generalised universal family $\widetilde{J_{\mathcal{M}}(p)} \subseteq \Bbbk[\boldsymbol{x}, y_i, z_i \,|\, i \in \mathfrak{r}(p)]$ of a reduced component V_p containing $\mathcal{M}$. Then $\widetilde{J_{\mathcal{M}}(p)}$ is homogeneous with respect to a strictly positive grading and the normalisation of the component V_p is the toric variety defined by the normal fan of the state polytope* $\operatorname{state}(\widetilde{J_{\mathcal{M}}(p)})$*, i.e. the Gröbner fan of $\widetilde{J_{\mathcal{M}}(p)}$.*

Before we prove the theorem we have to show four steps we will use in the proof.

Lemma IV.3.5. *Let $\mathcal{M}' \subset \Bbbk[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ be an initial monomial ideal of $\widetilde{J_{\mathcal{M}}(p)}$ with respect to a term order on $\Bbbk[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$. Then $\mathcal{M}_1 := \mathcal{M}'_{(\boldsymbol{y})=(\boldsymbol{z})=1}$ is a monomial $\mathcal{A}$-graded ideal in the component V_p. Furthermore, $\mathcal{M}'$ is the only initial monomial ideal of $\widetilde{J_{\mathcal{M}}(p)}$ with $\mathcal{M}_1 = \mathcal{M}'_{(\boldsymbol{y})=(\boldsymbol{z})=1}$.*

Proof. Recall that the number of $\boldsymbol{x}$ variables is n. We denote by $n_{\boldsymbol{y}}$ and $n_{\boldsymbol{z}}$ the number of $\boldsymbol{y}$ and $\boldsymbol{z}$ variables, respectively. Then the torus

$$\mathcal{T} := (\mathbb{k}^*)^{n+n_{\boldsymbol{y}}+n_{\boldsymbol{z}}}$$

acts on $\mathbb{k}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ coordinate-wise. Let $\lambda = (\lambda_1, ..., \lambda_n, \lambda_{\boldsymbol{y}}, \lambda_{\boldsymbol{z}}) \in \mathcal{T}$ be arbitrary, where $\lambda_{\boldsymbol{y}}$ and $\lambda_{\boldsymbol{z}}$ denote the coordinates acting respectively on $\boldsymbol{y}$ and $\boldsymbol{z}$. Note that $(\lambda_1, ..., \lambda_n)$ is in fact an element of the torus T that acts coordinate-wise on $S = \mathbb{k}[x_1, ..., x_n]$. By trivial extension to $\boldsymbol{y}$ and $\boldsymbol{z}$ such an element $(\lambda_1, ..., \lambda_n)$ also acts on $\mathbb{k}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$. Then one can easily check that

$$\begin{aligned}\left(\lambda.\widetilde{J_{\mathcal{M}}(p)}\right)_{(\boldsymbol{y})=(\boldsymbol{z})=1} &= \left((\lambda_1, ..., \lambda_n)\,.\widetilde{J_{\mathcal{M}}(p)}\right)_{(\boldsymbol{y})=\lambda_{\boldsymbol{y}},(\boldsymbol{z})=\lambda_{\boldsymbol{z}}} \\ &= \left((\lambda_1, ..., \lambda_n)\,.\left(\widetilde{J_{\mathcal{M}}(p)}\right)_{(\boldsymbol{y})=\lambda_{\boldsymbol{y}},(\boldsymbol{z})=\lambda_{\boldsymbol{z}}}\right)\end{aligned}$$

holds. Since $\left(\widetilde{J_{\mathcal{M}}(p)}\right)_{(\boldsymbol{y})=\lambda_{\boldsymbol{y}},(\boldsymbol{z})=\lambda_{\boldsymbol{z}}}$ is $\mathcal{A}$-graded and in V_p for every $\lambda \in \mathcal{T}$ and the n-torus orbit of an $\mathcal{A}$-graded ideal in V_p is an $\mathcal{A}$-graded ideal in V_p, we also have that

$$\left(\lambda.\widetilde{J_{\mathcal{M}}(p)}\right)_{(\boldsymbol{y})=(\boldsymbol{z})=1}$$

is an $\mathcal{A}$-graded ideal in V_p for every $\lambda \in \mathcal{T}$. Moreover, the monomial ideal $\mathcal{M}'$ is given as the initial ideal with respect to a weight vector $\omega \in \mathbb{N}^{n+n_{\boldsymbol{y}}+n_{\boldsymbol{z}}}$. Hence, by using [Eis95, Theorem 15.17] we get that

$$\mathcal{M}_1 = \mathcal{M}'_{(\boldsymbol{y})=(\boldsymbol{z})=1} = \left(\operatorname{in}_\omega\left(\widetilde{J_{\mathcal{M}}(p)}\right)\right)_{(\boldsymbol{y})=(\boldsymbol{z})=1}$$

is an $\mathcal{A}$-graded ideal in V_p and since $\mathcal{M}'$ is monomial, $\mathcal{M}_1$ is too.

For the second part, we fix the minimal set of generators $\{f_1, ..., f_l\}$ of $\mathcal{M}_1$ and denote by s_i the standard monomial in the degree of f_i. Since $\mathcal{M}_1$ is a monomial $\mathcal{A}$-graded ideal in V_p, we get another generalised universal family

$$\widetilde{J_{\mathcal{M}_1}(p_1)} = \left\langle \boldsymbol{z}^{\boldsymbol{c}_i^+}\boldsymbol{y}^{\boldsymbol{c}_i^-} f_i - \boldsymbol{z}^{\boldsymbol{c}_i^-}\boldsymbol{y}^{\boldsymbol{c}_i^+} s_i \,\middle|\, i \in J_{p_1} \right\rangle + \left\langle f_i \mid i \notin J_{p_1} \right\rangle,$$

which also gives V_p. Thus $\widetilde{J_{\mathcal{M}_1}(p_1)}$ is isomorphic to $\widetilde{J_{\mathcal{M}}(p)}$, so that in fact

$$\widetilde{J_{\mathcal{M}}(p)} = \left\langle \boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-} f_i - \boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+} s_i \,\middle|\, i \in J_{p_1} \right\rangle + \left\langle f_i \mid i \notin J_{p_1} \right\rangle$$

holds after a suitable change of $\boldsymbol{y}$ and $\boldsymbol{z}$ coordinates in $\widetilde{J_{\mathcal{M}_1}(p_1)}$ as in Theorem IV.2.5. Because all s_i are not in $\mathcal{M}_1$ the $\boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+} s_i$ are also not in $\mathcal{M}'$, so that we can choose a term order $\prec$ on $\mathbb{k}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ such that

$$\boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+} s_i \prec \boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-} f_i.$$

Then we claim that

$$G_{\mathcal{M}_1} := \left\{ \boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-} f_i - \boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+} s_i \,\middle|\, i \in J_{p_1} \right\} \cup \left\{ f_i \,\middle|\, i \notin J_{p_1} \right\}$$

together with a possibly empty set of monomials in $\Bbbk[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ is the reduced Gröbner basis of $\widetilde{J_{\mathcal{M}}(p)}$ with respect to the term order $\prec$. To show this, we start with a pair of binomials

$$\boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-}f_i - \boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+}s_i,\ \boldsymbol{z}^{\boldsymbol{b}_j^+}\boldsymbol{y}^{\boldsymbol{b}_j^-}f_j - \boldsymbol{z}^{\boldsymbol{b}_j^-}\boldsymbol{y}^{\boldsymbol{b}_j^+}s_j \text{ in } G_{\mathcal{M}_1}$$

and compute their S-polynomial $\boldsymbol{z}^{\boldsymbol{b}^+}\boldsymbol{y}^{\boldsymbol{b}^-}\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{z}^{\boldsymbol{b}^-}\boldsymbol{y}^{\boldsymbol{b}^+}\boldsymbol{x}^{\boldsymbol{n}}$. Then there are two cases, either we have that $\boldsymbol{z}^{\boldsymbol{b}^+}\boldsymbol{y}^{\boldsymbol{b}^-}\boldsymbol{x}^{\boldsymbol{m}}$ and $\boldsymbol{z}^{\boldsymbol{b}^-}\boldsymbol{y}^{\boldsymbol{b}^+}\boldsymbol{x}^{\boldsymbol{n}}$ are not in $\widetilde{J_{\mathcal{M}}(p)}$ or they both are. First assume they are not. Then by using Construction III.3.6 if $\mathcal{M}_1$ is coherent or Construction III.3.12 if $\mathcal{M}_1$ is non-coherent, $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ reduces to zero via the binomials $f_i - s_i$ for $i \in J_{p_1}$ because both monomials reduce to the standard monomial in their common degree. But then we can use the telescoping sum (IV.3) again to reduce $\boldsymbol{z}^{\boldsymbol{b}^+}\boldsymbol{y}^{\boldsymbol{b}^-}\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{z}^{\boldsymbol{b}^-}\boldsymbol{y}^{\boldsymbol{b}^+}\boldsymbol{x}^{\boldsymbol{n}}$ via the $\boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-}f_i - \boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+}s_i$ in $G_{\mathcal{M}_1}$ to zero because the exponents $\boldsymbol{b}_i$ satisfy the local equations.

On the other hand, if $\boldsymbol{z}^{\boldsymbol{b}^+}\boldsymbol{y}^{\boldsymbol{b}^-}\boldsymbol{x}^{\boldsymbol{m}}$ and $\boldsymbol{z}^{\boldsymbol{b}^-}\boldsymbol{y}^{\boldsymbol{b}^+}\boldsymbol{x}^{\boldsymbol{n}}$ are both in $\widetilde{J_{\mathcal{M}}(p)}$ then their difference reduces either to a monomial in $\Bbbk[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ or to zero. Therefore, $G_{\mathcal{M}_1}$ together with a possibly empty set of monomials is a Gröbner basis for $\prec$. Furthermore, the set $G_{\mathcal{M}_1}$ itself is reduced, because no f_i divides any s_j and hence no $\boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-}f_i$ divides any $\boldsymbol{z}^{\boldsymbol{b}_j^-}\boldsymbol{y}^{\boldsymbol{b}_j^+}s_j$. The additional monomials do also not reduce any element of $G_{\mathcal{M}_1}$, because on the one side reducing one of the binomials would result in $\boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+}s_i$ being in $\widetilde{J_{\mathcal{M}}(p)}$ which is a contradiction, and on the other side the remaining f_i are minimal generators of $\mathcal{M}_1$ so that there cannot be any $\boldsymbol{x}^{\boldsymbol{u}} \in \widetilde{J_{\mathcal{M}}(p)}$ dividing f_i. Thus, $G_{\mathcal{M}_1} \cup \{\text{monomials in } \Bbbk[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]\}$ is the reduced Gröbner basis of $\widetilde{J_{\mathcal{M}}(p)}$ with respect to the term order $\prec$. Therefore, $\mathcal{M}'$ is the unique initial monomial ideal of $\widetilde{J_{\mathcal{M}}(p)}$ that gives $\mathcal{M}_1$ when substituting 1, because the reduced Gröbner basis is given by $\mathcal{M}_1$. □

Since we know the correspondence between initial monomial ideals of $\widetilde{J_{\mathcal{M}}(p)}$ and monomial ideals in V_p now, we are interested in the Gröbner cone of these initial ideals.

Proposition IV.3.6. *Let $\mathcal{M}'$ be an initial monomial ideal of $\widetilde{J_{\mathcal{M}}(p)}$ with monomial $\mathcal{A}$-graded ideal $\mathcal{M}_1 = \mathcal{M}'_{(\boldsymbol{y})=(\boldsymbol{z})=1} = \langle \boldsymbol{x}^{\boldsymbol{m}_1}, ..., \boldsymbol{x}^{\boldsymbol{m}_l}\rangle$. Then the cone of maximal dimension in the Gröbner fan of $\widetilde{J_{\mathcal{M}}(p)}$ corresponding to $\mathcal{M}'$ is*

$$\sigma = \left\{\omega \,\middle|\, \langle\omega, v_i\rangle \geq 0,\, \boldsymbol{x}^{\boldsymbol{m}_i} \notin \widetilde{J_{\mathcal{M}}(p)}\right\} \qquad \text{for}$$

$$v_i = \begin{pmatrix} \boldsymbol{m}_i - \boldsymbol{n}_i \\ -\boldsymbol{b}_i \\ \boldsymbol{b}_i \end{pmatrix},$$

where $\boldsymbol{x}^{\boldsymbol{n}_i}$ is the standard monomial in the degree of $\boldsymbol{x}^{\boldsymbol{m}_i}$ and $\boldsymbol{b}_i \in \mathbb{Z}^{\#\mathfrak{r}(p)}$ unique such that $\boldsymbol{z}^{\boldsymbol{b}_i^+} \cdot \boldsymbol{y}^{\boldsymbol{b}_i^-} \cdot \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-} \cdot \boldsymbol{y}^{\boldsymbol{b}_i^+} \cdot \boldsymbol{x}^{\boldsymbol{n}_i} \in \widetilde{J_{\mathcal{M}}(p)}$.

Proof. Let $\prec$ be a term order on $\Bbbk[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ such that $\mathcal{M}' = \operatorname{in}_\prec\left(\widetilde{J_{\mathcal{M}}(p)}\right)$. Then as shown in the proof of Lemma IV.3.5

$$\left\{\boldsymbol{z}^{\boldsymbol{b}_i^+} \cdot \boldsymbol{y}^{\boldsymbol{b}_i^-} \cdot \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-} \cdot \boldsymbol{y}^{\boldsymbol{b}_i^+} \cdot \boldsymbol{x}^{\boldsymbol{n}_i} \,\middle|\, \boldsymbol{x}^{\boldsymbol{m}_i} \notin \widetilde{J_{\mathcal{M}}(p)}\right\} \cup \left\{\boldsymbol{x}^{\boldsymbol{m}_i} \in \widetilde{J_{\mathcal{M}}(p)}\right\}$$

is a subset of the reduced Gröbner basis of $\widetilde{J_{\mathcal{M}}(p)}$ with respect to $\prec$ that contains all binomials of the reduced Gröbner basis. Thus, the relative interior of the corresponding Gröbner cone σ is given by all $\omega \in \mathbb{Z}^{n+n_{\boldsymbol{y}}+n_{\boldsymbol{z}}}$ such that

$$\operatorname{in}_\omega\left(\boldsymbol{z}^{\boldsymbol{b}_i^+}\cdot \boldsymbol{y}^{\boldsymbol{b}_i^-}\cdot \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-}\cdot \boldsymbol{y}^{\boldsymbol{b}_i^+}\cdot \boldsymbol{x}^{\boldsymbol{n}_i}\right) = \boldsymbol{z}^{\boldsymbol{b}_i^+}\cdot \boldsymbol{y}^{\boldsymbol{b}_i^-}\cdot \boldsymbol{x}^{\boldsymbol{m}_i}$$

for all $\boldsymbol{x}^{\boldsymbol{m}_i} \notin \widetilde{J_{\mathcal{M}}(p)}$. But this holds exactly if

$$\sigma = \left\{\omega \,\middle|\, \langle \omega, v_i\rangle \geq 0,\, \boldsymbol{x}^{\boldsymbol{m}_i} \notin \widetilde{J_{\mathcal{M}}(p)}\right\} \qquad \text{for}$$

$$v_i = \begin{pmatrix} \boldsymbol{m}_i - \boldsymbol{n}_i \\ -\boldsymbol{b}_i \\ \boldsymbol{b}_i \end{pmatrix}.$$

□

Before we give the last lemma needed for the proof of the theorem we state a short proposition we need for this lemma.

Proposition IV.3.7. *Let*

$$\widetilde{J_{\mathcal{M}}(p)} = \left\langle \boldsymbol{z}^{\boldsymbol{b}_i^+}\boldsymbol{y}^{\boldsymbol{b}_i^-}\cdot \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-}\boldsymbol{y}^{\boldsymbol{b}_i^+}\cdot \boldsymbol{x}^{\boldsymbol{n}_i} \,\middle|\, i \in J_p\right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_i} \mid i \notin J_p\right\rangle$$

be the generalised universal family of an irreducible component V_p containing $\mathcal{M}$. Then we have

$$\Bbbk[\boldsymbol{y}^{\boldsymbol{b}_i} \mid i \in J_p, \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}] \cong \Bbbk[\boldsymbol{z}^{\boldsymbol{b}_i}\boldsymbol{y}^{-\boldsymbol{b}_i}\boldsymbol{x}^{\boldsymbol{m}_i-\boldsymbol{n}_i} \mid i \in J_p, \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}].$$

Proof. The first step is that for $\boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}$ and $\boldsymbol{z} = (z_j)_{j\in\mathfrak{r}(p)}$

$$\Bbbk[\boldsymbol{y}^{\boldsymbol{b}_i} \mid i \in J_p] \cong \Bbbk[\boldsymbol{z}^{\boldsymbol{b}_i}\boldsymbol{y}^{-\boldsymbol{b}_i} \mid i \in J_p]$$

holds, since this is just the diagonal embedding. Secondly,

$$\Bbbk[\boldsymbol{z}^{\boldsymbol{b}_i}\boldsymbol{y}^{-\boldsymbol{b}_i} \mid i \in J_p] \cong \Bbbk[\boldsymbol{z}^{\boldsymbol{b}_i}\boldsymbol{y}^{-\boldsymbol{b}_i}\boldsymbol{x}^{\boldsymbol{m}_i-\boldsymbol{n}_i} \mid i \in J_p]$$

holds, because the relations on the $\boldsymbol{x}^{\boldsymbol{m}_i-\boldsymbol{n}_i}$ are the same as the relations on the $\boldsymbol{y}^{\boldsymbol{b}_i}$. In fact, any relation between the $\boldsymbol{x}^{\boldsymbol{m}_i-\boldsymbol{n}_i}$ can be rewritten to taking an S-polynomial of two of them and reducing this. But this is exactly the construction of the local equations which have been used for the base change to the $\boldsymbol{y}^{\boldsymbol{b}_i}$, so that these also satisfy these relation. □

Lemma IV.3.8. *Let V_p be a reduced irreducible component containing an $\mathcal{A}$-graded monomial ideal $\mathcal{M}$ and let $\{\boldsymbol{b}_i \mid i \in J_p\}$ be the exponent vectors in the generalised universal family obtained from $\mathcal{M}$ for this component. Then*

$$(\mathcal{U}_{\mathcal{M}})_{red} \cap V_p = \operatorname{Spec}(\Bbbk[\boldsymbol{z}^{\boldsymbol{b}_i}\boldsymbol{y}^{-\boldsymbol{b}_i}\boldsymbol{x}^{\boldsymbol{m}_i-\boldsymbol{n}_i} \mid i \in J_p, \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}])$$

holds.

Proof. To prove this we will go through the construction of the $\boldsymbol{b}_i$ exponents of $\boldsymbol{y}$ and $\boldsymbol{z}$ again to illustrate the claim of the lemma. The affine chart of the reduced structure of the irreducible component determined by p that contains $\mathcal{M}$ is given by

$$(\mathcal{U}_{\mathcal{M}})_{\text{red}} \cap V_p \cong \operatorname{Spec}\left(\Bbbk[y_i \,|\, i \in \mathfrak{r}']/p'\right),$$

by Remark IV.1.16. Now let $A_1, ..., A_h$ be the matrices used for the reduction as in Construction IV.1.17 to remove the binomials in p' and let $A := A_h \cdot ... \cdot A_1$ be their product, then we set $\tilde{\boldsymbol{b}_i}$ to be the i-th column of A. Thus, we get the surjective morphism

$$\begin{array}{rccc} \Phi_A: & \Bbbk[y_i \,|\, i \in \mathfrak{r}'] & \to & \Bbbk[\boldsymbol{y}^{\tilde{\boldsymbol{b}_i}} \,|\, i \in \mathfrak{r}', \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}'}]/_{\langle y_k - 1\rangle} \\ & y_i & \mapsto & \boldsymbol{y}^{\tilde{\boldsymbol{b}_i}} \end{array}$$

for $k \in \mathfrak{r}' \setminus \mathfrak{r}(p)$. By construction of A the kernel of Φ_A is exactly p' so that we have

$$\Bbbk[y_i \,|\, i \in \mathfrak{r}']/p' \cong \Bbbk[\boldsymbol{y}^{\tilde{\boldsymbol{b}_i}} \,|\, i \in \mathfrak{r}', \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}'}]/_{\langle y_k - 1\rangle}.$$

Now we set y_k to 1 for $k \in \mathfrak{r}' \setminus \mathfrak{r}(p)$ by projecting the $\tilde{\boldsymbol{b}_i}$ to the $\mathfrak{r}(p)$ variables. *I.e.* if π is the projection from the $\mathfrak{r}'$ variables to the $\mathfrak{r}(p)$ variables, then with $\boldsymbol{b}_i := \pi(\tilde{\boldsymbol{b}_i})$ we have

$$\Bbbk[y_i \,|\, i \in \mathfrak{r}']/p' \cong \Bbbk[\boldsymbol{y}^{\boldsymbol{b}_i} \,|\, i \in \mathfrak{r}', \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}],$$

where the $\boldsymbol{b}_i$ are precisely as defined before. Note that the indices of the removed redundant variables are $J_p \setminus \mathfrak{r}'$. But for $i \in J_p \setminus \mathfrak{r}'$ the resulting exponent of $J_{\mathcal{M}}$ is $\boldsymbol{b}_i = \sum_{j\in\mathfrak{r}'} \lambda_j \boldsymbol{b}_j$ for $\lambda_j \in \mathbb{N}$, because y_i was a redundant variable. Thus, adding $\boldsymbol{y}^{\boldsymbol{b}_i}$ for $i \in J_p \setminus \mathfrak{r}'$ to the generators of the ring does not change the ring, so that

$$\Bbbk[y_i \,|\, i \in \mathfrak{r}']/p' \cong \Bbbk[\boldsymbol{y}^{\boldsymbol{b}_i} \,|\, i \in J_p, \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}].$$

Finally, by using Proposition IV.3.7 we get

$$\Bbbk[\boldsymbol{y}_i \,|\, i \in \mathfrak{r}']/p' \cong \Bbbk[\boldsymbol{z}^{\boldsymbol{b}_i} \cdot \boldsymbol{y}^{-\boldsymbol{b}_i} \cdot \boldsymbol{x}^{\boldsymbol{m}_i - \boldsymbol{n}_i} \,|\, i \in J_p, \boldsymbol{y} = (y_j)_{j\in\mathfrak{r}(p)}].$$

□

Note that for the coherent component this is similar to the construction in the proof of [SST02, Theorem 4.1].

Now we have collected all steps and can prove the theorem.

Proof of Theorem IV.3.4. First of all, note that

$$\widetilde{J_{\mathcal{M}}(p)} = \left\langle \boldsymbol{z}^{\boldsymbol{b}_j^+} \cdot \boldsymbol{y}^{\boldsymbol{b}_j^-} \cdot \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{z}^{\boldsymbol{b}_j^-} \cdot \boldsymbol{y}^{\boldsymbol{b}_j^+} \cdot \boldsymbol{x}^{\boldsymbol{n}_j} \,\middle|\, j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \,|\, j \notin J_p \right\rangle$$

is homogeneous with respect to a strictly positive grading for the degree vector $(\boldsymbol{a}, \boldsymbol{1}, \boldsymbol{1})$, where $\boldsymbol{a}$ is a strictly positive vector in the row span of $\mathcal{A}$ and $\boldsymbol{1}$ is the classical degree vector on $\boldsymbol{y}$ and $\boldsymbol{z}$. Thus, by Theorem II.2.17 there exists a state polytope $P = \operatorname{state}(\widetilde{J_{\mathcal{M}}(p)})$. Now let σ be a maximal cone in the normal fan of P, *i.e.* in the Gröbner fan. Then this gives an initial monomial ideal $\mathcal{M}' \subseteq \Bbbk[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ of $\widetilde{J_{\mathcal{M}}(p)}$, which in turn by Lemma IV.3.5 when substituting $(\boldsymbol{y}) = (\boldsymbol{z}) = 1$ gives a monomial $\mathcal{A}$-graded ideal $\mathcal{M}_1$ in V_p. Now there are two cases, either $\mathcal{M}_1$ is the original monomial ideal $\mathcal{M}$ we used to construct $\widetilde{J_{\mathcal{M}}(p)}$ or $\mathcal{M}_1$ is some other monomial ideal on that component.

For the first case, assume this monomial is $\mathcal{M}$. Then by Proposition IV.3.6 the cone σ is given by $\{\omega \,|\, \langle \omega, v_i \rangle \geq 0\}$ for

$$v_i := \begin{pmatrix} \boldsymbol{m}_i - \boldsymbol{n}_i \\ -\boldsymbol{b}_i \\ \boldsymbol{b}_i \end{pmatrix}, i \in J_p.$$

Hence, we have that $\sigma^\vee$ is the positive hull over $\mathbb{Q}$ of $\left\{v_i \,|\, i \in J_p\right\}$. The affine chart of V_p, that contains $\mathcal{M}$, is given by

$$(\mathcal{U}_{\mathcal{M}})_{\text{red}} \cap V_p \cong \operatorname{Spec}\left(\mathbb{k}[y_i \,|\, i \in \mathfrak{r}']/p'\right),$$

see Remark IV.1.16. But by Lemma IV.3.8 we have

$$(\mathcal{U}_{\mathcal{M}})_{\text{red}} \cap V_p \cong \operatorname{Spec}(\mathbb{k}[\boldsymbol{z}^{\boldsymbol{b}_i}\boldsymbol{y}^{-\boldsymbol{b}_i}\boldsymbol{x}^{\boldsymbol{m}_i - \boldsymbol{n}_i} \,|\, i \in J_p, \boldsymbol{y} = (y_j)_{j \in \mathfrak{r}(p)}])$$

and the exponent vectors on the right hand side are the v_i, the generators of the cone $\sigma^\vee$. Thus, if we denote by $M_\sigma := (\mathbb{Q} \cdot \sigma^\vee) \cap \mathbb{Z}^{n+n_{\boldsymbol{y}}+n_{\boldsymbol{z}}}$ the lattice of $\sigma^\vee$, we conclude that

$$\operatorname{Spec}\left(\mathbb{k}[\sigma^\vee \cap M_\sigma]\right)$$

is the normalisation of

$$\operatorname{Spec}\left(\mathbb{k}[y_i \,|\, i \in \mathfrak{r}']/p'\right),$$

the affine chart of V_p containing $\mathcal{M}$.

Secondly, let $\mathcal{M}_1 \neq \mathcal{M}$. Then for the monomial ideal $\mathcal{M}_1$ we get another universal family

$$J_{\mathcal{M}_1} = \left\langle \boldsymbol{x}^{\boldsymbol{u}_j} - \boldsymbol{y}'^{\boldsymbol{c}_j} \cdot \boldsymbol{x}^{\boldsymbol{v}_j} \mid j \in J_{p_1} \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{u}_j} \mid j \notin J_{p_1} \right\rangle$$

where p_1 is the prime ideal that gives V_p for $\mathcal{M}_1$. By Theorem IV.2.5 there is an isomorphism $\Phi : \mathbb{k}[\boldsymbol{y}'^{\pm 1}] \to \mathbb{k}[\boldsymbol{y}^{\pm 1}]$, such that $\Phi(J_{\mathcal{M}_1}) = J_{\mathcal{M}}$. Thus, if we apply Φ (extended to $\boldsymbol{z}'$) to the general universal family we get

$$\begin{aligned} \Phi(\widetilde{J_{\mathcal{M}_1}(p_1)}) &= \left\langle \boldsymbol{z}^{\Phi(\boldsymbol{c}_j)^+} \cdot \boldsymbol{y}^{\Phi(\boldsymbol{c}_j)^-} \cdot \boldsymbol{x}^{\boldsymbol{u}_j} - \boldsymbol{z}^{\Phi(\boldsymbol{c}_j)^-} \cdot \boldsymbol{y}^{\Phi(\boldsymbol{c}_j)^+} \cdot \boldsymbol{x}^{\boldsymbol{v}_j} \,\middle|\, j \in J_{p_1} \right\rangle \\ &\quad + \left\langle \boldsymbol{x}^{\boldsymbol{u}_j} \mid j \notin J_{p_1} \right\rangle \\ &= \widetilde{J_{\mathcal{M}}(p)}. \end{aligned}$$

Then Proposition IV.3.6 implies, as it did for $\mathcal{M}$, that $\sigma^\vee$ is generated by

$$v_i' := \begin{pmatrix} \boldsymbol{u}_i - \boldsymbol{v}_i \\ -\Phi(\boldsymbol{c}_i) \\ \Phi(\boldsymbol{c}_i) \end{pmatrix}, i \in J_{p_1}.$$

On the other hand we have $(\mathcal{U}_{\mathcal{M}_1})_{\text{red}} \cap V_p \cong \operatorname{Spec}\left(\mathbb{k}[\boldsymbol{y}']/p_1'\right)$ by Remark IV.1.16 and by the same argumentation as above we get

$$\mathbb{k}[\boldsymbol{y}']/p_1' \cong \mathbb{k}[\boldsymbol{z}'^{\boldsymbol{c}_i} \cdot \boldsymbol{y}'^{-\boldsymbol{c}_i} \cdot \boldsymbol{x}^{\boldsymbol{u}_i - \boldsymbol{v}_i} \,|\, i \in J_{p_1}, \boldsymbol{y} = (y_j)_{j \in \mathfrak{r}(p_1)}].$$

But Φ is an isomorphism over $\mathbb{Z}$ so if we apply $\operatorname{Id}_{\mathbb{k}[\boldsymbol{x}]} \otimes \Phi$ (again extended to $\boldsymbol{z}'$) to the right hand side we get

$$\mathbb{k}[\boldsymbol{z}'^{\Phi(\boldsymbol{c}_i)} \cdot \boldsymbol{y}'^{-\Phi(\boldsymbol{c}_i)} \cdot \boldsymbol{x}^{\boldsymbol{u}_i - \boldsymbol{v}_i} \,|\, i \in J_{p_1}, \boldsymbol{y} = (y_j)_{j \in \mathfrak{r}(p_1)}],$$

which implies that $\operatorname{Spec}(\Bbbk[\sigma^\vee \cap M_\sigma])$ is the normalisation of the affine chart of V_p containing $\mathcal{M}_1$. Note that for every maximal cone σ we get the same lattice $M_\sigma =: M_p$.

The normalisation maps for all maximal cones which we have just constructed map the identity point to the ideal

$$J_{\mathcal{M}\,(\boldsymbol{y}=(\mathbf{1}))} = \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j} \mid j \in J_p \right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \mid j \notin J_p \right\rangle \subset V_p.$$

Clearly, all these normalisation maps are equivariant under the action of the torus $\operatorname{Hom}(M_p, \Bbbk^*)$. Hence, there exists a unique $\operatorname{Hom}(M_p, \Bbbk^*)$-equivariant morphism Ψ from the projective toric variety given by the Gröbner fan of $\widetilde{J_{\mathcal{M}}(p)}$ onto the non-coherent component $V_p \subseteq (\mathcal{H}_{\mathcal{A}})_{\text{red}}$, that restricts to the normalisation maps constructed above on each affine open chart. Hence, Ψ is the normalisation morphism from the projective toric variety, given by the normal fan of the polytope $\operatorname{state}(\widetilde{J_{\mathcal{M}}(p)})$, to V_p. □

Definition IV.3.9. We call a state polytope $\operatorname{state}\left(\widetilde{J_{\mathcal{M}}(p)}\right)$ of a generalised universal family a *generalised state polytope of* $\mathcal{A}$.

Corollary IV.3.10. *Let*

$$\widetilde{J_{\mathcal{M}}(p)} = \left\langle \boldsymbol{z}^{\boldsymbol{b}_i^+} \boldsymbol{y}^{\boldsymbol{b}_i^-} \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-} \boldsymbol{y}^{\boldsymbol{b}_i^+} \boldsymbol{x}^{\boldsymbol{n}_i} \right\rangle + \langle \boldsymbol{x}^{\boldsymbol{u}_j} \rangle$$

be a generalised universal family of a non-coherent component V of a toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$. Furthermore, let F be a face of $\operatorname{state}\left(\widetilde{J_{\mathcal{M}}(p)}\right)$ *with defining normal vector $\omega \in \mathbb{Z}^{n+2\dim(V)}$. Then the orbit in V corresponding to F is given by*

$$\operatorname{in}_\omega\left(\widetilde{J_{\mathcal{M}}(p)}\right)_{(\boldsymbol{y})=\lambda,(\boldsymbol{z})=\mu}$$

for all $\lambda, \mu \in (\Bbbk^*)^{\dim(V)}$.

Proof. First, note that the edges of $\operatorname{state}\left(\widetilde{J_{\mathcal{M}}(p)}\right)$ correspond to wall ideals which are generated by just one primitive binomial and monomials. Thus, a face F corresponds to the orbit, that contains those $\mathcal{A}$-graded ideals that contain the primitive binomials corresponding to its edges with coefficients and monomials, and has exactly these wall ideals as one-dimensional orbits in its closure. But these are precisely the binomials in $\widetilde{J_{\mathcal{M}}(p)}$ with

$$\operatorname{in}_\omega\left(\boldsymbol{z}^{\boldsymbol{b}_i^+} \boldsymbol{y}^{\boldsymbol{b}_i^-} \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-} \boldsymbol{y}^{\boldsymbol{b}_i^+} \boldsymbol{x}^{\boldsymbol{n}_i}\right) = \boldsymbol{z}^{\boldsymbol{b}_i^+} \boldsymbol{y}^{\boldsymbol{b}_i^-} \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{b}_i^-} \boldsymbol{y}^{\boldsymbol{b}_i^+} \boldsymbol{x}^{\boldsymbol{n}_i}.$$

Thus the claim follows. □

Definition IV.3.11. For a face F with normal vector ω of $\operatorname{state}\left(\widetilde{J_{\mathcal{M}}(p)}\right)$ we call the general point which is given by the identity in the orbit corresponding to F

$$I_F := \left(\operatorname{in}_\omega\left(\widetilde{J_{\mathcal{M}}(p)}\right)\right)_{(\boldsymbol{y})=(\boldsymbol{z})=\mathbf{1}}$$

the *general ideal of F*.

Remark IV.3.12. An ideal I in the orbit corresponding to a face F with binomial generators with trivial coefficients is the general ideal of F.

Example IV.3.13 (continuing **IV.1.4**). For $\mathcal{A} = \{1, 3, 4, 7\}$ and the monomial $\mathcal{A}$-graded ideal $\mathcal{M} = \langle a^3, ab, b^2, bc, ad, a^2c^2, bd^2, ac^5, d^4\rangle$ we have computed the generalised universal family

$$\widetilde{J_{\mathcal{M}}(p)} = \langle z_1b^2 - y_1a^2c, z_2bd^2 - y_2ac^4, z_3d^4 - y_3c^7, a^3, ab, bc, ad, a^2c^2, ac^5\rangle$$

of the reduced non-coherent irreducible component V. This component V contains the eight monomial ideals $\mathcal{M}, \mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_6$. A state polytope for $\widetilde{J_{\mathcal{M}}(p)}$ is a cube in $\mathbb{Q}^{10}$ (there are 10 variables in total) with the vertices corresponding to the monomial ideals in the following way:

$$\begin{array}{rcl}
\mathcal{M} & \leftrightarrow & (1, -1, 4, -2, 0, -1, 0, 0, 1, 0)^t \\
\mathcal{M}_0 & \leftrightarrow & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^t \\
\mathcal{M}_1 & \leftrightarrow & (0, 0, -7, 4, 0, 0, 1, 0, 0, -1)^t \\
\mathcal{M}_2 & \leftrightarrow & (-1, 1, 3, -2, -1, -1, 0, 1, 1, 0)^t \\
\mathcal{M}_3 & \leftrightarrow & (1, -1, -3, 2, 0, -1, 1, 0, 1, -1)^t \\
\mathcal{M}_4 & \leftrightarrow & (-2, 2, -1, 0, -1, 0, 0, 1, 0, 0)^t \\
\mathcal{M}_5 & \leftrightarrow & (-2, 2, -8, 4, -1, 0, 1, 1, 0, -1)^t \\
\mathcal{M}_6 & \leftrightarrow & (-1, 1, -4, 2, -1, -1, 1, 1, 1, -1)^t
\end{array}$$

A sketch of the polytope in its affine hull is given in Figure IV.1. Note that the

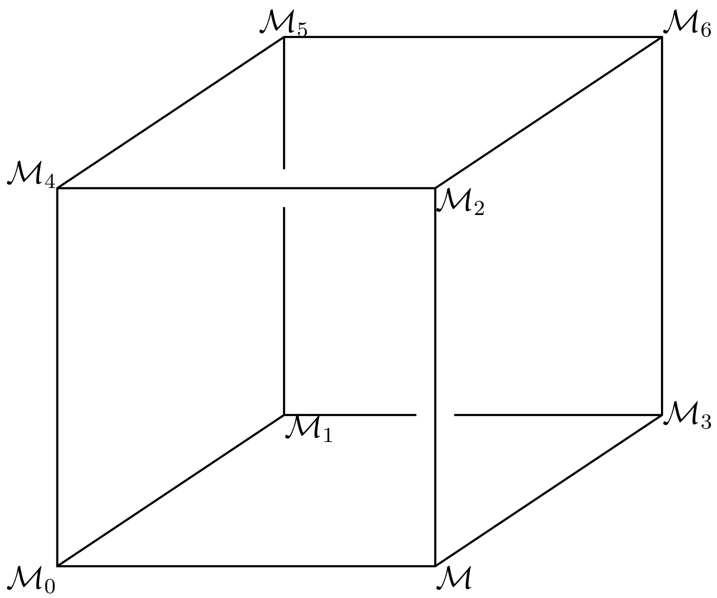

Figure IV.1: state($\widetilde{J_{\mathcal{M}}(p)}$)

two non-coherent vertices $\mathcal{M}_0$ and $\mathcal{M}_6$ are on opposing sides of the polytope, thus the intersection with the coherent component is not given by a face of the this polytope. The polytope of the coherent component is three-dimensional and has 51 vertices (the coherent monomial ideals). One can compute that the intersection is not a face of this polytope either. See Figure IV.2 for a sketch of the two-dimensional faces of the state polytope of $I_{\mathcal{A}}$ that contain the coherent monomial ideals of the non-coherent component.

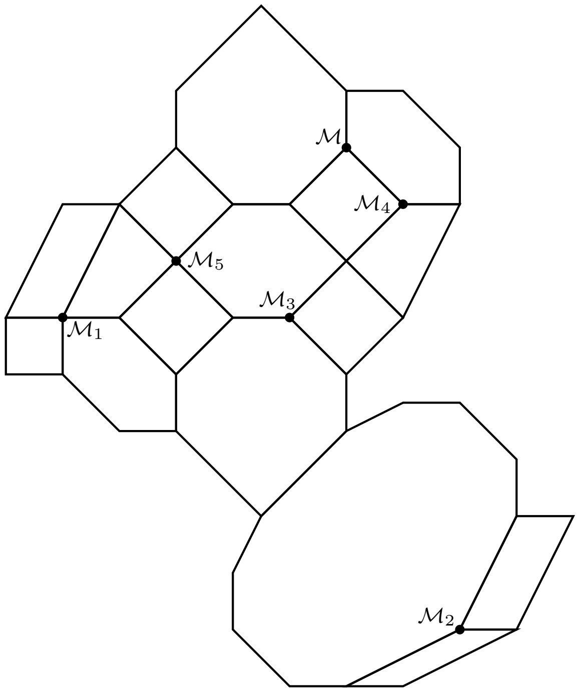

Figure IV.2: The two-dimensional faces of state($I_{\mathcal{A}}$) that contain the coherent monomial ideals of the non-coherent component

Furthermore, in [Stu96] Sturmfels has computed a family of $\mathcal{A}$-graded ideals for this $\mathcal{A}$ in the proof of [Stu96, Theorem 10.4]:

$$\begin{aligned}&\langle x_1^2x_3 - c_1x_2^2, x_1x_3^4 - c_2x_2x_4^2, x_3^7 - c_3x_4^4,\\&x_1^3, x_1x_2, x_1x_4, x_2^3, x_2^2x_4, x_2x_3, x_2x_4^3\rangle\end{aligned} \tag{IV.4}$$

for $c_1, c_2, c_3 \in \Bbbk^*$. Moreover, he shows that there is a $\Bbbk^*$ parametrisation of this family identifying all $\mathcal{A}$-graded ideals with c_1c_3/c_2^2 constant, because these are isomorphic as $\mathcal{A}$-graded ideals. The family (IV.4) is exactly the ambient torus of the non-coherent component constructed above if we identify x_1, x_2, x_3, x_4 with a, b, c, d, respectively, and c_1, c_2, c_3 with $\frac{z_1}{y_1}, \frac{z_2}{y_2}, \frac{z_3}{y_3}$, respectively. Recall that the primary decomposition (IV.1) of the defining ideal of $\mathcal{M}$ was

$$I'_{\mathcal{M}} = \langle y_6, y_8\rangle \cap \left\langle y_7^2 - y_3y_9, y_6y_7 - y_8y_9, y_3y_6 - y_7y_8\right\rangle .$$

The first primary ideal yields the non-coherent component V and the second ideal the coherent component. Hence,

$$\left\langle y_7^2 - y_3y_9\right\rangle$$

gives the intersection of V with the coherent component. Note, that we have identified y_3 with $\frac{y_1}{z_1}$, y_7 with $\frac{y_2}{z_2}$, and y_9 with $\frac{y_3}{z_3}$. Thus, the isomorphism class $c_1c_3/c_2^2 = 1$ corresponds exactly to the intersection of V with the coherent component.

$\diamondsuit$

Remark IV.3.14. This example implies that two $\mathcal{A}$-graded ideals, that correspond to points in the same orbit in a non-coherent component, need not be isomorphic as $\mathcal{A}$-graded ideals. This is in contrast to the coherent component, where the orbits are exactly the isomorphism classes.

Chapter V

Properties of Toric Hilbert Schemes

In this chapter we will state properties of toric Hilbert schemes which we have found and they will be shown using the generalised universal families and generalised state polytopes. We start with a stratification of the toric Hilbert scheme by considering actions of subtori. Then we provide some results on the intersection behaviour of the irreducible components of toric Hilbert schemes.

V.1 Stratification by Subtorus Actions

The goal of this section is to give a stratification of the toric Hilbert scheme induced by the maximal dimensional subtorus action that leaves the $\mathcal{A}$-graded ideals invariant for each stratum.

By construction the torus $(\Bbbk^*)^n$ acts on S by

$$\boldsymbol{\lambda}.x_i = \lambda_i \cdot x_i$$

for $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) \in (\Bbbk^*)^n$.

Definition V.1.1. A *k-dimensional subtorus* $T \subseteq (\Bbbk^*)^n$ is a torus $(\Bbbk^*)^k \cong T$ with the isomorphism given by n integer vectors $\{d_1, ..., d_n\} \subset \mathbb{Z}^k$ such that

$$\begin{array}{ccc} (\Bbbk^*)^k & \to & T \\ \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) & \mapsto & \left(\boldsymbol{\alpha}^{d_1}, ..., \boldsymbol{\alpha}^{d_n}\right), \end{array}$$

where $\boldsymbol{\alpha}^{d_i} = \alpha_1^{d_i^1} \cdot ... \cdot \alpha_k^{d_i^k}$.

Remark V.1.2. By this definition the action of T on S is given by

$$\boldsymbol{\alpha}.x_i = \boldsymbol{\alpha}^{d_i} \cdot x_i$$

for $\boldsymbol{\alpha} \in (\Bbbk^*)^k$. This is equivalent to a $\mathbb{Z}^k$-grading on S given by

$$\deg(x_i) = d_i \in \mathbb{Z}^k.$$

Definition V.1.3. We say that a subtorus $T \subseteq (\Bbbk^*)^n$ is given by a k times n matrix D so that the columns D_i define the action on S and equivalently the corresponding $\mathbb{Z}^k$-grading.

Now let $I \subseteq S$ be $\mathcal{A}$-graded and choose a subtorus T given by a matrix D. Then I is homogeneous with respect to the grading induced by D exactly if for every binomial $\alpha_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{u}} - \beta_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{v}} \in I$ we have $D\boldsymbol{u} = D\boldsymbol{v}$, since $\deg(\boldsymbol{x}^{\boldsymbol{u}}) = D\boldsymbol{u}$. But the binomial $\alpha_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{u}} - \beta_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{v}}$ is invariant under the action of T if and only if $\boldsymbol{x}^{\boldsymbol{u}}$ and $\boldsymbol{x}^{\boldsymbol{v}}$ have the same degree with respect to the D-grading, *i.e.* the binomial is homogeneous. Thus, we get the following proposition:

Proposition V.1.4. *An $\mathcal{A}$-graded ideal $I \subseteq S$ is invariant under the subtorus action induced by D if and only if*

$$D\boldsymbol{u} = D\boldsymbol{v} \quad \text{for all } \alpha_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{u}} - \beta_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{v}} \in I \tag{V.1}$$

holds. □

Condition (V.1) in Proposition V.1.4 only has to be checked for a set of generators. Hence, we use Lemma II.1.8 to write

$$I = \langle \boldsymbol{x}^{\boldsymbol{u}} - \alpha_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{v}} \mid \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}(\mathcal{A}) \text{ and } \boldsymbol{x}^{\boldsymbol{v}} \notin I\rangle$$

generated by primitive binomials. Thus, we have to check those primitive binomials where $\alpha_{\boldsymbol{uv}} \neq 0$.

Definition V.1.5. Let $I \subseteq S$ be $\mathcal{A}$-graded and $\alpha_{\boldsymbol{uv}} \in \Bbbk$ as in Lemma II.1.8 such that

$$I = \langle \boldsymbol{x}^{\boldsymbol{u}} - \alpha_{\boldsymbol{uv}}\boldsymbol{x}^{\boldsymbol{v}} \mid \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}(\mathcal{A}) \text{ and } \boldsymbol{x}^{\boldsymbol{v}} \notin I\rangle .$$

Then we denote the set of Graver binomials for which the coefficient is not zero by

$$\mathcal{G}_I = \langle \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}(\mathcal{A}) \mid \alpha_{\boldsymbol{uv}} \neq 0\rangle .$$

This directly implies the following:

Corollary V.1.6. *An $\mathcal{A}$-graded ideal $I \subseteq S$ is invariant under the subtorus action induced by D exactly if*

$$D\boldsymbol{u} = D\boldsymbol{v} \qquad \text{for all } \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}_I. \qquad \square$$

Definition V.1.7. For a given subset $\mathcal{G}_I$ of the Graver basis $\mathcal{G}(\mathcal{A})$ denote the submodule generated by the differences of the exponent vectors by

$$V_I = \operatorname{span}\{\boldsymbol{u} - \boldsymbol{v} \mid \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}_I\} .$$

Proposition V.1.8. *An $\mathcal{A}$-graded ideal $I \subseteq S$ is invariant under the subtorus action induced by D if and only if*

$$V_I \subseteq \operatorname{Ker}(D).$$

Proof. I is invariant exactly if

$$\begin{aligned} & D\boldsymbol{u} = D\boldsymbol{v} && \text{for all } \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}_I \\ \Leftrightarrow\ & D(\boldsymbol{u} - \boldsymbol{v}) = 0 && \text{for all } \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}_I \\ \Leftrightarrow\ & \operatorname{Ker}(D) \supseteq V_I. \end{aligned}$$

□

This means all possible subtorus actions under which I is invariant are determined by V_I. The smallest such subtorus is the trivial one $(\Bbbk^*)^0$, whereas the maximal subtorus given by D must satisfy

$$V_I = \operatorname{Ker}(D).$$

Proposition V.1.9. *Let I be an $\mathcal{A}$-graded ideal. Then the rank of a maximal subtorus under whose action I is invariant is*

$$n - \operatorname{rank}(V_I).$$

Proof. Let D induce a subtorus action with $V_I = \operatorname{Ker}(D)$. Then

$$\operatorname{rank}(V_I) = \operatorname{rank}(\operatorname{Ker}(D)) = n - \operatorname{rank}(D). \qquad \square$$

Now we have all that is needed to construct the desired stratification on the toric Hilbert scheme.

Definition V.1.10. Two $\mathcal{A}$-graded ideals I_1, I_2 lie in the same *subtorus rank stratum* if

$$\operatorname{rank}(V_{I_1}) = \operatorname{rank}(V_{I_2}).$$

Definition V.1.11. Two $\mathcal{A}$-graded ideals I_1, I_2 lie in the same *degree stratum* if they have the same module of maximal possible degrees, *i.e.*

$$V_{I_1} = V_{I_2}.$$

This stratification is a refinement of the coarser subtorus rank stratification.

Note that V_I becomes of maximal rank if $\mathcal{G}_I = \mathcal{G}(\mathcal{A})$, *i.e.* all coefficients are not zero. This is the case for example for the toric ideal $I_{\mathcal{A}}$. For any $\mathcal{A}$-graded ideal I we have $\mathcal{G}_I \subset \mathcal{G}(\mathcal{A})$ so that $V_I \subseteq V_{I_{\mathcal{A}}}$ and therefore the rank of $V_{I_{\mathcal{A}}}$ is the upper bound for all V_I. But since $\{\boldsymbol{u} - \boldsymbol{v} \mid \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}(\mathcal{A})\}$ spans $\operatorname{Ker}(\mathcal{A})$, the maximal dimension for V_I is $n - d$. Thus, the maximal dimension of a subtorus is at least d.

Example V.1.12. We continue Example IV.1.4 with $\mathcal{A} = \{1, 3, 4, 7\}$ from Chapter IV. If we take the generalised universal family of the non-coherent component and substitute $(\mathbf{1})$ then we get the $\mathcal{A}$-graded ideal

$$I = \left\langle b^2 - a^2c, ac^4 - bd^2, d^4 - c^7, a^3, ab, bc, ad, a^2c^2, bd^3 \right\rangle$$

in the torus of the non-coherent component. For this ideal we get

$$\mathcal{G}_I = \left\{ b^2 - a^2c, ac^4 - bd^2, c^7 - d^4 \right\}$$

and

$$V_I = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ -4 \end{pmatrix} \right\}.$$

But V_I has rank 2, thus a maximal subtorus that leaves I invariant has rank 2 and must be given by a matrix D with

$$\operatorname{Ker}(D) = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3 \\ 2 \end{pmatrix} \right\}. \qquad \Diamond$$

Now we will construct descriptions of the strata that use the polytopes of the components. Let I be any $\mathcal{A}$-graded ideal. Then I lies on some component V of the toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$. By Theorem IV.3.4 there is a polytope P giving the normalisation of this component (if it is the coherent component one uses Theorem III.2.5).

Definition V.1.13. Let I be an $\mathcal{A}$-graded ideal and P the polytope corresponding to a component containing I. We denote by F_I the face of P that defines the torus orbit containing I, and by $F_I^{(1)}$ the set of primitive binomials that label an edge of F_I.

Note that F_I is not unique since I might be contained in several components, so that it is *some face corresponding to* I.

Let e be an edge of F_I. Then by Theorems II.3.10 and III.2.10 there is a Graver binomial $g_e \in \mathcal{G}(\mathcal{A})$ and an ideal $W_{g_e} \in \mathcal{H}_{\mathcal{A}}$ corresponding to that edge. Since e is an edge of F_I the wall ideal W_{g_e} is in the closure of the orbit containing I. Thus, if $g_e = \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ then without loss of generality we have

$$\boldsymbol{x}^{\boldsymbol{u}} - \alpha_{\boldsymbol{uv}} \boldsymbol{x}^{\boldsymbol{v}} \in I \quad \text{with } \alpha_{\boldsymbol{uv}} \neq 0. \tag{V.2}$$

Proposition V.1.14. *Let I be an $\mathcal{A}$-graded ideal with some corresponding face F_I. Then we have*

$$\{g_e \,|\, e \text{ is an edge of } F_I\} \subseteq \mathcal{G}_I.$$

Proof. For each edge e of F_I, the Graver binomial $g_e = \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ satisfies (V.2), hence the claim holds. □

Corollary V.1.15. *Let I be an $\mathcal{A}$-graded ideal with some corresponding face F_I. Then the span of*

$$\left\{\boldsymbol{u} - \boldsymbol{v} \,|\, \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in F_I^{(1)}\right\}$$

is a subspace of V_I. □

For a coherent ideal I there is a different description of the restriction module.

Lemma V.1.16. *Let I be a coherent $\mathcal{A}$-graded ideal and F_I the corresponding face of the state polytope of $I_{\mathcal{A}}$. Then the translation of the affine hull of F_I, so that it contains the origin, equals V_I.*

Proof. First note that if I is torus isomorphic to I' then $V_I = V_{I'}$, since non-zero $\alpha_{\boldsymbol{uv}}$ stay exactly non-zero under the torus isomorphism. Thus, we may assume $I = \text{in}_\omega(I_{\mathcal{A}})$ for all ω in the relative interior of the normal cone of F_I. Now let ω be in the relative interior of the normal cone of F_I. Then since $I = \text{in}_\omega(I_{\mathcal{A}})$ we get for all $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}(\mathcal{A})$ that

$$\omega \cdot (\boldsymbol{u} - \boldsymbol{v}) = 0 \quad \Leftrightarrow \quad \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}_I$$

holds. Furthermore, for every ω in the normal cone of F_I, *i.e.* even for the extremal rays, we have

$$\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in \mathcal{G}_I \quad \Rightarrow \quad \omega \cdot (\boldsymbol{u} - \boldsymbol{v}) = 0. \tag{V.3}$$

On the other hand, take a generating set of rays $\omega_1, ..., \omega_l$ for the normal cone of F_I and write them in a matrix W as rows. Then the kernel of W is exactly the affine hull of F_I translated so that it contains the origin. Combining this with (V.3) we get that V_I is contained in the translation of the affine hull of F_I. But by Corollary V.1.15 we have

$$\operatorname{span}\left\{\boldsymbol{u}-\boldsymbol{v} \,|\, \boldsymbol{x}^{\boldsymbol{u}}-\boldsymbol{x}^{\boldsymbol{v}} \in F_I^{(1)}\right\} \subseteq V_I \subseteq \operatorname{span}\left\{\boldsymbol{u}-\boldsymbol{v} \,|\, \boldsymbol{x}^{\boldsymbol{u}}-\boldsymbol{x}^{\boldsymbol{v}} \in F_I^{(1)}\right\}$$

and thus the equality holds. □

Now consider a non-coherent $\mathcal{A}$-graded ideal I. Then the edges of the generalised state polytope are not parallel to the difference of the exponent vectors of the corresponding Graver binomials, because they are embedded into a higher dimensional space due to the new variables $\boldsymbol{y}$ and $\boldsymbol{z}$. But still, by Corollary V.1.15 the span of the exponent vectors is contained in the restriction space. Furthermore, we even get a similar result as in the coherent case.

Lemma V.1.17. *Let I be a non-coherent $\mathcal{A}$-graded ideal and F_I the corresponding face of the polytope of some non-coherent component containing I. Then V_I is generated by the exponents of the Graver binomials corresponding to the edges of F_I.*

Proof. Let v be a vertex of F_I and $\mathcal{M}$ the corresponding monomial $\mathcal{A}$-graded ideal. We compute the generalised universal family

$$\widetilde{J_{\mathcal{M}}(p)} = \left\langle \boldsymbol{z}^{\boldsymbol{e}_j^+}\boldsymbol{y}^{\boldsymbol{e}_j^-}\cdot\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{z}^{\boldsymbol{e}_j^-}\boldsymbol{y}^{\boldsymbol{e}_j^+}\cdot\boldsymbol{x}^{\boldsymbol{n}_j} \,|\, j \in J_p\right\rangle + \left\langle \boldsymbol{x}^{\boldsymbol{m}_j} \,|\, j \notin J_p\right\rangle$$

for the component containing I and choose a weight vector ω in the relative interior of the normal cone of F_I. Then we have $I = \left(\operatorname{in}_\omega\left(\widetilde{J_{\mathcal{M}}(p)}\right)\right)_{(\boldsymbol{y})=\lambda,(\boldsymbol{z})=\mu}$ for some λ and μ in the torus by Corollary IV.3.10. Denote by

$$\mathcal{G}_\omega = \left\{\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j} \,|\, j \in J_p,\, \omega\cdot\begin{pmatrix}\boldsymbol{m}_j-\boldsymbol{n}_j\\ -\boldsymbol{e}_j\\ \boldsymbol{e}_j\end{pmatrix} = 0\right\}$$

the subset of binomial generators of $\widetilde{J_{\mathcal{M}}(p)}$, that are homogeneous with respect to ω. Note that by definition of J_p we have for each of the $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j}$ that neither $\boldsymbol{x}^{\boldsymbol{m}_j} \in I$ nor $\boldsymbol{x}^{\boldsymbol{n}_j} \in I$. Then I is generated by some monomials and the binomials in $\mathcal{G}_\omega$ with some non-zero coefficient, so that

$$V_I = \operatorname{span}\left\{\boldsymbol{m}_j - \boldsymbol{n}_j \,|\, \boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j} \in \mathcal{G}_\omega\right\}.$$

On the other hand, consider the wall ideals $W_1, ..., W_l$ corresponding to the edges of F_I emerging from v. Then again for each wall ideal W_i there is a unique Graver binomial $g_i \in \mathcal{G}(\mathcal{A})$ that is the binomial generator of W_i. But the wall ideals are constructed by flips (see II.3), so that in fact the g_i are some of the $\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{x}^{\boldsymbol{n}_j}$ from the generators of $\widetilde{J_{\mathcal{M}}(p)}$. Furthermore, the wall ideals are in the closure of the orbit of I. Thus, ω is also equal on the two terms of the binomials $\boldsymbol{z}^{\boldsymbol{e}_j^+}\boldsymbol{y}^{\boldsymbol{e}_j^-}\cdot\boldsymbol{x}^{\boldsymbol{m}_j} - \boldsymbol{z}^{\boldsymbol{e}_j^-}\boldsymbol{y}^{\boldsymbol{e}_j^+}\cdot\boldsymbol{x}^{\boldsymbol{n}_j}$ corresponding to the g_i, so that

$$\{g_1, ..., g_l\} \subseteq \mathcal{G}_\omega.$$

Now take a binomial $g = \boldsymbol{x^m} - \boldsymbol{x^n} \in \mathcal{G}_\omega \setminus \{g_1, ..., g_l\}$. Then there is an exponent e such that

$$\boldsymbol{z}^{\boldsymbol{e}^+}\boldsymbol{y}^{\boldsymbol{e}^-} \cdot \boldsymbol{x^m} - \boldsymbol{z}^{\boldsymbol{e}^-}\boldsymbol{y}^{\boldsymbol{e}^+} \cdot \boldsymbol{x^n} \in \widetilde{J_{\mathcal{M}}(p)}$$

and hence $\begin{pmatrix} \boldsymbol{m}-\boldsymbol{n} \\ -\boldsymbol{e} \\ \boldsymbol{e} \end{pmatrix}$ is in the affine hull of F_I translated to the origin. But the exponent vectors

$$\left\{ \begin{pmatrix} \boldsymbol{m}_1-\boldsymbol{n}_1 \\ -\boldsymbol{e}_1 \\ \boldsymbol{e}_1 \end{pmatrix}, ..., \begin{pmatrix} \boldsymbol{m}_l-\boldsymbol{n}_l \\ -\boldsymbol{e}_l \\ \boldsymbol{e}_l \end{pmatrix} \right\}$$

of $\{g_1, ..., g_l\}$ span the translation of the affine hull of F_I so that $\boldsymbol{m}-\boldsymbol{n}$ is a linear combination of the $\boldsymbol{m}_i - \boldsymbol{n}_i$. Therefore, we get that

$$\operatorname{span}\{\boldsymbol{m}_i - \boldsymbol{n}_i \,|\, i = 1, ..., l\} = V_I$$

already holds for the edges emerging from one vertex and, moreover, holds for the set of all edges of F_I. □

Combining the previous two lemmata we get the following description of the restriction space.

Theorem V.1.18. *Let I be an $\mathcal{A}$-graded ideal, F_I some corresponding face and denote by $\mathcal{G}_{F_I}$ the set of Graver binomials corresponding to the edges of F_I. Then the restriction space is given by*

$$V_I = \operatorname{span}\{\boldsymbol{m} - \boldsymbol{n} \,|\, \boldsymbol{x^m} - \boldsymbol{x^n} \in \mathcal{G}_{F_I}\}.$$

Proof. If I is coherent use Corollary V.1.15 and Lemma V.1.16. If I is non-coherent use Lemma V.1.17. □

Corollary V.1.19. *Let I be an $\mathcal{A}$-graded ideal and F_I some corresponding face. Take any vertex of F_I. Then the exponent vectors of the Graver binomials of the edges emerging from that vertex span V_I.* □

Remark V.1.20. Theorem V.1.18 means that the degree stratification is given by the faces of the polytopes of the components. To be precise, each face corresponds to a subset of a stratum and each stratum is then given by the collection of all faces with the same span of their edges. Whereas, for the coherent component span means the affine hull of the face, and for the non-coherent components span means the span of the difference of the exponents of the Graver binomials of the edges.

For the coherent component we get that exactly all parallel faces belong to the same degree stratum. In a non-coherent component also non-parallel faces may belong to the same stratum.

Corollary V.1.21. *Let I_1 and I_2 be two $\mathcal{A}$-graded ideals such that F_{I_1} is a face of F_{I_2}. Then V_{I_1} is a subspace of V_{I_2}.* □

Example V.1.22. We continue Example II.1.6 from Chapter II with the grading induced by $\mathcal{A} = \{1, 2, 3\}$. For this $\mathcal{A}$ there was only one component, the coherent component given by a hexagon (see Figure V.1). The degree stratification has 7 strata. The stratum S_1, that contains the monomial $\mathcal{A}$-graded ideals, on which the complete torus $(\Bbbk^*)^4$ acts invariantly, *i.e.*

$$S_1 = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6\}.$$

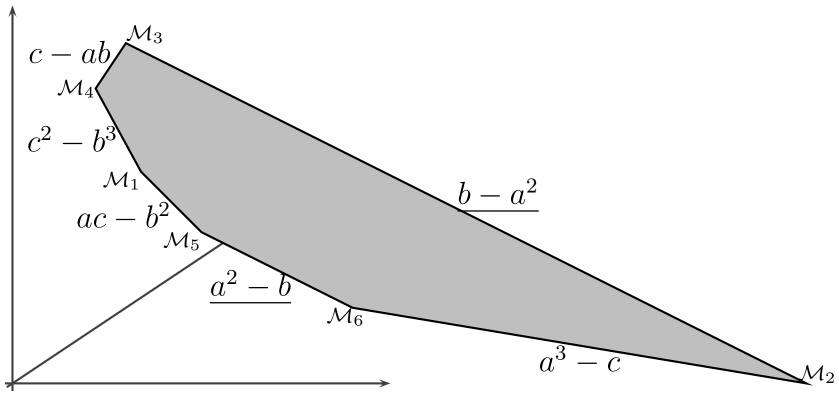

Figure V.1: The polytope $state(I_{\mathcal{A}})$ of the coherent component

The second stratum is given by the two parallel edges which are underlined in Figure V.1 so that

$$S_2 = \left\{ \left\langle a^2 - \lambda b, ab, ac, c^2 \right\rangle, \left\langle a^2 - \mu b, ab, b^2 \right\rangle \mid \lambda, \mu \in \Bbbk^* \right\},$$

i.e. it consists of the torus orbits of the two wall ideals. Thus, the restriction on the torus action to leave them invariant is given by $\left(\begin{smallmatrix} 2 \\ -1 \\ 0 \end{smallmatrix} \right)$. Then there are the strata of the remaining four edges

$$\begin{aligned} S_3 &= \left\{ \left\langle a^3 - \lambda c, b \right\rangle \mid \lambda \in \Bbbk^* \right\}, \\ S_4 &= \left\{ \left\langle c - \lambda ab, a^2 \right\rangle \mid \lambda \in \Bbbk^* \right\}, \\ S_5 &= \left\{ \left\langle c^2 - \lambda b^3, a^2, ab, ac \right\rangle \mid \lambda \in \Bbbk^* \right\}, \text{ and} \\ S_6 &= \left\{ \left\langle ac - \lambda b^2, b \right\rangle \mid \lambda \in \Bbbk^* \right\}, \end{aligned}$$

where the four restriction spaces are $\operatorname{span}\left(\begin{smallmatrix} 3 \\ 0 \\ -1 \end{smallmatrix} \right)$, $\operatorname{span}\left(\begin{smallmatrix} -1 \\ -1 \\ 1 \end{smallmatrix} \right)$, $\operatorname{span}\left(\begin{smallmatrix} 0 \\ -3 \\ 2 \end{smallmatrix} \right)$, and $\operatorname{span}\left(\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix} \right)$, respectively. And last, the two-dimensional stratum

$$S_7 = \left\{ \left\langle a^2 - \lambda b, ab - \mu c \right\rangle \mid \lambda, \mu \in \Bbbk^* \right\}$$

which is the orbit of the toric ideal, where the only non-trivial subtorus actions are one-dimensional and given by multiples of $\mathcal{A}$.

$\diamondsuit$

Now we will transfer the description of the degree stratification to the subtorus rank stratification.

Lemma V.1.23. *Let I be a coherent $\mathcal{A}$-graded ideal with some corresponding face F_I. Then the rank of a maximal subtorus action is $n - \dim(F_I)$.*

Proof. The maximal rank of a subtorus is $n - \operatorname{rank}(V_I)$. But by Lemma V.1.16 the latter is $\dim(F_I)$. □

Lemma V.1.24. *Let I be a non-coherent $\mathcal{A}$-graded ideal with some corresponding face F_I of a generalised state polytope. Then the rank of a maximal subtorus action is greater or equal to $n - \dim(F_I)$.*

Proof. Denote the edges of F_I by

$$\left\{ \begin{pmatrix} \boldsymbol{m}_1 - \boldsymbol{n}_1 \\ -\boldsymbol{e}_1 \\ \boldsymbol{e}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{m}_l - \boldsymbol{n}_l \\ -\boldsymbol{e}_l \\ \boldsymbol{e}_l \end{pmatrix} \right\}.$$

Then we have

$$\begin{aligned} \operatorname{rank}(V_I) &= \operatorname{rank}(\operatorname{span}\{\boldsymbol{m}_1 - \boldsymbol{n}_1, \ldots, \boldsymbol{m}_l - \boldsymbol{n}_l\}) \\ &\leq \operatorname{rank}\left(\operatorname{span}\left\{ \begin{pmatrix} \boldsymbol{m}_1 - \boldsymbol{n}_1 \\ -\boldsymbol{e}_1 \\ \boldsymbol{e}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{m}_l - \boldsymbol{n}_l \\ -\boldsymbol{e}_l \\ \boldsymbol{e}_l \end{pmatrix} \right\}\right) \\ &= \dim(F_I) \end{aligned}$$

and thus the claim follows. □

Remark V.1.25. For the coherent component the part of the subtorus rank stratum of rank k corresponds to the interior of all faces of codimension k. Note that it is the codimension of the face within the ambient space, not within the polytope.

For the non-coherent component this does not hold. First of all, the ambient space does not have dimension n. There the stratum corresponding to an $(n-k)$-dimensional face may have maximal subtorus rank greater than k. But the maximal rank is at least k.

Example V.1.26 (continuing **V.1.22**)**.** The subtorus rank stratification consists of three strata of maximal ranks 3, 2, and 1, corresponding to the codimension of the faces of the state polytope state$(I_{\mathcal{A}})$:

$$\begin{aligned} S_3^r &= S_1 \\ S_2^r &= S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \\ S_1^r &= S_7 \end{aligned}$$

◊

Corollary V.1.27. *Let I be a non-coherent $\mathcal{A}$-graded ideal with corresponding face F_I. If the dimension of F_I is either* 0, 1, *or* 2, *then the rank of a maximal subtorus action is $n - \dim(F_I)$.*

Proof. Denote by k the maximal rank of a subtorus action on I. We have $n \geq k$ and by Lemma V.1.24 $k \geq n - \dim(F_I)$. The case $\dim(F_I) = 0$ is trivial. For $\dim(F_I) = 1$ there is just one edge and hence exactly one Graver binomial g, whose exponent vector generates V_I. Thus, we have $\operatorname{rank}(V_I) = \dim(F_I)$. Finally, if $\dim(F_I) = 2$ then take the edge directions

$$\left\{ \begin{pmatrix} \boldsymbol{m}_1 - \boldsymbol{n}_1 \\ -\boldsymbol{e}_1 \\ \boldsymbol{e}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \boldsymbol{m}_l - \boldsymbol{n}_l \\ -\boldsymbol{e}_l \\ \boldsymbol{e}_l \end{pmatrix} \right\}$$

of F_I. Assume $\dim(\operatorname{span}\{\boldsymbol{m}_1 - \boldsymbol{n}_1, \ldots, \boldsymbol{m}_l - \boldsymbol{n}_l\}) = 1$. Then for each pair $i \neq j$ we get $\boldsymbol{m}_i - \boldsymbol{n}_i = \lambda(\boldsymbol{m}_j - \boldsymbol{n}_j)$ for $\lambda \neq 0$. But this is a contradiction since there are at least two distinct Graver binomials. □

Example V.1.28 (continuing **V.1.12**)**.** The non-coherent component was given by a three-dimensional cube. Since the opposing facets are in fact parallel one would assume that it has eight degree strata. One for the vertices, one for each of the three different edge directions, one for each of the three different pairs of parallel facets and one for the cube itself. But it turns out that it only has five degree strata. One stratum is the collection of monomial ideals

$$S_1 = \{\mathcal{M}, \mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_6\},$$

where the complete torus action of $(\mathbb{k}^*)^4$ is invariant. Then there are three strata S_2, S_3, and S_4, where each stratum contains the orbits of the four wall ideals containing $b^2 - a^2c$, $ac^4 - bd^2$, and $d^4 - c^7$, respectively. The maximal torus actions on each stratum have rank 3 and are restricted by $\begin{pmatrix} -2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -7 \\ 4 \end{pmatrix}$, respectively. The fifth stratum S_5 is given by the six two-dimensional faces and the complete cube, because each two-dimensional face and the cube have the same restriction space

$$\begin{aligned} \operatorname{span}\left\{\begin{pmatrix} -2 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \\ -2 \end{pmatrix}\right\} &= \operatorname{span}\left\{\begin{pmatrix} -2 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -7 \\ 4 \end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -7 \\ 4 \end{pmatrix}\right\} \\ &= \operatorname{span}\left\{\begin{pmatrix} -2 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -7 \\ 4 \end{pmatrix}\right\}. \end{aligned}$$

The subtorus rank stratification has only three strata, because S_1 and S_5 remain strata of their own and S_2, S_3, and S_4 become a new stratum. ♢

Therefore, we see that the maximal rank subtorus action on the torus of a non-coherent component might be higher than the maximal rank of a subtorus action on the toric ideal, *i.e.* on the torus of the coherent component. But that the maximal restriction space $V_{I_\mathcal{A}}$ can also occur on a non-coherent component, shows the next example.

Example V.1.29. We continue Example IV.1.27 from Chapter IV, where we had $\mathcal{A} = \{\binom{0}{6}, \binom{2}{4}, \binom{3}{0}, \binom{3}{7}, \binom{4}{2}, \binom{6}{1}\} \subset \mathbb{Z}^2$. Consider the non-coherent component C given by the following universal family of that component

$$\begin{aligned} \widetilde{J_\mathcal{M}(p)} = \; & \langle b^2 - y_2^2y_3^2y_4ae, cd - y_3y_4af, df^5 - y_1y_3y_4c^3e^6, \\ & a^4bce^4 - y_2y_3^2y_4d^5f, a^5be^4 - y_2y_3d^6\rangle + \\ & \langle bc^2, ac^2, c^3d, c^8e^3, d^2f^4, c^6e^5, d^3f^3, d^4f^2\rangle. \end{aligned}$$

Then for the $\mathcal{A}$-graded ideal

$$\begin{aligned} I = \; & \langle b^2 - ae, cd - af, df^5 - c^3e^6, a^4bce^4 - d^5f, a^5be^4 - d^6\rangle + \\ & \langle bc^2, ac^2, c^3d, c^8e^3, d^2f^4, c^6e^5, d^3f^3, d^4f^2\rangle. \end{aligned}$$

in the interior of the component we have

$$\mathcal{G}_\omega = \left\{b^2 - ae, cd - af, df^5 - c^3e^6, a^4bce^4 - d^5f, a^5be^4 - d^6\right\}$$

and thus

$$\begin{aligned} V_I &= \operatorname{span}\left\{\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ -6 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \\ -5 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \\ -6 \\ 4 \\ 0 \end{pmatrix}\right\} \\ &= \operatorname{span}\left\{\begin{pmatrix} 2 \\ -3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -9 \\ 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}. \end{aligned}$$

Hence, $\operatorname{rank}(V_I) = 4$ and a maximal subtorus, that has invariant action, has rank 2. But that means $V_I = V_{I_\mathcal{A}}$. ♢

V.2 Intersection Behaviour

In this section we will present some cases of intersection behaviour of components in toric Hilbert schemes. They will all be shown by examples where this behaviour appears.

Example V.2.1. In this example we will present two non-coherent components that contain exactly the same coherent monomial $\mathcal{A}$-graded ideals and their intersection is given by a facet of each of the two generalised state polytopes for the two components. For this we take $\mathcal{A} = \{\binom{0}{6}, \binom{2}{4}, \binom{3}{0}, \binom{3}{7}, \binom{4}{2}, \binom{6}{1}\}$ in $\mathbb{Z}^2$. We consider the coherent monomial $\mathcal{A}$-graded ideal

$$\begin{aligned}\mathcal{M} = \langle & e^2, ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2, b^5c^8, ad^3f^3, \\ & d^5f, b^7c^4e, d^6, b^9c^4, a^4d^2f^4, b^{10}c^3e, b^{12}c^3, a^7df^5, b^{13}c^2e, b^{15}c^2, b^{16}e, a^{11}f^6\rangle .\end{aligned}$$

The universal family $J_{\mathcal{M}}$ has 26 binomial generators and hence we have 26 new variables $y_1, .., y_{26}$ and the defining ideal with removed redundant variables is

$$\begin{aligned}I'_{\mathcal{M}} = \quad & \langle y_{18}y_{19} - y_{21}y_{22}, y_{11}y_{14} - y_{21}y_{22}, y_{14}y_{18}y_{25}y_{26} - y_{19}y_{21}y_{25}y_{26}, \\ & y_1y_{21}y_{22} - y_{25}y_{26}, y_{21}y_{22}^2y_{25}^2y_{26} - y_{19}y_{21}y_{25}y_{26}, \\ & y_{19}y_{21}y_{22}y_{25}^2y_{26} - y_{14}y_{21}y_{25}y_{26}\rangle\end{aligned}$$

in the ring $\Bbbk[y_1, y_{11}, y_{14}, y_{18}, y_{19}, y_{21}, y_{22}, y_{25}, y_{26}]$ over the remaining nine variables. Then

$$\begin{aligned}I'_{\mathcal{M}} = \langle & y_{19}^2 - y_{14}y_{22}, y_{18}y_{19} - y_{21}y_{22}, y_{11}y_{19} - y_{18}y_{22}, y_{18}^2 - y_{11}y_{21}, \\ & y_{14}y_{18} - y_{19}y_{21}, y_{11}y_{14} - y_{21}y_{22}, y_{22}^2y_{25} - y_{19}, y_{19}y_{22}y_{25} - y_{14}, \\ & y_{18}y_{22}y_{25} - y_{21}, y_{11}y_{22}y_{25} - y_{18}, y_1y_{21}y_{22} - y_{25}y_{26}, y_{22}y_{25}^2y_{26} - y_1y_{19}y_{21}, \\ & y_{19}y_{25}^2y_{26} - y_1y_{14}y_{21}, y_1y_{18}y_{22}^2 - y_{26}, y_1y_{18}y_{25}^2y_{26} - y_1^2y_{21}^2, \\ & y_1y_{11}y_{25}^2y_{26} - y_1^2y_{18}y_{21}, y_1^3y_{19}y_{21}^2y_{25} - y_1y_{25}^4y_{26}^2\rangle \cap \\ & \langle y_{11}, y_{19}, y_{22}, y_{26}\rangle \cap \langle y_{11}, y_{18}, y_{22}, y_{26}\rangle \cap \langle y_{14}, y_{18}, y_{22}, y_{26}\rangle \cap \\ & \langle y_1, y_{26}, y_{11}y_{14} - y_{21}y_{22}, y_{18}y_{19} - y_{21}y_{22}\rangle \cap \ldots\end{aligned}$$

is a primary decomposition, where we omitted the remaining 12 primary ideals. The first primary ideal gives the coherent component. We take the two primary ideals $\mathfrak{q}_1 = \langle y_{11}, y_{19}, y_{22}, y_{26}\rangle$ and $\mathfrak{q}_2 = \langle y_{11}, y_{18}, y_{22}, y_{26}\rangle$. Then we get the two non-coherent components V_1 and V_2, respectively, containing $\mathcal{M}$ which are already reduced, since the two ideals are in fact prime. Thus, we get the universal families

$$\begin{aligned}J_1 \quad = \quad & \left\langle e^2 - y_1bc^2, ad^3f^3 - y_2b^7c^3e, b^9c^4 - y_3a^3d^2f^4, b^{12}c^3 - y_5a^6df^5, \right. \\ & \left. b^{16}e - y_6a^{10}f^6\right\rangle + \\ & \left\langle ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2, \right. \\ & \left. b^5c^8, d^5f, b^7c^4e, d^6, a^4d^2f^4, b^{10}c^3e, a^7df^5, b^{13}c^2e, b^{15}c^2, a^{11}f^6\right\rangle \quad \text{and} \\ J_2 \quad = \quad & \left\langle e^2 - y_1bc^2, ad^3f^3 - y_2b^7c^3e, a^4d^2f^4 - y_4b^{10}c^2e, b^{12}c^3 - y_5a^6df^5, \right. \\ & \left. b^{16}e - y_6a^{10}f^6\right\rangle + \\ & \left\langle ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2, \right. \\ & \left. b^5c^8, d^5f, b^7c^4e, d^6, d^9c^4, b^{10}c^3e, a^7df^5, b^{13}c^2e, b^{15}c^2, a^{11}f^6\right\rangle ,\end{aligned}$$

after replacing $y_1, y_{14}, y_{18}, y_{19}, y_{21}, y_{25}$ by $y_1, y_2, y_3, y_4, y_5, y_6$. Then, on V_1 there are 4 coherent monomial ideals: $\mathcal{M}$,

$$\begin{aligned}\mathcal{M}_1 \;=\; &\langle ae, bc^2, ac^2, cd, de^2, bde, b^3d, c^8e^3, c^3e^6, be^7, c^6e^5, ce^8, b^4ce^4, d^4f^2, e^9,\\ &b^3e^6, ad^3f^3, d^5f, b^5e^5, d^6, b^7e^4, a^4d^2f^4, b^9ce^3, b^{11}ce^2, a^7df^5, b^{12}e^3,\\ &b^{14}e^2, b^{16}e, a^{11}f^6\rangle\,,\end{aligned}$$

given by $y_1 \to \infty, y_2 = 0, y_3 = 1, y_5 = 1, y_6 = 0$,

$$\begin{aligned}\mathcal{M}_2 \;=\; &\langle ae, bc^2, ac^2, cd, de^2, bde, b^3d, c^8e^3, c^3e^6, be^7, c^6e^5, ce^8, b^4ce^4, d^4f^2, e^9,\\ &b^3e^6, ad^3f^3, d^5f, b^5e^5, d^6, b^7e^4, a^4d^2f^4, b^9ce^3, b^{11}ce^2, a^7df^5, b^{12}e^3,\\ &b^{14}e^2, a^{10}f^6\rangle\,,\end{aligned}$$

given by $y_1 \to \infty, y_2 = 0, y_3 = 1, y_5 = 1, y_6 \to \infty$, and

$$\begin{aligned}\mathcal{M}_3 \;=\; &\langle e^2, ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2, b^5c^8, ad^3f^3,\\ &d^5f, b^7c^4e, d^6, b^9c^4, a^4d^2f^4, b^{10}c^3e, b^{12}c^3, a^7df^5, b^{13}c^2e, b^{15}c^2, a^{10}f^6\rangle\,,\end{aligned}$$

given by $y_1 = 0, y_2 = 0, y_3 = 0, y_5 = 0, y_6 \to \infty$. Moreover, on V_1 there are 28 non-coherent monomial ideals

$$\mathcal{M}'_1, ..., \mathcal{M}'_{28}.$$

The non-coherent monomial ideals on V_2 are also 28,

$$\mathcal{M}''_1, ..., \mathcal{M}''_{28}.$$

The two components have only 16 non-coherent monomial ideals in common, the other 12 are different on V_1 and V_2, for example we have $\mathcal{M}'_1 \notin V_2$ and $\mathcal{M}''_1 \notin V_1$, for

$$\begin{aligned}\mathcal{M}'_1 \;=\; &\langle e^2, ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2, b^5c^8, d^5f,\\ &b^7c^3e, d^6, a^2d^3f^3, b^{12}c^3, a^7df^5, b^{13}c^2e, b^{15}c^2, b^{16}e, a^{11}f^6\rangle \quad \text{and}\\ \mathcal{M}''_1 \;=\; &\langle e^2, ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2, b^5c^8, d^5f,\\ &b^7c^3e, d^6, a^2d^3f^3, b^9c^4, b^{10}c^2e, a^5d^2f^4, b^{12}c^3, a^7df^5, b^{15}c^2, b^{16}e, a^{11}f^6\rangle\,.\end{aligned}$$

But the coherent monomial ideals on V_2 are the same as on V_1, i.e

$$\{\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}\,,$$

because they are given by the same values for y_1, y_2, y_5, y_6 and for $y_4 = 0$.

Moreover, the 20 common monomial ideals are all the monomial ideals in the closure of the orbit of

$$\begin{aligned}I = \;\; &\langle e^2 - bc^2, ad^3f^3 - b^7c^3e, b^9c^4, b^{12}c^3 - a^6df^5, b^{16}e - a^{10}f^6\rangle +\\ &\langle ae, ac^2, cd, bde, b^3d, bc^{10}e, b^3c^9, b^4c^6e, b^2c^{12}, b^6c^5, d^4f^2,\\ &b^5c^8, d^5f, b^7c^4e, d^6, a^4d^2f^4, b^{10}c^3e, a^7df^5, b^{13}c^2e, b^{15}c^2, a^{11}f^6\rangle\;.\end{aligned}$$

Furthermore, the ideal I itself lies on both components, because it is given by $y_1 = y_2 = y_5 = y_6 = 1, y_3 = y_4 = 0$. Thus, the intersection of V_1 and V_2 is the closure of the orbit of I, which is a codimension-one face of both generalised state polytope. In fact, one can compute that the two cones in the normal fans of the state polytopes are one-dimensional up to the respective lineality space of the two normal fans. $\diamondsuit$

Example V.2.2. Now we give an example of an embedded component whose generalised state polytope is a facet of the generalised state polytope of the component in which it is embedded. For this we take again the grading by $\mathcal{A} = \{\binom{0}{6}, \binom{2}{4}, \binom{3}{0}, \binom{3}{7}, \binom{4}{2}, \binom{6}{1}\}$ in $\mathbb{Z}^2$. We consider the non-coherent monomial $\mathcal{A}$-graded ideal

$$\begin{aligned}\mathcal{M} = \ & \langle be, bc^2, b^2, cd, ac^4, abcf, a^2c^3f, a^2bf^2, a^3c^2f^2, c^8e^3, c^3e^6, bf^6, a^4cf^3, \\ & ac^2e^6, bdf^5, a^2ce^6, bd^2f^4, e^9, a^3e^6, a^3c^3e^5, a^5c^3e^3, d^4e^3, bd^4f^2, a^6c^2e^3, \\ & bd^5f, bd^6, a^7ce^3f, bd^6, a^7ce^3f, a^8e^3f^2, d^{12}\rangle\,.\end{aligned}$$

The universal family $J_{\mathcal{M}}$ has 29 binomial generators and thus there are 29 new variables $y_1, .., y_{29}$. After removing the redundant variables, the defining ideal

$$\begin{aligned}I'_{\mathcal{M}} \ = \ & \langle y_{13}y_{28}y_{29} - y_{20}y_{23}, y_{12}y_{20}^4y_{23}^4y_{28}^2y_{29} - y_{23}y_{28}^2y_{29}, \\ & y_{12}y_{13}y_{20}^5y_{23}^5 - y_{13}y_{20}y_{23}^2\rangle\end{aligned}$$

in $\Bbbk[y_{12}, y_{13}, y_{20}, y_{23}, y_{28}, y_{29}]$ has the following primary decomposition

$$\begin{aligned}I'_{\mathcal{M}} = \ & \langle y_{20}^2, y_{20}y_{28}, y_{28}^2, y_{13}y_{20}, y_{13}^2, y_{13}y_{28}y_{29} - y_{20}y_{23}\rangle \cap \langle y_{20}, y_{28}\rangle \cap \\ & \langle y_{20}, y_{29}\rangle \cap \langle y_{23}^2, y_{23}y_{28}, y_{28}^2, y_{13}y_{28}y_{29} - y_{20}y_{23}\rangle \cap \\ & \langle y_{13}y_{28}y_{29} - y_{20}y_{23}, y_{12}y_{20}^4y_{23}^3 - 1\rangle \cap \langle y_{13}, y_{23}\rangle \cap \\ & \langle y_{13}^3, y_{13}^2y_{23}, y_{13}y_{23}^2, y_{23}^3y_{23}^2y_{28}, y_{23}y_{28}^2, y_{28}^3, y_{13}y_{28}y_{29} - y_{20}y_{23}\rangle\,.\end{aligned}$$

Let $q_1 = \langle y_{20}^2, y_{20}y_{28}, y_{28}^2, y_{13}y_{20}, y_{13}^2, y_{13}y_{28}y_{29} - y_{20}y_{23}\rangle$ and $q_2 = \langle y_{20}, y_{28}\rangle$ be two primary ideals of the decomposition. They give two non-coherent components V_1 and V_2 containing $\mathcal{M}$ and we compute their reduction. Hence, we take their radicals:

$$\begin{aligned}p_1 &= \sqrt{q_1} = \langle y_{13}, y_{20}, y_{28}\rangle \\ p_2 &= \sqrt{q_2} = q_2 = \langle y_{20}, y_{28}\rangle\end{aligned}$$

Thus, the two universal families are

$$\begin{aligned}J_1 \ = \ & \left\langle bf^6 - y_1c^6e^5, bd^4f^2 - y_3a^4c^2e^5, d^{12} - y_4a^{13}f^6\right\rangle + \\ & \left\langle be, bc^2, b^2, cd, ac^4, abcf, a^2c^3f, a^2bf^2, a^3c^2f^2, c^8e^3, c^3e^6, a^4cf^3, ac^2e^6, \right. \\ & bdf^5, a^2ce^6, bd^2f^4, e^9, a^3e^6, a^3c^3e^5, a^5c^3e^3, d^4e^3, a^6c^2e^3, bd^5f, bd^6, \\ & \left. a^7ce^3f, bd^6, a^7ce^3f, a^8e^3f^2\right\rangle \quad \text{and} \\ J_2 \ = \ & \left\langle bf^6 - y_1c^6e^5, a^4cf^3 - y_2d^3e^3, bd^4f^2 - y_3a^4c^2e^5, d^{12} - y_4a^{13}f^6\right\rangle + \\ & \left\langle be, bc^2, b^2, cd, ac^4, abcf, a^2c^3f, a^2bf^2, a^3c^2f^2, c^8e^3, c^3e^6, ac^2e^6, \right. \\ & bdf^5, a^2ce^6, bd^2f^4, e^9, a^3e^6, a^3c^3e^5, a^5c^3e^3, d^4e^3, a^6c^2e^3, bd^5f, bd^6, \\ & \left. a^7ce^3f, bd^6, a^7ce^3f, a^8e^3f^2\right\rangle,\end{aligned}$$

respectively, after substituting the remaining four variables $y_{12}, y_{13}, y_{23}, y_{29}$ by y_1, y_2, y_3, y_4. But this means V_1 is an embedded component in V_2. Furthermore, one can compute that the state polytope of J_2 is a four-dimensional hypercube and that the state polytope of J_1 is a three-dimensional cube which is a facet of state(J_2).

$\lozenge$

Example V.2.3. Here we give an example of an embedded component in the coherent component. Again, let $\mathcal{A} = \{\binom{0}{6}, \binom{2}{4}, \binom{3}{0}, \binom{3}{7}, \binom{4}{2}, \binom{6}{1}\}$ in $\mathbb{Z}^2$. We consider the coherent monomial $\mathcal{A}$-graded ideal

$$\mathcal{M} = \langle ae, bc^2, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, ab^3f^2, c^8e^3, b^5f^2, c^3e^6, d^2f^4, c^6e^5, adf^5, ce^8, b^4ce^4, a^2f^6, d^4f^2, b^6ce^3, b^9c, ab^2f^6, ab^9, b^8ce^2, b^4f^6, d^6e\rangle .$$

The universal family $J_{\mathcal{M}}$ has 27 binomial generators and hence we have 27 new variables $y_1, .., y_{27}$. The defining ideal with removed redundant variables is

$$I'_{\mathcal{M}} = \langle y_{16}y_{17} - y_{24}y_{27}, y_{24}^2y_{25}y_{27}^2 - y_{12}y_{17}, y_{12}y_{16}y_{24}y_{25}y_{27}^2 - y_{12}^2y_{27}\rangle$$

in the ring $\Bbbk[y_{12}, y_{16}, y_{17}, y_{24}, y_{25}, y_{27}]$ over the remaining six variables. Then

$$I'_{\mathcal{M}} = \langle y_{16}y_{17} - y_{24}y_{27}, y_{16}y_{24}y_{25}y_{27} - y_{12}\rangle \cap \langle y_{17}, y_{27}\rangle \cap \langle y_{12}^2, y_{17}, y_{24}\rangle$$

is a primary decomposition. Note that the first primary ideal gives the coherent component. We take

$$\mathfrak{q} = \langle y_{12}^2, y_{17}, y_{24}\rangle .$$

Then we get a non-coherent component V which is not reduced. If we take the radical $\mathfrak{p} := \sqrt{\mathfrak{q}} = \langle y_{12}, y_{17}, y_{24}\rangle$ then the reduced structure of V is given by the universal family

$$J_{\mathcal{M}}(\mathfrak{p}) = \langle adf^5 - y_1bce^7, b^8ce^2 - y_2a^2d^3f^3, d^6e - y_3b^{11}\rangle + \langle ae, bc^2, ac^2, cd, abcf, a^2cf, b^3cf, a^2bf^2, a^3f^2, ab^3f^2, c^8e^3, b^5f^2, c^3e^6, d^2f^4, c^6e^5, ce^8, b^4ce^4, a^2f^6, d^4f^2, b^6ce^3, b^9c, ab^2f^6, ab^9, b^4f^6\rangle ,$$

after replacing y_{16}, y_{25}, y_{27} by y_1, y_2, y_3. The closure of the orbit of the general point

$$I = J_{\mathcal{M}}(\mathfrak{p})_{(y_1=y_2=y_3=1)}$$

in V contains 8 monomial ideals. These are given by the following limits in the universal family:

$$\begin{aligned} \mathcal{M} &= J_{\mathcal{M}}(\mathfrak{p})_{(y_1=y_2=y_3=0)} & \mathcal{M}_1 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_1=y_2=0, y_3\to\infty)}, \\ \mathcal{M}_2 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_1=y_3=0, y_2\to\infty)}, & \mathcal{M}_3 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_2=y_3=0, y_1\to\infty)}, \\ \mathcal{M}_4 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_1=0, y_2, y_3\to\infty)}, & \mathcal{M}_5 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_2=0, y_1, y_3\to\infty)}, \\ \mathcal{M}_6 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_3=0, y_1, y_2\to\infty)}, \text{ and } & \mathcal{M}_7 &= J_{\mathcal{M}}(\mathfrak{p})_{(y_1, y_2, y_3\to\infty)} \end{aligned}$$

One can also describe these monomial ideals as initial ideals of the generalised universal family $\widetilde{J_{\mathcal{M}}(\mathfrak{p})}$. Then we get the following correspondence between weight vectors on $\Bbbk[a, ..., f, y_1, y_2, y_3, z_1, z_2, z_3]$ and monomial ideals:

$$\begin{aligned} \mathcal{M} &\leftrightarrow (\mathbf{0},0,0,0,1,1,1), & \mathcal{M}_1 &\leftrightarrow (\mathbf{0},0,0,1,1,1,0), \\ \mathcal{M}_2 &\leftrightarrow (\mathbf{0},0,1,0,1,0,1), & \mathcal{M}_3 &\leftrightarrow (\mathbf{0},1,0,0,0,1,1), \\ \mathcal{M}_4 &\leftrightarrow (\mathbf{0},0,1,1,1,0,0), & \mathcal{M}_5 &\leftrightarrow (\mathbf{0},1,0,1,0,1,0), \\ \mathcal{M}_6 &\leftrightarrow (\mathbf{0},1,1,0,0,0,1), \text{ and } & \mathcal{M}_7 &\leftrightarrow (\mathbf{0},1,1,1,0,0,0), \end{aligned}$$

where $\mathbf{0}$ is the zero weight vector on $a, ..., f$. Note that all of these ideals are coherent. In fact, even I is coherent, and it corresponds to a three-dimensional face, *i.e.* a facet, of the state polytope state$(I_{\mathcal{A}})$. Therefore, the coherent component contains an embedded component which is given by a facet of the state polytope.

$\diamondsuit$

V.3 Outlook

So far, we have computed amongst others the above examples and found the presented characteristics. While doing this we have also been looking for other characteristics while going through different $\mathcal{A}$-gradings. For example: Are there non-coherent components consisting just of one monomial ideal? Are there two non-coherent components that have exactly the same monomial ideals? This would in particular mean that a non-coherent component would not be determined by its monomial $\mathcal{A}$-graded ideals.

We have seen that the coherent component is given by the Gröbner fan, *i.e.* a state polytope, of the toric ideal $I_{\mathcal{A}}$, and each non-coherent component V_p by the Gröbner fan of a generalised universal family $\widetilde{J_{\mathcal{M}}(p)}$. This means the normalisation of V_p is given by the toric variety associated to a state polytope of the binomial ideal $\widetilde{J_{\mathcal{M}}(p)}$. Then the question arises whether this component may also be realised as the coherent component of the toric Hilbert scheme of some other $\mathcal{A}'$. If this is the case it would be interesting to know if there is a connection between the non-coherent components of $\mathcal{H}_{\mathcal{A}'}$ and the other components of $\mathcal{H}_{\mathcal{A}}$ besides V_p. For example, $\mathcal{H}_{\mathcal{A}'}$ could be a local chart around V_p.

To investigate on this there are several steps to study. First of all, the natural candidate for the toric ideal of $\mathcal{A}'$ would be

$$I := \left\langle \boldsymbol{z}^{\boldsymbol{e}_i^+} \boldsymbol{y}^{\boldsymbol{e}_i^-} \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{e}_i^-} \boldsymbol{y}^{\boldsymbol{e}_i^+} \boldsymbol{x}^{\boldsymbol{n}_i} \right\rangle \subseteq \widetilde{J_{\mathcal{M}}(p)},$$

where these are the binomial generators of $\widetilde{J_{\mathcal{M}}(p)}$. Then one has to check under which conditions

$$\operatorname{state}(I) = \operatorname{state}(\widetilde{J_{\mathcal{M}}(p)})$$

holds. Furthermore, I need not be a toric ideal. To check if I is a toric ideal and find the corresponding $\mathcal{A}'$ in this case one can use the criterion in [Alt00].

Conjecture V.3.1. *If for a generalised universal family $\widetilde{J_{\mathcal{M}}(p)}$ the ideal generated by its binomial generators*

$$I := \left\langle \boldsymbol{z}^{\boldsymbol{e}_i^+} \boldsymbol{y}^{\boldsymbol{e}_i^-} \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{e}_i^-} \boldsymbol{y}^{\boldsymbol{e}_i^+} \boldsymbol{x}^{\boldsymbol{n}_i} \right\rangle,$$

is a toric ideal for some $\mathcal{A}'$ and $\operatorname{state}(I) = \operatorname{state}(\widetilde{J_{\mathcal{M}}(p)})$ *then*

$$\mathcal{H}_{\mathcal{A}'} \cong V_p,$$

i.e. the toric Hilbert scheme of $\mathcal{A}'$ consists only of the coherent component which is isomorphic to the reduced non-coherent component V_p.

Chapter VI

Stable Toric Pairs

In his work *Complete moduli in the presence of semiabelian group action* [Ale02] Alexeev constructs the so-called *stable toric pairs*. Such a stable toric pair consists of a polarised stable toric variety together with a Cartier divisor on it. Polarised stable toric varieties arise from convex rational polytopes as the projective variety associated to the cone over the polytope. For a polytope he constructs a proper Artin stack of stable toric pairs and shows that this stack has a coarse moduli space. We will show that a subset of the toric Hilbert scheme consists of certain stable toric pairs. Furthermore, we construct a connection between the toric Hilbert scheme and a coarse moduli space of stable toric pairs.

For most of the first part of this chapter we will cite [Ale02] and follow his definitions and constructions. First of all, we start by introducing affine stable toric varieties as they are the basis for constructing stable toric pairs. Throughout this chapter we use the following notations:

Let $M' \cong \mathbb{Z}^{d-1}$ be a lattice with associated vector space $M'_{\mathbb{Q}} := M' \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. This lattice is embedded in the lattice $M := M' \oplus \mathbb{Z} \cong \mathbb{Z}^d$ with its associated vector space $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. We denote the embedding of M' into M on height 1 by

$$\begin{array}{rccc} \Phi : & M' & \to & M \\ & m & \mapsto & (m,1)\,. \end{array}$$

By abuse of notation we will also write Φ for $\Phi \otimes_{\mathbb{Z}} \mathbb{Q} : M'_{\mathbb{Q}} \to M_{\mathbb{Q}}$, the associated embedding over $\mathbb{Q}$.

Furthermore, we will work with cones and polytopes. A convex rational polyhedral cone is the positive hull (denoted by Cone) of a finite set of points in $M_{\mathbb{Q}}$ and a convex rational polytope is the convex hull (denoted by ConvHull) of a finite set of points in $M'_{\mathbb{Q}}$. For more details on convex polyhedral objects see [Zie95].

VI.1 Affine Stable Toric Varieties

Definition VI.1.1. A *complex Δ of lattice polytopes* in $M'_{\mathbb{Q}}$ is a finite collection $\Delta = \{\delta_i\} \subset M'_{\mathbb{Q}}$ of polytopes such that

- all vertices are in M',
- for any $\delta \in \Delta$ all faces of δ are in Δ, and

- for $\delta_1, \delta_2 \in \Delta$ we have that $\delta_1 \cap \delta_2$ is a face of each.

A *fan* Σ in $M_{\mathbb{Q}}$ is a finite collection $\Sigma = \{\sigma_i\} \subset M_{\mathbb{Q}}$ of convex rational polyhedral cones such that

- for any $\sigma \in \Sigma$ all faces of σ are in Σ, and
- for $\sigma_1, \sigma_2 \in \Sigma$ we have that $\sigma_1 \cap \sigma_2$ is a face of each.

Note that every complex of lattice polytopes and every fan is equipped with a partial order, *i.e.* for two polytopes δ_1, δ_2 we say $\delta_1 \geq \delta_2$ if δ_2 is face of δ_1. A fan is equipped with the same partial order. This defines a poset structure on complexes of lattice polytopes and on fans.

Definition VI.1.2. Let Δ be a complex of lattice polytopes or a fan. Then we denote the *support* of Δ by

$$|\Delta| := \bigcup_{\delta \in \Delta} \delta.$$

Note that the support of a complex or a fan need not be convex.

Definition VI.1.3. A *pointed cell complex* is a complex of lattice polytopes Δ together with a set of points $P \subset M' \cap \left(\bigcup_{\delta \in \Delta} \delta\right)$ such that for each $\delta \in \Delta$ the vertices of δ are contained in $P \cap \delta$.

A *(pointed) cell decomposition of a polytope* Q is a (pointed) cell complex Δ such that the support of Δ equals Q.

Recall from toric geometry (see [Oda88], [Ful93], or [CLS]), that for a rational polyhedral cone $\sigma \subset M_{\mathbb{Q}}$ the intersection of that cone with the lattice M gives a semigroup with corresponding semigroup algebra:

$$S_\sigma := M \cap \sigma \quad R_\sigma := \mathbb{k}[S_\sigma].$$

Definition VI.1.4. Let $\sigma \subset M_{\mathbb{Q}}$ be a cone. Then we denote the associated lattice by $M_\sigma := M \cap (\mathbb{Q} \cdot \sigma)$ and the associated torus by $T_\sigma := \operatorname{Hom}(M_\sigma, \mathbb{k}^*)$. For a polytope $\delta \subset M'$ we define by $M_\delta := M \cap (\mathbb{Q} \cdot \Phi(\delta))$ the associated sublattice in M and the associated torus by $T_\delta := \operatorname{Hom}(M_\delta, \mathbb{k}^*)$. For a complex of lattice polytope or a fan we get an associated torus for each polytope or cone and projection maps $\phi_{j,i} : T_{\sigma_j} \to T_{\sigma_i}$, whenever $\sigma_j \geq \sigma_i$, induced by $M_{\sigma_i} \subseteq M_{\sigma_j}$.

Definition VI.1.5. Let $\Sigma = \{\sigma_i\}$ be a fan in $M_{\mathbb{Q}}$. Then we define the sheaf $\underline{R}[\Sigma]$ on the poset structure of Σ by

$$\underline{R}[\Sigma](\sigma) = R_\sigma = \mathbb{k}[M \cap \sigma] \quad \text{for each } \sigma \in \Sigma$$

with epimorphisms

$$\begin{aligned} \mathrm{pr}_{i,j} : \; R_{\sigma_j} &\to R_{\sigma_i} \\ \boldsymbol{x}^{\boldsymbol{s}} &\mapsto \begin{cases} \boldsymbol{x}^{\boldsymbol{s}} & \boldsymbol{s} \in S_{\sigma_i} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for all $\sigma_j \geq \sigma_i$.

Lemma VI.1.6. *The module of global sections $H^0(\Sigma, \underline{R}[\Sigma])$ is a free module with basis*

$$\left\{\boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{s} \in \lim_{\to \sigma} S_\sigma\right\}$$

and multiplication

$$\boldsymbol{x}^{\boldsymbol{s}} \cdot \boldsymbol{x}^{\boldsymbol{t}} = \begin{cases} \boldsymbol{x}^{\boldsymbol{s}+\boldsymbol{t}} & \boldsymbol{s}, \boldsymbol{t} \in S_\sigma \text{ for some } \sigma \in \Sigma \\ 0 & \text{otherwise} \end{cases}.$$

Proof. See [Ale02, Lemma 2.3.2]. □

Remark VI.1.7. Note that the basis of $H^0(\Sigma, \underline{R}[\Sigma])$ is exactly $\{\boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{s} \in |\sigma|\}$, since the direct limit of the S_σ just identifies lattice points on the intersection of cones.

Definition VI.1.8. Let Σ be a fan and $\underline{R}[\Sigma]$ be the sheaf constructed in Definition VI.1.5. Then we define an algebra $R[\Sigma] := H^0(\Sigma, \underline{R}[\Sigma])$ and the affine variety $\mathbb{A}[\Sigma] := \operatorname{Spec} R[\Sigma]$.

Example VI.1.9. Let $M = \mathbb{Z}^2$ and consider the fan Σ given by the two cones $\sigma_1 := \operatorname{Cone}\{\binom{1}{0}, \binom{1}{1}\}$, $\sigma_2 := \operatorname{Cone}\{\binom{1}{1}, \binom{0}{1}\}$, and all their faces. Denote their intersection by $\tau = \operatorname{Cone}\{\binom{1}{1}\}$ and the two other faces by $\tau_1 \subset \sigma_1$ and $\tau_2 \subset \sigma_2$, see Figure VI.1. Then we have

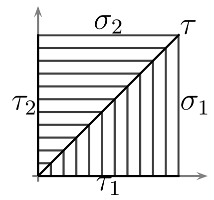

Figure VI.1: The fan Σ

$$R_{\sigma_1} = \Bbbk[x_1, x_2], \quad R_{\sigma_2} = \Bbbk[x_2, x_3],$$
$$R_{\tau_1} = \Bbbk[x_1], \quad R_\tau = \Bbbk[x_2], \quad \text{and } R_{\tau_2} = \Bbbk[x_3].$$

Thus we get

$$R[\Sigma] = \Bbbk[x_1, x_2, x_3] / (x_1 x_3)$$

and hence that $\mathbb{A}[\Sigma]$ is the union of two planes which intersect in a line.

◊

Now we will define a twisted version of $R[\Sigma]$ by using cocycles in the torus.

Definition VI.1.10. Let Σ be a fan in $M_{\mathbb{Q}}$ or a complex of lattice polytopes in $M'_{\mathbb{Q}}$. Then a collection $t := \left\{t_{i,j} \in T_{\sigma_j} \mid \sigma_j \geq \sigma_i\right\}$ of torus elements is called a *cocycle for* Σ if

$$t_{k,i} = \phi_{j,i}(t_{k,j}) \cdot t_{j,i} \quad \text{in } T_{\sigma_i}$$

for all i, j, k with $\sigma_k \geq \sigma_j \geq \sigma_i$.

Definition VI.1.11. Let Σ be a fan in $M_{\mathbb{Q}}$ and t a cocycle for Σ. Then we define the twisted sheaf $\underline{R}[\Sigma, t]$ with the same sections $\underline{R}[\Sigma, t](\sigma) := \underline{R}[\Sigma](\sigma) = R_\sigma$ as before, but with twisted homomorphisms

$$p_{j,i} := t_{j,i} \cdot \mathrm{pr}_{j,i} .$$

Furthermore, we define the global section algebra and corresponding variety

$$R[\Sigma, t] = H^0 (\Sigma, \underline{R}[\Sigma, t]) , \quad \mathbb{A}[\Sigma, t] = \operatorname{Spec} R[\Sigma, t].$$

In fact the original algebra and variety from Definition VI.1.8 are just a special case of these since $R[\Sigma] = R[\Sigma, 1]$.

Remark VI.1.12. Note that by construction we have

$$\dim_{\Bbbk} (R[\Sigma, t]_a) = \begin{cases} 1 & a \in |\Sigma| \cap M \\ 0 & \text{otherwise} \end{cases} .$$

The varieties $\mathbb{A}[\Sigma, t]$ are a certain type of affine toric varieties, the so-called affine stable toric varieties. For this, Alexeev denotes the notion of a seminormal variety in [Ale02, Definition 1.1.6].

Definition VI.1.13. A reduced variety X is called *seminormal* if every finite bijective morphism to X is an isomorphism.

Then he uses this notion of seminormality to define stable semiabelic varieties [Ale02, Definition 1.1.5], where we state the definition in the toric case as we only need this one.

Definition VI.1.14. A *stable toric variety* is a seminormal toric variety X with

- finitely many orbits and
- the stabiliser of every point is connected.

Theorem VI.1.15. *For every fan Σ and a cocycle t on it the variety $\mathbb{A}[\Sigma, t]$ is an affine stable toric variety. Furthermore, every affine stable toric variety is isomorphic to $\mathbb{A}[\Sigma, t]$ for some Σ and t.*

Proof. See Lemma 2.3.11 and Theorem 2.3.14 in [Ale02]. □

VI.2 Projective Stable Toric Varieties

Before we will construct the projective version of the stable toric varieties, recall the following.

Definition VI.2.1. A *polarised toric variety* is a projective toric variety with an ample invertible sheaf.

To construct the projective version we will recall the correspondence between polytopes and projective toric varieties. In fact, each lattice polytope $\delta \subset M'_{\mathbb{Q}}$ defines a projective toric variety with a line bundle L and an action of the torus $T = \operatorname{Hom}(M, \Bbbk^*)$ on L. For this, embed δ with Φ into $M_{\mathbb{Q}}$ on height 1 and take the cone over the embedded polytope.

Definition VI.2.2. Let δ be a polytope in $M'_{\mathbb{Q}}$. Then we denote the cone over the embedded polytope $\Phi(\delta)$ in $M_{\mathbb{Q}}$ by

$$C_\delta = \{\lambda \cdot (d, 1) \mid d \in \delta, \lambda \in \mathbb{Q}_{\geq 0}\} \subset M_{\mathbb{Q}}.$$

For a complex of polytopes we define the *associated fan* by

$$C_\Delta = \{C_\delta \mid \delta \in \Delta\}.$$

Using the cone C_δ over the polytope δ we get the semigroup algebra

$$S_\delta := \Bbbk[C_\delta \cap M]$$

and the associated projective toric variety

$$\mathbb{P}(\delta) := \operatorname{Proj} \Bbbk[S_\delta]$$

with the linearised ample invertible sheaf $L_\delta = \mathcal{O}(1)$.

Definition VI.2.3. We say that a polytope $\delta \subset M'_{\mathbb{Q}}$ is *normal* if the semigroup $C_\delta \cap M$ is generated by

$$\{(p, 1) \mid p \in \delta \cap M'\},$$

i.e. by the lattice points at height 1 of C_δ.

Before we extend the construction to complexes of lattice polytopes, note that a cocycle for Δ is also a cocyle for C_Δ.

Definition VI.2.4. Let Δ be a complex of lattice polytopes in $M'_{\mathbb{Q}}$ and t a cocycle for Δ. Then we define an algebra

$$R[\Delta, t] := R[C_\Delta, t]$$

with associated projective variety

$$\mathbb{P}[\Delta, t] := \operatorname{Proj} R[\Delta, t]$$

and an ample sheaf $L[\Delta, t]$ on it as $\mathcal{O}(1)$.

Remark. In [Ale02], Alexeev considers the torus T as a sheaf on the poset Δ and he uses cohomology classes $[\tau] \in H^1(\Delta, T)$ instead of the cocycles t. But for the algebra he then uses a representative τ which is a cocycle for C_Δ.

Remark VI.2.5. The polytopes $\delta \in \Delta$ need not be normal so that the algebras $\Bbbk[C_\delta \cap M]$ and $R[\Delta, t]$ are not necessarily generated in degree 1. But they are finite over the elements of degree 1 though and thus $\mathcal{O}(1)$ is an invertible sheaf, *i.e.* a line bundle.

Example VI.2.6. Let $M' = \mathbb{Z}^2$ and consider the complex Δ consisting of the two polytopes

$$\delta_1 := \operatorname{ConvHull}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)\} \text{ and}$$
$$\delta_2 := \operatorname{ConvHull}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)\},$$

and all their faces, which is the two-dimensional crosspolytope divided by a diagonal (See Figure VI.2). Then C_Δ is given by the two cones $C_{\delta_1}, C_{\delta_2}$ and all their

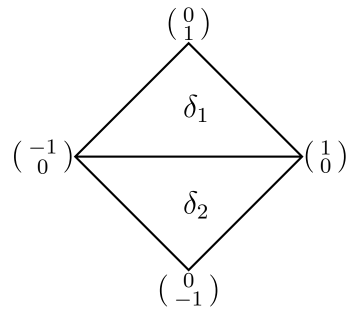

Figure VI.2: The cell complex Δ

faces. The semigroup $C_{\delta_1} \cap M$ is generated by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and the semigroup $C_{\delta_2} \cap M$ by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. If we identify $x_1, ..., x_5$ with $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively, we get

$$S_{\delta_1} = \Bbbk[x_1, x_2, x_3, x_5] / \left(x_1 x_3 - x_5^2 \right), \quad S_{\delta_2} = \Bbbk[x_1, x_3, x_4, x_5] / \left(x_1 x_3 - x_5^2 \right),$$
$$\text{and} \qquad S_{\delta_1 \cap \delta_2} = \Bbbk[x_1, x_3, x_5] / \left(x_1 x_3 - x_5^2 \right).$$

Thus, we have

$$R[\Delta, 1] = \Bbbk[x_1, x_2, x_3, x_4, x_5] / \left(x_2 x_4, x_1 x_3 - x_5^2 \right).$$

$\Diamond$

The analogon of Theorem VI.1.15 follows now for the projective case.

Theorem VI.2.7. *For every complex of lattice polytopes Δ with cocycle t on it the projective variety $\mathbb{P}[\Delta, t]$ together with the ample sheaf $L[\Delta, t]$ is a polarised stable toric variety with T-linearised action on $L[\Delta, t]$. Furthermore, as a topological space we have*

$$\mathbb{P}[\Delta, t] = \lim_{\to \delta \in \Delta} \mathbb{P}(\delta).$$

Proof. See Theorem 2.4.4 and Corollary 2.4.5 in [Ale02]. □

This means as in the affine case the projective variety $\mathbb{P}[\Delta, t]$ is glued from the irreducible varieties $\mathbb{P}(\delta)$ along the morphisms arising from the poset structure of the complex. Moreover, we get the same characterisation of stable toric varieties.

Theorem VI.2.8. *Every polarised stable toric variety $(\mathbb{P}, L)$ with linearised ample sheaf L is isomorphic to $\mathbb{P}[\Delta, t]$ with $L[\Delta, t]$ for some complex of lattice polytopes Δ with a cocycle t.*

Proof. See [Ale02, Theorem 2.4.7]. □

VI.3 Stable Toric Pairs and their Moduli Space

Now the polarised stable toric varieties will be specified further. For this we consider pairs $(\mathbb{P}, \Theta)$ of a projective toric variety $\mathbb{P}$ and a Cartier divisor Θ on it which is equivalent to a triple $(\mathbb{P}, L, \theta)$, where $L = \mathcal{O}(\Theta)$ and $\theta \neq 0$ in $H^0(\mathbb{P}, L)$ is an equation of Θ. This means we have to add a section θ to a polarised stable toric variety to get such a pair. However, in order that Θ does not contain any T-orbits we need to formulate a condition on θ.

We start with an irreducible $\mathbb{P}$. Hence, $(\mathbb{P}, L) = (\mathbb{P}(\delta), L_\delta)$ for some lattice polytope δ. This means we have the canonical decomposition of the module of global sections

$$H^0(\mathbb{P}(\delta), L_\delta) = \bigoplus_{m \in M \cap \delta} \Bbbk \cdot \boldsymbol{x}^m$$

and thus an equation θ of a Cartier divisor can be written as

$$\theta = \sum_{m \in M \cap \delta} e_m \boldsymbol{x}^m.$$

This allows us to formulate the condition on θ.

Lemma VI.3.1. *The divisor Θ does not contain any T-orbit if and only if $e_m \neq 0$ for all vertices of δ.*

Proof. See [Ale02, Lemma 2.6.1]. □

To have a more geometric version which we will also be using in the general case we define the following subset of lattice points given by a divisor.

Definition VI.3.2. Denote the set of non-zero coefficients of a divisor Θ by $C(\Theta) = \{m \in M \mid e_m \neq 0\}$.

With this description Lemma VI.3.1 reformulates to the following:

Corollary VI.3.3. *The divisor Θ does not contain any T-orbit entirely if and only if the vertices of δ are contained in $C(\Theta)$ or equivalently δ is the convex hull of $C(\Theta)$.* □

Now we take a general $(\mathbb{P}, L)$, *i.e.* $(\mathbb{P}, L)[\Delta, t]$. Then the equation θ of a divisor can be written as

$$\theta = \sum_{m \in M \cap \Delta} e_m \boldsymbol{x}^m$$

and thus the definition of the set $C(\Theta)$ as before is also suitable in the general case. Then again we get a similar condition for the divisor Θ not to contain any T-orbits.

Corollary VI.3.4. *A divisor Θ on a polarised stable toric variety $(\mathbb{P}, L)[\Delta, t]$ does not contain any T-orbits exactly if all vertices of the polytopes of the complex Δ are contained in $C(\Theta)$.* □

This is a geometrically suitable criterion, so that we can define what stable toric pairs are.

Definition VI.3.5. A *stable toric pair* $(\mathbb{P}, \Theta)$ is a polarised stable toric variety $(\mathbb{P}, L)$ together with a Cartier divisor Θ, that does not contain any T-orbits, such that $\mathcal{O}_{\mathbb{P}}(\Theta) \cong L$.

For these stable toric pairs Alexeev constructs a moduli space. But first of all he shows a connection between geometric fibers of stable toric pairs and cell decompositions of a polytope.

Lemma VI.3.6. *Let $f : (\mathbb{P}, \Theta) \to S$ be a flat family of stable toric pairs with a linearised action of a split torus T/S over a connected locally Noetherian base S. If one geometric fiber $(\mathbb{P}_s, \Theta_s)$ corresponds to a cell decomposition of a polytope $Q \subset M'_{\mathbb{Q}}$ then any other geometric fiber corresponds to a (possibly different) cell decomposition of the same polytope Q.*

Proof. See [Ale02, Lemma 2.10.1]. □

This basically means that flat families of stable toric pairs consist of varieties arising from cell decompositions of the same polytope. This enables one to construct moduli stacks with substacks that have a connection to toric Hilbert schemes, as we will show later. For more details on stacks see [Fan01].

Definition VI.3.7. Fix a lattice M' together with its corresponding split torus $T' = \operatorname{Spec} \mathbb{Z}[M']$. Denote by $\mathcal{TP}^{\mathrm{fr}}$ the moduli stack on the category of locally Noetherian schemes associating to each scheme S the groupoid of flat families of stable toric pairs $(\mathbb{P}, \Theta)/S$ together with a linearised action of T_S. By the previous lemma, for a fixed polytope $Q \subset M'_{\mathbb{Q}}$ we have a connected substack $\mathcal{TP}^{\mathrm{fr}}[Q]$ for which every geometric fiber in a family corresponds to a cell decomposition of Q.

The substack $\mathcal{TP}^{\mathrm{fr}}[Q]$ is of interest for this work so we will focus on this one.

Theorem VI.3.8. *The substack $\mathcal{TP}^{fr}[Q]$ is a proper Artin stack of finite presentation over $\mathbb{Z}$ with finite stabilisers. It has a coarse moduli space $TP^{fr}[Q]$ which is a proper scheme over $\mathbb{Z}$.*

Proof. See [Ale02, Theorem 2.10.10]. □

VI.4 Approximation of the Moduli Space

Alexeev has constructed a simplification of the moduli space $TP^{\mathrm{fr}}[Q]$, *i.e.* there is a finite morphism from $TP^{\mathrm{fr}}[Q]$ to its simplification which is projective over the base scheme. This simplification is constructed by first subdividing Q into cells and then defining a projective normal toric scheme for each cell.

Definition VI.4.1. Let $P_{\max} := M' \cap Q$ be the set of lattice points of the polytope Q, $M'' := \operatorname{Hom}(P_{\max}, \mathbb{Z})$ a lattice and define a semigroup homomorphism by

$$\begin{array}{rccc} \phi : & \operatorname{Hom}(P_{\max}, \mathbb{Z}_{\geq 0}) & \to & C_Q \cap M \\ & 1_p & \mapsto & (p, 1) \end{array},$$

where 1_p is 1 on p and 0 elsewhere. For each point $q \in \Phi(Q)$ take the minimal natural number N such that Nq is integral and define the semigroup

$$H_q := \phi^{-1}(\mathbb{Z}_{\geq 0} N q).$$

Example VI.4.2. Consider the lattice $M' = \mathbb{Z}^2$ and let Q be the crosspolytope in $M'_{\mathbb{Q}}$. Then the lattice points of Q are

$$P_{\max} = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$$

which we identify with $\{p_1, ..., p_5\}$ in that order. In this example we take the point $q = \left(\begin{smallmatrix}0\\1/2\\1\end{smallmatrix}\right) \in \Phi(Q)$. Then $N = 2$ is the minimal natural number with Nq integral. The homomorphism ϕ is given by the matrix $\left(\begin{smallmatrix}1&0&-1&0&0\\0&1&0&-1&0\\1&1&1&1&1\end{smallmatrix}\right)$ so that $\phi^{-1}(Nq) = \{1_{p_2} + 1_{p_5}\}$ and

$$\phi^{-1}(2 \cdot Nq) = \{2 \cdot 1_{p_2} + 2 \cdot 1_{p_5}, 1_{p_1} + 2 \cdot 1_{p_2} + 1_{p_3}, 3 \cdot 1_{p_2} + 1_{p_4}\}.$$

Further computation shows that H_q is generated by

$$\{1_{p_2} + 1_{p_5}, 1_{p_1} + 2 \cdot 1_{p_2} + 1_{p_3}, 3 \cdot 1_{p_2} + 1_{p_4}\}.$$

$\diamondsuit$

Definition VI.4.3. Denote by σ the simplex in $\operatorname{Hom}(P_{\max}, \mathbb{Q}_{\geq 0})$ with vertices 1_p for $p \in P_{\max}$ and let ψ be the restriction of $\phi \otimes_{\mathbb{Z}} \mathbb{Q}$ to σ, *i.e.* ψ maps σ to Q. For every $q \in Q$ we denote the fiber of this map by $\delta_q := \psi^{-1}(q)$.

Remark VI.4.4. Since ϕ is linear over $\mathbb{Z}$ and N was chosen minimal, H_q is a saturated semigroup. By construction of δ_q it follows that H_q consists of the lattice points in C_{δ_q}, the cone over δ_q.

Example VI.4.5 (continuing **VI.4.2**)**.** For the crosspolytope Q the simplex σ has five vertices and by mapping $1_{p_i} \mapsto e_i$ we identify $\operatorname{Hom}(P_{\max}, \mathbb{Z})$ with $\mathbb{Z}^5$. Then again ψ is given by $\left(\begin{smallmatrix}1&0&-1&0&0\\0&1&0&-1&0\\1&1&1&1&1\end{smallmatrix}\right)$. Thus, δ_q is the intersection of $\operatorname{ConvHull}\{e_1, ..., e_5\}$ with $\psi_{\mathbb{Q}}^{-1}(q)$. This is given by

$$\operatorname{ConvHull}\left\{\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0\right), \left(0, \frac{3}{4}, 0, \frac{1}{4}, 0\right), \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}\right)\right\},$$

i.e. it is a triangle. $\diamondsuit$

Fix a face F of δ_q. By construction, F is given by the intersection of a face F_σ of σ with $(\phi \otimes \mathbb{Q})^{-1}(q)$. Denote the vertices of F_σ by $\{1_{p_1}, ..., 1_{p_l}\}$. Then every point of F is a convex linear combination of the points $\{1_{p_1}, ..., 1_{p_l}\}$ so that q lies in the convex hull of $\{p_1, ..., p_l\}$ which is a pointed lattice subpolytope of Q. On the other hand, each pointed lattice subpolytope of Q determines a set of vertices of σ and thus a face of δ_q. Therefore, we get the following lemma:

Lemma VI.4.6. *The faces of δ_q are in one-to-one correspondence with pointed lattice subpolytopes $\delta \subset Q$ that contain q in their relative interior. Simplicial subpolytopes correspond to vertices of δ_q.* $\square$

Example VI.4.7 (continuing **VI.4.2**)**.** The fiber of $q = \left(\begin{smallmatrix}0\\1/2\\1\end{smallmatrix}\right)$ was a triangle in $\mathbb{Q}^5$ with the vertices

$$v_1 := \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}\right), v_2 := \left(0, \frac{3}{4}, 0, \frac{1}{4}, 0\right), v_3 := \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0\right)$$

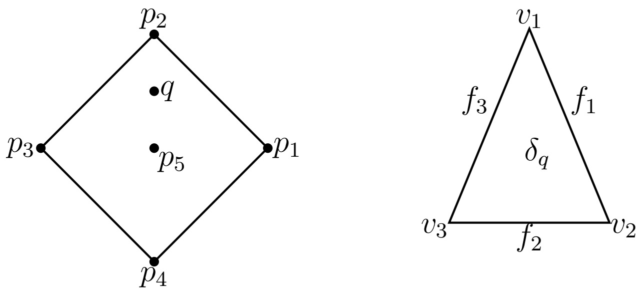

Figure VI.3: The point q in Q and its fiber polytope δ_q

and the facets f_1, f_2, f_3 as pictured in Figure VI.3. Then Lemma VI.4.6 gives the following correspondence between the faces of δ_q and sets of lattice points: The vertices correspond to the sets of lattice points that appear as non zero coordinates, *i.e.* $v_1 \leftrightarrow \{p_2, p_5\}$, $v_2 \leftrightarrow \{p_2, p_4\}$, and $v_3 \leftrightarrow \{p_1, p_2, p_3\}$. The edges correspond to the union of the sets from the two vertices in that edge, *i.e.* $f_1 \leftrightarrow \{p_2, p_4, p_5\}$, $f_2 \leftrightarrow \{p_1, p_2, p_3, p_4\}$, and $f_3 \leftrightarrow \{p_1, p_2, p_3, p_5\}$. Finally, the complete polytope δ_q corresponds to the set of all points $\{p_1, p_2, p_3, p_4, p_5\}$. The pointed lattice subpolytopes containing q in their interior are shown in Figure VI.4 together with the corresponding face.

$\diamondsuit$

Definition VI.4.8. Define the projective normal toric scheme associated to a point $q \in Q$ as

$$F_q := \operatorname{Proj} H_q.$$

Remark VI.4.9. As a scheme, F_q is the toric variety defined by the normal fan of δ_q, because by Remark VI.4.4 H_q was the lattice semigroup of the cone over δ_q.

If two polytopes δ_{q_1} and δ_{q_2} are combinatorial equivalent with parallel faces then their normal fans are equal and thus $F_{q_1} = F_{q_2}$ as schemes. On the other hand, that δ_{q_1} and δ_{q_2} are combinatorial equivalent and have parallel faces is by Lemma VI.4.6 equivalent to q_1 and q_2 lying in the relative interior of exactly the same lattice subpolytopes of Q. Furthermore, if δ_{q_1} is obtained from δ_{q_2} by contracting some faces, *i.e.* a degeneration, then F_{q_1} is a refinement of F_{q_2} which gives a natural morphism $F_{q_1} \to F_{q_2}$. This map is birational if both polytopes have the same dimension.

This means Q can be naturally subdivided into locally closed polytopal domains D_α. Each domain consists of all points q with the same F_q. By construction of δ_q this means if $D_{\alpha_1} \subset \overline{D_{\alpha_2}}$ then for each pair of points $q_1 \in D_{\alpha_1}, q_2 \in D_{\alpha_2}$ the fiber δ_{q_1} is a degeneration of δ_{q_2}. Note that by Lemma VI.4.6 and the above two points are in the same domain D_α exactly if they are contained in the relative interiors of the same lattice subpolytopes of Q. Hence, the subdivision can be obtained by intersecting the images under ψ of all subsets of faces of σ. By this we get a cell decomposition of Q which does not necessarily consist of lattice polytopes. The domains D_α are then the relative interiors of the polytopes in this cell decomposition.

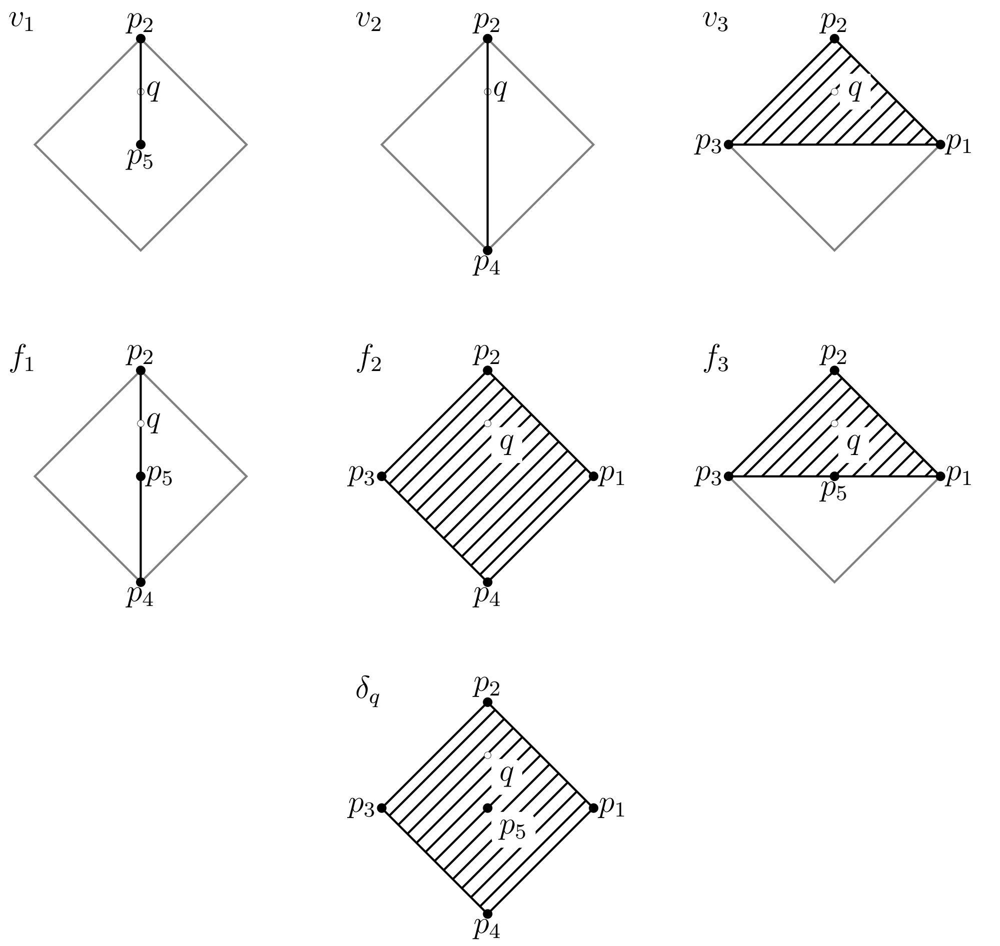

Figure VI.4: The pointed subpolytopes containing q in their relative interior

Example VI.4.10 (continuing **VI.4.2**)**.** If we intersect all subpolytopes of Q we get a decomposition into four two-dimensional polytopes

$$D_1 := \mathrm{ConvHull}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\right\}, \qquad D_2 := \mathrm{ConvHull}\left\{\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\right\},$$
$$D_3 := \mathrm{ConvHull}\left\{\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\right\}, \qquad D_4 := \mathrm{ConvHull}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\right\},$$

and all their faces (See Figure VI.5). For each of the four two-dimensional domains the fiber polytope is a triangle. If we take the centroids as representatives for each domain, then we get the following fiber polytopes:

$$\begin{aligned}
\begin{pmatrix}1/3\\1/3\end{pmatrix} &\leftrightarrow \quad \delta_1 = \quad \mathrm{ConvHull}\left\{\left(\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{6},0,0\right), \left(\tfrac{1}{3},\tfrac{1}{2},0,\tfrac{1}{6},0\right), \left(\tfrac{1}{3},\tfrac{1}{3},0,0,\tfrac{1}{3}\right)\right\}\\
\begin{pmatrix}-1/3\\1/3\end{pmatrix} &\leftrightarrow \quad \delta_2 = \quad \mathrm{ConvHull}\left\{\left(\tfrac{1}{6},\tfrac{1}{3},\tfrac{1}{2},0,0\right), \left(0,\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{6},0\right), \left(0,\tfrac{1}{3},\tfrac{1}{3},0,\tfrac{1}{3}\right)\right\}\\
\begin{pmatrix}-1/3\\-1/3\end{pmatrix} &\leftrightarrow \quad \delta_3 = \quad \mathrm{ConvHull}\left\{\left(\tfrac{1}{6},0,\tfrac{1}{2},\tfrac{1}{3},0\right), \left(0,\tfrac{1}{6},\tfrac{1}{3},\tfrac{1}{2},0\right), \left(0,0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)\right\}\\
\begin{pmatrix}1/3\\-1/3\end{pmatrix} &\leftrightarrow \quad \delta_4 = \quad \mathrm{ConvHull}\left\{\left(\tfrac{1}{2},0,\tfrac{1}{6},\tfrac{1}{3},0\right), \left(\tfrac{1}{3},\tfrac{1}{6},0,\tfrac{1}{2},0\right), \left(\tfrac{1}{3},0,0,\tfrac{1}{3},\tfrac{1}{3}\right)\right\}
\end{aligned}$$

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The result of this partition is that for the subdivision $\{D_\alpha\}$ we have a projective normal toric scheme F_α for each domain D_α, and whenever $D_{\alpha_1} \subset \overline{D_{\alpha_2}}$

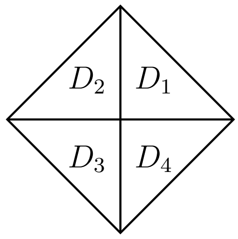

Figure VI.5: The decomposition into domains $\{D_\alpha\}$

we have a natural morphism $F_{\alpha_1} \to F_{\alpha_2}$. Hence, this defines an inverse system of projective normal toric varieties on the partially ordered set given by $\{D_\alpha\}$.

Definition VI.4.11. Define the *simplification scheme*

$$T_{\text{simp}} := \lim_{\leftarrow \{D_\alpha\}} F_\alpha.$$

Theorem VI.4.12. *For each D_α there is a natural morphism $TP^{fr}[Q] \to F_\alpha$ compatible with the morphisms $F_{\alpha_1} \to F_{\alpha_2}$ from the poset structure of the subdivision and, hence, a morphism $f : TP^{fr}[Q] \to T_{simp}$.*

Proof. See [Ale02, Theorem 2.11.8]. □

Alexeev shows even more properties of $TP^{\text{fr}}[Q]$, of the simplification scheme T_{simp}, and of the morphism between them which explain the term simplification.

Corollary VI.4.13. *The schemes $TP^{fr}[Q]$ and T_{simp} have natural stratifications with strata in one-to-one correspondence with cell decompositions Δ of Q. For every Δ these strata are isogeneous. Moreover, the morphism $f : TP^{fr}[Q] \to T_{simp}$ is finite and $TP^{fr}[Q]$ is projective over the base scheme.*

Proof. See corollaries 2.11.10 and 2.11.11 in [Ale02]. □

VI.5 Secondary Polytopes and their Generalisation

The polytopes δ_α of the domains $\{D_\alpha\}$ are all parallel to the kernel of the morphism $\text{Hom}(P_{\max}, \mathbb{Q}) \to M'_{\mathbb{Q}}$. Denote the Minkowski sum of these polytopes by $\sum(Q, P_{\max}) := \sum \delta_\alpha$. Then the normal fan of $\sum(Q, P_{\max})$ is a refinement of each normal fan of some δ_α. Hence, the projective toric variety $F_{\sum(Q,P_{\max})}$ given by this fan maps to every F_α. Thus, there is a natural map to T_{simp}.

Theorem VI.5.1. *The projective toric variety $F_{\sum(Q,P_{max})}$ maps isomorphically onto an irreducible component of T_{simp}. Moreover, it maps isomorphically onto an irreducible component of $TP^{fr}[Q]$.*

Proof. See Theorem 2.12.2 and Corollary 2.12.3 in [Ale02]. □

Remark VI.5.2. The Minkowski sum $\sum(Q, P_{\max})$ is exactly the *secondary polytope* of Q defined in [GKZ08, Chapter 7] by Gelfand, Kapranov, and Zelevinsky which was later described as a special case of the *fiber polytopes* defined by Billera and Sturmfels in [BS92, Section 2] when the source polytope is a simplex.

Example VI.5.3 (continuing **VI.4.2**). The four polytopes $\delta_1, ..., \delta_4$ are all obviously combinatorial equivalent (they are all triangles) and they also have parallel faces, *i.e.* the edge directions are

$$\{(1,-1,1,-1,0),(1,0,1,0,-2),(0,1,0,1,-2)\}$$

for each polytope δ_i. Thus, the Minkowski sum $\sum(Q, P_{\max})$ is again a triangle and it is given by

$$\mathrm{ConvHull}\left\{\left(\frac{4}{3},\frac{2}{3},\frac{4}{3},\frac{2}{3},0\right),\left(\frac{2}{3},\frac{4}{3},\frac{2}{3},\frac{4}{3},0\right),\left(\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{4}{3}\right)\right\}.$$

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As for $\mathcal{A}$-graded ideals there is also a notion of coherency for cell decompositions of polytopes.

Definition VI.5.4. A map $\psi \in \mathrm{Hom}(P_{\max}, \mathbb{Q})$ is called a *height function on Q*.

For a height function ψ on Q consider the convex hull Q_ψ of the rays

$$\{(p,h)\,|\,p \in P_{\max}, h \geq \psi(p)\} \subseteq M_{\mathbb{Q}}.$$

Then the projections to $M'_{\mathbb{Q}}$ of the compact faces of this polyhedron form a cell decomposition Δ_ψ of lattice polytopes of Q. Denote by P_ψ the subset of lattice points $p \in P_{\max}$ such that $(p, \psi(p))$ lies on the boundary of Q_ψ. Thus, for a height function ψ the pair (Δ_ψ, P_ψ) is a pointed cell decomposition of Q.

Definition VI.5.5. A pointed cell decomposition (Δ, P) is called *coherent* (or *regular* by some authors) if $(\Delta, P) = (\Delta_\psi, P_\psi)$ for a height function ψ on Q (See [GKZ08, Chapter 7.C]).

Now we describe the correspondence between the secondary polytope of Q and the coherent pointed cell decompositions in detail as we will use this later. For this, fix a coherent decomposition (Δ_ψ, P_ψ) with corresponding height function ψ on Q. Then set $\overline{\psi} := \psi \circ \phi$. This function lies in the dual of the vector space over $\mathbb{Q}$ associated to the lattice $M'' = \mathrm{Hom}(P_{\max}, \mathbb{Z})$ which contains σ, *i.e.* $\overline{\psi} \in \mathrm{Hom}\left(M''_{\mathbb{Q}}, \mathbb{Q}\right)$. Consider a point $q \in Q$ and denote by $F_{q,\overline{\psi}}$ the face of δ_q on which $\overline{\psi}$ attains its minimum. Denote this minimum by $m_{\psi,q}$. As before, $F_{q,\overline{\psi}}$ is the intersection of $(\phi \otimes \mathbb{Q})^{-1}(q)$ with a face of σ. This face of σ is spanned by a set of points,

$$\{1_{p_1}, ..., 1_{p_l}\}.$$

For each choice of non zero coefficients $\lambda_1, ..., \lambda_l \in \mathbb{Q}$ such that $q = \sum_{i=1}^{l} \lambda_i p_i$ we have

$$\sum_{i=1}^{l} \lambda_i \psi(p_i) = m_{\psi,q}$$

since $\overline{\psi}$ attains its minimum on this face.

On the other hand, consider any other set of points $\{p'_1, ..., p'_j\}$, that is not contained in $\{1_{p_1}, ..., 1_{p_l}\}$, and a choice of non zero coefficients $\mu_1, ..., \mu_j \in \mathbb{Q}$ such that $q = \sum_{i=1}^{j} \mu_i p'_i$. Then due to the minimality of ψ it follows that

$$\sum_{i=1}^{j} \mu_i \psi(p'_i) > m_{\psi,q}$$

holds. Thus, the lattice polytope of the pointed cell decomposition (Δ_ψ, P_ψ) containing q in its interior is exactly the convex hull of $\{1_{p_1}, ..., 1_{p_l}\}$ and the points in P_ψ, that lie in this polytope, are also $\{1_{p_1}, ..., 1_{p_l}\}$.

Note that this means that the face $F_{q,\overline{\psi}}$ of δ_q does not depend on the choice of ψ. If we take a different height function ψ' such that $(\Delta_{\psi'}, P_{\psi'}) = (\Delta_\psi, P_\psi)$ then the above shows that the face of δ_q on which $\overline{\psi'}$ attains its minimum must be the same, since the polytope containing q is identical.

Moreover, if we pick another point q' in the same domain as q then $F_{q',\overline{\psi}}$ is the same face up to the combinatorial equivalence of δ_q and $\delta_{q'}$. Hence, for each domain D_α the height function ψ defines a unique face $F_{\alpha,\psi}$ of δ_α.

Lemma VI.5.6. *The faces of the secondary polytope $\sum(Q, P_{max})$ correspond bijectively to the coherent pointed cell decompositions of Q. In particular, the vertices correspond to the coherent triangulations.*

Proof. Let (Δ_ψ, P_ψ) be a coherent pointed cell decomposition of Q with height function $\overline{\psi} \in \operatorname{Hom}\left(M''_{\mathbb{Q}}, \mathbb{Q}\right)$. Denote by $F_{\overline{\psi}}$ the face of $\sum(Q, P_{\max})$ on which $\overline{\psi}$ attains its minimum. But the face of a Minkowski sum on which $\overline{\psi}$ attains its minimum is the Minkowski sum of the faces of the summands on which $\overline{\psi}$ attains its minimum, *i.e.*

$$F_{\overline{\psi}} = \sum_{\{D_\alpha\}} F_{\alpha,\overline{\psi}}.$$

On the other hand, every $\overline{\psi'} \in \operatorname{Hom}\left(M''_{\mathbb{Q}}, \mathbb{Q}\right)$ comes from a height function ψ' on Q. Therefore, if $\overline{\psi'}$ attains its minimum on $F_{\overline{\psi}}$ then it follows that $F_{\alpha,\psi} = F_{\alpha,\psi'}$ for every domain D_α and thus

$$(\Delta_\psi, P_\psi) = (\Delta_{\psi'}, P_{\psi'}),$$

which shows the claimed correspondence.

Finally, $F_{\overline{\psi}}$ is a vertex of the generalised state polytope if and only if each $F_{\alpha,\psi}$ is a vertex of δ_α and hence exactly if every point in Q lies in a simplex in the decomposition by Lemma VI.4.6. This is equivalent to Δ_ψ being a triangulation. □

Example VI.5.7 (continuing **VI.4.2**). For the two-dimensional crosspolytope the secondary polytope was

$$\sum(Q, P_{\max}) = \operatorname{ConvHull}\left\{\left(\tfrac{4}{3},\tfrac{2}{3},\tfrac{4}{3},\tfrac{2}{3},0\right), \left(\tfrac{2}{3},\tfrac{4}{3},\tfrac{2}{3},\tfrac{4}{3},0\right), \left(\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},\tfrac{4}{3}\right)\right\}.$$

If we identify the above vertices with v_1, v_2, v_3 in that order we get for each of these vertices a vector in the interior of the normal cone which we name n_1, n_2, n_3 respectively, *i.e.*

$$n_1 = (1,2,1,2,2), n_2 = (2,1,2,1,1), \text{ and } n_3 = (2,2,2,2,1).$$

This means the vertex v_1 corresponds to the pointed cell decomposition given by the height function defined by n_1. In Figure VI.6 we can see the height function, the compact faces of Q_{n_1}, and the resulting pointed cell decomposition. Note

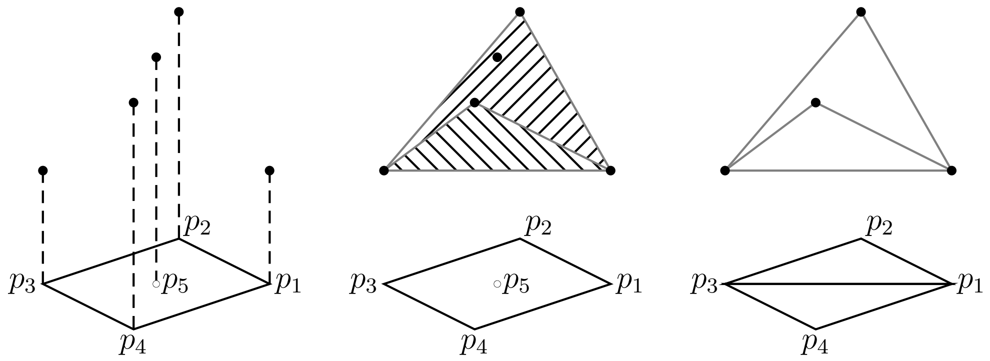

Figure VI.6: The height function, the compact faces of Q_{n_1}, and the resulting pointed cell decomposition

that this decomposition does not contain p_5.

If we do the same for n_2 we get the decomposition into two triangles by taking the other diagonal and this one does also not contain p_5. For n_3 we get the pointed cell decomposition into four triangles (See Figure VI.7). Note that this

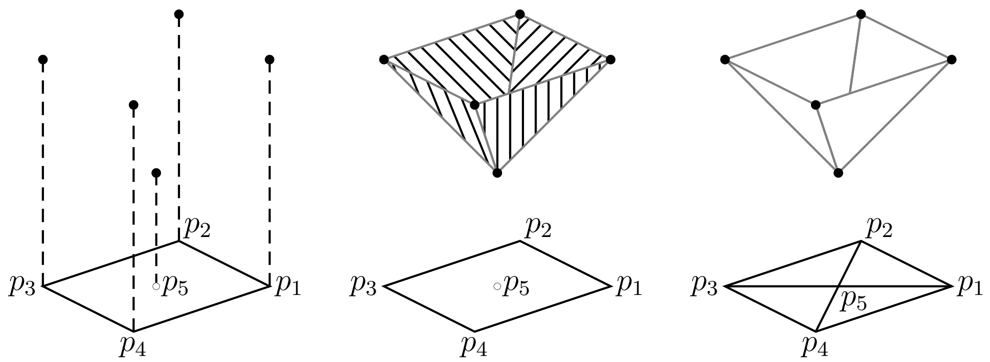

Figure VI.7: The height function, the compact faces of Q_{n_3}, and the resulting pointed cell decomposition

time the pointed cell decomposition contains all five points. Recalling that the height functions for the facets are in the facets of the normal cones of the vertices we get the pointed cell decompositions for the edges. The decomposition corresponding to $\sum(Q, P_{\max})$ itself is the complete Q with all points, since this can be decomposed into all the others. Therefore, if we denote the edges of $\sum(Q, P_{\max})$ by $f_1 = \text{ConvHull}\{v_1, v_3\}$, $f_2 = \text{ConvHull}\{v_2, v_3\}$, and $f_3 = \text{ConvHull}\{v_3, v_1\}$ we get the complete correspondence as shown in Figure VI.8.

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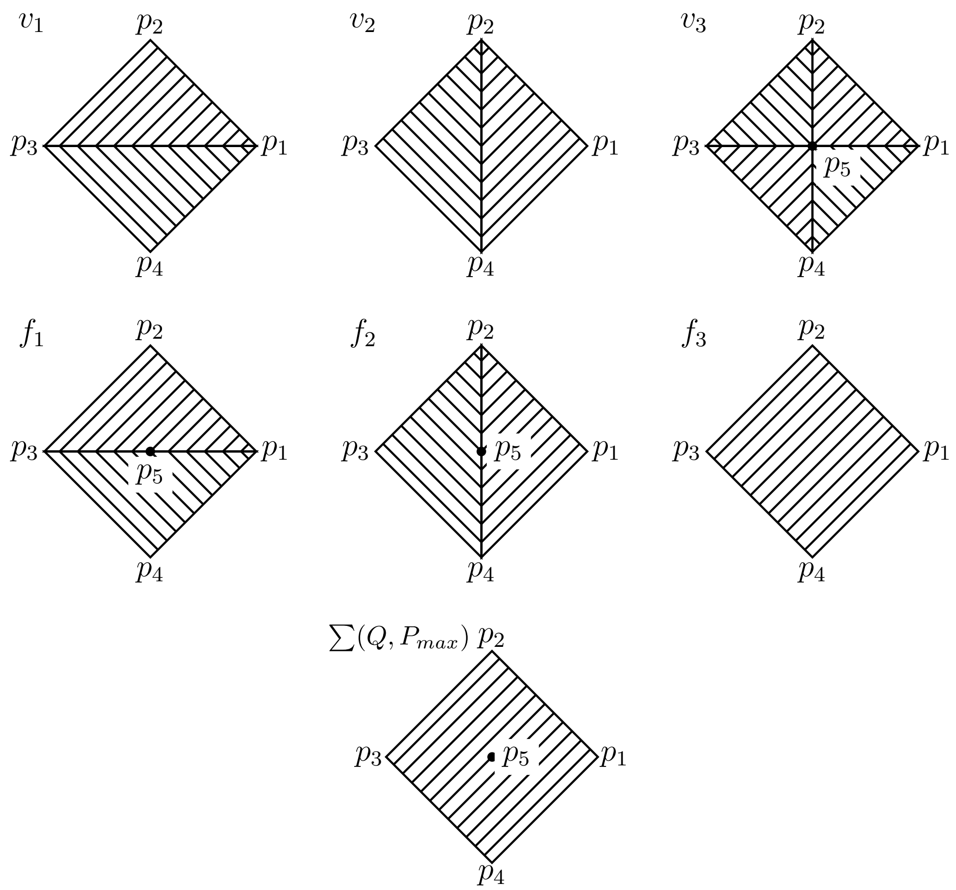

Figure VI.8: The correspondence between pointed cell decompositions of the crosspolytope Q and the faces of $\sum(Q, P_{\max})$

Before comparing certain moduli spaces of stable toric pairs with toric Hilbert schemes, we will give Alexeev's construction of his generalised secondary polytopes in [Ale02, Section 2.12].

For this we fix a polytope Q together with a pointed cell decomposition (Δ, P).

Definition VI.5.8. Define the *sheaf of height functions* $\underline{\text{Hom}}$ on the poset structure of (Δ, P) by

$$\underline{\text{Hom}}\,(\delta) := \text{Hom}\,(P_\delta, \mathbb{Z})\,,$$

where $P_\delta = P \cap \delta$. For each $\delta \in \Delta$ set the map

$$\begin{array}{rccc} \phi_\delta : & \text{Hom}\,(P_\delta, \mathbb{Z}) & \to & M_\delta \\ & (\lambda_p) & \mapsto & \sum_{p \in P_\delta} \lambda_p\,(1, p) \end{array}$$

and define the *kernel sheaf* $\mathbb{L}$ on the poset structure of (Δ, P) by

$$\mathbb{L}\,(\delta) := \text{Ker}\,(\phi_\delta)\,.$$

Again, for both we have the associated sheafes $\underline{\text{Hom}}_{\mathbb{Q}} := \underline{Hom} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{L}_{\mathbb{Q}} := \mathbb{L} \otimes \mathbb{Q}$ over $\mathbb{Q}$. For the generalised secondary polytopes we need the spaces $C_0\left(\underline{\text{Hom}}_{\mathbb{Q}}\right)$ and $B_0\,(\mathbb{L}_{\mathbb{Q}})$ so we will first recall them. If Q has dimension d then the 0-th group in the Čech complex of $\underline{\mathrm{Hom}}_{\mathbb{Q}}$ is just

$$C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right) = \bigoplus_{\delta\in\Delta^{(d)}} \mathrm{Hom}\left(P_\delta, \mathbb{Q}\right),$$

where $\Delta^{(d)}$ are the d-dimensional polytopes in the cell decomposition. This means that for each maximal polytope we have the space of height functions. Furthermore, the first group in the Čech complex of $\mathbb{L}_{\mathbb{Q}}$ is

$$\begin{aligned} C_1\left(\mathbb{L}_{\mathbb{Q}}\right) &= \bigoplus_{\delta\in\Delta^{(d-1)}} \mathbb{L}_{\mathbb{Q}}\left(\delta\right) \\ &= \bigoplus_{\delta\in\Delta^{(d-1)}} \mathrm{Ker}\left(\mathrm{Hom}\left(P_\delta, \mathbb{Q}\right) \to \left(M_\delta\right)_{\mathbb{Q}}\right). \end{aligned}$$

In other words, for each codimension-one polytope we get the space of relations among the points of the pointed cell decomposition contained in this polytope. Finally, $B_0\left(\mathbb{L}_{\mathbb{Q}}\right)$ is the image of $C_1\left(\mathbb{L}_{\mathbb{Q}}\right)$ in $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)$. This means every relation is mapped into the two d-dimensional polytopes that contain the corresponding codimension-one polytope with positive sign and negative sign, respectively.

Definition VI.5.9. Let (Δ, P) be a pointed cell decomposition of Q. Then we define the *generalised secondary polytope* $\Sigma\left(\Delta, P\right)$ to be the projection in $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)/B_0\left(\mathbb{L}_{\mathbb{Q}}\right)$ of the direct product of the secondary polytopes $\Sigma\left(\delta, P_\delta\right)$ in $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)$.

Example VI.5.10 (continuing **VI.4.2**)**.** Take the pointed cell decomposition

$$(\Delta, P) = \left(\{\delta_1, \delta_2\}, \{p_1, p_2, p_3, p_4, p_5\}\right),$$

with $\delta_1 = \mathrm{ConvHull}\{p_1, p_2, p_3, p_5\}$ and $\delta_2 = \mathrm{ConvHull}\{p_1, p_3, p_4, p_5\}$. Then we have $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right) = \underline{\mathrm{Hom}}_{\mathbb{Q}}(\delta_1) \oplus \underline{\mathrm{Hom}}_{\mathbb{Q}}(\delta_2)$ and if we identify $1_{p_1}, 1_{p_2}, 1_{p_3}, 1_{p_5}$ in that order with the canonical basis in $\mathbb{Q}^4 = \underline{\mathrm{Hom}}_{\mathbb{Q}}(\delta_1)$ and $1_{p_1}, 1_{p_4}, 1_{p_3}, 1_{p_5}$ with the one in $\mathbb{Q}^4 = \underline{\mathrm{Hom}}_{\mathbb{Q}}(\delta_2)$ we get $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right) = \mathbb{Q}^8$.

A triangle with an additional point on one of the edges has three pointed cell decompositions: The whole triangle with the point, the triangle without the point and the subdivision into two triangles. Thus, the secondary polytope of δ_1 and that of δ_2 are both just a line segment, where the interior corresponds to the first decomposition and the vertices to the other two. The secondary polytopes can be computed as

$$\begin{aligned} \Sigma\left(\delta_1, M' \cap \delta_1\right) &= \mathrm{ConvHull}\left\{\left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0, 0, 0, 0, 0\right), \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, 0, 0, 0, 0\right)\right\} \\ &\text{and} \\ \Sigma\left(\delta_2, M' \cap \delta_2\right) &= \mathrm{ConvHull}\left\{\left(0, 0, 0, 0, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0\right), \left(0, 0, 0, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)\right\}, \end{aligned}$$

so that direct sum of the two is the tetragon

$$\begin{aligned} \mathrm{ConvHull} \quad &\left\{\left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0\right), \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right),\right. \\ &\left.\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0\right), \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)\right\}. \end{aligned} \tag{VI.1}$$

There is just one pointed face of dimension one in the complex Δ, that has non trivial relations, $\delta_0 := \mathrm{ConvHull}\{p_1, p_3, p_5\}$, so that the first group in the Čech complex of $\mathbb{L}_{\mathbb{Q}}$ is just

$$C_1\left(\mathbb{L}_{\mathbb{Q}}\right) = \mathbb{L}_{\mathbb{Q}}\left(\delta_0\right) = \mathbb{Q}^1,$$

because there is just the relation $1_{p_1} + 1_{p_3} - 2 \cdot 1_{p_5}$. Thus, the image of this group in $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)$ is the one-dimensional subgroup

$$B_0\left(\mathbb{L}_{\mathbb{Q}}\right) = \mathrm{span}\left\{(1,0,1,-2,-1,0,-1,2)\right\}.$$

The affine hull of (VI.1) contains $B_0\left(\mathbb{L}_{\mathbb{Q}}\right)$ so that the generalised secondary polytope is a line segment. ◊

Remark VI.5.11. If a polytope $\delta \in \Delta^{(d-1)}$ is a simplex and P_δ are only the vertices then $\mathbb{L}_{\mathbb{Q}}(\mathbb{Q}) = 0$. Thus, if every codimension-one polytope of the pointed cell decomposition is such a simplex we get $C_1\left(\mathbb{L}_{\mathbb{Q}}\right) = 0$ and hence $B_0\left(\mathbb{L}_{\mathbb{Q}}\right) = 0$.

As in Definition VI.5.4 we can define an analogon to a height function on Q on a pointed cell decomposition.

Definition VI.5.12. A *system of height functions* ψ *on* (Δ, P) is a collection of height functions $\psi_i \in \mathrm{Hom}\left(P_{\delta_i}, \mathbb{Q}\right)$ for each $\delta_i \in \Delta$ such that $\psi_i - \psi_j$ is linear on $\delta_i \cap \delta_j$, *i.e.* an element

$$\psi \in \mathrm{Hom}\left(C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)/B_0\left(\mathbb{L}_{\mathbb{Q}}\right), \mathbb{Q}\right),$$

because the functions that are orthogonal on $B_0\left(\mathbb{L}_{\mathbb{Q}}\right)$ are exactly those which are linear on the intersections.

As for the height functions, a system of height functions ψ on (Δ, P) defines a pointed cell decomposition $(\delta_{i,\psi_i}, P_{\delta_{i,\psi_i}})$ of each $\delta_i \in \Delta$. Since the ψ_i differ by linear functions on the intersection of two polytopes δ_i and δ_j the pointed cell decompositions δ_{i,ψ_i} and δ_{j,ψ_j} satisfy the decomposition condition. This means a pair of polytopes, one from each decomposition, intersects in a common face. Hence,

$$(\Delta_\psi, P_\psi) := \left(\bigcup_{\delta_i \in \Delta} \delta_{i,\psi_i}, \bigcup_{\delta_i \in \Delta} P_{\delta_{i,\psi_i}}\right)$$

is a pointed cell decomposition of Q and a subdecomposition of(Δ, P).

Definition VI.5.13. A subdecomposition of a pointed cell decomposition (Δ, P) of Q is called *regular* if it is given by (Δ_ψ, P_ψ) for a system of height functions ψ on (Δ, P).

Lemma VI.5.14. *The faces of the generalised secondary polytope* $\Sigma(\Delta, P)$ *are in one-to-one correspondence to the regular subdecompositions of* (Δ, P).

Proof. Let (Δ_ψ, P_ψ) be a regular subdecomposition of (Δ, P). Then as in Lemma VI.5.6 the coherent pointed cell decomposition of $\delta \in \Delta$ corresponds to a unique face F_δ of $\Sigma(\delta, P_\delta)$. Hence, we get a face

$$F = \bigoplus_{\delta \in \Delta} F_\delta \quad \text{of} \quad \bigoplus_{\delta \in \Delta} \left(\Sigma(\delta, P_\delta)\right)$$

which maps to a face F_ψ of $\Sigma(\Delta, P)$ because the ψ_i differ by linear functions on the intersections. Also, ψ attains its minimum over $\Sigma(\Delta, P)$ on F_ψ.

On the other hand, every $\psi' \in \mathrm{Hom}\left(C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)/B_0\left(\mathbb{L}_{\mathbb{Q}}\right), \mathbb{Q}\right)$ that attains its minimum over $\Sigma(\Delta, P)$ on F_ψ must attain its minimum on F_δ for each $\Sigma(\delta, P_\delta)$. Thus, it follows that

$$(\Delta_{\psi'}, P_{\psi'}) = (\Delta_\psi, P_\psi).$$

□

Example VI.5.15 (continuing **VI.4.2**). The generalised secondary polytope of the pointed cell decomposition

$$(\Delta, P) = (\{\delta_1, \delta_2\}, \{p_1, p_2, p_3, p_4, p_5\})$$

is a line segment. The interior of the segment corresponds to the pointed cell decomposition (Δ, P) itself. One vertex corresponds to the pointed cell decomposition $(\Delta, \{p_1, p_2, p_3, p_4\})$ where the non-vertex point is not contained. The other vertex corresponds to the subdecomposition into four triangles. This is shown in Figure VI.9.

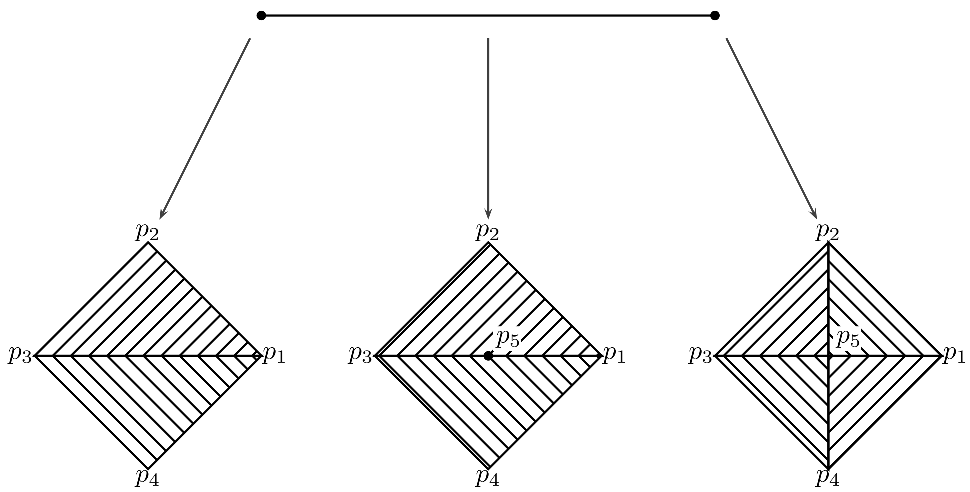

Figure VI.9: The subdecompositions and their correspondence to the faces of the generalised secondary polytope

$\Diamond$

Equip $C_0\left(\underline{\mathrm{Hom}}_{\mathbb{Q}}\right)/B_0\left(\mathbb{L}_{\mathbb{Q}}\right)$ with the lattice $C_0\left(\underline{\mathrm{Hom}}\right)/B_0\left(\mathbb{L}\right)$ modulo torsion. Then we set $F_{\Sigma(\Delta,P)}$ to be the toric scheme given by $\Sigma\left(\Delta, P\right)$ with respect to this lattice.

Lemma VI.5.16. *There is a natural morphism*

$$F_{\Sigma(\Delta,P)} \to T_{simp}$$

which is finite to the image.

Proof. See [Ale02, 2.12.13]. $\square$

VI.6 Correspondence to Toric Hilbert Schemes

We have seen in Remark VI.1.12 that the multigraded Hilbert function of an algebra $R[\Delta, t]$ looks like the multigraded Hilbert function of a toric ideal. From this the conjecture arouses that for a suitable choice of a polytope Q each $R[\Delta, t]$ for a pointed cell decomposition of Q might be an $\mathcal{A}$-graded algebra where $\mathcal{A}$ are the lattice points of Q. As we will see, this does not hold for general polytopes and it need not hold for all decompositions of a polytope Q. In this last section we will investigate on this conjecture and give criteria on Q, as well as on the $\mathcal{A}$-graded ideals, where $\mathcal{A}$ are the lattice points of Q, and on the pointed cell decompositions of Q, so that these $\mathcal{A}$-graded algebras are in fact stable toric varieties of the type $R[\Delta, 1]$ and vice versa.

First of all, recall from Definition VI.2.3 that a polytope δ is called normal if the semigroup $C_\delta \cap M$ is generated by the elements at height 1.

Definition VI.6.1. We call a pointed cell decomposition (Δ, P) *normal* if every $\delta \in \Delta$ is normal and $P_\delta = \delta \cap M'$.

Now fix a normal polytope $Q \subset M'_{\mathbb{Q}}$ and let $Q \cap M' = \{p_1, ..., p_n\}$ be its lattice points. Denote the embedded lattice points of Q in M by $\mathcal{A} = \{a_1, ..., a_n\}$, *i.e.* $a_i = (p_i, 1)$, and by d the rank of M. Then the matrix $\mathcal{A}$ satisfies the condition $\operatorname{Ker}(\mathcal{A}) \cap \mathbb{N}^n = 0$, because the last row is $(\mathbf{1}) \in \mathbb{Z}^n$. As in the previous chapters, set $S := \Bbbk[x_1, ..., x_n]$ with $\deg(x_i) = a_i$.

Let I be an $\mathcal{A}$-graded ideal. Then I lies on some orbit on a component of $\mathcal{H}_{\mathcal{A}}$. The general ideal of this orbit contains just pure binomials and monomials.

Lemma VI.6.2. *Let I be a reduced $\mathcal{A}$-graded ideal which is the general point of an orbit, i.e.*

$$I = \langle \boldsymbol{x}^{\boldsymbol{m}_1} - \boldsymbol{x}^{\boldsymbol{n}_1}, ..., \boldsymbol{x}^{\boldsymbol{m}_l} - \boldsymbol{x}^{\boldsymbol{n}_l} \rangle + \langle \boldsymbol{x}^{\boldsymbol{u}_1}, ..., \boldsymbol{x}^{\boldsymbol{u}_j} \rangle .$$

Then $\operatorname{Proj}(S/I)$ *is seminormal.*

Proof. Let $I = \bigcap q_j$ be a minimal primary decomposition of I. Because I is reduced, the q_j are in fact prime. We fix one q_j. As in Section III.2, q_j is a binomial prime ideal so that the irreducible component of $\operatorname{Proj}(S/I)$ given by S/q_j is a toric variety. We will show that it is even a normal toric variety. Let $x_{i_1}, ..., x_{i_l}$ be the variables not appearing in q_j with respective degrees $a_{i_1}, ..., a_{i_l}$. Thus, S/q_j is normal exactly if $\mathbb{N} \cdot (a_{i_1}, ..., a_{i_l}) \subseteq \mathbb{N}\mathcal{A}$ is saturated.

Assume $\mathbb{N} \cdot (a_{i_1}, ..., a_{i_l})$ is not saturated. Then there exists some lattice point $a \in (\mathbb{Q}_{\geq 0} \cdot (a_{i_1}, ..., a_{i_l})) \cap M$ such that for all $\boldsymbol{x}^{\boldsymbol{m}_0} \in \Bbbk[\boldsymbol{x}]$ of degree a with $\boldsymbol{x}^{\boldsymbol{m}_0} \notin I$ we have $\boldsymbol{x}^{\boldsymbol{m}_0} \in q_j$ and because $a \in \mathbb{N}\mathcal{A}$ such $\boldsymbol{m}_0$ exists. Since $a \in \mathbb{Q}_{\geq 0} \cdot (a_{i_1}, ..., a_{i_l})$, there exists a strictly positive $N \in \mathbb{N}$ such that $N \cdot a \in \mathbb{N} \cdot (a_{i_1}, ..., a_{i_l})$. Thus, we have some $\boldsymbol{x}^{\boldsymbol{m}} \notin q_j$ of degree $N \cdot a$. Since I is reduced we also have $(\boldsymbol{x}^{\boldsymbol{m}_0})^N \notin I$ of degree $N \cdot a$. Hence, because I is $\mathcal{A}$-graded and both $\boldsymbol{x}^{\boldsymbol{m}}$ and $(\boldsymbol{x}^{\boldsymbol{m}_0})^N$ are not in I we get

$$\boldsymbol{x}^{\boldsymbol{m}} - (\boldsymbol{x}^{\boldsymbol{m}_0})^N \in I.$$

On the other hand, $I = \bigcap q_j$ is a primary decomposition, so that there exists some $\boldsymbol{x}^{\boldsymbol{n}} \notin q_j$ such that $\boldsymbol{x}^{\boldsymbol{m}_0 + \boldsymbol{n}} \in I$. But by multiplying with $\boldsymbol{x}^{\boldsymbol{n}}$ we get

$$\boldsymbol{x}^{\boldsymbol{n}} \cdot \left(\boldsymbol{x}^{\boldsymbol{m}} - (\boldsymbol{x}^{\boldsymbol{m}_0})^N \right) = \boldsymbol{x}^{\boldsymbol{m}+\boldsymbol{n}} - \boldsymbol{x}^{\boldsymbol{m}_0+\boldsymbol{n}} \cdot (\boldsymbol{x}^{\boldsymbol{m}_0})^{N-1} \in I,$$

which implies $\boldsymbol{x}^{\boldsymbol{m}+\boldsymbol{n}} \in I$. Thus, we also have $\boldsymbol{x}^{\boldsymbol{m}+\boldsymbol{n}} \in q_j$ which is a contradiction to q_j being prime. Hence, the desired normality follows. Then $\operatorname{Proj}(S/I)$ is seminormal by [Pic98, Proposition 5.14]. □

This means if I is a reduced $\mathcal{A}$-graded ideal we get that $\operatorname{Proj}(S/I)$ is a seminormal toric variety. Furthermore, one can easily see that it has finitely many orbits and the stabiliser of each point is connected. Thus,

$$\mathbb{P}_I := \operatorname{Proj}(S/I)$$

is a stable projective toric variety. Because the degree of x_i is a_i there must be a pointed cell decomposition (Δ, P) of Q such that $\mathbb{P}_I = \mathbb{P}[\Delta, 1]$. To be precise, $\mathbb{P}_I$ is only isomorphic to $\mathbb{P}[\Delta, 1]$ by Theorem VI.2.8 where we omit the sheaf. But the binomials in $\mathbb{P}[\Delta, 1]$ are also pure binomials by construction so that the isomorphism is in fact the identity.

Remark VI.6.3. The proof of Lemma VI.6.2 even shows that each polytope δ in the decomposition Δ corresponding to I is given by one of the associated primes $\mathfrak{q}_j$. This means δ is the convex hull of the points $p_{i_1}, ..., p_{i_l}$ corresponding to the variables not in $\mathfrak{q}_j$ as in the proof for some j. Furthermore, the points $p_{i_1}, ..., p_{i_l}$ are exactly the lattice points of δ and δ is normal.

On the other side, consider a stable toric pair $(\mathbb{P}, \Theta)$ with $\mathbb{P} = \mathbb{P}[\Delta, 1]$ and Δ a pointed cell decomposition of Q. Assume that Δ is normal. Then for each $\delta \in \Delta$ the subset $\delta \cap \{a_1, ..., a_n\}$ generates the semigroup $C_\delta \cap M$. Hence,

$$R[\Delta, 1] = \Bbbk[x_1, ..., x_n]/I_\Delta$$

for some homogeneous ideal I_Δ and $\deg(x_i) = a_i$. In particular, I_Δ is generated by the binomials given by the relations on each cone C_δ and the monomials

$$\left\{ \boldsymbol{x}^{\boldsymbol{a}} \cdot \boldsymbol{x}^{\boldsymbol{b}} \mid \nexists \delta \in \Delta : \boldsymbol{a}, \boldsymbol{b} \in \delta \right\}.$$

By Remark VI.1.12 we get

$$\dim_{\Bbbk}(R[\Delta, t]_a) = \begin{cases} 1 & a \in |C_\delta| \cap M \\ 0 & \text{otherwise} \end{cases}.$$

Thus, since $|C_\delta| \cap M = \mathbb{N}\mathcal{A}$, it follows that I_Δ is a reduced $\mathcal{A}$-graded ideal.

Theorem VI.6.4. *There is a one-to-one correspondence between general points of orbits of reduced $\mathcal{A}$-graded ideals and unions of all strata of stable toric pairs with the same normal cell decomposition.*

Proof. For a normal cell decomposition Δ of Q let I_Δ be the ideal in

$$R[\Delta, 1] = S/I_\Delta.$$

Then, by the above, I_Δ is a reduced $\mathcal{A}$-graded ideal and the general ideal in some orbit of the toric Hilbert scheme $\mathcal{H}_\mathcal{A}$. Note, that since I_Δ is the general point of that orbit this general point is unique.

On the other hand, consider a reduced $\mathcal{A}$-graded ideal I which is the general point in its orbit. Then

$$\operatorname{Proj}(S/I) = \operatorname{Proj} R[\Delta, 1]$$

for some cell decomposition Δ of Q. This decomposition is normal, because by Remark VI.6.3 every $\delta \in \Delta$ is normal. □

Before we specify the correspondence even further we need a detail about height functions and their decompositions.

Lemma VI.6.5. *Let Q be a lattice polytope with lattice points $\{p_1, ..., p_n\}$ and ψ a height function on Q with corresponding pointed cell decomposition (Δ_ψ, P_ψ) of Q. Take a lattice point p in the relative interior of some C_δ with a representation $p = \sum_{p_i \in P_\delta} \lambda_i(p_i, 1)$. Then for every representation*

$$p = \sum_{\{p'_i\} \not\subseteq P_\delta} \mu_i(p'_i, 1)$$

it follows that

$$\sum_{\{p'_i\} \not\subseteq P_\delta} \mu_i \psi(p'_i) > \sum_{p_i \in P_\delta} \lambda_i \psi(p_i).$$

Proof. If comparing the last coordinate of the two representations we get that $\sum \mu_i = \sum \lambda_i =: \lambda$. Let $p' := \frac{1}{\lambda} p$. By Definition VI.5.4 the convex hull of

$$\{(p_i, \psi(p_i)) \mid p_i \in P_\delta\}$$

is a compact facet of Q_ψ. Let h be a normal vector to this facet with positive last coordinate and m_δ the value h attains on this facet. Then Q_ψ lies in the half-space

$$\{m \mid h \cdot m \geq m_\delta\} \subset M_\mathbb{Q}.$$

Thus, for every point $p'_i \in P_{\max} \setminus P_\delta$ the lift $(p'_i, \psi(p'_i))$ does not lie in the boundary of the half-space, because otherwise it would lie in the facet. But all of these points lie in the interior of this half-space. Therefore, the point $\sum_{\{p'_i\} \not\subseteq P_\delta} \frac{\mu_i}{\lambda} (p'_i, \psi(p'_i))$ is also not contained in the boundary. Hence, it follows that

$$\begin{aligned} h \cdot \left(\sum_{\{p'_i\} \not\subseteq P_\delta} \frac{\mu_i}{\lambda} (p'_i, \psi(p'_i)) \right) &> \left(\sum_{\{p'_i\} \not\subseteq P_\delta} \frac{\mu_i}{\lambda} m_\delta \right) \\ &= m_\delta \\ &= h \cdot \left(p', \sum_{p_i \in P_\delta} \frac{\lambda_i}{\lambda} \psi(p_i) \right), \end{aligned}$$

which implies

$$h \cdot \left(p', \sum_{\{p'_i\} \not\subseteq P_\delta} \frac{\mu_i}{\lambda} \psi(p'_i) \right) > h \cdot \left(p', \sum_{p_i \in P_\delta} \frac{\lambda_i}{\lambda} \psi(p_i) \right).$$

But the last coordinate of h was chosen positive so that

$$\sum_{\{p'_i\} \not\subseteq P_\delta} \mu_i \psi(p'_i) > \sum_{p_i \in P_\delta} \lambda_i \psi(p_i)$$

holds. □

Theorem VI.6.6. *In the correspondence of Theorem VI.6.4 orbits of coherent $\mathcal{A}$-graded ideals correspond to strata with coherent cell decompositions.*

Proof. Let Δ be a coherent normal cell decomposition of Q and choose a $\mathbb{Z}$-valued height function ψ such that $\Delta = \Delta_\psi$ and $P_\psi = P_{\max}$. Then ψ is also a weight vector $\psi = \omega := (w_1, ..., w_n) \in \mathbb{Z}^n$. As in Theorem VI.6.4 we have the ideal I_Δ such that $R[\Delta, 1] = S/I_\Delta$ which is an $\mathcal{A}$-graded ideal. On the other hand $\mathrm{in}_\omega(I_\mathcal{A})$ is also an $\mathcal{A}$-graded ideal.

Let $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in I_\Delta$. Then there is some $\delta \in \Delta$ such that $p_i \in \delta$ for all i with $m_i \neq 0$ or $n_i \neq 0$. Recall that x_i corresponds to $a_i = (p_i, 1)$. Thus, since P_δ contains all lattice points of δ, the restriction of ψ to P_δ extends to a linear function on δ. But this means for $l := \sum m_i = \sum n_i$ that we have

$$\begin{aligned} \omega \cdot \boldsymbol{m} &= \sum m_i \psi(p_i) \\ &= l \cdot \psi\left(\sum \frac{m_i}{l} p_i\right) \\ &= l \cdot \psi\left(\sum \frac{n_i}{l} p_i\right) \\ &= \sum n_i \psi(p_i) = \omega \cdot \boldsymbol{n}. \end{aligned}$$

Hence, we get $\mathrm{in}_\omega(\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}) = \boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}$ so that

$$\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in \mathrm{in}_\omega(I_\mathcal{A}).$$

Now consider $\boldsymbol{x}^{\boldsymbol{m}} \in I_\Delta$. Because I_Δ is $\mathcal{A}$-graded there exists a unique $\boldsymbol{x}^{\boldsymbol{n}} \notin I_\Delta$ such that $\deg(\boldsymbol{x}^{\boldsymbol{n}}) = \deg(\boldsymbol{x}^{\boldsymbol{m}})$. This implies $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in I_\mathcal{A}$ and that there exists a $\delta \in \Delta$ such that $p_i \in \delta$ for all i with $n_i \neq 0$. But $\boldsymbol{x}^{\boldsymbol{m}} \in I_\Delta$ induces in particular that there exists $p_j \notin \delta$ such that $m_j \neq 0$. Hence, by Lemma VI.6.5 we have

$$\omega \cdot \boldsymbol{m} = \sum m_i \psi(p_i) > \sum n_i \psi(p_i) = \omega \cdot \boldsymbol{n}$$

and therefore

$$\mathrm{in}_\omega(\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}}) = \boldsymbol{x}^{\boldsymbol{m}}.$$

Thus, we get $I_\Delta \subseteq \mathrm{in}_\omega(I_\mathcal{A})$, but because both are $\mathcal{A}$-graded they must be equal.

For the other direction, let $I = \mathrm{in}_\omega(I_\mathcal{A})$ be a reduced coherent $\mathcal{A}$-graded ideal. Then $\omega = (w_1, ..., w_n)$ is also a height function ψ on Q with $\psi(p_i) = w_i$. Denote by $(\Delta, P_{\max})$ the induced pointed cell decomposition for which we have $I_\Delta = \mathrm{in}_\omega(I_\mathcal{A}) = I$. Then for every $\delta \in \Delta^{(d)}$ the restriction of ψ to $\delta \cap M'$ extends to a linear function on δ. In fact, every affine dependency of lattice points in δ corresponds to a binomial $\boldsymbol{x}^{\boldsymbol{m}} - \boldsymbol{x}^{\boldsymbol{n}} \in I_\Delta$. But this means $\omega \cdot \boldsymbol{m} = \omega \cdot \boldsymbol{n}$ so that ψ extends linearly.

Now take a point $p_j \in P_{\max} \setminus \delta$. Because δ is of maximal dimension, C_δ is, so that there exists a lattice point $q = \sum m_i(p_i, 1) \in C_\delta$ such that $p' := (p_j, 1) + q$ is in the relative interior of C_δ. Since δ is normal we can choose m_i such that $m_i = 0$ for $p_i \notin \delta$ and a representation $p' = \sum n_i(p_i, 1)$ with the same property for the n_i. Then consider the binomial

$$\boldsymbol{x}^{\boldsymbol{m}+1_j} - \boldsymbol{x}^{\boldsymbol{n}} \in I_\mathcal{A}.$$

The first monomial $\boldsymbol{x}^{\boldsymbol{m}+1_j}$ is in $I_\Delta = \mathrm{in}_\omega(I_\mathcal{A})$ because there is no polytope in Δ that contains p_j and all other p_i with $m_i \neq 0$. In fact, this polytope would be different from δ so that the cone over it could not contain p' which is a contradiction. Hence, we get

$$\omega \cdot (\boldsymbol{m} + 1_j) > \omega \cdot \boldsymbol{n}.$$

But this means that $(p_j, \psi(p_j)) = (p_j, w_j)$ is above the affine hull of

$$\{(p_i, \psi(p_i)) \mid p_i \in \delta\}.$$

Therefore, we get $(\Delta, P_{\max}) = (\Delta_\psi, P_\psi)$. □

Remark VI.6.7. Theorem VI.6.6 has some similarities to Theorem [Stu96, Theorem 10.10]. Note that we are not considering all $\mathcal{A}$-graded ideals, but instead the orbits of the reduced ones. Furthermore, our correspondence outlines, that coherent on the toric Hilbert scheme corresponds to coherent on Alexeev's moduli space. This points out that the correspondence might even be component-wide.

Corollary VI.6.8. *Let I be a reduced coherent $\mathcal{A}$-graded ideal and (Δ, P_{max}) the corresponding pointed cell decomposition of Q. Then the normal cone of* state$(I_\mathcal{A})$ *corresponding to I and the normal cone of the secondary polytope $\sum(Q, P_{max})$ corresponding to (Δ, P_{max}) are equal.*

Proof. Just note that the proof of Theorem VI.6.6 shows that each weight vector giving I as an initial ideal of $I_\mathcal{A}$ is a height vector giving the pointed cell decomposition $(\Delta, P_{\max})$, and vice versa. □

Example VI.6.9. We will use all the previous results from the running Example VI.4.2. For the crosspolytope $Q = \mathrm{ConvHull}\{p_1, p_2, p_3, p_4, p_5\}$ we thus set

$$\begin{aligned}\mathcal{A} \;:=\; & \Big\{a_1 = (p_1, 1) = \begin{pmatrix}1\\0\\1\end{pmatrix}, a_2 = (p_2, 1) = \begin{pmatrix}0\\1\\1\end{pmatrix}, a_3 = (p_3, 1) = \begin{pmatrix}-1\\0\\1\end{pmatrix}, \\ & a_4 = (p_4, 1) = \begin{pmatrix}0\\-1\\1\end{pmatrix}, a_5 = (p_5, 1) = \begin{pmatrix}0\\0\\1\end{pmatrix}\Big\}.\end{aligned}$$

Then the toric ideal is

$$I_\mathcal{A} = \langle x_1x_3 - x_5^2, x_2x_4 - x_5^2\rangle \subset S = \Bbbk[x_1, ..., x_5]$$

which has three initial monomial ideals

$$\mathcal{M}_1 = \langle x_1x_3, x_2x_4\rangle, \mathcal{M}_2 = \langle x_1x_3, x_5^2\rangle, \mathcal{M}_3 = \langle x_2x_4, x_5^2\rangle.$$

These are the only monomial $\mathcal{A}$-graded ideals and one can check that there is only the coherent component. Thus, the state polytope of $I_\mathcal{A}$ must be a triangle, and in fact computation shows that

$$\mathrm{state}(I_\mathcal{A}) = \mathrm{ConvHull}\{w_1, w_2, w_3\},$$

with $w_1 = (6, 6, 6, 6, 7), w_2 = (6, 7, 6, 7, 5)$, and $w_3 = (7, 6, 7, 6, 5)$. We have ordered the vertices so that $\mathcal{M}_i$ corresponds to w_i for $i = 1, 2, 3$. If we denote the edges of state$(I_\mathcal{A})$ by $e_1 := \mathrm{ConvHull}\{w_1, w_2\}, e_2 := \mathrm{ConvHull}\{w_2, w_3\}$, and

$e_3 := \mathrm{ConvHull}\{w_1, w_3\}$, we get the correspondence to the orbits of the wall ideals as

$$\begin{array}{rcl} e_1 & \leftrightarrow & W_1 := \left\{\left\langle x_2x_4 - \lambda x_5^2, x_1x_3\right\rangle \mid \lambda \in \Bbbk^*\right\}, \\ e_2 & \leftrightarrow & W_2 := \left\{\left\langle x_1x_3 - \lambda x_2x_4, x_5^2\right\rangle \mid \lambda \in \Bbbk^*\right\}, \\ e_3 & \leftrightarrow & W_3 := \left\{\left\langle x_1x_3 - \lambda x_5^2, x_2x_4\right\rangle \mid \lambda \in \Bbbk^*\right\}, \end{array}$$

and the whole polytope corresponds to the orbit of the toric ideal $I_{\mathcal{A}}$. The reduced ideals are $\mathcal{M}_1$, the two orbits corresponding to e_1 and e_3, and the orbit of the toric ideal. Therefore, in the correspondence of Theorem VI.6.4 these ideals correspond to cell decompositions of Q. These decompositions are shown in Figure VI.8. Since the monomial ideal must correspond to a decomposition into simplices with no interior lattice points, $\mathcal{M}_1$ corresponds to the decomposition into four triangles. The two one-dimensional orbits must hence give the decompositions into two triangles, so that they contain the one binomial relation. Thus, W_1 gives the division by the vertical diagonal and W_3 the division by the horizontal diagonal. Finally, the complete polytope corresponds to the trivial decomposition. Note that all decompositions appear in the correspondence, since all of them are normal.

For the coherency correspondence in Theorem VI.6.6 we have to be more precise. Here we only get the pointed decompositions. Thus, we can allocate the faces $\{w_1, e_1, e_3, \mathrm{state}(I_{\mathcal{A}})\}$ to faces of $\sum(Q, P_{\max})$ and get

$$\begin{array}{ll} w_1 \leftrightarrow v_3 & \mathrm{state}(I_{\mathcal{A}}) \leftrightarrow \sum(Q, P_{\max}) \\ e_1 \leftrightarrow f_2 & \qquad e_2 \leftrightarrow f_1 \end{array} .$$

$\diamondsuit$

Now that we have described the correspondence in the coherent case we will generalise the results to general points on the toric Hilbert scheme and general decompositions.

For this let I be any reduced $\mathcal{A}$-graded ideal with trivial binomial coefficients, *i.e.*

$$I = \langle \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{x}^{\boldsymbol{n}_i} \rangle + \langle \boldsymbol{x}^{\boldsymbol{u}_j} \rangle .$$

Then I defines a normal pointed cell decomposition $(\Delta, P_{\max})$ of Q by Theorem VI.6.4 with $I = I_\Delta$. On the other hand, I lies on some (not necessarily unique) component V of the toric Hilbert scheme $\mathcal{H}_{\mathcal{A}}$. Assume that V is a non-coherent component and denote by $\widetilde{J_{\mathcal{M}}(\boldsymbol{p})}$ the generalised universal family and by P_V the generalised state polytope of this component. Then there is a face F_I of P_V corresponding to I and since it is generated by pure binomials I is in fact the general ideal of F_I. By Corollary IV.3.10 the torus orbit of I in V is thus given by

$$\mathrm{in}_\omega\left(\widetilde{J_{\mathcal{M}}(\boldsymbol{p})}\right)_{(\boldsymbol{y})=\lambda,(\boldsymbol{z})=\mu}$$

for a normal vector ω on F_I and $\lambda, \mu \in (\Bbbk^*)^{\dim(F_I)}$. But because I is the general ideal of F_I we get that the torus is parametrised by the universal family

$$J_I = \left\langle \boldsymbol{z}^{\boldsymbol{e}_i^+} \boldsymbol{y}^{\boldsymbol{e}_i^-} \boldsymbol{x}^{\boldsymbol{m}_i} - \boldsymbol{z}^{\boldsymbol{e}_i^-} \boldsymbol{y}^{\boldsymbol{e}_i^+} \boldsymbol{x}^{\boldsymbol{n}_i} \right\rangle + \langle \boldsymbol{x}^{\boldsymbol{u}_i} \rangle \qquad \text{(VI.2)}$$

for all $(\boldsymbol{y}), (\boldsymbol{z}) \in (\Bbbk^*)^{\dim(F_I)}$.

The orbits in the closure of the orbit of I correspond to the faces of F_I, or equivalently to the initial ideals of the generalised universal family $\widetilde{J_{\mathcal{M}}(p)}$ of the component V, that are also initial ideals of J_I. Now let I_1 be the general ideal of one of the faces F_1 of F_I. Then, since I_1 corresponds to a face of F_I and thus in particular to a face of P, we get that there exists a weight vector ω such that

$$I_1 = \left(\mathrm{in}_\omega\left(\widetilde{J_{\mathcal{M}}(p)}\right)\right)_{(\boldsymbol{y})=(\boldsymbol{z})=1}.$$

However, because F_1 is a face of F we get

$$I_1 = (\mathrm{in}_\omega(J_I))_{(\boldsymbol{y})=(\boldsymbol{z})=1}.$$

Proposition VI.6.10. *Let I be a reduced $\mathcal{A}$-graded ideal which lies on a non-coherent component V with corresponding face F_I of the polytope P_V. Denote by (Δ, P_{max}) the corresponding pointed cell decomposition of Q. Let I_1 be the general ideal of a face F_1 of F_I that is reduced. Then I_1 corresponds to a pointed cell decomposition (Δ_1, P_{max}) which is a sub decomposition of (Δ, P_{max}).*

Proof. By Theorem VI.6.4 the ideal I_1 corresponds to a pointed cell decomposition $(\Delta_1, P_{\max})$ and since I_1 is the general ideal of F_1 we have

$$I_1 = I_{\Delta_1}.$$

Now take $\delta \in \Delta_1$ and denote its lattice points by $\{p_{i_1}, ..., p_{i_l}\} = \delta \cap M'$. Then we have $x_{i_1} \cdot ... \cdot x_{i_l} \notin I_{\Delta_1}$, because the corresponding points are contained in the common polytope δ. Furthermore, we have

$$I_1 = (\mathrm{in}_\omega(J_I))_{(\boldsymbol{y})=(\boldsymbol{z})=1}, \tag{VI.3}$$

so it follows that $x_{i_1} \cdot ... \cdot x_{i_l} \notin I$ as well, because (VI.3) implies

$$\boldsymbol{x}^{\boldsymbol{m}} \in I \Rightarrow \boldsymbol{x}^{\boldsymbol{m}} \in I_1.$$

Finally, this concludes that there exists $\delta_0 \in \Delta$ such that $\{p_{i_1}, ..., p_{i_l}\} \subset \delta_0$. Thus, Δ_1 is a subdecomposition of Δ. □

Now we will show what the universal family (VI.2) has to do with the toric Hilbert scheme of the lattice points of one of the polytopes in the corresponding cell decomposition. As before, denote by $(\Delta, P_{\max})$ the pointed cell decomposition given by I. Take a polytope $\delta \in \Delta$ and denote its lattice points by $\{p_{i_1}, ..., p_{i_l}\}$. Then consider the ideal

$$J_I|_\delta := J_{I\,(x_i=0\,|\,p_i\notin\delta)} \subseteq \Bbbk[x_{i_1}, ..., x_{i_l}, \boldsymbol{y}, \boldsymbol{z}]$$

which we get by substituting 0 for all variables not corresponding to points in δ.

Lemma VI.6.11. *The family of ideals in $\Bbbk[x_{i_1}, ..., x_{i_l}]$ parametrised by $J_I|_\delta$ for $(\boldsymbol{y}), (\boldsymbol{z}) \in (\Bbbk^*)^{\dim(F_I)}$ is the torus of the coherent component of the toric Hilbert scheme of $\mathcal{A}' = \{(p_{i_1}, 1), ..., (p_{i_l}, 1)\}$.*

Proof. Note that when substituting $(\boldsymbol{y}) = (\boldsymbol{z}) = 1$ we get

$$\left(J_I|_{\delta} \right)_{(\boldsymbol{y})=(\boldsymbol{z})=1} = I_{(x_i=0, p_i \notin \delta)}.$$

By construction of Δ_I it follows that we get the toric ideal, *i.e.*

$$I_{(x_i=0, p_i \notin \delta)} = I_{\mathcal{A}'}.$$

Moreover, for every $a \in \mathbb{N}\mathcal{A}'$ we have

$$\dim_{\Bbbk} \left(S/ \left(J_I \right)_{(\boldsymbol{y})_a=\lambda, (\boldsymbol{z})=\mu} \right) = 1$$

for every $\lambda, \mu \in (\Bbbk^*)^{\dim(F_I)}$ and there exists $\boldsymbol{x}^{\boldsymbol{m}} \in \Bbbk[x_{i_1}, ..., x_{i_l}]$ such that $\boldsymbol{x}^{\boldsymbol{m}} \notin J_I$. Thus, after the restriction we get that

$$\left(J_I|_{\delta} \right)_{(\boldsymbol{y})=\lambda, (\boldsymbol{z})=\mu} \tag{VI.4}$$

is $\mathcal{A}'$-graded for every $\lambda, \mu \in (\Bbbk^*)^{\dim(F_I)}$. Since they all contain no monomials they are all isomorphic as $\mathcal{A}'$-graded ideals to $I_{\mathcal{A}'}$. Therefore, all ideals given by VI.4 lie in the torus of the coherent component of $\mathcal{H}_{\mathcal{A}'}$.

On the other hand, let I_1 be an $\mathcal{A}'$-graded ideal in the torus of the coherent component of $\mathcal{H}_{\mathcal{A}'}$. Then there are $\nu_{i_1}, ..., \nu_{i_l} \in \Bbbk^*$ such that I_1 is isomorphic to $I_{\mathcal{A}'}$ via $x_{i_j} \mapsto \nu_{i_j} x_{i_j}$. Extend this to $\nu = (\nu_1, ..., \nu_n)$ by setting the $\nu_j = 1$ for $p_j \notin \delta$. Then we get that I is isomorphic as an $\mathcal{A}$-graded ideal to $\Phi_\nu(I)$ which lies on the orbit of I. Thus, there exist $\lambda, \mu \in (\Bbbk^*)^{\dim(F_I)}$ such that

$$\Phi_\nu(I) = \left(J_I \right)_{(\boldsymbol{y})=\lambda, (\boldsymbol{z})=\mu}.$$

But this implies for the restriction that

$$I_1 = \left(\Phi_\nu(I) \right)_{(x_i=0, p_i \notin \delta)} = \left(J_I|_{\delta} \right)_{(\boldsymbol{y})=\lambda, (\boldsymbol{z})=\mu}$$

holds, so that I_1 is given by the parametrisation. □

Theorem VI.6.12. *Let I be a reduced $\mathcal{A}$-graded ideal with corresponding face F_I of the polytope P_V of a non-coherent component V and let (Δ, P_{max}) be the corresponding pointed cell decomposition of Q. Furthermore, let I_1 be the general ideal of a face F_1 of F_I that is reduced. Then the subdecomposition (Δ_1, P_{max}) is a coherent cell decomposition on every $\delta \in \Delta$.*

Proof. Fix $\delta \in \Delta$. Then by Proposition VI.6.10 I_1 defines a normal cell decomposition $\{\delta_i\}$ of δ. By Lemma VI.6.11

$$\left(J_I|_{\delta} \right)_{(\boldsymbol{y})=\lambda, (\boldsymbol{z})=\mu} \tag{VI.5}$$

for $\lambda, \mu \in (\Bbbk^*)^{\dim(F_I)}$ gives the torus of the coherent component of $\mathcal{H}_{\mathcal{A}'}$ for the grading $\mathcal{A}' = \{(p_{i_1}, 1) ..., (p_{i_l}, 1)\}$. Take a normal vector ω of F_I in P_V. Then we have

$$I_1 = \left(J_{I_1} \right)_{((\boldsymbol{y})=(\boldsymbol{z})=1)} = \left(\mathrm{in}_\omega \left(J_I \right) \right)_{((\boldsymbol{y})=(\boldsymbol{z})=1)}$$

and restricting this to δ results in

$$\begin{aligned} I_{1\,(x_i=0,p_i\notin\delta)} &= \Big((\mathrm{in}_\omega\,(J_I))_{((\boldsymbol{y})=(\boldsymbol{z})=1)}\Big)_{(x_i=0,p_i\notin\delta)} \\ &= \Big((\mathrm{in}_\omega\,(J_I))_{(x_i=0,p_i\notin\delta)}\Big)_{((\boldsymbol{y})=(\boldsymbol{z})=1)} \\ &= \left(\mathrm{in}_\omega\,(\,J_I|_\delta)\right)_{((\boldsymbol{y})=(\boldsymbol{z})=1)}. \end{aligned}$$

Thus, it follows that $I_{1\,(x_i=0,p_i\notin\delta)}$ lies in the closure of the torus given by equation (VI.5) and hence in the coherent component of $\mathcal{H}_{\mathcal{A}'}$. Therefore, there exists a weight vector $\omega' = \left(\omega'_{i_1}, ..., \omega'_{i_l}\right)$ such that

$$I_{1\,(x_i=0,p_i\notin\delta)} = \mathrm{in}_{\omega'}\,(I_{\mathcal{A}'})\,,$$

since its binomial generators have trivial coefficients. Hence, the cell decomposition $(\{\delta_i\}, P_{\max})$ of δ is by Corollary VI.6.8 given by the height function ω' so that $(\{\delta_i\}, P_{\max})$ is coherent. □

Remark VI.6.13. Note that we have not proven, that $(\Delta_1, P_{\max})$ is a regular subdecomposition, just that on each $\delta \in \Delta$ we get a height function that gives the decomposition $\Delta_1 \cap \delta$, *i.e.* a coherent decomposition of δ.

This means, if the general ideal I of a non-coherent is reduced with corresponding decomposition $(\Delta, P_{\max})$, then the orbits of this component, that consist of reduced $\mathcal{A}$-graded ideals correspond to subdecompositions of the pointed cell decomposition $(\Delta, P_{\max})$. The subdecompositions are coherent on each polytope δ of the original decomposition $(\Delta, P_{\max})$ and fit on the intersections. Although, these two properties also hold for regular subdecompositions it does not imply that the subdecompositions given by the above orbits are in fact regular.

Moreover, the conclusion that a subdecomposition of $(\Delta, P_{\max})$, that is coherent on every $\delta \in \Delta$ and fits on the intersections, is in fact regular is false as the next example shows.

Example VI.6.14. We consider the pointed cell complex of lattice polytopes $(\Delta, P_{\max}) = (\{\delta_1, \delta_2\}, \{p_1, ..., p_{13}\})$ of two triangles in the plane with

$$\begin{aligned} \delta_1 &= \mathrm{ConvHull}\,\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9\}\,, \\ \delta_2 &= \mathrm{ConvHull}\,\{p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}\} \end{aligned}$$

and

$$\begin{aligned} &p_1 = \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right), p_2 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), p_3 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), p_4 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right), p_5 = \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right), p_6 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), p_7 = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \\ &p_8 = \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right), p_9 = \left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right), p_{10} = \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right), p_{11} = \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), p_{12} = \left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right), p_{13} = \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right), \end{aligned}$$

as shown in Figure VI.10.

Now we take the subdecomposition (Δ', P) consisting of the ten polytopes

$$\begin{aligned} \delta'_1 &= \mathrm{ConvHull}\,\{p_1, p_2, p_5, p_6\}\,, & \delta'_2 &= \mathrm{ConvHull}\,\{p_1, p_3, p_6\}\,, \\ \delta'_3 &= \mathrm{ConvHull}\,\{p_1, p_3, p_4, p_9\}\,, & \delta'_4 &= \mathrm{ConvHull}\,\{p_3, p_6, p_7, p_8\}\,, \\ \delta'_5 &= \mathrm{ConvHull}\,\{p_3, p_8, p_9\}\,, & \delta'_6 &= \mathrm{ConvHull}\,\{p_5, p_6, p_{11}\}\,, \\ \delta'_7 &= \mathrm{ConvHull}\,\{p_5, p_{10}, p_{11}, p_{13}\}\,, & \delta'_8 &= \mathrm{ConvHull}\,\{p_6, p_7, p_8, p_{11}\}\,, \\ \delta'_9 &= \mathrm{ConvHull}\,\{p_8, p_9, p_{12}, p_{13}\}\,, & \delta'_2 &= \mathrm{ConvHull}\,\{p_8, p_{11}, p_{13}\}\,, \end{aligned}$$

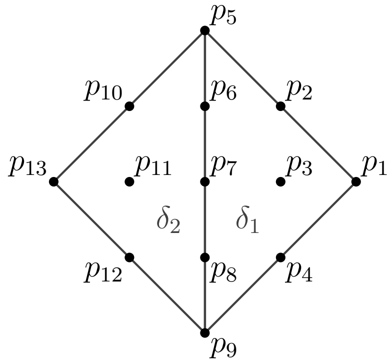

Figure VI.10: The pointed cell complex (Δ, P)

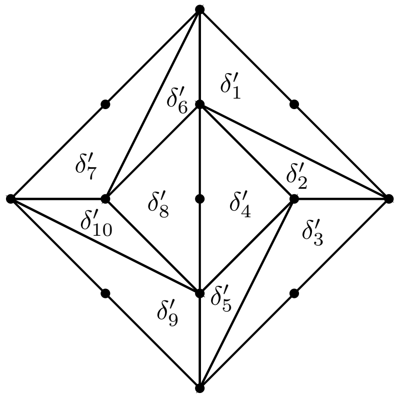

Figure VI.11: The subdecomposition (Δ_1, P)

see Figure VI.11. The subdecompositions of δ_1 and δ_2 are coherent on each polytope by using the two height functions $\phi_1 = (4, 3, 1, 5, 2, 1, 1, 1, 6)$ and respectively $\phi_2 = (10, 2, 2, 2, 6, 9, 2, 7, 8)$.

Assume that a system of height functions Φ exists that gives the subdecomposition (Δ', P). This system is given by heights

$$(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9)$$

on the lattice points of δ_1 and heights

$$(w'_5, w'_6, w'_7, w'_8, w'_9, w'_{10}, w'_{11}, w'_{12}, w'_{13})$$

on the lattice points of δ_2. The values w_3, w_6, w_7, w_8 define a linear function on δ_1 and the values $w'_6, w'_7, w'_8, w'_{11}$ define a linear function on δ_2 which we can subtract from δ_1 and δ_2, respectively, without changing the subdecomposition. Thus, we may assume $\Phi_1 = \Phi_2$ on $\delta_1 \cap \delta_2$ and

$$w_3 = w_6 = w_7 = w_8 = w'_6 = w'_7 = w'_8 = w'_{11} = 0, w_5 = w'_5, \text{ and } w_9 = w'_9.$$

The subdivision of the quadrangle ConvHull $\{p_1, p_3, p_5, p_6\}$ into δ'_1 and δ'_2 implies $w_1 + w_6 < w_3 + w_5$ and since $w_3 = w_6$ we get $w_1 < w_5$. The same argument for the other four outer quadrangles results in $w_9 < w_1$, $w'_{13} < w'_9$, and $w'_5 < w'_{13}$. But this leads to the contradiction

$$w_9 < w_1 < w_5 = w'_5 < w'_{13} < w'_9 = w_9.$$

Hence the subdecomposition can not be regular.

On the other hand the question arises if this subdecomposition comes from an initial ideal of the $\mathcal{A}$-graded ideal corresponding to the cell complex (Δ, P). To investigate on that take a monomial $\mathcal{A}$-graded ideal that corresponds to a subdecomposition, *i.e.* triangulation of (Δ_1, P) and one that is also an initial ideal of the ideal corresponding to (Δ_1, P). The only triangulation of Δ_1 is shown in Figure VI.12 and when identifying $(p_i, 1)$ with x_i this corresponds to the non-coherent monomial $\mathcal{A}$-graded ideal

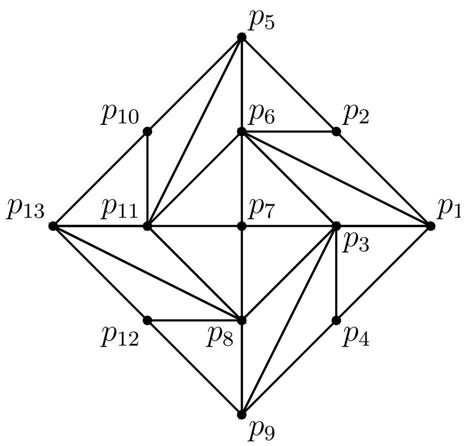

Figure VI.12: The triangulation of (Δ_1, P)

$$\begin{aligned}\mathcal{M} \;=\; &\langle x_1x_5, x_1x_7, x_1x_8, x_1x_9, x_1x_{10}, x_1x_{11}, x_1x_{12}, x_1x_{13}, x_2x_3, x_2x_4, x_2x_7,\\ &x_2x_8, x_2x_9, x_2x_{10}, x_2x_{11}, x_2x_{12}, x_2x_{13}, x_3x_5, x_3x_{10}, x_3x_{11}, x_3x_{12},\\ &x_3x_{13}, x_4x_5, x_4x_6, x_4x_7, x_4x_8, x_4x_{10}, x_4x_{11}, x_4x_{12}, x_4x_{13}, x_5x_7, x_5x_8,\\ &x_5x_9, x_5x_{12}, x_5x_{13}, x_6x_8, x_6x_9, x_6x_{10}, x_6x_{12}, x_6x_{13}, x_7x_9, x_7x_{10},\\ &x_7x_{12}, x_7x_{13}, x_8x_{10}, x_9x_{10}, x_9x_{11}, x_9x_{13}, x_{10}x_{12}, x_{11}x_{12}\rangle\end{aligned}$$

with 50 generators, where the M-grading on $S = \Bbbk[x_1, ..., x_{13}]$ is given by

$$\mathcal{A} := \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 & 0 & -1 & -2 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We compute the universal family $J_{\mathcal{M}}$ which has 50 binomial generators and hence 50 new variables $y_1, .., y_{50}$ and compute the defining ideal with removed redundant variables

$$I'_{\mathcal{M}} = \langle y_1y_4y_9^2y_{26}^2y_{35}y_{38}^2y_{43}y_{48}y_{50}^2 - y_{43}\rangle$$

in the ring $\Bbbk[y_1, y_4, y_9, y_{20}, y_{26}, y_{35}, y_{38}, y_{43}, y_{48}, y_{50}]$ over the remaining ten variables. Then

$$I'_{\mathcal{M}} = \langle y_1y_4y_9^2y_{26}^2y_{35}y_{38}^2y_{48}y_{50}^2 - 1\rangle \cap \langle y_{43}\rangle$$

is a primary decomposition where the first ideal corresponds to the coherent component. Thus, we consider $q = \langle y_{43} \rangle$ since $\mathcal{M}$ is an initial ideal of I_{Δ_1} so that they have to lie in the same component. Hence, we get the already reduced non-coherent component V which is given by the universal family

$$\begin{aligned} \widetilde{J_{\mathcal{M}}(p)} = & \big\langle x_1x_5 - y_1x_2^2, x_1x_8 - y_4y_{26}x_3x_4, x_1x_9 - y_4x_4^2, x_2x_3 - y_9x_1x_6, \\ & x_3x_5 - y_1y_9x_2x_6, x_3x_{11} - y_{20}x_7^2, x_4x_8 - y_{26}x_3x_9, x_5x_{13} - y_{35}x_{10}^2, \\ & x_6x_8 - y_{36}x_7^2, x_6x_{10} - y_{38}x_5x_{11}, x_6x_{13} - y_{35}y_{38}x_{10}x_{11}, \\ & x_9x_{11} - y_{48}y_{50}x_8x_{12}, x_9x_{13} - y_{48}x_{12}^2, x_{11}x_{12} - y_{50}x_8x_{13}\big\rangle + \\ & \langle \boldsymbol{x}^{\boldsymbol{u}_i} \rangle \end{aligned}$$

where the $\boldsymbol{x}^{\boldsymbol{u}_i}$ are the monomial generators of $\mathcal{M}$ not appearing in one of the binomials. Then one can check that the point

$$(y_1, y_4, y_9, y_{20}, y_{26}, y_{35}, y_{36}, y_{38}, y_{48}, y_{50}) = (1, 1, 0, 0, 0, 1, 1, 0, 1, 0)$$

gives the ideal I_{Δ_1} corresponding to the subdecomposition Δ_1 (see Figure VI.11). On the other hand,

$$(y_1, y_4, y_9, y_{20}, y_{26}, y_{35}, y_{36}, y_{38}, y_{48}, y_{50}) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

corresponds to the cell decomposition Δ_2 into the five tetragons

$$\begin{aligned} &\mathrm{ConvHull}\{p_1, p_2, p_3, p_5, p_6\}, \quad \mathrm{ConvHull}\{p_1, p_3, p_4, p_8, p_9\}, \\ &\mathrm{ConvHull}\{p_3, p_6, p_7, p_8, p_{11}\}, \quad \mathrm{ConvHull}\{p_5, p_6, p_{10}, p_{11}, p_{13}\}, \text{ and} \\ &\mathrm{ConvHull}\{p_8, p_9, p_{11}, p_{12}, p_{13}\}, \end{aligned}$$

shown in Figure VI.13. But this means that I_Δ, arising from the decomposition

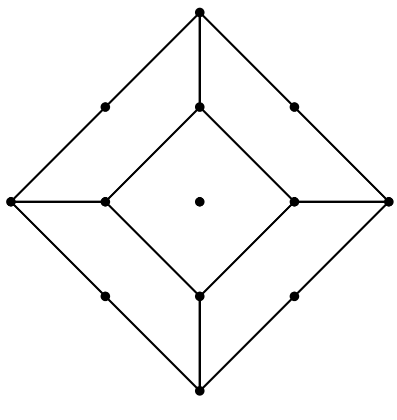

Figure VI.13: The cell decomposition corresponding to the torus of V

Δ of Q into two triangles, does not lie on the component V because Δ is not a subdecomposition of Δ_2.

◊

On the other hand, it is not clear if every subdecomposition of $(\Delta, P_{\max})$, that is coherent on each $\delta \in \Delta$ arises from the general ideal of a face of F_I. Moreover, it could be that the subdecomposition arising from the general ideal of a face of F_I as above is a regular subdecomposition. The open question remains if there is an analog to the coherent case, *i.e.* to Theorem VI.6.6.

Conjecture VI.6.15. *The subdecomposition (Δ_1, P_{max}) of (Δ, P_{max}) given by a reduced general ideal of a face F_1 of F_I is in fact regular. Moreover, for every normal regular subdecomposition (Δ_1, P_{max}) of (Δ, P_{max}) the ideal I_{Δ_1} is the general ideal of a face of F_I.*

This would mean that the correspondence from Theorem VI.6.4 maps components in the toric Hilbert scheme to components in the moduli space of stable toric pairs. We have already shown that this is true for the coherent components of both sides.

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Acknowledgements

I am very grateful to my advisor Klaus Altmann for his help and advice, he always provided me with feedback and constructive criticism when needed. Further, my gratitude goes to the International Research Training Group IGK 870: Arithmetic and Geometry for their support and the scholarship. Also I would like to thank Nathan O. Ilten for all the fruitful discussions and for the proofreading of this thesis. I owe many thanks to Diane Maclagan for her feedback and remarks on my ideas and constructions. I am grateful to Gregory G. Smith for his comments on my results. Furthermore, Daniel R. Grayson and Michael E. Stillman deserve my gratitude for their help on my Macaulay2 codes that proved to be invaluable for finding and computing all my examples. Moreover, I could even use one of Mike's machines to calculate them.

I thank all my colleagues, Andreas Hochenegger, Nathan Ilten, Lars Kastner, Lars Petersen, Hendrik Süß, and Robert Vollmert, for all their collaboration and help, and especially our secretary Mary Metzler-Kliegl for her reliable assistance. I would like to thank Christian Haase for pointing out to me the non regular subdecomposition I was looking for. I am also grateful to Swantje Gährs, Lars Kastner and Matthias Lenz for reading and commenting on various versions of my work. I am very thankful to my parents, Irene and Roland for everything they have made possible for me. Finally, I thank Julia for her support, for understanding my enthusiasm, and for all her help.

Zusammenfassung

Diese Dissertation befasst sich mit der Analyse und Konstruktion der irreduziblen Komponenten von torischen Hilbert Schemata.

Das torische Hilbert Schema parametrisiert alle Ideale in einem gegebenen multigraduierten Ring, die dieselbe multigraduierte Hilbertfunktion haben wie das torische Ideal. Dieses Schema wurde von V. Arnol′d, B. Sturmfels, I. Peeva, M. Stillman, D. Maclagan, R. Thomas und anderen entwickelt und untersucht. Es ist bekannt, dass das torische Ideal auf einer eindeutigen und somit ausgezeichneten irreduziblen Komponente des torischen Hilbert Schemas liegt, der sogenannten kohärenten Komponente. Die Normalisierung der kohärenten Komponente ist die torische Varietät, die durch den Gröbner Fächer des torischen Ideals gegeben ist.

Darüber hinaus kann ein torisches Hilbert Schema weitere irreduzible Komponenten, die sogenannten nicht-kohärenten Komponenten besitzen. Durch Ergebnisse von M. Haiman und B. Sturmfels wird gezeigt, dass die zugrunde liegende reduzierte Struktur jeder nicht-kohärenten Komponente eine projektive torische Varietät ist. Das bedeutet, dass auch die nicht-kohärenten Komponenten, respektive deren Normalisierungen durch polyedrische Fächer gegeben sind.

Der Hauptteil dieser Dissertation besteht aus der expliziten Konstruktion von sogenannten verallgemeinerten universellen Familien, die die nicht-kohärenten Komponenten parametrisieren. Überdies zeigen wir, dass die Normalisierung der zugehörigen nicht-kohärenten Komponente die torische Varietät assoziiert zum Gröbner Fächer der verallgemeinerten universellen Familie ist.

Diese Konstruktion ermöglicht die Berechnung konkreter Beispiele, die unter anderem zeigen, dass es torische Hilbert Schemata gibt, bei denen der Schnitt zweier Komponenten die Vereinigung von Orbits sein kann aber nicht muss. Darüber hinaus gibt es Schemata mit eingebetteten Komponenten, sogar in der kohärenten Komponente. Eine weitere Anwendung der Konstruktion ist die Stratifizierung des torischen Hilbert Schemas anhand der möglichen Subtorus Wirkungen. Wir zeigen, dass jedes Stratum durch die Vereinigung von Seiten der Polytope gegeben ist, die die Komponenten beschreiben.

Abschließend geben wir für ein Gitterpolytop Q die Beschreibung des Modulraums der stabilen torischen Paare mit punktierter Zerlegung von Q nach V. Alexeev an. Dann stellen wir eine Beziehung zwischen Orbits des torischen Hilbert Schemas, gegeben durch die Gitterpunkte von Q, und Strata im Modulraum der stabilen torischen Paare, die der gleichen Unterteilung von Q entsprechen, her. Insbesondere verbindet diese Beziehung genau die kohärenten Komponenten beider Räume.