5 Dirichlet problem for the inhomogeneous polyharmonic equation in the upper half plane

The representation formula (2.17) suggests a Dirichlet problem for the inhomogeneous polyharmonic equation. For treating this problem the kernel functions in (2.17) have to be calculated first. This is done by using the next rule.

Lemma 10 *For $f, g \in C^{2\rho}(D;\mathbb{C})$, $1 \leqslant \rho$, $D \subset \mathbb{C}$ open and g harmonic, i.e. $\partial_z\partial_{\bar z} g = 0$ in D*

$$(\partial_z\partial_{\bar z})^\rho fg = g(\partial_z\partial_{\bar z})^\rho f + \sum_{\tau=0}^{\rho-1}\binom{\rho}{\tau}\{\partial_z^\tau\partial_{\bar z}^\rho f\partial_z^{\rho-\tau}g + \partial_z^\rho\partial_{\bar z}^\tau f\partial_{\bar z}^{\rho-\tau}g\}.$$

The proof follows by induction.
By the symmetry of the Green functions Lemma 3 can be reformulated.

Lemma 11 *For $1 \leqslant \rho < n$, $z, \zeta \in \overline{\mathbb{H}}$, $z \neq \zeta$,*

$$(\partial_\zeta\partial_{\bar\zeta})^\rho G_n(z,\zeta) = G_{n-\rho}(z,\zeta)$$

$$+\sum_{\mu=0}^{\rho-1}\frac{(z-\bar z)^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_\zeta\partial_{\bar\zeta})^{\rho-\mu}\breve{G}_{n-\mu}(z,\zeta)$$

with

$$\partial_\zeta\partial_{\bar\zeta}\breve{G}_\tau(z,\zeta) = (\zeta-\bar\zeta)^{\tau-2}g_1(z,\zeta)$$

and

$$g_1(z,\zeta) = \frac{1}{\bar\zeta - z} - \frac{1}{\zeta - \bar z},$$

for $2 \leqslant \tau$.

Lemma 12 *For $t \in \mathbb{R}$, $z \in \mathbb{H}$ and $0 \leqslant 2\nu \leqslant n-2$*

$$(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-1}G_n(z,t) = \sum_{\mu=0}^{n-2\nu-2}(-1)^{n-\mu-\nu}\frac{(n-\mu-\nu-2)!}{\nu!(n-\mu-1)!}$$
$$\times(z-\bar z)^{n-\mu-1}g_{n-\mu-2\nu-1}(z,t), \tag{5.1}$$

where $g_\alpha(z,\zeta)$ *is given in (1.9) for* $1 \leqslant \alpha$.

Proof For $1 \leqslant k$

$$\partial_\zeta^\rho\partial_{\bar\zeta}^\tau(\zeta-\bar\zeta)^k = \begin{cases}(-1)^\tau\frac{k!}{(k-\rho-\tau)!}(\zeta-\bar\zeta)^{k-\rho-\tau}, & \rho+\tau\leqslant k,\\ 0, & k<\rho+\tau.\end{cases}$$

Moreover,

$$\partial_\zeta^\sigma g_1(z,\zeta) = (-1)^{\sigma+1}\frac{\sigma!}{(\zeta-\bar z)^{\sigma+1}},$$
$$\partial_{\bar\zeta}^\sigma g_1(z,\zeta) = (-1)^{\sigma}\frac{\sigma!}{(\bar\zeta-z)^{\sigma+1}}.$$

Thus applying Lemma 10

$$(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-\mu-1}\breve G_{n-\mu}(z,\zeta) = g_1(z,\zeta)(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-\mu-2}(\zeta-\bar\zeta)^{n-\mu-2}$$
$$+\sum_{\tau=0}^{min\{\nu,n-\mu-\nu-3\}}(-1)^{n-\nu-\mu}\binom{n-\mu-\nu-2}{\tau}$$
$$\times\frac{(n-\mu-2)!(n-\mu-\nu-\tau-2)!}{(\nu-\tau)!}(\zeta-\bar\zeta)^{\nu-\tau}g_{n-\mu-\nu-\tau-1}(z,\zeta).$$

For $\zeta=\bar\zeta$ this is

$$(\partial_\zeta\partial_{\bar\zeta})^{n-\nu-\mu-1}\breve G_n(z,\zeta) = \begin{cases}(-1)^{n-\nu-\mu}\frac{(n-\mu-2)!(n-\mu-2-\nu)!}{\nu!} & \\ \times g_{n-\mu-2\nu-1}(z,\zeta), & 2\nu+\mu+2\leqslant n,\\ 0, & n<2\nu+\mu+2,\end{cases}$$

so that for $\zeta = \bar{\zeta}$

$$(\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-1} G_n(z,\zeta) = \sum_{\mu=0}^{n-2\nu-2} (-1)^{n-\nu-\mu} \frac{(n-\nu-\mu-2)!}{\nu!(n-\mu-1)!}$$

$$\times (z-\bar{z})^{n-\mu-1} g_{n-\mu-2\nu-1}(z,\zeta).$$

Lemma 13 *For $t \in \mathbb{R}$, $z \in \mathbb{H}$ and $2 \leqslant 2\nu \leqslant n-2$*

$$\partial_\zeta^{n-\nu} \partial_{\bar{\zeta}}^{n-\nu-1} G_n(z,t) = \sum_{\mu=0}^{n-2\nu-1} (-1)^{n-\nu-\mu} \frac{(n-\nu-\mu-2)!}{(\nu-1)!(n-\mu-1)!}$$

$$\times (z-\bar{z})^{n-\mu-1} g_{n-\mu-2\nu}(z,t) + \sum_{\mu=0}^{n-2\nu-2} (-1)^{n-\nu-\mu} \frac{(n-\nu-\mu-2)!}{\nu!(n-\mu-1)!}$$

$$\times (z-\bar{z})^{n-\mu-1} \partial_\zeta g_{n-\mu-2\nu-1}(z,t) \tag{5.2}$$

and

$$\partial_\zeta^{n} \partial_{\bar{\zeta}}^{n-1} G_n(z,t) = -\left(\frac{z-\bar{z}}{t-\bar{z}}\right)^n \frac{1}{t-z}. \tag{5.3}$$

Proof For proper μ and ν

$$\partial_\zeta^{n-\nu-\mu-1} \partial_{\bar{\zeta}}^{n-\nu-\mu-2} (\zeta-\bar{\zeta})^{n-\mu-2} = \begin{cases} (-1)^{n-\nu-\mu} \frac{(n-\mu-2)!}{(2\nu+\mu+1-n)!} & \\ \times (\zeta-\bar{\zeta})^{2\nu+\mu+1-n}, & n \leqslant 2\nu+\mu+1, \\ 0, & 2\nu+\mu+1 < n. \end{cases}$$

Lemma 10 applied shows

$$\partial_\zeta^{n-\nu-\mu} \partial_{\bar{\zeta}}^{n-\nu-\mu-1} \breve{G}_{n-\mu}(z,\zeta) = g_1(z,\zeta) \partial_\zeta (\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-\mu-2}$$

$$+ \partial_\zeta g_1(z,\zeta) (\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-\mu-2} (\zeta-\bar{\zeta})^{n-\mu-2} + \sum_{\tau=0}^{min\{\nu, n-\nu-\mu-3\}} (-1)^{n-\mu-1}$$

$$\times \binom{n-\mu-\nu-2}{\tau} \frac{(n-\mu-2)!(n-\mu-\nu-\tau-2)!}{(\nu-\tau)!}$$

$$\times\{(\nu-\tau)(\zeta-\bar{\zeta})^{\nu-\tau-1} g_{n-\mu-\nu-\tau-1}(z,\zeta)+(\zeta-\bar{\zeta})^{\nu-\tau}\partial_\zeta g_{n-\mu-\nu-\tau-1}(z,\zeta)\}.$$

Arguing as in the preceding proof this is for $\zeta=\bar{\zeta}$ and $1\leqslant\nu$

$$\partial_\zeta^{n-\nu-\mu}\partial_{\bar{\zeta}}^{n-\nu-\mu-1}\breve{G}_{n-\mu}(z,\zeta)$$
$$=\begin{cases}(-1)^{n-\mu-\nu}(n-\mu-2)!g_1(z,\zeta), & n=2\nu+\mu+1,\\ (-1)^{n-\mu-\nu}\frac{(n-\mu-2)!(n-\mu-\nu-2)!}{\nu!}\{\nu g_{n-\mu-2\nu}(z,\zeta) & \\ +\partial_\zeta g_{n-\mu-2\nu-1}(z,\zeta)\}, & 2\nu+\mu+2\leqslant n,\\ 0, & n<2\nu+\mu+1,\end{cases}$$

and for $\nu=0$

$$\partial_\zeta^{n-\mu}\partial_{\bar{\zeta}}^{n-\mu-1}\breve{G}_{n-\mu}(z,\zeta)$$
$$=\begin{cases}(-1)^{n-\mu}(n-\mu-2)!g_1(z,\zeta), & n=\mu+1,\\ (-1)^{n-\mu}(n-\mu-2)!^2\partial_\zeta g_{n-\mu-1}(z,\zeta), & \mu+2\leqslant n,\\ 0, & n<\mu+1.\end{cases}$$

Hence for $1\leqslant\nu$ and $\zeta=\bar{\zeta}$

$$\partial_\zeta^{n-\nu}\partial_{\bar{\zeta}}^{n-\nu-1}G_n(z,\zeta)=(-1)^{\nu-1}\frac{(z-\bar{z})^{2\nu}}{(2\nu)!}g_1(z,\zeta)$$

$$+\sum_{\mu=0}^{n-2\nu-2}(-1)^{n-\mu-\nu}\frac{(n-\mu-\nu-2)!}{\nu!(n-\mu-1)!}\{\nu g_{n-\mu-2\nu}(z,\zeta)+\partial_\zeta g_{n-\mu-2\nu-1}(z,\zeta)\}.$$

Observing for $\zeta=\bar{\zeta}$

$$\partial_\zeta G_1(z,\zeta)=-g_1(z,\zeta),$$

$$\partial_\zeta G_{\nu+1}(z,\zeta) = 0$$

for $1 \leqslant \nu$, then for $\zeta = \bar{\zeta}$

$$\partial_\zeta^n \partial_{\bar{\zeta}}^{n-1} G_n(z,\zeta) = \frac{1}{\zeta - \bar{z}} - \frac{1}{\zeta - z} + \sum_{\mu=0}^{n-2} \frac{(z-\bar{z})^{n-\mu-1}}{(\zeta-\bar{z})^{n-\mu}}$$
$$= -\left(\frac{z-\bar{z}}{\zeta-\bar{z}}\right)^n \frac{1}{\zeta - z}.$$

This last formula can be gained from differentiating (1.5) with z interchanged with ζ.

For checking the boundary behaviour of the function given in (2.17) besides the kernel functions (5.1), (5.2), (5.3) also their proper derivatives have to be calculated. On the basis of Lemma 12 and Lemma 13 denote for $1 \leqslant n$, $t \in \mathbb{R}$, $z \in \mathbb{H}$

$$A = \partial_\zeta^n \partial_{\bar{\zeta}}^{n-1} G_n(z,t) = -\left(\frac{z-\bar{z}}{t-\bar{z}}\right)^n \frac{1}{t-z}, \tag{5.4}$$

$$B = (\partial_\zeta \partial_{\bar{\zeta}})^{n-1} G_n(z,t) = \sum_{\mu=1}^{n-1} (-1)^{\mu+1} \frac{1}{\mu} (z-\bar{z})^\mu g_\mu(z,t), \tag{5.5}$$

$$C_\nu = \partial_\zeta^{n-\nu} \partial_{\bar{\zeta}}^{n-\nu-1} G_n(z,t) = \sum_{\mu=2\nu}^{n-1} (-1)^{\mu-\nu-1} \frac{(\mu-\nu-1)!}{\mu!(\nu-1)!} \frac{(z-\bar{z})^\mu}{(t-z)^{\mu-2\nu+1}}$$
$$+ \sum_{\mu=2\nu}^{n-1} (-1)^\nu \frac{(\mu-\nu)!}{\mu!\nu!} \frac{(z-\bar{z})^\mu}{(t-\bar{z})^{\mu-2\nu+1}} \tag{5.6}$$

for $2 \leqslant 2\nu \leqslant n-1$,

$$D_\nu = (\partial_\zeta \partial_{\bar{\zeta}})^{n-\nu-1} G_n(z,t) = \sum_{\mu=2\nu+1}^{n-1} (-1)^{\mu-\nu-1} \frac{(\mu-\nu-1)!}{\mu!\nu!}$$
$$\times (z-\bar{z})^\mu g_{\mu-2\nu}(z,t) \tag{5.7}$$

for $2 \leqslant 2\nu \leqslant n-2$.

Lemma 14 *For* $1 \leqslant n$, $t \in \mathbb{R}$ *and* $z \in \mathbb{H}$

$$\partial_{\bar z} A = -n \frac{(z-\bar z)^{n-1}}{(t-\bar z)^{n+1}},$$

$$\partial_z \partial_{\bar z} A = -n(n-1) \frac{(z-\bar z)^{n-2}}{(t-\bar z)^{n+1}},$$

$$\partial_z^{\nu} \partial_{\bar z}^{\nu+1} A = \sum_{\tau=0}^{\nu} (-1)^{\tau+1} \binom{\nu}{\tau} \frac{(n+\nu-\tau)!}{(n-\nu-\tau-1)!} \frac{(z-\bar z)^{n-\nu-\tau-1}}{(t-\bar z)^{n+\nu-\tau+1}}$$

for $2 \leqslant 2\nu \leqslant n-2$,

$$\partial_z^{\nu} \partial_{\bar z}^{\nu} A = \sum_{\tau=0}^{\nu-1} (-1)^{\tau+1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} \frac{(z-\bar z)^{n-\nu-\tau-1}}{(t-\bar z)^{n+\nu-\tau}}$$

for $2 \leqslant 2\nu \leqslant n-1$.

Proof Rewriting

$$A = \frac{z-\bar z}{|t-z|^2} - \sum_{\nu=1}^{n-1} \frac{(z-\bar z)^{\nu}}{(t-\bar z)^{\nu+1}}$$

and differentiating shows the first two formulas.
Applying $(\partial_z \partial_{\bar z})^{\nu}$ to the first formula and applying Lemma 10 shows

$$\partial_z^{\nu} \partial_{\bar z}^{\nu+1} A = -\sum_{\tau=0}^{\nu} \binom{\nu}{\tau} \frac{n!(-1)^{\tau}}{(n-1-\nu-\tau)!} \frac{(z-\bar z)^{n-1-\nu-\tau}}{(t-\bar z)^{n+1+\nu-\tau}} \frac{(n+\nu-\tau)!}{n!}.$$

Similarly from the second formula

$$\partial_z^{\nu} \partial_{\bar z}^{\nu} A = -\sum_{\tau=0}^{\nu-1} \binom{\nu-1}{\tau} \frac{n!(-1)^{\tau}}{(n-1-\nu-\tau)!} \frac{(z-\bar z)^{n-1-\nu-\tau}}{(t-\bar z)^{n+\nu-\tau}} \frac{(n+\nu-\tau-1)!}{n!}$$

follows.

Lemma 15 *For* $1 \leqslant n$, $t \in \mathbb{R}$ *and* $z \in \mathbb{H}$

$$\partial_{\bar{z}} B = -g_1(z,t) - \frac{(z-\bar{z})^{n-1}}{(t-\bar{z})^n} - \sum_{\mu=1}^{n-2} (-1)^\mu \frac{(z-\bar{z})^\mu}{(t-z)^{\mu+1}}$$

$$= -\frac{(z-\bar{z})^{n-1}}{(t-\bar{z})^n} + \left(\frac{\bar{z}-z}{t-z}\right)^{n-1} \frac{1}{t-\bar{z}},$$

$$\partial_z \partial_{\bar{z}} B = -(n-1)(z-\bar{z})^{n-2} g_n(z,t),$$

$$\partial_z^\nu \partial_{\bar{z}}^{\nu+1} B = \sum_{\tau=0}^{\nu-1} (-1)^{\nu-1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-2)!} \frac{(z-\bar{z})^{n-\nu-\tau-2}}{(t-z)^{n+\nu-\tau-1}}$$

$$+ \sum_{\tau=0}^{\nu} (-1)^{n+1-\tau} \binom{\nu}{\tau} \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} \frac{(z-\bar{z})^{n-\nu-\tau-1}}{(t-\bar{z})^{n+\nu-\tau}}$$

for $2 \leqslant 2\nu \leqslant n-2$,

$$\partial_z^\nu \partial_{\bar{z}}^\nu B = \sum_{\tau=0}^{\nu-1} (-1)^\nu \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-1)!} (z-\bar{z})^{n-\nu-\tau-1} g_{n+\nu-\tau-1}(z,t)$$

for $2 \leqslant 2\nu \leqslant n-1$.

Proof Differentiating (5.5) gives

$$\partial_{\bar{z}} B = \sum_{\mu=1}^{n-1} \{(-1)^\mu (z-\bar{z})^{\mu-1} g_\mu(z,t) - \frac{(z-\bar{z})^\mu}{(t-\bar{z})^{\mu+1}}\}$$

$$= -g_1(z,t) + \sum_{\mu=2}^{n-1} \{(-1)^\mu \frac{(z-\bar{z})^{\mu-1}}{(t-z)^\mu} + \frac{(z-\bar{z})^{\mu-1}}{(t-\bar{z})^\mu}\} - \sum_{\mu=2}^{n} \frac{(z-\bar{z})^{\mu-1}}{(t-\bar{z})^\mu}$$

which is the first expression. Differentiating again shows

$$\partial_z \partial_{\bar{z}} B = -(n-1) \frac{(z-\bar{z})^{n-2}}{(t-\bar{z})^n} - (n-1) \left(\frac{\bar{z}-z}{t-z}\right)^{n-2} \frac{1}{(t-z)^2}$$

$$= -(n-1)(z-\bar{z})^{n-2} g_n(z,t).$$

Proceeding as in the preceding proof leads to

$$\partial_z^\nu \partial_{\bar{z}}^\nu B = -\sum_{\tau=0}^{\nu-1} \binom{\nu-1}{\tau} \frac{(n-1)!(-1)^\nu}{(n-1-\tau-\nu)!} \frac{(z-\bar{z})^{n-1-\tau-\nu}}{(t-z)^{n+\nu-1-\tau}}$$

$$\times \frac{(n-1+\nu-1-\tau)!}{(n-1)!} - \sum_{\tau=0}^{\nu-1} \binom{\nu-1}{\tau} \frac{(n-1)!(-1)^{\tau+n}}{(n-1-\tau-\nu)!} \frac{(z-\bar{z})^{n-1-\tau-\nu}}{(t-\bar{z})^{n+\nu-1-\tau}}$$

$$\times \frac{(n+\nu-\tau-2)!}{(n-1)!} = \sum_{\tau=0}^{\nu-1} (-1)^\nu \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-1)!}$$

$$\times (z-\bar{z})^{n-\nu-\tau-1} g_{n+\nu-\tau-1}(z,t)$$

and

$$\partial_z^\nu \partial_{\bar{z}}^{\nu+1} B = \sum_{\tau=0}^{\nu-1} (-1)^{\nu+1} \binom{\nu-1}{\tau} \frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-2)!}$$

$$\times (z-\bar{z})^{n-\nu-\tau-2} g_{n+\nu-\tau-1}(z,t) + \sum_{\tau=0}^{\nu-1} (-1)^{n-\tau-1} \binom{\nu-1}{\tau}$$

$$\times \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} \frac{(z-\bar{z})^{n-\nu-\tau-1}}{(t-\bar{z})^{n+\nu-\tau}} = \sum_{\tau=1}^{\nu} (-1)^{\nu+1} \binom{\nu-1}{\tau-1}$$

$$\times \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} (z-\bar{z})^{n-\nu-\tau-1} \frac{(-1)^{n+\nu-\tau}}{(t-\bar{z})^{n+\nu-\tau}} + \sum_{\tau=0}^{\nu-1} (-1)^{\nu+1} \binom{\nu-1}{\tau}$$

$$\times \frac{(n+\nu-\tau-1)!}{(n-\nu-\tau-1)!} (z-\bar{z})^{n-\nu-\tau-1} \frac{(-1)^{n+\nu-\tau}}{(t-\bar{z})^{n+\nu-\tau}} + \sum_{\tau=0}^{\nu-1} (-1)^{\nu+1} \binom{\nu-1}{\tau}$$

$$\times\frac{(n+\nu-\tau-2)!}{(n-\nu-\tau-2)!}\frac{(z-\bar{z})^{n-\nu-\tau-2}}{(t-z)^{n+\nu-\tau-1}}.$$

There are the formulas in the lemma.

Lemma 16 *For* $2\leqslant 2\nu\leqslant n-1$, $t\in\mathbb{R}$, $z\in\mathbb{H}$

$$(\partial_z\partial_{\bar{z}})^\rho C_\nu=\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\rho+\mu-\nu-1}\binom{\rho}{\tau}\frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau)!}{(\nu-1)!(\mu-\rho-\tau)!(\mu-2\nu)!}$$

$$\times\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-z)^{\mu-2\nu+\rho-\tau+1}}+\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\nu+\tau}\binom{\rho}{\tau}$$

$$\times\frac{(\mu-\nu)!(\mu-2\nu+\rho-\tau)!}{\nu!(\mu-\rho-\tau)!(\mu-2\nu)!}\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-2\nu+\rho-\tau+1}}$$

for $0\leqslant\rho\leqslant\nu$,

$$(\partial_z\partial_{\bar{z}})^\nu C_\nu=\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\nu}\{(-1)^{\mu-1}\binom{\nu}{\tau}\binom{\mu-\nu-1}{\nu-1}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-z)^{\mu-\nu-\tau+1}}$$

$$+(-1)^{\nu+\tau}\binom{\nu}{\tau}\binom{\mu-\nu}{\nu}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}}\}$$

$$=-\sum_{\tau=0}^{\nu}\binom{\nu}{\tau}(z-\bar{z})^{\nu-\tau}g_{\nu-\tau+1}(z,t)$$

$$+\sum_{\mu=2\nu+1}^{n-1}\sum_{\tau=0}^{\nu}(-1)^{\mu-1}\binom{\nu}{\tau}\{\binom{\mu-\nu-1}{\nu-1}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-z)^{\mu-\nu-\tau+1}}$$

$$+(-1)^{\mu-\nu-\tau+1}\binom{\mu-\nu}{\nu}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}}\}$$

$$=\sum_{\mu=2\nu}^{n-1}\{(-1)^{\mu-1}\binom{\mu-\nu-1}{\nu-1}\frac{(z-\bar{z})^{\mu-2\nu}(t-\bar{z})^{\nu}}{(t-z)^{\mu-\nu+1}}$$

$$+\binom{\mu-\nu}{\nu}\frac{(z-\bar{z})^{\mu-2\nu}(t-z)^{\nu}}{(t-\bar{z})^{\mu-\nu+1}}\},$$

$$\partial_z^{\rho}\partial_{\bar{z}}^{\rho+1}C_\nu=\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\rho+\mu-\nu}\binom{\rho}{\tau}\frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau)!}{(\nu-1)!(\mu-\rho-\tau-1)!(\mu-2\nu)!}$$

$$\times\frac{(z-\bar{z})^{\mu-\rho-\tau-1}}{(t-z)^{\mu-2\nu+\rho-\tau+1}}+\sum_{\mu=2\nu}^{n-1}\sum_{\tau=0}^{\rho+1}(-1)^{\tau+\nu}\binom{\rho+1}{\tau}\frac{(\mu-\nu)!(\mu-2\nu+\rho-\tau+1)!}{\nu!(\mu-\rho-\tau)!(\mu-2\nu)!}$$

$$\times\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-2\nu+\rho-\tau+2}}$$

for $0\leqslant\rho<\nu$,

$$\partial_z^{\nu}\partial_{\bar{z}}^{\nu+1}C_\nu=(-1)^{n-1}\frac{(n-\nu-1)!}{(\nu-1)!(n-2\nu-1)!}\frac{(z-\bar{z})^{n-2\nu-1}(t-\bar{z})^{\nu-1}}{(t-z)^{n-\nu}}$$

$$+\frac{(\mu-\nu)!}{\nu!(\mu-2\nu-1)!}\frac{(z-\bar{z})^{n-2\nu-1}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+1}},$$

$$(\partial_z\partial_{\bar{z}})^{\nu+1}C_\nu=(-1)^{n-1}\frac{(n-\nu-1)!}{(\nu-1)!(n-2\nu-2)!}\frac{(z-\bar{z})^{n-2\nu-2}(t-\bar{z})^{\nu-1}}{(t-z)^{n-\nu}}$$

$$+(-1)^{n-1}\frac{(n-\nu)!}{(\nu-1)!(n-2\nu-1)!}\frac{(z-\bar{z})^{n-2\nu-1}(t-\bar{z})^{\nu-1}}{(t-z)^{n-\nu+1}}$$

$$+\frac{(n-\nu)!}{\nu!(n-2\nu-2)!}\frac{(z-\bar{z})^{n-2\nu-2}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+1}}$$

$$-\frac{(n-\nu)!}{(\nu-1)!(n-2\nu-1)!}\frac{(z-\bar{z})^{n-2\nu-1}(t-z)^{\nu-1}}{(t-\bar{z})^{n-\nu+1}},$$

$$(\partial_z\partial_{\bar{z}})^{\nu+\rho}C_\nu=\sum_{\tau=0}^{min\{\rho,n-2\nu-1\}}\sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho-1,n-2\nu-1-\tau\}}(-1)^{n-\rho}\binom{\rho}{\tau}\binom{\rho-1}{\sigma}$$

$$\times \frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-\bar{z})^{\nu+\sigma-\rho}}{(t-z)^{n-\nu+\rho-\tau}}$$

$$+ \sum_{\tau=0}^{min\{\rho-1,n-2\nu-1\}} \sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho,n-2\nu-1-\tau\}} (-1)^{\rho-\sigma+\tau} \binom{\rho-1}{\tau} \binom{\rho}{\sigma}$$

$$\times \frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{\nu+\sigma-\rho}}{(t-\bar{z})^{n-\nu+\rho-\tau}}$$

for $2(\nu+\rho) \leqslant n-1$,

$$\partial_z^{\nu+\rho} \partial_{\bar{z}}^{\nu+\rho+1} C_\nu = \sum_{\tau=0}^{min\{\rho,n-2\nu-1\}} \sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho,n-2\nu-1-\tau\}} (-1)^{n-1-\rho} \binom{\rho}{\tau} \binom{\rho}{\sigma}$$

$$\times \frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma-1)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-\bar{z})^{\nu-\rho+\sigma-1}}{(t-z)^{n-\nu+\rho-\tau}}$$

$$+ \sum_{\tau=0}^{min\{\rho,n-2\nu-1\}} \sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho,n-2\nu-1-\tau\}} (-1)^{\rho-\sigma+\tau} \binom{\rho}{\tau} \binom{\rho}{\sigma}$$

$$\times \frac{(n-\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{\nu-\rho+\sigma}}{(t-\bar{z})^{n-\nu+\rho-\tau+1}}$$

for $2(\nu+\rho) \leqslant n-1$.

Lemma 17 *For* $0 \leqslant 2\nu \leqslant n-2$, $t \in \mathbb{R}$ *and* $z \in \mathbb{H}$

$$(\partial_z \partial_{\bar{z}})^\rho D_\nu = \sum_{\mu=2\nu+1}^{n-1} \sum_{\tau=0}^{\rho} (-1)^{\mu-\nu-1+\rho} \binom{\rho}{\tau}$$

$$\times \frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau-1)!}{\nu!(\mu-\tau-\rho)!(\mu-2\nu-1)!} (z-\bar{z})^{\mu-\tau-\rho} g_{\mu-2\nu+\rho-\tau}(z,t)$$

for $0 \leqslant \rho \leqslant \nu$,

$$\partial_z^{\rho}\partial_{\bar{z}}^{\rho+1}D_\nu = \sum_{\mu=2\nu+1}^{n-1}\sum_{\tau=0}^{\rho}(-1)^{\mu-\nu+\rho}\binom{\rho}{\tau}$$

$$\times\frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau-1)!}{\nu!(\mu-\tau-\rho-1)!(\mu-2\nu-1)!}\frac{(z-\bar{z})^{\mu-\rho-\tau-1}}{(t-z)^{\mu-2\nu+\rho-\tau}}$$

$$+\sum_{\mu=2\nu+1}^{n-1}\sum_{\tau=0}^{\rho+1}(-1)^{\tau+\nu-1}\binom{\rho+1}{\tau}\frac{(\mu-\nu-1)!(\mu-2\nu+\rho-\tau)!}{\nu!(\mu-\tau-\rho)!(\mu-2\nu-1)!}$$

$$\times\frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-2\nu+\rho-\tau+1}}$$

for $0 \leqslant \rho \leqslant \nu$,

$$\partial_z^{\nu}\partial_{\bar{z}}^{\nu+1}D_\nu = \sum_{\mu=2\nu+1}^{n-1}\{\sum_{\tau=0}^{\nu}(-1)^{\mu}\binom{\nu}{\tau}\binom{\mu-\nu-1}{\nu}\frac{(z-\bar{z})^{\mu-\tau-\nu-1}}{(t-z)^{\mu-\nu-\tau}}$$

$$+\sum_{\tau=0}^{\nu+1}(-1)^{\tau+\nu-1}\binom{\nu+1}{\tau}\binom{\mu-\nu-1}{\nu}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}}\}$$

$$= -g_1(z,t)-\sum_{\tau=0}^{\nu-1}\binom{\nu}{\tau}\frac{(z-\bar{z})^{\nu-\tau}}{(t-z)^{\nu-\tau+1}}+\sum_{\tau=0}^{\nu}(-1)^{\nu-\tau-1}\binom{\nu+1}{\tau}\frac{(z-\bar{z})^{\nu-\tau+1}}{(t-\bar{z})^{\nu-\tau+2}}$$

$$+\sum_{\mu=2\nu+2}^{n-1}\{\sum_{\tau=0}^{\nu}(-1)^{\mu}\binom{\nu}{\tau}\binom{\mu-\nu-1}{\nu}\frac{(z-\bar{z})^{\mu-\tau-\nu-1}}{(t-z)^{\mu-\nu-\tau}}$$

$$+\sum_{\tau=0}^{\nu+1}(-1)^{\tau+\nu-1}\binom{\nu+1}{\tau}\binom{\mu-\nu-1}{\nu}\frac{(z-\bar{z})^{\mu-\nu-\tau}}{(t-\bar{z})^{\mu-\nu-\tau+1}}\}$$

$$= \sum_{\mu=2\nu+1}^{n-1}\binom{\mu-\nu-1}{\nu}\{(-1)^{\mu}\frac{(z-\bar{z})^{\mu-2\nu-1}(t-\bar{z})^{\nu}}{(t-z)^{\mu-\nu}}$$

$$+\frac{(z-\bar{z})^{\mu-2\nu-1}(t-z)^{\nu+1}}{(t-\bar{z})^{\mu-\nu+1}}\},$$

$$(\partial_z\partial_{\bar{z}})^{\nu+1}D_\nu = \sum_{\mu=2\nu+1}^{n-1}\sum_{\tau=0}^{\nu+1}(-1)^\mu\binom{\nu+1}{\tau}\binom{\mu-\nu-1}{\nu}$$

$$\times(\mu-\tau-\nu)(z-\bar{z})^{\mu-\tau-\nu-1}g_{\mu-\nu-\tau+1}(z,t) = (-1)^{n-1}\frac{(n-\nu-1)!}{\nu!(n-2\nu-2)!}$$

$$\times\frac{(z-\bar{z})^{n-2\nu-2}(t-\bar{z})^{\nu}}{(t-z)^{n-\nu}} - \frac{(n-\nu-1)!}{\nu!(n-2\nu-2)!}\frac{(z-\bar{z})^{n-2\nu-1}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+1}},$$

$$\partial_z^{\nu+1}\partial_{\bar{z}}^{\nu+2}D_\nu = (-1)^n\frac{(n-\nu-1)!}{\nu!(n-2\nu-3)!}\frac{(z-\bar{z})^{n-2\nu-3}(t-\bar{z})^{\nu}}{(t-z)^{n-\nu}}$$

$$+(-1)^n\frac{(n-\nu-1)!}{(\nu-1)!(n-2\nu-2)!}\frac{(z-\bar{z})^{n-2\nu-2}(t-\bar{z})^{\nu-1}}{(t-z)^{n-\nu}}$$

$$+\frac{(n-1-\nu)!(n-2\nu-1)}{\nu!(n-2\nu-2)!}\frac{(z-\bar{z})^{n-2\nu-2}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+1}}$$

$$-\frac{(n-1-\nu)!(n-\nu+1)}{\nu!(n-2\nu-2)!}\frac{(z-\bar{z})^{n-2\nu-1}(t-z)^{\nu}}{(t-\bar{z})^{n-\nu+2}},$$

$$(\partial_z\partial_{\bar{z}})^{\nu+\rho}D_\nu = \sum_{\tau=0}^{min\{\rho-1,n-2\nu-2\}}\sum_{\sigma=max\{0,\rho-\nu-1\}}^{min\{\rho-1,n-2\nu-2-\tau\}}(-1)^{n+\rho}\binom{\rho-1}{\tau}\binom{\rho-1}{\sigma}$$

$$\times\frac{(n-\nu+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(\nu-\rho+\sigma+1)!}\frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-\bar{z})^{\nu-\rho+\sigma+1}}{(t-z)^{n-\nu+\rho-\tau-1}}$$

$$+\sum_{\tau=0}^{min\{\rho-1,n-2\nu-2\}}\sum_{\sigma=max\{0,\rho-\nu-1\}}^{min\{\rho-1,n-2\nu-2-\tau\}}(-1)^{\rho+\tau-\sigma}\binom{\rho-1}{\tau}\binom{\rho-1}{\sigma}\frac{1}{n-\nu}$$

$$\times \frac{(n-\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-2)!(\nu-\rho+\sigma+1)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-z)^{\nu-\rho+\sigma+1}}{(t-\bar{z})^{n-\nu+\rho-\tau-2}}$$

for $2(\nu+\rho) \leqslant n-2$,

$$\partial_z^{\nu+\rho}\partial_{\bar{z}}^{\nu+\rho+1} D_\nu = \sum_{\tau=0}^{min\{\rho-1,n-2\nu-2\}} \sum_{\sigma=max\{0,\rho-\nu-1\}}^{min\{\rho,n-2\nu-2-\tau\}} (-1)^{n-\rho+1} \binom{\rho-1}{\tau}\binom{\rho}{\sigma}$$

$$\times \frac{(n-\nu+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(\nu-\rho+\sigma)!} \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-2}(t-\bar{z})^{\nu-\rho+\sigma}}{(t-z)^{n-\nu+\rho-\tau-1}}$$

$$+ \sum_{\tau=0}^{min\{\rho,n-2\nu-1\}} \sum_{\sigma=max\{0,\rho-\nu-1\}}^{min\{\rho-1,n-2\nu-1-\tau\}} (-1)^{\nu+\rho-\sigma} \binom{\rho}{\tau}\binom{\rho-1}{\sigma}$$

$$\times \frac{n-2\nu-1}{n-\nu} \frac{(n-\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(\nu-\rho+\sigma+1)!}$$

$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{\nu-\rho+\sigma+1}}{(t-\bar{z})^{n-\nu+\rho-\tau+1}}$$

for $2(\nu+\rho) \leqslant n-2$.

The proof of the last two lemmas follow step by step as indicated in the formulations of them by direct differentiation using Lemma 10 in the same way as for Lemma 14 and Lemma 15.

The boundary behaviour of the function in (2.17) is checked via the property of the Poisson integral (3.10).

Lemma 18 *Let* $\gamma \in C(\mathbb{R};\mathbb{C})$ *be bounded and* $\alpha, \beta \in \mathbb{R}$ *with* $\beta < \alpha+1$. *Then*

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma(t)\frac{(z-\bar{z})^\alpha}{(t-z)^\beta}dt = 0.$$

Proof Rewriting

$$\frac{(z-\bar{z})^\alpha}{(t-z)^\beta} = \frac{(z-\bar{z})^{\alpha-1}}{(t-z)^{\beta-2}} \frac{t-\bar{z}}{t-z} \frac{z-\bar{z}}{|t-z|^2}$$

$$= (z-\bar{z})^{\alpha+1-\beta} \left(\frac{t-\bar{z}}{t-z} - 1\right)^{\beta-2} \frac{t-\bar{z}}{t-z} \frac{z-\bar{z}}{|t-z|^2},$$

where the first factor on the right-hand side tends to zero with z tending to $\bar{z}$ with the two middle terms are bounded and the last is the Poisson kernel.

Theorem 21 *Let* $1 \leqslant n$ *and* $|z|^{2(n-1)} f \in L_1(\mathbb{H};\mathbb{C})$, $\gamma_\nu \in C(\mathbb{R};\mathbb{C})$ *for* $0 \leqslant 2\nu \leqslant n-1$ *and* $\hat{\gamma_\nu} \in C(\mathbb{R};\mathbb{C})$ *for* $0 \leqslant 2\nu \leqslant n-2$. *Then the Dirichlet problem* $(\partial_z \partial_{\bar{z}})^n w = f$ *in* $\mathbb{H}$, $(\partial_z \partial_{\bar{z}})^\nu w = \gamma_\nu$, $0 \leqslant 2\nu \leqslant n-1$, $\partial_z^\nu \partial_{\bar{z}}^{\nu+1} w = \hat{\gamma_\nu}$, $0 \leqslant 2\nu \leqslant n-2$ *on* $\mathbb{R}$ *is uniquely solvable if and only if*

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{\nu=1}^{n-1} \frac{(z-\bar{z})^\nu}{(t-\bar{z})^{\nu+1}} \gamma_0(t) = 0, \tag{5.8}$$

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(z-\bar{z})^{n-1}}{(t-\bar{z})^{n+1}} \gamma_0(t)dt = 0, \tag{5.9}$$

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{\tau=0}^{\rho-1} (-1)^\tau \binom{\rho-1}{\tau} \frac{(n+\rho-\tau-1)!}{(n-\rho-\tau-1)!} \times \frac{(z-\bar{z})^{n-\rho-\tau-1}}{(t-\bar{z})^{n+\rho-\tau}} \gamma_0(t)dt = 0 \tag{5.10}$$

for $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to\bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{\tau=0}^{\rho} (-1)^\tau \binom{\rho}{\tau} \frac{(n+\rho-\tau)!}{(n-\rho-\tau-1)!} \times \frac{(z-\bar{z})^{n-\rho-\tau-1}}{(t-\bar{z})^{n+\rho-\tau+1}} \gamma_0(t)dt = 0 \tag{5.11}$$

for $2 \leqslant 2\rho \leqslant n-1$,

$$\lim_{z\to\bar{z}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\sum_{\mu=1}^{n-2}(-1)^{\mu}\frac{(z-\bar{z})^{\mu}}{(t-z)^{\mu+1}}+\frac{(z-\bar{z})^{n-1}}{(t-\bar{z})^{n}}\}\hat{\gamma}_0(t)dt=0, \quad (5.12)$$

$$\lim_{z\to\bar{z}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sum_{\tau=0}^{\rho-1}(-1)^{\rho}\binom{\rho-1}{\tau}\frac{(n+\rho-\tau-2)!}{(n-\rho-\tau-1)!}$$
$$\times(z-\bar{z})^{n-\rho-\tau-1}g_{n+\rho-\tau-1}(z,t)\hat{\gamma}_0(t)dt=0 \quad (5.13)$$

for $2\leqslant 2\rho\leqslant n-1$,

$$\lim_{z\to\bar{z}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\sum_{\tau=0}^{\rho-1}(-1)^{\rho}\binom{\rho-1}{\tau}\frac{(n+\rho-\tau-2)!}{(n-\rho-\tau-2)!}$$
$$\times\frac{(z-\bar{z})^{n-\rho-\tau-2}}{(t-z)^{n+\rho-\tau-1}}+\sum_{\tau=0}^{\rho}(-1)^{n-\tau}\binom{\rho}{\tau}\frac{(n+\rho-\tau-1)!}{(n-\rho-\tau-1)!}$$
$$\times\frac{(z-\bar{z})^{n-\rho-\tau-1}}{(t-\bar{z})^{n+\rho-\tau}}\}\hat{\gamma}_0(t)dt=0 \quad (5.14)$$

for $2\leqslant 2\rho\leqslant n-2$,

$$\lim_{z\to z_0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\sum_{\tau=0}^{min\{\rho-\nu,n-2\nu-1\}}\sum_{\sigma=max\{0,\rho-2\nu\}}^{min\{\rho-\nu-1,n-2\nu-1-\tau\}}(-1)^{n-\rho+\nu}\binom{\rho-\nu}{\tau}$$
$$\times\binom{\rho-\nu-1}{\sigma}\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma)!}$$
$$\times\frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-\bar{z})^{2\nu+\sigma-\rho}}{(t-z)^{n-2\nu+\rho-\tau}}$$
$$+\sum_{\tau=0}^{min\{\rho-\nu-1,n-2\nu-1\}}\sum_{\sigma=max\{0,\rho-2\nu\}}^{min\{\rho-\nu,n-2\nu-1-\tau\}}(-1)^{\rho-\nu-\sigma+\tau}\binom{\rho-\nu-1}{\tau}$$
$$\times\binom{\rho-\nu}{\sigma}\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma)!}$$
$$\times\frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{2\nu+\sigma-\rho}}{(t-\bar{z})^{n-2\nu+\rho-\tau}}\}\gamma_{\nu}(t)dt=0 \quad (5.15)$$

for $1 \leqslant \nu \leqslant \rho - 1,\ 2 \leqslant 2\rho \leqslant n-1,$

$$\lim_{z \to \bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \{ \sum_{\tau=0}^{\rho-1} \binom{\rho}{\tau} (z-\bar{z})^{\rho-\tau} g_{\rho-\tau+1}(z,t) + \sum_{\mu=2\rho+1}^{n-1} \sum_{\tau=0}^{\rho} (-1)^{\mu}$$
$$\times \binom{\rho}{\tau} \{ \binom{\mu-\rho-1}{\rho-1} \frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-z)^{\mu-\rho-\tau+1}} + (-1)^{\mu-\rho-\tau} \binom{\mu-\rho}{\rho}$$
$$\times \frac{(z-\bar{z})^{\mu-\rho-\tau}}{(t-\bar{z})^{\mu-\rho-\tau+1}} \} \} \gamma_{\rho}(t) dt = 0 \tag{5.16}$$

for $2 \leqslant 2\rho \leqslant n-1,$

$$\lim_{z \to \bar{z}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \{ \sum_{\tau=0}^{min\{\rho-\nu, n-2\nu-1\}} \sum_{\sigma=max\{0,\rho-2\nu+1\}}^{min\{\rho-\nu, n-2\nu-1-\tau\}} (-1)^{n-\rho+\nu} \binom{\rho-\nu}{\tau}$$
$$\times \binom{\rho-\nu}{\sigma} \frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma-1)!}$$
$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-\bar{z})^{2\nu-\rho+\sigma-1}}{(t-z)^{n-2\nu+\rho-\tau}}$$
$$+ \sum_{\tau=0}^{min\{\rho-\nu, n-2\nu-1\}} \sum_{\sigma=max\{0,\rho-2\nu\}}^{min\{\rho-\nu, n-2\nu-1-\tau\}} (-1)^{\rho-\nu-\sigma+\tau-1} \binom{\rho-\nu}{\tau}$$
$$\times \binom{\rho-\nu}{\sigma} \frac{(n-2\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma)!}$$
$$\times \frac{(z-\bar{z})^{n-2\nu-\tau-\sigma-1}(t-z)^{2\nu-\rho+\sigma}}{(t-\bar{z})^{n-2\nu+\rho-\tau+1}} \} \gamma_{\nu}(t) dt = 0 \tag{5.17}$$

for $1 \leqslant \nu \leqslant \rho,\ 2 \leqslant 2\rho \leqslant n-2,$

$$
\begin{aligned}
&\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\sum_{\tau=0}^{min\{\rho-\nu-1,n-2\nu-2\}}\ \sum_{\sigma=max\{0,\rho-2\nu-1\}}^{min\{\rho-\nu-1,n-2\nu-2-\tau\}}(-1)^{n+\rho-\nu}\\
&\times\binom{\rho-\nu-1}{\tau}\binom{\rho-\nu-1}{\sigma}\frac{(n-2\sigma+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(2\nu-\rho+\sigma+1)!}\\
&\times\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-2}(t-\bar z)^{2\nu-\rho+\sigma+1}}{(t-z)^{n-2\nu+\rho-\tau-1}}\\
&+\sum_{\tau=0}^{min\{\rho-\nu-1,n-2\nu-2\}}\ \sum_{\sigma=max\{0,\rho-2\nu-1\}}^{min\{\rho-\nu-1,n-2\nu-2-\tau\}}(-1)^{\rho-\nu+\tau-\sigma}\binom{\rho-\nu-1}{\tau}\\
&\times\binom{\rho-\nu-1}{\sigma}\frac{1}{n-\nu}\frac{(n-2\nu+\rho-\tau-1)!}{(n-2\nu-\tau-\sigma-2)!(2\nu-\rho+\sigma+1)!}\\
&\times\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-2}(t-z)^{2\nu-\rho+\sigma+1}}{(t-\bar z)^{n-2\nu+\rho-\tau-2}}\}\hat\gamma_\nu(t)dt=0 \qquad (5.18)
\end{aligned}
$$

for $1\leqslant\nu\leqslant\rho-1,\ 2\leqslant 2\rho\leqslant n-1,$

$$
\begin{aligned}
&\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\sum_{\tau=0}^{\rho-1}\binom{\rho}{\tau}\frac{(z-\bar z)^{\rho-\tau}}{(t-z)^{\rho-\tau+1}}+\sum_{\tau=0}^{\rho}(-1)^{\rho-\tau}\binom{\rho+1}{\tau}\\
&\times\frac{(z-\bar z)^{\rho-\tau+1}}{(t-\bar z)^{\rho-\tau+2}}+\sum_{\mu=2\rho+2}^{n-1}\{\sum_{\tau=0}^{\rho}(-1)^{\mu+1}\binom{\rho}{\tau}\\
&\times\binom{\mu-\rho-1}{\rho}\frac{(z-\bar z)^{\mu-\tau-\rho-1}}{(t-z)^{\mu-\rho-\tau}}+\sum_{\tau=0}^{\rho+1}(-1)^{\tau+\rho}\binom{\rho+1}{\tau}\\
&\times\binom{\mu-\rho-1}{\rho}\frac{(z-\bar z)^{\mu-\rho-\tau}}{(t-\bar z)^{\mu-\rho-\tau+1}}\}\}\hat\gamma_\rho(t)=0 \qquad (5.19)
\end{aligned}
$$

for $2\leqslant 2\rho\leqslant n-2,$

$$\lim_{z\to\bar z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\sum_{\tau=0}^{min\{\rho-\nu-1,n-2\nu-2\}}\sum_{\sigma=max\{0,\rho-2\nu-1\}}^{min\{\rho-\nu,n-2\nu-2-\tau\}}(-1)^{n-\rho}$$
$$\times\binom{\rho-\nu-1}{\tau}\binom{\rho-\nu}{\sigma}\frac{(n-2\nu+\rho-\tau-2)!}{(n-2\nu-\tau-\sigma-2)!(2\nu-\rho+\sigma)!}$$
$$\times\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-2}(t-\bar z)^{2\nu-\rho+\sigma}}{(t-z)^{n-2\nu+\rho-\tau-1}}$$
$$+\sum_{\tau=0}^{min\{\rho-\nu,n-2\nu-1\}}\sum_{\sigma=max\{0,\rho-2\nu-1\}}^{min\{\rho-\nu-1,n-2\nu-1-\tau\}}(-1)^{\rho-\sigma-1}\binom{\rho-\nu}{\tau}$$
$$\times\binom{\rho-\nu-1}{\sigma}\frac{n-2\nu-1}{n-\nu}\frac{(n-2\nu+\rho-\tau)!}{(n-2\nu-\tau-\sigma-1)!(2\nu-\rho+\sigma+1)!}$$
$$\times\frac{(z-\bar z)^{n-2\nu-\tau-\sigma-1}(t-z)^{2\nu-\rho+\sigma+1}}{(t-\bar z)^{n-2\nu+\rho-\tau+1}}\}\hat{\gamma_\nu}(t)dt=0 \tag{5.20}$$

for $1\leqslant\nu\leqslant\rho-1$, $2\leqslant 2\rho\leqslant n-2$.
The solution then is

$$\begin{aligned}
w(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\left(\frac{z-\bar z}{t-\bar z}\right)^n\frac{\gamma_0(t)}{t-z}+\sum_{\mu=1}^{n-1}(-1)^\mu\frac{(z-\bar z)^\mu}{\mu}g_\mu(z,t)\hat{\gamma_0}(t)\\
&+ \sum_{\nu=1}^{[\frac{n-1}{2}]}\{\sum_{\mu=2\nu}^{n-1}(-1)^{\mu-\nu}\frac{(\mu-\nu-1)!}{\mu!(\nu-1)!}\frac{(z-\bar z)^\mu}{(t-z)^{\mu-2\nu+1}}\\
&+ \sum_{\mu=2\nu}^{n-1}(-1)^{\nu-1}\frac{(\mu-\nu)!}{\mu!\nu!}\frac{(z-\bar z)^\mu}{(t-\bar z)^{\mu-2\nu+1}}\}\gamma_\nu(t)\\
&+ \sum_{\nu=1}^{[\frac{n}{2}]-1}\sum_{\mu=2\nu+1}^{n-1}(-1)^{\mu-\nu}\frac{(\mu-\nu-1)!}{\mu!\nu!}(z-\bar z)^\mu g_{\mu-2\nu}(z,t)\hat{\gamma_\nu}(t)\}dt\\
&- \frac{1}{\pi}\int_{\mathbb{H}}G_n(z,\zeta)f(\zeta)d\xi d\eta.
\end{aligned} \tag{5.21}$$

Proof The solution (5.21) is shortly represented as

$$w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{A\gamma_0(t) + B\hat{\gamma}_0(t) + \sum_{\nu=1}^{[\frac{n-1}{2}]} C_\nu\gamma_\nu(t)$$

$$+ \sum_{\nu=1}^{[\frac{n}{2}]-1} D_\nu\hat{\gamma}_\nu(t)\}dt - \frac{1}{\pi}\int_{\mathbb{H}} G_n(z,\zeta)f(\zeta)d\xi d\eta.$$

If the problem has a solution it is representable as (5.21) because of the representation (2.17). From this also the uniqueness follows. It remains to verify that (5.21) is a solution. That it satisfies the differential equation is obvious as all the boundary integrals form a polyharmonic function of order n while the area integral is a particular weak solution of the equation by the respective property of the Pompeiu operator.
Using Lemma 17 for $t_0 \in \mathbb{R}$

$$\lim_{z\to t_0} w(z) = -\lim_{z\to t_0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} A\gamma_0(t)dt$$

is seen. Rewriting A as

$$A = -\frac{z-\bar{z}}{|t-z|^2} + \sum_{\nu=1}^{n-1}\frac{(z-\bar{z})^\nu}{(t-\bar{z})^{\nu+1}}$$

and using (3.10)

$$\lim_{t\to t_0} w(z) = \gamma_0(t)$$

if and only if (5.8) holds.
Similarly

$$\lim_{z\to t_0} w_{\bar{z}}(z) = -\lim_{z\to t_0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{A_{\bar{z}}\gamma_0(t) + B_{\bar{z}}\hat{\gamma}_0(t)\}dt = \hat{\gamma}_0(t_0)$$

if and only if (5.9) and (5.12) are satisfied.
Moreover, for $2 \leqslant 2\rho \leqslant n-1$

$$\lim_{z\to t_0}(\partial_z\partial_{\bar z})^\rho w(z) = -\lim_{z\to t_0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{(\partial_z\partial_{\bar z})^\rho A\gamma_0(t) + (\partial_z\partial_{\bar z})^\rho B\hat\gamma_0(t)$$

$$+\sum_{\nu=1}^{\rho}(\partial_z\partial_{\bar z})^\rho C_\nu\gamma_\nu(t) + \sum_{\nu=1}^{\rho-1}(\partial_z\partial_{\bar z})^\rho D_\nu\hat\gamma_\nu(t)\}dt = \gamma_\rho(t_0)$$

if and only if (5.10), (5.13), (5.15), (5.16) and (5.18) are valid.
Also for $2 \leqslant 2\rho \leqslant n-2$

$$\lim_{z\to t_0}\partial_z^\rho\partial_{\bar z}^{\rho+1}w(z) = -\lim_{z\to t_0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\partial_z^\rho\partial_{\bar z}^{\rho+1}A\gamma_0(t) + \partial_z^\rho\partial_{\bar z}^{\rho+1}B\hat\gamma_0(t)$$

$$+\sum_{\nu=1}^{\rho}\partial_z^\rho\partial_{\bar z}^{\rho+1}C_\nu\gamma_\nu(t) + \sum_{\nu=1}^{\rho}\partial_z^\rho\partial_{\bar z}^{\rho+1}D_\nu\hat\gamma_\nu(t)\}dt = \hat\gamma_\rho(t_0)$$

if and only if (5.11), (5.14), (5.17), (5.19) and (5.20) hold.

Remark 11 This Dirichlet problem is uniquely solvable only for $n = 1$. For $1 < n$ the conditions (5.8) - (5.11) are satisfied e.g. if γ_0 is the boundary value of a function analytic in $\mathbb{H}$. All other solvability conditions are e.g. satisfied if the respective boundary functions are as well boundary values of functions analytic in $\mathbb{H}$ as antianalytic in the complement $\mathbb{C}\setminus\overline{\mathbb{H}}$.
If e.g. ω is an entire analytic function, then denoting $\omega(t) = \gamma(t)$ for $t \in \mathbb{R}$

$$\lim_{\substack{z\to t,\\ z\in\mathbb{H}}}\omega(z) = \gamma(t)$$

for $t \in \mathbb{R}$. Moreover, the function $\omega(\bar z)$ for $z \in \mathbb{C}$ is antianalytic satisfying

$$\lim_{\substack{\bar z\to t,\\ z\in\mathbb{H}}}\omega(\bar z) = \lim_{\substack{\zeta\to t,\\ \zeta\in\mathbb{C}\setminus\overline{\mathbb{H}}}}\omega(\zeta) = \gamma(t).$$