Summary

In analysis integral representations for functions play a significant role. This in the case in particular in view of ordinary and partial differential equations. Those differential equations appear in most problems of applied mathematics. Generally integral representations are formulated for regular domains, i.e. for limited domains with (piece-wise) smooth boundary.

In the center of my investigations are integral representations in complex analysis. Here are the Cauchy formula and (as a special case) the Schwarz formula in general known. These representations refer to analytic functions. Considered from the theory of differential equations they are the solutions of the Cauchy-Riemann differential equation

$$\partial_{\bar{z}}\omega = \frac{1}{2}(\partial_x\omega + i\partial_y\omega) = 0.$$

More generally functions from the Sobolev space W_1^1 are representable with the help of the Cauchy-Pompeiu formula, which in the case of the Cauchy-Riemann equation considers with the Cauchy formula. Generalized Cauchy-Pompeiu integral representations have been developed for functions from $W^{n,p}(D;\mathbb{C})$ for $n \in \mathbb{N}$ and $1 \leqslant p$.

These representations include besides the classical Cauchy-Pompeiu formula also the Green representation formula for the Poisson equation.

This hierarchy of integral representations is developed from the Gauss integral theorem and arises in an iterative way. Its use is exemplarily explained: The simplest Cauchy-Pompeiu formula has the form $\omega = C\omega + Tf$, where

$$C\omega(z) = \frac{1}{2\pi i}\int_{\partial D}\omega(\zeta)\frac{d\zeta}{\zeta - z},$$

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

and $\omega_{\bar{z}} = f$. The Pompeiu operator T has the properties $\partial_{\bar{z}}Tf = f$, $\partial_z Tf = \Pi f$ where

$$\Pi f(z) = -\frac{1}{\pi}\int_D f(\xi)\frac{d\xi d\eta}{(\xi - z)^2} \tag{1}$$

Here the derivative are in distributional sense and (1) is to be understood as a Cauchy main value integral.
For the solution of the inhomogeneous Beltrami equation

$$\omega_{\bar{z}} + \mu\omega_z = f, \tag{2}$$

where $|\mu(z)| \leqslant q_0 < 1$, we search a particular solution in the form $\omega = T\rho$ with unknown ρ. Inserting this in the differential equation (2) this gives

$$\rho + \mu\Pi\rho = f.$$

This is a unique integral equation for ρ, where $\mu\Pi$ is a contraction. Thus this integral equation is uniquely solvable. This principle can extended to move complicated partial elliptic differential equation of any order. Including the general solution of the related homogeneous problem (in the linear case) a solution for correctly posed boundary value problem can be attained.
The Cauchy-Pompeiu integral representations have proved useful still in other regard. The kernel functions, which are basic solutions of certain differential operators are expressible as derivations of suitable polyharmonic Green functions. Thus separation of the integral representation is achieved is a sum of terms, one belonging to the kernel of the involved differential operator the other from the orthogonal complement of this kernel in $L^2(D;\mathbb{C})$. In principle, this orthogonal decomposition is available for differential operators of the form $\partial_z^\mu \partial_{\bar{z}}^\nu$, where $\mu, \nu \geq 0$. This is done for regular domains, see [26]. Applications and generalizations (in view Quaternionic- and Clifford analysis) have been found, see [30].
For special domains the integral representations are explicit, as soon as the Green functions of any order are known. This is the case for the unit circle. In my work the situation is examined for the upper half plane in $\mathbb{C}$. This thesis consists of 5 chapters. In chapter

1 the Green functions of higher order is introduced and its properties explained. The investigation is based on [3] and [28]. The Gauss theorems, Cauchy-Pompeiu representations, Neumann functions are examined in chapter 2. The basic boundary value problems for analytic functions in the upper half plane are studied in chapter 3. In chapter 4 these boundary value problems are solved for the inhomogeneous Cauchy-Riemann equation in the upper half plane. Finally in chapter 5 Dirichlet problem for the inhomogeneous polyharmonic equation is treated in the upper half plane. To this end the Cauchy-Pompeiu formula is modified with the help of the Green function from chapter 1. Under certain necessary and sufficient solvability conditions this representation is shown to be the uniquely given solution.