3 Basic boundary value problems for analytic function in the upper half plane

3.1 Poisson representation formulas for the half plane

Let f be an analytic function of z throughout the half plane $Imz > 0$, continuous such that f satisfies the growth property

$$\lim_{R\to\infty} M(R, f) = 0, \tag{3.1}$$

where $M(R, f) = max_{\substack{|z|=R \\ 0\leq Imz}} |f(z)|$.

For any fixed point z above the real axis let C_R denote the upper half of the positively oriented circle of radius R centered at the origin, where $R > |z|$. Then, according to the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(s)ds}{s-z} + \frac{1}{2\pi i}\int_{-R}^{R} \frac{f(t)dt}{t-z} \tag{3.2}$$

We find that the first of these integrals approaches 0 as R tends to ∞ consequently,

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}dt, \tag{3.3}$$

where $Imz > 0$ and the integral is understood as a Cauchy principal value integral. Representation (3.3) is the *Cauchy integral formula for the half plane* $Imz > 0$.

When the point z lies below the real axis, the right-hand side of equation (3.2) is zero; hence integral (3.3) is zero for such a point. Consequently, when z is above the real axis, we have the formula

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left(\frac{1}{t-z} + \frac{c}{t-\bar{z}}\right) f(t)dt, \tag{3.4}$$

where c is an arbitrary constant and $Imz > 0$.

In the two cases $c = -1$ and $c = 1$ this formula reduces, respectively, to

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)}{|t-z|^2}dt, \tag{3.5}$$

where $y > 0$ and to

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(t-x)f(t)}{|t-z|^2} dt, \tag{3.6}$$

where $y > 0$. Here $z = x + iy$ is used.

If $f(z) = u(x,y) + iv(x,y)$, it follows from formulas (3.5) and (3.6) that the harmonic functions u and v are represented in the half plane $y > 0$ in terms of the boundary values of u by the formulas

$$\begin{aligned} u(x,y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(t,0)}{|t-z|^2} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(t,0)}{(t-x)^2 + y^2} dt \end{aligned} \tag{3.7}$$

where $y > 0$,

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-t)u(t,0)}{(t-x)^2 + y^2} dt \tag{3.8}$$

where $y > 0$

Formula (3.7) is known *as the Poisson integral formula for the upper half plane* and

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(t,0)}{t-z} dt + ic_0$$

as the Schwarz integral formula, where c_0 is an arbitrary real constant. The constant c_0 can be determined e.g. by $\mathrm{Im} f(i) = c$. Then

$$c = \mathrm{Im} \frac{1}{\pi i} \int_{-\infty}^{\infty} u(t,0) \frac{t+i}{t^2+1} dt + c_0 = -\frac{1}{\pi} \int_{-\infty}^{\infty} u(t,0) \frac{t}{t^2+1} + c_0$$

and

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} u(t,0) \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) dt + ic.$$

3.2 Schwarz problem for the half plane

Let F denote a real-valued function of x that is bounded for all x and continuous except for at most a finite number of finite jumps. When $y \geqq \varepsilon$ and $|x| \leqq \frac{1}{\varepsilon}$, where ε is any positive constant, the integral

$$I(x,y)=\int_{-\infty}^{\infty}\frac{F(t)dt}{(t-x)^2+y^2}$$

converges uniformly with respect to x and y, as do the integrals of the partial derivatives of the integrand with respect to x and y. Each of these integrals is the sum of a finite number of improper or definite integrals over intervals where F is continuous; hence the integrand of each component integral is a continuous function of t, x and y when $y \geqq \varepsilon$. Moreover, each partial derivative of $I(x,y)$ is represented by the integral of the corresponding derivative of the integrand whenever $y > 0$.

We write $U(x,y) = yI(x,y)/\pi$. Thus U is the Poisson integral transform of F, suggested by equation (3.7).

$$U(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yF(t)}{(t-x)^2+y^2}dt, \tag{3.9}$$

where $y > 0$.

Except for the factor $1/\pi$, the kernel here is $y/|t-z|^2$. It is the imaginary part of the function $1/(t-z)$ which is analytic in z when $y > 0$. It follows that the kernel is harmonic, and so it satisfies Laplace's equation in x and y. Because the order of differentiation and integration can be interchanged, function (3.9) then satisfies that Laplace equation. Consequently, *U is harmonic when $y > 0$.*

To prove that

$$\lim_{\substack{y\to 0\\ y>0}} U(x,y) = F(x) \tag{3.10}$$

for each fixed x at which F is continuous, we substitute $t = x + y\tan r$ in formula (3.9) and write

$$U(x,y)=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}F(x+y\tan r)dr, \tag{3.11}$$

where $y > 0$.

If

$$G(x,y,r)=F(x+y\tan r)-F(x)$$

and α is some small positive constant, then

$$\pi[U(x,y)-F(x)] = \int_{-\pi/2}^{\pi/2} G(x,y,r)dr = I_1(y)+I_2(y)+I_3(y), \quad (3.12)$$

where

$$I_1 = \int_{-\pi/2}^{-\pi/2+\alpha} G(x,y,r)dr$$

$$I_2 = \int_{-\pi/2+\alpha}^{\pi/2-\alpha} G(x,y,r)dr$$

$$I_3 = \int_{\pi/2-\alpha}^{\pi/2} G(x,y,r)dr.$$

If M denotes an upper bound for $|F(x)|$, then $|G(x,y,r)| \leqq 2M$. For a given positive number ε we select α so that $6M\alpha < \varepsilon$; then

$$|I_1(y)| \leqq 2M\alpha < \varepsilon/3$$

and

$$|I_3(y)| \leqq 2M\alpha < \varepsilon/3$$

We next show that corresponding to ε there is a positive number δ such that

$$|I_2(y)| < \varepsilon/3,$$

when $0 < y < \delta$.

Since F is continuous at x, there is a positive number γ such that

$$|G(x,y,r)| < \frac{\varepsilon}{3\pi},$$

when $0 < y|\tan r| < \gamma$.

Note that the maximum value of $|\tan r|$ as r ranges between $-\pi/2+\alpha$ and $\pi/2-\alpha$ is $\tan(\pi/2-\alpha) = \cot\alpha$. Hence if we write $\delta = \gamma\tan\alpha$, it follows that

$$|I_2(y)| < \frac{\varepsilon}{3\pi}(\pi - 2\alpha) < \frac{\varepsilon}{3},$$

when $0 < y < \delta$.
We have thus shown that

$$|I_1(y)| + |I_2(y)| + |I_3(y)| < \varepsilon,$$

when $0 < y < \delta$.
Condition (3.10) now follows from this result and equation (3.12). On the basis of (3.9) then for any real constant c the function

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t - z}dt + ic$$

therefore solves *the Schwarz problem for analytic functions for the half plane* $y > 0$ with the boundary condition

$$\lim_{\substack{z \to t \\ 0 < Imz}} \mathrm{Re} f(z) = F(t), \quad t \in \mathbb{R}.$$

It is evident from the form (3.11) of formula (3.9) that $|U(x,y)| \leqq M$ in the half plane where M is an upper bound of $|F(x)|$; that is, U is bounded. We note that $U(x,y) = F_0$ when $F(x) = F_0$, where F_0 is a constant.
According to formula (3.8) of the preceding section, under certain conditions on F the function

$$V(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-t)F(t)}{(t-x)^2 + y^2}dt + c, \tag{3.13}$$

where c is an arbitrary real constant and $y > 0$, is a harmonic conjugate of the function U given by formula (3.9). Actually, *formula (3.13) furnishes a harmonic conjugate of* U *if* F *is everywhere continuous, except for at most a finite number of finite jumps, and if* F *satisfies the growth property* $|x^k F(x)| < M$ *for some* $k > 0$. For under those conditions we find that U and V satisfy the Cauchy-Riemann equations when $y > 0$.

Remark 9 The boundary behaviour (3.10) can be proved also in the following way. Consider for $z \in \mathbb{H}$

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t-z} dt.$$

We assert, that

$$\lim_{z \to t_0} \mathrm{Re} f(z) = F(t_0),$$

where $t_0 \in \mathbb{R}$.

Proof Denoting $u = \mathrm{Re} f$, then

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yF(t)}{|t-z|^2} dt.$$

From (3.7) applied to $u(t, y) \equiv 1$ it follows

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{|t-z|^2} dt = 1.$$

Hence,

$$u(z) - F(t_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} y \frac{F(t) - F(t_0)}{|t-z|^2} dt.$$

Let

$$|t_0 - x| < \frac{1}{2}\delta$$

and $0 < y < \frac{1}{2}\delta$. By the continuity of F at the point t_0 for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$|F(t) - F(t_0)| < \varepsilon,$$

for $|t - t_0| < \delta$. Decomposing

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{t_0-\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^{\infty}$$

and observing the estimates

$$\left|y\int_{t_0-\delta}^{t_0+\delta}\frac{F(t)-F(t_0)}{|t-z|^2}dt\right|\le y\int_{t_0-\delta}^{t_0+\delta}\frac{\varepsilon dt}{|t-z|^2}\le\varepsilon\int_{-\infty}^{\infty}\frac{ydt}{|t-z|^2}=\varepsilon$$

and

$$\begin{aligned}
&\left|\left(\int_{-\infty}^{t_0-\delta}+\int_{t_0+\delta}^{\infty}\right)\left(y\frac{F(t)-F(t_0)}{|t-z|^2}\right)dt\right| \\
&\le y2M\left(\int_{-\infty}^{t_0-\delta}+\int_{t_0+\delta}^{\infty}\right)\frac{dt}{|t-z|^2} \\
&= 2My\left(\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right)\frac{dt}{(|t|-|t_0-x|)^2} \\
&= 4My\frac{1}{\delta-|t_0-x|}\le\frac{8My}{\delta}
\end{aligned}$$

we have

$$|u(z)-F(t_0)|\le\varepsilon+\frac{8My}{\delta}.$$

Hence,

$$\lim_{z\to t_0}u(z)=F(t_0).$$

3.3 Dirichlet problem for the half plane

Theorem 13 *Let w be an analytic function in $\mathbb{H}$, and a function $G\in L_2(\mathbb{R};\mathbb{C})\cap C(\mathbb{R};\mathbb{C})$ be given. Then*

$$w(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}G(t)\frac{1}{t-z}dt \tag{3.14}$$

is the uniquely given solution to the Dirichlet problem $w=G$ on $\mathbb{R}$ if and only if for $z\in\mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}G(t)\frac{dt}{t-\bar{z}}=0. \tag{3.15}$$

Proof The proof of this theorem consists of two parts.
1)The sufficiency of (3.15) follows at once from subtracting (3.15) from (3.14) leading to

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-\bar{z}}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} G(t)\frac{y}{(t-x)^2+y^2}dt$$

Thus for $z \longrightarrow t_0$ and $t_0 \in \mathbb{R}$, $\lim_{z\rightarrow t_0} w(z) = G(t_0)$ follows.
2)The formula (3.15) is shown to be necessary. Let w be a solution to the Dirichlet problem. Then w is an analytic function in $\mathbb{H}$ having continuous boundary values

$$\lim_{z\rightarrow t_0} w(z) = G(t_0), \tag{3.16}$$

where $t_0 \in \mathbb{R}$.
Consider for $z \notin \mathbb{R}$ the function

$$\widetilde{w}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-z}$$

and

$$\widetilde{w}(\bar{z}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-\bar{z}},$$

where $z \in \mathbb{H}$.
From

$$\widetilde{w}(z) - \widetilde{w}(\bar{z}) = \frac{1}{\pi}\int_{-\infty}^{\infty} G(t)\frac{ydt}{(t-x)^2+y^2} \tag{3.17}$$

and the properties of the Poisson kernel [see formula (3.7)]

$$\lim_{z\rightarrow t_0}(w(z) - \widetilde{w}(\bar{z})) = G(t_0) \tag{3.18}$$

follows. The formula (3.17) tends to $G(t_0)$ when y tends to 0 and x tends to t_0. Then for the validity of

$$\lim_{z\to t_0} w(z) = G(t_0),$$

where $t_0 \in \mathbb{R}$, follows

$$\lim_{z\to t_0} \widetilde{w}(\bar{z}) = 0.$$

Moreover, from

$$\widetilde{w}(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(|\zeta|t)\frac{dt}{t-\frac{\zeta}{|\zeta|}},$$

$$|\widetilde{w}(\zeta)| \leqslant \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}|G(|\zeta|t)|^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{dt}{|t-\frac{\zeta}{|\zeta|}|^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2\pi|\zeta|^{\frac{1}{2}}}\left(\int_{-\infty}^{\infty}|G(t)|^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{dt}{|t-\frac{\zeta}{|\zeta|}|^2}\right)^{\frac{1}{2}}$$

it is seen

$$\lim_{\substack{\zeta\to\infty\\ Im\zeta<0}} \widetilde{w}(\zeta) = 0.$$

Thus

$$\widetilde{w}(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-\zeta}$$

is an analytic function in $Im\zeta < 0$ with vanishing boundary values on $\mathbb{R}$. As also

$$\lim_{\substack{\zeta\to\infty\\ Im\zeta<0}} \widetilde{w}(\zeta) = 0$$

then $\widetilde{w}$ vanishes identically in $Im\zeta < 0$.
This in fact was required to prove.

3.4 Neumann problem for the half plane

Let $w_{\bar{z}} = 0$ in $\mathbb{H}$ and $\partial_y w = iw' = iG$ on $\mathbb{R}$, $w(i) = c$, where $G \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, satisfying $G(t)\log(1+t^2) \in L_1(\mathbb{R};\mathbb{C})$ and for $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-\bar{z}} = 0$$

i.e.

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \overline{G(t)}\frac{dt}{t-z} = 0,$$

and $c \in \mathbb{C}$.
Then the Dirichlet problem $w' = G$ on $\mathbb{R}$ is solvable by, see 3.3,

$$w'(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yG(t)}{|t-z|^2}dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\left(\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right)dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-z}$$

if and only if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-\bar{z}} = 0$$

for $z \in \mathbb{H}$. Integration leads to

$$w(z) = c - \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\log|t-z|^2 dt + \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\log(1+t^2)dt.$$

From

$$w_{\bar{z}}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-\bar{z}} = 0$$

w is seen to be analytic. Moreover,

$$w^{'}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\frac{dt}{t-z} = \frac{1}{\pi}\int_{-\infty}^{\infty} G(t)\frac{ydt}{|t-z|^2}, \quad w(i) = c.$$

Summing up we have the following theorem.

Theorem 14 *The Neumann problem $w_{\bar{z}} = 0$ in $\mathbb{H}$, $\partial_y w = iG$ on $\mathbb{R}$, $w(i) = c$ with $G \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ and $(1+t^2)G(t) \in L_1(\mathbb{R};\mathbb{C})$ is solvable if and only if*

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G(t)}{t-\bar{z}}dt = 0$$

for $z \in \mathbb{H}$. Then the solution is

$$w(z) = c + \frac{1}{2\pi i}\int_{-\infty}^{\infty} G(t)\log|\frac{t-i}{t-z}|^2 dt.$$

3.5 Robin boundary value problem

At first a particular case is studied, see [13].
Special Robin problem: Find an analytic function in the upper half plane $\mathbb{H}$, satisfying the boundary condition

$$w + \partial_\nu w = \gamma \tag{3.19}$$

on $\mathbb{R}$ for given $\gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$.
Note that in our case for the boundary condition (3.19) we have

$$\partial_\nu w = \partial_y w = iw^{'}.$$

Thus (3.19) can be written as

$$w + iw^{'} = \gamma$$

on $\mathbb{R}$. As we know from Section 3.3

$$w(z) + iw^{'}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma(t)\frac{dt}{t-z} = \frac{1}{\pi}\int_{-\infty}^{\infty} \gamma(t)\frac{ydt}{|t-z|^2} \tag{3.20}$$

if and only if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\bar{z}}=0,$$

for $z \in \mathbb{H}$. The solution to the differential equation (3.20) is

$$w(z)=C_0e^{iz}-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z\frac{e^{i(z-s)}\mathrm{Im}s}{|t-s|^2}dsdt. \tag{3.21}$$

Remark 10 As the right-hand side of (3.20) is analytic the integral in (3.21) is analytic in $\mathbb{H}$.

Verification of (3.21). By differentiating (3.21)

$$w^{'}(z)=iC_0e^{iz}+\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z\frac{e^{i(z-s)}\mathrm{Im}s}{|t-s|^2}dsdt-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{ydt}{|t-z|^2}$$

follows. This gives together with (3.21)

$$w(z)+iw^{'}(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}y\gamma(t)\frac{dt}{|t-z|^2}.$$

In order to determine C_0 consider

$$w(i)=C_0e^{-1},$$

hence

$$C_0=ew(i),$$

$$w(z)=w(i)e^{1+iz}-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z\frac{e^{i(z-s)}\mathrm{Im}s}{|t-s|^2}dsdt.$$

General Robin problem: Find an analytic function in the upper half plane $\mathbb{H}$, satisfying the boundary condition

$$\alpha w+\partial_\nu w=\gamma \quad \text{on} \quad \mathbb{R},$$

where $\alpha \in C(\mathbb{R};\mathbb{C})$ and $w(0) = C_0$. This boundary condition on $\mathbb{R}$ is

$$\alpha w + iw^{'} = \gamma,$$

which is

$$w^{'} - i\alpha w = -i\gamma \quad \text{on} \quad \mathbb{R}. \tag{3.22}$$

If $\gamma = 0$, then

$$w(t) = Ce^{i\int_0^t \alpha(\zeta)d\zeta}$$

We can verify this for $t \in \mathbb{R}$:

$$w^{'}(t) = iC\alpha(t)e^{i\int_0^t \alpha(\zeta)d\zeta} = i\alpha(t)w(t).$$

If $\gamma \neq 0$, then by the method of varying the constant for $t \in \mathbb{R}$

$$w(t) = C(t)e^{i\int_0^t \alpha(\zeta)d\zeta}$$

and

$$w^{'}(t) = C^{'}(t)e^{i\int_0^t \alpha(\zeta)d\zeta} + i\alpha(t)C(t)e^{i\int_0^t \alpha(\zeta)d\zeta}$$

so that on $\mathbb{R}$

$$w^{'}(t) - i\alpha(t)w(t) = C^{'}(t)e^{i\int_0^t \alpha(\zeta)d\zeta} + iC(t)\alpha(t)e^{i\int_0^t \alpha(\zeta)d\zeta}$$

$$-i\alpha C(t)e^{i\int_0^t \alpha(\zeta)d\zeta} = -i\gamma(t).$$

Thus

$$C^{'}(t) = -i\gamma(t)e^{-i\int_0^t \alpha(\zeta)d\zeta}.$$

By integration

$$C(t) = -i\int_0^t \gamma(s)e^{-i\int_0^s \alpha(\zeta)d\zeta}ds + C_0$$

follows with arbitrary $C_0 \in \mathbb{C}$. Hence, on $\mathbb{R}$

$$w(t) = [-i\int_0^t \gamma(s)e^{-i\int_0^s \alpha(\zeta)d\zeta}ds + C_0]e^{i\int_0^t \alpha(\zeta)d\zeta}$$

$$= C_0 e^{i\int_0^t \alpha(\zeta)d\zeta} - i\int_0^t \gamma(s)e^{i\int_s^t \alpha(\zeta)d\zeta}ds.$$

These are the Dirichlet boundary values for the analytic function w. The development of Section 3.3 leads to the next result.

Theorem 15 *The Robin problem $w_{\bar{z}} = 0$ in $\mathbb{H}$, $\alpha w + \partial_\nu w = \gamma$ on $\mathbb{R}$, $w(0) = C_0$ for $\alpha, \gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $w(0) = C_0$, $C_0 \in \mathbb{C}$, is uniquely solvable if and only if*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{C_0 e^{i\int_0^t \alpha(s)ds} - i\int_0^t \gamma(\sigma)e^{i\int_\sigma^t \alpha(s)ds}d\sigma\}\frac{dt}{t-\bar{z}} = 0,$$

where $z \in \mathbb{H}$. The solution is given by

$$w(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\{C_0 e^{i\int_0^t \alpha(s)ds} - i\int_0^t \gamma(\sigma)e^{i\int_\sigma^t \alpha(s)ds}d\sigma\}\frac{ydt}{|t-z|^2}.$$

A special case can be treated in another way. If $\alpha(t)$ are the boundary values of a function α analytic in $\mathbb{H}$, then

$$w' - i\alpha w$$

is analytic in $\mathbb{H}$ so that

$$w' - i\alpha w = -i\gamma$$

on $\mathbb{R}$ is a Dirichlet problem for an analytic function on $\mathbb{H}$. Thus

$$\begin{aligned} w'(z) - i\alpha(z)w(z) &= \frac{-i}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-z} \\ &= -\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{ydt}{|t-z|^2} \end{aligned} \tag{3.23}$$

is an analytic solution in $\mathbb{H}$ if and only if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\bar{z}}=0,$$

where $z\in\mathbb{H}$.
If $\gamma=0$, then $w(z)=Ce^{i\int_i^z\alpha(\zeta)d\zeta}$.
If $\gamma\neq 0$,then the solution to (3.28) is of the form $w(z)=C(z)e^{i\int_i^z\alpha(\zeta)d\zeta}$.
Therefore

$$w^{'}=C^{'}e^{i\int_i^z\alpha(\zeta)d\zeta}+i\alpha Ce^{i\int_i^z\alpha(\zeta)d\zeta},$$

$$w^{'}-i\alpha w=C^{'}e^{i\int_i^z\alpha(\zeta)d\zeta}+i\alpha Ce^{i\int_i^z\alpha(\zeta)d\zeta}-i\alpha Ce^{i\int_i^z\alpha(\zeta)d\zeta}$$

$$=-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{ydt}{|t-z|^2},$$

i.e.

$$C^{'}=-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{ydt}{|t-z|^2}e^{-i\int_i^z\alpha(\zeta)d\zeta}.$$

By integration

$$C(z)=C_0-\frac{i}{\pi}\int_i^z\int_{-\infty}^{\infty}\frac{\mathrm{Im}s\gamma(t)}{|t-s|^2}e^{-i\int_i^s\alpha(\zeta)d\zeta}dtds$$

and

$$w(z)=C_0e^{i\int_i^z\alpha(\zeta)d\zeta}-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z\frac{\mathrm{Im}s}{|t-s|^2}e^{i\int_s^z\alpha(\zeta)d\zeta}dsdt$$

follow. Obviously $w(i)=C_0$.
This can be easily checked by

$$w^{'}=C_0i\alpha e^{i\int_i^z\alpha(\zeta)d\zeta}-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{y\gamma(t)}{|t-z|^2}dt$$

$$-\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z \frac{i\alpha(z)\mathrm{Im}s}{|t-s|^2}e^{i\int_s^z \alpha(\zeta)d\zeta}dsdt$$
$$= i\alpha w - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{y\gamma(t)}{|t-z|^2}dt$$

from which

$$w' - i\alpha w = -i\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\gamma(t)}{|t-z|^2}dt$$

follows.

Theorem 16 *The Robin problem $w_{\bar{z}} = 0$ in $\mathbb{H}$, $\alpha w + \partial_\nu w = \gamma$ on $\mathbb{R}$, $w(i) = C_0$ for $\alpha \in \mathcal{O}(\mathbb{H}) \cap C(\overline{\mathbb{H}};\mathbb{C})$, $\gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ and $C_0 \in \mathbb{C}$ is uniquely solvable if and only if*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\gamma(t)}{t-\bar{z}}dt = 0$$

for $z \in \mathbb{H}$. The solution is given by

$$w(z) = C_0 e^{i\int_i^z \alpha(\zeta)d\zeta} - \frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z \frac{1}{|t-s|^2}e^{i\int_s^z \alpha(\zeta)d\zeta}dsdt. \quad (3.24)$$

Here $\mathcal{O}(\mathbb{H})$ denotes the set of analytic functions in $\mathbb{H}$.

3.6 Neumann problem for harmonic functions in the upper half plane

The Neumann problem for the Poisson equation is solved in Section 2.4. Here this problem is treated again on the basis of the Poisson formula for the Laplace equation.

Let $G(x)$ be continuous for all real x, except for at most a finite number of finite jumps, and $xG(x) \in L_2(\mathbb{R};\mathbb{C})$. For each fixed real number t the function $\log|z-t|$ is harmonic in the half plane $\mathrm{Im}z > 0$. Consequently, the function

$$\begin{aligned} U(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\log|z-t|G(t)dt + U_0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\log[(t-x)^2+y^2]G(t)dt + U_0, \end{aligned} \quad (3.25)$$

where $y > 0$ and U_0 is a real constant, is harmonic in that half plane. Formula (3.25) is given with the Poisson formula (3.9) in mind; for it follows from formula (3.25) that

$$U_y(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yG(t)}{(t-x)^2 + y^2} dt, \tag{3.26}$$

where $y > 0$.
In view of equations (3.9) and (3.10), then,

$$\lim_{y \to 0} U_y(x, y) = G(x), y > 0, \tag{3.27}$$

at each point x where G is continuous.
Integral formula (3.26) therefore solves *the Neumann problem for the half plane* $y > 0$ with the boundary condition $\partial_y U = G$. But we have not presented conditions on G that are sufficient to ensure that the harmonic function U is bounded as $|z|$ increases.
When G is an even function, formula (3.25) can be written as

$$U(x, y) = \frac{1}{2\pi} \int_{0}^{\infty} \log \frac{(t-x)^2 + y^2}{(t+x)^2 + y^2} G(t) dt + U_0, \tag{3.28}$$

where $x > 0$, $y > 0$.
This represents a function which is harmonic in particular in the first quadrant $x > 0$, $y > 0$ and which satisfies the boundary conditions

$$U(0, y) = U_0, y > 0$$

$$\lim_{\substack{y \to 0 \\ y > 0}} U_y(x, y) = G(x), x > 0.$$